A GENERALIZATION OF THE INEQUALITY OF HÖLDER

Florentin Smarandache  
University of New Mexico  
200 College Road  
Gallup, NM 87301, USA  
E-mail: smarand@unm.edu

One generalizes the inequality of Hölder thanks to a reasoning by recurrence. As particular cases, one obtains a generalization of the inequality of Cauchy-Buniakowski-Schwartz, and some interesting applications.

**Theorem:** If \( a_{i}^{(k)} \in \mathbb{R}_{+} \) and \( p_k \in ]1, +\infty[ \), \( i \in \{1,2,\ldots,n\} \), \( k \in \{1,2,\ldots,m\} \), such that: \( \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1 \), then:

\[
\sum_{i=1}^{n} \prod_{k=1}^{m} a_{i}^{(k)} \leq \prod_{k=1}^{m} \left( \sum_{i=1}^{n} a_{i}^{(k)} \right)^{\frac{1}{p_k}} \\
\text{with } m \geq 2.
\]

**Proof:**

For \( m = 2 \) one obtains exactly the inequality of Hölder, which is true. One supposes that the inequality is true for the values which are strictly smaller than a certain \( m \).

Then:

\[
\sum_{i=1}^{n} \prod_{k=1}^{m} a_{i}^{(k)} = \sum_{i=1}^{n} \left( \prod_{k=1}^{m-2} a_{i}^{k} \cdot (a_{i}^{(m-1)} \cdot a_{i}^{(m)}) \right) \leq \prod_{k=1}^{m-2} \left( \sum_{i=1}^{n} a_{i}^{(k)} \right)^{\frac{1}{p_k}} \cdot \left( \sum_{i=1}^{n} (a_{i}^{(m-1)})^{p} \cdot (a_{i}^{(m)})^{p} \right)^{\frac{1}{p}}
\]

where \( \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{m-2}} + \frac{1}{p} = 1 \) and \( p_h > 1, 1 \leq h \leq m-2, \ p \ > 1 \);

but

\[
\sum_{i=1}^{n} (a_{i}^{(m-1)})^{p} \cdot (a_{i}^{(m)})^{p} \leq \left( \sum_{i=1}^{n} (a_{i}^{(m-1)})^{p} \right)^{\frac{1}{t_1}} \cdot \left( \sum_{i=1}^{n} (a_{i}^{(m)})^{p} \right)^{\frac{1}{t_2}}
\]

where \( \frac{1}{t_1} + \frac{1}{t_2} = 1 \) and \( t_1 > 1, \ t_2 > 1 \).

It results from it:
\[
\sum_{i=1}^{n} (a_i^{(m-1)})^p \cdot (a_i^{(m)})^p \leq \left( \sum_{i=1}^{n} (a_i^{(m-1)})^{p_{t_1}} \right)^{\frac{1}{p_{t_1}}} \cdot \left( \sum_{i=1}^{n} (a_i^{(m)})^{p_{t_2}} \right)^{\frac{1}{p_{t_2}}}
\]

with \( \frac{1}{p_{t_1}} + \frac{1}{p_{t_2}} = \frac{1}{p} \).

Let us note \( pt_1 = p_{m-1} \) and \( pt_2 = p_m \). Then \( \frac{1}{p_{t_1}} + \frac{1}{p_{t_2}} + \ldots + \frac{1}{p_m} = 1 \) is true and one has \( p_j > 1 \) for \( 1 \leq j \leq m \) and it results the inequality from the theorem.

Note: If one poses \( p_j = m \) for \( 1 \leq j \leq m \) and if one raises to the power \( m \) this inequality, one obtains a generalization of the inequality of Cauchy-Buniakovski-Schwartz:

\[
\left( \sum_{i=1}^{n} \prod_{k=1}^{m} a_i^{(k)} \right)^m \leq \prod_{k=1}^{m} \sum_{i=1}^{n} (a_i^{(k)})^m.
\]

Application:
Let \( a_1, a_2, b_1, b_2, c_1, c_2 \) be positive real numbers.
Show that:
\[
(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \leq 8(a_1^6 + a_2^6)(b_1^6 + b_2^6)(c_1^6 + c_2^6)
\]

Solution:
We will use the previous theorem. Let us choose \( p_1 = 2, p_2 = 3, p_3 = 6 \); we will obtain the following:
\[
a_1 b_1 c_1 + a_2 b_2 c_2 \leq (a_1^2 + a_2^2) \frac{1}{2} (b_1^3 + b_2^3) \frac{1}{3} (c_1^6 + c_2^6) \frac{1}{6},
\]
or more:
\[
(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \leq (a_1^2 + a_2^2)^3 (b_1^3 + b_2^3)^2 (c_1^6 + c_2^6),
\]
and knowing that
\[
(b_1^3 + b_2^3)^2 \leq 2(b_1^6 + b_2^6)
\]
and that
\[
(a_1^2 + a_2^2)^3 = a_1^6 + a_2^6 + 3(a_1^4 a_2^2 + a_1^2 a_2^4) \leq 4(a_1^6 + a_2^6)
\]
since
\[
a_1^4 a_2^2 + a_1^2 a_2^4 \leq a_1^6 + a_2^6 \text{ (because: } -(a_2^2 - a_1^2)^3 (a_1^2 + a_2^2) \leq 0 \)
\]

it results the exercise which was proposed.