Evolutionary sequence of spacetime/intrinsic spacetime and associated sequence of geometry in a metric force field. Part II.

Akindele J. Adekugbe

Center for The Fundamental Theory, P. O. Box 2575, Akure, Ondo State 340001, Nigeria.
E-mail: adekugbe@alum.mit.edu

Graphical analysis of the geometry of a curved 'three-dimensional' absolute intrinsic metric space, (an absolute intrinsic Riemannian metric space) $\phi\hat{M}^3$, which is curved onto the absolute time/absolute intrinsic time 'dimensions' (along the vertical), as a curved hyper-surface, and projects a flat three-dimensional proper intrinsic metric space $\phi E'^3$ underlying its outward manifestation namely, the flat proper physical Euclidean 3-space E'^3, both as flat hyper-surfaces along the horizontal, isolated in part one of this paper, is done. Two absolute intrinsic tensor equations, one of which is of the divergenceless form of Einstein free-space field equations and the other which is a tensorial statement of local Euclidean invariance on $\phi\hat{M}^3$, are derived. Simultaneous (algebraic) solution of the equations yields the absolute intrinsic metric tensor and absolute intrinsic Ricci tensor of absolute intrinsic Riemann geometry on the curved absolute intrinsic metric space $\phi\hat{M}^3$, in terms of an isolated absolute intrinsic curvature parameter. Relations for absolute intrinsic coordinate projections into the underlying flat proper intrinsic space are derived. A superposition procedure that yields resultant absolute intrinsic metric tensor and resultant absolute intrinsic Ricci tensor, as well as resultant absolute intrinsic coordinate projection relations when two or a larger number of absolute intrinsic Riemannian metric spaces co-exist, are developed. Finally the fact that a curved 'three-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$ is perfectly isotropic (that is, all directions are perfectly the same) and is consequently contracted to a 'one-dimensional' absolute intrinsic metric space denoted by $\phi\hat{\rho}$, which is curved onto the absolute time/absolute intrinsic time 'dimensions' along the vertical and that the underlying projective three-dimensional flat proper intrinsic metric space $\phi E'^3$ is perfectly isotropic and is consequently contracted to a straight line one-dimensional isotropic proper intrinsic metric space $\phi\rho'$ along the horizontal, with respect to observers in the physical proper Euclidean 3-space E'^3 that overlies $\phi\rho'$, are deduced.

1 Graphical analysis of absolute intrinsic Riemann geometry of curved absolute intrinsic metric spaces

Let us start with a curved 'two-dimensional' absolute intrinsic metric space (an absolute intrinsic Riemannian metric space) $\phi\hat{M}^2$ with extended absolute intrinsic 'dimensions' $\phi\hat{x}^1$ and $\phi\hat{x}^2$, a sub-space of the 'three-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$ in Fig. 5 of part one of this paper [1]. The extended curved absolute intrinsic 'dimensions' of $\phi\hat{M}^2$ originate from a point O($\phi\hat{x}^1_{(0)}, \phi\hat{x}^2_{(0)}$) of the underlying two-dimensional proper intrinsic metric space $\phi E'^2$, with extended straight line proper intrinsic dimensions $\phi x'^1$ and $\phi x'^2$, as illustrated in Fig. 1. We shall temporarily make the following changes of notation of intrinsic dimensions for convenience:

$$\phi\hat{x}^i \to \hat{u}^i \text{ and } \phi x'^i \to u'^i,$$

as already implemented in Fig. 1. On the other hand, the notations $\phi\hat{M}^3$, $\phi\hat{E}^3$ and $\phi E'^3$ for the intrinsic spaces shall be retained in order to avoid confusion.

Let us take a short segment AB $\equiv \Delta\hat{u}^1 (\equiv \Delta\phi\hat{x}^1)$ about point $\hat{u}^1_{(1)} (\equiv \phi\hat{x}^1_{(1)})$ along the 'dimension' $\hat{u}^1$. Then in the limit as $\Delta\hat{u}^1$ becomes very small, that is, in the limit as A$\to$B, we must let $\Delta\hat{u}^1 \to d\hat{u}^1$ and $\Delta u'^1 \to du'^1$ in Fig. 1. We require in this limit that the length of the arc AGB be equal to the length of the hypotenuse AB of the triangle ABC, then the absolute intrinsic angle $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)$ is single-valued, being equal to $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})$ over the arc AGB in this limit.

Similarly by taking a short segment, DE$=\Delta\hat{u}^2 (\equiv \Delta\phi\hat{x}^2)$, about point $\hat{u}^2_{(1)} (\equiv \phi\hat{x}^2_{(1)})$ along the curved 'dimension' $\hat{u}^2$ we have, in the limit as $\Delta\hat{u}^2$ becomes very small, that is, in the limit as D $\to$ E, we must let $\Delta\hat{u}^2 \to d\hat{u}^2$ and $\Delta u'^2 \to du'^2$ in Fig. 1. We also require in this limit that the length of the arc DHE be equal to the length of the hypotenuse DE of the triangle DEF. Then the absolute intrinsic angle $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2)$ is single-valued, being equal to $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})$ over the arc DHE.

Thus by displacing the limiting constant elementary intervals $d\hat{u}^1$ and $d\hat{u}^2$ defined above along the curved 'dimensions' $\hat{u}^1$ and $\hat{u}^2$ respectively, one can attach a locally flat manifold of elementary 'dimensions' $d\hat{u}^1$ and $d\hat{u}^2$ to every point of the '2-dimensional' curved absolute intrinsic space $\phi\hat{M}^2$. One can then construct geometry, that is, derive single absolute intrinsic metric tensor, single absolute intrinsic Ricci tensor, single absolute intrinsic Riemann scalar, etc, (in a lumped parameter fashion), which are valid at every point

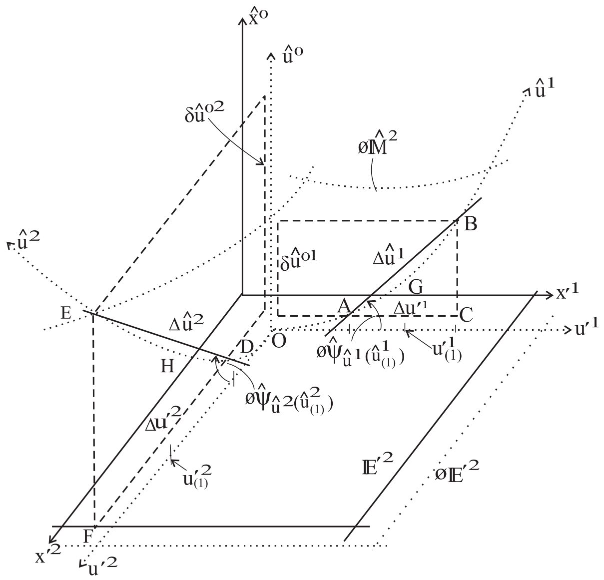

Fig. 1: A curved 'two-dimensional' absolute intrinsic metric space $\phi\hat{M}^2$, (an absolute intrinsic Riemannian metric space), and its underlying projective flat 2-dimensional proper intrinsic metric space $\phi E'^2$, lying underneath flat proper physical Euclidean 2-space E'^2.

within the locally flat neighborhood, with respect to Euclidean observers in the underlying flat proper physical Euclidean 2-space E'^2 and repeat this about every point of the curved absolute intrinsic metric space $\phi\hat{M}^2$. This is the graphical approach to the absolute intrinsic Riemann geometry of a curved '2-dimensional' absolute intrinsic metric space, which has no counterpart in conventional Riemann geometry. The derivation can be easily extended to a curved '3-dimensional' absolute intrinsic metric space - a '3-dimensional' absolute intrinsic Riemannian metric space.

The elementary intervals $d\hat{u}^1$ and $d\hat{u}^2$ defined about point $(\hat{u}^1_{(1)}, \hat{u}^2_{(1)})$ of the '2-dimensional' absolute intrinsic Riemann space $\phi\hat{M}^2$, project intervals du'^1 and du'^2 respectively about the corresponding point $(u'^1_{(1)}, u'^2_{(1)})$ of the underlying proper intrinsic space $\phi E'^2$ in Fig. 1. One obtains the following from elementary coordinate geometry,

$$du'^1 = d\hat{u}^1 \cos\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) \text{ and } du'^2 = d\hat{u}^2 \cos\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^2_{(1)}) \quad (1)$$

A Riemannian observer at the point $(\hat{u}^1_{(1)}, \hat{u}^2_{(1)})$ on $\phi\hat{M}^2$; (this is the proper Riemannian observer), observes Euclidean metric tensor locally about his position. He constructs Euclidean line element in terms of the intervals $d\hat{u}^1$ and $d\hat{u}^2$ as follows:

$$(d\phi\hat{l})^2 = (d\hat{u}^1)^2 + (d\hat{u}^2)^2 = \sum_{i,k=1}^{2} \delta_{ik} d\hat{u}^i d\hat{u}^k \qquad (2a)$$

This local Euclidean line element on $\phi\hat{M}^2$ can be written equivalently in terms of the coordinate intervals of the underlying proper intrinsic metric space $\phi E'^2$, by virtue of system (1), as follows:

$$(d\phi\hat{l})^2 = (d\hat{u}^1)^2 + (d\hat{u}^2)^2$$
$$= \sec^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})(du'^1)^2 + \sec\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})(du'^2)^2 \qquad (2b)$$

or

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{2} \sec\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i_{(1)}) \sec\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k_{(1)}) \delta_{ik} du'^i d\hat{u}'^k \qquad (2c)$$

or

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{2} \phi\hat{g}_{ik}(\hat{u}^1_{(1)}, \hat{u}^2_{(1)}) du'^i du'^k \qquad (2d)$$

The absolute intrinsic metric tensor $\phi\hat{g}_{ik}$ is purely diagonal, given in terms of absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})$ and $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(2)})$ as follows:

$$\begin{aligned} \phi\hat{g}_{ik} &= \sec\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i_{(1)}) \sec\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k_{(1)}) \delta_{ik} \\ &= \begin{pmatrix} \sec^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) & 0 \\ 0 & \sec^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \end{pmatrix} \end{aligned} \qquad (3)$$

Thus the locally flat region of the curved absolute intrinsic metric space $\phi\hat{M}^2$ bounded by elementary coordinate intervals $d\hat{u}^1$ and $d\hat{u}^2$ about point $(\hat{u}^1_{(1)}, \hat{u}^2_{(1)})$ of $\phi\hat{M}^2$, which possesses Euclidean metric tensor δ_{ik} with respect to a Riemannian observer at the location of this locally flat region of $\phi\hat{M}^2$, possesses the absolute intrinsic sub-Riemannian metric tensor $\phi\hat{g}_{ik}$ with respect to Euclidean 3-observers in the underlying flat proper physical space E'^2 in Fig. 1.

However the absolute intrinsic line element (2c) or (2d) given in terms of the proper intrinsic coordinate intervals du'^1 and du'^2 of $\phi E'^2$, cannot be used to write absolute intrinsic geodesics on the curved absolute intrinsic metric space $\phi\hat{M}^2$. Rather the absolute intrinsic geodesic on $\phi\hat{M}^2$ must be written in terms of locally straight elementary absolute intrinsic coordinate intervals $d\hat{u}^1$ and $d\hat{u}^2$ on $\phi\hat{M}^2$ and the absolute intrinsic metric tensor $\phi\hat{g}_{ik}$ of Eq. (3) with respect to these observers (in E'^2 in Fig. 1).

As shall be demonstrated shortly in this paper, there is local Euclidean invariance on $\phi\hat{M}^2$ with respect to observers in E'^2, which allows us to write,

$$\sum_{i,k}^{2} \delta_{ik} du'^i du'^k = \sum_{i,k}^{2} \delta_{ik} d\hat{u}^i d\hat{u}^k \qquad (4)$$

This is the discrete version (in the graphical approach) of intrinsic local Euclidean invariance (ϕLEI) on $\phi\hat{M}^2$ with respect to Euclidean observers in E'^2 in Fig. 1.

Eq. (4) then allows us to replace $du'^i du'^k$ by $d\hat{u}^i d\hat{u}^k$ in Eqs. (2c) and (2d) yielding the following respectively:

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{2} \sec\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i_{(1)}) \sec\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k_{(1)})\delta_{ik} d\hat{u}^i d\hat{u}^k \qquad (5a)$$

or

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{2} \phi\hat{g}_{ik}(\hat{u}^1_{(1)}, \hat{u}^2_{(1)}) d\hat{u}^i d\hat{u}^k \qquad (5b)$$

The absolute intrinsic line element (5a) or (5b) and the absolute intrinsic metric tensor (3) admit of easy generalizations to the case of '3-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$. The absolute intrinsic geodesic is given at an arbitrary point $(\hat{u}^1, \hat{u}^2, \hat{u}^3)$ on $\phi\hat{M}^3$, which corresponds to point (u'^1, u'^2, u'^3) in the underlying flat proper intrinsic metric space $\phi E'^3$ with respect to observers in E'^3 as follows:

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{3} \sec\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i_{(1)}) \sec\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k_{(1)})\delta_{ik} d\hat{u}^i d\hat{u}^k \qquad (6a)$$

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{3} \phi\hat{g}_{ik}(\hat{u}^1_{(1)}, \hat{u}^2_{(1)}, \hat{u}^3_{(1)}) d\hat{u}^i d\hat{u}^k \qquad (6b)$$

The absolute intrinsic metric tensor is a 3×3 diagonal matrix containing elements $\sec^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})$, $\sec^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})$ and $\sec^2\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3_{(1)})$ in this case.

In the graphical approach to the absolute intrinsic Riemann geometry of curved absolute intrinsic metric spaces, once one measures the absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i)$ on $\phi\hat{M}^2$ or $\phi\hat{M}^3$, of the inclination of intervals $d\hat{u}^i$ of the curved absolute intrinsic 'dimension' $\hat{u}^i$ to the respective underlying projective straight line proper intrinsic dimensions u'^i at a given point, one then obtains the absolute intrinsic metric tensor from Eq. (7) of part one of this paper [1] at that point. There is no correspondence to this in conventional Riemann geometry, as far as I know.

The absolute intrinsic metric tensor of the absolute intrinsic Riemann geometry of a curved absolute intrinsic metric space $\phi\hat{M}^3$ is purely diagonal always. This is so since all the absolute intrinsic 'dimensions' of $\phi\hat{M}^3$ span the absolute intrinsic time 'dimension' $\hat{u}^0 \equiv \phi\hat{c}\phi\hat{t}$ along the vertical only, such that each curved absolute intrinsic 'dimension' $\hat{u}^k$ of $\phi\hat{M}^3$ lies above its projective straight line proper intrinsic dimension u'^k in $\phi E'^3$ (along the horizontal). Consequently each curved absolute intrinsic 'dimension' $\hat{u}^k$ lies on the vertical $u'^k\hat{u}^0-$plane. The cross terms, $d\hat{u}^1 d\hat{u}^2$, $d\hat{u}^1 d\hat{u}^3$ and $d\hat{u}^1 d\hat{u}^3$ are therefore precluded in the absolute intrinsic line elements (5a) or (5b) and (6a) or (6b).

1.1 The absolute intrinsic dimensionless curvature parameter of a curved 'one-dimensional' absolute intrinsic space on a vertical proper intrinsic space - absolute intrinsic time plane

Let us consider a curve s on the $u'^1u'^2-$plane in $\phi E'^3$ shown in Fig. 2(a). The Eulerian curvature κ_{Eul}, (in honor of Euler), of the curve s at point P in Fig. 2(a) is given from definition [5, see ch.1] as follows:

$$\frac{d\hat{t}}{ds} = \kappa_{\text{Eul}}\hat{n} \qquad (7)$$

where $\hat{t}$ and $\hat{n}$ are unit tangent vector and unit normal vector respectively, to the curve s at P. Hence,

$$\mid \frac{d\hat{t}}{ds} \mid = \mid \kappa_{\text{Eul}}\hat{n} \mid = \frac{d\phi}{ds} = \kappa_{\text{Eul}} \qquad (8)$$

Now let this same curve s be on the vertical $u'^1\hat{u}^0-$plane as shown in Fig. 2b, where it has been re-denoted by $\hat{u}^1$. It now carries a hat label since it is now a 'one-dimensional absolute intrinsic metric space (or an absolute intrinsic metric space 'dimension') on the $u'^1\hat{u}^0-$plane. Again the curvature of $\hat{u}^1$ is given by Eq. (7), except that the unit normal vector $\hat{n}$ projects components $\hat{n}\sin\phi\hat{\psi}(\hat{u}^1_{(1)})$ into the proper intrinsic dimension u'^1 along the horizontal. Hence the curvature of $\hat{u}^1$ that is valid with respect to observers in (all frames in) the underlying physical Euclidean space in Fig. 2b is the following

$$\frac{d\hat{t}}{d\hat{s}} = \hat{n}\sin\phi\hat{\psi}(\hat{u}^1_{(1)})\kappa_{\text{Eul}} \qquad (9)$$

Let us define the absolute intrinsic Riemannian curvature $\phi\hat{\kappa}_{\text{Riem}}(\hat{u}^1_{(1)})$ of the plane curve $\hat{u}^1$, (which is a one-dimensional absolute intrinsic metric space (an absolute intrinsic Riemann space) at point P in Fig. 2b as follows:

$$\phi\hat{\kappa}_{\text{Riem}}(\hat{u}^1_{(1)}) = \mid \frac{d\hat{t}}{d\hat{s}} \mid = \mid \hat{n} \mid \sin\phi\hat{\psi}(\hat{u}^1_{(1)})\kappa_{\text{Eul}} \qquad (10)$$

or

$$\phi\hat{\kappa}_{\text{Riem}}(\hat{u}^1_{(1)}) = \sin\phi\hat{\psi}(\hat{u}^1_{(1)})\kappa_{\text{Eul}} \qquad (11)$$

The dimensionless intrinsic parameter $\sin\phi\hat{\psi}(\hat{u}^1_{(1)})$ shall be referred to as absolute intrinsic curvature parameter at the given point $\hat{u}^1_{(1)}$ along the curved 'one-dimensional' absolute intrinsic metric space $\hat{u}^1$ and denoted by $\phi\hat{k}(\hat{u}^1_{(1)})$. It is an absolute intrinsic parameter since the $\hat{u}^1$ is a 'one-dimensional' absolute intrinsic metric space (or a 'dimension' of 'three-dimensional' absolute intrinsic metric space). Hence Eq. (26) shall be re-written as follows:

$$\phi\hat{\kappa}_{\text{Riem}}(\hat{u}^1_{(1)}) = \phi\hat{k}(\hat{u}^1_{(1)})\kappa_{\text{Eul}} \qquad (12)$$

where

$$\phi\hat{k}(\hat{u}^1_{(1)}) = \sin\phi\hat{\psi}(\hat{u}^1_{(1)}) \qquad (13)$$

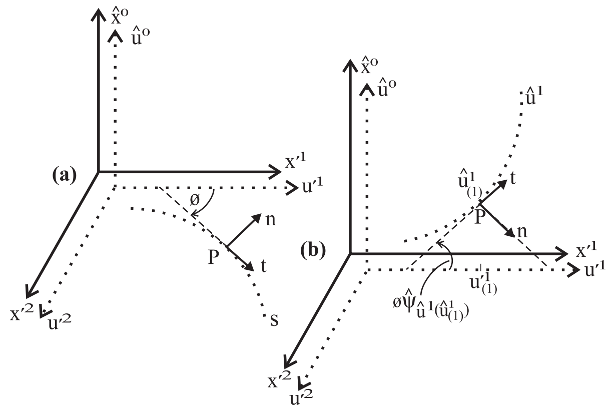

Fig. 2: Deriving the absolute intrinsic curvature parameter of a curved 'one-dimensional' absolute intrinsic metric space on a vertical proper intrinsic space - absolute intrinsic time plane.

Since the absolute intrinsic angle $\phi\hat{\psi}$ has constant zero value along plane curves in the underlying proper intrinsic $\phi E'^3$, the absolute intrinsic Riemannian curvature of a plane curve in $\phi E'^3$ is zero.

The absolute intrinsic curvature parameters $\phi\hat{k}_{\hat{u}^1}$ and $\phi\hat{k}_{\hat{u}^2}$ at points $\hat{u}^1_{(1)}$ and $\hat{u}^2_{(1)}$ of the curved absolute intrinsic 'dimensions' $\hat{u}^1$ and $\hat{u}^2$ respectively in Fig. 1 are given as follows by virtue of definition (13):

$$\left.\begin{aligned} \phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)}) &= \sin\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) \\ \phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)}) &= \sin\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \end{aligned}\right\} \tag{14}$$

Since the absolute intrinsic angle $\phi\hat{\psi}_{\hat{u}^i}$ measures the inclination of the curved absolute intrinsic 'dimension' $\hat{u}^i$ on the vertical plane $u'^i\hat{u}^0$ relative to the underlying flat proper intrinsic space $\phi E'^3$ (as a hyper-surface) along the horizontal, it has the same value with respect to all frames (or all observers) in the underlying flat proper physical Euclidean 3-space E'^3 (also as a hyper-surface) along the horizontal overlying $\phi E'^3$. Hence the absolute intrinsic curvature parameter $\phi\hat{k}_{\hat{u}^i}$ has the same value with respect to all frames (or all observers) in the underlying flat proper physical Euclidean 3-space E'^3

1.2 Expressing the components of the absolute intrinsic metric tensor in terms of absolute intrinsic curvature parameters in absolute intrinsic Riemann geometry

From the components of the absolute intrinsic metric tensor in Eq. (3), one obtains the following

$$\left.\begin{aligned} \phi\hat{g}_{11} &= \sec^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) = (1-\sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}))^{-1}; \\ \phi\hat{g}_{22} &= \sec^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) = (1-\sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}))^{-1}; \\ \phi\hat{g}_{12} &= \phi\hat{g}_{21} = 0 \end{aligned}\right\} \tag{15}$$

Then by using system (14) in system (15), the components of the absolute intrinsic metric tensor are given in terms of absolute intrinsic curvature parameters as follows:

$$\left.\begin{aligned} \phi\hat{g}_{11} &= (1-\phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)})^2)^{-1}; \\ \phi\hat{g}_{22} &= (1-\phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)})^2)^{-1}; \\ \phi\hat{g}_{21} &= \phi\hat{g}_{12} = 0 \end{aligned}\right\} \tag{16}$$

Hence,

$$\phi\hat{g}_{ik} = \begin{pmatrix} \dfrac{1}{1-\phi\hat{k}_{\hat{u}^1}(\hat{u}_{(1)})^2} & 0 \\ 0 & \dfrac{1}{1-\phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)})^2} \end{pmatrix} \tag{17}$$

Extension to the case of a '3-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$ is straight forward, in which case the 2×2 diagonal matrix of Eq. (18) becomes a 3×3 diagonal matrix.

Thus the absolute intrinsic metric tensor has parametric dependence on the square of the absolute intrinsic curvature parameters in absolute intrinsic Riemann geometry. The absolute intrinsic curvature parameters shall ultimately be related to the absolute intrinsic parameter(s) of the absolute intrinsic metric force field that gives rise to absolute intrinsic Riemann geometry with further development.

1.3 Establishing local Euclidean invariance on a curved absolute intrinsic metric space

The '2-dimensional' absolute intrinsic metric space $\phi\hat{M}^2$ (a 'two-dimensional absolute intrinsic Riemannian manifold) in Fig. 1, is locally Euclidean. Hence the Riemannian observer located at point $(\hat{u}^1_{(1)}, \hat{u}^2_{(1)})$ on $\phi\hat{M}^2$ writes the absolute intrinsic local Euclidean line element (17a) at that point. On the other hand, the absolute intrinsic Euclidean line element (2a) with respect to a Riemannian observer at point $(\hat{u}^1_{(1)}, \hat{u}^2_{(1)})$ on $\phi\hat{M}^2$ is equivalent to the absolute intrinsic sub-Riemannian line element (2c) or (2d) with respect to Euclidean observers in the underlying flat proper physical space E'^2. An extra term shall be added to the right-hand side of Eq. (2c) or (2d) in order to recover the absolute intrinsic local Euclidean line element on $\phi\hat{M}^2$ with respect to observers in E'^2 in this subsection.

One observes from Fig. 1 that the interval $d\hat{u}^1$ of the curved absolute intrinsic 'dimension' $\hat{u}^1$ projects component du'^1 into the underlying proper intrinsic dimension u'^1 of $\phi E'^2$ and component $\delta\hat{u}^{01}$ into the vertical absolute intrinsic time 'dimension' $\hat{u}^0$. Similarly the interval $d\hat{u}^2$ of the curved absolute intrinsic 'dimension' $\hat{u}^2$ of $\phi\hat{M}^2$, projects component du'^2 into the underlying proper intrinsic dimension u'^2 and component $\delta\hat{u}^{02}$ into the vertical absolute intrinsic time 'dimension' $\hat{u}^0$.

We have made use of the components du'^1 and du'^2 projected into the underlying flat proper intrinsic space $\phi E'^2$ in deriving the absolute intrinsic metric line element (2c) or (2d) with respect to observers in E'^2, while the components $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ have been left out. This is so because the absolute time and absolute intrinsic time 'dimensions' along the vertical, being absolute, are not dimension and intrinsic dimension respectively with respect to observers the relative proper physical Euclidean 2-space E'^2. Hence the absolute intrinsic coordinate intervals $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ projected along $\hat{u}^0$ are not absolute intrinsic metric components. Rather they are to be referred to as 'non-metric' components, while the components du'^1 and du'^2 projected into the underlying flat proper intrinsic metric space $\phi E'^2$, which have been used in deriving the absolute intrinsic metric line element (2c) or (2d), shall be referred to as metric components, with respect to observers in the proper physical Euclidean 2-space E'^2.

Although the 'non-metric' absolute intrinsic coordinate intervals $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ projected along the absolute intrinsic time 'dimension' $\hat{u}^0$ are elusive and must be disregarded in deriving absolute intrinsic metric line element on the curved 'two-dimensional' absolute intrinsic metric space $\phi\hat{M}^2$ with respect to observers in E'^2 in Fig. 1, as done in obtaining the absolute intrinsic line element (2c) and (2d) and the absolute intrinsic sub-Riemannian metric tensor (17), let us temporarily put both the metric components du'^1 and du'^2 and the 'non-metric' components $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ into consideration in order to recover the absolute intrinsic Euclidean line element and the absolute intrinsic Euclidean metric tensor on $\phi\hat{M}^2$ with respect to observers in E'^2. Thus let us apply the Pythagorean formula to triangles ABC and DEF in Fig. 1 to have the following

$$(d\hat{u}^1)^2 = (du'^1)^2 + (\delta\hat{u}^{01})^2 \text{ and } (d\hat{u}^2)^2 = (du'^2)^2 + (\delta\hat{u}^{02})^2 \tag{18}$$

But $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ are given in terms of absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})$ and $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})$ and intervals $d\hat{u}^1$ and $d\hat{u}^2$ respectively as follows:

$$d\hat{u}^{01} = d\hat{u}^1 \sin\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) \text{ and } d\hat{u}^{02} = d\hat{u}^2 \sin\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \tag{19}$$

Then from systems (18) and (19) we have the following

$$\left.\begin{aligned} (du'^1)^2 &= (d\hat{u}^1)^2 - (d\hat{u}^1)^2\sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}); \\ (du'^2)^2 &= (d\hat{u}^2)^2 - (d\hat{u}^2)^2\sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \end{aligned}\right\} \tag{20}$$

And from system (20) we construct Euclidean line element in terms of components du'^1 and du'^2 projected into $\phi E'^2$ as follows:

$$\begin{aligned} (d\phi l')^2 &= (du'^1)^2 + (du'^2)^2 \\ &= (d\hat{u}^1)^2 - (d\hat{u}^1)^2\sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) + \\ &\quad + (d\hat{u}^2)^2 - (d\hat{u}^2)^2\sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \end{aligned} \tag{21}$$

Then by using the relations $d\hat{u}^i = du'^i \sec\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i_{(1)}); i = 1, 2$, which follows from system (1), Eq. (21) becomes the following

$$\begin{aligned} (d\phi l')^2 &= (du'^1)^2(\sec^2\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) - \tan^2\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})) \\ &\quad + (du'^2)^2(\sec^2\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) - \tan^2\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})) \end{aligned} \tag{22}$$

which upon using, $\sec^2\psi - \tan^2\psi = 1$, gives $(d\phi l')^2 = (du'^1)^2 + (du'^2)^2$.

Thus by considering the 'non-metric' components $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ projected into the absolute intrinsic time 'dimension' $\hat{u}^0$ along the vertical along with the metric components du'^1 and du'^2 projected into the proper intrinsic space $\phi E'^2$ along the horizontal by the intervals $d\hat{u}^1$ and $d\hat{u}^2$ of the absolute intrinsic metric space 'dimensions' $\hat{u}^1$ and $\hat{u}^2$ in Fig. 1, in constructing the absolute intrinsic line element on $\phi\hat{M}^2$ with respect to observers in E'^2, the absolute intrinsic local Euclidean line element is recovered at every point on $\phi\hat{M}^2$ with respect to observers in E'^2. This is the same as saying that there is intrinsic local Euclidean invariance (ϕLEI) on $\phi\hat{M}^2$ with respect to observers in E'^2 when the projective 'non-metric' coordinate intervals $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ and the projective metric coordinate intervals du'^1 and du'^2 are put into consideration in constructing the absolute intrinsic line element on $\phi\hat{M}^2$, which has been stated mathematically without proof as Eq. (4) earlier.

1.4 Tensorial statement of intrinsic local Euclidean invariance on absolute intrinsic Riemann spaces

We shall, by virtue of absolute intrinsic local Euclidean invariance on $\phi\hat{M}^2$, (when the projective 'non-metric' coordinate intervals $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ and the projective metric coordinate intervals du'^1 and du'^2 are put into consideration in constructing the absolute intrinsic line element on $\phi\hat{M}^2$), established above, replace the elementary proper intrinsic coordinate intervals du'^1 and du'^2 by absolute intrinsic coordinate intervals $d\hat{u}^1$ and $d\hat{u}^2$ respectively at the right-hand side of Eq. (22) to have as follows:

$$\begin{aligned} (d\phi l')^2 &= (du'^1)^2 + (du'^2)^2 \\ &= (d\hat{u}^1)^2(\sec^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) - \tan^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})) \\ &\quad + (d\hat{u}^2)^2(\sec^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) - \tan^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})) \end{aligned} \tag{23}$$

Equation (23) states formally intrinsic local Euclidean invariance, $(du'^1)^2 + (du'^2)^2 = (d\hat{u}^1)^2 + (d\hat{u}^2)^2$ on $\phi\hat{M}^2$, which has already been stated without deriving it by Eq (4). Thus the absolute intrinsic line element recovered at every point of $\phi\hat{M}^2$ with respect to observers in E'^2 when both the projective metric and 'non-metric' intrinsic coordinate intervals are put into consideration is the following

$$\begin{aligned} (d\phi\hat{l})^2 &= (d\hat{u}^1)^2(\sec^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) - \tan^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})) \\ &\quad + (d\hat{u}^2)^2(\sec^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) - \tan^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})) \end{aligned} \tag{24}$$

It is for the purpose of recovering the absolute intrinsic Euclidean line element (24) on the curved absolute intrinsic metric space $\phi\hat{M}^2$ with respect to observers in the underlying proper physical Euclidean space E'^2 in Fig. 1 that the 'non-metric' intrinsic coordinate intervals $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ projected along the absolute intrinsic time 'dimension' $\hat{u}^0$ have been considered along with the metric intrinsic coordinate intervals du'^1 and du'^2 projected into $\phi E'^2$ in that figure in deriving the intrinsic line element in Eq. (18)-(22). However observers in the proper physical Euclidean space E'^2 must actually make use of the metric intrinsic coordinate intervals du'^1 and du'^2 projected into $\phi E'^2$ solely in deriving the absolute intrinsic sub-Riemannian line element (3) on $\phi\hat{M}^2$ with respect to themselves, since the 'non-metric' intrinsic coordinate intervals are metrically elusive to these observers.

Now, by subtracting the absolute intrinsic metric line element (2b) (obtained by using the metric intrinsic coordinate intervals only) from the absolute intrinsic Euclidean line element (24), one obtains the absolute intrinsic line element $d\phi\hat{l}^2_{\text{nm}}$ on the 'non-metric' sub-space formed by the 'non-metric' components $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ projected into the absolute intrinsic time 'dimension' $\hat{u}^0$ along the vertical in Fig. 1 as follows:

$$(d\phi\hat{l}_{\text{nm}})^2 = \tan^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)(du'^1)^2 + \tan^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2)(du'^2)^2 \tag{25}$$

Observe that $(d\phi\hat{l}_{\text{nm}})^2$ vanishes for $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1) = \phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2) = 0$, which will be the case if the absolute intrinsic 'dimensions' $\hat{u}^1$ and $\hat{u}^2$ were along the horizontal in Fig. 1. That is, if the $\hat{u}$ and $\hat{u}^2$ were not curving onto the absolute intrinsic time 'dimension' $\hat{u}^0$ along the vertical in that figure.

Now let us rewrite the line element $(d\phi\hat{l}_{\text{nm}})^2$ of Eq. (25) as follows:

$$(d\phi\hat{l}_{\text{nm}})^2 = \sum_{i,k=1}^{2} \tan^2\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i)\tan^2\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k)\delta_{ik}du'^i du'^k \tag{26}$$

Eq. (26) is the same as the following by virtue of the now validated intrinsic local Euclidean invariance (4) of part one of this paper [1]:

$$(d\phi\hat{l}_{\text{nm}})^2 = \sum_{i,k=1}^{2} \tan^2\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i)\tan^2\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k)\delta_{ik}d\hat{u}^i d\hat{u}^k \tag{27}$$

Then let us introduce another absolute intrinsic tensor to be denoted by $\phi\hat{R}_{ik}$ and rewrite Eq. (27) as follows:

$$(d\phi\hat{l}_{\text{nm}})^2 = \sum_{i,k=1}^{2} \phi\hat{R}_{ik}d\hat{u}^i d\hat{u}^k \tag{28}$$

where

$$\phi\hat{R}_{ik} = \tan^2\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i)\tan^2\phi\hat{\psi}_{\hat{u}^k}(\hat{u}^k)\delta_{ik} \tag{29}$$

or

$$\phi\hat{R}_{ik} = \begin{pmatrix} \tan^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1) & 0 \\ 0 & \tan^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2) \end{pmatrix} \tag{30}$$

Again the absolute intrinsic tensor $\phi\hat{R}_{ik}$ vanishes for absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^i}(\hat{u}^i) = 0; i = 1, 2$, which will be so if none of the 'dimensions' $\hat{u}^i; i = 1, 2$, was curving towards the absolute intrinsic 'dimension' $\hat{u}^0$ along the vertical in Fig. 1. Certainly the absolute intrinsic tensor $\phi\hat{R}_{ik}$ conveys information about the absolute intrinsic curvature of the absolute intrinsic Riemannian metric space $\phi\hat{M}^2$.

The absolute intrinsic local Euclidean line element (25) on the absolute intrinsic metric space $\phi\hat{M}^2$ can then be written in terms of the absolute intrinsic metric tensor $\phi\hat{g}_{ik}$ and the new absolute intrinsic (curvature) tensor $\phi\hat{R}_{ik}$ as follows:

$$(d\phi\hat{l})^2 = \sum_{i,k=1}^{2}(\phi\hat{g}_{ik} - \phi\hat{R}_{ik})d\hat{u}^i d\hat{u}^k = \sum_{i,k=1}^{2}\delta_{ik}d\hat{u}^i d\hat{u}^k \tag{31}$$

The absolute intrinsic Euclidean line element of Eq. (31) obtains at every point on $\phi\hat{M}^2$, once the projective 'non-metric' intrinsic coordinate intervals $\delta\hat{u}^{01}$ and $\delta\hat{u}^{02}$ and the projective metric coordinate intervals du'^1 and du'^2 are put into consideration in constructing the absolute intrinsic metric line element on $\phi\hat{M}^2$ with respect to observers in E'^2 in Fig. 1. Eq. (31) can therefore be said to express intrinsic local Euclidean invariance on $\phi\hat{M}^2$. Thus the tensorial statement of intrinsic local Euclidean invariance (ϕLEI) on a curved 'two-dimensional' absolute intrinsic metric space $\phi\hat{M}^2$ − a 'two-dimensional' absolute intrinsic Riemannian metric space − in Fig. 1, which is also valid for $\phi\hat{M}^3$, is the following

$$\phi\hat{g}_{ik} - \phi\hat{R}_{ik} = \delta_{ik} \quad (\phi\text{LEI}) \tag{32}$$

1.5 The absolute intrinsic matrix (or scalar) $\phi\hat{\text{C}}$

Now let us introduce a 2×2 absolute intrinsic matrix $\phi\hat{\text{C}}$ through the following relation,

$$\phi\hat{R}_{ik} - \phi\hat{\text{C}}\phi\hat{g}_{ik} = 0 \tag{33}$$

Then from the definitions of the absolute intrinsic tensors $\phi\hat{g}_{ik}$ and $\phi\hat{R}_{ik}$ in Eq. (3) and (30), the absolute intrinsic matrix $\phi\hat{\text{C}}$ is given in the case of '2-dimensional' absolute intrinsic Riemann space $\phi\hat{M}^2$ as follows:

$$\phi\hat{\text{C}} = \begin{pmatrix} \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1) & 0 \\ 0 & \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2) \end{pmatrix} \tag{34}$$

And from system (14), the matrix $\phi\hat{\text{C}}$ is given in terms of absolute intrinsic curvature parameters as follows:

$$\phi\hat{\text{C}} = \begin{pmatrix} \phi\hat{k}_{\hat{u}^1}(\hat{u}^1)^2 & 0 \\ 0 & \phi\hat{k}_{\hat{u}^2}(\hat{u}^2)^2 \end{pmatrix} \tag{35}$$

Eqs. (34) and (35) become the following respectively for '3-dimensional' absolute intrinsic Riemann space $\phi\hat{M}^3$:

$$\phi\hat{\mathrm{C}} = \begin{pmatrix} \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1) & 0 & 0 \\ 0 & \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2) & 0 \\ 0 & 0 & \sin^2\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3) \end{pmatrix} \tag{36}$$

and

$$\phi\hat{\mathrm{C}} = \begin{pmatrix} \phi\hat{k}_{\hat{u}^1}(\hat{u}^1)^2 & 0 & 0 \\ 0 & \phi\hat{k}_{\hat{u}^2}(\hat{u}^2)^2 & 0 \\ 0 & 0 & \phi\hat{k}_{\hat{u}^3}(\hat{u}^3)^2 \end{pmatrix} \tag{37}$$

Now by multiplying through Eq. (33) from the left by $\phi\hat{g}^{ik}$ one obtains the following

$$\phi\hat{g}^{ik}\phi\hat{R}_{ik} - \phi\hat{g}^{ik}\phi\hat{C}\phi\hat{g}_{ik} = 0 \tag{38}$$

Then by applying the known rules for raising and lowering of the indices of a tensor in Riemann geometry, $g^{\alpha\beta}R_{\alpha\delta} = R^{\beta}_{\delta}$, and $g^{\alpha\beta}g_{\alpha\gamma} = \delta^{\beta}_{\gamma}$; so that $\phi\hat{g}^{ik}\phi\hat{R}_{ik} = \phi\hat{R}^i_k$, and $\phi\hat{g}^{ik}\phi\hat{g}_{ik} = \delta^i_i$, Eq. (38) simplifies as follows:

$$\phi\hat{R}^i_i - \phi\hat{C} = 0$$

or

$$\phi\hat{C} = \phi\hat{R}^i_i \tag{39}$$

Thus Eq. (33) can be re-written in terms of $\phi\hat{R}^i_i$ as follows:

$$\phi\hat{R}_{ik} - \phi\hat{R}^i_i\phi\hat{g}_{ik} = 0 \tag{40}$$

In a situation where $\sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1) = \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2) = \sin^2\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3) \equiv \sin^2\phi\hat{\psi}$ (or $\phi\hat{k}_{\hat{u}^1}(\hat{u}^1) = \phi\hat{k}_{\hat{u}^2}(\hat{u}^2) = \phi\hat{k}_{\hat{u}^3}(\hat{u}^3) \equiv \phi\hat{k}$), in Eq. (37), as will be the case for an isotropic absolute intrinsic metric space $\phi\hat{M}^2$ or $\phi\hat{M}^3$, the purely diagonal matrix $\phi\hat{R}^i_i$ or $\phi\hat{C}$ can be replaced by a number namely, $\mathrm{Tr}\,\phi\hat{C}/n$ or $\mathrm{Tr}\,\phi\hat{R}^i_i/n$ in Eq. (33) or (40) to have

$$\phi\hat{R}_{ik} - \frac{1}{n}\mathrm{Tr}\,\phi\hat{C}\phi\hat{g}_{ik} = 0 \tag{41}$$

or

$$\phi\hat{R}_{ik} - \frac{1}{n}\mathrm{Tr}\,\phi\hat{R}^i_i\phi\hat{g}_{ik} = 0 \tag{42}$$

Eq. (42) becomes its familiar form in conventional Riemann geometry for $n = 2$ namely,

$$\phi\hat{R}_{ik} - \frac{1}{2}\phi\hat{R}\phi\hat{g}_{ik} = 0 \tag{43}$$

where $\phi\hat{R}$ is any one of the equal entries of the diagonal matrix $\phi\hat{R}^i_i$ or $\phi\hat{C}$. Obviously the absolute intrinsic tensor $\phi\hat{R}_{ik}$ defined by Eq. (30), (referred to as absolute intrinsic curvature tensor earlier), is the absolute intrinsic Ricci tensor in absolute intrinsic Riemann geometry (of curved absolute intrinsic metric spaces).

It must be noted that Eqs. (41), (42) or (43) are possible for the restrictive situation in which all the curved absolute intrinsic 'dimensions' $\hat{u}^q$ of $\phi\hat{M}^3$ have identical absolute intrinsic curvatures or identical absolute intrinsic curvature parameters, $\phi\hat{k}_{\hat{u}^q}(\hat{u}^q) = \phi\hat{k}$; $q = 1, 2, 3$, at each point of $\phi\hat{M}^3$, as stated earlier. Interestingly it is this restrictive situation that pertains to isotropic absolute intrinsic metric spaces, which shall be of relevance in absolute intrinsic Riemann geometry ultimately. Thus let us re-write Eq. (41), (42) or (43) in the following final form in which it shall be found most useful for application later,

$$\phi\hat{R}_{ik} - \phi\hat{k}^2\phi\hat{g}_{ik} = 0 \tag{44}$$

where $\phi\hat{k}$ is the identical absolute intrinsic curvature parameter of all the absolute intrinsic 'dimensions' of $\phi\hat{M}^3$ at each point of $\phi\hat{M}^3$.

What we have achieved in this section is that we have formulated the absolute intrinsic Riemann geometry of the curved absolute intrinsic metric space $\phi\hat{M}^3$ − an absolute intrinsic Riemann space − relative to 3-observers in its underlying flat proper physical Euclidean 3-space E'^3 and have derived the two important absolute intrinsic tensor equations (32) and (44). While Eq. (32) is a tensorial statement of intrinsic local Euclidean invariance (ϕLEI) on $\phi\hat{M}^3$, as stated earlier, the corresponding significance of Eq. (44) shall be derived elsewhere with further development.

Equations (32) and (44) apply to the curved absolute intrinsic metric space $\phi\hat{M}^3$. They must be solved algebraically to obtain the absolute intrinsic metric tensor $\phi\hat{g}_{ik}$ and absolute intrinsic Ricci tensor $\phi\hat{R}_{ik}$ on $\phi\hat{M}^3$ with respect to 3-observers in the underlying proper physical Euclidean 3-space E'^3 to have as follows:

$$\phi\hat{g}_{ik} = \begin{pmatrix} (1-\phi\hat{k}^2)^{-1} & 0 & 0 \\ 0 & (1-\phi\hat{k}^2)^{-1} & 0 \\ 0 & 0 & (1-\phi\hat{k}^2)^{-1} \end{pmatrix} \tag{45}$$

and

$$\phi\hat{R}_{ik} = \begin{pmatrix} \dfrac{\phi\hat{k}^2}{1-\phi\hat{k}^2} & 0 & 0 \\ 0 & \dfrac{\phi\hat{k}^2}{1-\phi\hat{k}^2} & 0 \\ 0 & 0 & \dfrac{\phi\hat{k}}{1-\phi\hat{k}^2} \end{pmatrix}, \tag{46}$$

where it must be noted that the situation in which all the absolute intrinsic 'dimensions' of $\phi\hat{M}^3$ possess identical absolute intrinsic curvature parameter $\phi\hat{k}$ at each point of $\phi\hat{M}^3$, which pertains to isotropic absolute intrinsic metric spaces that shall be the only relevant situation in absolute intrinsic Riemann geometry, has been considered.

While a Riemannian observer located at a point on the absolute intrinsic Riemann space $\phi\hat{M}^3$ constructs absolute intrinsic Euclidean line element (2a) at that point, since $\phi\hat{M}^3$ is locally Euclidean, $\phi\hat{M}^3$ possesses unique absolute intrinsic sub-Riemannian metric tensor $\phi\hat{g}_{ik}$ of Eq. (45) with respect to Euclidean observers in E'^3. Hence the Euclidean observers write absolute intrinsic line element on $\phi\hat{M}^3$ in terms of $\phi\hat{g}_{ik}$ in the Gaussian form involving isotropic coordinates as follows:

$$d\phi\hat{s}^2 = (d\hat{u}^0)^2 - \sum_{i,k=1}^{3} \phi\hat{g}_{ik} d\hat{u}^i d\hat{u}^k \tag{47}$$

or

$$d\phi\hat{s}^2 = (d\hat{u}^0)^2 - \frac{(d\hat{u}^1)^2 + (d\hat{u}^2)^2 + (d\hat{u}^3)^2}{1 - \phi\hat{k}^2} \tag{48}$$

We have accomplished in this section the two stages of formulation of the absolute intrinsic Riemann geometry of curved absolute intrinsic metric spaces (or of curved absolute metric nospaces) isolated in part one of this paper namely, (i) derivation of the projections of the curved absolute intrinsic 'dimensions' of an absolute intrinsic metric space $\phi\hat{M}^3$ into its underlying projective proper intrinsic metric space $\phi E'^3$, and (ii) formulation of absolute intrinsic Riemann geometry on the curved absolute intrinsic metric space from the projections. The derived projection relations (1) for $\phi\hat{M}^2$, which is directly extendable to $\phi\hat{M}^3$, with respect to Euclidean observers in the underlying proper physical Euclidean 3-space E'^3, is accomplishment of stage one. On the other hand, the derivation of the two absolute intrinsic tensor equations (32) and (44) by starting from system (18) and the absolute intrinsic metric tensor (45), absolute intrinsic Ricci tensor (46) and the absolute line element (48) by solving equations (32) and (44) simultaneously, is accomplishment of stage two. We shall proceed to the accomplishment of the two stages of formulation of absolute intrinsic Riemann geometry on curved absolute intrinsic metric space in the next sub-section in a situation where two or a larger number of absolute intrinsic metric spaces co-exist or are superposed.

1.6 Superposition of absolute intrinsic Riemann spaces

Although superposition of Riemann spaces may be unknown or meaningless in conventional Riemann geometry, it is definitely of important relevance in absolute intrinsic Riemann geometry. The 'two-dimensional' absolute intrinsic Riemann space $\phi\hat{M}^2$, to be re-denoted by $\phi\hat{M}^2_{(1)}$, with curved absolute intrinsic 'dimensions' $\hat{u}^1$ and $\hat{u}^2$ in Fig. 1, is curved relative to its underlying projective flat proper intrinsic space $\phi E'^2$. If another 'two-dimensional' absolute intrinsic Riemann space $\phi\hat{M}^2_{(2)}$ with curved absolute intrinsic 'dimensions' $\hat{v}^1$ and $\hat{v}^2$, say, is brought into the location of $\phi\hat{M}^2_{(1)}$, so that $\phi\hat{M}^2_{(2)}$ and $\phi\hat{M}^2_{(2)}$ co-exist, then $\phi\hat{M}^2_{(2)}$ will be curved relative to $\hat{M}^2_{(1)}$.

The resultant absolute intrinsic curvature parameter $\phi\hat{\hat{k}}$ of the absolute intrinsic space $\hat{M}^2_{(2)}$ relative to the underlying flat proper intrinsic space $\phi E'^2$ can then be derived, and the resultant absolute intrinsic metric tensor $\phi\hat{\hat{g}}_{ik}$, the resultant absolute intrinsic Ricci tensor $\phi\hat{\hat{R}}_{ik}$ and the resultant absolute intrinsic line element $d\phi\hat{\hat{s}}^2$ can be written straight away in terms of $\phi\hat{\hat{k}}$, by simply replacing $\phi\hat{k}$ by $\phi\hat{\hat{k}}$ in equations (45), (46) and (48) to accomplish stage two. The resultant projections into $\phi E'^2$ of the curved absolute intrinsic 'dimensions' of $\hat{M}^2_{(2)}$ can also be derived. The procedure can be extended to situations where three, four and larger number of absolute intrinsic metric spaces coexist.

1.6.1 The resultant absolute intrinsic metric tensor and resultant absolute intrinsic Ricci tensor when two or a larger number of parallel absolute intrinsic metric spaces coexist

Let us consider a pair of 'two-dimensional' absolute intrinsic Riemann spaces denoted by $\phi\hat{M}^2_{(1)}$ and $\phi\hat{M}^2_{(2)}$ with absolute intrinsic 'dimensions' $\hat{u}^1$, $\hat{u}^2$ and $\hat{v}^1$, $\hat{v}^2$ respectively. Let these 'dimensions' of the two absolute intrinsic metric spaces be curved relative to the same proper intrinsic dimensions u'^1 and u'^2 respectively of their underlying global flat proper intrinsic space $\phi E'^2$ prior to their superposition. In other words, as the two absolute intrinsic metric spaces existed at their separate locations before superposing them, the following intrinsic coordinate transformations existed:

$$\left.\begin{aligned} u'^1 &= f^1(\hat{u}^1);\;\; u'^2 = f^2(\hat{u}^2); \\ u'^1 &= g^1(\hat{v}^1);\;\; u'^2 = g^2(\hat{v}^2) \end{aligned}\right\} \tag{49}$$

The absolute intrinsic metric spaces $\phi\hat{M}^2_{(1)}$ and $\phi\hat{M}^2_{(2)}$ in this situation in which $\hat{u}^1$ of $\phi\hat{M}^2_{(1)}$ and $\hat{v}^1$ of $\phi\hat{M}^2_{(2)}$ are both curved relative to u'^1 of $\phi E'^2$ and $\hat{u}^2$ of $\phi\hat{M}^2_{(1)}$ and $\hat{v}^2$ of $\phi\hat{M}^2_{(2)}$ are both curved relative to u'^2 of $\phi E'^2$ at their different locations, as illustrated in Figs. 3a and 3b, shall be referred to as parallel absolute intrinsic metric spaces (or parallel absolute intrinsic Riemannian metric spaces).

Now let us superpose the absolute intrinsic metric spaces $\phi\hat{M}^2_{(1)}$ and $\phi\hat{M}^2_{(2)}$ in Figs. 3a and 3b by bringing $\phi\hat{M}^2_{(2)}$ to the location of $\phi\hat{M}^2_{(1)}$. The origin P of $\phi\hat{M}^2_{(2)}$ does not have to coincide with the origin O of $\phi\hat{M}^2_{(1)}$ in doing this. Since the curved absolute intrinsic 'dimensions' $\hat{u}^1$ of $\phi\hat{M}^2_{(1)}$ and $\hat{v}^1$ of $\phi\hat{M}^2_{(2)}$ both lie above the same proper intrinsic dimension u'^1 of $\phi E'^2$ (and dimension x'^1 of E'^2) and the curved absolute intrinsic 'dimensions' $\hat{u}^2$ of $\phi\hat{M}^2_{(1)}$ and $\hat{v}^2$ of $\phi\hat{M}^2_{(2)}$ both lie above the same proper intrinsic dimension u'^2 of $\phi E'^2$ (and dimension x'^2 of E'^2) prior to their superposition, the curved absolute intrinsic 'dimension' $\hat{v}^1$ will be naturally curved relative to the curved absolute intrinsic 'dimension' $\hat{u}^1$ on the

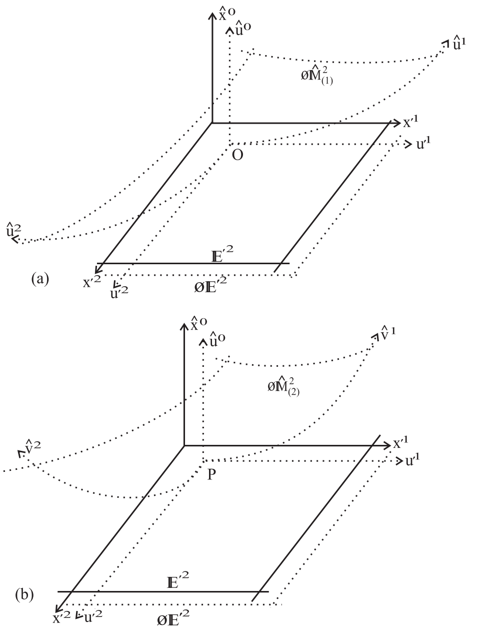

Fig. 3: A pair of parallel 'two-dimensional' absolute intrinsic metric spaces (or absolute intrinsic Riemannian metric spaces) are underlied by the global two-dimensional flat proper intrinsic space and proper physical Euclidean 2-space, prior to their superposition.

vertical $u'^1\hat{u}^0-$plane, and the curved absolute intrinsic 'dimension' $\hat{v}^2$ will be naturally curved relative to the curved absolute intrinsic 'dimension' $\hat{u}^2$ on the vertical $u'^2\hat{u}^0-$plane upon bringing them to the same location (or upon superposing them), as illustrated in Fig. 4. This case shall be referred to as superposition of parallel absolute intrinsic Riemann spaces.

The point $\hat{v}^1_{(1)}$ measured from point P of $\phi\hat{M}^2_{(2)}$ lies above point $\hat{u}^1_{(1)}$ measured from point O of $\phi\hat{M}^2_{(1)}$, and they both lie vertically above point $u'^1_{(1)}$ of $\phi E'^2$ (and point $x'^1_{(1)}$ of E'^2). Likewise point $\hat{v}^2_{(1)}$ of $\phi\hat{M}^2_{(2)}$ lies vertically above point $\hat{u}^2_{(1)}$ of $\phi\hat{M}^2_{(1)}$ and they both lie vertically above point $u'^2_{(1)}$ of $\phi E'^2$ (and point $x'^2_{(1)}$ of E'^2). The curved absolute intrinsic space 'dimension' $\hat{v}^1$ has known absolute intrinsic curvature parameter $\phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)})$ at point $\hat{v}^1_{(1)}$ relative to $\phi E'^2$, and the curved absolute intrinsic 'dimension' $\hat{v}^2$ has known absolute intrinsic curvature parameter $\phi\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)})$ at point $\hat{v}^2_{(1)}$ relative to $\phi E'^2$ from Fig. 3b. Likewise the curved absolute intrinsic 'dimension' $\hat{u}^1$ has known absolute intrinsic curvature parameter $\phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)})$ at point $\hat{u}^1_{(1)}$ relative to $\phi E'^2$ and the curved absolute intrinsic 'dimension' $\hat{u}^2$ has known absolute intrinsic curvature parameter $\phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)})$ at point $\hat{u}^2_{(1)}$ relative to $\phi E'^2$ from Fig. 3a.

We wish to obtain the resultant absolute intrinsic curvature parameters of the curved absolute intrinsic 'dimension' $\hat{v}^1$ at point $\hat{v}^1_{(1)}$ and of the curved absolute intrinsic 'dimension' $\hat{v}^2$ at point $\hat{v}^2_{(1)}$ relative to relative to $\phi E'^2$ or with respect to observers in E'^2, when the absolute intrinsic metric space $\phi\hat{M}^2_{(2)}$ is curved relative to the absolute intrinsic metric space $\phi\hat{M}^2_{(1)}$ as in Fig. 4 and then write the resultant absolute intrinsic metric tensor, resultant absolute intrinsic Ricci tensor and resultant absolute intrinsic line element at point $(\hat{v}^1_{(1)}, \hat{v}^2_{(1)})$ of $\phi\hat{M}^2_{(2)}$ in terms of the resultant absolute intrinsic curvature parameters relative to these observers.

Now the resultant absolute intrinsic metric tensor, $\phi\hat{\bar{g}}_{ik}$ at point $(\hat{v}^1_{(1)}, \hat{v}^2_{(1)})$ of $\phi\hat{M}^2_{(2)}$ is given in terms of the absolute intrinsic angles $\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)})$ and $\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)})$ of inclination of the curved absolute intrinsic 'dimension' $\hat{v}^1$ relative to the straight line proper intrinsic dimension u'^1 and of the curved absolute intrinsic 'dimension' $\hat{v}^2$ relative to the straight line proper intrinsic dimension u'^2 respectively in Fig. 3b as follows:

$$\phi\hat{g}^{(2)}_{ik} = \begin{pmatrix} \dfrac{1}{1-\sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)})} & 0 \\ 0 & \dfrac{1}{1-\sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)})} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-\phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)})^2} & 0 \\ 0 & \dfrac{1}{1-\phi\hat{k}_{\hat{v}^2}(\hat{v}^2)^2_{(1)}} \end{pmatrix} \quad (50)$$

Likewise the absolute intrinsic metric tensor is given at point $(\hat{u}^1_{(1)}, \hat{u}^2_{(1)})$ of $\phi\hat{M}_{(1)}$ in Fig. 7a as follows:

$$\phi\hat{g}^{(1)}_{ik} = \begin{pmatrix} \dfrac{1}{1-\sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})} & 0 \\ 0 & \dfrac{1}{1-\sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-\phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)})^2} & 0 \\ 0 & \dfrac{1}{1-\phi\hat{k}_{\hat{u}^2}(\hat{u}^2)^2_{(1)}} \end{pmatrix} \quad (51)$$

When the two parallel absolute intrinsic Riemann spaces coexist, as illustrated in Fig. 4, then the resultant absolute intrinsic metric tensor $\phi\hat{\bar{g}}_{ik}$ of the upper absolute intrinsic metric space $\phi\hat{M}^{(2)}_2$ relative to the underlying proper intrinsic metric space $\phi E'^2$ and proper physical Euclidean space E'^2, is given in terms of the resultant absolute intrinsic angles $\phi\hat{\bar{\psi}}_{\hat{v}^1}(\hat{v}^1)$ and $\phi\hat{\bar{\psi}}_{\hat{v}^2}(\hat{v}^2)$, and in terms of the resultant absolute intrinsic curvature parameters $\phi\hat{\bar{k}}_{\hat{v}^1}(\hat{v}^1)$ and $\phi\hat{\bar{k}}_{\hat{v}^2}(\hat{v}^2)$

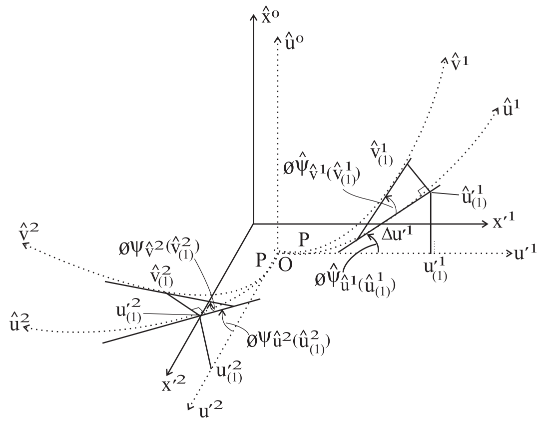

Fig. 4: Two co-existing parallel absolute intrinsic metric spaces.

respectively as follows:

$$\phi\hat{\bar{g}}_{ik} = \begin{pmatrix} \dfrac{1}{1-\sin^2\phi\hat{\bar{\psi}}_{\hat{v}^1}(\hat{v}^1)} & 0 \\ 0 & \dfrac{1}{1-\sin^2\phi\hat{\bar{\psi}}_{\hat{v}^2}(\hat{v}^2)} \end{pmatrix} \tag{52}$$

and

$$\phi\hat{\bar{g}}_{ik} = \begin{pmatrix} \dfrac{1}{1-\phi\hat{\bar{k}}_{\hat{v}^1}(\hat{v}^1)^2} & 0 \\ 0 & \dfrac{1}{1-\phi\hat{\bar{k}}_{\hat{v}^2}(\hat{v}^2)^2} \end{pmatrix} \tag{53}$$

where, as can be observed from Fig. 4,

$$\phi\hat{\bar{\psi}}_{\hat{v}^1}(\hat{v}^1) = \phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1) + \phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)$$

and

$$\phi\hat{\bar{\psi}}_{\hat{v}^2}(\hat{v}^2) = \phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2) + \phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2).$$

Now the absolute intrinsic Riemann spaces $\phi\hat{M}^2_{(2)}$ and $\phi\hat{M}^2_{(1)}$ are curved relative to the flat proper intrinsic space $\phi E'^2$, with the Euclidean metric δ_{ik}, (in Figs. 3a and 3b), prior to their superposition. Hence the components of their absolute intrinsic metric tensors can be written in terms of the components of the Euclidean metric prior to their superposition respectively as follows:

$$\left.\begin{aligned} (\phi\hat{g}^{(2)}_{11})^{-1} &= \delta_{11} - \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1) = \delta_{11} - \phi\hat{k}_{\hat{v}^1}(\hat{v}^1)^2; \\ (\phi\hat{g}^{(2)}_{22})^{-1} &= \delta_{22} - \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2) = \delta_{22} - \phi\hat{k}_{\hat{v}^2}(\hat{v}^2)^2; \\ (\phi\hat{g}^{(2)}_{12})^{-1} &= (\phi\hat{g}^{(2)}_{21})^{-1} = 0; \;\; (\text{for } \phi\hat{M}^2_{(2)}). \end{aligned}\right\} \tag{54}$$

and

$$\left.\begin{aligned} (\phi\hat{g}^{(1)}_{11})^{-1} &= \delta_{11} - \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1) = \delta_{11} - \phi\hat{k}_{\hat{u}^1}(\hat{u}^1)^2; \\ (\phi\hat{g}^{(1)}_{22})^{-1} &= \delta_{22} - \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2) = \delta_{22} - \phi\hat{k}_{\hat{u}^2}(\hat{u}^2)^2; \\ (\phi\hat{g}^{(1)}_{12})^{-1} &= (\phi\hat{g}^{(1)}_{21})^{-1} = 0; \;\; (\text{for } \phi\hat{M}^2_{(1)}). \end{aligned}\right\} \tag{55}$$

Upon the two absolute intrinsic Riemann spaces co-existing as in Fig. 4, on the other hand, while the $\phi\hat{M}^2_{(1)}$ is still curved relative to the flat proper intrinsic space $\phi E'^2$, such that the tangent to the curved absolute intrinsic 'dimension' $\hat{u}^1$ at point $\hat{u}^1_{(1)}$ is inclined to the straight line proper intrinsic dimension u'^1 at absolute intrinsic angle $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)})$ and the tangent to absolute intrinsic 'dimension' $\hat{u}^2$ at point $\hat{u}^2_{(1)}$ is inclined to u'^2 at absolute angle intrinsic $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)})$, the absolute intrinsic Riemann space $\phi\hat{M}^2_{(2)}$ is curved relative to the absolute intrinsic Riemann space $\phi\hat{M}^2_{(1)}$ with absolute intrinsic metric tensor $\phi\hat{g}^{(1)}_{ik}$. Consequently the tangent to the absolute intrinsic 'dimension' $\hat{v}^1$ at point $\hat{v}^1_{(1)}$ of $\phi\hat{M}^2_{(2)}$ is now inclined at absolute intrinsic angle $\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)})$ relative to the tangent to absolute intrinsic 'dimension' $\hat{u}^1$ at point $\hat{u}^1_{(1)}$ of $\phi\hat{M}^2_{(1)}$ and the tangent to the absolute intrinsic 'dimension' $\hat{v}^2$ at point $\hat{v}^2_{(1)}$ of $\phi\hat{M}^2_{(2)}$ is now inclined at absolute intrinsic angle $\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)})$ relative to the tangent to the curved absolute intrinsic 'dimension' $\hat{u}^2$ at point $\hat{u}^2_{(1)}$ of $\phi\hat{M}^2_{(1)}$. In the present situation, the absolute intrinsic metric tensor $\phi\hat{g}^{(1)}_{ik}$ of the absolute intrinsic metric space $\phi\hat{M}^2_{(1)}$ serves as the foundation absolute intrinsic metric tensor upon which the absolute intrinsic metric tensor of absolute intrinsic metric space $\phi\hat{M}^2_{(2)}$ must be constructed.

The components of the resultant absolute intrinsic metric tensor, (i.e. of the upper curved absolute intrinsic metric space $\phi\hat{M}^2_{(2)}$ relative to the flat proper intrinsic space $\phi E'^2$ in Fig. 4), are therefore given in terms of the components of $\phi\hat{g}^{(1)}_{ik}$, (like system (54) or (55) is written relative to the Euclidean metric δ_{ik}) as follows:

$$\left.\begin{aligned} (\phi\hat{\bar{g}}_{11})^{-1} &= \phi\hat{g}^{(1)}_{11} - \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}) \\ &= \phi\hat{g}^{(1)}_{11} - \phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)})^2; \\ (\phi\hat{\bar{g}}_{22})^{-1} &= \phi\hat{g}^{(1)}_{22} - \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \\ &= \phi\hat{g}^{(1)}_{22} - \phi\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)})^2; \\ (\phi\hat{\bar{g}}_{12})^{-1} &= (\phi\hat{\bar{g}}_{21})^{-1} = 0 \end{aligned}\right\} \tag{56}$$

It is appropriate to further elucidate system (56). The components $\phi\hat{g}^{(1)}_{11}$ and $\phi\hat{g}^{(1)}_{22}$ of the absolute intrinsic metric tensor $\phi\hat{g}_{ik}^{(1)}$ of the curved absolute intrinsic metric space $\phi\hat{M}_{(1)}^2$ have been written relative to the intrinsic Euclidean metric tensor reference in system (53) (by virtue of the appearance of the components δ_{11} and δ_{22} of the Euclidean metric tensor in (53)), because $\phi\hat{M}_{(1)}^2$ is curved relative to the proper intrinsic metric space $\phi E'^2$ with intrinsic Euclidean metric tensor in Fig. 3a. The components $\phi\hat{g}_{11}^{(2)}$ and $\phi\hat{g}_{22}^{(2)}$ of the absolute intrinsic metric tensor $\phi\hat{g}_{ik}^{(2)}$ of the curved absolute intrinsic metric space $\phi\hat{M}_{(2)}^2$ have likewise been written relative to the absolute intrinsic Euclidean metric tensor reference in system (54), because $\phi\hat{M}_{(2)}^2$ is curved relative to the proper intrinsic metric space $\phi E'^2$ in Fig. 3b. The components δ_{11} and δ_{22} of the intrinsic Euclidean metric tensor of $\phi E'^2$ that appear in systems (53) and (54) are related to the constant zero absolute intrinsic angle ($\phi\hat{\psi} = 0$) of inclination to the horizontal of the intrinsic dimensions u'^1 ($\equiv \phi x'^1$) and u'^2 ($\equiv \phi x'^2$) of $\phi E'^2$ in Figs. 3a and 3b as, $\delta_{11} = \delta_{22} = \cos^2(\phi\hat{\psi} = 0) = 1$.

On the other hand, the components $\phi\hat{\hat{g}}_{11}^{(2)}$ and $\phi\hat{\hat{g}}_{22}^{(2)}$ of the resultant absolute intrinsic metric tensor $\phi\hat{\hat{g}}_{ik}^{(2)}$ of the curved absolute intrinsic metric space $\phi\hat{M}_{(2)}^2$ relative to the proper intrinsic Euclidean space $\phi E'^2$ in Fig. 4 have been written relative to absolute intrinsic sub-Riemannian metric tensor reference in system (55). This is so because $\phi\hat{M}_{(2)}^2$ is curved relative to the intermediate curved absolute intrinsic metric space $\phi\hat{M}_{(1)}^2$, which is, in turn, curved relative to the proper intrinsic metric space $\phi E'^2$ in Fig. 4. The components $\phi\hat{g}_{11}^{(1)}$ and $\phi\hat{g}_{22}^{(1)}$ of the absolute intrinsic metric metric tensor on the absolute intrinsic metric space $\phi\hat{M}_{(1)}^2$ that appear in system (55) are related to the varying absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)$ and $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2)$ of inclinations of the curved absolute intrinsic 'dimensions' $\hat{u}^1$ ($\equiv \phi\hat{x}^1$) and $\hat{u}^2$ ($\equiv \phi\hat{x}^2$) of $\phi\hat{M}_{(1)}^2$ relative to the proper intrinsic dimensions u'^1 and u'^2 of $\phi E'^2$ respectively at an arbitrary point on $\phi\hat{M}_{(1)}^2$ as, $\phi\hat{g}_{11}^{(1)} = \cos^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)$ and $\phi\hat{g}_{22}^{(1)} = \cos^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2)$.

In other words, the constant zero absolute intrinsic angle ($\phi\hat{\psi} = 0$) of inclination to the horizontal of the proper intrinsic dimensions u'^1 and u'^2 of the reference proper intrinsic Euclidean space $\phi E'^2$ in Figs. 3a and 3b have been replaced by the varying absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)$ and $\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2)$ of inclinations to the horizontal of the absolute intrinsic 'dimensions' $\hat{u}^1$ and $\hat{u}^2$ of the intermediate curved absolute intrinsic metric space $\phi\hat{M}_{(1)}^2$ in Fig. 4. Consequently, $\delta_{11} = \delta_{22} = \cos^2(\phi\hat{\psi} = 0) = 1$ in systems (53) and (54) have been replaced by $\phi\hat{g}_{11}^{(1)} = \cos^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1)$ and $\phi\hat{g}_{22}^{(1)} = \cos^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2)$ respectively in system (56).

By substituting system (55) into system (56) we have the following

$$\left.\begin{aligned}
(\phi\hat{\hat{g}}_{11})^{-1} &= 1 - \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}_{(1)}^1) - \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}_{(1)}^1) \\
&= 1 - \phi\hat{k}_{\hat{u}^1}(\hat{u}_{(1)}^1)^2 - \phi\hat{k}_{\hat{v}^1}(\hat{v}_{(1)}^1)^2; \\
(\phi\hat{\hat{g}}_{22})^{-1} &= 1 - \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}_{(1)}^2) - \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}_{(1)}^2) \\
&= 1 - \phi\hat{k}_{\hat{u}^2}(\hat{u}_{(1)}^2)^2 - \phi\hat{k}_{\hat{v}^2}(\hat{v}_{(1)}^2)^2; \\
(\phi\hat{\hat{g}}_{12})^{-1} &= (\phi\hat{\hat{g}}_{21})^{-1} = 0
\end{aligned}\right\} \quad (57)$$

The components of the resultant absolute intrinsic metric tensor in system (59) are the same as in Eqs. (39) and (40). Hence we obtain expressions for the resultant absolute intrinsic angles $\phi\hat{\hat{\psi}}$ and the resultant absolute intrinsic curvature parameter $\phi\hat{\hat{k}}$ in terms of the absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}}$ and $\phi\hat{\psi}_{\hat{v}}$ and absolute intrinsic curvature parameters $\phi\hat{k}_{\hat{u}}$ and $\phi\hat{k}_{\hat{v}}$ of the individual absolute intrinsic metric spaces prior to their superposition respectively as follows:

$$\begin{aligned}
\phi\hat{\hat{g}}_{11} &= \left(1 - \sin^2\phi\hat{\hat{\psi}}_{\hat{v}^1}(\hat{v}^1)\right)^{-1} \\
&= \left(1 - \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}_{(1)}^1) - \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}_{(1)}^1)\right)^{-1}
\end{aligned}$$

Hence

$$\begin{aligned}
\sin^2\phi\hat{\hat{\psi}}_1(\hat{v}_{(1)}^1) &= \sin^2\left(\phi\hat{\psi}_{\hat{u}^1}(\hat{u}_{(1)}^1) + \phi\hat{\psi}_{\hat{v}^1}(\hat{v}_{(1)}^1)\right); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}_{(1)}^1) + \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}_{(1)}^1) \qquad (58a)
\end{aligned}$$

$$\begin{aligned}
\phi\hat{\hat{g}}_{22} &= \left(1 - \sin^2\phi\hat{\hat{\psi}}_{\hat{v}^2}(\hat{v}^2)\right)^{-1} \\
&= \left(1 - \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}_{(1)}^2) - \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}_{(1)}^2)\right)^{-1}
\end{aligned}$$

Hence

$$\begin{aligned}
\sin^2\phi\hat{\hat{\psi}}_2(\hat{v}_{(1)}^2) &= \sin^2\left(\phi\hat{\psi}_{\hat{u}^2}(\hat{u}_{(1)}^2) + \hat{\psi}_{\phi\hat{v}^2}(\hat{v}_{(1)}^2)\right); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}_{(1)}^2) + \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}_{(1)}^2) \qquad (58b)
\end{aligned}$$

Consequently,

$$\phi\hat{\hat{k}}_1(\hat{v}_{(1)}^1)^2 = \phi\hat{k}_{\hat{u}^1}(\hat{u}_{(1)}^1)^2 + \phi\hat{k}_{\hat{v}^1}(\hat{v}_{(1)}^1)^2 \qquad (59a)$$

$$\phi\hat{\hat{k}}_2(\hat{v}_{(1)}^2)^2 = \phi\hat{k}_{\hat{u}^2}(\hat{u}_{(1)}^2)^2 + \phi\hat{k}_{\hat{v}^2}(\hat{v}_{(1)}^2)^2 \qquad (59b)$$

Equations (58a) and (58b) give the rules for the composition of two absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}}$ and $\phi\hat{\psi}_{\hat{v}}$, while Eqs. (59a) and (59b) give the corresponding rule for composition two absolute intrinsic curvature parameters for the purpose of writing the resultant absolute intrinsic metric tensor and resultant absolute intrinsic Ricci tensor in absolute intrinsic Riemann geometry, for the situation where a pair of parallel 'two-dimensional' absolute intrinsic metric spaces coexist, as illustrated in Fig. 4. They can be extended to the situation where a pair of parallel 'three-dimensional' absolute intrinsic Riemann spaces coexist as follows:

$$\left.\begin{aligned}
\sin^2\phi\hat{\bar{\psi}}_1(\hat{v}^1_{(1)}) &= \sin^2\left(\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) + \phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)})\right); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) + \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}); \\
\sin^2\phi\hat{\bar{\psi}}_2(\hat{v}^2_{(1)}) &= \sin^2\left(\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) + \phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)})\right); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) + \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}); \\
\sin^2\phi\hat{\bar{\psi}}_3(\hat{v}^3_{(1)}) &= \sin^2\left(\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3_{(1)}) + \phi\hat{\psi}_{\hat{v}^3}(\hat{v}^3_{(1)})\right); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3_{(1)}) + \sin^2\phi\hat{\psi}_{\hat{v}^3}(\hat{v}^3_{(1)})
\end{aligned}\right\} \tag{60}$$

And

$$\left.\begin{aligned}
\phi\hat{\bar{k}}_1(\hat{v}^1_{(1)})^2 &= \phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)})^2 + \phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)})^2; \\
\phi\hat{\bar{k}}_2(\hat{v}^2_{(1)})^2 &= \phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)})^2 + \phi\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)})^2; \\
\phi\hat{\bar{k}}_3(\hat{v}^3_{(1)})^2 &= \phi\hat{k}_{\hat{u}^3}(\hat{u}^3_{(1)})^2 + \phi\hat{k}_{\hat{v}^3}(\hat{v}^3_{(1)})^2
\end{aligned}\right\} \tag{61}$$

Systems (60) and (61) admit of generalization to a situation where N parallel '3-dimensional' absolute intrinsic Riemann spaces coexist, (where the Nth absolute intrinsic metric space $\phi\hat{M}^3_{(N)}$ has curved absolute intrinsic 'dimensions' $\hat{w}^1$, $\hat{w}^2$ and $\hat{w}^3$), as follows:

$$\left.\begin{aligned}
\sin^2\phi\hat{\bar{\psi}}_1(\hat{w}^1_{(1)}) &= \sin^2\Big(\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) + \phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}) \\
&\qquad + \cdots + \phi\hat{\psi}_{\hat{w}^1}(\hat{w}^1_{(1)})\Big) \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) + \sin^2\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}) \\
&\qquad + \cdots + \sin^2\phi\hat{\psi}_{\hat{w}^1}(\hat{w}^1_{(1)}); \\
\sin^2\phi\hat{\bar{\psi}}_2(\hat{w}^2_{(1)}) &= \sin^2\Big(\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) + \phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \\
&\qquad + \cdots + \phi\hat{\psi}_{\hat{w}^2}(\hat{w}^2_{(1)})\Big); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) + \sin^2\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \\
&\qquad + \cdots + \sin^2\phi\hat{\psi}_{\hat{w}^2}(\hat{w}^2_{(1)}); \\
\sin^2\phi\hat{\bar{\psi}}_3(\hat{w}^3_{(1)}) &= \sin^2\Big(\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3_{(1)}) + \phi\hat{\psi}_{\hat{v}^3}(\hat{v}^3_{(1)}) \\
&\qquad + \cdots + \phi\hat{\psi}_{\hat{w}^3}(\hat{w}^3_{(1)})\Big); \\
&= \sin^2\phi\hat{\psi}_{\hat{u}^3}(\hat{u}^3_{(1)}) + \sin^2\phi\hat{\psi}_{\hat{v}^3}(\hat{v}^3_{(1)}) \\
&\qquad + \cdots + \sin^2\phi\hat{\psi}_{\hat{w}^3}(\hat{w}^3_{(1)})
\end{aligned}\right\} \tag{62}$$

and

$$\left.\begin{aligned}
\phi\hat{\bar{k}}_1(\hat{w}^1_{(1)})^2 &= \phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)})^2 + \phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)})^2 + \cdots \\
&\quad \cdots + \phi\hat{k}_{\hat{w}^1}(\hat{w}^1_{(1)})^2; \\
\phi\hat{\bar{k}}_2(\hat{w}^2_{(1)})^2 &= \phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)})^2 + \phi\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)})^2 + \cdots \\
&\quad \cdots + \phi\hat{k}_{\hat{w}^2}(\hat{w}^2_{(1)})^2; \\
\phi\hat{\bar{k}}_3(\hat{w}^3_{(1)})^2 &= \phi\hat{k}_{\hat{u}^3}(\hat{u}^3_{(1)})^2 + \phi\hat{k}_{\hat{v}^3}(\hat{v}^3_{(1)})^2 + \cdots \\
&\quad \cdots + \phi\hat{k}_{\hat{w}^3}(\hat{w}^3_{(1)})^2
\end{aligned}\right\} \tag{63}$$

Although equations (60) - (63) show no ceiling on the resultant absolute intrinsic angle $\phi\hat{\bar{\psi}}_q$; $q = 1, 2$ or 3, we know that $\phi\hat{\bar{\psi}}_q$ has a maximum value of $\phi\hat{\bar{\psi}}_q = \frac{\pi}{2}$, since then the curved absolute intrinsic 'dimension' $\hat{w}^q$ of the last (i.e. the Nth) absolute intrinsic metric space will lie along the vertical, parallel to the absolute intrinsic time 'dimension' $\hat{u}^0$. This implies that there is a ceiling on the number of absolute intrinsic metric spaces that can be superposed. The implication of going beyond the ceiling, that is, for making $\phi\hat{\bar{\psi}}_q > \frac{\pi}{2}$, will be derived elsewhere with further development.

The resultant absolute intrinsic metric tensor $\phi\hat{\bar{g}}_{ik}$ of the last (i.e. the Nth) absolute intrinsic metric space $\phi\hat{M}^3_{(N)}$ relative to the underlying proper intrinsic metric space $\phi E'^3$ is given in terms of the resultant absolute intrinsic angles and the resultant absolute intrinsic curvature parameters respectively as follows:

$$\phi\hat{\bar{g}}^{(2)}_{ik} = \begin{pmatrix} \dfrac{1}{1-\sin^2\phi\hat{\bar{\psi}}_1} & 0 & 0 \\ 0 & \dfrac{1}{1-\sin^2\phi\hat{\bar{\psi}}_2} & 0 \\ 0 & 0 & \dfrac{1}{1-\sin^2\phi\hat{\bar{\psi}}_3} \end{pmatrix} \tag{64}$$

or

$$\phi\hat{\bar{g}}^{(2)}_{ik} = \begin{pmatrix} \dfrac{1}{1-(\phi\hat{\bar{k}}_1)^2} & 0 & 0 \\ 0 & \dfrac{1}{1-(\phi\hat{\bar{k}}_2)^2} & 0 \\ 0 & 0 & \dfrac{1}{1-(\phi\hat{\bar{k}}_3)^2} \end{pmatrix} \tag{65}$$

The resultant absolute intrinsic Ricci tensor is likewise given in terms of resultant absolute intrinsic angles and resultant absolute intrinsic curvature parameters as follows:

$$\phi\hat{\bar{R}}_{ik} = \begin{pmatrix} \frac{\sin^2\phi\hat{\bar{\psi}}_1}{1-\sin^2\phi\hat{\bar{\psi}}_1} & 0 & 0 \\ 0 & \frac{\sin^2\phi\hat{\bar{\psi}}_2}{1-\sin^2\phi\hat{\bar{\psi}}_2} & 0 \\ 0 & 0 & \frac{\sin^2\phi\hat{\bar{\psi}}_3}{1-\sin^2\phi\hat{\bar{\psi}}_3} \end{pmatrix} \quad (66)$$

or

$$\phi\hat{\bar{R}}_{ik} = \begin{pmatrix} \frac{(\phi\hat{\bar{k}}_1)^2}{1-(\phi\hat{\bar{k}}_1)^2} & 0 & 0 \\ 0 & \frac{(\phi\hat{\bar{k}}_2)^2}{1-(\phi\hat{\bar{k}}_2)^2} & 0 \\ 0 & 0 & \frac{(\phi\hat{\bar{k}}_3)^2}{1-(\phi\hat{\bar{k}}_3)^2} \end{pmatrix} \quad (67)$$

And the resultant absolute intrinsic line element is the following

$$(d\phi\hat{\bar{s}})^2 = (d\hat{u}^0)^2 - \frac{\left((d\hat{u}^1)^2 + (d\hat{u}^2)^2 + (d\hat{u}^3)^2\right)}{1-(\phi\hat{\bar{k}})^2} \quad (68)$$

where $\phi\hat{\bar{k}} = \phi\hat{\bar{k}}_1 = \phi\hat{\bar{k}}_2 = \phi\hat{\bar{k}}_3$ has been assumed.

As mentioned earlier, only the situation where $\phi\hat{\bar{k}}_1(\hat{w}^1)$, $\phi\hat{\bar{k}}_2(\hat{w}^2)$ and $\phi\hat{\bar{k}}_3(\hat{w}^3)$ are all identical to $\phi\hat{\bar{k}}$, as assumed in writing Eq. (68), shall be of relevance in absolute intrinsic Riemann geometry ultimately. For that situation, the two absolute intrinsic tensor equations (32) and (44) derived in the context of absolute intrinsic Riemann geometry earlier are given in terms of the resultant absolute intrinsic tensors $\phi\hat{\bar{g}}_{ik}$ and $\phi\hat{\bar{R}}_{ik}$ and resultant absolute intrinsic curvature parameter as follows:

$$\phi\hat{\bar{g}}_{ik} - \phi\hat{\bar{R}}_{ik} = \delta_{ik} \quad (69)$$

and

$$\phi\hat{\bar{R}}_{ik} - (\phi\hat{\bar{k}})^2\phi\hat{\bar{g}}_{ik} = 0 \quad (70)$$

The solution to equations (69) and (70) are equations (65) and (67) with $(\phi\hat{\bar{k}})^2 = (\phi\hat{\bar{k}}_1)^2 = (\phi\hat{\bar{k}}_2)^2 = (\phi\hat{\bar{k}}_3)^2$ assumed.

We have again accomplished in this sub-section the first stage of the formulation of absolute intrinsic Riemann geometry in a situation where two or a larger number of parallel absolute intrinsic metric spaces co-exist. We shall now proceed to the second stage namely, obtaining resultant absolute intrinsic coordinate projection relations, when two or a larger number of parallel absolute intrinsic metric spaces co-exist.

1.6.2 The resultant absolute intrinsic coordinate projection relations when two or a larger number of parallel absolute intrinsic metric spaces co-exist

Now let us redraw Fig. 4 while showing certain detail required for this sub-sub-section as Fig. 5. Let us consider elementary absolute intrinsic metric coordinate intervals $d\hat{v}^1$ and $d\hat{v}^2$ defined about point $(\hat{v}^1_{(1)}, \hat{v}^2_{(1)})$ of $\phi\hat{M}^2_{(2)}$ to be the dimensions of a locally flat frame on $\phi\hat{M}^2_{(2)}$. The interval $d\hat{v}^1$ about point $\hat{v}^1_{(1)}$ of $\phi\hat{M}^2_{(2)}$ projects component $d\hat{u}^1$ into the underlying curved absolute intrinsic 'dimension' $\hat{u}^1$ at point $\hat{u}^1_{(1)}$ of $\hat{u}^1$, as shown in Fig. 5. Likewise the interval $d\hat{v}^2$ about point $\hat{v}^2_{(1)}$ of $\phi\hat{M}^2_{(2)}$ projects component $d\hat{u}^2$ into the underlying curved absolute intrinsic 'dimension' $\hat{u}^2$ of $\phi\hat{M}^2_{(1)}$, which lies along the tangent to $\hat{u}^2$ at point $\hat{u}^2_{(1)}$ of $\hat{u}^2$, as also shown in Fig. 5. The following projection relations obtain from elementary coordinate geometry:

$$d\hat{u}^1 = d\hat{v}^1 \cos\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}); \;\; d\hat{u}^2 = d\hat{v}^2 \cos\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \quad (71)$$

In turn, the component $d\hat{u}^1$ projected about point $\hat{u}^1_{(1)}$ of the curved absolute intrinsic 'dimension' $\hat{u}^1$ of $\phi\hat{M}^2_{(1)}$ projects component du'^1 about the corresponding point $u'^1_{(1)}$ of its underlying straight line proper intrinsic dimension u'^1 of $\phi E'^2$ and the component $d\hat{u}^2$ projected about point $\hat{u}^2_{(1)}$ of the curved absolute intrinsic 'dimension' $\hat{u}^2$ projects component du'^2 about the corresponding point $u'^2_{(1)}$ of its underlying straight line proper intrinsic dimension u'^2 of $\phi E'^2$, as shown in Fig. 5.

Again the following coordinate projection relations obtain from Fig. 5 from elementary coordinate geometry:

$$du'^1 = d\hat{u}^1 \cos\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}); \;\; du'^2 = d\hat{u}^2 \cos\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \quad (72)$$

Then by combining systems (71) and (72) we obtain the following:

$$\left.\begin{aligned} du'^1 &= d\hat{v}^1 \cos\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) \cos\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}); \\ du'^2 &= d\hat{v}^2 \cos\phi\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \cos\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \end{aligned}\right\} \quad (73)$$

System (73) gives the resultant length contraction relations of the absolute intrinsic metric coordinate intervals of the absolute intrinsic metric space $\phi\hat{M}^2_{(2)}$ with respect to observers in E'^2. They become the following in terms of absolute intrinsic curvature parameters:

$$\left.\begin{aligned} du'^1 &= d\hat{v}^1(1-\phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)}))^{1/2}(1-\phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)}))^{1/2}; \\ du'^2 &= d\hat{v}^2(1-\phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)}))^{1/2}(1-\phi\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)}))^{1/2} \end{aligned}\right\} \quad (74)$$

where the definitions of the absolute intrinsic curvature parameters of system (14) have been used.

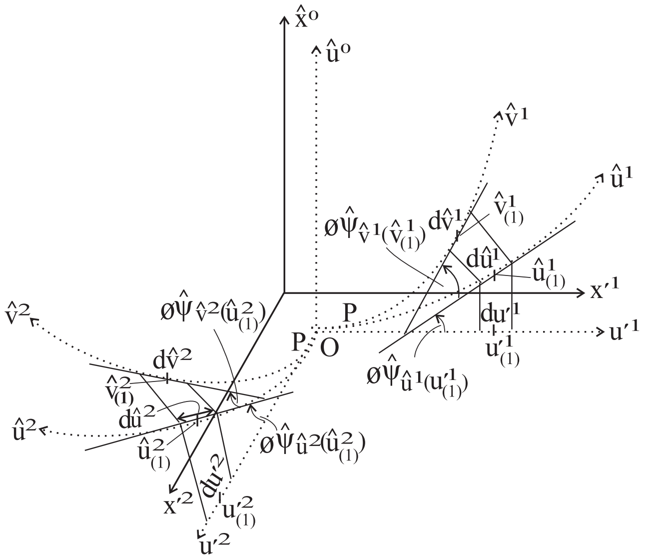

Fig. 5: Obtaining the resultant intrinsic coordinate projections when a pair of parallel 'two-dimensional' absolute intrinsic metric spaces co-exist.

Systems (73) and (74) admit of generalization to a situation where any number N of 'two-dimensional' parallel absolute intrinsic Riemann spaces co-exist, in which case they become the following respectively:

$$\left.\begin{aligned} du'^1 &= d\hat{w}^1 \cos\phi\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) \cos\phi\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}) \cdots \\ &\quad \cdots \cos\phi\hat{\psi}_{\hat{w}^1}(\hat{w}^1_{(1)}); \\ du'^2 &= d\hat{w}^2 \cos\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \cos\phi\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \cdots \\ &\quad \cdots \cos\phi\hat{\psi}_{\hat{w}^2}(\hat{w}^2_{(1)}) \end{aligned}\right\} \quad (75)$$

and

$$\left.\begin{aligned} du'^1 &= d\hat{w}^1(1-\phi\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)}))^{1/2}(1-\phi\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)}))^{1/2}\times \\ &\quad \cdots (1-\phi\hat{k}_{\hat{w}^1}(\hat{w}^1_{(1)}))^{1/2}; \\ du'^2 &= d\hat{w}^2(1-\phi\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)}))^{1/2}(1-\phi\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)}))^{1/2}\times \\ &\quad \cdots (1-\phi\hat{k}_{\hat{w}^2}(\hat{w}^2_{(1)}))^{1/2} \end{aligned}\right\} \quad (76)$$

Systems (72) through (76) for superposition of 'two-dimensional' parallel absolute intrinsic metric spaces admit of easy and direct extension to superposition of 'three-dimensional' parallel absolute intrinsic metric spaces, in which case, a third expression for du'^3 must be added to each of the systems.

Now the two-dimensional proper physical Euclidean space E'^2 is the outward manifestation of $\phi E'^2$ in Fig. 1 through Fig. 5 of this paper. The intrinsic coordinate projection formulae relating the proper intrinsic coordinate intervals du'^1 and du'^2 of $\phi E'^2$ to the absolute intrinsic coordinate intervals $d\hat{w}^1$ and $d\hat{w}^2$ of $\phi\hat{M}^2_{(N)}$ respectively, likewise have their outward manifestations. The outward manifestations of systems (75) and (76) obtained by simply removing the symbol ϕ are the following respectively,

$$\left.\begin{aligned} dx'^1 &= d\hat{x}^1 \cos\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}) \cos\hat{\psi}_{\hat{v}^1}(\hat{v}^1_{(1)}) \cdots \\ &\quad \cdots \cos\hat{\psi}_{\hat{w}^1}(\hat{w}^1_{(1)}); \\ dx'^2 &= d\hat{x}^2 \cos\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \cos\hat{\psi}_{\hat{v}^2}(\hat{v}^2_{(1)}) \cdots \\ &\quad \cdots \cos\hat{\psi}_{\hat{w}^2}(\hat{w}^2_{(1)}) \end{aligned}\right\} \quad (77)$$

and

$$\left.\begin{aligned} dx'^1 &= d\hat{x}^1(1-\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)}))^{1/2}(1-\hat{k}_{\hat{v}^1}(\hat{v}^1_{(1)}))^{1/2} \cdots \\ &\quad \cdots (1-\hat{k}_{\hat{w}^1}(\hat{w}^1_{(1)}))^{1/2}; \\ dx'^2 &= d\hat{x}^2(1-\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)}))^{1/2}(1-\hat{k}_{\hat{v}^2}(\hat{v}^2_{(1)}))^{1/2} \cdots \\ &\quad \cdots (1-\hat{k}_{\hat{w}^2}(\hat{w}^2_{(1)}))^{1/2} \end{aligned}\right\} \quad (78)$$

It must be recalled, as stated at the beginning of this section, that $\hat{u}^i$ stands for $\phi\hat{x}^i$ and u'^i stands for $\phi x'^i$, and $\hat{x}^i$ – a coordinate of absolute space $\hat{E}^2$ – is the outward manifestation of $\phi\hat{x}^i$.

In the case of a singular absolute intrinsic metric space, systems (77) and (78) simplify respectively as follows:

$$dx'^1 = d\hat{x}^1 \cos\hat{\psi}_{\hat{u}^1}(\hat{u}^1_{(1)}); \;\; dx'^2 = d\hat{x}^2 \cos\hat{\psi}_{\hat{u}^2}(\hat{u}^2_{(1)}) \quad (79)$$

and

$$dx'^1 = d\hat{x}^1(1-\hat{k}_{\hat{u}^1}(\hat{u}^1_{(1)}))^{1/2}; \; dx'^2 = d\hat{x}^2(1-\hat{k}_{\hat{u}^2}(\hat{u}^2_{(1)}))^{1/2} \quad (80)$$

Systems (79) or (80) is the outward or physical manifestation of system (1) derived from Fig. 1. System (79) or (80) expresses the evolution of the flat two-dimensional proper physical space E'^2 from the flat 'two-dimensional' absolute space $\hat{E}^2$ with the presence of absolute intrinsic Riemann geometry. It must be noted that the coordinates $\hat{x}^1$ and $\hat{x}^2$ of the absolute space $\hat{E}^2$ are not curved despite systems (79) and (80).

Once the absolute intrinsic 'dimensions' $\hat{u}^1(\equiv \phi\hat{x}^1)$ and $\hat{u}^2(\equiv \phi\hat{x}^2)$ of the initially flat absolute intrinsic space $\phi\hat{E}^2$ underlying $\hat{E}^2$ become curved to form the absolute intrinsic Riemann space $\phi\hat{M}^2$ and project proper intrinsic dimensions $u'^1(\equiv \phi x'^1)$ and $u'^2(\equiv \phi x'^2)$ respectively along the horizontal in Fig. 1, then the projective proper intrinsic dimensions u'^1 and u'^2 of $\phi E'^2$ along the horizontal are made manifest in the proper dimensions x'^1 and x'^2 respectively of the flat proper physical space E'^2 along the horizontal, without any need to prescribe the curvature of the 'dimensions' $\hat{x}^1$ and $\hat{x}^2$ of the initially flat absolute space $\hat{E}^2$ overlying the initially flat absolute intrinsic space $\phi\hat{E}^2$. The flat proper physical space E'^2 has simply evolved from the initial flat absolute space $\hat{E}^2$ by virtue of the evolution of curved absolute intrinsic metric space $\phi\hat{M}^2$ (or evolution of absolute intrinsic Riemann geometry).

The results of the first stage of the formulation of absolute intrinsic Riemann geometry for a singular 'two-dimensional' or 'three-dimensional' absolute intrinsic metric space in sub-sections 1.1 through 1.5 and for two or a larger number of co-existing parallel 'two-dimensional' or 'three-dimensional' absolute intrinsic metric spaces in sub-section 1.6 are valid with respect to observers in the flat proper physical space E'^2 or E'^3.

It is the intervals of intrinsic dimensions du'^1, du'^2 and du'^3 of the proper intrinsic metric space $\phi E'^3$ projected by the corresponding intervals of the curved absolute intrinsic 'dimensions' $d\hat{v}^1, d\hat{v}^2$ and $d\hat{v}^3$ of the absolute intrinsic metric space $\phi\hat{M}^3_{(2)}$, expressed by system (73) or (74) for $N = 2$ or by system (75) or (76) for N co-existing curved absolute intrinsic metric spaces, and indeed the entire proper intrinsic space $\phi E'^3$ projected by $\phi\hat{M}^3$ in absolute intrinsic Riemann geometry that intrinsic observers located in $\phi E'^3$ could observe. And it is the outward manifestations of intervals du'^1, du'^2 and du'^3 namely, dx'^1, dx'^2 and dx'^3 expressed by system (79) or (80) or system (77) or (78) in general, and indeed the entire flat proper physical 3-space E'^3 that evolved from flat absolute space $\hat{E}^3$ in the context of absolute intrinsic Riemann geometry, which observers in E'^3 could observe.

Now let us rewrite system (75) in terms of resultant absolute intrinsic angles $\phi\hat{\psi}_{1\text{res}}$ and $\phi\hat{\psi}_{2\text{res}}$ as follows:

$$du'^1 = d\hat{w}^1 \cos\phi\hat{\psi}_{1\text{res}}; \;\; du'^2 = d\hat{w}^2 \cos\phi\hat{\psi}_{2\text{res}} \tag{81}$$

And let us rewrite system (76) in terms of resultant absolute intrinsic curvature parameters $\phi\hat{k}_{1\text{res}}$ and $\hat{k}_{2\text{res}}$ as follows:

$$du'^1 = d\hat{w}^1(1 - \phi\hat{k}_{1\text{res}})^{1/2}; \;\; du'^2 = d\hat{w}^2(1 - \phi\hat{k}_{2\text{res}})^{1/2} \tag{82}$$

Then as follows from systems (75) and (81)

$$\left.\begin{aligned} \cos\phi\hat{\psi}_{1\text{res}} &= \cos(\phi\hat{\psi}_{\hat{u}^1} + \phi\hat{\psi}_{\hat{v}^1} + \cdots + \phi\hat{\psi}_{\hat{w}^1}); \\ &= \cos\phi\hat{\psi}_{\hat{u}^1}\cos\phi\hat{\psi}_{\hat{v}^1}\cdots\cos\phi\hat{\psi}_{\hat{w}^1} \\ \cos\phi\hat{\psi}_{2\text{res}} &= \cos(\phi\hat{\psi}_{\hat{u}^2} + \phi\hat{\psi}_{\hat{v}^2} + \cdots + \phi\hat{\psi}_{\hat{w}^2}); \\ &= \cos\phi\hat{\psi}_{\hat{u}^2}\cos\phi\hat{\psi}_{\hat{v}^2}\cdots\cos\phi\hat{\psi}_{\hat{w}^2} \end{aligned}\right\} \tag{83}$$

And as follows from systems (78) and (82),

$$\left.\begin{aligned} \phi\hat{k}^2_{1\text{res}} &= 1 - (1 - \phi\hat{k}^2_{\hat{u}^1})(1 - \phi\hat{k}^2_{\hat{v}^1})\cdots \\ &\quad \cdots(1 - \phi\hat{k}_{\hat{w}^1})^2; \\ \phi\hat{k}^2_{2\text{res}} &= 1 - (1 - \phi\hat{k}^2_{\hat{u}^2})(1 - \phi\hat{k}^2_{\hat{v}^2})\cdots \\ &\quad \cdots(1 - \phi\hat{k}_{\hat{w}^2})^2 \end{aligned}\right\} \tag{84}$$

System (83) expresses the rule for composition of the absolute intrinsic angles $\phi\hat{\psi}_{\hat{u}^1}, \phi\hat{\psi}_{\hat{v}^1}, \ldots, \phi\hat{\psi}_{\hat{w}^1}$ and $\phi\hat{\psi}_{\hat{u}^2}$, $\phi\hat{\psi}_{\hat{v}^2}, \ldots, \phi\hat{\psi}_{\hat{w}^2}$ for the purpose of obtaining resultant intrinsic coordinate projections (or resultant intrinsic length contraction formulae) in the context of absolute intrinsic Riemann geometry, while system (84) expresses the corresponding rule for composition of absolute intrinsic curvature parameters $\phi\hat{k}_{\hat{u}^1}$, $\phi\hat{k}_{\hat{v}^1}$, ..., $\phi\hat{k}_{\hat{w}^1}$ and $\phi\hat{k}_{\hat{u}^2}$, $\phi\hat{k}_{\hat{v}^2}$, ..., $\phi\hat{k}_{\hat{w}^2}$. These rules shall be written more compactly as follows:

$$\begin{aligned} \cos\phi\hat{\psi}_{i\text{res}} &= \cos(\phi\hat{\psi}_{i1} + \phi\hat{\psi}_{i2} + \cdots + \phi\hat{\psi}_{iN}) \\ &= \cos\phi\hat{\psi}_{i1}\cos\phi\hat{\psi}_{i2}\cdots\cos\phi\hat{\psi}_{iN} \end{aligned} \tag{85}$$

and

$$\phi\hat{k}^2_{i\text{res}} = 1 - (1 - \phi\hat{k}^2_{i1})(1 - \phi\hat{k}^2_{i2})\cdots(1 - \phi\hat{k}^2_{iN}) \tag{86}$$

where $i = 1, 2$ or 3 refers to the three curved absolute intrinsic 'dimensions' of the absolute intrinsic metric spaces superposed, and N is the number of absolute intrinsic metric spaces superposed.

One observes from Eq. (85) that if $\phi\hat{\psi}_{iq} = 90^\text{o}$; $q = 1$ or 2 or 3 ... or N, then $\cos\phi\hat{\psi}_{iq} = 0$ and $\cos\phi\hat{\psi}_{i\text{res}} = 0$. Hence $\phi\hat{\psi}_{i\text{res}} = 90^\text{o}$ too. Also if $\phi\hat{k}_{iq} = 1$; $q = 1, 2, 3, \ldots$ or N, which corresponds to $\phi\hat{\psi}_{iq} = 90^\text{o}$ from $\phi\hat{k}_{iq} = \sin\phi\hat{\psi}_{iq}$, then $\phi\hat{k}_{1\text{res}} = 1$ too. These results show that the rules for composition of absolute intrinsic angles and absolute intrinsic curvature parameters for the purpose of obtaining resultant intrinsic coordinate projections, (or resultant intrinsic length contraction formulae), in absolute intrinsic Riemann geometry do not lead to values of resultant absolute intrinsic angles larger than 90^o or resultant absolute intrinsic curvature parameters larger than unity. In other words, absolute intrinsic angle, $\phi\hat{\psi} = 90^\text{o}$, is an invariant absolute intrinsic angle and absolute intrinsic curvature parameter, $\phi\hat{k} = 1$, is an invariant absolute intrinsic curvature parameter in the rules for composition of absolute intrinsic angles and absolute intrinsic curvature parameters, for the purpose of obtaining resultant absolute intrinsic coordinate projection relations or resultant intrinsic length contraction formulae with respect to observers in the underlying proper physical Euclidean 3-space E'^3, when two or a larger number of absolute intrinsic metric spaces co-exist.

Finally it is important to remark the major difference between the rule for composition of absolute intrinsic angles of system (62) (or the equivalent rule for composition of absolute intrinsic curvature parameters of system (63)) for the purpose of writing the resultant absolute intrinsic metric tensor and resultant absolute intrinsic Ricci tensor and the corresponding rule for composition of absolute intrinsic angles of system (83) (or its equivalent rule for composition of absolute intrinsic curvature parameters of system (84)) for the purpose of writing the resultant intrinsic coordinate projection relations or resultant intrinsic length contraction formulae in the context of absolute intrinsic Riemann geometry. These rules are valid with respect to all observers in the underlying proper physical Euclidean 3-space E'^3, when a general N parallel 'three-dimensional' absolute intrinsic metric spaces (or N parallel 'three-dimensional' absolute intrinsic metric spaces) are superposed.

1.6.3 Parallelism of all absolute intrinsic metric spaces in the universe

We have considered so far in this sub-section the highly ordered situation of the co-existence of parallel absolute intrinsic metric spaces. As defined previously, a pair of 'three-dimensional' absolute intrinsic metric spaces $\phi\hat{M}^3_{(1)}$ with absolute intrinsic 'dimensions' $\hat{u}^1, \hat{u}^2, \hat{u}^3$ and $\phi\hat{M}^3_{(2)}$ with absolute intrinsic 'dimensions' $\hat{v}^1, \hat{v}^2, \hat{v}^3$, are parallel if each curved 'dimension' $\hat{v}^i$ of $\phi\hat{M}^3_{(2)}$ and the corresponding curved 'dimension' $\hat{u}^i$ of $\phi\hat{M}^3_{(1)}$ lie on the same vertical $u'^i\hat{u}^0$-plane. In this situation, the curved absolute intrinsic 'dimensions' $\hat{v}^1, \hat{v}^2, \hat{v}^3$ of $\phi\hat{M}^3_{(2)}$ and $\hat{u}^1, \hat{u}^2, \hat{u}^3$ of $\phi\hat{M}^3_{(1)}$ are parameterized in the same set of intrinsic dimensions u'^1, u'^2, u'^3 of the underlying global proper intrinsic space $\phi E'^3$ prior to their superposition as,

$$u'^1 = g^1(\hat{v}^1);\ u'^2 = g^2(\hat{v}^2);\ u'^3 = g^3(\hat{v}^3) \qquad (87a)$$

and

$$u'^1 = f^1(\hat{u}^1);\ u'^2 = f^2(\hat{u}^2);\ u'^3 = f^3(\hat{u}^3) \qquad (87b)$$

When $\phi\hat{M}^3_{(2)}$ and $\phi\hat{M}^3_{(1)}$ coexist, the curved absolute intrinsic 'dimension' $\hat{v}^q$ lies above the curved absolute intrinsic 'dimension' $\hat{u}^q$ on the vertical $u'^q\hat{u}^0-$plane, for $q = 1, 2$ and 3, as illustrated in Fig. 4 for the pair of parallel 'two-dimensional' absolute intrinsic metric spaces in Figs. 3a and 10b prior to their superposition.

Now let us consider the chaotic situation of the coexistence of non-parallel absolute intrinsic metric spaces. In this situation, some or all of the curved absolute intrinsic 'dimensions' $\hat{v}^q$ of $\phi\hat{M}^3_{(2)}$ do not lie above the corresponding curved absolute intrinsic 'dimensions' $\hat{u}^q$ of $\phi\hat{M}^3_{(1)}$ on the vertical $u'^q\hat{u}^0-$plane. In this situation, while the absolute intrinsic 'dimensions' $\hat{u}^1, \hat{u}^2$ and $\hat{u}^3$ of $\phi\hat{M}^3_{(1)}$ are parameterized in terms of a proper intrinsic coordinate set (or frame) (ξ'^1, ξ'^2, ξ'^3) in the underlying global proper intrinsic space $\phi E'^3$, the curved absolute intrinsic 'dimensions' $\hat{v}^1, \hat{v}^2$ and $\hat{v}^3$ of $\phi\hat{M}^3_{(2)}$ are parameterized in terms of a different proper intrinsic coordinate set (or frame) $(\eta'^1, \eta'^2, \eta'^3)$ in the underlying global proper intrinsic space $\phi E'^3$ in general prior to the superposition of $\phi\hat{M}^3_{(1)}$ and $\phi\hat{M}^3_{(2)}$. In other words, the following transformations of intrinsic coordinates obtain in general prior to the superposition of $\phi\hat{M}^3_{(1)}$ and $\phi\hat{M}^3_{(2)}$:

$$\eta'^1 = \mathrm{f}^1(\hat{v}^1);\ \eta'^2 = \mathrm{f}^2(\hat{v}^2);\ \eta'^3 = \mathrm{f}^3(\hat{v}^3) \qquad (88a)$$

and

$$\xi'^1 = \mathrm{g}^1(\hat{u}^1);\ \xi'^2 = \mathrm{g}^2(\hat{u}^2);\ \xi'^3 = \mathrm{g}^3(\hat{u}^3) \qquad (88b)$$

When $\phi\hat{M}^3_{(2)}$ and $\phi\hat{M}^3_{(1)}$ coexist, or are superposed, they are both underlied by the global flat proper intrinsic space $\phi E'^3$. However the absolute intrinsic 'dimensions' $\hat{v}^1, \hat{v}^2$ and $\hat{v}^3$ of $\phi\hat{M}^3_{(2)}$ are curved relative to the proper intrinsic coordinates η'^1, η'^2 and η'^3 respectively of one frame in $\phi E'^3$, while the absolute intrinsic 'dimension' $\hat{u}^1, \hat{u}^2$ and $\hat{u}^3$ of $\phi\hat{M}^3_{(1)}$ are curved relative to the proper intrinsic coordinates ξ'^1, ξ'^2 and ξ'^3 of another frame in $\phi E'^3$.

Having described the superposition of parallel absolute intrinsic metric spaces and the superposition of non-parallel absolute intrinsic metric spaces above, it shall now be shown that non-parallel absolute intrinsic metric spaces do not exist in nature. As deduced from the consistent arguments leading to the isolation of absolute intrinsic metric spaces in section 4 of part one of this paper [1], all local absolute intrinsic coordinate sets (or local absolute intrinsic frames) $(\hat{u}^1, \hat{u}^2, \hat{u}^3)$, $(\hat{v}^1, \hat{v}^2, \hat{v}^3)$, $(\hat{w}'^1, \hat{w}^2, \hat{w}^3)$, etc, at a point in an absolute intrinsic metric space $\phi\hat{M}^3$ are equivalent to a singular local absolute intrinsic coordinate set (or local absolute intrinsic frame) $(\hat{u}^1, \hat{u}^2, \hat{u}^3)$ with respect to observers in the proper physical Euclidean 3-space E'^3 underlying $\phi\hat{M}^3$. All the projective local proper intrinsic coordinate sets (or local intrinsic frames) (u'^1, u'^2, u'^3), (v'^1, v'^2, v'^3), (w'^1, w'^2, w'^3), etc, at the corresponding point in the underlying projective proper intrinsic space $\phi E'^3$ are equivalent to a singular local intrinsic coordinate set (or local intrinsic frame) (u'^1, u'^2, u'^3), with respect to all observers in E'^3, as a consequence.

It follows from the foregoing paragraph that the two frames (ξ'^1, ξ'^2, ξ'^3) and $(\eta'^1, \eta'^2, \eta'^3)$ in the flat proper intrinsic space $\phi E'^3$ that lie underneath two co-existing non-parallel absolute intrinsic metric spaces $\phi\hat{M}^3_{(1)}$ and $\phi\hat{M}^3_{(2)}$ respectively in our discussion above, are equivalent to the singular intrinsic coordinate set (or frame) (u'^1, u'^2, u'^3) in $\phi E'^3$, (where u'^1, u'^2 and u'^3 are actually the intrinsic dimensions of $\phi E'^3$). It then follows that the curved absolute intrinsic 'dimensions' $\hat{u}^2, \hat{u}^2, \hat{u}^3$ and $\hat{v}^2, \hat{v}^2, \hat{v}^3$ of the co-existing non-parallel absolute intrinsic metric spaces $\phi\hat{M}^3_{(1)}$ and $\phi\hat{M}^3_{(2)}$ respectively, are actually curved relative to the singular proper intrinsic coordinate set (or frame) (u'^1, u'^2, u'^3) of the underlying proper intrinsic metric space $\phi E'^3$, which makes them parallel. The local coordinate set $(\eta'^1, \eta'^2, \eta'^3)$ in (103a) and local coordinate set ξ'^1, ξ'^2, ξ'^3 in system (103b) must be replaced by the same local coordinate set (u'^1, u'^2, u'^3). The conclusion that follows from this is that all absolute intrinsic metric spaces in the universe are parallel, all lying above the singular coordinate set (or frame) (u'^1, u'^2, u'^3) of the universal isotropic proper intrinsic space $\phi E'^3$ that lies underneath all absolute intrinsic metric spaces.

The programme of this sub-section, which is to formulate absolute intrinsic Riemann geometry when two or a larger number of absolute intrinsic metric spaces co-exist, (both the first and second stages of the formulation), has been accomplished. We shall proceed to the next and concluding section of this paper to discuss an interesting and dramatic aspect of absolute intrinsic Riemann geometry.

2 Perfect isotropy and implied 'one-dimensionality' of each of curved absolute intrinsic metric space and its underlying proper intrinsic metric space with respect to observers in the proper physical Euclidean 3-space

We shall for the purpose of discussion in this section change from the notation $\hat{u}^1, \hat{u}^2, \hat{u}^3$ for the absolute intrinsic 'dimensions' of an absolute intrinsic metric space $\phi\hat{M}^3$ and the straight line proper intrinsic dimensions u'^1, u'^2, u'^3 of the global flat proper intrinsic space $\phi E'^3$ that lies underneath all absolute intrinsic metric spaces, adopted for convenience to this point in this paper, to the natural notations $\phi\hat{x}^1, \phi\hat{x}^2, \phi\hat{x}^3$ for 'dimensions' of absolute intrinsic metric space $\phi\hat{M}^3$ and $\phi x'^1, \phi x'^2, \phi x'^3$ for the intrinsic dimensions of the underlying flat proper intrinsic space $\phi E'^3$.

As deduced from the fact that both the absolute intrinsic line element and absolute intrinsic metric tensor on an absolute intrinsic metric space $\phi\hat{M}^3$ are invariant with change of local absolute intrinsic coordinate set in section 4 of part one of this paper [1], different local absolute intrinsic coordinate sets ($\phi\hat{u}^1$, $\phi\hat{u}^2, \phi\hat{u}^3$), ($\phi\hat{v}^1, \phi\hat{v}^2, \phi\hat{v}^3$), ($\phi\hat{w}^1, \phi\hat{w}^2, \phi\hat{w}^3$), etc, that are arbitrarily orientated relative to one another at a point P on the curved absolute intrinsic metric space $\phi\hat{M}^3$ with respect to a Riemannian observer located at the point P on $\phi\hat{M}^3$, are identical to a singular local absolute intrinsic coordinate set ($\phi\hat{\xi}^1, \phi\hat{\xi}^2, \phi\hat{\xi}^3$) at the point P on $\phi\hat{M}^3$ with respect to Euclidean observers in the proper physical Euclidean 3-space E'^3 underlying $\phi\hat{M}^3$.

An implication of the foregoing paragraph is that the local absolute intrinsic coordinates $\phi\hat{u}^1$, $\phi\hat{v}^1$ and $\phi\hat{w}^1$, etc, of the different local absolute intrinsic frames, which are orientated along different directions about point P on $\phi\hat{M}^3$ with respect to a Riemannian observer at this point, are all identical to a singular local absolute intrinsic coordinate $\phi\hat{\eta}^1$ at the point P on $\phi\hat{M}^3$ with respect to observers in E'^3 underlying $\phi\hat{M}^3$. It thus follows that the different absolute intrinsic angles $\phi\hat{\alpha}, \phi\hat{\beta}, \phi\hat{\gamma}$, etc, at which the local absolute intrinsic coordinates $\phi\hat{u}^1, \phi\hat{v}^1, \phi\hat{w}^1$, etc, of different local absolute intrinsic coordinate sets are inclined relative to each other at the point P on $\phi\hat{M}^3$ with respect to a Riemannian observer at this point, all vanish, that is, $\phi\hat{\alpha} = \phi\hat{\beta} = \phi\hat{\gamma} = 0$, with respect to observers in the proper physical Euclidean 3-space E'^3 underlying $\phi\hat{M}^3$. The different absolute intrinsic coordinates $\phi\hat{u}^1, \phi\hat{v}^1, \phi\hat{w}^1$, etc, are all aligned along a singular direction, thereby constituting a singular local absolute intrinsic coordinate $\phi\hat{\xi}^1$ at the point P on $\phi\hat{M}^3$ with respect to observers in the proper physical Euclidean 3-space E'^3 consequently.

The different absolute intrinsic angles $\phi\hat{\delta}, \phi\hat{\theta}, \phi\hat{\varphi}$, etc, at which the local absolute intrinsic coordinates $\phi\hat{u}^2, \phi\hat{v}^2, \phi\hat{w}^2$, etc, of different local absolute intrinsic coordinate sets (or frames) are inclined relative to one another at point P on $\phi\hat{M}^3$ with respect to a Riemannian observer at this point all vanish, that is, $\phi\hat{\delta} = \phi\hat{\theta} = \phi\hat{\varphi} = 0$, with respect to Euclidean observers in E'^3 underlying $\phi\hat{M}^3$. Consequently the different local absolute intrinsic coordinates $\phi\hat{u}^2, \phi\hat{v}^2, \phi\hat{w}^2$, etc, are all aligned along a singular direction thereby constituting a singular local absolute intrinsic coordinate $\phi\hat{\xi}^2$ at the point P on $\phi\hat{M}^3$ with respect to Euclidean observers in E'^3. Likewise the different local absolute intrinsic coordinates $\phi\hat{u}^3, \phi\hat{v}^3, \phi\hat{w}^3$, etc, are all aligned along a singular direction thereby constituting a singular local absolute intrinsic coordinate $\phi\hat{\xi}^3$ at point P on $\phi\hat{M}^3$ with respect to all Euclidean observers in E'^3.

We find from the foregoing two paragraphs that two directions within an approximately flat infinitesimal local neighborhood about a point P of the curved absolute intrinsic metric space $\phi\hat{M}^3$, which are distinct directions separated by absolute intrinsic Euler angles $\phi\hat{\alpha}, \phi\hat{\beta}$ and $\phi\hat{\gamma}$ with respect to a Riemannian observer located at point P on $\phi\hat{M}^3$, are the same direction with respect to observers in the proper physical Euclidean 3-space E'^3 underlying $\phi\hat{M}^3$. This is so since any magnitudes of the absolute intrinsic angles $\phi\hat{\alpha}, \phi\hat{\beta}$ and $\phi\hat{\gamma}$ in $\phi\hat{M}^3$ are equivalent to zero magnitudes of the corresponding angles α', β' and γ' in the proper physical Euclidean 3-space E'^3.

It then follows (from the preceding paragraph) that a singular local absolute intrinsic frame ($\phi\hat{\xi}^1$, $\phi\hat{\xi}^2$, $\phi\hat{\xi}^3$) at point P on $\phi\hat{M}^3$, with mutually perpendicular local absolute intrinsic coordinates $\phi\hat{\xi}^1$, $\phi\hat{\xi}^2$ and $\phi\hat{\xi}^3$, to which different local absolute intrinsic coordinate sets ($\phi\hat{u}^1, \phi\hat{u}^2, \phi\hat{u}^3$), ($\phi\hat{v}^1$, $\phi\hat{v}^2$, $\phi\hat{v}^3$), ($\phi\hat{w}^1$, $\phi\hat{w}^2$, $\phi\hat{w}^3$), etc, at point P on $\phi\hat{M}^3$ are identical with respect to Euclidean observers in E'^3, as known until now in this paper, is impossible. This is so because the absolute intrinsic angle $\phi\hat{\varphi} = \frac{\phi\pi}{2}$ separating the local absolute intrinsic 'dimensions' $\phi\hat{\xi}^1$ and $\phi\hat{\xi}^2$ and the absolute intrinsic angle $\phi\hat{\theta} = \frac{\phi\pi}{2}$ separating the local absolute intrinsic 'dimensions' $\phi\hat{\xi}^2$ and $\phi\hat{\xi}^3$ with respect to the Riemannian observer at point P on $\phi\hat{M}^3$, both vanish with respect to Euclidean observers in E'^3, thereby causing $\phi\hat{\xi}^2$, $\phi\hat{\xi}^2$ and $\phi\hat{\xi}^3$ to be aligned along a singular direction. They thereby constitute a singular local absolute intrinsic coordinate $\phi\hat{\xi}_P$ at point P on $\phi\hat{M}^3$ with respect to all observers in E'^3.

The result derived at point P on the absolute intrinsic metric space $\phi\hat{M}^3$ in the foregoing paragraph obtains at every other point on $\phi\hat{M}^3$. In other words, only singular local absolute intrinsic coordinates $\phi\hat{\xi}_Q$, $\phi\hat{\xi}_R$, $\phi\hat{\xi}_S$, $\phi\hat{\xi}_T$, etc, exist at points Q, R, S, T, etc, on $\phi\hat{M}^3$ with respect to Euclidean observers in the proper physical Euclidean 3-space E'^3 underlying $\phi\hat{M}^3$. When the singular indefinitely short local absolute intrinsic coordinates at every point on $\phi\hat{M}^3$ are joined together, one obtains a continuous curved 'one-dimensional' absolute intrinsic metric space (a 'one-dimensional' absolute intrinsic Riemannian metric space) be denoted by $\phi\hat{\rho}$, with respect to Euclidean observers in E'^3.

We have arrived at an important conclusion in the foregoing paragraph that the curved absolute intrinsic metric spaces (or absolute intrinsic Riemannian metric spaces), which have been considered to be 'two-dimensional' $\phi\hat{M}^2$ or 'three-dimensional' $\phi\hat{M}^3$ with respect to Euclidean observers in the underlying proper physical Euclidean space E'^2 or E'^3 so far in this paper, are actually 'one-dimensional' curved absolute intrinsic metric spaces (or 'one-dimensional' curved absolute intrinsic Riemannian metric spaces) denoted by $\phi\hat{\rho}$, with respect to all 3-observers in the underlying proper physical Euclidean 3-space E'^3.

The 'one-dimensional' absolute intrinsic metric space $\phi\hat{\rho}$ curving towards the absolute intrinsic time 'dimension' $\phi\hat{c}\phi\hat{t}$ along the vertical with respect to Euclidean observers in E'^3, will naturally project a one-dimensional straight line proper intrinsic space, to be denoted by $\phi\rho'$ underneath the proper physical Euclidean 3-space E'^3 with respect to observers in E'^3. However we shall for completeness show below that a three-dimensional flat proper intrinsic space $\phi E'^3$ considered to be projected underneath the proper physical Euclidean 3-space E'^3 by curved 'three-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$ previously in this paper, naturally contracts to a one-dimensional proper intrinsic metric space $\phi\rho'$ with respect to observers in E'^3.

Now any magnitudes of proper intrinsic Euler angles $\phi\alpha'$, $\phi\beta'$ and $\phi\gamma'$ in the flat proper intrinsic space $\phi E'^3$ are equivalent to zero magnitude of proper physical Euler angles α', β' and γ' respectively in the proper physical Euclidean 3-space E'^3 with respect to observers in E'^3 overlying $\phi E'^3$; knowing that $\phi\alpha' \equiv 0 \times \alpha'$, $\phi\beta' \equiv 0 \times \beta'$, and $\phi\gamma' \equiv 0 \times \gamma'$. Consequently any two distinct directions, which are separated by non-zero intrinsic angles $\phi\alpha', \phi\beta'$ and $\phi\gamma'$ in the flat three-dimensional proper intrinsic space $\phi E'^3$ with respect to intrinsic observers in $\phi E'^3$, are the same direction with respect to observers in the proper physical Euclidean 3-space E'^3

A consequence of the foregoing paragraph is that mutually perpendicular proper intrinsic dimensions $\phi\eta'^1, \phi\eta'^2$ and $\phi\eta'^3$ of $\phi E'^3$ with respect to intrinsic 3-observers in $\phi E'^3$ are impossible with respect to 3-observers in E'^3. This is so because the intrinsic angle $\phi\varphi' = \frac{\phi\pi}{2}$ between intrinsic dimensions $\phi\eta'^1$ and $\phi\eta'^2$ and $\phi\theta' = \frac{\phi\pi}{2}$ between intrinsic dimensions $\phi\eta'^2$ and $\phi\eta'^3$ with respect to intrinsic observers in $\phi E'^3$ both vanish with respect to observers in E'^3. The three intrinsic dimensions $\phi\eta'^1, \phi\eta'^2$ and $\phi\eta'^3$ of $\phi E'^3$ are consequently aligned along a singular direction, thereby constituting a singular intrinsic space denoted by $\phi\rho'$ above, which underlies the proper physical Euclidean 3-space E'^3 with respect to all observers in E'^3.

The one-dimensional proper intrinsic space $\phi\rho'$ has no unique orientation in the flat three-dimensional proper intrinsic space $\phi E'^3$ that contracts to it. Consequently it has no unique orientation in the proper physical Euclidean 3-space E'^3. Thus $\phi\rho'$ is an isotropic intrinsic space (or intrinsic dimension) in E'^3. It can be considered to lie along any direction in E'^3 by observers in E'^3.

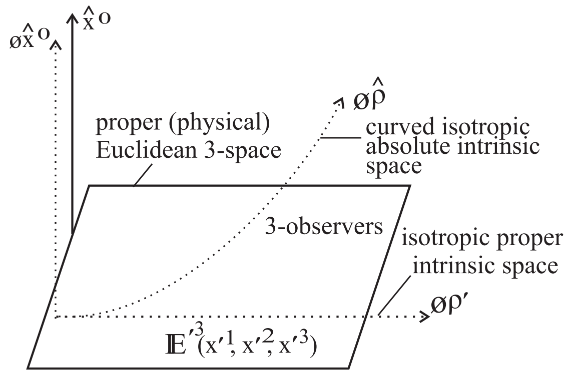

Fig. 6: The 'three-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$, which is curved onto the absolute intrinsic time 'dimension' $\phi\hat{x}^0$ along the vertical and its underlying projective flat three-dimensional proper intrinsic metric space $\phi E'^3$ (in Fig. 1 of part one of this paper [1]), are naturally contracted into 'one-dimensional' absolute intrinsic metric space $\phi\hat{\rho}$ and one-dimensional proper intrinsic metric space $\phi\rho'$ respectively, where $\phi\hat{\rho}$ is curved onto straight line absolute intrinsic time 'dimension' along the vertical and $\phi\rho'$ is a straight line isotropic one-dimensional proper intrinsic space underlying the proper physical Euclidean 3-space E'^3 along the horizontal with respect to 3-observers in E'^3.

The curved 'one-dimensional' absolute intrinsic metric space $\phi\hat{\rho}$ to which the curved 'three-dimensional' absolute intrinsic metric space $\phi\hat{M}^3$ is naturally contracted with respect to observers in the physical proper Euclidean 3-space E'^3, is curved onto the straight line absolute intrinsic time 'dimension' $\phi\hat{x}^0 = \phi\hat{c}\phi\hat{t}$ along the vertical and projects a straight line isotropic proper intrinsic metric space $\phi\rho'$ underneath the proper physical Euclidean 3-space E'^3 along the horizontal with respect to observers in E'^3, thereby yielding Fig. 6.

Thus the 'three-dimensional' absolute intrinsic metric spaces $\phi\hat{M}^3$, (which are 'three-dimensional' absolute intrinsic metric spaces), underlied by flat three-dimensional proper intrinsic metric space $\phi E'^3$, which we have carried along from the beginning of this paper to this point, have now been found to be naturally contracted to curved 'one-dimensional' absolute intrinsic metric spaces, (which are 'one-dimensional' absolute intrinsic Riemannian metric spaces) $\phi\hat{M}^1$, underneath which lies its projective one-dimensional isotropic proper intrinsic metric $\phi\rho'$, with respect to observers in E'^3.

The proper physical Euclidean 3-space E'^3 that has been known to be the outward manifestation of the 3-dimensional proper intrinsic metric space $\phi E'^3$ until now in this paper is now the outward manifestation of the one-dimensional isotropic proper intrinsic space $\phi\rho'$ in Fig. 6. It may be recalled that this fact has been stated as *ansatz* in sub-section 4.4 of [3], prior to formal validation of the existence of the proper intrinsic space $\phi\rho'$ underlying the proper physical Euclidean 3-space E'^3 in nature in section 1 of [4].

The absolute intrinsic metric tensors $\phi\hat{g}_{ik}$ of absolute intrinsic Riemann geometry on curved 'three-dimensional' absolute intrinsic metric spaces $\phi\hat{M}^3$, which are 3×3 diagonal matrices in section 1, are actually 1×1 matrices or numbers $\phi\hat{g}_{11}$ or $\phi\hat{\hat{g}}_{11}$ on curved 'one-dimensional' absolute intrinsic metric spaces $\phi\hat{\rho}$. Likewise for the absolute intrinsic Ricci tensors. The absolute intrinsic Gaussian line element written in terms of elementary intervals of three absolute intrinsic metric 'dimensions' $\hat{u}^1, \hat{u}^2$ and $\hat{u}^3$, (which are the same as $\phi\hat{x}^1, \phi\hat{x}^2$ and $\phi\hat{x}^3$), of $\phi\hat{M}^3$ as Eq. (47) or (48), is actually the following absolute intrinsic Gaussian line element in terms of the interval of the one-dimensional absolute intrinsic space $\phi\hat{\rho}$:

$$(d\phi\hat{s})^2 = (d\phi\hat{x}^0)^2 - \phi\hat{g}_{11}(d\phi\hat{\rho})^2 = (d\phi\hat{x}^0)^2 - \frac{(d\phi\hat{\rho})^2}{1-(\phi\hat{k})^2} \tag{89}$$

It must be remembered that $\phi\hat{\rho}$ has been formed by bundling together into 'one-dimensional' absolute intrinsic space of the curved absolute intrinsic space 'dimensions' $\phi\hat{x}^1, \phi\hat{x}^2$ and $\phi\hat{x}^3$ of $\phi\hat{M}^3$ with respect to observers in the proper physical Euclidean 3-space E'^3. We have, in effect, simply replaced $(d\phi\hat{x}^1)^2 + (d\phi\hat{x}^2)^2 + (d\phi\hat{x}^3)^2$ in Eq. (48) by $(d\phi\hat{\rho})^2$ in Eq. (93).

All absolute intrinsic Riemannian metric spaces in the universe are curved 'one-dimensional' absolute intrinsic metric spaces, $\phi\hat{\rho}, \phi\hat{\rho}', \phi\hat{\rho}''$, etc, which are all curved relative to the singular universal isotropic proper intrinsic metric space $\phi\rho'$, (with no unique orientation in the universal proper physical Euclidean 3-space E'^3) with respect to observers in E'^3. Hence they are all parallel absolute intrinsic metric spaces with respect to observers in E'^3.

Illustrated in Fig. 7 is a situation where two absolute intrinsic metric spaces $\phi\hat{\rho}$ and $\phi\hat{\rho}'$ co-exist (or are superposed), such that $\phi\hat{\rho}$ is curved relative to curved $\phi\hat{\rho}'$ and $\phi\hat{\rho}'$ is curved relative to the proper intrinsic metric space $\phi\rho'$ along the horizontal. For the purpose of writing absolute intrinsic metric tensor and absolute intrinsic Ricci tensor on the upper curved 'one-dimensional' absolute intrinsic metric space $\phi\hat{\rho}$ with respect to observers in E'^3, the resultant absolute intrinsic angle $\phi\hat{\hat{\psi}}$ of inclination of $\phi\hat{\rho}$ relative to $\phi\rho'$ at point $\phi\hat{\rho}_{(1)}$ along $\phi\hat{\rho}$, which corresponds to point $\phi\hat{\rho}'_{(1)}$ along $\phi\hat{\rho}'$ and point $\phi\rho'_{(1)}$ along $\phi\rho'$, is given in terms of the absolute intrinsic angles $\phi\hat{\psi}_2(\phi\hat{\rho}_{(1)})$ and $\phi\hat{\psi}_1(\phi\hat{\rho}'_{(1)})$ as follows, as derived in sub-sub-section 1.6.1, (see Eqs. (58a) and (58b)):

$$\begin{aligned}\sin^2\phi\hat{\hat{\psi}} &= \sin^2\Big(\phi\hat{\psi}_1(\phi\hat{\rho}'_{(1)}) + \phi\hat{\psi}_2(\phi\hat{\rho}_{(1)})\Big)\\ &= \sin^2\phi\hat{\psi}_1(\phi\hat{\rho}'_{(1)}) + \sin^2\phi\hat{\psi}_2(\phi\hat{\rho}_{(1)})\end{aligned} \tag{90}$$

Hence the resultant absolute intrinsic curvature parameter $\phi\hat{\hat{k}}$ of the upper curved absolute intrinsic space $\phi\hat{\rho}$ at point $\phi\hat{\rho}_{(1)}$ along $\phi\hat{\rho}$ in Fig. 7 to appear in the component of the resultant absolute intrinsic metric tensor $\phi\hat{\hat{g}}_{11}$ at this point is

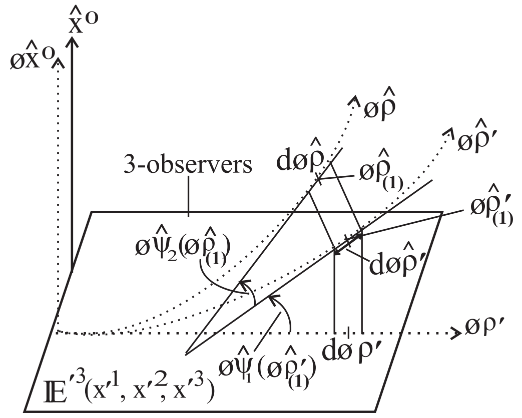

Fig. 7: Co-existing pair of 'one-dimensional' absolute intrinsic metric spaces with respect to 3-observers in the underlying proper physical Euclidean 3-space.

given in terms of the absolute intrinsic curvature parameters $\phi\hat{k}_2(\phi\hat{\rho}_{(1)})$ and $\phi\hat{k}_1(\phi\hat{\rho}'_{(1)})$ of the curved absolute intrinsic metric spaces $\phi\hat{\rho}$ and $\phi\hat{\rho}'$ respectively relative to $\phi\rho'$ determined prior to their superposition as follows:

$$(\phi\hat{\hat{k}})^2 = \phi\hat{k}_1(\phi\hat{\rho}'_{(1)})^2 + \phi\hat{k}_2(\phi\hat{\rho}_{(1)})^2 \tag{91}$$

The component of the resultant absolute intrinsic metric tensor $\phi\hat{\hat{g}}_{11}$ at point $\phi\hat{\rho}_{(1)}$ on the upper curved absolute intrinsic space $\phi\hat{\rho}$ is then given in terms of $(\phi\hat{\hat{k}})^2$ as follows:

$$\begin{aligned}\phi\hat{\hat{g}}_{11} &= \Big(1-(\phi\hat{\hat{k}})^2\Big)^{-1}\\ &= \Big(1-\phi\hat{k}_1(\phi\hat{\rho}'_{(1)})^2 - \phi\hat{k}_2(\phi\hat{\rho}_{(1)})^2\Big)^{-1}\end{aligned} \tag{92}$$

And the resultant absolute intrinsic line element must be written by simply replacing $(\phi\hat{k})^2$ by $(\phi\hat{\hat{k}})^2$ in Eq. (89).

Finally, while the curved 'one-dimensional' absolute intrinsic metric space, (or absolute intrinsic metric space), $\phi\hat{\rho}$ is absolute, hence with hat label, the underlying proper intrinsic metric $\phi\rho'$, (without hat label), is relative. (This is similar to the fact that the absolute time parameter $\hat{t}$ is absolute, while the proper time t' (or τ) that evolves from it is relative). Thus while $\phi\hat{\rho}$ and $\phi\hat{\rho}'$ must not be counted as extra intrinsic dimensions in physics, (being mere absolute intrinsic parameters), in Fig. 7, their underlying isotropic proper intrinsic metric space $\phi\rho'$ is an extra intrinsic, (that is, a non-observable and non-detectable) dimension in physics, where its intrinsic (or non-detectable nature) is accounted for by the symbol ϕ attached to it.

There are four dimensions already in Fig. 7 namely, the dimensions x'^1, x'^2, x'^3 of the proper physical Euclidean 3-space E'^3 and the proper intrinsic space (or intrinsic dimension) $\phi\rho'$. The isotropic intrinsic space (or dimension) $\phi\rho'$ is a straight line, just as the proper physical 3-space E'^3 overlying it is flat or Euclidean. The isotropic proper intrinsic space

$\phi\rho'$ has no unique orientation or basis in the physical Euclidean 3-space E'^3.

This first part of this paper shall be ended at this point, while the development of absolute intrinsic Riemann geometry shall be extended to curved 'two-dimensional' absolute intrinsic metric spacetime $(\phi\hat{\rho}, \phi\hat{c}\phi\hat{t})$, which is underlied by its projective proper intrinsic metric spacetime $(\phi\rho', \phi c\phi t')$ and flat four-dimensional proper physical spacetime (E'^3, ct') in the second part.

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