Arithmetic information in particle mixing

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Abstract

Quantum information theory motivates certain choices for parameterizations of the CKM and MNS mixing matrices. In particular, we consider the rephasing invariant parameterization of [1], which is given by a sum of real circulant matrices. Noting the relation of this parameterization to complex rotation matrices, we find a reduction in the degrees of freedom required for the CKM matrix.

1 Introduction

Many parameterizations for a unitary mixing matrix V exist. The standard one uses three rotation angles plus a CP phase. In 1985, Jarlskog [2] defined the natural CP invariant J in terms of the complex entries of V. The invariant is fixed by all rephasing of rows and columns in V, begin given by

$$J \equiv \mathrm{Im}(V_{ij} V_{kl} \overline{V_{kj} V_{il}}) \tag{1}$$

for any choice of 2×2 submatrix. A rephasing invariant parameterization for V was given in [1], but in it the Jarlskog invariant is not directly related to the complex entries of V, being instead a function of the entry norms. These norms are neatly expressed as a sum of real circulant matrices.

Since the Standard Model cannot itself elaborate on the CKM or MNS parameters, we look to quantum information theory for guidance. Complex 3×3 circulant matrices are fundamental to information theory for the following reasons. For Hilbert spaces of finite dimension d, the discrete (or quantum) Fourier transform operator F_d [3] may be considered one of a set of *mutually unbiased* bases [4][5][6]. In dimension 3,

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \overline{\omega} & 1 \\ \overline{\omega} & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{2}$$

where $\omega = \exp(2i\pi/3)$ is the complex cubed root of unity. The columns of F_3 form a basis, namely the eigenvector set of a 3×3 Pauli matrix. Two bases are unbiased if all possible inner products between two elements, one in each basis, have the same norm. A collection of mutually unbiased basis sets is one such that every basis is unbiased with respect to each other one. In dimension 3, there are four such sets, two of which are given by F_3 and the identity matrix I_3. This generalises the three Pauli matrices to dimension 3.

In any prime dimension d, a complete set of mutually unbiased bases for the Hilbert space defines a special set of $d+1$ unitary operators. For $d = 2$, a complete set of three mutually unbiased bases is given by

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{3}$$

and the identity matrix I_2. Observe that $R_2^8 = I_2$ makes R_2 a unitary root of the identity I_2, representing the phase $\pi/4$. In any dimension d, such a circulant R_d defines a basis that is mutually unbiased with respect to F_d [7][8]. For dimension 3, the complete set of four mutually unbiased bases is given by the collection $\{F_3, R_3, R_3^2, I\}$ where

$$R_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \overline{\omega} & 1 \\ 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \end{pmatrix}, \tag{4}$$

is a unitary root such that R_3^3 is a multiple of I_3 and $R_3^{12} = I_3$. The Fourier operator naturally satisfies $F_d^4 = I_d$.

No non trivial mixture of I_3 and F_3 can reproduce a unitary matrix, so we look at mixtures of the set $\{I_3, R_3, {R_3}^2\}$ of circulants. The circulant form of the rephasing invariant V in [1] suggests that complex circulants might provide an interesting reduction of the parameter set required for V. Note that in some sense a mixing matrix should be a deformation of F_3, because the circulant Hermitian mass matrices of Koide form [9] are diagonalised by F_3.

Consider now a general circulant. By definition, an m-circulant matrix is specified by its first row, the other rows being equal to the first except for a shift of each entry m places to the right. Thus the discrete Fourier transform diagonalises all 1-circulants. For 3×3 matrices, the 2-circulants are co-diagonalized by F_3. That is, a codiagonal is a 2-circulant M with non zero entries M_{13}, M_{22} and M_{31}. The sum of a diagonal and a codiagonal essentially results in a matrix with 2×2 and 1×1 blocks. It follows that the

F_3 transform of a sum $A+B$, where A is a 1-circulant and B is a 2-circulant, results in a matrix with 2×2 and 1×1 blocks.

Consider a 3×3 unitary matrix of the form $A + iB$, for A and B real. For example, the block form

$$R_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix} \tag{5}$$

embeds the circulant R_2 in dimension 3. The operators R_{ij} are similarly defined for $(ij) = (12)$ and (31). Note that the qubit 1-circulant R_2 in dimension 2 must become a 1-circulant and 2-circulant sum in dimension 3, and we demand that all 3×3 rotation matrices take a similar form. The idea is that the particle generations which index our matrices should correspond to Pauli directions in space, and impose perfect cyclicity in dimension 3.

The cyclic product $R_{12}R_{23}R_{31}$ is also a circulant sum of the form $A+iB$. More general decompositions of the form $A + iB$ use factors like

$$R_{23}(r) = \frac{1}{\sqrt{r^2+1}} \begin{pmatrix} r+i & 0 & 0 \\ 0 & r & i \\ 0 & i & r \end{pmatrix} \tag{6}$$

for a real parameter r. Note that the contributions of the two circulants to the resulting set of nine probabilities do not interfere, since the norm is just the sum of square entries. These triple products leave the probability set invariant under permutations of rows and columns. Define a normalised three parameter product by

$$M = NR_{12}(a)R_{23}(b)R_{31}(c), \tag{7}$$

where a, b and c are real, and

$$N^{-2} = (a^2+1)(b^2+1)(c^2+1) \tag{8}$$

is the normalisation constant. As a circulant sum, M then takes the form

$$\begin{aligned} M = N & \begin{pmatrix} abc & -a-c & -b \\ -b & abc & -a-c \\ -a-c & -b & abc \end{pmatrix} \\ +iN & \begin{pmatrix} bc & ac-1 & ab \\ ac-1 & ab & bc \\ ab & bc & ac-1 \end{pmatrix}. \end{aligned} \tag{9}$$

Each $R_{ij}(r)$ factor is just a complex rotation matrix, but it is convenient to use the single tangent parameters. Thus M resembles the rotation part of the standard parameterization for the CKM matrix, while the CP phase is missing. This is the general form for a unitary cyclic decomposition, because one is always free to scale the imaginary entries of R_{ij} to unit norm. It is similar, for instance, to the tribimaximal extension considered in [10] for the MNS matrix.

For a normalised matrix of the form $A + iB$, the Fourier transform of $A + iB$ results in a 2×2 block of determinant 1 and a 1×1 block that is the row sum phase. That is, the complex matrix M is transformed into an element of $SU(2) \times U(1)$. In the triple product (7) the ordering of the three parameters matters. This is in contrast to a two parameter product, where a swapping of factors results in a transpose but does not alter the final set of norms.

Section 2 defines the known rephasing invariant mixing matrix [1], which is a real sum of circulants. Finally we look at the empirical MNS and CKM mixing matrices, noting that physical CKM values are recovered by the complex circulants with only three parameters.

2 The Rephasing Invariant Parameterization

In [1], the six invariants $\Gamma_{ijk} = V_{1i}V_{2j}V_{3k}$ are analysed. The Jarlskog invariant J gives the imaginary part of all the Γ_{ijk}. This leaves six real numbers to parameterise the norms of a mixing matrix, as in

$$\begin{pmatrix} |V_{11}|^2 & |V_{12}|^2 & |V_{13}|^2 \\ |V_{21}|^2 & |V_{22}|^2 & |V_{23}|^2 \\ |V_{31}|^2 & |V_{32}|^2 & |V_{33}|^2 \end{pmatrix} = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix}, \tag{10}$$

which is a sum of a 1-circulant and a 2-circulant. Simple constraints, such as $\det V = 1$, reduce this set to the four parameters required for a 3×3 unitary matrix. Moreover, we have

$$J^2 = x_1 x_2 x_3 - y_1 y_2 y_3. \tag{11}$$

Let us compare these parameters to the three parameters of (9), assuming that special mixing matrices take this reduced form. First set $x_i = N^2 {X_i}^2$ and $-y_i = N^2 {Y_i}^2$, for $i = 1, 2, 3$. Then we have

$$X_1 = abc \quad X_2 = -a - c \quad X_3 = -b \tag{12}$$

$$Y_1 = bc \quad Y_2 = ac - 1 \quad Y_3 = ab.$$

From (11),

$$J^2 = [(X_1X_2X_3)^2 + (Y_1Y_2Y_3)^2]N^6 \qquad (13)$$
$$= a^2b^4c^2(a^2+1)(c^2+1)\cdot N^6,$$

so that

$$J = \pm\frac{ab^2c}{(a^2+1)(c^2+1)\cdot(b^2+1)^{3/2}}. \qquad (14)$$

The question is, can the set (a, b, c) actually recover J? This would reduce the number of parameters in a mixing matrix from four to three. Perfect cyclicity is an appealing theoretical ansatz, so we apply it to the empirical mixing matrices in the next two sections.

The cofactor variables $x_i + y_j$ in [1] correspond to our cofactors $({X_i}^2 - {Y_j}^2)N^2$, up to a phase. This is a close connection between the full rephasing parameterization and the reduced complex form. The determinant condition from the real matrix is

$${X_1}^2 + {X_2}^2 + {X_3}^2 - {Y_1}^2 - {Y_2}^2 - {Y_3}^2 = N^{-2} \qquad (15)$$

and the secondary constraint is

$$\sum_{i\neq j, i<j} {X_i}^2{X_j}^2 = \sum_{i\neq j, i<j} {Y_i}^2{Y_j}^2. \qquad (16)$$

These constraints are not necessarily satisfied by the empirical matrices considered below, which typically have non trivial phases as determinants. However, one is free to rephase any unitary matrix without affecting the Jarlskog invariant or any other physical quantity.

3 The CKM Quark Mixing Matrix

Experimental estimates [11] of the unsquared CKM amplitudes are given by

$$\|V\| = \begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.001 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.001 & 0.97334 \pm 0.00023 & 0.0415 \pm 0.0011 \\ 0.00874 \pm 0.00026 & 0.0407 \pm 0.001 & 0.999133 \pm 0.000043 \end{pmatrix}. \qquad (17)$$

This is closely approximated by a three parameter product

$$NR_{12}(a)R_{23}(b)R_{31}(c) \qquad (18)$$

for $a = -0.2314$, $b = 24.0$ and $c = 0.0036$. The sign of a makes only a small difference to the resulting probabilities. Note that $V_{tb} = V_{33}$ is now expressed as $24(24^2 + 1)^{-1/2}$ in terms of b.

There is CP violation in the complex matrix, because none of the (a, b, c) are equal to zero and the Jarlskog invariant should be given by (14). Using the $|V_{ij}|$ we obtain $J = 3.2 \times 10^{-5}$, in good agreement with current constraints [11].

Since a zero parameter results in a lack of CP violation, and the two factor products are essentially unordered, we interpret the noncommutativity of the three rotation factors as the source of CP violation in quarks. Thus CP violation depends directly on the known Euler angles that parameterise the rotation components. Checking the B_s physics parameter

$$2\beta_s \equiv 2\arg(V_{ts}\overline{V}_{tb}\overline{V}_{cs}V_{cb}) \tag{19}$$

using the phases of (18), we obtain a value $2\beta_s = -0.0388$, in agreement with Standard Model fits [11].

4 The MNS Neutrino Mixing Matrix

CP violation in neutrino mixing is a more theoretical issue at this point, but some form of quark lepton complementarity is expected in the underlying physics. Neutrino mass matrices in the normal hierarchy have been defined as circulant Hermitian matrices [9], suggesting a democratic Fourier mixing matrix. The empirical MNS matrix is certainly more democratic than the CKM case, which vaguely resembles the identity matrix I_3, but not exactly so. Until recently, the MNS matrix [12] closely resembled the exact tribimaximal mixing matrix [13][14][15], which has the norm square form

$$\|T\|^2 \equiv \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \tag{20}$$

This matrix is obtained from the three parameter product

$$M_T = NR_{12}(1)R_{23}(\sqrt{2})R_{31}(0) = NR_{12}(1)R_{23}(1/\sqrt{2})R_{31}(0) \tag{21}$$

with one zero value. However, Daya Bay [16] and other recent neutrino experiments indicate a non zero value for all three parameters. This now directly imposes CP violation on the neutrino sector, given the form (7).

For the candidate parameter set $(a, b, c) = (1.414, 1.0, 0.144)$ the Jarlskog invariant (14) is $J = 0.024$.

The real tribimaximal form is quite robust, defined also as the product $F_3 F_2$ and by a range of similar, essentially parameter free products involving mutually unbiased bases. It would seem then that CP violation occurs in any true three dimensional extension of the two dimensional concept of quantum bit, just as braid diagrams extend the planar diagrams of symmetric structures. Note that the Fourier matrix factorisation may be related to the A_4 discrete symmetry of [13][14][15] and other work.

5 Conclusions

The complex product parameterisation discussed here recovers the CKM mixing matrix for quarks with only three real parameters, in association with the real rephasing invariant matrix. This tightens the standard parameterisation of this matrix in terms of Euler angles and a CP phase, by defining J in terms of the Euler parameters. Both the DZero [17] and CDF [18] experiments have reported a narrow range for β_s that is consistent with a Standard Model value of 0.019, in agreement with the reduced complex parameterisation.

The Jarlskog invariant [2] for the CKM matrix is closely related to the 3×3 mass matrices for quarks. Thus the Euler angles of the CKM matrix tightly constrain the four quark mass ratios, via (14). A similar result is expected for the neutrinos.

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This result first appeared in a rather incoherent form on my old blog *Arcadian Pseudofunctor*. There were many discussions with Carl Brannen, whose work on the complex parameterization will appear elsewhere. If this parameter reduction was already in the literature, I remain unaware of it, and its absence from PDG reviews would be a bit bizarre.

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