Fine Structure Constant $\alpha \sim 1/137.036$ and Blackbody Radiation Constant $\alpha_{\mathbf{R}} \sim 1/157.555$

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Abstract

The fine structure constant $\alpha = \mathbf{e}^2/\hbar c \approx 1/137.036$ and the blackbody radiation constant $\alpha_R = \mathbf{e}^2(a_R/k_B^4)^{1/3} \approx 1/157.555$ are linked by prime numbers. The blackbody radiation constant is a new method to measure the fine structure constant. It also links the fine structure constant to the Boltzmann constant.

Planck and Einstein noted respectively in 1905 and 1909 that $\mathbf{e}^2/c \sim h$ have the same order and dimension.[1, 2] This was before the introduction of the fine structure constant $\alpha = \mathbf{e}^2/\hbar c$ by Sommerfeld in 1916.[3] Therefore, the search for a mathematical relationship between $\mathbf{e}^2/c \sim h$ was started from the blackbody radiation. The Stefan-Boltzmann law states that the radiative flux density or irradiance is $J = \sigma T^4$ [$\mathrm{erg \cdot cm^{-2} \cdot s^{-1}}$] in CGS units or [$\mathrm{W \cdot m^{-2}}$] in SI units. From the Planck law, the Stefan-Boltzmann constant $\sigma = 5.670400(40) \times 10^{-5}$ [$\mathrm{erg \cdot cm^{-2} K^{-4} s^{-1}}$] is

$$\begin{aligned} \sigma &= 2\pi \int_0^\infty \frac{x^3 dx}{e^x - 1} \frac{c k_B^4}{(hc)^3} = \frac{2\pi^5}{15} \frac{c k_B^4}{(hc)^3} \\ &= 2\pi \Gamma(4) \zeta(4) \frac{c k_B^4}{(hc)^3} = \frac{4^2 \pi^5}{5!} \frac{c k_B^4}{(hc)^3} \end{aligned} \tag{1}$$

The Stefan-Boltzmann law can be expressed as the volume energy density of the blackbody $\varepsilon_T = \boldsymbol{a}_R T^4$ [$\mathrm{erg \cdot cm^{-3}}$], where the radiation density constant $\boldsymbol{a}_R$ is linked to the Stefan-Boltzmann constant

$$\boldsymbol{a}_R = \frac{4\sigma}{c} = \frac{4^3 \pi^5}{5!} \frac{k_B^4}{(hc)^3} \tag{2}$$

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In 1914, Lewis and Adams noticed that the dimension of the radiation density constant divided by the 4$^{\text{th}}$ power Boltzmann constant $\boldsymbol{a}_R/k_B^4$ is (energy $\times$ length)$^{-3}$, while $\mathbf{e}^2$ is (energy $\times$ length). However, they obtained an incorrect result equivalent to $\alpha^{-1} = \hbar c/\mathbf{e}^2 = 32\pi \left(\pi^5/5!\right)^{1/3} = 137.348$.[4] In 1915, Allen rewrote it as $\alpha = \mathbf{e}^2/\hbar c = (15/\pi^2)^{1/3}/(4\pi)^2$.[5]

In CGS units, $\mathbf{e}^2 = \left(4.80320427(12) \times 10^{-10}\right)^2$ [erg·cm], $\boldsymbol{a}_R = 7.56576738 \times 10^{-15}$ [erg·cm^{-3}K^{-4}], and $k_B^4 = \left(1.3806504(24) \times 10^{-16}\right)^4$ [erg^{4}K^{-4}]. We get the experimental dimensionless constant[6]

$$\begin{aligned} \alpha_R &= \mathbf{e}^2 \left(\frac{\boldsymbol{a}_R}{k_B^4}\right)^{1/3} = \frac{1}{\mathbf{157.5548787}} \\ &= \mathbf{0.00634699482} \end{aligned} \tag{3}$$

This is the blackbody radiation constant α_R, which is on the same order of the fine structure constant α= $\mathbf{e}^2/\hbar c$

$$\begin{aligned} \alpha_R &= \frac{2}{\pi}\left(\frac{\pi^5}{5!}\right)^{1/3} \alpha = \left(\frac{\pi^2}{15}\right)^{1/3} \alpha \\ &= \left(\frac{\Gamma(4)\zeta\left(4\right)}{\pi^2}\right)^{1/3} = 0.8697668 \cdot \alpha \end{aligned} \tag{4}$$

Therefore, $\alpha_R \neq \alpha$, both α and α_R are experimental results incapable of producing the α math formula. Physically, the fine structure constant α is obtained from the atomic *discrete* spectra, while the blackbody radiation constant α_R is obtained from the thermal radiation of a 3D cavity in the *continuous* spectra. However, their relationship can be given by the Riemann zeta-function or by the modification of Euler's product formula (1737)

$$\begin{aligned} \frac{\alpha_R^3}{\alpha^3} &= \frac{\pi^2}{15} = \frac{\zeta\left(4\right)}{\zeta\left(2\right)} = \frac{\sum_{n=1}^{\infty} \frac{1}{n^4}}{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \frac{\prod_{\mathrm{p}} \left(1 - \frac{1}{\mathrm{p}^2}\right)}{\prod_{\mathrm{p}} \left(1 - \frac{1}{\mathrm{p}^4}\right)} = \prod_{\mathrm{p}} \left(\frac{\mathrm{p}^2}{\mathrm{p}^2+1}\right) \\ &= \frac{2^2}{(2^2+1)} \frac{3^2}{(3^2+1)} \frac{5^2}{(5^2+1)} \frac{7^2}{(7^2+1)} \cdots = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdots \end{aligned} \tag{5}$$

where the Euler product extends over all the *prime* numbers. In other words, the fine structure constant and the blackbody radiation constant can be linked by the prime numbers. From (5), the Stefan-Boltzmann law as the volume energy density of the blackbody ε_T is related to the fine structure constant α and the oscillating charge $\mathbf{e}^2$ with the different resonating frequencies in a cavity

$$\begin{aligned}\varepsilon_T &= \boldsymbol{a}_R T^4 = \frac{4\sigma}{c}T^4 = \frac{4^3\pi^5}{5!}\frac{k_B^4}{(hc)^3}T^4 \\ &= \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 T^4 = \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4\end{aligned} \tag{6}$$

and the radiative flux density is

$$J = \sigma T^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{c}{4} k_B^4 T^4 = \frac{c}{4}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{7}$$

and the total brightness of a blackbody is

$$B = \frac{J}{\pi} = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{c}{4\pi} k_B^4 T^4 = \frac{c}{4\pi}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{8}$$

and the inner wall pressure of the blackbody cavity is

$$P = \frac{4\sigma}{3c}T^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{1}{3} k_B^4 T^4 = \frac{1}{3}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{9}$$

According to the Bose-Einstein model of photon-gas,[7] the free energy of the thermodynamics is

$$F = -PV = -\frac{4\sigma}{3c}VT^4 = -\frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{V}{3} k_B^4 T^4 = -\frac{V}{3}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{10}$$

and the total radiation energy is

$$E = -3F = 3PV = \frac{4\sigma}{c}VT^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 V k_B^4 T^4 = V\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{11}$$

where the photon gas $E = 3PV$ is the same as the extreme relativistic electron gas, and the entropy is

$$S = -\frac{\partial F}{\partial T} = \frac{16\sigma}{3c}VT^3 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{4V}{3} k_B^4 T^3 = \frac{4V}{3}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^3 \tag{12}$$

and the specific heat of the radiation is

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{16\sigma}{c}VT^3 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 4V k_B^4 T^3 = 4V\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^3 \tag{13}$$

Landau assumed that the volume V in (10)∼(13) must be sufficiently large in order to change from *discrete* to *continuous* spectra. Experimentally, solids or dense-gas have the continuous spectra, and hot low-density gas emits the discrete atomic spectra. The pattern of Planck spectra is given by $f(x) = x^3/(e^x - 1)$

where photon $h\nu$ is hidden in $x = h\nu/k_BT$. The *photon* integral in (1) is equal to a dimensionless constant (Fig. 1)

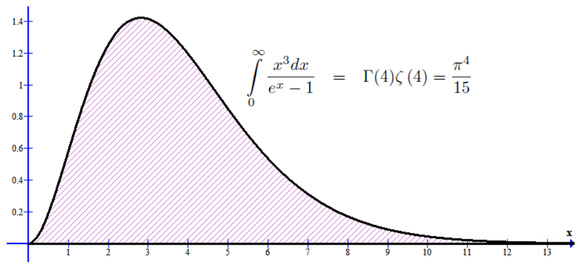

Figure 1: Photon integral is a dimensionless number $\Gamma(4)\zeta(4) = \frac{\pi^4}{15} = 6.4939394$

$$\begin{aligned}
\int_0^\infty \frac{x^3dx}{e^x-1} &= \Gamma(4)\zeta(4) = \frac{\pi^4}{15} = 2\cdot 3\cdot \prod_{\mathrm{p}}\left(\frac{\mathrm{p}^4}{\mathrm{p}^4-1}\right) \\
&= 2\cdot 3\cdot \frac{2^4}{(2^4-1)}\frac{3^4}{(3^4-1)}\frac{5^4}{(5^4-1)}\frac{7^4}{(7^4-1)}\cdots \\
&= \frac{3^3 5^2}{2^3\cdot 13}\cdot\frac{7^4}{(7^4-1)}\frac{11^4}{(11^4-1)}\cdots = \frac{3^3 5^2}{2^3\cdot 13}\prod_{\mathrm{p}>5}\left(\frac{\mathrm{p}^4}{\mathrm{p}^4-1}\right)
\end{aligned} \tag{14}$$

where the Euler product extends over all the *prime* numbers. The photon distribution integral (14) yields a zeta-function that is linked to the Euler prime products, therefore, there is no $h\nu$ in (6)∼(13). (14) shows clearly how the fine structure constant α for the discrete spectra in (4) is converted to the blackbody radiation constant α_R for the continuous spectra by multiplying a dimensionless constant. (5) and (14) indicate that this dimensionless constant can be expressed as the Euler infinite prime number product.

In (6)∼(13), the oscillators of the thermal electrons $\alpha/\mathbf{e}^2 = 1/\hbar c$ or $\alpha_R/\mathbf{e}^2$ play a critical role in the electromagnetic coupling on a 3D surface (Fig. 2).

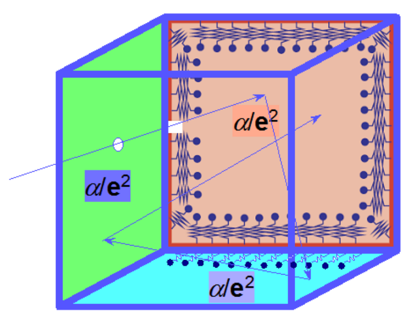

Figure 2: Blackbody radiation is related to α and $\mathbf{e}^2$ in a 3D cavity.

This 3D box (or sphere) model does not necessarily have solid walls, the plasma gas layer of a star can have the same effect. This links the quantum theory to the classical theory of blackbody radiation with or without using the Planck constant

$$\begin{aligned} \boldsymbol{a}_R &= \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 = \frac{\zeta(4)}{\zeta(2)}\left(\frac{2\pi}{hc}\right)^3 k_B^4 \\ &= \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 \cdot \frac{4}{5}\cdot\frac{9}{10}\cdot\frac{25}{26}\cdot\frac{49}{50}\cdot\frac{121}{122}\cdot\frac{169}{170}\cdot\frac{289}{290}\cdot\frac{361}{362}\cdots \end{aligned} \tag{15}$$

The Planck constant h with the revolutionary concept of energy quanta is a bridge between classical physics and quantum physics. Einstein's proposal of the light quanta $h\nu$ in 1905 was based on the Planck constant, however, Planck always had reservations due to the continuous spectra of blackbody radiation and the wave-particle duality. In 1951, Einstein said that, "All these fifty years of conscious brooding have brought me no nearer to the answer to the question, 'What are light quanta?'"[7] In QED, the photon is treated as a gauge boson, and the perturbation theory involves the finite power series in α. The discrete-continuous spectra is bridged by the Bose-Einstein distribution, and the *prime* sequences link the fine structure constant α to the blackbody radiation constant α_R. The blackbody radiation constant is a new method to measure the fine structure constant. It also links the fine structure constant to the Boltzmann constant.

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