A TREATY OF SYMMETRIC FUNCTION

An Approach in Deriving General Formulation for Sums of Power for an Arbitrary Arithmetic Progression and Applying the Method Formulated for Expressing Fermat's Last theorem and Riemann Zeta Function into Symmetric Function. The Generalize equation also leads to the formulation of a new set of Prime Numbers in which Mersenne and Wagstaff numbers fall under it

Paper Part I

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In remembrance of my beloved father who passed away on the 23rd of June 2009 and my special thanks to my brother Mohd Yunus Abd Shukor for introducing me Fermat's Last Theorem when I was a teenager. Although I didn't get the proof for this theorem, it enhanced my understanding towards developing the generalized equations for Symmetric Function for Sums of Powers and expressing Riemann Zeta Function using Sum of Power. The finding also contributes to a formulation of a new conjecture of Prime Number of a Power Sum Origin. Lastly to my sister Nazirah Abd Shukor, thanks for all the supports and patience.

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Abstract. Sum of Power had gathered interest of many classical mathematicians more than two thousand years ago. The quests of finding sum of power or discrete sum of numerical power can be traced back from the time of Archimedes in third BC then to Faulhaber in the sixteen century CE [1] & [2]. A new approach in deriving Sum of Power series using reverse look up method, a ,method where a mathematical formulation is constructed from set of data. Faulhaber [1] derived a general equation for Power sums and calculated the terms up to p=17 (i.e. $\sum_{i=1}^{n} x_i^p$). However, these formulae only work for integers from $x_1 = 1$ to $x_n = n$. A depth study on Power series revealed a systematic general equation which applicable for all numbers with a condition that the series should be in an arithmetic progression without the power p (i.e. $\sum_{i=1}^{n} x_i$). The general formulation is given as follows

$$\sum_{i=1}^{n} x_i^p = \sum_{j=0}^{u} \left[\phi_j s^{2j} \frac{\left[\sum_{i=1}^{n} x_i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \qquad [1]$$

Where: $p-(2j+1) \geq -1$ if p is even, $p-(2j+1) \geq 0$ if p is odd, $s = x_{i+1} - x_i$, ϕ_k is a coefficient and $\phi_0 = 1$ and $u = \begin{cases} \frac{p-1}{2} \; for\_odd\_p \\ \frac{p}{2} \; for\_even\_p \end{cases}$ [2]

These equations show that when n=2, it reduces into $x_1^p + x_2^p$ indicating that the sum of two powers could be expanded into a polynomial equation with variable ($x_1 + x_2$) which is useful for those who are familiar with Fermat Last Theorem. This formula shows that when p is odd, the summation of power series has a factor of arithmetic sum (i.e. $\left[\sum_{i=1}^{n} x_i^p\right] \equiv 0 \bmod \left(\sum_{i=1}^{n} x\right)$).

For both odd and even p, Fermat's last theorem could be expressed as follows

$$\sum_{i=1}^{2} x_i^p = x_1^p + x_2^p = \frac{\left[\sum_{i=1}^{2} x_i\right]^p}{2^{p-1}} + f\left(\left[\sum_{i=1}^{2} x_i\right], n, s\right) \text{ or } \sum_{i=1}^{2} x_i^p = x_1^p + x_2^p = \frac{(x_1+x_2)^p}{2^{p-1}} + f\left((x_1+x_2), n, s\right). \qquad [3]$$

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1 Introduction.

Faulhaber [1] derived an equation for summation of power series for power sums of positive integers up to first *n*; his formula is given as follows

$$\sum_{k=1}^{n} k^{p} = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j} B_j n^{p+1-j} \qquad [4]$$

Where, $\binom{p+1}{j}$ is a binomial coefficient, and B_j is the *j*-th Bernoulli number.

The sums for $p = 1 \cdots 6$ is given as follows

$$\sum_{k=1}^{n} k = \frac{1}{2}\left(n^2 + n\right) \qquad [5]$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}\left(2n^3 + 3n^2 + n\right) \qquad [6]$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) \qquad [7]$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{12}\left(6n^5 + 15n^4 + 10n^2 - n\right) \qquad [8]$$

$$\sum_{k=1}^{n} k^5 = \frac{1}{42}\left(6n^7 + 21n^6 + 21n^5 - 7n^3 + n\right) \qquad [9]$$

William et al [3] discovered the formulae for sums of odd powers by adopting classical Faulhaber's theorem. This was done by considering an arithmetic progression in this form:

$$(a+b),(a+2b)\ldots(a+nb) \qquad [10]$$

By letting, *a*=*x* and *b*=1, the arithmetic progression can be written as follows:

$$(x+1),(x+2)\ldots(x+n) \qquad [11]$$

Let $\lambda = n(n+2x+1)$

be the sum of arithmetic series. The sum of power for this progression is gives as follows:

$$S_{2m-1} = (x+1)^{2m-1} + (x+2)^{2m-1} + \ldots + (x+n)^{2m-1} \qquad [12]$$

For some odd *p* (i.e. *p*=2*m*-1), the formulae for sum of power are given as follows:

$$S_3 = \frac{1}{4}[\lambda]^2 + \frac{1}{2}\left(x^2 + x\right)[\lambda] \qquad [13]$$

$$S_5 = \frac{1}{6}[\lambda]^3 + \frac{1}{12}\left(6x^2 + 6x + 1\right)[\lambda]^2 + \frac{1}{6}\left(3x^4 + 6x^3 + 2x^2 - x\right)[\lambda] \qquad [14]$$

$$S_7 = \frac{1}{8}[\lambda]^4 + \frac{1}{6}\left(3x^2 + 3x - 1\right)[\lambda]^3 + \frac{1}{12}\left(9x^4 + 18x^3 + 3x^2 - 6x + 1\right)[\lambda]^2$$
$$+\frac{1}{6}\left(3x^6 + 9x^5 + 6x^4 - 3x^3 - 2x^2 + x\right) \qquad [15]$$

Adopting Yoshinari Inaba's matrix method [4] for computing the *m*-th sum of power for the first *n* terms of arithmetic progression, N. Gauthier [5] derived a formula for computing the sum of *m*-th power of *n* successive terms of an arithmetic sequence gives as follows:

$$S_m = b^m + (a+b)^m + (2a+b)^m + \cdots + ((n-1)a+b)^m \qquad [16]$$

His result for *m*=2 is given as follows:

$$S_2 = \frac{1}{3}\left[a^2n^3 + 3a\left(1-\frac{1}{2}a\right)n^2 + \left(3-3a+\frac{1}{2}a^2\right)n\right] \qquad [17]$$

The search of a simpler general formulation for sum of power for arithmetic progression had attracted many mathematicians and different methods had been proposed to represent the summation for years [2]-[5]. This paper is to present an elegant method for the sum of power of *p*-th for first *n* term of arithmetic progression. The purpose of this method is to construct a simpler equation and use a non-complicated derivation technique.

2 An Alternative Derivation and Formulation of the Sum of Power for *p*-th Arithmetic Progression.

The idea of this paper is to expand the sum of power term into basic symmetric function $\left[\sum_{i=1}^{n} x_i\right]$ with repetitious coefficients. By expanding the general equation [1] for first *p*=10, yields

$$\left[\sum_{i=1}^{n} x_i\right] = \phi_0\left[\sum_{i=1}^{n} x_i\right] \qquad [18]$$

$$\left[\sum_{i=1}^{n} x_i^2\right] = \phi_0\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_1 n \qquad [19]$$

$$\left[\sum_{i=1}^{n} x_i^3\right] = \phi_0\frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_1\left[\sum_{i=1}^{n} x_i\right] \qquad [20]$$

$$\left[\sum_{i=1}^{n} x_i^4\right] = \phi_0\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_1\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_2 n \qquad [21]$$

$$\left[\sum_{i=1}^{n} x_i^5\right] = \phi_0\frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + \phi_1\frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_2\left[\sum_{i=1}^{n} x_i\right] \qquad [22]$$

$$\left[\sum_{i=1}^{n} x_i^6\right] = \phi_0\frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + \phi_1\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_2\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_3 n \qquad [23]$$

$$\left[\sum_{i=1}^{n} x_i^7\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_3 \left[\sum_{i=1}^{n} x_i\right] \qquad [24]$$

$$\left[\sum_{i=1}^{n} x_i^8\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^7} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_3 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_4 n \qquad [25]$$

$$\left[\sum_{i=1}^{n} x_i^9\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^9}{n^8} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^5}{n^4} + \phi_3 \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \phi_4 \frac{\left[\sum_{i=1}^{n} x_i\right]}{n} \qquad [26]$$

$$\left[\sum_{i=1}^{n} x_i^{10}\right] = \phi_0 \frac{\left[\sum_{i=1}^{n} x_i\right]^{10}}{n^9} + \phi_1 \frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^7} + \phi_2 \frac{\left[\sum_{i=1}^{n} x_i\right]^6}{n^5} + \phi_3 \frac{\left[\sum_{i=1}^{n} x_i\right]^4}{n^3} + \phi_4 \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \phi_5 n \qquad [21]$$

2.1 Data Analysis Method.

This method is about data analysis and using the result to construct the equation needed for each of *p*-th term.

Table 1 Data for *n*=2

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2
1	2	3	1	4	5
2	3	5	4	9	13
3	4	7	9	16	25
4	5	9	16	25	41
5	6	11	25	36	61
6	7	13	36	49	85
7	8	15	49	64	113
8	9	17	64	81	145
9	10	19	81	100	181
10	11	21	100	121	221
11	12	23	121	144	265
12	13	25	144	169	313
13	14	27	169	196	365
14	15	29	196	225	421
15	16	31	225	256	481

Table 2 Data for n=3

x_1	x_2	x_3	Sum(x_i)	x_1^2	x_2^2	x_3^2	Sum(x_i^2)_n=3
1	2	3	6	1	4	9	14
2	3	4	9	4	9	16	29
3	4	5	12	9	16	25	50
4	5	6	15	16	25	36	77
5	6	7	18	25	36	49	110
6	7	8	21	36	49	64	149
7	8	9	24	49	64	81	194
8	9	10	27	64	81	100	245
9	10	11	30	81	100	121	302
10	11	12	33	100	121	144	365
11	12	13	36	121	144	169	434
12	13	14	39	144	169	196	509
13	14	15	42	169	196	225	590
14	15	16	45	196	225	256	677
15	16	17	48	225	256	289	770

The plot of Sum(x) versus Sum(x^2) for various values of "*n*" is given as follows:

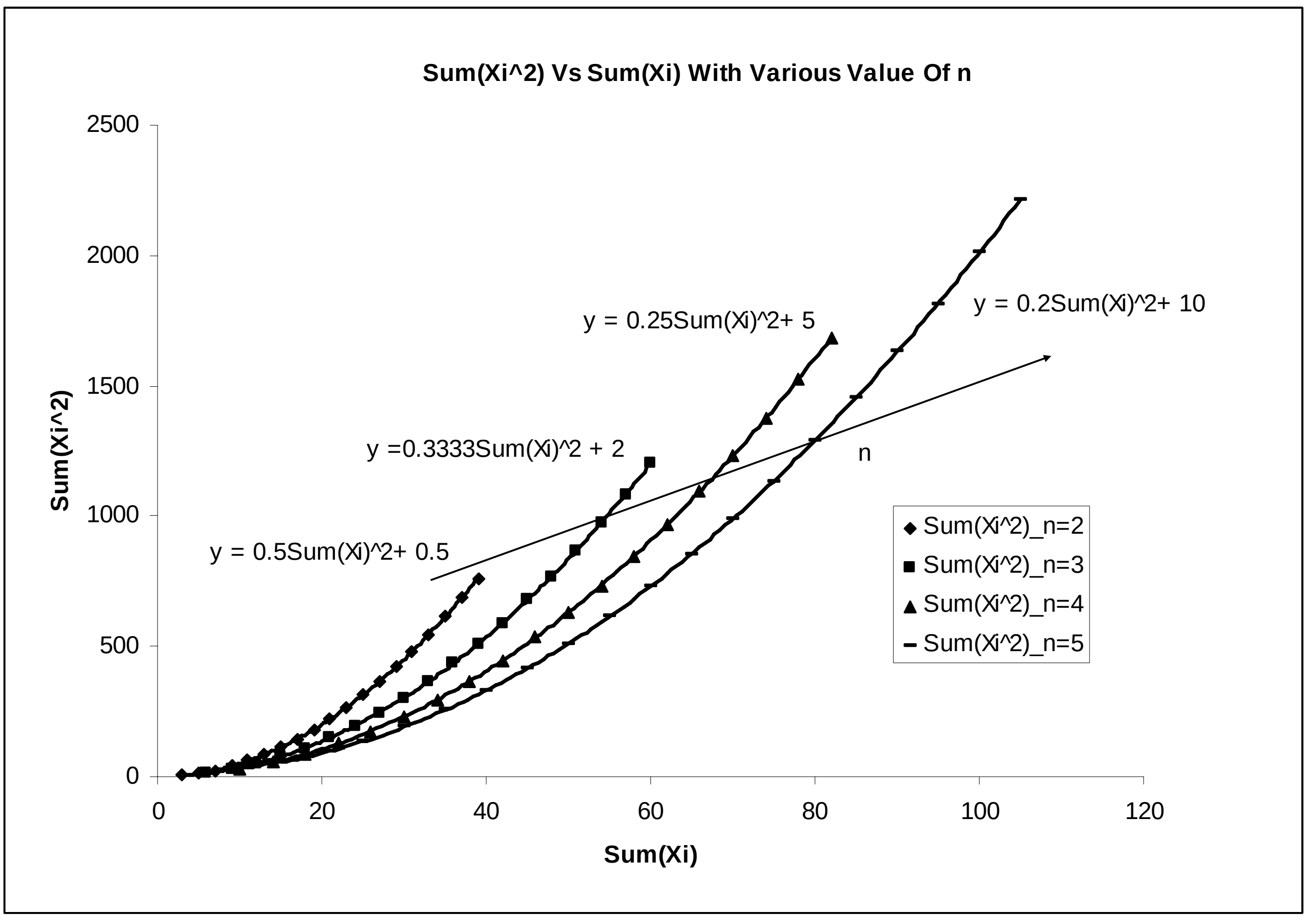

Figure 1.0 The curve for Sum(x^2) versus sum(x).

The curve coefficients for each "*n*" is tabulated as in the Table 3.

Table 3 Coefficient for a and b at various n

n	a	b
2	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{3}$	2
4	$\frac{1}{4}$	5
5	$\frac{1}{5}$	10
6	$\frac{1}{6}$	$\frac{35}{2}$
7	$\frac{1}{7}$	28
8	$\frac{1}{8}$	42
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\frac{n(n^2-1)}{12}$

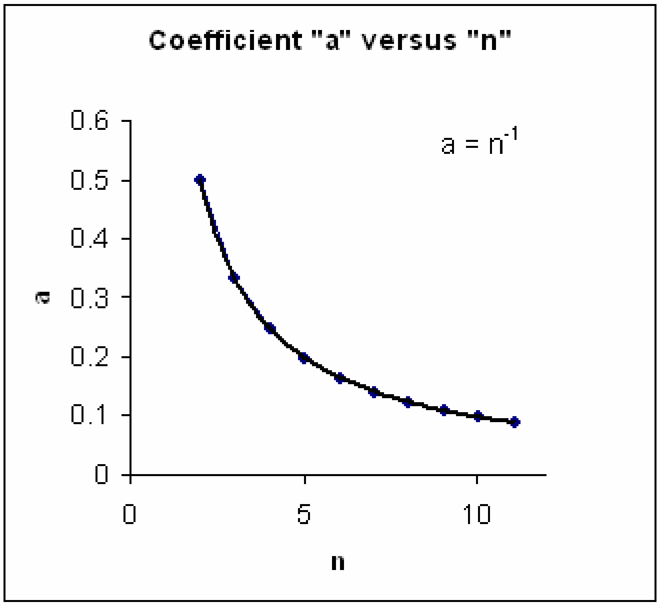

Figure 2.0 Curve for a versus n.

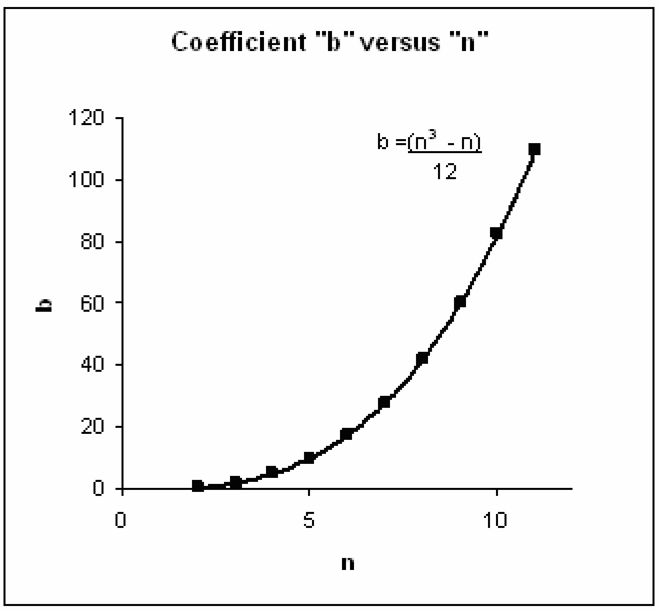

Figure 3.0 Curve for b versus n.

Therefore, for sum of power for p=2 it is given as follows:

$$\sum_{i=1}^{n} x_i^2 = \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \frac{n(n^2-1)}{12} \qquad [27]$$

This equation applicable only for the integers. The equation for sum of power for arbitrary arithmetic progression for p=2 can be obtained by tabulating the data of the arithmetic progression x_i with difference s. The tabulated data are given as in Table 4 to Table 6 as follows:

Table 4 Tabulated data for *s*=1.

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2_s=1
1	2	3	1	4	5
2	3	5	4	9	13
3	4	7	9	16	25
4	5	9	16	25	41
5	6	11	25	36	61
6	7	13	36	49	85
7	8	15	49	64	113
8	9	17	64	81	145
9	10	19	81	100	181
10	11	21	100	121	221
11	12	23	121	144	265
12	13	25	144	169	313
13	14	27	169	196	365
14	15	29	196	225	421
15	16	31	225	256	481

Table 5 Tabulated data with *s*=10

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2_s=11
1	12	13	1	144	145
2	13	15	4	169	173
3	14	17	9	196	205
4	15	19	16	225	241
5	16	21	25	256	281
6	17	23	36	289	325
7	18	25	49	324	373
8	19	27	64	361	425
9	20	29	81	400	481
10	21	31	100	441	541
11	22	33	121	484	605
12	23	35	144	529	673
13	24	37	169	576	745
14	25	39	196	625	821
15	26	41	225	676	901

Table 6 Tabulated data with *s*=26

x_1	x_2	Sum(x_i)	x_1^2	x_2^2	Sum(x_i^2)_n=2_s=26
1	27	28	1	729	730
2	28	30	4	784	788
3	29	32	9	841	850
4	30	34	16	900	916
5	31	36	25	961	986
6	32	38	36	1024	1060
7	33	40	49	1089	1138
8	34	42	64	1156	1220
9	35	44	81	1225	1306
10	36	46	100	1296	1396
11	37	48	121	1369	1490
12	38	50	144	1444	1588
13	39	52	169	1521	1690
14	40	54	196	1600	1796
15	41	56	225	1681	1906

The plot for various values of *s* is given as follows:

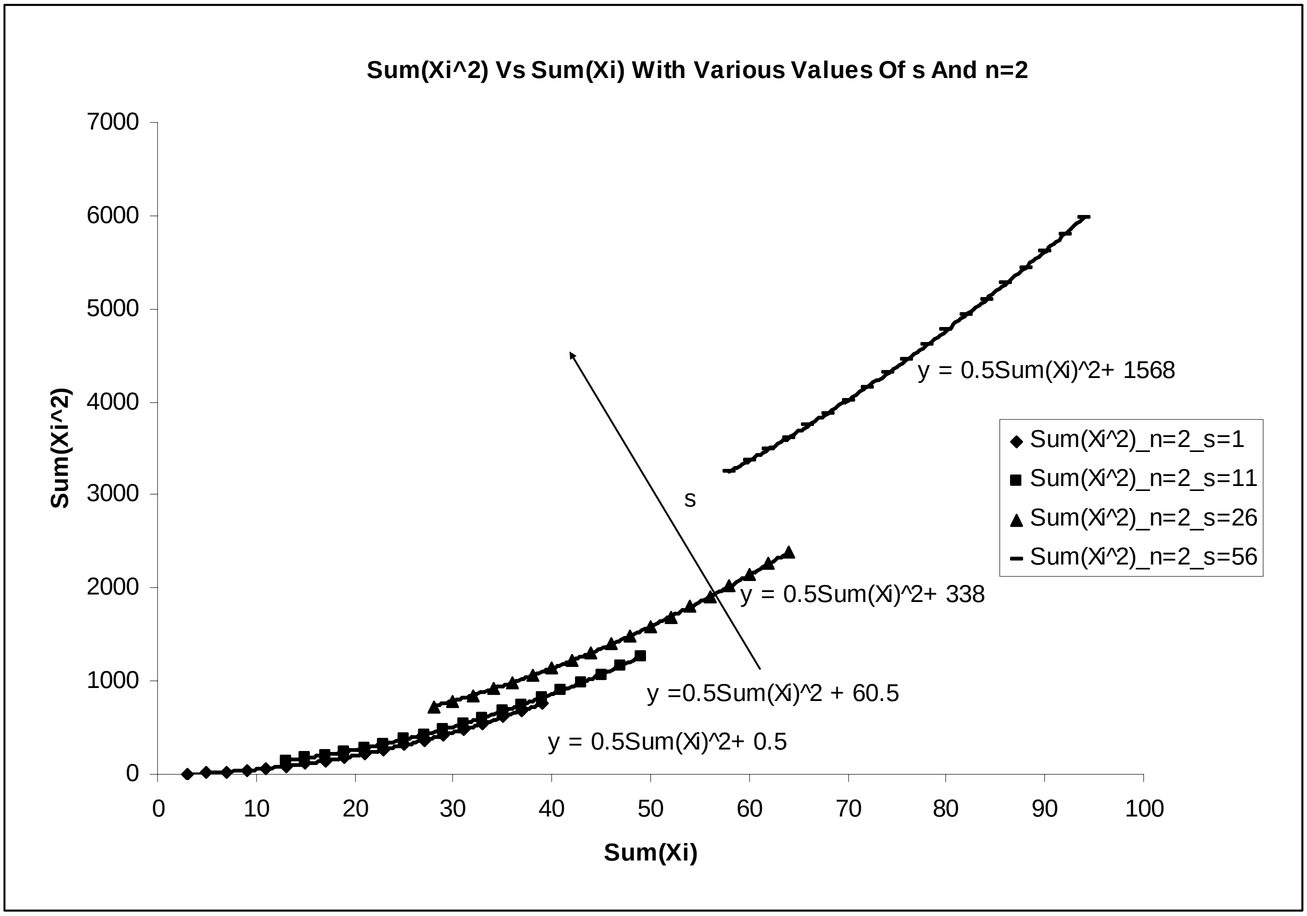

Figure 4.0 The curve for Sum(x_i^2) versus Sum(x) for various values of *s* and *n*=2.

Table 7 Coefficients "a" and "b" at various value of "*s*" with *n*=2.

s	a	b	$c = \frac{n(n^2-1)}{12}$	$\frac{b}{c}$	$\frac{b}{c} = s^2$
1	0.5	0.5	0.5	1	1
2	0.5	2	0.5	4	4
3	0.5	4.5	0.5	9	9
4	0.5	8	0.5	16	16
5	0.5	12.5	0.5	25	25
6	0.5	18	0.5	36	36
7	0.5	24.5	0.5	49	49
11	0.5	60.5	0.5	121	121
26	0.5	338	0.5	676	676
56	0.5	1568	0.5	3136	3136

Calculating and analyzing for some values of "*n*" and "*s*" yields

$$\sum_{i=1}^{n} x_i^2 = \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \frac{n(n^2-1)s^2}{12} \qquad [28]$$

The data analysis method can be expressed in a matrix form given as follows:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}_n = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^p & \left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 & \left(\sum_{i=1}^{n} x_i\right)^{p-4} s^4 & \cdots & s^p \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^p & \left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} s^2 & \left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} s^4 & \cdots & s^p \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\sum_{i=1}^{n} x_{i+m}\right)^p & \left(\sum_{i=1}^{n} x_{i+m}\right)^{p-2} s^2 & \left(\sum_{i=1}^{n} x_{i+m}\right)^{p-4} s^4 & \cdots & s^p \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i=1}^{n} x_i^p \\ \sum_{i=1}^{n} x_{i+1}^p \\ \vdots \\ \sum_{i=1}^{n} x_{i+m}^p \end{pmatrix} \text{ for even } p \qquad [29]$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}_n = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^p & \left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 & \left(\sum_{i=1}^{n} x_i\right)^{p-4} s^4 & \cdots & \left(\sum_{i=1}^{n} x_i\right) s^{p-1} \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^p & \left(\sum_{i=1}^{n} x_{i+1}\right)^{p-2} s^2 & \left(\sum_{i=1}^{n} x_{i+1}\right)^{p-4} s^4 & \cdots & \left(\sum_{i=1}^{n} x_i\right) s^{p-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \left(\sum_{i=1}^{n} x_{i+m}\right)^p & \left(\sum_{i=1}^{n} x_{i+m}\right)^{p-2} s^2 & \left(\sum_{i=1}^{n} x_{i+m}\right)^{p-4} s^4 & \cdots & \left(\sum_{i=1}^{n} x_i\right) s^{p-1} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i=1}^{n} x_i^p \\ \sum_{i=1}^{n} x_{i+1}^p \\ \vdots \\ \sum_{i=1}^{n} x_{i+m}^p \end{pmatrix} \text{ for odd } p \qquad [30]$$

The conclusion is that the method can be used to generate arithmetic *p-th* terms for any value of *p*. However, the larger the value of *p* the more tedious the calculation would be. Since Microsoft Excel having maximum precision of 15 digits, the error in calculation will occurs for numbers more than 15 digits. In order to overcome this problem an addon should be installed on the Microsoft Excel, this research was done using Xnumbers [6] which leads to precession of up to 200 digits.

2 Algebraic Manipulation Method.

For small *p* the sum of power can be derived using simple algebraic manipulation of arithmetic terms. The formulation for some small *p* can be obtained as follows:

For p=2 and n=2

Let $(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$ [31]

and $(x_2 - x_1)^2 = x_1^2 + x_2^2 - 2x_1x_2$ [32]

Since the series is an arithmetic progression, thus

$$(x_2 - x_1) = s \qquad [33]$$

Substituting [32] into [31], yields

$$2x_1x_2 = x_1^2 + x_2^2 - s^2 \qquad [34]$$

Substituting [34] into [31] yields

$$(x_1 + x_2)^2 = 2(x_1^2 + x_2^2) - s^2 \qquad [35]$$

Rearranging [35], yields

$$(x_1^2 + x_2^2) = \frac{(x_1 + x_2)^2}{2} + \frac{s^2}{2} \qquad [36]$$

or

$$\sum_{i=1}^{2} x_i^2 = \frac{\left[\sum_{i=1}^{2} x_i\right]^2}{2} + \frac{s^2}{2} \qquad [37]$$

Now consider *p=2* and *n*=3,

$$(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2) + 2x_3x_1 + 2x_3x_2 + 2x_1x_2 \qquad [38]$$

Since $s = (x_3 - x_2) = (x_2 - x_1)$

repeating for term $(x_3 - x_2)$, yields

$$(x_3 - x_2)^2 = x_3^2 + x_2^2 - 2x_3x_2 \qquad [39]$$

Therefore, $2x_3x_2 = x_3^2 + x_2^2 - s^2$ [40]

Since $s = (x_3 - x_2) = (x_2 - x_1)$ then

$$(x_3 - x_2) = s \text{ and} \qquad [41]$$

adding [41] and [33], yields

$$(x_3 - x_1) = 2s \qquad [42]$$

squaring both sides [42] and rearranging it, yields

$$2x_3x_1 = x_3^2 + x_1^2 - 4s^2 \qquad [43]$$

Substituting [43], [40] and [34] into [38], yields

$$(x_1 + x_2 + x_3)^2 = 3(x_1^2 + x_2^2 + x_3^2) - 6s^2 \qquad [44]$$

rearranging [44], yields

$$(x_1^2 + x_2^2 + x_3^2) = \frac{(x_1 + x_2 + x_3)^2}{3} + 2s^2 \qquad [45]$$

or

$$\sum_{i=1}^{3} x_i^2 = \frac{\left[\sum_{i=1}^{3} x_i\right]^2}{3} + 2s^2 \qquad [46]$$

repeating the same procedure for terms from 2 to *n*, and by considering a general formulation for $p = 2$ of this form

$$\sum_{i=1}^{n} x_i^2 = a\left[\sum_{i=1}^{n} x_i\right]^2 + bs^2 \qquad [47]$$

and then calculating for some *n* and tabulating the data in a table, the table can be seen as table [8]. The curves constructed from the tabulated data can be seen as in figure [5] and figure [6].

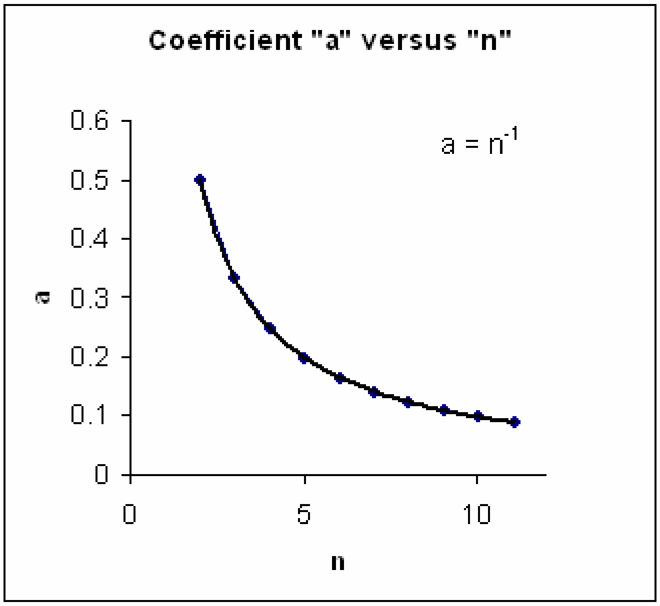

Figure 5.0 Curve for *a* versus *n*.

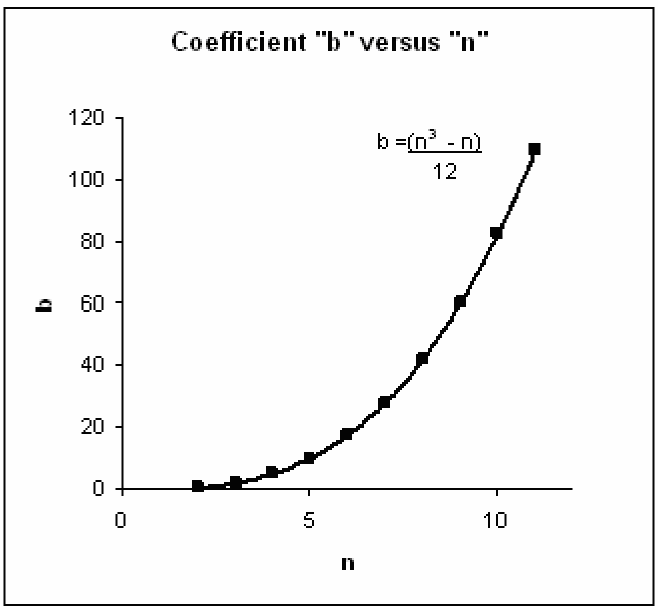

Figure 6.0 Curve for *b* versus *n*.

Table 8 Coefficient for *a* and *b* at various *n*.

n	a	b
2	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{3}$	2
4	$\frac{1}{4}$	5
5	$\frac{1}{5}$	10
6	$\frac{1}{6}$	$\frac{35}{2}$
7	$\frac{1}{7}$	28
8	$\frac{1}{8}$	42
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\frac{n(n^2-1)}{12}$

Consequently,

$$\sum_{i=1}^{n} x_i^2 = \frac{\left[\sum_{i=1}^{n} x_i\right]^2}{n} + \frac{n(n^2-1)}{12} s^2 \qquad [48]$$

Now consider p=*3* and n=2, thus:

$$(x_1 + x_2)^3 = (x_1^3 + x_2^3) + 3x_1x_2(x_1 + x_2) \qquad [49]$$

since $(y - x) = s$, then

$$2x_1x_2 = x_1^2 + x_2^2 - s^2 \qquad [50]$$

multiplying both sides [50] with $(x_1 + x_2)$, yields,

$$2x_1x_2(x_1+x_2)=(x_1^2+x_2^2)(x_1+x_2)-s^2(x_1+x_2) \qquad [51]$$

multiplying both sides [49] with 2 and substituting [51] into the equation, yields

$$2(x_1+x_2)^3=5(x_1^3+x_2^3)+3x_1x_2(x_1+x_2)-3s^2(x_1+x_2) \qquad [52]$$

subtracting equation [52] with equation [49], yields

$$(x_1+x_2)^3=4(x_1^3+x_2^3)-3s^2(x_1+x_2) \qquad [53]$$

rearranging [53], yields

$$(x_1^3+x_2^3)=\frac{(x_1+x_2)^3}{4}+\frac{3s^2(x_1+x_2)}{4} \qquad [54]$$

or

$$\sum_{i=1}^{2}x_i^3=\frac{\left[\sum_{i=1}^{2}x_i\right]^3}{4}+\frac{3s^2}{4}\left[\sum_{i=1}^{2}x_i\right] \qquad [55]$$

For $p=3$ and n=3,

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+3x_2(x_1^2+x_3^2)+3x_1(x_2^2+x_3^2)+3x_3(x_1^2+x_2^2)+6x_1x_2x_3 \qquad [56]$$

Since,

$$x_3-x_2=s \qquad [57]$$

$$x_2-x_1=s \qquad [58]$$

Adding [57] to [58], yields:

$$x_3-x_1=2s \qquad [59]$$

Squaring both sides of equations [57] to [59] and rearrange their terms, yields:

$$\left(x_3^2+x_2^2\right)=s^2+2x_3x_2 \qquad [60]$$

$$\left(x_2^2+x_1^2\right)=s^2+2x_2x_1 \qquad [61]$$

$$\left(x_3^2+x_1^2\right)=4s^2+2x_3x_1 \qquad [62]$$

Substituting equations [60] to [62] into [56] and simplifying it, yields

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+3s^2(4x_2+x_1+x_3)+24x_1x_2x_3 \qquad [63]$$

Manipulating equation [63], yields

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+3s^2(3x_2+(x_1+x_2+x_3))+24x_1x_2x_3 \qquad [64]$$

Since,

$$x_2=\frac{\left(x_1+x_2+x_3\right)}{3} \qquad [65]$$

Substituting equation [65] into [64] and simplifying it, yields:

$$(x_1+x_2+x_3)^3=(x_1^3+x_2^3+x_3^3)+6s^2(x_1+x_2+x_3)+24x_1x_2x_3 \qquad [66]$$

Now consider Product Identity for arithmetic progression for n=3 and it is given as follows:

$$\prod_{i=1}^{3}x_i=x_1x_2x_3=\frac{1}{3^3}\left(x_1+x_2+x_3\right)\left(\left(x_1+x_2+x_3\right)-3s\right)\left(\left(x_1+x_2+x_3\right)+3s\right) \qquad [67]$$

Substituting equation [67] into [66] and rearranging the terms, yields

$$(x_1^3+x_2^3+x_3^3)=\frac{(x_1+x_2+x_3)^3}{9}+2s^2(x_1+x_2+x_3) \qquad [68]$$

or

$$\sum_{i=1}^{3}x_i^3\,\frac{\left[\sum_{i=1}^{3}x_1\right]^3}{9}+2s^2\sum_{i=1}^{3}x_i \qquad [69]$$

repeating the same procedure for terms from 2 to n, and by considering a general formulation for $p = 3$ of this form

$$\sum_{i=1}^{n} x_i^3 = a\left[\sum_{i=1}^{n} x_i\right]^3 + bs^2\left[\sum_{i=1}^{n} x_i\right] \qquad [70]$$

and then calculating for some n terms yields:

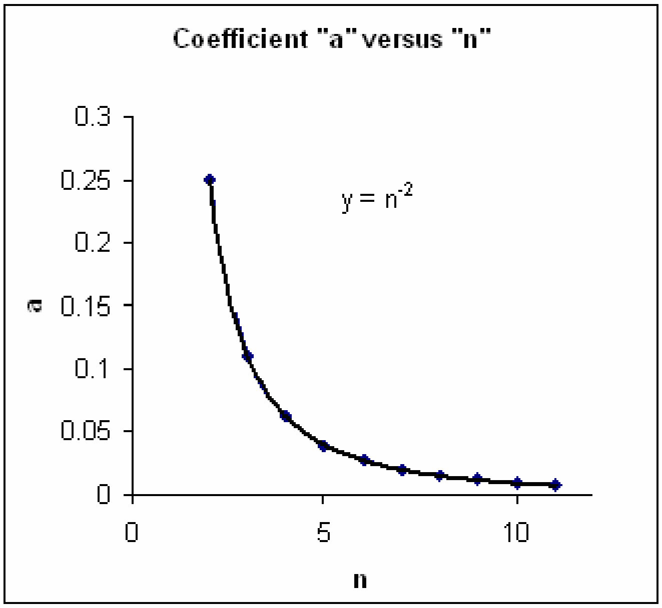

Figure 7.0 Curve for a versus n.

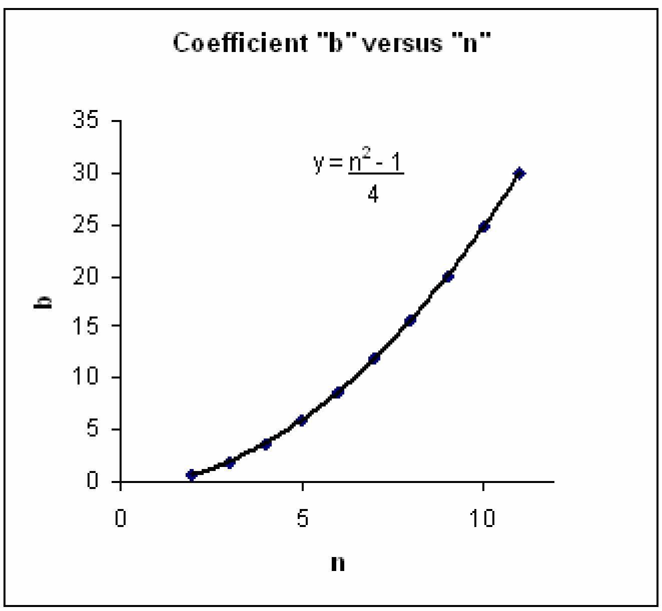

Figure 8.0 Curve for b versus n.

Table 9 Coefficient for a and b at various n

n	a	b
2	$\frac{1}{4}$	$\frac{3}{4}$
3	$\frac{1}{9}$	2
4	$\frac{1}{16}$	$\frac{15}{4}$
5	$\frac{1}{25}$	6
6	$\frac{1}{36}$	$\frac{35}{4}$
7	$\frac{1}{49}$	12
8	$\frac{1}{64}$	$\frac{63}{4}$
9	$\frac{1}{81}$	20
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n^2}$	$\frac{(n^2-1)}{4}$

As a result,

$$\sum_{i=1}^{n} x_i^3 = \frac{\left[\sum_{i=1}^{n} x_i\right]^3}{n^2} + \frac{(n^2-1)s^2}{4}\left[\sum_{i=1}^{n} x_i\right] \qquad [71]$$

For $p=4$ and n=2,

Let $$\left(x_1+x_2\right)^4=x_1^4+4x_2x_1^3+6x_1^2x_2^2+4x_1x_2^3+x_2^4 \qquad [72]$$

and rearranging [72] into symmetric function form, yields:

$$\left(x_1+x_2\right)^4=\left(x_1^4+x_2^4\right)+4x_1x_2\left(x_1^2+x_2^2\right)+6\left(x_1x_2\right)^2 \qquad [73]$$

From equation [48] when n=2, it gives

$$\sum_{i=1}^{2}x_i^2=\left(x_1^2+x_2^2\right)=\frac{\left[\sum_{i=1}^{2}x_1\right]^2}{2}+\frac{s^2}{2} \qquad [74]$$

Using product identity for arithmetic progression yields:

$$\prod_{i=1}^{2}x_i=\frac{1}{2^2}\left[\left[\sum_{i=1}^{2}x_i\right]-s^2\right] \qquad [75]$$

Substituting equation [75] into [73] and expressing them in a summation notations yields:

$$\left[\sum_{i=1}^{2}x_i\right]^4=\left[\sum_{i=1}^{2}x_1^4\right]+\frac{1}{2}\left(\left[\sum_{i=1}^{2}x_i\right]^2-s^2\right)\left(\left[\sum_{i=1}^{2}x_i\right]^2+s^2\right)+\frac{3}{8}\left(\left[\sum_{i=1}^{2}x_i\right]^2-s^2\right)^2 \qquad [76]$$

Simplifying and rearranging the equation [76], yields:

$$\left[\sum_{i=1}^{2}x_i^4\right]=\frac{1}{8}\left[\sum_{i=1}^{2}x_i\right]^4+\frac{3s^2}{4}\left[\sum_{i=1}^{2}x_i\right]^2+\frac{s^4}{8} \qquad [77]$$

Calculating the other terms and simplifying for term n, yields:

$$\left[\sum_{i=1}^{n}x_i^4\right]=\frac{\left[\sum_{i=1}^{n}x_i\right]^4}{n^3}+(n^2-1)s^2\frac{\left[\sum_{i=1}^{n}x_i\right]^2}{2n}+\frac{n(3n^2-7)(n^2-1)s^4}{240} \qquad [78]$$

Calculating the coefficients for the rest of the equations yields

$$\left[\sum_{i=1}^{n}x_i^5\right]=\frac{\left[\sum_{i=1}^{n}x_i\right]^5}{n^4}+5(n^2-1)s^2\frac{\left[\sum_{i=1}^{n}x_i\right]^3}{6n^2}+\frac{(3n^2-7)(n^2-1)s^4}{48}\left[\sum_{i=1}^{n}x_i\right] \qquad [79]$$

$$\left[\sum_{i=1}^{n}x_i^6\right]=\frac{\left[\sum_{i=1}^{n}x_i\right]^6}{n^5}+5(n^2-1)s^2\frac{\left[\sum_{i=1}^{n}x_i\right]^4}{4n^3}+(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n}x_i\right]^2}{16n}+\frac{n(3n^4-18n^2+31)(n^2-1)s^6}{1344} \qquad [80]$$

$$\left[\sum_{i=1}^{n}x_i^7\right]=\frac{\left[\sum_{i=1}^{n}x_i\right]^7}{n^6}+7(n^2-1)s^2\frac{\left[\sum_{i=1}^{n}x_i\right]^5}{4n^4}+7(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n}x_i\right]^3}{48n^2}+(3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n}x_i\right]}{192}$$

[81]

$$\left[\sum_{i=1}^{n} x_i^8\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^8}{n^7} + 7(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^6}{3n^5} + 7(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{24n^3} + (3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{48n}$$

$$+\frac{n(5n^6-55n^4+239n^2-381)(n^2-1)s^8}{11520}$$

[82]

$$\left[\sum_{i=1}^{n} x_i^9\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^9}{n^8} + 3(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^7}{n^6} + 21(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^5}{40n^4} + (3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]^3}{16n^2}$$

$$+(5n^6-55n^4+239n^2-381)(n^2-1)s^8\frac{\left[\sum_{i=1}^{n} x_i\right]}{1280}$$

[83]

$$\left[\sum_{i=1}^{n} x_i^{10}\right] = \frac{\left[\sum_{i=1}^{n} x_i\right]^{10}}{n^9} + 15(n^2-1)s^2\frac{\left[\sum_{i=1}^{n} x_i\right]^8}{4n^7} + 7(3n^2-7)(n^2-1)s^4\frac{\left[\sum_{i=1}^{n} x_i\right]^6}{8n^5} + 5(3n^4-18n^2+31)(n^2-1)s^6\frac{\left[\sum_{i=1}^{n} x_i\right]^4}{32n^3}$$

$$+(5n^6-55n^4+239n^2-381)(n^2-1)s^8\frac{\left[\sum_{i=1}^{n} x_i\right]^2}{256n} + \frac{n(3n^{10}-55n^8+462n^6-2046n^4+4191n^2-2555)s^{10}}{33792}$$

[84]

3 Product of Arithmetic Terms in Form of the Basic Elementary Symmetric Function (i.e. $\sum_{i=1}^{n} x_i$)

Theorem 1.0

Let $$P_n = x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i = \frac{1}{n^n}\prod_{t=0}^{\frac{n-2}{2}}\left(\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\frac{n}{2}(1+2t)s\right)^2\right) \text{ for even } n. \quad [85]$$

and $$P_n = x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i \frac{1}{n^n}\prod_{t=1}^{\frac{n-1}{2}}\left(\left(\sum_{i=1}^{n} x_i\right)^2 - (nts)^2\right) \text{ for odd } n. \quad [86]$$

Proof: By considering an arithmetic summation of n terms (i.e. $\sum_{i=1}^{n} x_i = \frac{n(2x_1+(n-1)s)}{2}$), and by rearranging it yields:

$$x_1 = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right) \quad [87]$$

Since $$x_n = \left(x_1 - (n-1)s\right) \qquad [88]$$

Substituting [87] into [88] yields:

$$x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right) \qquad [89]$$

Also,

$$x_{n-1} = \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right) - s = \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right) \qquad [90]$$

By taking product of x_1 to x_n for even *n* yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-3)s}{2}\right) \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{s}{2}\right) \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right) \qquad [91]$$

$$P_n = x_1 \cdot x_2 \cdots x_n = \frac{1}{n^n}\left(\sum_{i=1}^{n} x_i - \frac{n(n-1)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i - \frac{n(n-3)s}{2}\right) \cdots \left(\sum_{i=1}^{n} x_i - \frac{ns}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i + \frac{ns}{2}\right) \cdots \left(\sum_{i=1}^{n} x_i + \frac{n(n-3)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i + \frac{n(n-1)s}{2}\right) \qquad [92]$$

Simplifying [92], yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \frac{1}{n^n} \prod_{t=0}^{\frac{n-2}{2}} \left(\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\frac{n}{2}(1+2t)s\right)^2 \right) \text{ for even } n. \qquad [93]$$

and product of x_1 to x_n for odd *n* yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-1)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(n-3)s}{2}\right) \cdots \frac{\sum_{i=1}^{n} x_i}{n} \cdots \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-3)s}{2}\right) \cdot \left(\frac{\sum_{i=1}^{n} x_i}{n} + \frac{(n-1)s}{2}\right) \qquad [94]$$

$$P_n = x_1 \cdot x_2 \cdots x_n = \frac{1}{n^n}\left(\sum_{i=1}^{n} x_i - \frac{n(n-1)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i - \frac{n(n-3)s}{2}\right) \cdots \sum_{i=1}^{n} x_i \cdots \left(\sum_{i=1}^{n} x_i + \frac{n(n-3)s}{2}\right) \cdot \left(\sum_{i=1}^{n} x_i + \frac{n(n-1)s}{2}\right) \qquad [95]$$

Simplifying [95], yields:

$$P_n = x_1 \cdot x_2 \cdots x_n = \sum_{i=1}^{n} x_i \, \frac{1}{n^n} \prod_{t=1}^{\frac{n-1}{2}} \left(\left(\sum_{i=1}^{n} x_i\right)^2 - (nts)^2 \right) \text{ for odd } n. \qquad [96]$$

4 Elementary Symmetric Function for Alternating Permutation of Arithmetic Terms Through Quantitative Method.

Since Sum of Power is the basic building blocks for symmetric polynomials, therefore it can always be expressed as product and sum of symmetric functions with rational coefficients. Consider a set of symmetric functions of arbitrary arithmetic terms as follows:

$$(x_1, x_2, \cdots, x_{n-1}, x_n)$$

The elementary symmetric polynomials of n variables in form of n and symmetric function $\sum_{i=1}^{n} x_i$ are given as follows:

1st Order

$$O_1(x_1, x_2, \cdots, x_{n-1}, x_n) = x_1 + x_2 + \cdots + x_{n-1} + x_n = \sum_{i=1}^{n} x_i \qquad [97]$$

The second order can be calculated using quantitative method as follows:

Let the second order be

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = x_1x_2 + x_1x_3 + \cdots + x_ix_j = \sum_{i<j}^{n} x_ix_j$$

Consider an arithmetic term with s=1 and n=2, the tabulated data is given as follows:

Table 10 The values of $\sum_{1\le i<j}^{2} x_ix_j$ when n=2

x_1	x_2	$\sum_{i=1}^{2} x_i$	$\sum_{1\le i<j}^{2} x_ix_j$ _n=2
1	2	3	2
2	3	5	6
3	4	7	12
4	5	9	20
5	6	11	30
6	7	13	42
7	8	15	56
8	9	17	72
9	10	19	90
10	11	21	110
11	12	23	132
12	13	25	156
13	14	27	182
14	15	29	210
15	16	31	240
16	17	33	272
17	18	35	306
18	19	37	342
19	20	39	380

Plotting the data for some values of n yields graph as in Figure 9.

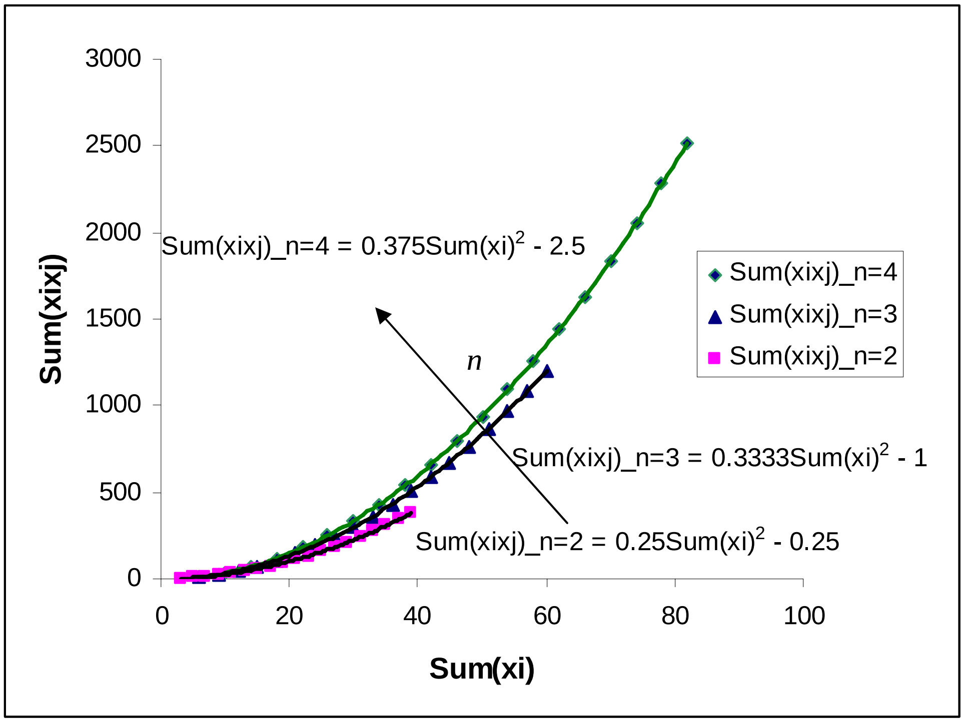

Figure 9 Graph of $\sum_{i<j}^{n} x_i x_j$ versus $\sum_{i=1}^{n} x_i$ for some values of n

Let the 2nd Order be as follows:

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j}^{n} x_i x_j = \phi_1 \left(\sum_{i=1}^{n} x_i \right)^2 - \phi_2$$

Collecting the coefficients of $\sum_{i<j}^{n} x_i x_j$ for some values of n yields Table 11.

Table 11 The values of ϕ_1 and ϕ_2 at various values of *n*.

n	ϕ_1	ϕ_2
2	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{9}$	$\frac{1}{3}$
4	$\frac{1}{16}$	$\frac{5}{12}$
5	$\frac{1}{25}$	$\frac{1}{2}$
6	$\frac{1}{36}$	$\frac{7}{12}$
7	$\frac{1}{49}$	$\frac{2}{3}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n^2}$	$\frac{(n+1)}{12}$

Therefore the 2nd Order for s=1 can be written as follows:

$$O_2(x_1, x_2, \cdots, x_{n-1}, x_n) = \sum_{i<j}^{n} x_i x_j = \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{(n+1)}{12}$$

Repeating the same process for various "*s*" yields:

Table 12 The value of coefficient ϕ_1 with various value of "*s*".

n	ϕ_1 (*s*=1)	ϕ_1 (*s*=2)	ϕ_1 (*s*=3)	ϕ_1 (*s*=4)
2	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
5	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
7	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$	$\frac{1}{49}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$	$\frac{1}{n^2}$

Table 13 The value of coefficient ϕ_1 with various values of "*s*".

n	ϕ_2 (*s*=1)	ϕ_2 (*s*=2)	ϕ_2 (*s*=3)	ϕ_2 (*s*=4)
2	$\frac{1}{4}$	1	$\frac{9}{4}$	4
3	$\frac{1}{3}$	$\frac{4}{3}$	3	$\frac{16}{3}$
4	$\frac{5}{12}$	$\frac{5}{3}$	$\frac{15}{4}$	$\frac{20}{3}$
5	$\frac{1}{2}$	2	$\frac{9}{2}$	8
6	$\frac{7}{12}$	$\frac{7}{3}$	$\frac{21}{4}$	$\frac{28}{3}$
7	$\frac{2}{3}$	$\frac{8}{3}$	6	$\frac{32}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{(n+1)}{12}$	$\frac{(n+1)2^2}{12}$	$\frac{(n+1)3^2}{12}$	$\frac{(n+1)4^2}{12}$

From the Table 13, it can be deduced that ϕ_2 can be written as follows

$$\phi_2 = \frac{(n+1)s^2}{12}$$

Therefore, the second order can be written as follows:

2nd Order

$$O_2\left(x_1, x_2, \cdots, x_{n-1}, x_n\right) = x_1x_2 + x_1x_3 + \cdots + x_ix_j = \sum_{i<j}^{n} x_ix_j = \binom{n}{2}\left(\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 - \frac{(n+1)s^2}{12}\right) \quad [98]$$

3rd Order

$$O_3\left(x_1, x_2, \cdots, x_{n-1}, x_n\right) = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_ix_jx_k = \sum_{i<j<k}^{n} x_ix_jx_k = \binom{n}{3}\left(\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 - \frac{(n+1)s^2}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\right) \quad [99]$$

4th Order

$$O_4\left(x_1, x_2, \cdots, x_{n-1}, x_n\right) = \sum_{i<j<k<l}^{n} x_ix_jx_kx_l = \binom{n}{4}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^4 - \frac{(n+1)s^2}{2}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 + \frac{(n+1)(5n+7)s^4}{240}\right] \quad [100]$$

5th Order

$$O_5\left(x_1, x_2, \cdots, x_{n-1}, x_n\right) = \sum_{i<j<k<l<m}^{n} x_ix_jx_kx_lx_m = \binom{n}{5}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^5 - \frac{5(n+1)s^2}{6}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 + \frac{(n+1)(5n+7)s^4}{48}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\right] \quad [101]$$

6th Order

$$O_6\left(x_1, x_2, \cdots, x_{n-1}, x_n\right) = \sum_{i<j<k<l<m<n}^{n} x_ix_jx_kx_lx_mx_n = \binom{n}{6}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^6 - \frac{5(n+1)s^2}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^4 + \frac{(n+1)(5n+7)s^4}{16}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 - \frac{(n+1)(35 \cdot n^2 + 112 \cdot n + 93)s^6}{4032}\right] \quad [102]$$

7th Order

$$O_7\left(x_1, x_2, \cdots, x_{n-1}, x_n\right) = \sum_{i<j<k<l<m<n<p}^{n} x_ix_jx_kx_lx_mx_nx_p = \binom{n}{7}\left[\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^7 - \frac{7(n+1)s^2}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^5 + \frac{7(n+1)(5n+7)s^4}{48}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^3 - \frac{(n+1)(35 \cdot n^2 + 112 \cdot n + 93)s^6}{576}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]\right] \quad [103]$$

8th Order

$$Q_8(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{i<j<k<l<m<n<o<p}^{n} x_i x_j x_k x_l x_m x_n x_o x_p=\binom{n}{8}\left[\begin{array}{l}\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^8-\dfrac{7(n+1)s^2}{3}\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^6+\dfrac{7(n+1)(5n+7)s^4}{24}\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^4\\ -\dfrac{(n+1)(35\cdot n^2+112\cdot n+93)s^6}{144}\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^2+\\ \dfrac{(n+1)(5n+9)(35\cdot n^2+126\,n+127)s^8}{34560}\end{array}\right] \quad [104]$$

9th Order can be calculated by using the same coefficients used in Order 8th, it is given as follows:

$$Q_9(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{\mathrm{i<j<k<l<m<n<o<p<q}}^{n} x_i x_j x_k x_l x_m x_n x_o x_p x_q=\binom{n}{9}\left[\begin{array}{l}\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^9-Q_1(n+1)s^2\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^7+Q_2(n+1)(5n+7)s^4\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^5\\ -Q_3(n+1)(35\,n^2+112n+93)s^6\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]^3+\\ Q_4(n+1)(5n+9)(35\,n^2+126n+127)s^8\left[\dfrac{\sum_{i=1}^{n}x_i}{n}\right]\end{array}\right] \quad [105]$$

Coefficients $[Q_1\cdots Q_2]$ can be calculated by using product identity of an arithmetic progression when n=9, the calculation is given as follows:

$$P_9=x_1\cdot x_2\cdots x_9=\frac{1}{9^9}\left(\left(\sum_{i=1}^{n}x_i\right)^2-(9s)^2\right)\cdot\left(\left(\sum_{i=1}^{n}x_i\right)^2-(18s)^2\right)\left(\left(\sum_{i=1}^{n}x_i\right)^2-(27s)^2\right)\left(\left(\sum_{i=1}^{n}x_i\right)^2-(36s)^2\right)\sum_{i=1}^{n}x_i \quad [106]$$

$$P_9=x_1\cdot x_2\cdots x_9=\frac{\left(\sum_{i=1}^{n}x_i\right)^9}{387420489}-\frac{10\left(\sum_{i=1}^{n}x_i\right)^7 s^2}{1594323}+\frac{91\left(\sum_{i=1}^{n}x_i\right)^5 s^4}{19683}-\frac{820\left(\sum_{i=1}^{n}x_i\right)^3 s^6}{729}+64\left(\sum_{i=1}^{n}x_i\right)s^8 \quad [107]$$

Comparing the coefficients yields:

$$Q_1=3,\ Q_2=\frac{21}{40},\ Q_3=\frac{1}{48}\text{ and }Q_4=\frac{1}{3840}$$

The last coefficient for each order can be found by solving the matrix of least square method as follows:

The 10th Order can be calculated as follows:

$$O_{10}(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{i<j<k<l<m<n<o<p<q<r}^{n} x_i x_j \cdots x_q x_r = \binom{n}{10}\left[\begin{array}{l} \left[\dfrac{\sum_{i=1}^{n} x_i}{n}\right]^{10} - Q_1(n+1)s^2\left[\dfrac{\sum_{i=1}^{n} x_i}{n}\right]^{8} + Q_2(n+1)(5n+7)s^4\left[\dfrac{\sum_{i=1}^{n} x_i}{n}\right]^{6} \\ -Q_3(n+1)(35\cdot n^2+112\cdot n+93)s^6\left[\dfrac{\sum_{i=1}^{n} x_i}{n}\right]^{4} + \\ Q_4(n+1)(5n+9)\left(35\cdot n^2+126\,n+127\right)s^8\left[\dfrac{\sum_{i=1}^{n} x_i}{n}\right]^{2} - T_k \end{array}\right] \quad [108]$$

The Generalised Equation can be written as follows:

$$O_m(x_1,x_2,\cdots,x_{n-1},x_n)=\sum_{i<j<\cdots<z}^{n} x_i x_j \cdots x_z = \binom{n}{m}\sum_{v=0}^{k}\left[(-1)^v Q_v T_v s^{2v}\left[\dfrac{\sum_{i=1}^{n} x_i}{n}\right]^{(m-2v)}\right] \quad [109]$$

Where coefficient $Q_0 = 1$ and $k = \begin{cases} \dfrac{m-1}{2}\ for\_odd\_m \\ \dfrac{m}{2}\ for\_even\_m \end{cases}$ [110]

The last coefficient T_k can be calculated by performing least square method analysis on various values of "n".

Let the last coefficient in this form:

$$T_k = \sum_{j=0}^{k} a_k n^{k-j} \quad [111]$$

The value of T_k at various values of n can be calculated as follows:

$$T_k = \frac{\sum_{v=0}^{\frac{k-1}{2}}\left[(-1)^v Q_v T_v s^{2v}\left(\dfrac{\sum_{i=0}^{n} x_i}{n}\right)^{(m-2v)}\right] - \dfrac{O_m}{\binom{n}{m}}}{Q_k s^{2k}} \qquad \text{for even } m \quad [112]$$

$$T_k = \frac{\sum_{v=0}^{\frac{k-1}{2}} \left[(-1)^v Q_v T_v s^{2v} \left(\frac{\sum_{i=0}^{n} x_i}{n} \right)^{(m-2v)} \right] - \frac{O_m}{\binom{n}{m}}}{Q_k s^{2k} \left(\frac{\sum_{i=0}^{n} x_i}{n} \right)} \qquad \text{for odd } m \qquad [113]$$

Where Q_k is a coefficient and is given as follows:

$$Q_k = \left(\frac{1}{a}\right)\left(\frac{1}{C_1}\right)\left(\frac{1}{2^{2k}}\right)\left(\frac{1}{2k+1}\right)\binom{p}{2k}$$

Where a and C_1 are the coefficients. The coefficient C_1 is the coefficient of first term of equation P_m as given in the equation [173]. While a is found to be 3 for $3 \le k \ge 5$, further study needed to be done in order to find the value of a for $k > 5$. For simplicity in calculation it is advisable to use the value of $O_m = P_n$. In which the elementary symmetric function of m-th order reduces to product identity of order n-th.

For some value of n, we can construct the matrix to solve the equation

$$T_k = \left(a_0 n^k + a_1 n^{k-1} + a_2 n^{k-2} \cdots a_{k-1} n + a_k\right) \qquad [114]$$

Let the equations at various value of n as a matrix in this form,

$$\begin{pmatrix} T_i \\ T_j \\ \vdots \\ T_t \\ T_u \end{pmatrix} = \begin{pmatrix} n_i^k & n_i^{k-1} & \cdots & n_i & 1 \\ n_j^k & n_j^{k-1} & \cdots & n_j & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ n_t^k & n_t^{k-1} & \cdots & n_t & 1 \\ n_u^k & n_u^{k-1} & \cdots & n_u & 1 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \\ a_k \end{pmatrix} \qquad [115]$$

Or

$$T = N \cdot a \Rightarrow a = N^{-1} \cdot T \qquad [116]$$

These orders are useful to construct sums of power of the arithmetic progression, as the sums of power of the arithmetic progression can always be expressed into elementary symmetric polynomials.

4 Using Multinomial Theorem and Product Arithmetic Terms in Generating Sums of Power for an Arbitrary Arithmetic term.

The Multinomial Theorem states that if *p* is nonnegative integers then

$$(x_1 + x_2 + \cdots + x_k)^p = \sum \binom{p}{r_1, r_2 \ldots r_k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} \qquad [117]$$

In this research it is proposed that The Multinomial Theorem for arbitrary arithmetic progression can be expressed as the power of arithmetic sum descending by 2 for each subsequent term (i.e *p*-2*j*). The equation is given as follows:

$$(x_1 + x_2 + \cdots + x_k)^p = \sum \binom{p}{r_1, r_2 \ldots r_k} x_1^{r_1} x_2^{r_2} \cdots x = \sum_{j=0}^{k} \left[\phi_k s^{2k} \frac{\left[\sum_{i=1}^{n} x_i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \qquad [118]$$

This relationship is actually the building block for sum of power of arbitrary arithmetic progression. The sum of powers can be calculated directly from this relationship; however for larger *p* the calculation would be tedious. Each monomial term in the multinomial can also be expressed as follows:

$$\sum_{i<j<\cdots<n}^{n} x_i^{r_1} x_j^{r_2} \cdots x_n^{r_n} = \sum_{j=0}^{k} \left[\ddot{\phi}_k s^{2k} \frac{\left[\sum_{i=1}^{n} x_i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \qquad [119]$$

Where: $p-(2j+1) \geq -1$ if p is even, $p-(2j+1) \geq 0$ if p is odd, $s = x_{i+1} - x_i$, ϕ_k is a coefficient,

$$k = \begin{cases} \frac{p-1}{2} \; for\_odd\_p \\ \frac{p}{2} \; for\_even\_p \end{cases} \quad \text{and} \quad p = \sum_{i=1}^{n} r_i \qquad [120]$$

Therefore let's consider the equation below for *p*=2,

$$\sum_{i<j}^{n} x_i x_j = \alpha_1 \left(\sum_{i=1}^{n} x_i \right)^2 + \alpha_2 s^2 \qquad [121]$$

Solving the coefficients yields,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_n = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^2 & s^2 \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^2 & s^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i<j}^{n} x_i x_j \\ \sum_{i<j}^{n} x_{i+1} x_{j+1} \end{pmatrix} \qquad [122]$$

When $n = 2$, the solution is given as follows:

$$\sum_{i<j}^{2} x_i x_j = \frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{1}{4} s^2 \qquad [123]$$

Calculating for some values of *n* and the tabulated data is given as follows:

Table 14 The values of α_1 and α_2 at various values of *n*.

n	α_1	α_2
2	$\frac{1}{4}$	$-\frac{1}{4}$
3	$\frac{1}{3}$	-1
4	$\frac{3}{8}$	$-\frac{5}{2}$
5	$\frac{2}{5}$	-5
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(n-1)}{2n}$	$-\frac{n(n^2-1)}{24}$

Therefore, for all *n* the equation is given as follows:

$$\sum_{i<j}^{n} x_i x_j = \frac{(n-1)}{2n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{24} s^2 \qquad [124]$$

When *p*=3, consider the equation below,

$$\sum_{i<j}^{n} x_i x_j^2 = \beta_1\left(\sum_{i=1}^{n} x_i\right)^3 + \beta_2\left(\sum_{i=1}^{n} x_i\right) s^2 \qquad [125]$$

Solving the coefficients yields,

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}_n = \begin{pmatrix} \left(\sum_{i=1}^{n} x_i\right)^3 & \left(\sum_{i=1}^{n} x_i\right)s^2 \\ \left(\sum_{i=1}^{n} x_{i+1}\right)^3 & \left(\sum_{i=1}^{n} x_{i+1}\right)s^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{i<j}^{n} x_i x_j^2 \\ \sum_{i<j}^{n} x_{i+1} x_{j+1}^2 \end{pmatrix} \qquad [126]$$

When $n = 2$, the solution is given as follows:

$$\sum_{i<j}^{2} x_i x_j^2 = \frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)s^2 \qquad [127]$$

Calculating for some values of *n* and the tabulated data is given as follows:

Table 15 The values of β_1 and β_2 at various values of *n*.

n	β_1	β_2
2	$\frac{1}{4}$	$-\frac{1}{4}$
3	$\frac{2}{9}$	0
4	$\frac{3}{16}$	1.25
5	$\frac{4}{25}$	4
$\vdots$	$\vdots$	$\vdots$
n	$\frac{(n-1)}{n^2}$	$\frac{(n^2-1)(n-3)}{12}$

Therefore, for all *n* the equation is given as follows:

$$\sum_{i<j}^{n} x_i x_j^2 = \frac{(n-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 + \frac{(n^2-1)(n-3)}{12}\left(\sum_{i=1}^{n} x_i\right)s^2 \qquad [128]$$

Applying the same procedures, we get equations as follows:

$$\sum_{i<j}^{n} x_i x_j^3 = \frac{(n-1)}{n^3}\left(\sum_{i=1}^{n} x_i\right)^4 + \frac{(n^2-1)(n-2)}{4n}\left(\sum_{i=1}^{n} x_i\right)s^2 - \frac{n(n^2-1)(3n^2-7)}{240}s^4 \qquad [129]$$

$$\sum_{i<j}^{n} x_i x_j^4 = \frac{(n-1)}{n^4}\left(\sum_{i=1}^{n} x_i\right)^5 + \frac{(n^2-1)(3n-5)}{6n^2}\left(\sum_{i=1}^{n} x_i\right)^3 s^2 + \frac{(n^2-1)(3n^2-7)(n-5)}{240}\left(\sum_{i=1}^{n} x_i\right)s^4 \qquad [130]$$

$$\sum_{i<j}^{n} x_i x_j^5 = \frac{(n-1)}{n^5}\left(\sum_{i=1}^{n} x_i\right)^6 + \frac{5(n^2-1)(2n-3)}{12n^3}\left(\sum_{i=1}^{n} x_i\right)^4 s^2 + \frac{(n^2-1)(3n^2-7)(n-3)}{48n}\left(\sum_{i=1}^{n} x_i\right)^2 s^4 - \frac{n(n^2-1)(3n^4-18n^2+31)}{1344}s^6 \qquad [131]$$

$$\sum_{i<j}^{n} x_i x_j^6 = \frac{(n-1)}{n^6}\left(\sum_{i=1}^{n} x_i\right)^7 + \frac{(n^2-1)(5n-7)}{4n^4}\left(\sum_{i=1}^{n} x_i\right)^5 s^2 + \frac{(n^2-1)(3n^2-7)(3n-7)}{48n^2}\left(\sum_{i=1}^{n} x_i\right)^3 s^4 + \frac{n(n^2-1)(3n^4-18n^2+31)(n-7)}{1344}\left(\sum_{i=1}^{n} x_i\right)s^6 \qquad [132]$$

Combining the results for all n yields

For odd q,

$$\sum_{i<j}^{n} x_i x_j^q = \frac{(n-1)}{n^q}\left(\sum_{i=1}^{n} x_i\right)^{q+1} + \frac{\phi_1}{n^{q-2}}\left(n-\frac{q+1}{q-1}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-1} s^2 + \frac{\phi_2}{n^{q-4}}\left(n-\frac{q+1}{q-3}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-3} s^4 + \cdots + \frac{\phi_{\frac{q-1}{2}}}{n}\left(n-\frac{q+1}{2}\right)\left(\sum_{i=1}^{n} x_i\right)^2 s^{q-1} - n\phi_{\frac{q+1}{2}}s^{q+1} \qquad [133]$$

$$\sum_{i<j}^{n} x_i x_j^q = \sum_{j=0}^{\frac{q-1}{2}}\left[\frac{\phi_j s^{2j}}{n^{q-2j}}\left(n-\frac{(q+1)}{q+(1-2j)}\right)\left(\sum_{i=1}^{n} x_i\right)^{q+(1-2j)}\right] - n\phi_{\frac{q+1}{2}}s^{q+1} \qquad [134]$$

Or

For even q,

$$\sum_{i<j}^{n} x_i x_j^q = \frac{(n-1)}{n^q}\left(\sum_{i=1}^{n} x_i\right)^{q+1} + \frac{\phi_1}{n^{q-2}}\left(n-\frac{q+1}{q-1}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-1} s^2 + \frac{\phi_2}{n^{q-4}}\left(n-\frac{q+1}{q-3}\right)\left(\sum_{i=1}^{n} x_i\right)^{q-3} s^4 + \cdots + \phi_{\frac{q}{2}}\left(n-(q+1)\right)\left(\sum_{i=1}^{n} x_i\right)s^q \qquad [135]$$

$$\sum_{i<j}^{n} x_i x_j^q = \sum_{j=0}^{\frac{q}{2}}\left[\frac{\phi_j s^{2j}}{n^{q-2j}}\left(n-\frac{(q+1)}{q+(1-2j)}\right)\left(\sum_{i=1}^{n} x_i\right)^{q+(1-2j)}\right] \qquad [136]$$

Where, ϕ_j is a function of Bernoulli numbers and $\left(q+(1-2j)\right)\neq 0$ (i.e. the denominator of $(q+1)$ is not zero) and if denominator is zero, the expansion of the term takes the last forms of $n\phi_{\frac{q+1}{2}}s^{q+1}$ or $\phi_{\frac{q}{2}}\left(n-(q+1)\right)\left(\sum_{i=1}^{n} x_i\right)s^q$ for odd and even q respectively.

Consider when p=2,

$$\left(\sum_{i=1}^{n} x_i\right)^2 = (x_1 + x_2 + \cdots + x_{n-1} + x_n)^2 = (x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2) + \binom{2}{1\ \ 1}(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n)$$

$$\left(\sum_{i=1}^{n} x_i\right)^2 = (x_1 + x_2 + \cdots + x_{n-1} + x_n)^2 = \sum_{i=1}^{n} x_i^2 + \binom{2}{1\ \ 1}\sum_{i<j}^{n} x_i x_j \qquad [137]$$

Rearranging the equation [137], yields

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - \binom{2}{1\ \ 1}\sum_{i<j}^{n} x_i x_j$$

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\sum_{i<j}^{n} x_i x_j \qquad [138]$$

Since, $\sum_{i<j}^{n} x_i x_j = \frac{(n-1)}{2n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{24}s^2$, then [139]

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2\left(\frac{(n-1)}{2n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{24}s^2\right)$$

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - \left(\frac{(n-1)}{n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n(n^2-1)}{12}s^2\right)$$

$$\sum_{i=1}^{n} x_i^2 = \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n} + \frac{n(n^2-1)}{12}s^2 \qquad [140]$$

Consider when p=3,

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + \binom{3}{1\ \ 2}\sum_{i<j}^{n} x_i x_j^2 + \binom{3}{1\ \ 1\ \ 1}\sum_{i<j<k}^{n} x_i x_j x_k \qquad [141]$$

Since, $\sum_{i<j}^{n} x_i x_j^2 = \frac{(n-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 + \frac{(n^2-1)(n-3)}{12}\left(\sum_{i=1}^{n} x_i\right)s^2$ and $\sum_{i<j<k}^{n} x_i x_j x_k$ is the elementary symmetric function of 3rd order, then

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + 3\left(\frac{(n-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 + \frac{(n^2-1)(n-3)}{12}\left(\sum_{i=1}^{n} x_i\right)s^2\right) + 6\binom{n}{3}\left(\frac{1}{n^3}\left(\sum_{i=1}^{n} x_i\right)^3 - \frac{(n+1)s^2}{4n}\left(\sum_{i=1}^{n} x_i\right)\right)$$

$$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + \frac{(n^2-1)}{n^2}\left(\sum_{i=1}^{n} x_i\right)^3 - \frac{(n^2-1)}{4}s^2\left(\sum_{i=1}^{n} x_i\right) \quad [142]$$

Rearranging equation [142], yields

$$\left(\sum_{i=1}^{n} x_i^3\right) = \frac{\left(\sum_{i=1}^{n} x_i\right)^3}{n^2} + \frac{(n^2-1)s^2}{4}\left(\sum_{i=1}^{n} x_i\right) \quad [143]$$

The multinomial also can be expressed as follows:

$$\left(\sum_{i=1}^{n} x_i\right)^p = \left(\sum_{i=1}^{n} x_i^p\right) + \sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} + \sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} + \cdots + \sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_k^{r_3} \quad [144]$$

Rearranging equation [144], yields

$$\left(\sum_{i=1}^{n} x_i^p\right) = \left(\sum_{i=1}^{n} x_i\right)^p - \left[\sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} + \sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} + \cdots + \sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_k^{r_3}\right] \quad [145]$$

The proposed conjecture reiterates that for all monomials for an arbitrary arithmetic progression can always be expanded as follows:

$$\sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} = \alpha_0\left(\sum_{i=1}^{n} x_i\right)^p + \alpha_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \alpha_{\frac{p-1}{2}}\left(\sum_{i=1}^{n} x_i\right)s^{p-1} \text{ for odd } p \quad [146]$$

$$\sum\binom{p}{r_1 \quad r_2} x_i^{r_1} x_j^{r_2} = \alpha_0\left(\sum_{i=1}^{n} x_i\right)^p + \alpha_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \alpha_{\frac{p}{2}} s^p \text{ for even } p \quad [147]$$

$$\sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} = \beta_0\left(\sum_{i=1}^{n} x_i\right)^p + \beta_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \beta_{\frac{p-1}{2}}\left(\sum_{i=1}^{n} x_i\right)s^{p-1} \text{ for odd } p \quad [148]$$

$$\sum\binom{p}{r_1 \quad r_2 \quad r_3} x_i^{r_1} x_j^{r_2} x_k^{r_3} = \beta_0\left(\sum_{i=1}^{n} x_i\right)^p + \beta_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \beta_{\frac{p}{2}} s^p \text{ for even } p \quad [149]$$

$$\sum\binom{p}{r_1 \quad \cdots \quad r_k} x_i^{r_1} \cdots x_m^{r_k} = \gamma_0\left(\sum_{i=1}^{n} x_i\right)^p + \gamma_1\left(\sum_{i=1}^{n} x_i\right)^{p-2} s^2 + \cdots + \gamma_{\frac{p-1}{2}}\left(\sum_{i=1}^{n} x_i\right)s^{p-1} \text{ for odd } p \quad [150]$$

$$\sum \begin{pmatrix} & p & \\ r_1 & \cdots & r_k \end{pmatrix} x_i^{r_1} \cdots x_m^{r_k} = \gamma_0 \left(\sum_{i=1}^{n} x_i \right)^p + \gamma_1 \left(\sum_{i=1}^{n} x_i \right)^{p-2} s^2 + \cdots + \gamma_{\frac{p}{2}} s^p \quad \text{for even } p \qquad [151]$$

Collecting the coefficients yields:

$$\left(\sum_{i=1}^{n} x_i^p \right) = \left(1 - (\alpha_0 + \beta_0 + \cdots + \gamma_0)\right) \left(\sum_{i=1}^{n} x_i \right)^p - \left(\alpha_1 + \beta_1 + \cdots + \gamma_1\right) \left(\sum_{i=1}^{n} x_i \right)^{p-2} s^2 - \cdots - \left(\alpha_{\frac{p-1}{2}} + \beta_{\frac{p-1}{2}} + \cdots + \gamma_{\frac{p-1}{2}} \right) \left(\sum_{i=1}^{n} x_i \right) s^{p-1} \quad \text{for odd } p \qquad [152]$$

$$\left(\sum_{i=1}^{n} x_i^p \right) = \left(1 - (\alpha_0 + \beta_0 + \cdots + \gamma_0)\right) \left(\sum_{i=1}^{n} x_i \right)^p - \left(\alpha_1 + \beta_1 + \cdots + \gamma_1\right) \left(\sum_{i=1}^{n} x_i \right)^{p-2} s^2 - \cdots - \left(\alpha_{\frac{p}{2}} + \beta_{\frac{p}{2}} + \cdots + \gamma_{\frac{p}{2}} \right) s^p \quad \text{for even } p \qquad [153]$$

Where $\left(1 - (\alpha_0 + \beta_0 + \cdots + \gamma_0)\right) = \dfrac{1}{n^{p-1}}$.

5 Reverse Look-up Method for Generating Sums of Power of Arithmetic terms.

The coefficients ϕ_m involved in the polynomials up to p=12 can be simplified as follows

$$\phi_0 = 1 \qquad [154]$$

$$\phi_1 = \frac{p(p-1)}{24}(n^2 - 1) \qquad [155]$$

$$\phi 2 = \frac{p(p-1)(p-2)(p-3)}{24^2(10)}(3n^2 - 7)(n^2 - 1) \qquad [156]$$

$$\phi_3 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{24^3(70)}(3n^4 - 18n^2 + 31)(n^2 - 1) \qquad [157]$$

$$\phi_4 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)}{24^4(1400)}(5n^6 - 55n^4 + 239n^2 - 381)(n^2 - 1) \qquad [158]$$

$$\phi_5 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)}{24^5(15400)}(3n^{10} - 55n^8 + 462n^6 - 2046n^4 + 4191n^2 - 2555) \text{ or}$$

$$\phi_5 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)}{24^5(15400)}(3n^6 - 37n^4 + 225n^2 - 511)(n^2 - 5)(n^2 - 1) \qquad [159]$$

$$\phi_6 = \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)(p-10)(p-11)}{24^6(400400)}(105n^{12} - 2730n^{10} + 35035n^8 - 265980n^6 + 1144143n^4 - 2325050n^2 + 1414477) \qquad [160]$$

or in binomial expansion forms

$$\phi_1 = \frac{1}{4}\frac{1}{(2m+1)}\binom{p}{2m}(n^2 - 1) \qquad [161]$$

$$\phi 2 = \frac{1}{48}\frac{1}{(2m+1)}\binom{p}{2m}(3n^2 - 7)(n^2 - 1) \qquad [162]$$

$$\phi_3 = \frac{1}{192}\frac{1}{(2m+1)}\binom{p}{2m}(3n^4 - 18n^2 + 31)(n^2 - 1) \quad [163]$$

$$\phi_4 = \frac{1}{1280}\frac{1}{(2m+1)}\binom{p}{2m}(5n^6 - 55n^4 + 239n^2 - 381)(n^2 - 1) \quad [164]$$

$$\phi_5 = \frac{1}{3072}\frac{1}{(2m+1)}\binom{p}{2m}(3n^6 - 37n^4 + 225n^6 - 511)(n^2 - 5)(n^2 - 1) \quad [165]$$

$$\phi_6 = \frac{1}{430080}\frac{1}{(2m+1)}\binom{p}{2m}(105n^{10} - 2625n^8 + 32410n^6 - 233570n^4 + 910573n^2 - 1414477)(n^2 - 1) \quad [166]$$

The generalize form of ϕ_m can be written as follows

$$\phi_m = \frac{1}{c_1 2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}\sum_{t=0}^{m} c_{t+1} n^{2(m-t)}(-1)^t \quad [167]$$

Or

$$\phi_m = \frac{1}{2^{2m}}\frac{1}{(2m+1)}\binom{p}{2m}P_m \quad [168]$$

where $(m-t) \geq 0$. The polynomials can be expressed as $P_m = \sum_{t=0}^{m} C_{t+1} n^{2(m-t)}(-1)^t$. In order to identify how the coefficients are formed, each term in the polynomial is tabulated in a table. The tabulated data is gives as in the Table 15.

Table 15 The terms values for P_m.

P_m	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term	7^{th} term		
P_1	n^2	-1							
P_2	$3n^4$	$-10n^2$	7						
P_3	$3n^6$	$-21n^4$	$49n^2$	-31					
P_4	$5n^8$	$-60n^6$	$294n^4$	$-620n^2$	381				
P_5	$3n^{10}$	$-55n^8$	$462n^6$	$-2046n^4$	$4191n^2$	-2555			
P_6	$105n^{12}$	$-2730n^{10}$	$35035n^8$	$-265980n^6$	$1144143n^4$	$-2325050n^2$	1414477		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
P_m	C_1n^m	C_2n^{2m-2}	C_3n^{2m-4}	C_4n^{2m-6}	C_5n^{2m-8}	C_6n^{2m-19}	C_7n^{2m-12}	….	C_m

By tabulating the value of $\frac{\left|\sum_{t=0}^{m} C_{t+1}\right|}{C_1}$, yields new data and it is given as in the Table 16.

Table 16 The terms normalized values for P_m .

P_m	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term	7^{th} term		
P_1	1	1							
P_2	1	$\frac{10}{3}$	$\frac{7}{3}$						
P_3	1	7	$\frac{49}{3}$	$\frac{31}{3}$					
P_4	1	12	$\frac{294}{5}$	124	$\frac{381}{5}$				
P_5	1	$\frac{55}{3}$	154	682	1397	$\frac{2555}{3}$			
P_6	1	26	$\frac{1001}{3}$	$\frac{17732}{7}$	$\frac{54483}{5}$	$\frac{66430}{3}$	$\frac{1414477}{3}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
P_m	1	$\frac{C_2}{C_1}$	$\frac{C_3}{C_1}$	$\frac{C_4}{C_1}$	$\frac{C_5}{C_1}$	$\frac{C_6}{C_1}$	$\frac{C_7}{C_1}$	….	$\frac{C_m}{C_1}$

Plotting the P_m curves for some *m* yields,

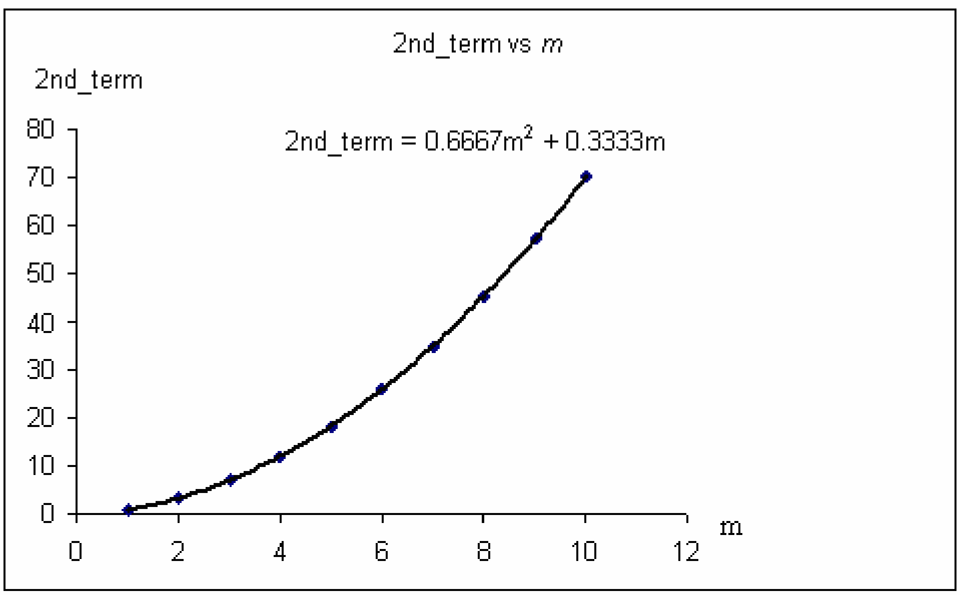

Figure 10 Graph of 2nd term versus *m*.

From this curve we can deduce that

$$2nd\_term = \frac{m}{3}(2m+1) \qquad [169]$$

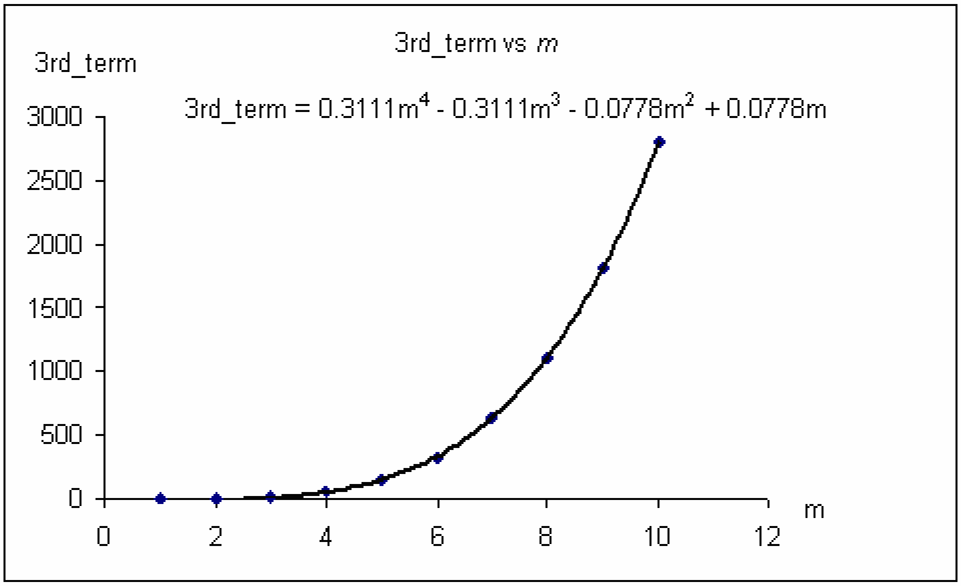

Figure 11 Graph of 3rd term versus *m.*

From this curve we can deduce that

$$3nd\_term = \frac{7m}{3^2 x10}(m-1)(2m+1)(2m-1) \qquad [170]$$

The term can be rewritten as $T_m = \gamma_m \cdot f(m)$

where $f(m)$ is a function of *m* and γ_m is a coefficient which depends on the Bernoulli number and $\left(2^{2m-1}-1\right)$.

Analyzing γ_m for some terms yields:

Table 17 The values of γ_m and B_m at various values of *m*.

m	γ_m	B_m	$\gamma_m = \zeta\left(2^{2m-1}-1\right)B_m$
0	1	-1	$-\left(2^{-1}-1\right)\cdot 2B_0$
1	$\frac{1}{3}$	$\frac{1}{6}$	$\left(2^{1}-1\right)\cdot 2B_1$
2	$\frac{7}{90}$	$-\frac{1}{30}$	$-\left(2^{3}-1\right)\cdot \frac{B_2}{3}$
3	$\frac{31}{1890}$	$\frac{1}{42}$	$\left(2^{5}-1\right)\cdot \frac{B_3}{45}$
4	$\frac{381}{113400}$	$-\frac{1}{30}$	$-\left(2^{7}-1\right)\cdot \frac{B_4}{1260}$
5	$\frac{2555}{3742200}$	$\frac{5}{66}$	$\left(2^{9}-1\right)\cdot \frac{B_5}{56700}$
6	$\frac{1414477}{3^6\cdot 35\cdot 400400}$	$-\frac{691}{2730}$	$-\left(2^{11}-1\right)\cdot \frac{B_6}{3742200}$
7	$\frac{860055}{3^7\cdot 35\cdot 400400}$	$\frac{7}{6}$	$\left(2^{13}-1\right)\cdot \frac{B_7}{340540200}$
8	$\frac{118518239}{3^7\cdot 35\cdot 400400\cdot 680}$	$-\frac{3617}{510}$	$-\left(2^{15}-1\right)\cdot \frac{B_8}{40864824000}$

9	$\dfrac{5749691557}{3^9 \cdot 35 \cdot 18088 \cdot 400400}$	$\dfrac{43867}{798}$	$\left(2^{17}-1\right)\cdot\dfrac{B_9}{6252318072000}$
10	$\dfrac{1922471824497}{3^{10} \cdot 35 \cdot 400400 \cdot 9948400}$	$-\dfrac{174611}{330}$	$-\left(2^{19}-1\right)\cdot\dfrac{B_{10}}{1187940433680000}$
11	$\dfrac{8960213962315}{3^{10} \cdot 35 \cdot 400400 \cdot 228813200}$	$\dfrac{854513}{138}$	$\left(2^{21}-1\right)\cdot\dfrac{B_{11}}{27441424018008000}$
12	$\dfrac{1982765468311237}{3^{10} \cdot 35 \cdot 400400 \cdot 19752096 \cdot 12650}$	$-\dfrac{236364091}{2730}$	$-\left(2^{23}-1\right)\cdot\dfrac{B_{12}}{7573833028970208000}$

From the Table 17, apparently P_m can be formulated as follows:

$$f(m) = -2\sum_{t=1}^{m}\left[(2t+1)\binom{m}{t} n^{2(m-t)} \frac{\prod_{j=0}^{t-1}(2m-2j+1)}{\prod_{j=0}^{t}(2t-2j+1)}\right] \quad [171]$$

Since 1st term is n^{2m} then,

$$P_m = n^{2m} + f(m) \quad [172]$$

Therefore, P_m is given as follows:

$$P_m = \left[n^{2m} - 2\sum_{t=1}^{m}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t n^{2(m-t)} \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] \quad [173]$$

Where, B_t is the Bernoulli's number and $m \geq 1$.

When m=1

$$P_1 = \left[n^{2(1)} - 2\sum_{t=1}^{1}\left[(2t+1)(2^{2t-1}-1)\binom{1}{t} B_t n^{2(1-t)} \frac{\prod_{j=0}^{t-1}(1+2(1-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right]$$

$$= n^{2(1)} - 2(2(1)+1)(2^{2(1)-1}-1)\binom{1}{1} B_1 n^{2(1-1)} \frac{\prod_{j=0}^{0}(3-2j)}{\prod_{j=0}^{1}(3-2j)}$$

$$= n^2 - 6\left(\frac{1}{6}\right)\frac{\prod_{j=0}^{0}(3-2j)}{\prod_{j=0}^{1}(3-2j)} = n^2 - \frac{(3)}{(3)\cdot(1)} = n^2 - 1 \quad [174]$$

When m=2

$$P_2 = \left[n^{2(2)} - 2\sum_{t=1}^{2} \left[(2t+1)(2^{2t-1}-1)\binom{2}{t} B_t n^{2(2-t)} \frac{\prod_{j=0}^{t-1}(1+2(2-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] \right]$$

$$= \left[\begin{array}{l} n^4 - 2(2(1)+1)(2^{2(1)-1}-1)\binom{2}{1} B_1 n^{2(2-1)} \frac{\prod_{j=0}^{0}(1+2(2-j))}{\prod_{j=0}^{1}(1+2(t-j))} - \\ 2(2(2)+1)(2^{2(2)-1}-1)\binom{2}{2} B_2 n^{2(2-2)} \frac{\prod_{j=0}^{1}(1+2(2-j))}{\prod_{j=0}^{2}(1+2(2-j))} \end{array} \right]$$

$$= \left[n^4 - 12\left(\frac{1}{6}\right) n^2 \frac{\prod_{j=0}^{0}(5-2j)}{\prod_{j=0}^{1}(3-2j)} - 70\left(-\frac{1}{30}\right) n^0 \frac{\prod_{j=0}^{1}(5-2j)}{\prod_{j=0}^{2}(5-2j)} \right]$$

$$= \left[n^4 - 2n^2 \frac{(5)}{(3)(1)} + \frac{7}{3}\frac{(5)(3)}{(5)(3)(1)} \right] = n^4 - \frac{10}{3}n^2 + \frac{7}{3} = \frac{1}{3}\left(3n^4 - 10n^2 + 7\right) \qquad [175]$$

Therefore,

$$\phi_m = \frac{1}{2^{2m}} \frac{1}{(2m+1)} \binom{p}{2m} P_m$$

$$= \frac{1}{2^{2m}} \frac{1}{(2m+1)} \binom{p}{2m} \left[n^{2m} - 2\sum_{t=1}^{m} \left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t n^{2(m-t)} \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] \right] \qquad [176]$$

Setting $x_1 = 1$ and s=1, this equation reduces into classical Faulhaber's Sum of Power for integers.

Power sums for Integers generalize equation is given as follows:

$$\sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} i^p = \sum_{j=0}^{m} \left[\phi_m \frac{\left[\sum_{i=1}^{n} i\right]^{p-2j}}{n^{p-(2j+1)}} \right] \qquad [177]$$

Since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, equation [177] becomes:

$$\sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} i^p = \sum_{j=0}^{m}\left[\phi_j \frac{\left[\frac{n(n+1)}{2}\right]^{p-2j}}{n^{p-(2j+1)}}\right]$$

$$= n\sum_{j=0}^{m}\left[\phi_j \left[\frac{(n+1)}{2}\right]^{p-2j}\right] \qquad [178]$$

Where, $\phi_0 = 1$ and $m = \begin{cases} \frac{p-1}{2}\ for\_odd\_p \\ \frac{p}{2}\ for\_even\_p \end{cases}$

For p=2,

$$\sum_{i=1}^{n} i^2 = n\sum_{j=0}^{1}\left[\phi_j \left[\frac{(n+1)}{2}\right]^{2-2j}\right]$$

$$= n\left[\phi_0\left(\frac{n+1}{2}\right)^2 + \phi_2\right]$$

$$= n\left[\left(\frac{n+1}{2}\right)^2 + \left(\frac{n^2-1}{12}\right)\right]$$

$$= \frac{n^2(n+1)^2}{4n} + \frac{n(n^2-1)}{12} = \frac{n(n+1)}{12}\left(3(n+1)+(n-1)\right)$$

$$= \frac{n(n+1)}{12}(4n+2) = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(2n^3+3n^2+n\right)$$

or

$$\sum_{i=1}^{n} i^2 = \frac{\left[\sum_{i=1}^{n} i\right]^2}{n} + \frac{n(n^2-1)}{12} \qquad [179]$$

Since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Equation [179] reduces to:

$$\sum_{i=1}^{n} i^2 = \frac{n^2(n+1)^2}{4n} + \frac{n(n^2-1)}{12} = \frac{n(n+1)}{12}\left(3(n+1)+(n-1)\right)$$

$$= \frac{n(n+1)}{12}(4n+2) = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(2n^3+3n^2+n\right) \qquad [180]$$

6 Derivation of Bernoulli's number from the Sums of Power generalized equation.

It is known that the generalized equation P_m is zero when n=1. Therefore, the coefficients P_m can be used to find Bernoulli's number. Few Bernoulli's numbers calculation can be seen as follows:

Consider,

$$P_m = \left[n^{2m} - 2\sum_{t=1}^{m} \left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t n^{2(m-t)} \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right]\right]$$

Since $P_m = 0$ when n=1.

$$\left[1^{2m} - 2\sum_{t=1}^{m} \left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t (1)^{2(m-t)} \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right]\right] = 0$$

$$\left[1 - 2\sum_{t=1}^{m} \left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right]\right] = 0 \qquad [181]$$

Rewritting the equation [181] yields

$$f(m) = \sum_{t=1}^{m} \left[(2t+1)(2^{2t-1}-1)\binom{m}{t} B_t \frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] - \frac{1}{2} = 0 \text{ for all } m \in N \qquad [182]$$

When $m=1$,

$$f(1) = \sum_{t=1}^{1} \left[(2t+1)(2^{2t-1}-1)\binom{1}{t} B_t \frac{\prod_{j=0}^{t-1}(1+2(1-j))}{\prod_{j=0}^{t}(1+2(t-j))} \right] - \frac{1}{2} = 0$$

$$(2(1)+1)(2^{2(1)-1}-1)\binom{1}{1} B_1 \frac{\prod_{j=0}^{1-1}(1+2(1-j))}{\prod_{j=0}^{1}(1+2(1-j))} - \frac{1}{2} = 0$$

$$3B_1 \frac{\prod_{j=0}^{0}(3-2j)}{\prod_{j=0}^{1}(3-2j)} - \frac{1}{2} = 0$$

$$3B_1 \frac{3}{(3)(1)} - \frac{1}{2} = 0$$

$$3B_1 - \frac{1}{2} = 0$$

$$B_1 = \frac{1}{6} \qquad [183]$$

When $m = 2$,

$$f(2) = \sum_{t=1}^{2}\left[(2t+1)(2^{2t-1}-1)\binom{2}{t}B_t \frac{\prod_{j=0}^{t-1}(1+2(2-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right] - \frac{1}{2} = 0$$

$$(2(1)+1)(2^{2(1)-1}-1)\binom{2}{1}B_1 \frac{\prod_{j=0}^{1-1}(1+2(2-j))}{\prod_{j=0}^{1}(1+2(1-j))} + (2(2)+1)(2^{2(2)-1}-1)\binom{2}{2}B_2 \frac{\prod_{j=0}^{2-1}(1+2(2-j))}{\prod_{j=0}^{2}(1+2(2-j))} - \frac{1}{2} = 0$$

$$6B_1 \frac{\prod_{j=0}^{0}(5-2j)}{\prod_{j=0}^{1}(3-2j)} + 35B_2 \frac{\prod_{j=0}^{1}(5-2j)}{\prod_{j=0}^{2}(5-2j)} - \frac{1}{2} = 0$$

$$6\left(\frac{1}{6}\right)\frac{5}{(3)(1)} + 35B_2 \frac{(5)(3)}{(5)(3)(1)} - \frac{1}{2} = 0$$

$$\frac{5}{3} + 35B_2 - \frac{1}{2} = 0$$

$$B_2 = -\frac{1}{30} \qquad [184]$$

The Bernoulli's number for when $m > 1$ can be calculated by rewriting the equation [182]. The derivation is given as follows:

Expanding equation [182] yields:

$$f(m) = (2(1)+1)(2^{2(1)-1}-1)\binom{m}{1}B_1 \frac{\prod_{j=0}^{0}(1+2(m-j))}{\prod_{j=0}^{1}(1+2((1)-j))} + (2(2)+1)(2^{2(2)-1}-1)\binom{m}{2}B_1 \frac{\prod_{j=0}^{1}(1+2(m-j))}{\prod_{j=0}^{2}(1+2((2)-j))}$$

$$+\cdots+(2(m)+1)(2^{2(m)-1}-1)\binom{m}{m}B_m \frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2((m)-j))} - \frac{1}{2} = 0$$

[185]

Rewritting equation [185] yields:

$$\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]+(2(m)+1)(2^{2(m)-1}-1)\binom{m}{m}B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2((m)-j))}-\frac{1}{2}=0$$

$$(2m+1)(2^{2m-1}-1)B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2(m-j))}=\frac{1}{2}-\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]$$

$$B_m\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2(m-j))}=\frac{1}{(2m+1)(2^{2m-1}-1)}\left[\frac{1}{2}-\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right]$$

Since,

$$\frac{\prod_{j=0}^{m-1}(1+2(m-j))}{\prod_{j=0}^{m}(1+2(m-j))}=\frac{(2m+1)(2m-1)(2m-3)\cdots(3)}{(2m+1)(2m-1)(2m-3)\cdots(3)(1)}=1$$

Therefore,

$$B_m=\frac{1}{(2m+1)(2^{2m-1}-1)}\left[\frac{1}{2}-\sum_{t=1}^{m-1}\left[(2t+1)(2^{2t-1}-1)\binom{m}{t}B_t\frac{\prod_{j=0}^{t-1}(1+2(m-j))}{\prod_{j=0}^{t}(1+2(t-j))}\right]\right] \qquad [186]$$

Conclusion.

The general equation for Sum of Power presented in this paper can be extended on many other uses due to its simplicity and elegant formulation. This formula includes Faulhaber's sum of power and most of other formulae derived for sum of power because of its expression in form of the most basic elementary symmetric function of an arithmetic progression. Since integer is part of arithmetic progression, it offers new form of sum of power as alternative to Faulhaber formulation. Apart from that, this generalized equation can be extended to real number powers. This study would be given in the Part II. It was found that, in the formulation of sum of powers the generalized equation uses Bernoulli's numbers in the formation of its coefficients. However, in the Part III, the Euler or Secant numbers involved in the formation of alternating Sum of Powers.

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