A complete graph model of the Schwarzschild black hole in $\mathbb{R}^3$

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Where $\hbar = c = G = 1$, the following components will be used to model a Schwarzschild black hole of rest mass-energy $E$ in $\mathbb{R}^3$:

1. A 2-sphere (event horizon) $S_0$ at coordinate distance $R_0 = 2E$, upon which lies $N_0 = E$ uniformly distributed vertices $V_0$.
2. A complete graph’s worth of edges $E_0$ generated by $V_0$.
3. An exterior volume at $R > R_0$, upon which lies a countable number of vertices $V_{\text{ext}}$.
4. A countable number of (non-complete) graph edges $E_{\text{ext}}$ generated by the Delaunay tetrahedralization of $V_0$ and $V_{\text{ext}}$.

The following presumptions are made:

1. Neither the event horizon nor the black hole centre at $R = 0$ are singular in any way.
2. The complete graph edges $E_0$ define a universal edge coordinate length of

$$L = \frac{1}{\sqrt{1 - R_0/R}},$$

where $R$ is the coordinate distance between an edge’s midpoint and the black hole centre. Accordingly, edge proper length is $L^2$ (e.g., light travels across an edge at coordinate speed $1/L$).
3. The complete graph edges $E_0$ define a universal minimum edge coordinate length of $L = 1$ (e.g., the Planck length).

The following steps are used to construct the model’s components:

1. With regard to the 2-sphere $S_0$, numerically solve for the coordinate radial distance $R_1 > R_0$ of a second 2-sphere $S_1$. Using the formula for the height of a regular tetrahedron

$$H_{\text{tet}} = L\sqrt{2/3}$$

as a guide:

$$R_1 = R_0 + H_0,$$

$$\frac{H_0}{\sqrt{2/3}} \approx \frac{1}{\sqrt{1 - \frac{R_0}{R_0 + H_0/2}}}.$$  

2. Calculate the number of vertices $N_1$ that lie upon $S_1$. Using the formulas for the area of a regular triangle

$$A_{\text{tri}} = L^2(1/4)\sqrt{3}.$$
and the Euler characteristic of a closed convex polyhedron

\[ N_1 + F_1 - E_1 = 2, \quad F_1 = 2N_1 - 4 \]  

(6)
as a guide:

\[ L_1 = \frac{1}{\sqrt{1 - \frac{R_0}{R_1}}}, \]  

(7)\[ F_1 = \frac{4\pi R_1^2}{L_1^2}, \]  

(8)\[ N_1 = \frac{4 + F_1}{2}. \]  

(9)

3. Numerically solve for \( H_1 \):

\[ \frac{H_1}{\sqrt{2/3}} \approx \frac{1}{\sqrt{1 - \frac{R_0}{R_1 + H_1/2}}}. \]  

(10)

4. Repeat steps 2 and 3 for each subsequent 2-sphere \( S_{\geq 2} \):

\[ R_{\geq 2} = R_{\geq 1} + H_{\geq 1}, \]  

(11)\[ N_{\geq 2} = 2 + \frac{8\pi R_{\geq 2}(R_{\geq 2} - R_0)}{\sqrt{3}}, \]  

(12)\[ H_{\geq 2} \approx \frac{1}{\sqrt{\frac{3}{2} - \frac{3R_0}{2R_{\geq 2} + H_{\geq 2}}}}. \]  

(13)

5. Generate the vertices \( V_0 \) that lie upon \( S_0 \). Use Coulomb repulsion on \( S_0 \) to make the vertex distribution roughly uniform.

6. Obtain the complete graph edges \( E_0 \) generated by \( V_0 \).

7. Generate the vertices \( V_{\geq 1} \) that lie along each 2-sphere \( S_{\geq 1} \). Use Coulomb repulsion on each 2-sphere to make its vertex distribution roughly uniform, if desired.

8. Obtain the (non-complete) graph edges generated by the Delaunay tetrahedralization of all vertices \( V_{\geq 0} \).

Depending on how well the vertices \( V_{\text{ext}} \) are uniformly distributed along their respective shells, one will have to multiply \( H_{\geq 0} \) and \( N_{\geq 1} \) by some small constant values (e.g., roughly on the order of 1) in order to meet the edge coordinate length requirement given in Eq. 1 with accuracy.

See Ref. [1] for a public domain C++ code that generates this model’s vertices and edges. Edge analysis code is included. The default configuration produces an edge coordinate length accuracy of \( \sim 0.99 \). See Fig. 1 for an example manifold. As with all discretization models [2], edge coordinate length accuracy is based on an average.

Unlike most other discretization models, this model does not allow one to arbitrarily choose the scale of the tetrahedra (e.g., \( dx^{\mu} dx^\nu \equiv 1 \) here). As such, the manifold is geodesically complete by definition, not by choice.

Thank you to P. Gibbs for his work on complete graphs [3].

References


Figure 1: One half of the edges for 2-spheres $S_0$ through $S_{10}$, where $N_0 = 10$. The manifold is geodesically complete, since it does not contain any infinitely small or large edges, or “dead end” paths. Edge coordinate length decreases as $r$ increases. The figure was rendered using OpenGL / Microsoft Visual C++ Express 2010, and was post-processed using Rick Brewster’s Paint.NET. Space kitten is generally nonplussed by $\pi$. 