Lorentz Contraction of Space and the Gravitational Field

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By considering the gravitational field as an optical medium with a radially-dependent index of refraction, we are able to show that a physical model of space being radially compressed by mass, rather than curved by mass, as in general relativity, yields the same results predicted by Einstein's theory of general relativity. We are further able to show that this spatial compression is equivalent to the Lorentz contraction of special relativity. The predictions of general relativity are all derived with relatively basic mathematics without reliance on the grossly complex Riemannian geometry needed for Einstein's curved space-time model.

Keywords: General Relativity, Special Relativity, Gravitation, Lorentz Contraction, Gravitational Lensing, Index of Refraction, Refractive Index

I. Refractive Index of the Gravitational Field

It is relatively easy to show that the gravitational field acts as an optical medium having a radially-dependent index of refraction. The analysis begins with Fermat's principle for the propagation of light in a static gravitational field [1]:

$$\delta \int g_{00}^{-1/2} dl = 0 \qquad (1)$$

where *dl* is the local length of a light beam measured by an observer at distance *r* within a gravitational field (where *r* is also the radial distance from the gravitational mass *M*), and g_{00} is a component of the metric tensor $g_{\mu\nu}$. In the above, $g_{00}^{-1/2} dl$ is an element of the optical path length, such that $g_{00}^{-1/2} dl = \frac{dt}{d\tau}$, where $d\tau$ represents time measured by the local observer for the light beam passing through length *dl*, and *dt* is the time measured by the observer an infinite distance away.

Equation (1) can then be rewritten as

$$\delta \int g_{00}^{-1/2} dl = \delta \int \frac{dt}{d\tau} dl = \delta \int \frac{dt}{d\tau} \frac{dl}{ds} ds = \delta \int n ds = 0 \qquad (2)$$

where *ds* is the length measured by the observer at infinity. If the scale of both length and time measurements made at infinity are set as the standard scale for all of gravitationally-influenced space, then propagation of light through the gravitational field satisfies Fermat's principle with space having a refractive index given by

$$n = \frac{dt}{d\tau} \frac{dl}{ds} = n_1 n_2 \qquad (3)$$

where n_1 is the time-related index (i.e., the index of refraction component corresponding to time dilation), and n_2 is the space-related index (i.e., the index of refraction component corresponding to length contraction).

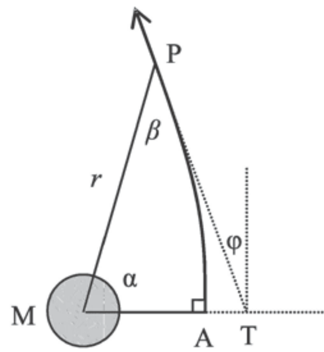

Fig. 1: Light deflection in a radially-dependent index of refraction

Fig. 1 illustrates refraction of light through a medium having this refractive index. In Fig. 1, the light beam travels from point A to point P, with β being the angle between the position vector $\boldsymbol{r}$ and a tangent at point P on the curved light beam, and α being the angle between position vector $\boldsymbol{r}$ and the horizontal (from the center of mass M to point A in this Figure). φ represent total angular deflection from the initial point A to the final point P. Assuming spherical symmetry for the most basic gravitational case, then we know that $nr\sin\beta$ must equal some constant [2], or

$$nr\sin\beta = n_0 r_0 \qquad (4)$$

where n_0 and r_0 are the index of refraction and the radius at point A, respectively. Since $\tan\beta = \frac{rd\alpha}{dr}$, then equation (4) yields:

$$d\alpha = \frac{dr}{r\sqrt{\left(\frac{nr}{n_0 r_0}\right)^2 - 1}} \quad (5)$$

which we may compare to the known solution [3], [4]

$$d\alpha = \frac{dR}{R\sqrt{\left[\left(\frac{R}{R_0}\right)^2\left(1 - \frac{2GM}{R_0 c^2}\right) + \frac{2GM}{rc^2}\right] - 1}} \quad (6)$$

where R is the radial coordinate of the Schwarzschild metric.

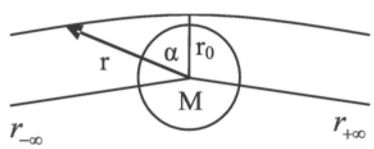

Fig. 2: Light deflection in a gravitational field

For a relatively weak gravitational field (i.e., $\frac{GM}{rc^2} << 1$), then $R = re^{GM/rc^2}$ and $n = e^{2GM/rc^2}$. For the light beam passing by massive object M in Fig. 2, the angular displacement of radial vector $\boldsymbol{r}$ is given by

$$\Delta\alpha = 2\int_{r_0}^{\infty} \frac{dr}{r\sqrt{\left(\frac{nr}{n_0 r_0}\right)^2 - 1}} \qquad (7)$$

where r_0 is the nearest distance to the center of massive body M. Substitution of n into equation (7) yields, to first order, $\Delta\alpha = \pi + \frac{4GM}{r_0 c^2}$. Thus, the total deflection angle of light in the gravitational field is given by $\Delta\varphi = \Delta\alpha - \pi = \frac{4GM}{r_0 c^2}$. Thus, beginning with the idea that space acts like an optical medium with a radial-dependent index of refraction n, the deflection angle of the light beam is found to be consistent with that predicted by general relativity [5], [6] and confirmed by experiment [7].

II. Relative Density of Space

In a general optical medium, the refractive index n is related to the fluid density by the Clausius-Mosotti equation which, for a gas, reduces to the simpler for of the Gladstone-Dale equation, given by

$$n - 1 = K\rho \qquad (8)$$

where ρ is the gas density and K is the Gladstone-Dale constant (having units of $1/\rho$), which is specific to the particular gas and depends weakly on the wavelength of the light [8].

Exploring the idea that space might have a "density" associated with it, substitution of $n = e^{2GM/rc^2}$, derived in section I, yields a density of $\rho = \frac{2GM}{c^2 Kr}$. We must now consider what this "density" represents. In a gas, the density is a representation of compression of the gas molecules. In the gas of a gravitational field, there are no "molecules" to consider, however there is a compression that we are familiar with: the spatial compression (and corresponding time dilation) of general relativity. $\rho = \frac{2GM}{c^2 Kr}$ can be considered an "effective density" of space if space is "compressed".

If we consider a distance measurement x_0, made in a region of space with no gravitational effects, and a distance measurement x representing the measurement made under gravitational spatial compression, then a corresponding unitless "density", measuring the degree of compression, can be given by $\rho = \frac{x_0 - x}{x}$. If we substitute a wavelength of light into this expression, we get $\rho = \frac{\lambda_0 - \lambda}{\lambda}$ or

$$\frac{\lambda_0 - \lambda}{\lambda} = \frac{2GM}{c^2 Kr} \qquad (9)$$

where λ is the "compressed" wavelength of light in the gravitational field, and λ_0 is the wavelength of the same light beam under no gravitational field (i.e., at infinity).

Equation (9), however, can be rewritten as

$$\lambda = \frac{\lambda_0}{\frac{2GM}{c^2 Kr} + 1} \qquad (10)$$

which is the exact formula for gravitational redshift given by general relativity if K is equal to 2.

Both gravitational redshift and gravitational lensing, exactly as predicted by general relativity, have been reproduced above from the perspective of a radially-dependent optical medium consisting of radially-dependent compressed space. In other words, rather than space being "curved" under the influence of a massive body, the same results may be produced by considering space as "compressed" by mass. It should be noted that the third "prong" of general relativistic predictions, the advance of the perihelion of Mercury, can also be predicted by a purely optical analog for space, independent of the concept of spatial curvature [9].

III. Lorentz Contraction and Relation to Special Relativity

From the above, we can see that a length measurement *x* made at a radius *r* from a massive body *M* is given by

$$x = \frac{x_0}{\frac{GM}{c^2 r} + 1} \quad (11)$$

where x_0 is the length measurement when *M* is zero (or if r is at infinity). In other words, the mass *M* can be viewed as causing a radially-dependent spatial compression, resulting in the same effects as those predicted by the geometric curvature of space of general relativity. Although it is common in relativity to discuss the variation in lengths of a theoretical "meter stick" (or variation in rates of time of a "clock"), it is important to keep in mind that it is not a physical meter stick which is contracting in length, it is space itself which experiences the compression (or time itself experiencing dilation) and not the physical body of the test object.

The gravitational situation is not the only place in relativity where spatial contraction can be found, of course. Special relativity provides Lorentz contraction following $x = x_0 \sqrt{1 - \frac{V^2}{c^2}}$, where x_0 represents the rest mass of the body and *V* is its velocity. The body in a gravitational field (relating to equation (11)) obviously has a potential energy associated with it, and this potential energy is linked to the spatial contraction. The body under motion, experiencing Lorentz contraction, obviously has a kinetic energy associated with it, and this kinetic energy is also linked to the spatial contraction. If we consider that the gravitational spatial contraction is the same effect as Lorentz contraction, and both are the result of the energy of the body, then we find that an object in a gravitational field caused by massive body *M*, at a distance *r*, but not moving in that field, experiences an equivalent length contraction if the object is moving at a velocity of $V = \sqrt{\frac{2GM}{r}}$ outside of gravitational effects. This comes from setting the potential energy of the former, given by $\frac{GMm}{r}$ equal to the kinetic energy of the latter, given by $\frac{1}{2}mV^2$, where *m* is the mass of the object.

From special relativity, we expect a mass increase for the moving body. Approaching this mass increase from the view that the increase in measured mass is actually the mass-equivalent of the object's kinetic energy, we can write

$$m = m_0 + \frac{1}{c^2}\left(\frac{1}{2}m_0 V^2\right) \qquad (12)$$

where m_0 is the object's rest mass. If we substitute $V = \sqrt{\frac{2GM}{r}}$ into equation (12), the result is

$$m = m_0 + \frac{GMm_0}{c^2 r} \qquad (13)$$

which is exactly what we expect (since equation (10) is equivalent to $\nu = \nu_0\left(1 + \frac{GM}{c^2 r}\right)$, where ν is the frequency corresponding to wavelength λ, and ν is directly proportional to the photon's energy, which is also directly proportional to its mass equivalent).

We can further write equation (13) as $m = m_0\left(1 + \frac{GM}{c^2 r}\right)$, but we have shown above that there is an equivalence between the contraction of equation (11) with the Lorentz contraction

$x = x_0\sqrt{1-\frac{V^2}{c^2}}$, thus we can substitute the special relativistic $\gamma = \sqrt{1-\frac{V^2}{c^2}}$ for the gravitational $\frac{1}{1+\frac{GM}{c^2 r}}$, which yields the familiar special relativistic equation $m = \frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}}$.

IV. Conclusion

Thus, we have shown that treating a massive body as radially compressing space, rather than causing a curvature in space, yields the same results as those predicted by general relativity. However, this compression of space is further shown to be equivalent to the Lorentz spatial contraction of special relativity, thus bringing both special relativity and gravitation into conformance by relying solely on relatively simple physical principles, rather than the immense mathematical complexities of Riemannian geometry.

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