Formal Proof that $c$ is the Limiting Speed*

If light is made of particles or waves which propagate at the same speed in all inertial frames, then the speed of light is the greatest speed possible.

*This argument originally was made as part of a course assignment by Professor Leonard Susskind in a physics class at the Stanford School of Continuing Studies in 2001. A slightly different version was posted online in 2007 at http://www.siuc.edu/~pulfrich/Pulfrich_Pages/lit_nonp/phys_astro/2007_cSpeed/LimitingVelocity.html.

Copyright (c) 2010, John Michael Williams. All Rights Reserved.
Introduction

We know from countless experiments in astronomy, satellite orbital dynamics, and particle and accelerator physics, that Einstein's formulation of special relativity is physically correct, and that any concept of a fixed, physically real, universally constant field or frame of reference must be either a coarse approximation or false.

Here is a derivation of the proposition that the speed of light, \( c \), is the limiting velocity. It assumes causality and is based on the observation that the speed of light is the same to all observers.

Assumptions

Naturally, it is not rational to claim that one can prove all one's assumptions. To reason logically, one must adopt axioms, which are assumptions accepted without proof.

In this work, we accept the usual assumptions of space and measurable distances, and we assume always that events are occurring in vacuum.

**Vacuum** is a complex concept. All we mean by it here, is a region of space devoid of all particles except those specified in the proofs. We know from quantum physics that this is not a reasonable assumption: The vacuum has a definite ground-state energy.

The ground-state, or zero-point, energy is ignored in this work. Computations based on observation indicate that the best laboratory vacuum must be seething and boiling with a sea of virtual particles which pop in and out of existence for microscopic periods of time. Notice the metaphors here ("sea", "boiling", "seething", etc.) which are taking the place of precise reasoning. We accept this without further consideration, because none of our arguments here depend on quantum theory.

We take "vacuum" for granted, although the precise value of the speed of light, \( c \), undoubtedly depends to some extent upon the zero-point energy density.

Preliminary Assumptions

We shall begin by assuming all operations to occur in vacuum and far from any object capable of exerting a measurable gravitational force.

We shall use the word "object" to refer to an elementary particle or other thing which may be localized and which may be affected as a whole by some other object, the interaction occurring at an identifiable location but without regard for rotation or resolution of substructure of any object.

We allow for clocks to synchronize events, an event being the coincidental observation of one or more objects in the same place as the clock. We allow distance to be measured (by units of length) between events.

We assume no object can move at infinite speed, or causality (ordering of effects) would not be possible.
We shall allow any object either to be accelerable or inacceleratable but never both.

**Definition of Accelerability**

An accelerable object is any one which can be made to move at different velocities, depending on what is done to affect its motion.

**Definition of an Inertial Frame**

An inertial frame, or briefly, a frame, is a coordinate system attached to the motion of an accelerable object so that the object is at rest in that frame. Given such an inertial frame, then, if a force be exerted on the object, it will be accelerated so it is in motion in that frame; if no force be exerted, the object will remain at rest in that frame. We use the word "force" here to refer to any physical operation which might be attempted to change the velocity of an object. We do not assume anything more specific, such as Newton's laws of motion.

**Definition of Inaccelerability**

Objects that are inacceleratable always travel at a fixed speed, possibly zero, relative to every accelerable object, and at a fixed velocity relative to any given inertial frame. In particular, photons are inacceleratable. Other particles such as gluons also may be inacceleratable. Because nothing in the present argument depends on quantum theory, it makes no difference whether one prefers to treat an inacceleratable object as a particle or as a wave.

**Observation: Light is Inacceleratable**

The experiment of Michelson and Morley, and the calculations of Maxwell, show that light (photons) is inacceleratable. We assume here that when light is reflected or refracted, photons may be affected by the reflecting or refracting object so that they vanish and are replaced after some little time by others with the same speed but different velocity (direction).

**Theorem I: An Inacceleratable Object Must Be Faster than any Accelerable One**

**Proof:** Suppose an inacceleratable object \( I \) which travels at some speed \( v_I \). Let \( I \) travel in some direction \( v \), passing close to an accelerable object \( C \) located at \( x_i \) (i = initial). Let \( I \) continue and farther on pass equally close to a different accelerable object \( D \) located at \( x_f \) (f = final), both accelerable objects being at rest in the same frame but separated by a considerable distance in the direction \( v \).

Now, let us repeat this observation but this time supposing that there might exist an
accelerable object $A$ allowing us exactly to copy the previous observation, with $A$ substituted for $I$ and travelling at speed $v_A$ which is greater than, or equal to, $v_I$. If so, we could repeat yet a third time with both $I$ and $A$ starting at $x_i$ together. But, $I$ must travel at speed $v_I$ in all inertial frames, including the rest frame of $A$, and so $I$ must arrive at $x_f$ before $A$. This contradicts the assumption that $I$ travels at the same speed in all frames; therefore, no accelerable object can travel at a speed as great as that of an inaccelerable object.

**Theorem II: The Speed of Every Inaccelerable Object is the Same**

First, proofs aside, if there were more than one speed for inaccelerable objects, they could change speed and therefore would not be inaccelerable. However, one might imagine that inaccelerable objects, like photons, never change between interactions and might then retain different speeds if created with different speeds. This very reasonable possibility is what requires the following:

**Coarse Proof:** Suppose two inaccelerable objects $I$ and $J$ could travel at different speeds $v_I$ and $v_J$, with $v_I$ greater than $v_J$. Start $I$, $J$, and an accelerable object $A$ at the same time at $x_i$ as above and in the direction $v$ as above. Now, there is no reason why $A$ could not be accelerated to a speed $v_A$ just below that of $I$; therefore, a $v_A$ could be reached such that $v_I$ was greater than $v_A$ and $v_A$ was greater than $v_J$. But, this would be the same as allowing an accelerable object to move at a speed greater than that of the inaccelerable object $J$. Therefore, there can be no room between $v_I$ and $v_J$ for $v_A$, and both $I$ and $J$ must move at the same speed.

**More Rigorous Proof:** Let there exist two accelerable objects $A_i$ and $A_f$ initially at points $x_i$ and $x_f$ respectively in the same frame, such that the distance, $L = x_f - x_i$ might be chosen to be large and well-defined in that frame. Let the direction of $x_f - x_i$ be represented by the vector $\vec{x}$.

Let there exist a different two accelerable objects $B_i$ and $B_f$ separated by the same distance $L$ along the same distance vector but in a different frame which is in motion in the direction of $\vec{x}$ at some arbitrarily high speed relative to the frame of the $A$'s. Representing $B_f$ as to the right of $B_i$, let $B_f$ pass close to $A_i$ and continue on, moving to the right. When $B_i$ then passes close to $A_i$, let there be an emission event as follows:

At the same instant in both frames, as defined by the same point on both the space and time axes, $A_i$ emits a pair of inaccelerable objects in the direction of $A_f$, and $B_i$ emits an identical pair in the direction of $B_f$. Of course, the two direction vectors are the same.
Let each pair of inacceleratable objects be called \( I \) and \( J \), with, by hypothesis, the speed of \( I \) greater than that of \( J \).

Now, because the distance between the \( A \)'s is equal to that between the \( B \)'s in their respective frames, if inacceleratable objects travel at the same (respective) speeds in every inertial frame, both of \( A_f \) and \( B_f \) would receive a pair of \( I \) and \( J \) separated by the same time interval, \( \Delta t \), as measured in their respective frames.

Because by hypothesis the speed of an \( I \) is greater than that of a \( J \), in both frames, the \( I \)'s must arrive at \( A_f \) and \( B_f \) first. This is indicated in Figure 1 by the shallower slope of \( I \), as compared with \( J \). But, the frame of the \( B \)'s is in motion relative to that of the \( A \)'s, in the direction of the inacceleratable propagation, so the distance from \( A_i \) to \( B_f \) must increase (in both frames) while the inacceleratable objects are propagating. Therefore, the difference in arrival times \( \Delta t \) between \( I \) and \( J \) at \( B_f \) must be greater than that at \( A_f \). See Figure 1.

![Figure 1](image.png)

**Fig. 1.** Showing the space-time relationship of two inacceleratable objects \( I \) and \( J \) propagated from objects moving at different speeds. The positions of accelerable object \( A_i \), and another accelerable object, \( B_i \), each of which emits an inacceleratable \( I \) and \( J \), also are shown. The \( A \)'s are at rest with respect to one another, and so are the \( B \)'s, but the distance to \( B_f \) increases while \( I \) and \( J \) are propagating.

The same quantity can not both be equal and not equal to \( \Delta t \), so the assumption implying \( \Delta t \) must be invalid: Inacceleratable objects can not exist which travel at different speeds. All differences between such speeds must be identically 0. This makes \( \Delta t \) equal to 0 regardless of relative motion of inertial frames, preventing the contradiction just derived.

**Conclusion**

We conclude that all inacceleratable objects move at the same speed \( c \), in the frame of every accelerable object. And no object, accelerable or inacceleratable, can exceed the speed \( c \) in
any frame.
The speed of light then equals this same \( c \).

**Postscript**
So, that proves it. It is easy to be confident, now, that the Lorentz transformations indeed can represent a physically valid principle. We find \( c \) in the Lorentz formulas,

\[
x' = \frac{x - vt}{\sqrt{1 - (v/c)^2}} \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - (v/c)^2}}
\]

Now we understand the meaning of \( c \); it truly is a constant. So, we can be more trusting in the inevitable correctness of these formulas.

But, to derive them from the assumptions of this proof, we assume the reader will take a course in modern physics!