A NEWTONIAN MESSAGE FOR QUANTIZATION

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Abstract. The dynamic equations related to Kepler motion are scale-invariant. This means that the dynamical model is universal: it works on the same principles at the micro level as well as at the macro level. Why then quantization? Is it telling something we couldn't read in the classical physics? The answer is negative: both the microcosm and the macrocosm show the same type of phenomena that could be taken into consideration by the classical theory. The only thing worth considering from the side of quantum revolution is the inspiration it could bring, in astrophysics for instance. This was lost, however, due to the artificial dichotomy of our spirit.

INTRODUCTION: A … LONG HISTORY

The history of scientific relationship between matter and light, in its modern view, starts with Heinrich Hertz. He is the one who succeeded in describing and realizing the action at distance in the form of electromagnetic signals, thus taking his place in history as the discoverer of *electromagnetic action at distance*. While this kind of action is never revealed by our senses and therefore one cannot say, for instance, that it is realized by forces, Hertz described it in the language of the Maxwell theory of electromagnetic waves, and this language is exclusively tributary to forces. The *Hertz dipole*, as the material structure capable of creating and receiving electromagnetic waves is usually called, was the cornerstone in all decisions on the material structures connected with the creation of light, starting very early in the last century. What are these material structures?

First of all it was the *electron*. It was discovered by J. J. Thomson in 1897 (**Thomson, 1907**). Based on this discovery, he proposed a model of matter. The first observation he took for granted was that the matter seems to be in a natural electrically neutral state. As the electron is charged with negative electric charge, J. J. Thomson advanced the idea that, if the matter is built-up from atoms, then these can be thought of as islands of negative charges in a confined continuum of positive charge. This is the so-called "plum pudding" model of the atom. Such an explanation of the structure of matter had, however, to be soon abandoned under the pressure of experimental facts, which its fundamental brick couldn't accommodate.

Towards the end of 19^{th} century became more and more obvious that the matter is unstable. Not from the known common viewpoint of chemistry but, we might say, from the point of view of alchemy: the elements themselves, therefore the atoms, are unstable, going over into other elements – the dream of alchemists! More to the point the heavy elements were decomposing themselves into lighter elements, with emission of three kinds of radiation, called by Ernest Rutherford alpha, beta and gamma. The first two of these proved to be radiations of particles, because they noticeably respond to electric and magnetic forces. The beta radiation was easily identified with the electrons of J. J. Thomson. The alpha radiation was electrically charged with positive charge, but was formed of particles much heavier than the electrons. These particles had the mass of a helium atom, the only difference being that they were electrically charged, while the helium atom was neutral. The gamma radiation was electrically neutral and very penetrant within matter. Its nature couldn't be decided quite so easily.

Mention should be made that here the natural philosophy used, maybe for the first time in the case of structural units of matter, the experimental logic ordinarily applied in experiments with light. Indeed, the quantitative properties of the units of matter, like those of light, are not offered directly to our senses, but through the intermediary of the effects induced by these units upon other parts of matter, which then are accessible to our senses. Thus, the experimental studies on the radiations mentioned above were done in the manner of J. J. Thomson, in high vacuum Crookes tubes, provided with electrodes, in order to create electrical deflecting fields. If the particle were not electrically charged they simply were not deflected, and that effect could be seen directly.

It was during such experimental studies, whose conclusions were drawn under the guidance of the Thomson atomic model, when Rutherford and Geiger noticed an interesting phenomenon. If one interposes a mica slate between the source of alpha radiation and the plate recording the particles, the recorded particle intensity is diminished. This means that not all of the alpha particles emitted by source succeeded in passing through matter; some are stopped within matter. It has immediately been initiated a study for quantitative evaluation of this effect in different materials, and probably just for the sake scientific completeness, mandatory through mere professional ethics, Marsden was asked to experimentally certify the fact that there are not scattering events at different angles with respect to the direction between the source of radiation and the target. This would completely confirm the theory based on the Thomson model of atom. The surprise was overwhelming: the alpha particles were scattered in any direction, even backwards! Of course, the majority of them went forward, however not all of them.

From these experimental results Rutherford figured out that the image of "plum pudding" for the atom is not quite correct. Much more realistic is the idea of an atom mostly empty, with the positive charge concentrated in a nucleus and the electrons at large distances from the nucleus. Because long time beforehand Charles Coulomb had already proved that the force between electric charges varies with the distance between them exactly like the Newtonian gravitational force, i.e. it is inversely proportional with the square of that distance, it was then only natural to assume that the electrons should move around positive nucleus just like the planets around Sun. Thus came out in the world the *planetary* or *nuclear model* of atom (**Rutherford, 1911**).

This was the historical moment when the theoretical physics happily met the astrophysics, a science in need of explanation of the process of producing light by the Sun as well as by the stars. The spectra of this light presented significant regularities, known as spectral series, with bright lines in definite positions in spectrum. The experimental physics had at its disposal means to associate the spectra of this light with chemical elements abundantly existing on Earth, especially the hydrogen, which seemed to be dominant in the spectral of celestial bodies. Therefore it was only natural to allocate the creation of light to the hydrogen atom in different environments, existing with certainty in Sun and stars.

As long as the light was conceived as an electromagnetic radiation, the situation was pretty clear and easy to explain: the hydrogen atom as imagined by Rutherford can produce naturally electromagnetic radiation, because its electron has acceleration and therefore it emits radiation just like a dipole antenna. Indeed, a Rutherford atom can be assimilated to a plane harmonic oscillator, which then can be treated as a Hertz dipole. The emission of radiation is then continuous. This is the point where it was discovered that the Rutherford atom doesn't work according to the usual classical rules. Indeed, the emission of radiation means emission of energy, and this cannot last but as long as this energy is available. When the energy is exhausted, the emission ceases. It will be therefore only natural that the atom vanishes due to the continuous emission of light, but there was no experimental fact to show that this is the case in reality.

By comparing the regularities of the hydrogen spectrum with the mechanical possibilities of the classical planetary model, Niels Bohr found indeed the answer in the energy of that model: it works in such a way that only certain orbits of the electron around nucleus are possible, and these have fixed energy (**Bohr, 1913**). *The light produced by an atom corresponds to some instantaneous transitions of the electron between different orbits, and carries with it the difference of energy of the orbits between which the transition is made*. In the construction of this image, Bohr used the idea, new in that epoch, that the light in such cases is not quite dense, and can be represented, according to Einstein, by an ideal gas of particles (**Einstein, 1905**), each one of them carrying a *quantum of energy*. In other words, at the atomic level, the light is emitted and absorbed in a form later baptized *photon* (**Lewis, 1926**).

Let's stop here with this long trip in history, in order to formulate actually our problem. From a classical point of view we have a dynamical model – the planetary model – under the demands of some new experimental facts. This model has to offer some reasons in order that those experimental facts be properly understood. The Bohr moment of knowledge shows that our spirit reacted by denying the classical way of thinking. Was it right in doing so? There are reasons to believe that the spirit was rather inconsistent with itself, in the sense that it has not used properly the concepts at its disposal. This work documents a part of these concepts and their use and misuse.

The Forces of Quantized Atom

In construction of the Rutherford model of atom, we have a typical example of analogy of knowledge which transcends the scale of contemplation of the universe. This analogy is first a consequence of the fact that, no matter of the scale of contemplation, the inverse square proportionality of forces with the distance is a property of some vacuum forces. Indeed, they were discovered between material points in vacuum, at two different scales of contemplation of the universe: planetary and secular scale. Only on the second place comes the problem that led to quantum mechanics. Indeed, the results that contradicted the experimental observations are obtained based on the classical equations of motion of the electron in atom. The first question, proper to ask, would then be: are we indeed allowed to assume these equations as valid to still another scale, not even accessible directly to our senses? In hindsight, the answer is yes: the dynamical equations of motion, for the specific type of force assumed in the planetary model, are explicitly invariant to a *scale transformation* for space and time (**Mariwalla, 1982**). Consequently, we are right in using the same equations of motion in two different worlds. We are therefore at liberty to use the classical equations of motion in order to describe the Rutherford atomic model. But then, the problem takes some interesting turns.

Still on the side of the classical theory we should then ask the very first question, whose answer generated the classical system itself: what kind of forces are responsible for the facts about the atomic system? Indeed, that was the question of Newton himself (*Principia*, Book I, Sections II and III): what kind of forces are responsible for the observed behavior of celestial bodies? This was the question that led to the discovery of the force inversely proportional with the square of distance in the first place. Isn't then just natural to question the cause first, in view of the fact that the observational data became richer? This was, indeed, the natural order of things, but it was destined to oblivion due to the impetus of technology.

Two works of Edwin Bidwell Wilson (**Wilson, 1919, 1924**) stand witness to the fact that Newtonian spirit was not completely lost into the avalanche of the new facts. Obviously, there are many more works on that side of spiritual movement, but we chose Wilson's works because they have a direct message for the triumphant quantization of today. In order to make our point we follow closely the second of the works cited above. This work is explicitly aligned to Newtonian natural philosophy, not only in that Wilson lists the facts that should lead to the expression of force responsible for them, but he is very critical about what should be considered as fact and what should be taken as product of imagination. But let's follow Wilson's idea.

The Wilson's list of facts we have to take into consideration for the classical treatments of the Rutherford atom is

$$m r \omega^2 = -F = dV/dr \tag{i}$$

$$m r^2 \omega = n \cdot \hbar \tag{ii}$$

$$E_2 - E_1 = 2\pi\hbar\nu \tag{iii}$$

$$\nu = N\left(1/n_1^2 - 1/n_2^2\right) \tag{iv}$$

$$E = V + m r^2 \omega^2 / 2 \tag{v}$$

The captions here are in Wilson's original order. The first condition is what he calls the "force condition", expressing the fact that for circular orbits the force is just the centripetal force deriving from a potential V, function only of the distance of the electron from the center of the orbit. The electron has mass 'm', angular velocity 'ω' and revolves on a circular orbit of radius r. The second condition represents "the quantum condition", whereby the kinetic moment is quantized. $\hbar$ is the (rationalized) Planck constant. The third condition is Bohr's "frequency condition", where ν is the frequency of light and 'E' the level of total energy of the orbit. The fourth condition is imposed by the "spectral law", where n_1 and n_2 are numbers, and 'N' is a constant (the Rydberg constant). The fifth condition is the expression of the total energy assignable to a certain orbit – the "energy equation".

Wilson recognizes that from among the five conditions listed above, only (iv) is an experimental fact. Two of them, namely (ii) and (iii), are simply hypotheses, while the other two, i.e. (i) and (v), are classically based definitions. We are now in position to accept these definitions, as they are part of a theory which is scale invariant: there should be nothing wrong with them. The assumptions, however, add constraints to the selection of possible forces responsible for the structure of the atom. Wilson inserts, as a last observation, the comment that the theory might not mean too much, but: "There is some advantage in replacing the hypothetical Coulomb law by an experimental fact." Truth be told, all the advantage should be there! However, the way in which the experimental fact enters our considerations about the force is not quite clean. In other words, the fact itself is not quite as pure as those which Newton had at his disposal, but obviously permeated by Bohr's hypotheses.

Wilson starts by trying to obtain an equation for the potential. First he notices that combining the hypothesis (iii) with the experimental condition (iv) a certain "conservation law" emerges, in the form

$$E_1 + \frac{2\pi\hbar N}{n_1^2} = E_2 + \frac{2\pi\hbar N}{n_2^2} \tag{1}$$

When combining this further with the definition (v), and using the fact that the potential energy is defined up to an additive constant, we have the equation

$$V + \frac{mr^2\omega^2}{2} + \frac{2\pi\hbar N}{n^2} = 0 \tag{2}$$

This equation is transformed by Wilson in a Clairaut-type equation, using the definition (i) and the quantization hypothesis (ii). It is

$$V = uV' + \frac{A}{V'}; \quad ur^2 = 1; \quad A \equiv \frac{\pi N\hbar^3}{m} \tag{3}$$

The *general solution* of this equation is function of a constant representing V':

$$V = Cu + \frac{A}{C} \tag{4}$$

Using the second of the equations (3), this gives a force going inversely with the third power of distance:

$$F = \frac{2C}{r^3} \tag{5}$$

The constant C can be calculated using, again, the definition (i) and the assumption (ii), so that the final result is

$$F(r) = \frac{\hbar^2}{m}\frac{n^2}{r^3} \tag{6}$$

Wilson gets therefore the important result that the experimental constraint (iv) combined with classical dynamics and the two Bohr hypotheses, *lead to the quantization of the magnitude of force*. The force is therefore quantized – not the orbit. However, this quantized force is not the

Coulomb force, but a force inversely proportional with the cube of the distance between the moving material point and the force center.

Nevertheless, the force from equation (6) is not the only solution of the Clairaut's equation (3); there is also the so-called *singular solution*, obtained by eliminating the constant C between (4) and its derivative with respect to C, when this derivative is zero:

$$V = Cu + \frac{A}{C} \quad \& \quad 0 = u - \frac{A}{C^2} \quad \Rightarrow \quad V(r) = 2\frac{\sqrt{A}}{r} \tag{7}$$

Obviously, this is the potential giving Newtonian forces inversely proportional with the square of distance, as it was to be expected due to the scale invariance of the classical equations of motion for the corresponding Kepler problem. The force will therefore include the Planck's constant in its final expression, but its value will not be quantized like the force having magnitude inversely proportional with the third power of distance:

$$F(r) = \frac{2\sqrt{A}}{r^2} \tag{8}$$

Here A is the constant defined in equation (3) above.

In conclusion, the new experimental fact asks for the existence not only of the force that generated the classical model, but also of an extra force whose magnitude varries inversely with the third power of the distance. The request is not classically direct but blemished by combining the fact with the hypotheses that later were taken as the sign of "new thinking". In spite of this though, we should take it in consideration. Indeed, the quantum hypotheses were insufficient as they were primarily formulated, and they have been subsequently improved. The two forces should then carry a fundamental message, inasmuch as they are closely related to the improved forms of the hypotheses. In fact, it turns out that these two forces were well known to Newton himself.

DISCUSSION OF THE TWO FORCES

If we would have to consider the classical problem of finding the force toward the center of the circular orbit, acting on the material point that moves along that orbit, the answer would be unambiguously that from equation (8). One can therefore see that the quantization conditions add a lot to the class of forces that might be compatible with them. While aware that two of those quantization conditions are simply assumptions, the results of Wilson deserves nevertheless a closer attention. They allow indeed unexpected connections, giving a clear message to modern formal quantization.

The two forces from equations (6) and (8) are special cases of *central forces*. First, their magnitude depends exclusively on distance; secondly they lead to special trajectories of the material points on which they act. The force from equation (6) generates trajectories having the shape of logarithmic spirals. These don't mean too much: intuitively a spiralling electron leads to the same conclusions as the exaustion of energy. However, the shape of the orbit is important for our argument. The Binet's equation for central forces is (**Whittaker, 1904**):

$$F(\vec{r}) = a^2\rho^2\left[\rho + \frac{d^2\rho}{d\theta^2}\right]; \quad \rho \equiv \frac{1}{r} \tag{9}$$

Here r and θ are the polar coordinates of the plane of motion, and ‘a’ is the area constant for that motion. Now, in the case of the force from equation (6) this equation reduces to

$$\frac{d^2\rho}{d\theta^2} + (a^2 - 2\sqrt{A})\rho = 0 \tag{10}$$

The resulting orbit is either a trigonometric spiral, that can be put in the form

$$r = \frac{r_0}{\cos(\lambda\theta + \theta_0)}; \quad \lambda^2 \equiv a^2 - 2\sqrt{A} \tag{11}$$

or a logarithmic spiral that can be put in the form

$$r = \frac{r_0}{\cosh(\lambda\theta + \theta_0)}; \quad \lambda^2 \equiv 2\sqrt{A} - a^2 \tag{12}$$

By the same token, the Binet equation corresponding to a force like that from equation (8) can be integrated, and shows that such force generates an elliptic orbits – a well known classical result. The parameters of that orbit are determined by the initial conditions of the motion. For the sake of completeness, we write it in the form

$$\frac{a}{r} = \frac{2\sqrt{A}}{a} + w_1\cos\theta + w_2\sin\theta \tag{13}$$

where w_1 and w_2 are the components of the initial velocity of the electron.

Nothing new over classical results until now, and were it for the question which ones of these orbits correspond to the quiet, stable atom, the classical dynamics would not be affected by anything. Indeed we know that the atom must emit radiation, and this would lead it to inevitable collapse. If the atom exists, and it is a classical structure, then there should be electronic orbits along which the electron doesn’t emit. Which one of the two orbits satisfies this requirement?

A TYPICALLY NEWTONIAN QUESTION

Therefore, we have to ask the question of stability of the atom within the classical framework. Given the energetical principle, we can discuss that stability in terms of the emitted radiation, as representing the new fact. As the radiation depends on the electronic orbit, the problem is to find that electronic orbit along which the radiation is not produced – the stable or “radiationless orbit”. Wilson has shown (**Wilson, 1919**) that the stable orbits *are not ellipses*, but they should be found among *the logarithmic spirals*. Indeed there is a class of spiralling electronic orbits along which the electron doesn’t emit radiation. Wilson judges the situation of the “radiationless orbits” by what is known today as radiative power of a nonuniformly moving charge (for a modern account of the electrodynamics involved here see **Jackson, 1998**, especially chapter 16). This power is null, therefore the charge doesn’t emit radiation in case the second time derivative of velocity is perpendicular to the velocity vector itself. In the Cartesian coordinates of the plane of motion this condition amounts to the differential equation

$$\dot{x}\cdot\dddot{x} + \dot{y}\cdot\dddot{y} = 0 \tag{14}$$

where the dot over a symbol represents the time derivative, as usual. According to this condition, an electron spiralling toward the center of force can, in some instances, be radiationless. Indeed, if the equations of motion of the electron are

$$x(t) = Ae^{-\lambda t}\cos(\mu t + \beta); \quad y(t) = Ae^{-\lambda t}\sin(\mu t + \beta) \tag{15}$$

then the dot product from equation (14) is

$$\dot{x}\cdot\ddot{x} + \dot{y}\cdot\ddot{y} = A^2(\lambda^4 - \mu^4)e^{-2\lambda t} \tag{16}$$

and becomes unconditionally zero for the cases where the ratio between λ and μ is one of the biquadratic roots of unity.

Assuming therefore a "fresh start" in the classical theory of atomic forces, as given by the radiationless condition (16), we are bound to discover a particular form of equations (15), viz.

$$x(t) = Ae^{-\varepsilon\mu t}\cos(\mu t + \beta); \quad y(t) = Ae^{-\varepsilon\mu t}\sin(\mu t + \beta); \quad \varepsilon^4 = 1 \tag{17}$$

This is therefore the classical expression of the electronic motion in a stable Rutherford atom. Now if, in a Newtonian spirit, we want to find what are the forces corresponding to these orbits, we just have to use the classical Binet formula in the reverse (see **Whittaker, 1904**; **Routh, 1898**). First eliminate the time from equation (17) in order to obtain the dependent variable 'u' from equation (9) as a function of the polar angle θ. We thus have

$$\rho = Be^{-\varepsilon\theta}; \quad B \equiv (Ae^{-\varepsilon\beta})^{-1} \tag{18}$$

Inserting this result into equation (9) gives the force responsible for determining the radiationless orbits of the electron

$$F(\vec{r}) = \frac{a^2B(1+\varepsilon^2)}{r^3} \tag{19}$$

In view of the fact that ε is the quartic root of unity, ε^2 should be the square root of unity, and therefore the equation (19) gives quite a comprehensive result, according to the very classic rules of natural philosophy. It shows that there are *two distinct radiationless cases*. First case occurs for $\varepsilon^2 = -1$, when the force is unconditionally zero, which is the obvious case of a free electron. According to the laws of classical dynamics this electron has no acceleration, and therefore cannot produce radiation. A second case occurs for $\varepsilon^2 = 1$, which is the proper case corresponding to spiralling orbits. There are two distinct possibilities for that force, because ε = ±1: one is for the invard spiralling, the other for outward spiralling.

This result shows that the atom does not emit radiation if the electron moves inward or outward along spiral orbits. Truth be told, this conclusion is totally in agreement with that based on the exaustion of energy. However, while by exaustion of energy we are led to conclude the atomic collapse, here our premise exactly the contrary. Not only this, but the non-radiation condition leads to a different premise from that introducing the Rutherford atom in the first place. Can the spiralling electron be part of a stable structure?

THE TWO FORCES IN HISTORY

The force varying inversely with the cube of distance was first deduced by Newton (*Principia*, Book I, Proposition IX). He certainly attached a great deal of importance to this law of force, due to another fact, known historically as the *problem of revolving orbits*. Only recently this problem started again attracting the attention of some theoreticians (**Lynden-Bell, 2006**). Here the force inversely proportional with the cube of distance is quite important, being a *transition force*: it actually characterizes the transition between two elliptic orbits. Newton himself, considered it as such (*Principia*, Book I, Proposition XLIV): a force which is difference between the force by means of which an elliptic orbit is described, and the force by means of which the same elliptic orbit, but rotated, is described. In modern terms (see **Lynden-Bell, 2006**; **Whittaker, 1917**; **Chandrasekhar, 1995**) if $r = f(\theta)$ is an orbit described under the force F, the the orbit $r = f(k\theta)$, where k is any constant, is described under the force

$$F'(\vec{r}) = F(\vec{r}) + \frac{c}{r^3} \tag{20}$$

where 'c' is a constant. This can be proved directly by starting from Binet's equation (9). What really changes in the case of primed system is the area constant (see **Whittaker, 1917**, p. 83).

The force from equation (6) is also the one assumed by Langmuir in his theory of static atoms (**Langmuir, 1921**). Probably Langmuir, who wanted to give rational basis to the static atom of Gilbert N. Lewis, followed the same line of argument as above in obtaining this force, specifically in order to avoid the quantum rules. G. C. Evans has shown that the two systems – Bohr's and Langmuir's – cannot be obtained from one another by a continuous contact transformations (**Evans, 1923**). They are, however, equivalent in a limited way so to speak: we have to limit the consideration to a *Bohr atom with circular orbits*. This brings us to the core of our argument.

EVOLUTION OF QUANTUM ASSUMPTIONS

Remember that there are two quantum hypotheses. In the form given by Bohr, these are totally unsecured, and actually proved insufficient along the time. The quantum condition related to energy still remained unsecured, but the one related to kinetic moment had to be improved in order to account *for revolving orbits* (**Sommerfeld, 1934**). By the beginning of the 20th century the problem of revolving orbits gained critical importance as related to the perihelion advance, for which the classical theory of Newton couldn't account correctly. This explaines the fact that Sommerfeld makes use of relativistic considerations in order to justify the quantum rules. However, fact is the the circular orbit is quite insufficient for quantum theory. No opinion has however been expressed toward the effect that it is also quite insufficient in rejecting the classical natural philosophy from the atomic realm.

SUFFICIENT CLASSICAL CONCEPTS: THE CASE OF ORBIT

Evans' observation is pointing therefore directly to a fundamental issue: *the circular orbits*. The incident reminds us that the development of science is almost exclusively done based on incomplete concepts. Probably this is inscribed in the very nature of science. The case of quantization is no exception: the classical theory was condemned only based on a circle, while the dynamics it endorses actually produces ellipses. By the choice of circle a confusion is made here between a physical parameter – the radius of orbit – and the radial distance.

In the case of ellipse we have at least three parameters, reflecting the size and the orientation of the ellipse. Therefore, a quantum jump between two trajectories would mean mathematically a transition between two triplets of numbers. As a matter of fact we can be more specific on this subject, even assuming that we have to do with material points. In Newtonian terms the "rotating orbits" are a special case of a family of orbits around *the same center of force*, located in a focus of the orbits. Given that focus, and taking it as the origin of the system of coordinates, the equation of the family of ellipses can be written in the form

$$r^2 = e^2(a_1 x + a_2 y + a_3)^2 \tag{21}$$

This equation is simply the definition of the eccentricity 'e': the ratio between the distance of a point of orbit to the focus, and the distance of the same point to the directrix corresponding to that focus. The equation of the directrix is understood here to be

$$a_1 x + a_2 y + a_3 = 0; \quad a_1^2 + a_2^2 = 1 \tag{22}$$

The orbits corresponding to the same center of force are therefore a four-parameter family. Consequently, when one says that "the electron has a quantum jump between two orbits", one really should understand that electron passes from a certain point of the orbit having the parameters (e, a_1, a_2, a_3) to a certain point of the orbit having the parameters (e', a'_1, a'_2, a'_3). The initial and final points are, again, completely arbitrary locations on the respective orbits.

Here the textbooks – especially the school manuals – usually depict the quantum jumps between circular orbits, suggesting that the real jump of the electron is between the point of the current orbit where the electron is located, and the closest point of the orbit on which it jumps. The image is even improved by depicting only the energy levels, as straight parallel lines, in order to forget about the orbit. This one became obsolete in quantum mechanics. The fact generated a great deal of discussion of principle, on the meaning of quantum mechanics and the kind of revolution it requires in thinking, etc. Indeed there is here a contradiction in terms in the first place: the original quantum condition is refering to jump between *orbits*, and is depicted as jump *between positions in space*. This portrayal was made possible due to the fact that the orbit of choice is a circle, and for the circle the position and the radial coordinate are coinciding. It is then hard to figure out anything else than the fact that the electron jumps between closest points.

And yet, there may be some rights in using this image of the "closest positions jumps", even within the classical mechanics. In order to show this, let's take the general equation of an orbit, represented as a conic section in the form

$$f(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \tag{23}$$

To make the path of operations more obvious, while formally simplify our calculations, we will use from now on the "Dirac notation". First, write equation (23) in the form

$$\langle x|\mathbf{a}|x\rangle + 2\langle a_3|x\rangle + a_{33} = 0 \tag{24}$$

where the notations are

$$\mathbf{a} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}; \quad |x\rangle \equiv \begin{pmatrix} x \\ y \end{pmatrix}; \quad |a_3\rangle \equiv \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} \tag{25}$$

These are the “ket” vectors. The corresponding “bra” vectors are obviously the transposed matrices. Another ellipse will have to be characterized here by another set of the five essential parameters from equation (23).

The ellipse (24) can be obtained by the integration of the following equations of motion (Hamilton equations)

$$|\dot{x}\rangle = \mathbf{I}(\mathbf{a}|x\rangle + |a_3\rangle); \quad \mathbf{I} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{26}$$

The vector from the right hand side of this equation is tangent to orbit. It makes therefore sense to ask, among other more complicated ones, the simple, purely Newtonian, question: what are the orbits corresponding to *the same direction of velocity*? Equation (26) shows that they are given by the following relation between the parameters of the orbits and the coordinates of the point:

$$d(\mathbf{a}|x\rangle + |a_3\rangle) = |0\rangle \quad \therefore \quad |dx\rangle + (\mathbf{a}^{-1}d\mathbf{a})|x\rangle + \mathbf{a}^{-1}|da_3\rangle = |0\rangle \tag{27}$$

The last of these equations represents a kind of geometrical “evolution” of the positions, that have the same velocity vector, in the sense that its integration will give the locus of those points. Let’s see what is that locus.

First calculate the matrix from the second equation; it can be written in the form:

$$(\mathbf{a}^{-1} \cdot d\mathbf{a}) = \begin{pmatrix} d\lambda - \dfrac{\omega_2}{2} & -\omega_3 \\ \omega_1 & d\lambda + \dfrac{\omega_2}{2} \end{pmatrix}; \quad \lambda \equiv \ln\sqrt{\det \mathbf{a}} \tag{28}$$

where $\omega_{1,2,3}$ are three differential 1-forms in the entries of the matrix **a**:

$$\omega_1 = \frac{a_{11}da_{12} - a_{12}da_{11}}{\Delta}; \quad \omega_2 = \frac{a_{11}da_{22} - a_{22}da_{11}}{\Delta}; \quad \omega_3 = \frac{a_{12}da_{22} - a_{22}da_{12}}{\Delta} \tag{29}$$

where Δ is the determinant of **a**. If the variation of the relative location of the center of ellipse, with respect to the center of force, is negligible when compared to the variation of the position of material point describing the ellipse, then the second of equations (27) can be assumed to be approximately ‘homogeneous’

$$|dx\rangle = -(\mathbf{a}^{-1}d\mathbf{a})|x\rangle \tag{30}$$

so that it can be integrated ‘in principle’ by an exponential procedure. One of the cases where the ‘principle’ works with fundamental results can be obtained when we refer the equations of the orbits to their asymptotic directions. These are complex, u±iv say, and the differential forms from (29) can be written as

$$\omega_1 = \frac{du}{v^2} \quad \omega_2 = 2\frac{udu + vdv}{v^2} \quad \omega_3 = \frac{(u^2 - v^2)du + 2uvdv}{v^2} \tag{31}$$

The equation of evolution (30) can therefore be written as

$$|dx\rangle = d(\ln\sqrt{\Delta^{-1}})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}|x\rangle + \begin{pmatrix} \dfrac{\omega_2}{2} & \omega_3 \\ -\omega_1 & -\dfrac{\omega_2}{2} \end{pmatrix}|x\rangle \tag{32}$$

Now, the exponentiation works perfectly in cases where the differentials in parameters are exact, and can be expresses as differentials of a certain parameter. Such cases can be obtained as follows. Obviously, the differential forms (29) give a coframe specific to SL(2, R) group algebra. This can even be made more explicit by the representation (31) of the differential forms. Indeed, the determinant of matrix from equation (32) is, up to a sign

$$\left(\frac{\omega_2}{2}\right)^2 - \omega_1\omega_3 \equiv \frac{(du)^2 + (dv)^2}{v^2} \tag{33}$$

and is obviously just the metric of the hyperbolic plane. Along the geodesics of this metric, taken in the form (**Guggenheimer, 1963**)

$$u(t) = u_0 + v_0 \tanh(t) \quad v(t) = \frac{v_0}{\cosh(t)} \tag{34}$$

the rates represented by the differential forms are constant:

$$\omega_1 = \frac{1}{v_0} dt \quad \omega_2 = 2\frac{u_0}{v_0} dt \quad \omega_3 = \frac{u_0^2 - v_0^2}{v_0} dt \tag{35}$$

Thus, if the entries of the matrix **a** are taken when u and v are given by equation (34), the equation (32) can be easily solved by exponentiation (**Bellman, 1960**) and gives

$$|x(t)\rangle = (\cosh t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}|x(0)\rangle + (\sinh t)\begin{pmatrix} \frac{u_0}{v_0} & \frac{u_0^2 - v_0^2}{v_0} \\ -\frac{1}{v_0} & -\frac{u_0}{v_0} \end{pmatrix}|x(0)\rangle \tag{36}$$

with appropriate initial conditions. The locus of these points can be obtained by eliminating the “hyperbolic time” ‘t’ between x and y. Obviously this gives a logarithmic spiral, or one of its transforms by projection.

In cases where the relative position of the center of trajectories varies significantly by comparison with the position of the points along the trajectories, the things get a little more complicated. In such cases it might help noticing the the variation of centers is almost always a stochastic process, and as a matter of fact there is a ‘parallelism’ between the variation of the centers and the variation of the trajectory itself. However, let’s not get into details that are irrelevant for the point we want to make here. Thus we have the following conclusion: assume all the possible elliptic orbits in a Newtonian central force. The points of different orbits characterized by the same direction of velocity follow a certain pattern in the plane of motion. In special conditions, this pattern is a logarithmic spiral.

SOMETHING EXTRA ABOUT $\mathcal{SL}$(2, R)

The above realization of the $\mathcal{sl}$(2, R) has a close connection with the quantum mechanics, as it was suggested to classical physics by its quantum-mechanical counterpart. Indeed, it was discovered toward the end of the last century (**Hannay, 1985**), apparently under the great impression made among physicists by the discovery of the quantum phase (**Berry, 1984**). Let’s briefly review the issue, as connected to the present task.

The algebraic structure is obvious if we differentiate exterior the 1-forms from equation (29). The result is

$$d\wedge\omega_1 = \frac{a_{11}}{\sqrt{\Delta}}\Omega;\quad d\wedge\omega_2 = \frac{2a_{12}}{\sqrt{\Delta}}\Omega;\quad d\wedge\omega_2 = \frac{a_{22}}{\sqrt{\Delta}}\Omega \tag{37}$$

where Δ is the determinant of the matrix **a** and **Ω** is the 2-form

$$\Omega \equiv \frac{a_{11}(da_{12}\wedge da_{22}) + a_{12}(da_{22}\wedge da_{11}) + a_{22}(da_{11}\wedge da_{12})}{\Delta^{3/2}} \tag{38}$$

This 2-form is closed: d∧**Ω** = 0. In fact it is exact. Indeed, it is the exterior differential of the 1-form

$$\Psi = \frac{a_{11}+a_{22}}{2\Delta^{1/2}}\, d\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right) \tag{39}$$

which is the Hannay angle proper (**Berry, 1985**) for this specific problem. Initially discovered in the problems related to the phase plane, it is nevertheless a universal tool wherever the families of conic sections are involved in the problems of physics.

CONCLUSIONS: THE MESSAGE PROPER

By his two quantum statements, Bohr went actually back at the times when Newton explained how the planetary system works: he explained how the atom works. In fact, both Newton and Bohr explained the very same structural model, but at different scales of contemplation of the universe. Newton's idea was perfect, and so was Bohr's. It is just that at the atomic level there should be a new explanation of the appearances, which seems to imply that the electron, unlike the planet in the planetary system proper, can have many replicas. In the case of a planet one cannot talk about replicas, it doesn't make sense. We can obviously talk of *many* planets, or of many stars in the case of galaxies, but this is an entirely different thing. The conclusion was that the Rutherford's atom is unstable only because *we assume that it works classically*, a fact that seemed at variance with reality. Although apparently dictated by the same type of forces, the mechanics of atom should therefore be different from Newtonian mechanics. It is, in point of fact, the future quantum mechanics developed by Heisenberg, Born, Jordan and Dirac, approximately a decade after Bohr.

However, a classical analysis of the atomic problem shouldn't be inappropriate, inasmuch as the dynamical model *per se transcends any scale*. As a matter of fact the analysis was performed by Wilson, and what was found? Well, there are two forces in the atom, not just one – the Coulomb force, the scale-analogous of the gravity. The other one, taken by itself, is responsible for the spiral motion. Wilson's results then can also be read as telling us the story of the way quantum jumps are taking place. Namely, the electron cannot jump from an orbit to another *but in points located along a logarithmic spiral*. In other words, the electron moves along an ellipse indeed, radiates uniformly and then moves along a spiral without radiating, moves again on an ellipse, then along a spiral, and so on. This is not a process of direct collapsing, but it can take ages in order to be accomplished. The force related to the spiral motion is quantized, not the orbit itself.

It was, indeed, this classical point of view that was missing when the Bohr quantum condition was critically ammended by Sommerfeld. As a matter of fact the ammendment was done by taking into consideration the problem of revolving orbits. This very problem involved, even from Newton himself, therefore from classical point of view, both types of forces discovered by Wilson. The chemists must have felt uneasy with the quantum jumps: this is why the static model of atom resurfaced again with Gilbert N. Lewis, and in order to be explained had to accommodate the quantum force of Langmuir.

These facts show that there shouldn't be that the spirit made a radical change in thinking by the "quantum revolution". The only revolution at this juncture can indeed be that we just realized that we have *to leave the concepts free from misappropriations*. The notorious case is that of orbits: an orbit is not necessarily a circle but, under the conditions of the classical dynamical model, it is an ellipse. If this is not taken into consideration, then we have to picture the quantum jumps as jumps between locations rather than jumps between orbits. This might not be bad at all, when it comes to the evolution of knowledge. For instance, we can imagine here the necessity of some hidden parameters in construction of a wave function: the position of the electron on a circular orbit is a random event. However, in reality this idea came painfully, as an opinion that had to be imposed upon the scientific community. As it were, it should be just a natural fact logically emerging from the theory, and should be valid no matter of the shape of the orbit.

However, if we are to consider the orbits as they were taken from Kepler, in order to tell us about forces, then the theory shows that the spirals are quite natural things, determined by the *transitions between orbits*: they represent the geometrical locus of the points where the transitions are made. They don't represent motions, but points of transitions. Consequently the force inversely proportional with the cube of distance, accidentally related to the spiral, is indeed a transition force as the classical theory conceived it. The transitions themselves are made in specific points of the elliptic trajectories. By this, the classical natural philosophy doesn't leave any room for hidden parameters. If there are indeed hidden parameters then they too have a precise place: the initial conditions of the Kepler problem.

But, it may seem like we are digressing. In point of fact we don't have the chance to "see" the spiralling orbits in the world of atom, because we cannot see there anything: the world of atom is out of reach for our senses. However, we can see the spirals in the sky, and a problem of our way of knowledge occurs. It is obvious that the model of Rutherford atom is obtained as a benefit of an analogy, but in its classical form this analogy operated in a unique sense: the atomic system is analogous to the solar system. We can rightly ask if the analogy shouldn't work also in reverse. Indeed, if the dynamical model transcends the scale, its theory should therefore be universal. To what extent then, the solar system is analogous to the atomic system? In other words, gaining experience on the model in the microcosm, can't we just improve our science of the macrocosm?

As the things show up nowadays, there is no hope indeed that the science will turn back to the reconsideration of the planetary model of the solar system, in order to reappraise Newton's ideas, as it rightfully should. According to our experience in the microscopic range, in such an “inverse” analogy, the essential connection with light would have to be taken into consideration. In this connection, something analogous to light would have to be produced by jumps of the planet from an orbit to another. Even if we don’t currently know what is that “something”, sudden jumps in planetary orbits are out of question: we have never noticed such events in our recorded experience. Thus, according to current opinion such an analogy, “inverse” to that which led to the Rutherford model for atom, can never be carried out. Indeed in the Kepler model for the solar system a connection with light in the manner of Bohr is unconceivable. For, the current scientific opinion is that light is electromagnetic by its nature and therefore it is intimately connected with the dynamics of electric charges. Therefore the Bohr’s rules have little or nothing in common with the solar system.

And yet, there is a struggle of our spirit to give quantization rules in the macroworld (**Christianto, 2006**), and we don’t think it is vain. Its task may be misplaced for the moment, but this is indeed due to the first perception of quantization and its inappropriate relation with the classical dynamics. Indeed, in the astrophysical world the spiral is an ordinary, apparently stable structure. True, it can be explained up to a point by density waves, for instance by a coincidence between the maximum density of the elliptic orbits and the maximum density of matter, but it can also be explained as a motion. However, this remains a purely qualitative association, because there is nothing to explain *what forces matter to follow just the spiral pattern*. There is therefore, again, room for some “hidden parameters”, we can think of as compelling the matter to follow just the logarithmic spiral, while orbiting gravitationally or electrically.

The present work then is capable to throw a bridge between past and present, between astrophysics and quantum theory. The struggle of spirit, we are talking about, goes back to the times when the internal measurements of the spiral nebulae were just done by Adriaan van Maanen. J. H. Jeans was the first to jump into a comprehensive theoretical analysis, plainly Newtonian style (**Jeans, 1923**). His conclusion was that there can be *no motion along the spiral arms of the nebulae*, at least it couldn’t be consistently read in the data of van Maanen. Then E. W. Brown found a vital information in that data (**Brown, 1925**): van Maanen’s data showed that *the vector speed at the spiral arm is tangent to the arm*. This suggests that the arm itself is the envelope of a family of gravitational orbits, just as we presented above. Therefore, we can think of the logarithmic spiral as of a quantum effect at the macro scale. The struggle for macroquantization is therefore not vain. Only the model chosen for analogy with the atom shouldn’t be the planetary model, but the galaxy (**Popescu, 1988**).

In hindsight, the only case we can think of, where the classical models were directly inspired by quantum findings, was the discovery of the Hannay angle (**Hannay, 1985**), as inspired by the quantal Berry phase (**Berry, 1984**). As we have shown here, the Hannay angle also bears witness of the first moment in history when the data about the cosmic forces were indeed showing a new fact about these forces: the Bohr moment. It is the moment that made us think of quantization. However, the way we do it shouldn’t be a revolution, but smooth logical consequence of some sound scientifical principles which, by the way, didn’t seem to be missing to Newton himself. And while we are on the subject, we might as well ask ourselves what surprises has the classical theory in store for us. Like, for instance: are, the classical forces, having magnitude exclusively depending on distance, or this is, again, an artifact that should require a "revolution" sometimes?

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