The fine structure constant and the quark and lepton mixing angles

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Abstract

The fine structure constant and the quark and lepton mixing angles are shown to arise naturally in the course of altering the symmetry of two algebraic identities. Specifically, the symmetry of the identity $x^2 = x^2$ is “broken” by making the substitution $x^n \rightarrow x^n - y^p$ on its left side, and the substitution $x \rightarrow x - z$ on its right side, where p equals the order of the identity; these substitutions convert the above identity into the equation $x^2 - y^2 = (x - z)^2$. These same substitutions are also applied to the only slightly more complicated identity $(x/a)^3 + x^2 = (x/a)^3 + x^2$ to produce this second equation $(x^3 - y^3)/a^3 + x^2 - y^3 = (x - z)^3/a^3 + (x - z)^2$. These two equations are then shown to share a mathematical property relating to dz/dy, where, on the second equation’s left side, this property helps define the special case $(x^3 - y^3)/a^3 + x^2 - y^3 = (10^3 - 0.1^3)/3^3 + 10^2 - 0.1^3 = 137.036$, an equation which incorporates a value close to the experimental fine structure constant inverse. Moreover, on the second equation’s right side, this same special case simultaneously produces values for the sines squared of the mixing angles. Specifically, the sines squared of the leptonic angles φ_{12}, φ_{23}, and φ_{13} appear as 0.3, 0.5, and not larger than roughly $1/30\,000$, respectively; and the sines squared of the quark mixing angles θ_{12} and θ_{13} appear as 0.05, and close to $1/90\,000$, respectively. Despite closely mirroring so many experimental values, including the precisely-known fine structure constant, the above mathematical model requires no free parameters adjusted to fit experiment.

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I. Introduction

The fine structure constant (FSC) and the quark and lepton mixing angles are shown to arise naturally in the course of an investigation of two algebraic identities whose symmetry is altered. Specifically, the symmetry of the identities

$$x^2 = x^2$$

and

$$\left(\frac{x}{a}\right)^3 + x^2 = \left(\frac{x}{a}\right)^3 + x^2$$

will be "broken" by making the substitution

$$x^n \rightarrow x^n - y^p$$

on their left side, and the substitution

$$x \rightarrow x - z$$

on their right side, where p equals the order of each identity. The resultant equations will then be shown to share a mathematical property relating to dz/dy, where for the second equation this property gives rise to values that are close to the experimental FSC and the sines squared of the quark and lepton mixing angles.

II. The fine structure constant

To generate the FSC start with the symmetric identity

$$x^2 = x^2$$

and break its symmetry by making the substitution

$$x^n \to x^n - y^p$$

on its left side, and the substitution

$$x \to x - z$$

on its right side, where $p = 2$. This produces

$$x^2 - y^2 = (x - z)^2 \quad . \tag{1}$$

If one assumes that

$$y = \frac{1}{x} \tag{2a}$$

and

$$x \gg 1 \quad , \tag{2b}$$

then for Eq. (1) the value for dz/dy turns out to be simply

$$\frac{dz}{dy} \approx y^p \quad , \tag{2c}$$

where $p = 2$, the order of Eq. (1) (see Appendix A for derivation). Because Eqs. (2a)–(2c) are all that will be needed to generate the FSC in the next example, they will be termed the *FSC Conditions*.

To produce the FSC, combine x^2 with the next higher-order term to form the expression

$$\left(\frac{x}{a}\right)^3 + x^2 \quad .$$

Set this expression equal to itself to form the new symmetric identity

$$\left(\frac{x}{a}\right)^3 + x^2 = \left(\frac{x}{a}\right)^3 + x^2$$

and apply the earlier substitutions

$$x \to x - z$$

$$x^n \to \; x^n - y^p$$

with p now equaling 3, the order of this new identity. This produces the *FSC Equation*

$$\frac{x^3 - y^3}{a^3} + x^2 - y^3 = \left(\frac{x}{a} - \frac{z}{a}\right)^3 + (x - z)^2 \quad . \tag{3}$$

For Eq. (3), if the first two FSC Conditions—Eqs. (2a) and (2b)—are assumed, then the third FSC Condition—Eq. (2c)—is also met if

$$x \approx \frac{a^3}{3} + 1 \tag{4}$$

(see Appendix B for derivation).

As it turns out, the smallest integers fulfilling these conditions are

$$x = 10$$

and

$$a = 3 \quad .$$

Substituting these integers into the *left* side of the FSC Equation gives

$$\frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = 137.036 \quad , \qquad (5a)$$

where 137.036 differs from the 2006 CODATA value of 137.035 999 679 by about 2.3 parts per billion [1]. In this way the FSC arises naturally from the analysis of the broken symmetry of two simple algebraic identities. (Note that the above FSC approximation is also returned by a "brute-force" computer search for precise approximations of the FSC reciprocal [2], and that the values 10 and 3 also arose independently during a study of how to reproduce economically the quark and lepton masses [3].)

III. The sines squared of the mixing angles

To see how the quark and lepton mixing angles (θ_{12}, θ_{23}, θ_{13}, and φ_{12}, φ_{23}, φ_{13}, respectively) are also contained within the above FSC Equation—or, more precisely, their sines squared—observe that the above values for x and a also determine that

$$z = \frac{1}{29\,999.93\ldots} \quad ,$$

so that the *right* side of the Eq. (3) equals

$$\left(\frac{10}{3} - \frac{1}{3 \times 29\,999.93\ldots}\right)^3 + \left(10 - \frac{1}{29\,999.93\ldots}\right)^2 = 137.036 \quad . \qquad (5b)$$

Now notice that the four constants above

$$\frac{10}{3} \quad ,$$

$$\frac{1}{3 \times 29\,999.93\ldots} \quad ,$$

$$10 \quad ,$$

$$\frac{1}{29\,999.93\ldots}$$

each relate to the sines squared of the mixing angles. Specifically, within the limits of experimental error [4]:

- The value $10/3$ equals the reciprocal of $sin^2\varphi_{12}$.
- The value $1/(3 \times 29\,999.93\ldots)$ equals $sin^2\theta_{13}$.
- The value 10 equals $sin^2\varphi_{23}/sin^2\theta_{12}$. (Note that if $sin^2\varphi_{23}$ is assumed to equal 0.5, representing maximal mixing, then for $0.5/sin^2\,\theta_{12}$ to produce the value 10, the value $sin^2\theta_{12}$ must equal 0.05. As it turns out, within the limits of experimental error the precisely-measured $sin^2\theta_{12}$ does equal 0.05 [4]).
- The value $1/29\,999.93\ldots$ equals $sin^2\varphi_{13}/sin^2\theta_{23}$, so that $sin^2\varphi_{13}$ cannot be larger than roughly $1/30\,000$.

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In this way the right side of the FSC Equation mirrors the sines squared of five of the six mixing angles. Note that three of these sines squared are predicted to be the "round" numbers 0.3, 0.5, and 0.05, whereas $sin^2\theta_{13}$ is predicted to be close to $1/90\,000$.

Admittedly, two of the above sines squared occur individually

- $\sin^2 \varphi_{12}$
- $\sin^2 \theta_{13}$

whereas the remaining four sines squared appear as *ratios*

- $\sin^2 \varphi_{23} / \sin^2 \theta_{12}$
- $\sin^2 \varphi_{13} / \sin^2 \theta_{23}$.

Importantly, however, while investigating the mixing angles via matrix algebra, the author arrived independently at this same combination of sines squared and ratios of sines squared. A detailed account of this alternative method is available, where it is shown that it yields values identical to the above five mixing angles, while generating precise values for $\sin^2 \varphi_{13}$ and $sin^2\theta_{23}$ [4]. This account also offers a comparison against experiment of the values predicted for all six mixing angles.

IV. The muon- and neutron-electron mass ratios

The calculation of the value of the FSC inverse depends inversely on the square of the electron's charge e

$$\frac{1}{\alpha} = 4\pi\varepsilon_0 \hbar c/e^2 \approx 137.036 \quad .$$

As this usage of electron charge is arbitrary, one could instead employ the square of the d-quark's charge of $-1/3$ to produce a "d-quark FSC inverse" that is ninefold larger the usual FSC inverse.

With this in mind let

$$k_2 = 100 \qquad\qquad k_3 = 1000$$

$$m_2 = 1/9 \qquad\qquad m_3 = 3$$

so that

$$\frac{k_3 - {k_2}^{-1}}{m_3} + \frac{(10 - (300 \times k_2)^{-1})^2}{m_2} = 9 \times 137.036\ 000\ 001\ 111\ \ldots \qquad (6a)$$

and

$$\frac{k_2 - {k_3}^{-1}}{m_2} + \frac{(10 - (300 \times k_3)^{-1})^3}{m_3} = 9 \times 137.036\ 000\ 000\ 012\ \ldots \qquad (6b)$$

reproduce nine times the value of the FSC inverse in two closely related ways. They are "closely related" in that:

- *The left sides of Eqs. (6a) and (6b) are identical after the* $2 \leftrightarrow 3$ *exchange of subscripts and powers.*

That is to say, in going from one equation to the other, the subscripts and powers that are 2 become 3, and those that are 3 become 2. Moreover, with the aid of two definitions that exploit the first term of Eq. (6a)

$$j_2 = \frac{k_3 - {k_2}^{-1}}{m_3} = 333.33 \quad , \qquad (7a)$$

and the second term of Eq. (6b)

$$j_3 = \frac{(10 - (300 \times k_3)^{-1})^3}{m_3} = 333.333\ 000\ 000\ 111\ \ldots \quad , \tag{7b}$$

the muon- and neutron-electron mass ratios can, perhaps surprisingly, be expressed economically as

$$\frac{m_\mu}{m_e} = k_2 k_3 \frac{4.1^3}{j_2 k_2 - 1} + k_2 \frac{\mathbf{-1}}{j_2 k_2 - 1} = \quad 206.768\ 270\ 7\ \ldots \tag{8a}$$

$$\frac{m_n}{m_e} = k_3 k_2 \frac{4.1^3 + \boldsymbol{6k_3}}{j_3 k_3 - 1} + k_3 \frac{\boldsymbol{6k_3}}{j_3 k_3 - 1} = 1\ 838.683\ 654\ 7\ \ldots \quad . \tag{8b}$$

- *Note that, with the exception of those values appearing in boldface, i.e.,* $\mathbf{-1}$ *and* $\boldsymbol{6k_3}$*, Eqs. (8a) and (8b) are also identical after the* $2 \leftrightarrow 3$ *exchange of subscripts.*

The above calculated mass ratios differ from their 2006 CODATA values of 206.768 282 3 (52) and 1 838.683 660 5 (11) by just 56 and 3 parts per billion, respectively, and are each close to their limits of experimental error [1].

Information theory has already been used to make the case that the success of equations similar to Eqs. (8a) and (8b) is unlikely to be coincidental (see Eqs. (14a) and (14b) in [5]). Essentially, they are too simple in comparison to the amount of mass data they correctly reproduce to be purely the result of accident. By way of contrast, here are two approximations found by a “brute force” computer search

$$\frac{M_\mu}{M_e} = \frac{\mathbf{596\ 113}}{2883} = 206.768\ 296\ 9 \qquad (9a)$$

$$\frac{M_n}{M_e} = \frac{\mathbf{5\ 300\ 925}}{2883} = 1\ 838.683\ 662\ 8 \qquad (9b)$$

(see Eqs. (12a) and (12b), also in [5]). In much the same way that Eqs. (8a) and (8b) achieve economy by exploiting the ad hoc constant 4.1^3 in their numerators, this search reproduced the muon- and neutron-electron mass ratios via a pair of rational approximations sharing an ad hoc common denominator of 2883; but there the similarity ends. Whereas Eqs. (9a) and (9b) require large additional ad hoc constants such as $\mathbf{596\ 113}$ and $\mathbf{5\ 300\ 925}$ (in boldface) to fit the mass ratio data with comparable accuracy, Eqs. (8a) and (8b) require only the relatively simple, easy-to-remember constants $-\mathbf{1}$ and $\boldsymbol{6k_3}$ to fit these same ratios. Accordingly, the value 4.1 is far more effective than 2883 at reproducing the above mass ratios, despite 2883 containing the same number of digits as $41/10$, and despite 2883 resulting from an exhaustive computer search [5]. It is just this economy that suggests that Eqs. (8a) and (8b) have a non-coincidental relationship with the muon- and neutron-electron mass ratios.

Moreover, the value 4.1 arose previously in connection with an examination of the electron, muon, and *tau* masses [3], which were shown to follow the simple proportion

$$m_e : m_\mu : m_\tau = 1 : 3 \times 4.1^3 : 3 \times 4.1^5 \quad . \qquad (10)$$

This proportion fits the experimental muon-electron mass ratio to about 1 part in 40 000, and the less precisely measured tau-electron mass ratio to about 1 part in 2000 [3]. All this is further evidence that 4.1 is a constant of *physical* significance, a point receiving additional support from Eqs. (1)−(5) in [3].

V. The muon-, neutron-, and proton-electron mass ratios

Extending the earlier definitions, we let j_1, j_2, and j_3 equal 333.3, 333.33, and 333.333, respectively; and let k_1 k_2, and k_3 equal 10, 100, and 1000, respectively. Now define

$$L_2 = \frac{4.1^3}{k_2^0} + \frac{4.1^3}{k_2^1} + \frac{4.1^3}{k_2^2} + \cdots \tag{11a}$$

$$L_3 = \frac{4.1^3}{k_3^0} + \frac{4.1^3}{k_3^1} + \frac{4.1^3}{k_3^2} + \cdots \tag{11b}$$

$$q_2 = 6k_3 \tag{11c}$$

so that the muon-, neutron-, and *proton*-electron mass ratios may be approximated

$$\frac{m_\mu}{m_e} = k_2 k_3 \frac{L_3}{j_2 k_2 - 1} - k_2 \frac{L_3 + 1}{j_2 k_2 - 1} = \quad 206.768\ 270\ 7\ \ldots \tag{12a}$$

$$\frac{m_n}{m_e} = k_3 k_2 \frac{\boldsymbol{L_2} + q_2}{j_3 k_3 - \mathbf{1}} - k_3 \frac{L_2 - q_2}{j_3 k_3 - \mathbf{1}} = 1\ 838.683\ 654\ 7\ \ldots \tag{12b}$$

$$\frac{m_p}{m_e} = k_3 k_2 \frac{q_2}{j_3 k_3 - \boldsymbol{j_1 k_1}} - k_3 \frac{L_2 - q_2}{j_3 k_3 - \boldsymbol{j_1 k_1}} = 1\ 836.152\ 675\ 23\ \ldots\ . \tag{12c}$$

- *Note that, except for the constants 1 and q_2 in their respective numerators, Eqs. (12a) and (12b) are identical after the $2 \leftrightarrow 3$ exchange of subscripts.*
- *Note that Eqs. (12b) and (12c) are also identical, except that in going from Eq. (12b) to (12c), the constant $\boldsymbol{L_2}$ in the first term of Eq. (12b)'s numerator vanishes, while the constant $\mathbf{1}$ in Eq. (12b)'s denominator becomes $\boldsymbol{j_1 k_1}$ (these differences appear in boldface).*

Also:

- The above muon-electron mass ratio m_μ/m_e is identical with its earlier value.
- The above neutron-electron mass ratio m_n/m_e is slightly larger than its earlier value, given that j_3 is now slightly smaller (see Eq. (7b), but the difference is too small to be of immediate experimental consequence.
- The above proton-electron mass ratio m_p/m_e differs from its 2006 CODATA value of 1 836.152 672 47 (80) by just 1.5 parts per billion [1].

Finally, observe that the denominator of Eq. (12a) equals

$$j_2 k_2 - 1 \;= 33333 - 1 \quad , \tag{13a}$$

whereas that of Eq. (12b) equals

$$j_3 k_3 - 1 = 333333 - 1 \quad , \tag{13b}$$

and that of Eq. (12c) equals

$$j_3 k_3 - j_1 k_1 = 333333 - 3333 \quad . \tag{13c}$$

This last denominator, belonging to Eq. (12c), the proton-electron mass ratio equation, is slightly more complicated than those belonging to the muon- and neutron-electron mass ratios. But with help from the definition

$$Q_2 = \frac{q_2}{k_2^0} + \frac{q_2}{k_2^1} + \frac{q_2}{k_2^2} + \cdots \quad , \tag{14}$$

which parallels Eq. (11a)'s definition of L_2, Eq. (12c) can be restated

$$\frac{m_p}{m_e} = 3Q_2\left(\mathrm{k}_2^{-\frac{1}{2}} + \mathrm{k}_2^{-\frac{3}{2}}\right) - 3L_2\left(\mathrm{k}_2^{-\frac{3}{2}} + \mathrm{k}_2^{-\frac{5}{2}} + \mathrm{k}_2^{-\frac{7}{2}} + \cdots\right) \tag{15}$$

so that Eq. (12c)'s more complicated denominator vanishes. Moreover, if

$$Q_2' = \frac{Q_2}{k_2^0} + \frac{Q_2}{k_2^1} + \frac{Q_2}{k_2^2} + \cdots \tag{16a}$$

$$L_2' = \frac{L_2}{k_2^0} + \frac{L_2}{k_2^1} + \frac{L_2}{k_2^2} + \cdots \tag{16b}$$

then

$$\frac{m_p}{m_e} = \frac{3}{k_3}\left(\mathrm{k}_2 Q_2' - L_2' - \mathrm{k}_2^{-1} Q_2'\right) \quad . \tag{17}$$

Equation (14) also allows Eqs. (12a)−(12c) to be written

$$\frac{m_\mu}{m_e} = k_2\,\frac{k_3 L_3 - L_3 - 1}{j_2 k_2\ - 1} \tag{18a}$$

$$\frac{m_n}{m_e} = k_3\,\frac{k_2\,L_2 + k_2\,Q_2 - L_2 - k_2^{-1} Q_2}{j_3 k_3 - 1} \tag{18b}$$

$$\frac{m_p}{m_e} = k_3\,\frac{k_2\,Q_2 - L_2 - k_2^{-1} Q_2}{j_3 k_3 - j_1 k_1} \quad . \tag{18c}$$

Lastly, Eqs. (12a)−(12c) can also be written

$$\frac{m_\mu}{m_e} = \frac{L_3 - \dfrac{L_3 + 1}{k_3}}{0.33332} \tag{19a}$$

$$\frac{m_n}{m_e} = \frac{L_2 + q_2 - \dfrac{L_2 - q_2}{k_2}}{3.33332} \tag{19b}$$

$$\frac{m_p}{m_e} = \frac{q_2 - \dfrac{L_2 - q_2}{k_2}}{3.3} \tag{19c}$$

or

$$\frac{m_\mu}{m_e} = 3\,\frac{4.1^3 - 0.1^3}{0.99996} \qquad (20a)$$

$$\frac{m_n}{m_e} = 3\,\frac{6060 + 4.1^3}{9.99996} \qquad (20b)$$

$$\frac{m_p}{m_e} = 3\,\frac{6060 - \frac{4.1^3}{99}}{9.9} \quad . \qquad (20c)$$

VI. Analysis and Conclusion

Even a conservative estimate of the number of digits correctly reproduced by Eq. (3), the FSC Equation—more than a dozen if those of the mixing angles are added to the more than eight digits reproduced of the FSC—suggests that it cannot be expected to compress such a quantity of information by accident. That there are limits to the compressibility of data has already been demonstrated in an article that shows how information theory can be used to distinguish coincidental, from non-coincidental, approximations [5]. Whereas it is always possible to present data in an *alternate* form—for example, a decimal number in binary form—it is not reasonable to expect that one can *compress* it at will, which is to say, one cannot logically expect to restate a 12-digit *decimal* number using just 12 *binary* digits without loss of information. (Actually, it would require about $12log_2(10)$ binary digits, which equals the number of bits of information that a 12-digit, base 10 number possesses.)

Thus, the quantity of information possessed by experimental data places limits on how simply that data may be mathematically summarized by chance, as is illustrated by Eqs. (9a) and (9b). Such limits, once understood, allow one to analyze experimental data with the aid of information theory to isolate non-coincidental—i.e., physically significant—regularities in the data [5], where a relationship is likely non-coincidental to the extent that the above limits are breached.

As it turns out, the FSC equation, Eq. (3), is especially unlikely to compress, purely by accident, the experimental data it models, given that:

- Its form derives entirely from an analysis of the "broken" symmetry of two simple algebraic identities.
- None of its four free variables are "adjusted to fit experimental data." Instead, its variables x and a are merely the smallest integers—10 and 3, respectively—that allow Eq. (3) to mimic Eq. (1) in fulfilling Eq. (2c), while Eq. (2a) allows y and z to be deduced from x and a.
- It reproduces more than a dozen digits of FSC and quark and lepton mixing data.
- Key terms of the FSC inverse Eqs. (7a) and (7b)—specifically, j_2 and j_3—can be used to help reproduce more than a dozen additional digits of data belonging to the muon- and neutron-electron mass ratios. Admittedly, mass Eqs. (8a) and (8b) accomplish this while requiring two constants chosen partly to fit the mass data and partly because they are round numbers: $\boldsymbol{-1}$ and $\boldsymbol{6k_3}$, but the information content of these numbers is far less than the information content of the high-precision mass ratio data they correctly reproduce.
- The "mirror image" FSC Eqs. (6a) and (6b) supply the key constants j_2 and j_3 that are in turn employed by the mirror image mass Eqs. (8a) and (8b); and yet, despite this restriction, they each still fit their respective experimental data near their experimental limits. For this reason these four equations appear especially unlikely to be merely the result of coincidence.

Accordingly, given that neither its form nor its content has been adjusted to fit either the FSC or quark and lepton mixing angles, it is logical to suspect that the FSC Equation, Eq. (3), compresses more than a dozen digits of experimental data for reasons that are not accidental, but physical. For these reasons it is logical to expect that experiment will for some time continue to mirror the "round numbers" predicted by the FSC Equation for the sines squared of the mixing angles.

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Appendix A

Proof that given Eq. (1), if Eqs. (2a) and (2b) are true, so is Eq. (2c).

Equation (1)

$$x^2 - y^2 = (x - z)^2$$

simplifies to

$$y^2 = 2xz - z^2$$

so that

$$2ydy = (2x - 2z)dz$$

$$\frac{y}{x - z} = \frac{dz}{dy} \quad .$$

Equation (2a) provides that $x = 1/y$ so that

$$\frac{y^2}{1 - yz} = \frac{dz}{dy} \quad .$$

Given Eqs. (2a) and (2b), the cross-terms on the right side of Eq. (1) guarantee that $z < y^3$, so that for small *y*

$$y^2 \approx \frac{dz}{dy} \ .$$

In this way Eq. (2c) is recovered for $p = 2$.

Appendix B

Proof that if Eq. (3)'s values for x and a are consistent with Eq. (4), then if Eqs. (2a) and (2b) are true, so is Eq. (2c).

If its higher-order terms are ignored then Eq. (3)

$$\frac{x^3 - y^3}{a^3} + x^2 - y^3 = \left(\frac{x}{a} - \frac{z}{a}\right)^3 + (x - z)^2$$

simplifies it to

$$-\frac{y^3}{a^3} - y^3 \approx -\frac{3x^2 z}{a^3} - 2xz$$

$$-1 - a^3 y^3 \approx -3x^2 z - 2a^3 xz$$

$$(a^3 + 1)y^3 \approx 3x^2 z + 2a^3 xz$$

so that

$$3(a^3 + 1)y^2 dy \approx (3x^2 + 2a^3 x)dz$$

$$\frac{3(a^3 + 1)y^2}{3x^2 + 2a^3 x} \approx \frac{dz}{dy} \quad . \tag{B1}$$

Although Eq. (B1) is approximate, the terms it ignores are on the order of z^2, where for quark and lepton mixing $z \approx 1/30\,000$.

In order to assure that Eq. (2c)

$$\frac{dz}{dy} \approx y^p$$

holds for Eq. (3), we rewrite Eq. (B1) as

$$\frac{3(a^3+1)y^2}{3x^2+2a^3x} \approx \frac{dz}{dy} \approx y^p \approx y^3$$

where $p = 3$, the order of Eq. (3). It follows that

$$\frac{a^3+1}{x^2+\frac{2}{3}a^3x} \approx y \quad .$$

Equation (2a) provides that $y = 1/x$ so that

$$\frac{a^3+1}{x+\frac{2}{3}a^3} \approx 1$$

$$a^3 + 1 \approx x + \frac{2}{3}a^3 \quad ,$$

which recovers Eq. (4)

$$x \approx \frac{a^3}{3} + 1 \quad .$$

Hence, Eq. (4) does constrain Eq. (3)'s values for x and a so that, if Eqs. (2a) and (2b) are true, then so is Eq. (2c).