Formal Proof that *c* is the Limiting Speed*

If light is made of particles or waves which propagate at the same speed in all inertial frames, then the speed of light is the greatest speed possible.

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* This argument originally was made as part of a course assignment by Professor Leonard Susskind in a physics class at the Stanford School of Continuing Studies in 2001. A slightly different version was posted online in 2007 at `http://www.siuc.edu/~pulfrich/Pulfrich_Pages/lit_nonp/phys_astro/2007_cSpeed/LimitingVelocity.html`.

Introduction

We know from countless experiments in astronomy, satellite orbital dynamics, and particle and accelerator physics, that Einstein's formulation of special relativity is physically correct, and that any concept of a fixed, physically real, universally constant field or frame of reference must be either a coarse approximation or false.

Here is a derivation of the proposition that the speed of light, c, is the limiting velocity. It assumes causality and is based on the observation that the speed of light is the same to all observers.

Assumptions

Naturally, it is not rational to claim that one can prove all ones assumptions. To reason logically, one must adopt axioms, which are assumptions accepted without proof.

In this work, we accept the usual assumptions of space and measurable distances, and we assume always that events are occurring in vacuum.

Vacuum is a complex concept. All we mean by it here, is a region of space devoid of all particles except those specified in the proofs. We know from quantum physics that this is not a reasonable assumption: The vacuum has a definite ground-state energy.

The ground-state, or zero-point, energy is ignored in this work. Computations based on observation indicate that the best laboratory vacuum must be seething and boiling with a sea of virtual particles which pop in and out of existence for microscopic periods of time. Notice the metaphors here ("sea", "boiling", "seething", etc.) which are taking the place of precise reasoning. We accept this without further consideration, because none of our arguments here depend on quantum theory.

We take "*vacuum*" for granted, although the precise value of the speed of light, c, undoubtedly depends to some extent upon the zero-point energy density.

Preliminary Assumptions

We shall begin by assuming all operations to occur in vacuum and far from any object capable of exerting a measurable gravitational force.

We shall use the word "object" to refer to an elementary particle or other thing which may be localized and which may be affected as a whole by some other object, the interaction occurring at an identifiable location but without regard for rotation or resolution of substructure of any object.

We allow for clocks to synchronize events, an event being the coincidental observation of one or more objects in the same place as the clock. We allow distance to be measured (by units of length) between events.

We assume no object can move at infinite speed, or causality (ordering of effects) would not be possible.

We shall allow any object either to be accelerable or inaccelerable but never both.

Definition of Accelerability

An accelerable object is any one which can be made to move at different velocities, depending on what is done to affect its motion.

Definition of an Inertial Frame

An inertial frame, or briefly, a *frame*, is a coordinate system attached to the motion of an accelerable object so that the object is at rest in that frame. Given such an inertial frame, then, if a force be exerted on the object, it will be accelerated so it is in motion in that frame; if no force be exerted, the object will remain at rest in that frame. We use the word "force" here to refer to any physical operation which might be attempted to change the velocity of an object. We do not assume anything more specific, such as Newton's laws of motion.

Definition of Inaccelerability

Objects that are inaccelerable always travel at a fixed speed, possibly zero, relative to every accelerable object, and at a fixed velocity relative to any given inertial frame. In particular, photons are inaccelerable. Other particles such as gluons also may be inaccelerable. Because nothing in the present argument depends on quantum theory, it makes no difference whether one prefers to treat an inaccelerable object as a particle or as a wave.

Observation: Light is Inaccelerable

The experiment of Michelson and Morley, and the calculations of Maxwell, show that light (photons) is inaccelerable. We assume here that when light is reflected or refracted, photons may be affected by the reflecting or refracting object so that they vanish and are replaced after some little time by others with the same speed but different velocity (direction).

Theorem I: An Inaccelerable Object Must Be Faster than any Accelerable One

Proof: Suppose an inaccelerable object I which travels at some speed v_I. Let I travel in some direction $\underline{v}$, passing close to an accelerable object C located at x_i (i = initial). Let I continue and farther on pass equally close to a different accelerable object D located at x_f (f = final), both accelerable objects being at rest in the same frame but separated by a considerable distance in the direction $\underline{v}$.

Now, let us repeat this observation but this time supposing that there might exist an accelerable object A allowing us exactly to copy the previous observation, with A substituted for I and travelling at speed v_A which is greater than, or equal to, v_I. If so, we could repeat yet a third time with both I and A starting at x_i together. But, I must travel at speed v_I in all inertial frames, including the rest frame of A, and so I must arrive at x_f before A. This contradicts the assumption that I travels at the same speed in all frames; therefore, no accelerable object can travel at a speed as great as that of an inaccelerable object.

Theorem II: The Speed of Every Inaccelerable Object is the Same

First, proofs aside, if there were more than one speed for inaccelerable objects, they could change speed and therefore would not be inaccelerable. However, one might imagine that inaccelerable objects, like photons, never change between interactions and might then retain different speeds if created with different speeds. This very reasonable possibility is what requires the following:

Coarse Proof: Suppose two inaccelerable objects I and J could travel at different speeds v_I and v_J, with v_I greater than v_J. Start I, J, and an accelerable object A at the same time at x_i as above and in the direction $\underline{v}$ as above. Now, there is no reason why A could not be accelerated to a speed v_A just below that of I; therefore, a v_A could be reached such that v_I was greater than v_A and v_A was greater than v_J. But, this would be the same as allowing an accelerable object to move at a speed greater than that of the inaccelerable object J. Therefore, there can be no room between v_I and v_J for v_A, and both I and J must move at the same speed.

More Rigorous Proof: Let there exist two accelerable objects A_i and A_f initially at points x_i and x_f respectively in the same frame, such that the distance, $L = x_f - x_i$ might be chosen to be large and well-defined in that frame. Let the direction of $x_f - x_i$ be represented by the vector $\underline{x}$.

Let there exist a different two accelerable objects B_i and B_f separated by the same distance L along the same distance vector but in a different frame which is in motion in the direction of $\underline{x}$ at some arbitrarily high speed relative to the frame of the A's. Representing B_f as to the right of B_i, let B_f pass close to A_i and continue on, moving to the right. When B_i then passes close to A_i, let there be an emission event as follows:

At the same instant in both frames, as defined by the same point on both the space and time axes, A_i emits a pair of inaccelerable objects in the direction of A_f, and B_i emits an identical pair in the direction of B_f. Of course, the two direction vectors are the same.

Let each pair of inaccelerable objects be called *I* and *J*, with, by hypothesis, the speed of *I* greater than that of *J*.

Now, because the distance between the *A*'s is equal to that between the *B*'s in their respective frames, if inaccelerable objects travel at the same (respective) speeds in every inertial frame, both of A_f and B_f would receive a pair of *I* and *J* separated by the same time interval, Δt, as measured in their respective frames.

Because by hypothesis the speed of an *I* is greater than that of a *J*, in both frames, the *I*'s must arrive at A_f and B_f first. This is indicated in Figure 1 by the shallower slope of *I*, as compared with *J*. But, the frame of the *B*'s is in motion relative to that of the *A*'s, in the direction of the inaccelerable propagation, so the distance from A_i to B_f must increase (in both frames) while the inaccelerable objects are propagating. Therefore, the difference in arrival times Δt between *I* and *J* at B_f must be greater than that at A_f. See Figure 1.

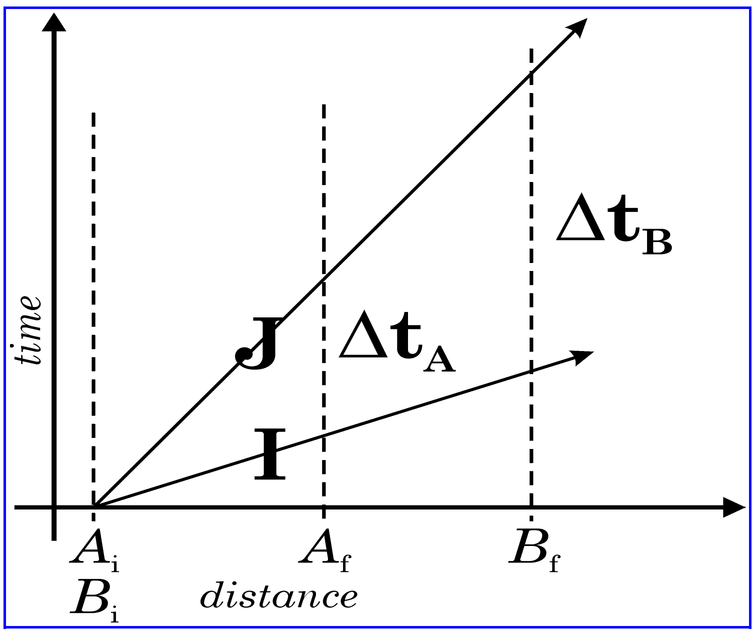

Fig. 1. Showing the space-time relationship of two inaccelerable objects *I* and *J* propagated from objects moving at different speeds. The positions of accelerable object A_i, and another accelerable object, B_i, each of which emits an inaccelerable *I* and *J*, also are shown. The *A*'s are at rest with respect to one another, and so are the *B*'s, but the distance to B_f increases while *I* and *J* are propagating.

The same quantity can not both be equal and not equal to Δt, so the assumption implying Δt must be invalid: Inaccelerable objects can not exist which travel at different speeds. All differences between such speeds must be identically 0. This makes Δt equal to 0 regardless of relative motion of inertial frames, preventing the contradiction just derived.

Conclusion

We conclude that all inaccelerable objects move at the same speed *c*, in the frame of every accelerable object. And no object, accelerable or inaccelerable, can exceed the speed *c* in any frame.

The speed of light then equals this same c.

Postscript

So, that proves it. It is easy to be confident, now, that the Lorentz transformations indeed can represent a physically valid principle. We find c in the Lorentz formulas,

$$x' = \frac{x - vt}{\sqrt{1-(v/c)^2}} \qquad t' = \frac{t-(v/c^2)x}{\sqrt{1-(v/c)^2}}$$

Now we understand the meaning of c; it truly is a constant. So, we can be more trusting in the inevitable correctness of these formulas.

But, to derive them from the assumptions of this proof, we assume the reader will take a course in modern physics!