Arithmetic information in particle mixing

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The mutually unbiased bases of quantum information theory are used to study the MNS and CKM mixing matrices. The resulting mixing matrix parameterisation requires only three real parameters, and is thus potentially more constraining than those in current use. We also discuss related results from MINOS and Fermilab.

I. INTRODUCTION

For Hilbert spaces of finite dimension d, the discrete (or quantum) Fourier transform operator F_d [1] may be considered one of a set of *mutually unbiased* bases [2][3][4]. In dimension 3,

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & \overline{\omega} & 1 \\ \overline{\omega} & \omega & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{1}$$

where $\omega = \exp(2i\pi/3)$ is the complex cubed root of unity. The columns of F_3 form a basis, namely the eigenvector set of a 3×3 Pauli matrix. Two bases are unbiased if all possible inner products between two elements, one in each basis, have the same norm. A collection of mutually unbiased basis sets is one such that every basis is unbiased with respect to each other one. In dimension 3, there are four such sets, two of which may be given by F_3 and the identity matrix.

Since the Fourier transform matrix is unitary, the Fourier transform of a unitary matrix is unitary. In a prime dimension d, a complete set of mutually unbiased bases for the Hilbert space defines a set of $d+1$ unitary operators. For $d = 2$, a complete set of three mutually unbiased bases is given by

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{2}$$

and the identity matrix I. Observe that $R_2^8 = I$ makes R_2 a unitary root of the identity I. In any dimension d, such a Hadamard circulant R_d defines a basis that is mutually unbiased with respect to F_d [5][6]. The collection of all such operators forms a finite set, in principle characterising the measurable quantities for quantum mechanics in a given dimension. For dimension 3, the complete set of four mutually unbiased bases is given [?] by the collection $\{F_3, R_3, R_3^2, I\}$ where

$$R_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \overline{\omega} & 1 \\ 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \end{pmatrix}, \tag{3}$$

is a unitary root such that R_3^3 is a multiple of I and $R_3^{12} = I$. The Fourier operator satisfies $F_d^4 = I$.

As Heisenberg and Schwinger originally envisaged, in describing basis sets this way we remove the emphasis on quantum *states* and consider instead the physical composition of measurement operators. This paper uses the R_d matrices to define a parameterisation of the mixing matrices of particle phenomenology.

This procedure supports the view that, in quantum gravity, the classical gauge groups should be derived from deeper measurement principles. After all, if spacetime is emergent, then presumably the symmetries that act upon it also are. Recent experimental hints that this view is reasonable include the MINOS observation [7] of a mass difference between neutrinos and antineutrinos. Carl Brannen has commented that these mass differences might be explained by a simple phase conjugation in Koide triplets [8].

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II. CIRCULANT SUMS AND PRODUCTS

An m-circulant matrix is defined by its first row, the other rows being equal to the first except for a shift of each entry m places to the right. In any dimension, the discrete Fourier transform diagonalises all 1-circulants.

In dimension 3, a codiagonal is a 2-circulant M with non zero entries M_{13}, M_{22} and M_{31}. The sum of a diagonal and a codiagonal essentially results in a matrix with 2×2 and 1×1 blocks. It follows that the Fourier transform of a sum $A + iB$, where A is a real 1-circulant and B is a real 2-circulant, results in a matrix with 2×2 and 1×1 blocks. We are interested in parameterisations of unitary matrices in dimension 3 in the form $A+iB$ [9]. In particular, consider the block forms

$$F_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad R_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix}. \tag{4}$$

The operators F_{ij} and R_{ij} are similarly defined for $(ij) = (12)$ and (31). Observe that the cyclic product $R_{12}R_{23}R_{31}$ is a sum of the form $A + iB$. More general decompositions of the form $A + iB$ are obtained using matrices such as

$$R_{23}(r) = \frac{1}{\sqrt{r^2+1}} \begin{pmatrix} r+i & 0 & 0 \\ 0 & r & i \\ 0 & i & r \end{pmatrix}, \tag{5}$$

with a real parameter r. In this case, the 1-circulant component A becomes the real part of the unitary matrix, while the 2-circulant B is the imaginary part, and their contributions to the associated matrix of probabilities do not have any interference. That is, it may be natural to decompose a magic probability matrix, such as a mixing matrix, into the sum of a real 1-circulant and a purely imaginary 2-circulant. Since swapping A and B results in the same set of probabilities, we consider such decompositions to manifest a duality of electric magnetic type.

These decompositions presume a cyclicity amongst particle generations, which leaves the probability set invariant under permutations of rows and columns. As an ansatz for complex mixing matrices, this potentially allows a reduction of the required number of parameters from four to three. The following sections consider whether the MNS and CKM matrices may be described by the normalised three parameter product form

$$M = NR_{12}(a)R_{23}(b)R_{31}(c), \tag{6}$$

where a, b and c are real, and

$$N^{-2} = (a^2+1)(b^2+1)(c^2+1) \tag{7}$$

is the normalisation constant. As a circulant sum, M takes the form

$$\begin{aligned} M = N & \begin{pmatrix} abc & -a-c & -b \\ -b & abc & -a-c \\ -a-c & -b & abc \end{pmatrix} \\ +iN & \begin{pmatrix} bc & ac-1 & ab \\ ac-1 & ab & bc \\ ab & bc & ac-1 \end{pmatrix}. \end{aligned} \tag{8}$$

This is a general form for a cyclic decomposition in terms of generation pairs, because one is always free to scale the imaginary entries of R_{ij} to unit norm. As seen in the following section, it is similar to the tribimaximal extension considered in [10], in terms of the (ub) term. A cyclic four parameter form for general unitary matrices is discussed in [9].

Markovitch [11] considers a symmetry condition for mixing matrices arising from the observation that the norm square matrix P of a rotation matrix satisfies $P - P^T = C(x)$, where $C(x)$ is a 1-circulant matrix with first row $(0, -x, x)$. For a three parameter product of the form M, the real circulant $M - M^T$ satisfies $x = a - b + c$. The real norm square of M gives a circulant with $x = b^2 - (-a-c)^2$.

For a normalised mixing matrix in the form $A + iB$, the Fourier transform of $A + iB$ results in a 2×2 block of determinant 1 and a 1×1 block that is the row sum phase. That is, the complex matrix M is transformed into an element of $SU(2) \times U(1)$, which is a four parameter Lie group obtained from an arbitrary $A + iB$ form. We consider a general four parameter mixing matrix as a three generational version of the three parameter product form M. Such parameterisations will be discussed in [9].

Observe that in the triple product M, the ordering of the three parameters matters. This is in contrast to a two parameter product, where a swapping of factors results in a transpose but does not alter the final set of probabilities. The three parameter ordering is viewed as a source of CP violation, since setting the third parameter to zero reduces CP phases to zero.

III. THE MNS LEPTON MIXING MATRIX

The MINOS collaboration have recently reported [7] a difference in the oscillations of neutrinos and antineutrinos, in particular a difference between the two mass triplets,

$$\text{neutrino:}\quad \Delta m^2 = 2.35 \times 10^{-3} \pm 0.11 \ \ \text{eV}^2$$

$$\text{antineutrino:}\quad \Delta \overline{m}^2 = 3.35 \times 10^{-3} \pm 0.45 \ \ \text{eV}^2$$

and two flavor mixing angles,

$$\text{neutrino:}\quad \sin^2(2\theta) > 0.91$$

$$\text{antineutrino:}\quad \sin^2(\overline{2\theta}) = 0.86 \pm 0.12$$

Observe that these mass parameters are consistent with two distinct Koide triplets [8] as follows. Each mass triplet arises as an eigenvalue set for a 1-circulant Hermitian matrix. If the single off diagonal complex phase in this mass matrix is chosen to be

$$\delta = \frac{2}{9} \pm \frac{\pi}{12} \tag{9}$$

then the three eigenvalues of interest are given by

$$\sqrt{m_i} = \mu(1 + \sqrt{2}\cos(\delta + \omega_i)) \tag{10}$$

for a suitable scale μ, and with ω_i the three cubed roots of unity. When the $+\pi/12$ phase corresponds to neutrinos and its conjugate to antineutrinos [8], these triplets give mass differences

$$\text{neutrino:}\quad \Delta m^2 = 2.49 \times 10^{-3} \ \ \text{eV}^2$$

$$\text{antineutrino:}\quad \Delta \overline{m}^2 = 3.39 \times 10^{-3} \ \ \text{eV}^2$$

in agreement with the MINOS results [7]. The phase 2/9 appears in both the charged lepton mass matrix and also in a large collection of Koide fits for hadron masses [8], and is thus independent of quantum numbers. The phase $\pm\pi/12$ appears as a geometric phase in connection with noncommutative paths built from operators associated to mutually unbiased bases (see [12]). For these spin path integrals, generation quantum number appears as a tripling of spin states.

The $\pi/12$ phase arises as a cubed root of $\pi/4$, which occurs for example in the 2×2 Fourier transform

$$F_2 R_2 F_2 = \begin{pmatrix} e^{\pi i/4} & 0 \\ 0 & e^{-\pi i/4} \end{pmatrix} \tag{11}$$

which shows that scaled $R_2(r)$ factors correspond to phases away from $\pi/4$. For $\pi/12$ the scaled R_2 matrix would be

$$\begin{pmatrix} 0.0673 & i \\ i & 0.0673 \end{pmatrix} \tag{12}$$

and this example may be used in three factor products to build simple 2×2 mixing matrices. This begs the question of whether or not the MNS neutrino mixing matrix is also fundamentally simple. The tribimaximal MNS mixing matrix [13] has exact values

$$T = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \tag{13}$$

This matrix is obtained from the norm squares of the three parameter product

$$M_T = N R_{12}(1) R_{23}(\sqrt{2}) R_{31}(0) = N R_{12}(1) R_{23}(1/\sqrt{2}) R_{31}(0), \tag{14}$$

corresponding to a general tribimaximal type probability matrix

$$P^2(a,b) = N^2 \begin{pmatrix} b^2(a^2+1) & a^2+1 & 0 \\ a^2 & a^2b^2 & b^2+1 \\ 1 & b^2 & a^2(b^2+1) \end{pmatrix}. \tag{15}$$

Observe that a slight rearrangement of the entries of $P^2(a,b)$ shows how T is built from multiples of one initial row. That is, if we replace the zero entry by the normalisation factor $(a^2+1)(b^2+1) = 6$ the entries may be arranged as multiples of the row $(1,1,2)$, by the numbers 1, 2 and 3. Taking these entries modulo 6 returns a non magic form T, obeying a sum rule along each row and column. The complex matrix M_T is also expressible as the circulant sum

$$M_T = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 0 & \sqrt{2} & 1 \\ 1 & 0 & \sqrt{2} \end{pmatrix} + \frac{i}{\sqrt{6}} \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}. \tag{16}$$

This may be viewed as a reduction to a two parameter form, since the zero parameter term $R_{31}(0)$ acts as a pseudo identity, whereby cycling generations or rephasing does not alter the probability matrix. Note that since R_{ij} is symmetric, the two parameter matrix product is given by the nine possible column products for the two R_{ij} components. The resulting probability matrix is thus a straightforward generalisation of the democratic representation of mutually unbiased probabilities, by a matrix with all entries equal.

The two flavor mixing angles observed by MINOS [7] are still potentially consistent with tribimaximal mixing in both the neutrino and antineutrino sectors. For example, the antineutrino matrix may be the transpose of the neutrino one, or the matrix obtained by conjugating all complex roots. Given that the a conjugate R_2 factor is its inverse, a composition of two inverse forms of M_T would result in the identity mixing matrix, obeyed by the charged leptons.

A robust operator, T also results from the matrix product F_3F_2 and a range of similar, essentially parameter free products. This Fourier matrix factorisation may be associated to the A_4 discrete symmetry [14][15][16] studied in the context of neutrino mass generation in the Standard Model, since this group is the product $\mathbb{Z}_3 \ltimes \mathbb{Z}_2^2$ of cyclic permutation groups. Here, however, the finite group appears as a discrete structure in quantum information theory, rather than as a subgroup of a large continuous symmetry.

IV. THE CKM QUARK MIXING MATRIX

Recent experimental estimates [13] of the (unsquared) CKM amplitudes, for a complex CKM matrix M_C, are given by

$$\begin{pmatrix} 0.97419 \pm 0.00022 & 0.2257 \pm 0.001 & 0.00359 \pm 0.00016 \\ 0.2256 \pm 0.001 & 0.97334 \pm 0.00023 & 0.0415 \pm 0.0011 \\ 0.00874 \pm 0.00026 & 0.0407 \pm 0.001 & 0.999133 \pm 0.000043 \end{pmatrix} \tag{17}$$

which is closely approximated by a three parameter product

$$M_C = N R_{12}(a) R_{23}(b) R_{31}(c) \tag{18}$$

for $a = -0.2314$, $b = 24.0$ and $c = 0.0035$. Note that the (td) term in this fit is a little outside current bounds, but these bounds rely on lattice computations and the data used in the fit here also relies on a largish value for the (td) term. These parameters are estimated using basic algebra, and may be improved upon further optimisation. The sign of a makes little difference to the resulting probabilities, but the negative sign gives a better fit, slightly reducing the size of the (td) term.

The product form is clearly unitary, and the smallness of c makes M_C similar to a matrix of MNS type. Note however that M_C is also closely approximated by a two factor product of type (15). In fact, the experimental bounds are all respected with the same parameters $a = 0.2314$ and $b = 24.0$, along with a small c dependent (ub) term at the top right of the adjusted magic form

$$|M_C|^2 = P^2(0.2314, 24.0) + c^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{19}$$

Observe that in either unitary form the three parameters (a,b,c) essentially come from three of the measured CKM entries, namely (td), (cs) and (ub). The factor ordering is therefore associated with an ordering of quark pairs.

Since $c = 0$ results in zero CP violation, and the two factor products are essentially unordered, we interpret the three factor $c \neq 0$ ordering as a source of CP violation in quarks. The B_s physics CP violation parameter

$$2\beta_s \equiv 2\arg(V_{ts}\overline{V}_{tb}\overline{V}_{cs}V_{cb}) \tag{20}$$

may easily be computed from M_C. Using the phases of (18), we obtain a value for $2\beta_s$ of -0.0388, in agreement with Standard Model fits [13]. Note that the value -0.0388 comes mainly from the (ts) term, with the other three phases roughly canceling out. In terms of the parameters, the angle β_s is closely approximated by the product abc.

V. CONCLUSIONS

The product parameterisation discussed here potentially constrains the required mixing parameters for both MNS and CKM matrices. Although the elements of the exact tribimaximal MNS matrix involve simple parameters from any point of view, the CKM matrix is usually assumed to be more complicated.

Do charged leptons also exhibit mass differences between particles and antiparticles? There are reasons to think not, such as tight limits on the electron positron mass difference [13], and the original prediction by Dirac of same mass antimatter. This perhaps suggests a relation between the identity mixing of charged leptons and their mass equality.

Note that in the categorical framework that motivates this paper, classical spacetime is emergent and the association of CPT violation with Lorentz violation, in the usual sense, does not arise in this purely information theoretic analysis of mass quantum numbers.

CPT violation is also a possible interpretation of recent results from Fermilab. Both the DZero [17] and CDF [18] experiments report a range for β_s that is consistent with a Standard Model value of -0.019, in agreement with our parameterisation. On the other hand, further sources of CP violation are indicated by the like sign muon pair asymmetry measured by DZero [19], which indicates new physics at 3.2σ. This asymmetry depends on oscillations in the neutral B_q^0 meson systems, and the standard analysis assumes CPT symmetry for the mixing Hamiltonian. CPT violation in information mixing is potentially consistent with both Fermilab observations.

Acknowledgements

I am grateful to Carl Brannen for valuable discussions and comments on the first draft.

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