Arithmetic information in particle mixing

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Abstract

The mutually unbiased bases of quantum information theory are used to study the MNS and CKM mixing matrices. The resulting mixing matrix parameterisation requires only three real parameters, and is thus potentially more constraining than those in current use. We also discuss related results from MINOS.

Keywords: CKM, tribimaximal, Koide matrix, discrete Fourier

1. Introduction

For Hilbert spaces of finite dimension d, the discrete (or quantum) Fourier transform operator F_d [1] may be considered one of a set of *mutually unbiased* bases [2][3][4]. In dimension 3,

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \overline{\omega} & 1 \\ \overline{\omega} & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{1}$$

where $\omega = \exp(2i\pi/3)$ is the complex cubed root of unity. The columns of F_3 form a basis, namely the eigenvector set of a 3×3 Pauli matrix. Two bases are unbiased if all possible inner products between two elements, one in each basis, have the same norm. A collection of mutually unbiased basis sets is one such that every basis is unbiased with respect to each other one. In dimension 3, there are four such sets, two of which may be given by F_3 and the identity matrix.

Since the Fourier transform matrix is unitary, the Fourier transform of a unitary matrix is unitary. In a prime dimension d, a complete set of mutually unbiased bases for the Hilbert space defines a set of $d + 1$ unitary operators.

For $d = 2$, a complete set of three mutually unbiased bases is given by

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{2}$$

and the identity matrix I. Observe that $R_2^8 = I$ makes R_2 a unitary root of the identity I. In any dimension d, such a Hadamard circulant R_d defines a basis that is mutually unbiased with respect to F_d [5][6]. The collection of all such operators forms a finite set, in principle characterising the measurable quantities for quantum mechanics in a given dimension. For dimension 3, the complete set of four mutually unbiased bases is given by the collection $\{F_3, R_3, R_3^2, I\}$ where

$$R_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \overline{\omega} & 1 \\ 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \end{pmatrix}, \tag{3}$$

is a unitary root such that R_3^3 is a multiple of I and $R_3^{12} = I$. The Fourier operator satisfies $F_d^4 = I$.

As Heisenberg and Schwinger originally envisaged, in describing basis sets this way we remove the emphasis on quantum *states* and consider instead the physical composition of measurement operations. This paper uses R_d type matrices to define a parameterisation of the mixing matrices of particle phenomenology.

This procedure supports the view that, in quantum gravity, the classical gauge groups should be derived from deeper measurement principles. After all, if spacetime is emergent, then presumably the symmetries that act upon it also are. Recent experimental hints of the need to abandon traditional continua for quantum states include the MINOS observation [7] of mass differences between neutrinos and antineutrinos. Carl Brannen has commented that these mass differences might be explained by a simple phase conjugation in Koide mass triplets [8]. A Koide mass matrix is a 3×3 Hermitian circulant matrix with eigenvalues the square roots of the rest masses. It depends on two parameters r and δ, as in

$$\sqrt{M} = \mu \begin{pmatrix} 1 & re^{i\delta} & re^{-i\delta} \\ re^{-i\delta} & 1 & re^{i\delta} \\ re^{i\delta} & re^{-i\delta} & 1 \end{pmatrix}, \tag{4}$$

where μ is a real scale parameter that selects the physical unit. As shown in [8], the value $r = \sqrt{2}$ accounts for both neutrino and charged lepton masses.

One may interpret $r^{-1} = \tan\theta$ geometrically in matrix space, in which case the lepton value corresponds to a phase $\theta = \pi/4$.

The neutrino and antineutrino masses are considered fundamental to a non local gravitational interaction, replacing the effective Higgs boson of the local theory. Neutrino mixing is presumed to carry gravitational charge from its generating (cosmological) horizon into the electroweak sector. Section 2 introduces the basic information matrices of interest, and then the lepton and quark mixing matrices are discussed in sections 3 and 4.

2. Circulant Sums and Products

An m-circulant matrix is defined by its first row, the other rows being equal to the first except for a shift of each entry m places to the right. In any dimension, the discrete Fourier transform diagonalises all 1-circulants.

In dimension 3, a codiagonal is a 2-circulant M with non zero entries M_{13}, M_{22} and M_{31}. The sum of a diagonal and a codiagonal essentially results in a matrix with 2×2 and 1×1 blocks. It follows that the Fourier transform of a sum $A+iB$, where A is a real 1-circulant and B is a real 2-circulant, results in a matrix with 2×2 and 1×1 blocks. We are interested in parameterisations of unitary matrices in dimension 3 in the form $A + iB$ [9]. In particular, consider the block forms

$$F_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad R_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix}. \tag{5}$$

The operators F_{ij} and R_{ij} are similarly defined for $(ij) = (12)$ and (31). Observe that the cyclic product $R_{12}R_{23}R_{31}$ is a sum of the form $A + iB$. More general decompositions of the form $A+iB$ are obtained using matrices such as

$$R_{23}(r) = \frac{1}{\sqrt{r^2+1}} \begin{pmatrix} r+i & 0 & 0 \\ 0 & r & i \\ 0 & i & r \end{pmatrix}, \tag{6}$$

with a real parameter r. In this case, the 1-circulant component A becomes the real part of the unitary matrix, while the 2-circulant B is the imaginary part, and their contributions to the associated matrix of probabilities do not have any interference. That is, it may be natural to decompose a magic probability matrix, such as a mixing matrix, into the sum of a real 1-circulant and a purely imaginary 2-circulant. Since swapping A and B results in the same set of probabilities, we consider such decompositions to manifest a duality of electric magnetic type.

These decompositions presume a cyclicity amongst particle generations, which leaves the probability set invariant under permutations of rows and columns. As an ansatz for complex mixing matrices, this potentially allows a reduction of the required number of parameters from four to three. The following sections consider whether the MNS and CKM matrices may be described by the normalised three parameter product form

$$M = NR_{12}(a)R_{23}(b)R_{31}(c), \tag{7}$$

where a, b and c are real, and

$$N^{-2} = (a^2+1)(b^2+1)(c^2+1) \tag{8}$$

is the normalisation constant. As a circulant sum, M takes the form

$$\begin{aligned} M = N & \begin{pmatrix} abc & -a-c & -b \\ -b & abc & -a-c \\ -a-c & -b & abc \end{pmatrix} \\ +iN & \begin{pmatrix} bc & ac-1 & ab \\ ac-1 & ab & bc \\ ab & bc & ac-1 \end{pmatrix}. \end{aligned} \tag{9}$$

This is a general form for a cyclic decomposition in terms of generation pairs, because one is always free to scale the imaginary entries of R_{ij} to unit norm. As seen in the following section, it is similar to the tribimaximal extension considered in [10], in terms of the (ub) term. A cyclic four parameter form for general unitary matrices is discussed in [9].

Markovitch [11] considers a symmetry condition for mixing matrices arising from the observation that the norm square matrix P of a rotation matrix satisfies $P - P^T = C(x)$, where $C(x)$ is a 1-circulant matrix with first row $(0, -x, x)$. For a three parameter product of the form M, the real circulant $M - M^T$ satisfies $x = a - b + c$. The real norm square of M gives a circulant with $x = b^2 - (-a-c)^2$.

For a normalised mixing matrix in the form $A + iB$, the Fourier transform of $A + iB$ results in a 2×2 block of determinant 1 and a 1×1 block that is the row sum phase. That is, the complex matrix M is transformed into an element of $SU(2) \times U(1)$, which is a four parameter Lie group obtained from an arbitrary $A+iB$ form. We consider a general four parameter mixing matrix as a three generational version of the three parameter product form M. Such parameterisations will be discussed in [9].

Observe that in the triple product M, the ordering of the three parameters matters. This is in contrast to a two parameter product, where a swapping of factors results in a transpose but does not alter the final set of probabilities. The three parameter ordering is viewed as a source of CP violation, since setting the third parameter to zero reduces CP phases to zero.

3. The Masses and MNS Lepton Mixing

We assume that the generation of rest mass is entropic, which is to say time asymmetric with respect to an observer's measurement clock. In an observer dependent cosmology, the rest mass emergence is associated to a minimal zero entropy horizon, and defined by Michael Rios [13] in terms of the distance in a Jordan algebra matrix space from the origin. Since the information content of rest mass operators is prior to the generation of a classical spacetime, and the cosmology is time asymmetric, one does not expect universal conservation of CPT symmetry.

The MINOS collaboration have recently reported [7] a difference in the oscillations of neutrinos and antineutrinos, with mass differences given by

$$\text{neutrino:} \;\; \Delta m^2 = 2.35 \times 10^{-3} \pm 0.11 \;\text{ eV}^2$$

$$\text{antineutrino:} \;\; \Delta \overline{m}^2 = 3.35 \times 10^{-3} \pm 0.45 \;\text{ eV}^2$$

and the two flavor mixing angles

$$\text{neutrino:} \;\; \sin^2(2\theta) > 0.91$$

$$\text{antineutrino:} \;\; \sin^2(\overline{2\theta}) = 0.86 \pm 0.12$$

Observe that these mass parameters are consistent with two distinct Koide triplets [8] as follows. Each mass triplet arises as an eigenvalue set for the 1-circulant Hermitian matrix. If the single off diagonal complex phase in this mass matrix is chosen to be

$$\delta = \frac{2}{9} \pm \frac{\pi}{12} \tag{10}$$

then the three eigenvalues of interest are given by

$$\sqrt{m_i} = \mu(1 + \sqrt{2}\cos(\delta + \omega_i)) \tag{11}$$

for a suitable scale μ, and with ω_i the three cubed roots of unity. Taking the $+\pi/12$ phase to correspond to neutrinos, and its conjugate to antineutrinos [8], and with a scale of $\mu = 0.01$, the eigenvalue triplets give mass differences of

$$\text{neutrino:} \quad \Delta m^2 = 2.49 \times 10^{-3} \ \text{eV}^2$$

$$\text{antineutrino:} \quad \Delta \overline{m}^2 = 3.39 \times 10^{-3} \ \text{eV}^2$$

Since there is flexibility in the choice of μ, this shows basic agreement with the MINOS results [7]. The universal phase 2/9 appears in both the charged lepton mass matrix and also in a large collection of Koide fits for hadron masses [8]. The phase $\pm\pi/12$ may be viewed as a geometric phase in connection with noncommutative paths built from operators associated to mutually unbiased bases (see [12]). Alternatively, it is the 24th root of unity that represents the 3×8 information dimensions of the full quantum number space for the low energy spectrum. The $\pi/12$ phase is thus a cubed root of $\pi/4$, which occurs for example in the 2×2 Fourier transform

$$F_2 R_2 F_2 = \begin{pmatrix} e^{\pi i/4} & 0 \\ 0 & e^{-\pi i/4} \end{pmatrix} \tag{12}$$

showing that scaled $R_2(r)$ factors correspond to phases θ away from $\pi/4$. Is there a connection between $R_2(r)$ factors and the Koide mass matrix? A two factor product $R_2(a)R_2(b)$ has the property that

$$R_2(b)R_2(a) = (R_2(a)R_2(b))^T \tag{13}$$

from which it follows that the Jordan algebra matrix product $X \circ Y \equiv 1/2(XY + YX)$ sends the 1-circulant piece of $R_2(a) \circ R_2(b)$ to the symmetrised norm of a Hermitian Koide matrix. Jordan algebra matrix models [13] are expected to play an important role in extensions of information theory to gravitational questions, so we view this transmutation of products as a natural means of extracting the positive energy mass sector from the pre spacetime mixing.

This begs the question of whether or not the MNS neutrino mixing matrix also involves simple parameters. The tribimaximal MNS mixing matrix [14] has exact values

$$T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \tag{14}$$

This matrix is obtained from the norm squares of the three parameter product

$$M_T = NR_{12}(1)R_{23}(\sqrt{2})R_{31}(0) = NR_{12}(1)R_{23}(1/\sqrt{2})R_{31}(0), \tag{15}$$

corresponding to a general tribimaximal type probability matrix, most easily expressed in the form

$$P^2(a,b) = N^2 \begin{pmatrix} b^2(a^2+1) & a^2+1 & 0 \\ a^2 & a^2b^2 & b^2+1 \\ 1 & b^2 & a^2(b^2+1) \end{pmatrix}. \tag{16}$$

Observe that a slight rearrangement of the entries of $P^2(a,b)$ shows how T is built from multiples of one initial row. That is, if we replace the zero entry by the normalisation factor $(a^2+1)(b^2+1) = 6$ the entries may be arranged as multiples of the row $(1,1,2)$, by the numbers 1, 2 and 3. Taking these entries modulo 6 returns a non magic form T, obeying a sum rule along each row and column. The complex matrix M_T is also expressible as the circulant sum

$$M_T = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 0 & \sqrt{2} & 1 \\ 1 & 0 & \sqrt{2} \end{pmatrix} + \frac{i}{\sqrt{6}} \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}. \tag{17}$$

This may be viewed as a reduction to a two parameter form, since the zero parameter term $R_{31}(0)$ acts as a pseudo identity, whereby cycling generations or rephasing does not alter the probability matrix. Note that since R_{ij} is symmetric, the two parameter matrix product is given by the nine possible column products for the two R_{ij} components. The resulting probability matrix is thus a straightforward generalisation of the democratic representation of mutually unbiased probabilities, by a matrix with all entries equal.

As pointed out some years ago by a number of people, such as Hans de Vries, we observe that the three Koide eigenvalues of $\sqrt{M}$ are expressible in terms of the nine amplitudes of M_T via

$$\lambda_1 = \sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}} \cos\delta + \sqrt{0} \sin\delta \tag{18}$$

$$\lambda_2 = \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\cos\delta - \sqrt{\frac{1}{2}}\sin\delta$$
$$\lambda_3 = \sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\cos\delta + \sqrt{\frac{1}{2}}\sin\delta$$

Through basic trigonometry, this rule holds for any value of the angle δ, including those for all lepton triplets. The three parameters (a, b, c) appear in the coefficients of the first expression λ_1, but the coefficients in the other two rows also result in the same tribimaximal mixing matrix. At this natural scale, the signs for the λ_i give a cosine rule that forces a mass sum of exactly 2.

The two flavor mixing angles observed by MINOS [7] are still potentially consistent with tribimaximal mixing in both the neutrino and antineutrino sectors. For example, the antineutrino matrix may be the transpose of the neutrino one, or the matrix obtained by conjugating all complex roots. Given that a conjugate R_2 factor is its inverse, a composition of two inverse forms of M_T would result in the identity mixing matrix, obeyed by the charged leptons.

A robust operator, T also results from the matrix product F_3F_2 and a range of similar, essentially parameter free products. This Fourier matrix factorisation may be associated to the A_4 discrete symmetry [15][16][17] studied in the context of neutrino mass generation in the Standard Model, since this group is the product $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^2$ of cyclic permutation groups. Here, however, the finite group appears as a discrete structure in quantum information theory, rather than as a subgroup of a large continuous symmetry.

4. The CKM Quark Mixing Matrix

Recent experimental estimates [14] of the (unsquared) CKM amplitudes, for a complex CKM matrix M_C, are given by

$$\begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.001 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.001 & 0.97334 \pm 0.00023 & 0.0415 \pm 0.0011 \\ 0.00874 \pm 0.00026 & 0.0407 \pm 0.001 & 0.999133 \pm 0.000043 \end{pmatrix} \quad (19)$$

which is closely approximated by a three parameter product

$$M_C = NR_{12}(a)R_{23}(b)R_{31}(c) \quad (20)$$

for $a = -0.2314$, $b = 24.0$ and $c = 0.0035$. In the form $\tan\theta$ these best fit parameters correspond to the three Euler angles of the standard parameterisation for M_C, although $b = 24$ must be inverted to obtain the usual angle. Kuo and Liu [18] have also considered circulant structure in M_C, showing that three parameters suffice.

Note that the (td) term in this fit is a little outside current bounds, but these bounds rely on lattice computations and the data [14] used in the fit here also relies on a largish value for the (td) term. The sign of a makes little difference to the resulting probabilities, but the negative sign gives a better fit, slightly reducing the size of the (td) term.

The product form is clearly unitary, and the smallness of c makes M_C similar to a matrix of MNS type. Note however that M_C is also closely approximated by a two factor product of type (16). In fact, the experimental bounds are all respected with the same parameters $a = 0.2314$ and $b = 24.0$, along with a small c dependent (ub) term at the top right of the adjusted magic form

$$|M_C|^2 = P^2(0.2314, 24.0) + c^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{21}$$

Observe that in either unitary form the three parameters (a, b, c) essentially come from three of the measured CKM entries, namely (td), (cs) and (ub). The factor ordering is therefore associated with an ordering of quark pairs.

Since $c = 0$ results in zero CP violation, and the two factor products are essentially unordered, we interpret the three factor $c \neq 0$ ordering as a source of CP violation in quarks. The B_s physics CP violation parameter

$$2\beta_s \equiv 2\arg(V_{ts}\overline{V}_{tb}\overline{V}_{cs}V_{cb}) \tag{22}$$

may easily be computed from M_C. Using the phases of (20), we obtain a value for $2\beta_s$ of -0.0388, in agreement with Standard Model fits [14]. Note that the value -0.0388 comes mainly from the (ts) term, with the other three phases roughly canceling out. In terms of the parameters, the angle β_s is closely approximated by the product abc.

As for tribimaximal mixing, we may now express candidate square root mass triplets for a phase δ in terms of the entries of the CKM matrix. The choice

$$\lambda_1 = 0.2252 + 0.9735 \cos\delta + 0.0406 \sin\delta \tag{23}$$

$$\lambda_2 = 0.9743 - 0.2250\cos\delta - 0.0098\sin\delta$$
$$\lambda_3 = 0.0035 - 0.0416\cos\delta + 0.9991\sin\delta$$

obeys the cosine probability rule that gives a mass sum of 2, as for the lepton triplets.

5. Conclusions

The product parameterisation discussed here potentially constrains the required mixing parameters for both MNS and CKM matrices. Although the elements of the exact tribimaximal MNS matrix involve simple parameters from any point of view, the CKM matrix is usually assumed to be more complicated.

Of course further study of the relation between mixing and rest mass is warranted. Do charged leptons also exhibit mass differences between particles and antiparticles? There are reasons to think not, given the tight limits on the electron positron mass difference [14]. This perhaps suggests a relation between the identity mixing of charged leptons and their mass equality, whereby only the light neutrinos and antineutrinos carry the gravitational charge into the electroweak sector.

In the abstract categorical framework that motivates this paper, classical spacetime is emergent and the mathematical association of CPT violation with Lorentz violation does not automatically arise in this purely information theoretic analysis of mass quantum numbers. It is possible that Lorentz symmetry is restored in a collective continuum derived from a richer spectrum of energy levels.

Both the DZero [19] and CDF [20] experiments report a range for β_s that is consistent with a Standard Model value of -0.019, in agreement with our parameterisation.

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