Smarandache’s Orthic Theorem

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Abstract In this paper we present the Smarandache’s Orthic Theorem in the geometry of the triangle.

Keywords Smarandache’s Orthic Theorem, triangle.

§1. The main result

Smarandache’s Orthic Theorem

Given a triangle $ABC$ whose angles are all acute (acute triangle), we consider $A'B'C'$, the triangle formed by the legs of its altitudes.

In which conditions the expression:

$$||A'B'|| \cdot ||B'C'|| + ||B'C'|| \cdot ||C'A'|| + ||C'A'|| \cdot ||A'B'||$$

is maximum?

Proof. We have

$$\triangle ABC \sim \triangle A'B'C' \triangle AB'C' \sim \triangle A'BC'. \quad (1)$$

We note

$$||BA'|| = x, ||CB'|| = y, ||AC'|| = z.$$

It results that

$$||A'C'|| = a - x, ||B'A'|| = b - y, ||C'B'|| = c - z.$$

$$\overline{BA'C} = \overline{B'A'C'} = \overline{BA'C'}; \overline{AB'C} = \overline{AB'C'}; \overline{B'C'A} = \overline{B'C'A}.$$

From these equalities it results the relation (1)
\[ \triangle A'B'C' \sim \triangle A'B'C \Rightarrow \frac{AC'}{a-x} = \frac{x}{\|A'B'\|}, \quad (2) \]

\[ \triangle A'B'C \sim \triangle AB'C' \Rightarrow \frac{AC'}{z} = \frac{c-z}{\|B'C'\|}, \quad (3) \]

\[ \triangle AB'C \sim \triangle A'B'C \Rightarrow \frac{BC'}{y} = \frac{b-y}{\|A'B'\|}, \quad (4) \]

From (2), (3) and (4) we observe that the sum of the products from the problem is equal to:

\[ x(a-x) + y(b-y) + z(c-z) = \frac{1}{4}(a^2 + b^2 + c^2) - (x - \frac{a}{2})^2 - (y - \frac{b}{2})^2 - (z - \frac{c}{2})^2, \]

which will reach its maximum as long as \( x = \frac{a}{2}, y = \frac{b}{2}, z = \frac{c}{2} \), that is when the altitudes’ legs are in the middle of the sides, therefore when the \( \triangle ABC \) is equilateral. The maximum of the expression is \( \frac{1}{4}(a^2 + b^2 + c^2) \).

\section*{§2. Conclusion (Smarandache’s Orthic Theorem)}

If we note the lengths of the sides of the triangle \( \triangle ABC \) by \( \|AB\| = c, \|BC\| = a, \|CA\| = b \), and the lengths of the sides of its orthic triangle \( \triangle A'B'C' \) by \( \|A'B'\| = c^*, \|B'C'\| = a^*, \|CA'\| = b^* \), then we proved that:

\[ 4(a^*b^* + b^*c^* + c^*a^*) \leq a^2 + b^2 + c^2. \]

\section*{§3. Open problems related to Smarandache’s Orthic Theorem}

1. Generalize this problem to polygons. Let \( A_1A_2 \cdots A_m \) be a polygon and \( P \) a point inside it. From \( P \) we draw perpendiculars on each side \( A_iA_{i+1} \) of the polygon and we note by \( A'_i \) the intersection between the perpendicular and the side \( A_iA_{i+1} \). A pedal polygon \( A'_1A'_2 \cdots A'_n \) is formed. What properties does this pedal polygon have?

2. Generalize this problem to polyhedrons. Let \( A_1A_2 \cdots A_n \) be a polyhedron and \( P \) a point inside it. From \( P \) we draw perpendiculars on each polyhedron face \( F_i \) and we note by \( A'_i \) the intersection between the perpendicular and the side \( F_i \). A pedal polyhedron \( A'_1A'_2 \cdots A'_p \) is formed, where \( p \) is the number of polyhedron’s faces. What properties does this pedal polyhedron have?

\section*{References}


