On the Gini Mean Difference Arc-Lengths Test for Circular Data

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Abstract

In this paper, we propose a new test of uniformity on the circle based on the Gini mean difference of the sample arc-lengths, i.e. the gaps between successive observations on the circumference of the circle. These sample arc-lengths are analogous to sample spacings, which are the gaps between successive observations on the real line. Such a Gini mean difference test is analogous to Rao's spacings test, which has been used to test the uniformity of circular data.

We obtain both the exact and asymptotic distributions of the Gini mean difference arc-lengths test, under the null hypothesis of circular uniformity. We also provide a table of upper percentile values of the exact distribution for small to moderate sample sizes. Some examples of circular data analysis are also considered. It is also seen that the Gini mean difference arc-lengths tests is more asymptotically efficient than Rao's test in the sense of Pitman asymptotic relative efficiency.

Keywords: Circular statistical inference, Directional data analysis, Goodness-of-fit tests, Spacings, Rao's spacings test, Gini mean difference

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1. Introduction

Sample observations representing directions in two dimensions can be modeled as random variables taking values on the circumference of the circle.

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We take this circle to be the unit circle with unit radius and a circumference of length 2π. A circular probability distribution is one whose support is this circumference.

The simple goodness-of-fit problem on the circle consists of testing fit to a single fixed circular distribution for a given data set. In particular, consider a random sample of angular measurements $\theta_1, \theta_2, \ldots, \theta_n$ with circular distribution function F defined on the interval $[0, 2\pi)$. We are interested in testing the null hypothesis

$$H_0 : F = F_0$$

where F_0 is a completely specified distribution function.

Without loss of generality, if F is assumed to be continuous as we shall do, by way of the probability integral transform, the goodness-of-fit problem reduces to one of testing circular uniformity, i.e. testing the null hypothesis

$$H_0 : F(\theta) = \tfrac{\theta}{2\pi} \cdot I(0 \leq \theta < 2\pi).$$

Let $0 \leq \theta_{(1)} \leq \theta_{(2)} \leq \ldots \leq \theta_{(n)} < 2\pi$ denote the sample order statistics. The *sample arc-lengths* are defined by the random variables

$$D_k = \theta_{(k)} - \theta_{(k-1)}, \text{ for } k = 1, 2, \ldots, n \tag{1.1}$$

where we take $\theta_{(0)} = \theta_{(n)} - 2\pi$ to make D_1 the natural gap between the first and last order statistics that straddle the origin. The sample arc-lengths $\{D_k\}$ represent the differences between successive observations on the circumference of the circle, and thus are analogous to spacings on the real line. When data are directions in two-dimensions and represented by angles, the sample arc-lengths $\{D_k\}$ are said to form the "maximal invariant," i.e. they remain invariant under the choice of zero-direction and sense of rotation. Tests based on these sample arc-lengths are studied here for testing the null hypothesis.

Under the null hypothesis of circular uniformity, the joint distribution of $(D_1, D_2, \ldots, D_n)$ is a Dirichlet$(1, 1, \ldots, 1; 1)$ distribution on the $(n-1)$-simplex with probability density function

$$\begin{aligned} &f_{D_1, D_2, \ldots, D_{n-1}}(d_1, d_2, \ldots, d_{n-1}) \\ &= \frac{(n-1)!}{(2\pi)^{n-1}} \cdot I\left(\bigcap_{k=1}^{n-1}(d_k > 0), \sum_{k=1}^{n-1} d_k \leq 2\pi\right). \end{aligned} \tag{1.2}$$

Moreover, under the null hypothesis, these sample arc-lengths are exchangeable random variables and have the same distribution as the spacings from a random sample of $(n-1)$ random variables from the Uniform distribution on the line segment $[0, 2\pi)$. This suggests that spacings tests on the real line, with some minor modifications, can be used for circular statistical inference. In fact, spacings tests are the only general class of goodness-of-fit tests that are directly applicable to both circular and linear data. The spacings or arc-lengths $\{D_k\}$ form the maximal invariant statistic under changes in origin so that every rotationally invariant statistic that is useful for the circular case can be expressed in terms of $\{D_k\}$ (cf. Jammalamadaka and SenGupta (2001)).

Most common among spacings tests are symmetric spacings tests, i.e. general test statistics of the form

$$V_n(g) = \frac{1}{n} \sum_{k=1}^{n} g(nD_k) \tag{1.3}$$

where $g(\cdot)$ is a real-valued function satisfying some regularity conditions, and

$$W_n(h) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} h(nD_i, nD_j) \tag{1.4}$$

where $h : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a symmetric function satisfying some regularity conditions. Test statistics of the form $V_n(g)$ are symmetric sum-functions of the sample spacings (e.g. cf. with Pyke (1965), Sethuraman and Rao (1970), and Rao and Sethuraman (1975)), and those of the form $W_n(h)$ are U-statistics of the sample spacings (cf. with Tung and Jammalamadaka (2010)). Moreover, such symmetric spacings tests are known to have asymptotic Normal distribution.

Among spacings tests of the form $V_n(g)$, Rao's spacings test (cf. with Rao (1969), Rao (1976))

$$J_n = \frac{1}{n} \sum_{k=1}^{n} \frac{|nD_k - 2\pi|}{2} = \frac{1}{2} \sum_{k=1}^{n} \left| D_k - \frac{2\pi}{n} \right| = \sum_{k=1}^{n} \left(D_k - \frac{2\pi}{n} \right)_+ \tag{1.5}$$

which corresponds to taking $g(t) = |t - 2\pi|/2$, has a nice interpretation for the circle. Since (1.5) is a modification of Rao's spacings test for the circular case, we will refer to it as Rao's arc-lengths test. Suppose n arcs, each of fixed length $(2\pi)/n$, are placed starting with each of the sample observations on the circumference. The uncovered part of the circumference contributed by observation θ_k is given by $\left(D_k - \frac{2\pi}{n}\right)_+$ whereas J_n gives the total uncovered portion of the circumference. The circumference is completely covered by these fixed arcs, resulting in $J_n = 0$, whenever the sample observations are uniformly distributed on the circumference. Large values of J_n indicate clustering of sample observations or evidence for directionality, and rejection of the null hypothesis of circular uniformity. Rao's test is a powerful statistic that can discriminate between uniform (isotropic) and concentrated (anisotropic) circular distributions, regardless of whether the distributions are unimodal or multimodal.

Under the null hypothesis of circular uniformity, the probability density function of J_n is

$$f_{J_n}(u) = \sum_{k=1}^{n-1} \binom{n}{k} \left(\frac{u}{2\pi}\right)^{n-k-1} \frac{\psi_k(nu) \cdot (n-1)! \cdot I[0 \le u \le 2\pi(1-1/n)]}{n^{k-1}(n-k-1)!} \tag{1.6}$$

where

$$\psi_k(x) = \frac{1}{2\pi(k-1)!} \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} \left(\frac{x}{2\pi} - j\right)_+^{k-1}. \tag{1.7}$$

A table of upper percentiles of the exact distribution for J_n was first given in Rao (1976), and extended tables of these critical values can be found in Russell and Levitin (1995).

Under the null hypothesis, J_n has an asymptotic Normal distribution, i.e. in the limit as $n \to \infty$,

$$\begin{aligned} \sqrt{n}\left(J_n - e^{-1}\right) &= \sqrt{n}\left(\frac{1}{2}\sum_{k=1}^{n}\left|D_k - \frac{2\pi}{n}\right| - e^{-1}\right) \\ &\xrightarrow{D} N_1\left(0, 2e^{-1} - 5e^{-2}\right). \end{aligned} \tag{1.8}$$

We introduce the Gini mean difference arc-lengths test in the next section and obtain both its exact and asymptotic distributions under the null hypothesis. Section 3 contains examples of circular data analysis featuring Rao's test and the Gini mean difference test, and Section 4 discusses their Pitman asymptotic relative efficiencies.

2. The Gini Mean Difference Arc-Lengths Test

Analogous to Rao's test is the Gini mean difference arc-lengths test

$$G_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|nD_i - nD_j|}{2} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} (nD_i - nD_j)_+}{n(n-1)} \tag{2.1}$$

which corresponds to taking $h(u, v) = |u - v|/2$. The statistic G_n is of the form $W_n(h)$ and an average over all pairs of absolute pairwise differences of the sample arc-lengths. The Gini mean difference spacings test was first proposed in Jammalamadaka and Goria (2004) for testing goodness-of-fit on the real line. There, under the goodness-of-fit null hypothesis (i.e. linear uniformity on $[0, 1]$), they derive both the exact and asymptotic distributions, and show that it has good performance based on Monte Carlo powers.

Under the null hypothesis of circular uniformity, the sample arc-lengths between successive observations should be approximately evenly spaced, about $(2\pi)/n$ apart, and G_n should be close to zero. Large values of G_n resulting from unusually large arc-lengths or unusually short arc-lengths between observations, are evidence for directionality, and rejection of the null hypothesis of circular uniformity.

Like Rao's test, the Gini mean difference test G_n is one of the few spacings-type tests for which both the exact and asymptotic distributions are known, under the null hypothesis. Here, we will adapt both the exact and asymptotic null distributions for the Gini mean difference spacings test on the real line to the case of the unit circle with circumference of length 2π.

Let $U_1, U_2, \ldots, U_{n-1}$ be independent Uniform($[0, 1]$) random variables, and let $\{X_k\} = \{(2\pi)U_k\}$ define $(n - 1)$ independent Uniform($[0, 2\pi)$) random variables. We define the *uniform spacings* on the unit interval $[0, 1]$ by the random variables

$$T_k = U_{(k)} - U_{(k-1)}, \text{ for } k = 1, 2, \ldots, n \tag{2.2}$$

where $0 \equiv U_{(0)} \leq U_{(1)} \leq U_{(2)} \leq \ldots \leq U_{(n-1)} \leq U_{(n)} \equiv 1$.

Under the null hypothesis, the sample arc-lengths $\{D_k\}$ are related to the uniform spacings $\{T_k\}$ by the relation

$$D_k \simeq X_{(k)} - X_{(k-1)} = (2\pi)[U_{(k)} - U_{(k-1)}] = (2\pi)T_k. \tag{2.3}$$

Here as elsewhere, we use $\simeq$ to denote the distributional equivalence of quantities on the left and right hand sides of the symbol. Since

$$\sum_{j=1}^{n}\sum_{j=1}^{n}|T_i - T_j| \simeq 2\sum_{k=1}^{n-1} U_k \tag{2.4}$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|D_i - D_j| \simeq (2\pi)\sum_{i=1}^{n}\sum_{j=1}^{n}|T_i - T_j| \simeq 2\sum_{k=1}^{n-1} X_k. \tag{2.5}$$

Thus, we have

$$G_n = \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}|nD_i - nD_j| \simeq \frac{S_{n-1}}{n-1} \tag{2.6}$$

where $S_{n-1} = \sum_{k=1}^{n-1} X_k$ is the sum of $(n-1)$ independent Uniform($[0, 2\pi)$) random variables. The probability distribution of S_{n-1} is a variation of the classical Irwin-Hall Uniform sum distribution, which was first derived by P.S. Laplace in 1814 (cf. Wilks (1962), Feller (1971, Theorem 1, I.9)). The probability density function of S_{n-1} has the form

$$f_{S_{n-1}}(s) = \frac{I[0 < s < 2\pi(n-1)]}{(2\pi)^{n-1}(n-2)!}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k(s - 2\pi k)_+^{n-2} \tag{2.7}$$

and can be derived via the Fourier inversion formula, and Cauchy's integral formula from complex analysis. The cumulative distribution function of S_{n-1} is

$$F_{S_{n-1}}(s) = \frac{I[0 < s < 2\pi(n-1)]}{(2\pi)^{n-1}(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k(s - 2\pi k)_+^{n-1}. \tag{2.8}$$

Under the null hypothesis of circular uniformity, the probability density function of G_n is

$$f_{G_n}(y) = \frac{(n-1)\cdot I(0 < y < 2\pi)}{(2\pi)^{n-1}(n-2)!}\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\left[(n-1)y - 2\pi k\right]_+^{n-2} \tag{2.9}$$

with cumulative distribution function

$$F_{G_n}(y) = \frac{I(0 < y < 2\pi)}{(2\pi)^{n-1}(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left[(n-1)y - 2\pi k\right]_+^{n-1} \tag{2.10}$$

and characteristic function

$$\varphi_{G_n}(t) = \int_{-\infty}^{\infty} e^{ity}\, dF_{G_n}(y) = (n-1)^{n-1} \left(\frac{\exp\left(\frac{2\pi it}{n-1}\right) - 1}{2\pi it} \right)^{n-1}, \ (i = \sqrt{-1}). \tag{2.11}$$

Under the null hypothesis, G_n has an asymptotic Normal distribution which is applicable to large sample situations. From the classical Central Limit Theorem, in the limit as $n \to \infty$,

$$\sqrt{n} \left(\frac{1}{n-1} \sum_{k=1}^{n-1} U_k - \frac{1}{2} \right) \xrightarrow{D} N_1 \left(0, \frac{1}{12} \right). \tag{2.12}$$

Since $G_n \simeq \frac{S_{n-1}}{n-1} = \frac{(2\pi)}{n-1} \sum_{k=1}^{n-1} U_k$, therefore we have in the limit as $n \to \infty$,

$$\sqrt{n}(G_n - \pi) = \sqrt{n} \left(\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |nD_i - nD_j|}{2n(n-1)} - \pi \right) \xrightarrow{D} N_1 \left(0, \frac{\pi^2}{3} \right). \tag{2.13}$$

Let α be the upper-tail probability corresponding to the critical value y_α of the test statistic G_n. Then

$$\alpha = \mathbb{P}(G_n > y_\alpha) = 1 - F_{G_n}(y_\alpha). \tag{2.14}$$

In Table 1, we give the upper percentiles of the exact distribution function for the statistic G_n for testing the null hypothesis of circular uniformity. The table gives these critical values, which have been given in degrees for immediate applicability, for small to moderate sample sizes. If for a given sample size n and significance level α, the observed value of the test statistic G_n, say y_{obs}, is greater than the tabulated critical value y_α, i.e. $y_{obs} > y_\alpha$, then we reject the null hypothesis of circular uniformity.

Note that, under the null hypothesis, the so-called "p-value" or observed significance level can be calculated by

$$p = \mathbb{P}(G_n > y_{obs}) = 1 - F_{G_n}(y_{obs}). \tag{2.15}$$

Equivalently, the null hypothesis is rejected, whenever $p < \alpha$.

n	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.04$	$\alpha = 0.05$	$\alpha = 0.10$
4	312.25	300.32	291.87	285.12	279.40	258.62
5	296.23	284.63	276.75	270.64	265.57	247.67
6	284.59	273.79	266.61	261.07	256.49	240.31
7	275.90	265.81	259.15	254.02	249.78	234.90
8	269.07	259.57	253.32	248.53	244.58	230.74
9	263.51	254.51	248.61	244.10	240.39	227.40
10	258.88	250.31	244.71	240.44	236.92	224.65
11	254.93	246.75	241.41	237.33	233.99	222.32
12	251.53	243.68	238.57	234.67	231.47	220.33
13	248.55	241.00	236.08	232.34	229.27	218.59
14	245.91	238.63	233.89	230.29	227.33	217.06
15	243.56	236.52	231.94	228.46	225.60	215.69
16	241.44	234.61	230.19	226.82	224.06	214.47
17	239.52	232.90	228.60	225.33	222.65	213.37
18	237.77	231.33	227.15	223.98	221.38	212.36
19	236.16	229.89	225.83	222.74	220.21	211.44
20	234.68	228.57	224.61	221.60	219.14	210.60
21	233.32	227.35	223.48	220.55	218.14	209.82
22	232.05	226.21	222.44	219.57	217.22	209.10
23	230.86	225.16	221.46	218.66	216.37	208.42
24	229.76	224.17	220.55	217.81	215.56	207.79
25	228.72	223.24	219.70	217.01	214.81	207.20
30	224.36	219.36	216.13	213.67	211.67	204.74
35	220.99	216.36	213.37	211.10	209.24	202.84
40	218.29	213.96	211.16	209.04	207.30	201.32
45	216.04	211.98	209.34	207.34	205.70	200.07
50	213.87	210.25	207.78	205.90	204.35	199.01

Table 1. Upper Percentiles of the Exact Distribution for the Gini Mean Difference Arc-Lengths Test G_n.

3. Some Circular Data Analysis Examples

In this section, we present a couple of examples of circular data analysis featuring Rao's test J_n and the Gini mean difference test G_n.

Example (Hospital Birth Times Data). Suppose one wants to know whether or not birth times at a hospital are uniformly distributed throughout the day. The alternative hypothesis is that there is a time (or times) when births are more frequent. Table 2 displays hypothetical data for delivery times collected across several days. This data can be found in both Russell and Levitin (1995) and Levitin and Russell (1997).

k	Delivery Time	$\theta_{(k)}$	D_k
1	12:20 am	5	34
2	12:40 am	10	5
3	12:40 am	10	0
4	12:48 am	12	2
5	1:08 am	17	5
6	5:40 am	85	68
7	6:00 am	90	5
8	6:36 am	99	9
9	6:40 am	100	1
10	7:20 am	110	10
11	10:12 am	153	43
12	3:32 pm	233	80
13	3:40 pm	235	2
14	7:44 pm	296	61
15	10:04 pm	331	35

Table 2. Hospital Birth Times Data.

These observed event times are modeled as realizations from a continuous circular distribution. The observations can be converted to angles around a circle in an obvious way, e.g. if we want the angular units in degrees, we use 1 hr. $= \frac{360 \text{ deg.}}{24} = 15$ deg. and 1 min. $= \frac{360 \text{ deg.}}{24 \text{ hr.}} \cdot \frac{1 \text{ hr.}}{60} = 0.25$ deg. Thus, 12:00 am $= 0$ deg., 6:00 am $= 90$ deg., 12:00 pm $= 180$ deg., 6pm $= 270$ deg., and 9:15 am $= 138.75$ deg., etc.

Rao's arc-lengths test statistic gives an observed value of $J_{15} = 177$ with a p-value between 0.01 and 0.05. At the 5% significance level, this is sufficient evidence to reject the null hypothesis of circular uniformity and conclude that there are times when births are more frequent.

On the other hand, the Gini mean difference arc-lengths test statistic gives an observed value of $G_{15} = 224.86$ with a p-value of 0.053. The results from the Gini test are borderline significant, and may possibly indicate there are times when births are more frequent.

Example (Homing Pigeon Data). 13 homing pigeons were released one at a time in the Toggenburg Valley in Switzerland under sub-Alpine conditions. They did not appear to have adjusted quickly to the homing direction, but preferred to fly in the axis of the valley, indicating a somewhat bimodal distribution. The vanishing angles are arranged here in increasing order as follows:

$$20, 135, 145, 165, 170, 200, 300, 325, 335, 350, 350, 350, 355.$$

Do these homing pigeons have a preferred direction of flight? (This example can also be found in Jammalamadaka and SenGupta (2001).)

The observed value of Rao's arc-lengths test statistic is $J_{13} = 161.92$ with a p-value between 0.05 and 0.10 (cf. with the table of upper percentiles of the distribution for J_n in Rao (1976)). On the basis of Rao's arc-lengths test, there is not enough evidence to reject the hypothesis of circular uniformity at the 5% significance level.

On the other hand, the observed value of the Gini mean difference arc-lengths test statistic is $G_{13} = 231.67$ with an observed significance level or p-value of $p = 0.043$. Therefore, the results of the Gini mean difference arc-lengths test are significant at the 5% significance level and we can reject the null hypothesis of circular uniformity. On the basis of Gini mean difference arc-lengths test, there is sufficient evidence that the homing pigeons have a preferred direction of flight.

4. Asymptotic Relative Efficiencies

In this section, we discuss the Pitman asymptotic relative efficiencies (ARE) of both the Gini mean difference test G_n and Rao's test J_n, as well as generalized versions of these statistics. In the case of large samples, comparing the Pitman ARE's of two test statistics is a way of making a quantitative comparison of two distinct tests for a statistical hypothesis of interest. The Pitman ARE of one sequence of tests against another is defined as the limit of the inverse ratio of sample sizes required for two tests to attain the same power at a sequence of alternatives which converges to the null hypothesis.

We define the generalized Rao's arc-lengths test

$$J_n(r) = \frac{1}{2n} \sum_{k=1}^{n} |nD_k - 2\pi|^r, \;\; r > 0 \tag{4.1}$$

and the generalized Gini mean difference arc-lengths test

$$G_n(r) = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |nD_i - nD_j|^r, \;\; r > 0. \tag{4.2}$$

For the special case $r = 1$, we have $J_n(1) = J_n$ and $G_n(1) = G_n$. Moreover, the special case of $J_n(2)$ corresponds to both $G_n(2)$ and the statistic $\frac{1}{n}\sum_{k=1}^{n}(nD_k)^2$, which will be called the Greenwood statistic.

Sethuraman and Rao (1970) have shown that among spacings tests of the form $V_n(g)$, the most asymptotically efficient, i.e. the asymptotically locally most powerful test (ALMP) is the Greenwood statistic. Tung and Jammalamadaka (2010) have shown that among spacings tests of the form $W_n(h)$, the ALMP test is the Gini mean squared difference test

$$G_n(2) = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |nD_i - nD_j|^2 \tag{4.3}$$

which also has the same efficiency as the Greenwood statistic.

Suppose the Pitman ARE of $J_n(2)$ and $G_n(2)$ is taken to be 1. The following Table 3, taken from Tung and Jammalamadaka (2010), lists the Pitman ARE of $J_n(r)$ and $G_n(r)$ with respect to various choices of $r > 0$. It is seen that the Pitman ARE's of $J_n(1)$ and $G_n(1)$ are respectively 0.572654 and 3/4, thus the Gini mean difference test $G_n(1)$ is more asymptotically efficient than Rao's test $J_n(1)$. Moreover, it is also seen that the generalized Gini mean difference test $G_n(r)$ is more Pitman efficient than the generalized Rao's test $J_n(r)$, except for the case $r = 2$, when both tests $G_n(2)$ and $J_n(2)$ correspond to the Greenwood statistic and have a Pitman ARE of 1.

r	Generalized Rao	Generalized Gini
1	0.572654	3/4
3/2	0.892135	0.946889
2	1	1
5/2	0.93921	0.96137
3	0.818649	0.867857
4	0.550562	0.615384

Table 3. Pitman Asymptotic Relative Efficiencies for $J_n(r)$ and $G_n(r)$.

5. Conclusion

We have introduced a new test of uniformity on the circle based on the Gini mean difference of the sample arc-lengths, and obtained both its exact and asymptotic distributions under the null hypothesis. The table of upper percentile values for this test will be of use to applied scientists employing it for circular data analysis. On the basis of Pitman asymptotic relative efficiency, the Gini mean difference test is more asymptotically efficient than Rao's test.

Extension of the Gini mean difference to the two-sample problem of testing for identical circular distributions involving "spacing-frequencies" (see e.g. Holst and Rao (1980), Holst and Rao (1981), and Rao and Mardia (1980)) will be investigated elsewhere.

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