THE GEOMETRY OF HEAVENLY MATTER FORMATIONS

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Abstract. Our visual cosmic spectacle is populated with matter formations of sophisticated geometrical shapes. Many of these are flat, or at least tend to be flat. The geometrical description of these started, historically, with circles, going through conic sections and ending with spirals. The flat formations, of matter both static and in motion, helped creating the modern theory of gravitation. They decided the modern image of the world we live in. Are they connected? Yes! The geometry of this connection is presented, starting from its main physical reasons: we cannot see them but in projections on the canopy.

Key Words: galaxies, solar system, Saturn rings, asteroid belt, comets, conic sections, logarithmic spirals, exponential curves, mass accretion, cosmology, cosmogony

INTRODUCTION

The Kepler synthesis of the planetary motion of Mars led to the three laws governing the motion of planets in general. The first two of these laws are directly related with the fact that the motion is a flat geometrical shape – the ellipse. These two laws of Kepler also helped creating the Newtonian synthesis of the force responsible for the motion of planets – the Newtonian gravitational force. From the very beginning it was realized that this force is universal, and therefore it should characterize the matter not only as a solar system, but in general. By the middle of 19^{th} century the scientific world became aware of still other matter formations in the sky – the nebulae (**Alexander, 1852**). However, not all of these appeared to be flat formations, so the application of the gravitation in order to describe them dynamically was not thought for immediately. What was here thought, however, as proper to apply was some kind of theory of fluids with gravitational forces acting between the stars (**Wilczynski, 1896, 1899**). The only material formations that could ever have a genuine Newtonian explanation, in the manner of solar system, were only the spiral nebulae.

In the earliest attempts to describe the spiral galaxies classically, based on the data of their internal motions, the use of Newtonian theory led to the result that the spiral arms as paths of motion of stars would not be an option firmly sustained by the experimental data at hand (**Jeans, 1923**). This could be counted, we suppose, as a first sign of divorce between the Newtonian theory and the cosmic data. Nevertheless, the data could sustain the idea that spiral arms are some envelopes of Kepler orbits of the stars around the center of the galaxy (**Brown, 1925**). Useless to say, none of these ideas had the chance to be clearly assessed by experimental data. In view of this, the spiral arms are most commonly described today as *density waves* in the spirit of the works of Wilczynski cited above (**Toomre, 1977**).

However, speaking of Kepler orbits and their envelopes, Newton himself had an important point on this issue, related to the so-called problem of "revolving orbits". In his system of natural philosophy, Newton was able to describe the spiral as an orbit determined by a central force with magnitude depending exclusively on distance, and being inversely proportional with the third power of distance (*Principia*, Book I, Proposition IX). Then he proved that this force has another important property – that of being a *force of transition* between two identical Kepler orbits rotated with respect to each other (*Principia*, Book I, Propositions XLIII and XLIV). Thus he succeeded in describing the revolving orbits by a central force with magnitude depending exclusively on distance. The magnitude of force is here a linear combination of the gravitation force proper and the transition force. It is well known that this description is not quite accurate, having to be corrected by the general relativity, which attracted the attention of physicists and astronomers in an entirely different theoretical direction. However, the Newtonian problem of revolving orbits has something of eternal ingenuity, so that it started attracting again the attention of some astrophysicists (**Lynden-Bell, 2006**).

The ingenuity of the Newtonian description of the problem of revolving orbits might come to light from another fact, this time related to quantum mechanics: it can classically explain the quantum jumps. Indeed, the process of motion of a particle along a revolving orbit can be decomposed in an infinitesimal motion along an ellipse, followed by an infinitesimal motion along a logarithmic spiral, and then by an infinitesimal motion along the next ellipse, etc. In other words the jump between Keplerian orbits is realized by a motion along the logarithmic spiral. In this case, the whole process of quantization can be described in terms of the *Hannay angle* related to the family of rotating ellipses (**Mazilu, 2010**).

Fact is that, in case we accept the idea of "enveloping", this problem of revolving orbits is entirely analogous to the problem of structure of spiral arms of galaxies, at least from geometrical point of view. One, as well as the other, is referring to *a family of Kepler orbits*. As the motions they indicate seem to be flat, at least by their tendency, one is led to the conclusion that, if they are described by forces in the Newtonian manner, then these forces *have to be central*. However, the Keplerian motion means more than the flatness of the motion: *it also requires the area law to be respected*. This issue, however, cannot be properly assessed in astronomical observations of galaxies, mostly because we don't have the chance to see much of their internal motion during our lifetime.

Nevertheless, with the advent of the spectroscopic technologies, the possibilities are open for measurements of internal velocities of the nebulae, at least for the cases where they are not perpendicular to the line of sight. Such results of van Maanen were used by Jeans and Brown in their theoretical speculations mentioned above. They seem to contradict the Newtonian wisdom. In time, the Newton's hypothesis regarding the relation between mass and gravitation, which is quite arbitrary by itself (**Poincaré, 1897**), came to be directly challenged. It was thus realized that there is a huge discrepancy between the "shining matter" and "opaque matter", as Newton used to call the kinds of matter we perceive in space (in the letters to Bishop Bentley). The specific issue is that, while the “shining matter” reveals certain speeds of revolution in the spirals, these don’t seem to agree with the speeds calculated by the Newtonian dynamics recipes, thus indicating that there may be more “opaque matter” located in the cosmic formation than “shining matter” (**Sofue, Rubin, 2001**). Thus, the whole Newtonian theory came under attack from this angle. Of course, there are reactions to such conclusions (**Bekenstein, 2005**), but none of them seem to address the problem of mass hypothesis of Newton, which seems to us to be essential in case we ever need to buid solidly upon previous achievements of science. Moreover, the mentioned reactions don’t even seem addressed to defending or criticizing the Newtonian natural philosophy per se, but mainly to show that the idea of dark matter isn’t really necessary: there are a lot of other possibilities opened to the physical speculation. So the dark mass, like the mass itself to Newton, is out of critical question. It seems to be merely a kind of option due to the infinite possibilities of our spirit.

In our opinion, a big problem here is if the Newtonian spirit is indeed respected in inferences from spectroscopic measurements of the internal velocities of spiral galaxies. Only after this issue was decided can one go to deeper problems, depending, of course, on the result of decision. Having this in mind as a general program, we aim to first decide if the flat matter formations which the heavens present to us do have something in common. In view of the universality of Newtonian gravitation they should indeed have something in common. Specifically, the task of the present work will be twofold: (1) to decide if the flat cosmic shapes we see in the sky are all of the same nature. The “same nature” will be specified in due time, as we proceed. The partial answer, involving only the geometrical point of view, is affirmative. Then comes the second task: (2) to determine the position of *the area law* in connection with this kind of data of our knowledge. It will be shown that, with respect to the area law, the shapes we see in heavens are divided in two groups: ones are obeying the area law, the others are not obeying it. The first group is related to evolving Kepler orbits. However, the evolution leads by no means to spirals, but to formations like the Saturn rings or the asteroid belt. The second group, not respecting the area law, comprises the spirals, general parabolic motions (like the ballistic motions), and the comets’ structure.

We apply here the Newtonian philosophy in its initial spirit. First, recall that Newton had at his disposal Kepler’s synthesis, with clear (*we don’t say precise!*) quantitative data. Therefore a synthesis is the first thing to do. We have precise data on the very same problem, that seem to baffle us on occasions. Then, we have clear (*again, we don’t say precise!*) data on the shape of our planetary system, galaxies, and structure of incoherent cosmic matter. A unitary geometrical synthesis of these is missing from the view of astrophysicists, and this seems to be reflected in the science today. Thus, for instance, if from astrometric measurements we can infer a geometrical shape, this one doesn’t come directly with it’s physical explanation. However, it never came with that physical explanation, even to Newton: the synthesis of geometrical shape, like that of the orbits of Kepler, preceded the physical explanation, in order that a physical theory may be assessed. Specifically, Newton had first the geometry well described and then, on that very base, he proceeded at the description of the forces. Let's therefore describe the geometry of flat heavenly formations of matter.

GEOMETRICAL SHAPES OF HEAVENLY FORMATIONS OF MATTER

First and foremost we need to see what is the common geometrical denominator among the flat formations of the solar system and galaxies, whose shapes are given by astrometric observational data. Such was the starting point of the Newton in the first place. And, indeed, there is one such denominator: the Keplerian trajectories and the shape of spirals are curves belonging to the same family, namely *the anharmonic plane curves*. The English literature on this subject is scarce (see **Reaves, 1914**, and the work cited there). However, the French and German literature is abundant (see **Fouret, 1874**; **Serret, 1886**, Vol. II pp. 425 – 432). We shall follow here the work of Mihăileanu (**Mihăileanu, 1972**). This author presents the anharmonic plane curves as curves defined by a homographic relation between the slope of the tangent and the slope of the secant in any one of their points, the secant being taken with respect to an arbitrary point in plane – the pole. This definition builds upon the main projective property ever to be considered for the astrometric observations: they give us *only projections*.

Assuming, for simplicity and in order to give a clear idea, that we take the pole for origin, in a system of coordinates (u, v), the condition from the definition of anharmonic curves can be put into equation in the form

$$v'(u) = \frac{av + bu}{cv + du} \tag{1}$$

where a, b, c, d are real coefficients. The characteristics of the 2×2 real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2}$$

more specifically, the *fixed points* of its homographic action and its *eigenvalues*, decide the shape of the curve described by equation (1). In order to show this, the best we can do is to integrate the equation (1). It can be put in the form

$$-\frac{1}{v}\frac{dv}{d\xi} = \frac{a\xi + b}{c\xi(\xi - \alpha_1)(\xi - \alpha_2)}; \quad \xi \equiv \frac{v}{u} \tag{3}$$

where $\alpha_{1,2}$ are the fixed points of the homographic action of the matrix from equation (2), i.e. the roots of the quadratic equation

$$c\xi^2 + (d - a)\xi - b = 0 \tag{4}$$

They are actually *the slopes of asymptotes* to the curve described by equation (1). In order to prove this, recall the definition of an asymptote: it is the tangent to curve in the point at infinity, i.e. for it the slope of the tangent to curve and that of the secant from the pole are the same:

$$\lim_{u\to\infty} v'(u) = \lim_{u\to\infty} \xi(u) \tag{5}$$

In view of the equation (1) this condition boils down to equation (4).

Therefore a plane anharmonic curve has always two asymptotic directions, with the slopes given by equation (4). The algebraic nature of the two roots of that equation decides the shape of the curve described by equation (1). If we denote $\omega_{1,2}$ the eigenvalues of the matrix (2), then we have the following relations between eigenvalues and the fixed points:

$$\omega_k \alpha_k = a\alpha_k + b; \quad \omega_k = c\alpha_k + d; \quad k = 1,2 \tag{6}$$

One can see therefore that the eigenvalues of the matrix have the same algebraical nature as the asymptotic slopes of the curve. With these we can easily integrate the equation (3).

Assuming that the matrix has *distinct eigenvalues*, the equation (3) gives the solution

$$\frac{|v - \alpha_2 u|^{\omega_2}}{|v - \alpha_1 u|^{\omega_1}} = K \tag{7}$$

with K an integration constant. Consequently the curve is a *real parabola* of degree ω_1/ω_2, if the eigenvalues *are real*. In cases where the eigenvalues *are complex*, we can write

$$\alpha_{1,2} = \alpha \pm i\beta \tag{8}$$

and the curve is a *logarithmic spiral*:

$$\rho = Ke^{-m\psi} \tag{9}$$

where K is a real constant, and we denoted

$$\rho^2 = \frac{(\beta u)^2 + (v - \alpha u)^2}{\beta}; \quad \tan\psi = \frac{\beta u}{v - \alpha u}; \quad m = \frac{d + \alpha c}{\beta c} \tag{10}$$

There is still another case, where the two eigenvalues of the matrix (2) *are identical*. Denote ω their common value. In this case, the anharmonic curve has only one asymptotic direction, say of slope α. The equation (3) can be written as

$$-\frac{1}{v}\frac{dv}{d\xi} = \frac{a\xi + b}{c\xi(\xi - \alpha)^2} \tag{11}$$

and can be integrated with the result

$$\ln|v - \alpha u| + \frac{\omega}{c}\left|\frac{u}{v - \alpha u}\right| = K \tag{12}$$

where K is, again, a real constant. This is a projection of an exponential function.

Thus, we got more than we bargained for: the anharmonic shapes contain not only the logarithmic spirals and the conic sections, obtained from them for m = 0 in equation (10), but also some other shapes, that don't seem to have obvious correspondent in the heavens. However, let's see where the geometry leads us.

THE CLASSICAL CONNECTION

Assume, in order to fix our ideas, that the parameter λ and the entries of the matrix **a** are constants. The equation (1) can be put into the form of a linear system:

$$|\dot{u}\rangle = \lambda \mathbf{a}|u\rangle; \quad |u\rangle \equiv \begin{pmatrix} u \\ v \end{pmatrix} \tag{13}$$

In order to make a connection with the classical dynamics we obviously need the second derivatives. These are, in our conditions

$$|\ddot{u}\rangle = (\lambda\mathbf{a})^2|u\rangle \equiv \mathbf{b}|u\rangle \tag{14}$$

In order to have the area constant a… constant, one needs that

$$v\ddot{u} - u\ddot{v} = 0 \tag{15}$$

This condition comes down to

$$b_{12}v^2 + (b_{11} - b_{22})uv - b_{21}u^2 = 0 \tag{16}$$

no matter of u and v. So the matrix **b** must be the identity 2×2 matrix, up to an arbitrary factor. Choosing the factor λ appropriately, we can have the matrix **a** in such a way that its square is the identity matrix:

$$\mathbf{a}^2 = \mathbf{e} \tag{17}$$

This means that the matrix **a** should have null trace and unit determinant. In adequate notations, it is

$$\mathbf{a} = \begin{pmatrix} -a_{12} & -a_{22} \\ a_{11} & a_{12} \end{pmatrix} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \tag{18}$$

Thus the system from equation (13) can be written as

$$(a_{11}u + a_{12}v)du + (a_{12}u + a_{22}v)dv = 0 \tag{19}$$

so that the corresponding curve is a conic section:

$$a_{11}u^2 + 2a_{12}uv + a_{22}v^2 = K \tag{20}$$

where K is an arbitrary constant. In this case the motion given by differential equation (19) is the Hamiltonian motion for which (20) is Hamiltonian, and even gives a law of conservation. We get the important result that, in order to have satisfied the area law, the anharmonic curves must be conic sections. Therefore the anharmonic motions are more general than the Kepler motions, in that they may or may not obey the area law. The first conclusion is that the galactic internal motions *should not be judged by the standards of the Kepler motions*. In particular, if the velocity profile doesn't fit the Keplerian standards, it doesn't necessarily follow that the mass is missing.

A Physical Connection Involving Initial Conditions

We do recognize, from what was just said, that all of the heavenly flat formations of matter we know visually or otherwise, can be geometrically described in connection with the anharmonic curves. In this description a 2×2 matrix with real entries is essential: it generates the motion kinematically describing the geometrical shape. We have a limited experience regarding the meaning of that matrix, and this amounts to its particular structure, as conferred by the classical dynamical treatment of the Kepler motion. Thus, classically, the coefficients of the matrix of the quadratic form from equation (20), should depend on the *physical parameters* of the matter assumed to generate the field, and also on the *initial conditions* of the motion. For instance, we cannot deal with ellipses if the initial velocities in the dynamical problem describing the Kepler motion don't have an *upper limit of their magnitude*.

Guided only by this case, we figure out that it makes sense to assume that a celestial motion is determined by a general 2×2 matrix with constant real entries. Then, in order to find the forces responsible for that motion, like Newton did, we need to find a relationship between the matrices characterizing the motion of different points from the constitution of a physical body. This amounts to starting the natural philosophy from the realistic assumption that a physical body is never a material point, but a collection of such material points. Only in limiting conditions can a body be considered a material point, and these conditions depend on the scale of the universe at our disposal.

Let, therefore, our matrix with real entries generating the anharmonic motion be the 2×2 matrix:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{21}$$

These coefficients contain, in a certain yet unspecified manner, both the physical parameters of the particle involved in motion, and the possible initial conditions of the motion. The set of these matrices can be assumed as germane to the physics of world matter, for instance as *a set of fundamental spinors* describing the matter in motion. This description is, for instance, analogous to that of skyrmions by means of the so-called "hedgehog ansatz" (**Manton, Sutcliffe, 2004**): it gives actually a family of matrices depending on two or three parameters.

In our case, the ansatz will say that a cosmic shape is described by the matrix from equation (21) through the differential equation:

$$v'(u) = \frac{\alpha v + \beta u}{\gamma v + \delta u} \tag{22}$$

This differential equation can be arranged in the form

$$(\delta + \alpha)(vdu - udv) = d\left\{\gamma v^2 + (\delta - \alpha)uv - \beta u^2\right\} \tag{23}$$

where we read an important conclusion. If the motion obeys the area law, then we have

$$vdu - udv = Adt \tag{24}$$

where A is the rate of area swept by the radius vector from the pole. The equation (23) can be hereby integrated with the result

$$\gamma v^2 + (\delta - \alpha)uv - \beta u^2 = (\delta + \alpha)At + B \tag{25}$$

Thus, when our spinor *is traceless*, the anharmonic motion it generates is indeed a conic section. Otherwise, if the spinor is not trace free, under the spell of the same area law, we still have a conic section, but with the parameters varying linearly in time – *an expanding conic section*. This should be the case of the Saturn rings or that of the asteroid belt from our solar system. Come to think of it: this is the real case of a revolving Kepler body in general, inasmuch as its constituent material points are all revolving around the same center of force in equidistant ellipses. A conclusion then imposes by itself: as the area law was discovered by observations upon extended bodies, it should be somehow a sort of average law also. Indeed, it is respected for the constituent particle as well as for their ensemble. The structural connections characterizing a solid cosmic body, or a body in general for that matter, should therefore take into account this important cosmological circumstance. It is reflected in physics nowadays only by speculations upon the inertial mass and charge of bodies. Mention should be made that in truth, reading the equation (25), the conclusion is that we have to do with a Keplerian motion proper in cases where the area rate A is zero, regardless of the matrix, or in cases where the matrix is traceless. The first case, however, represents just a radial motion. Only the second case represents an actual Kepler motion. Therefore, *the Kepler motion*, even if variable, *is indication of the fact that the area law is obeyed*. When considered as generated by a matrix, it corresponds to the case of a traceless matrix.

But the area law may not be respected in general. We can integrate the equation (23) directly, without the benefit of time, by referring it to the characteristics of the matrix (21). Let $m_{1,2}$ be the fixed points of its homographic action, and $\lambda_{1,2}$ its eigenvalues. Denoting

$$U = v - m_1 u; \quad V = v - m_2 u \tag{26}$$

the equation (23) can be reduced to the simple form

$$\lambda_2 \frac{dV}{V} - \lambda_1 \frac{dU}{U} = 0 \tag{27}$$

which can be integrated right away, with the result already obtained before:

$$\frac{V^{\lambda_2}}{U^{\lambda_1}} = K \tag{28}$$

where K is a constant. In the case where the matrix (21) has two identical eigenvalues, denoted λ, the equation (23) can be put in the form

$$(\delta + \alpha)(vdu - udv) = \gamma d(v - mu)^2 \tag{29}$$

where 'm' is the unique fixed point of the homographic action of matrix. Using the linear transformation

$$U = u; \quad V = v - mu \tag{30}$$

equation (29) becomes

$$\frac{\lambda}{\gamma} d\frac{U}{V} = \frac{dV}{V} \tag{31}$$

from which we can recover directly the result obtained above

$$\frac{\lambda}{\gamma}\frac{U}{V} = \ln\frac{V}{K} \tag{32}$$

where K is an integration constant. Encouraged by the idea that annular matter is quite natural among the anharmonic shapes, we venture to indicate a case where the exponential shape could be surely found. This is the case of *trailing matter*, for instance *the comets' tails*. Probably this shape could also be found, for instance, in Jupiter or Saturn structures.

Therefore, if we can assume that the 2×2 matrices are somehow germane to the problems of astrophysics, in the manner in which the spinors are considered fundamental, then they can generate all of the cosmic shapes of our visual experience, in the form of anharmonic curves. These are of the particular Kepler kind only in cases where the area law is obeyed, and even for these cases the Kepler motion proper is observed when the generating matrix is an involution. In that case the motion is Hamiltonian. In the cases where the area law is not satisfied and the matrix is not trace free, it generates spiral motions, parabolic motions as well as exponential motions. This shows, in particular, that Newton was right in treating the parabolic motion on a par with celestial motions. On the other hand, the exponential motions might be useful in astrophysics, for the description of the structure of comets, but mostly in cosmology, specifically in cosmogonical problems. They are certainly useful, for instance, in the theory of mass accretion, sketched even by Newton in his letters to Bentley. A particle falling toward an attracting mass illustrates such a case: it falls directly to the attracting mass while it is far from it, and then, in closer encounter, follows a curve having the direction of motion as asymptote.

CONCLUSIONS

Regarding our two tasks with this work, we have to report the following, already announced, results. The flat cosmic shapes of our perceived heavens are really near to kin with one another. Specifically, from a geometrical point of view *they are all anharmonic curves*. They can be Kepler orbits, or the like, only in case along them the area law is respected. In case the area law is not respected they can be logarithmic spirals or parabolas of different orders. From this point of view Newton was right in considering the ballistic motion as being of the same nature as the proper Kepler motion. However, the analogy is quite limited, inasmuch as for the ballistic motion the area law is not respected. But there is more to the conclusion than this unitary description of the flat heavenly and earthly shapes.

The 2×2 matrix generating the anharmonic plane shapes can have an objective meaning, inasmuch as the Newtonian dynamical description of the Kepler motion shows that it should be related to the initial conditions of the motion. From cosmological point of view this fact is very important. It means that the genesis of a flat cosmic formation can be read in its shape, at least for the cases where the area law is not satisfied. Apart from the known fact related to the age of galaxies by the Hubble sequence, this may mean that this way of appreciating the age of the galaxies has a theoretical basis just as firm, from the point of view of the natural philosophy, as the Newtonian forces themselves. Another point to be noticed here is that there is a class of anharmonic shapes that might be proper in astrophysical problems of mass accretion, or even in the cosmogonical problem of origin of the universe. On the other hand, for the cases where the area law is satisfied, i.e. for the annular cosmic formations, like the Saturn rings and asteroid belts, the distance with respect to center is a clear indication of the age of orbit. More to the point, the Saturn rings have the fine substructure recently discovered by the cosmic probes. By the same token we can also say that not all the asteroids, for instance, are of the same age.

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