Effective relativistic quantum mechanics on a causal net

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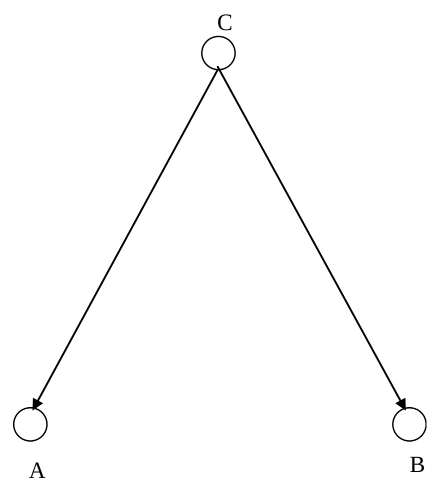
The observation of emergent particles in condensed matter physics such as phonons, holes, and recently magnetic monopoles [1] may lead one to speculate that real particles themselves could exist as emergent properties of a space-time lattice. There are many examples in the literature of the Dirac equation being supported by a lattice structure [2,3,4,5], related, for example, to the physics of ice [6] or of graphene [7]. In particular the case of the Dirac equation on the ice lattice is of potential practical interest as ice models can now be realised in a variety of new experimental systems, including spin ice materials [1] and artificial micromagnetic arrays [8]. We noticed that a Reichenbach causal net [9] has the same topology as an ice lattice and hence, if formulated in space-time, might support a version of relativistic quantum mechanics. In this paper we explore this idea at its simplest level. We propose a causal net for the free motion of a particle based on a relational concept of time as an ordered series of possible events that are linked by a principle of common cause. The causal net is 1+1 dimensional but applies along the direction of motion of the particle in three dimensions. It is reminiscent of the Feynman chequerboard [2] and the ice model [6] but differs from both in important respects. The causal net appears to support the 3+1 dimensional Dirac equation [10] for a free fermion and hence the Schrödinger equation in the low velocity limit. The causal net idea gives an intuitive picture of relativistic quantum mechanical motion and the origin of quantization although it is not a complete re-derivation of conventional quantum mechanics, partly because the space-time is discrete, and partly because it does not automatically imply a principle of superposition. If however we identify a causal net as representing a Dirac state vector then superposition of nets allows a development more consistent with conventional quantum mechanics. We therefore suggest that a more detailed study of effective quantum mechanics on causal nets, based on the transparent underlying concepts of causality and probability, might be a fruitful exercise.

The question of whether time is absolute or relational dates back to the days of Newton and Leibniz and is enshrined in the Leibniz–Clarke correspondence [11]. Here we adopt a relational view of time as an ordered series of closely spaced events. From this perspective a classical particle trajectory could appear as a statistically correlated series of events in space-time (for example, a series of actual observations). If the correlation is perfect then one may say that an event at one point in space "causes" the event at the next point, providing a Newtonian trajectory. If the correlation is imperfect, but greater than that resulting from statistical independence, then adjacent events in space are implied to have a common cause originating at previous time [9].

Reichenbach [9] argued in quite general terms that a satisfactory definition of time could only be obtained on the basis of a principle of common cause. By this he meant that if two events (A,B) occur repeatedly with greater frequency than complete statistical independence predicts (ie. P(A,B)>P(A)P(B)) then there exists an event C at previous time such that the fork ACB is "conjunctive", or has one side open (see Figure 1). A network of such conjunctive forks constitutes a causal net in which time is ordered and events may be considered simultaneous only when they share a common cause.

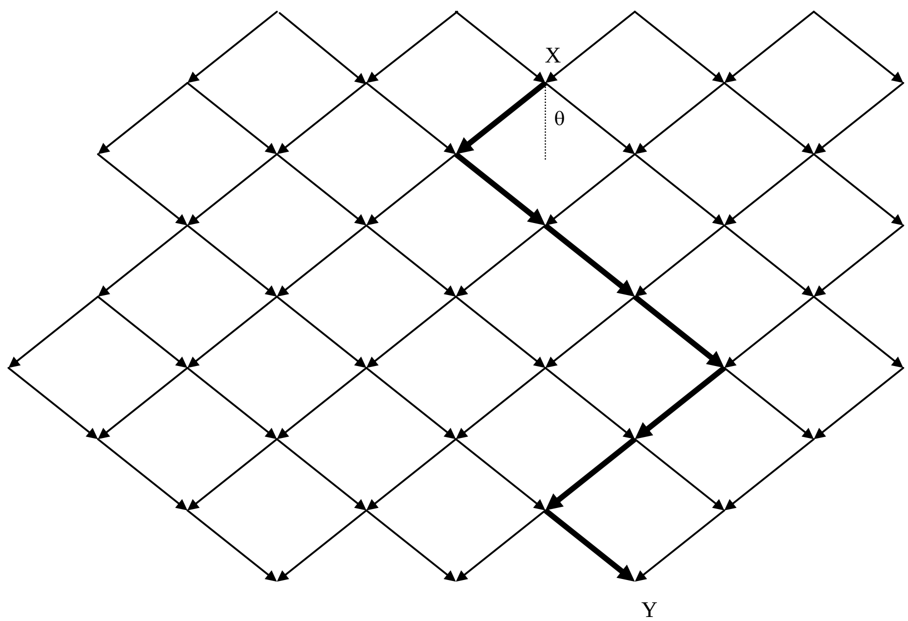
Figure 1: Reichenbach 'conjunctive' fork linking events A and B with common cause C



It is widely accepted that quantum mechanics cannot be developed from a basic application of Reichenbach's principle of common cause with a single conjunctive fork since general quantum mechanical statistics violate the principle [12]. Here we consider a modified framework where a complete causal network of possible events is comprised of conjunctive forks such that each possible event has two effective local common causes or screening factors. Adjacent possible events on the net that are simultaneous are thus considered to share a common cause. It appears that, at least in the simple case we consider, this allows our common cause principle - based on the simultaneity of neighboring *possible* events - to be applied consistently with quantum statistics.

To construct the causal net for a particle motion in space-time, we consider a 1 dimensional space aligned with the direction of particle motion, and embedded in 3 dimensional space. In this 1 dimensional space the simplest causal net that satisfies our definition of simultaneity is a 1+1 dimensional "diamond" lattice with causal links connecting the lattice points as in Figure 2. Note that this is topologically similar to an ordered ice type lattice [6]. Each causal connection is defined by a connecting arrow giving a definite lineal order and an associated probability. Each vertex on the causal net represents a possible event – meaning a possible observation of the particle - and has two incoming and two outgoing causal connections so that each point has an effective common cause. Starting at a vertex and following an outgoing arrow at random at each subsequent vertex describes a ‘causal chain’ as a series of possible events. The causal net thus describes the connectivity between all causal connections and can statistically model all future possibilities. Measurement or observation at a vertex or a region of the net provides, through Bayesian statistics, a re-evaluation of these probabilities after a measurement.

Figure 2: Causal net with common cause with causal chain from X to Y



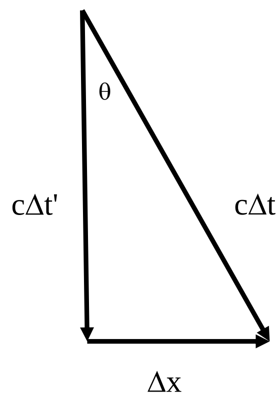
In this model time and space can be discretely “counted” by attaching an integer to each of the vertex points but there is no underlying continuous space-time. To relate to conventional mechanics we interpolate this set of integers by a set of real number coordinates. Expecting that space and time have different dimensions we need to introduce a constant c with dimensions [space/time]. The net is then made up of elementary triangles labeled with $(\Delta x, c\Delta t, c\Delta t')$ as shown in Figure 3. We have not yet added any specific interpretation to these quantities. To guarantee invariance of causality on the net we impose c the speed of light [13]. Since, from geometry, $\frac{\Delta x}{c\Delta t} = sin\theta \leq 1$, we then identify Δx and Δt as relativistic space-time intervals in an observer frame S and $\Delta t'$ as the particle proper time interval in its rest frame S'. The net geometry guarantees the invariant space-time interval

$$(c\Delta t')^2 = (c\Delta t)^2 - (\Delta x)^2. \tag{1}$$

Having abandoned the absolute concept of space-time we need to define the observed velocity in terms of finite differences. The definition adopted is $v = \Delta x/\Delta t$ which we equate to the expectation of the velocity on the causal net. The two time intervals are then related by $\Delta t' =$

$\Delta t/\gamma$ where γ is the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$, specifying the net angles $\cos\theta = \frac{1}{\gamma}$, $sin\theta = \frac{v}{c}$.

Figure 3: The space-time 'triangle' for the causal net

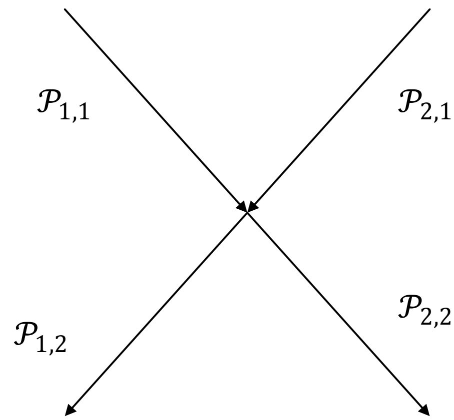


We now specialize to the case of the motion of a free particle. Clearly equation (1) can be scaled by a factor m which we interpret as the particle rest mass. Rearranging (1) we then have the relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$ where E is the particle energy $E = \gamma mc^2$ and $p = \gamma mv$ the momentum.

By construction the net describes only physically admissible motions of the particle. Experience shows that real particle trajectories obey a principle of least action – that is the integral $\int p\, dx$ is stationary. On our net if the action $\sum p\Delta x$ differed for different trajectories then this would rule some trajectories as physically inadmissible. Therefore we conclude that $\sum p\Delta x$ is the same on the net for all paths between two points which means that $p\Delta x$ is a constant η for valid nets. The possible paths followed by a particle are also analogous to Feynman paths [14] but the geometry of the net results in all vertices for constant t' having the equivalent action when the relativistic Lagrangian $-\sum \gamma mc^2\Delta t'$ is used. The construction of the net with simultaneous possible events occurring for equivalent proper time is also similar to Malament's [15] standard simultaneity condition where actual simultaneous events lie on a hyperplane orthogonal to the particle world-line and we shall see that many of the symmetries we derive for the causal net are consistent with his conditions for valid hypersurfaces.

This completes our geometrical description of the causal net. We see that the causal net is consistent with special relativity but that mechanical concepts such as mass, momentum and energy are projected onto an ordered set of possible observations or events. These events obey the normal rules of probability and are imperfectly correlated, hence giving an indeterminism to particle motion. Thus a particle in its own rest frame S' over interval $\Delta t'$ moving at a speed |v| in frame S can move to a position $\pm\Delta x$ in time Δt. This produces a random trajectory in space-time (see Figure 2).

Figure 4: A vertex (1,2) on the causal net with associated probabilities



Consider an individual vertex on the causal net and label the incoming probabilities on row 1 $\mathcal{P}_{1,1}$ and $\mathcal{P}_{2,1}$ and outgoing probabilities on row 2 $\mathcal{P}_{1,2}$ and $\mathcal{P}_{2,2}$ (Figure 4). Probability is conserved at the vertex and the total probability at a vertex is given by $\Omega_{1,2} = \mathcal{P}_{1,1} + \mathcal{P}_{2,1}$. If the average velocity measured on the net is uniform then $\mathcal{P}_{1,1} = \mathcal{P}_{2,2}$ and $\mathcal{P}_{1,2} = \mathcal{P}_{2,1}$. This implies that the probabilities 'cross' at each vertex without actually interfering although the probabilities are coupled. If we consider normalized branching probabilities at the vertex defined as $\hat{\mathcal{P}}_{1,1} + \hat{\mathcal{P}}_{2,1} = 1$ then the expected velocity at the vertex is

$$\mathbb{E}[v] = \gamma \frac{\Delta x}{\Delta t}\left[\hat{\mathcal{P}}_{1,1} - \hat{\mathcal{P}}_{2,1}\right] = v \tag{2}$$

The branching probabilities are then given by

$$\hat{\mathcal{P}}_{1,1} = \frac{E + mc^2}{2E} \ , \qquad \hat{\mathcal{P}}_{2,1} = \frac{E - mc^2}{2E} \tag{3}$$

From this we can see that in the low velocity limit $|v| \to 0$ then $\hat{\mathcal{P}}_{1,1} \to 1$ and $\hat{\mathcal{P}}_{2,1} \to 0$ and in the high velocity limit $|v| \to c$ then $\hat{\mathcal{P}}_{1,1} \to \hat{\mathcal{P}}_{2,1} \to 1/2$. The branching ratio Γ can be written as a function of γ or the net angle θ.

$$\Gamma = \frac{\hat{\mathcal{P}}_{1,1}}{\hat{\mathcal{P}}_{2,1}} = \frac{E + mc^2}{E - mc^2} = \frac{\gamma + 1}{\gamma - 1} = \frac{1 + \cos\theta}{1 - \cos\theta}$$

(4)

Using the branching probability (4) we can write a non trivial matrix equation linking the probabilities

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} \mathcal{P}_{1,1} \\ \mathcal{P}_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & \Gamma \\ 1/\Gamma & 0 \end{pmatrix} \begin{pmatrix} \mathcal{P}_{1,1} \\ \mathcal{P}_{2,1} \end{pmatrix}$$

(5)

We now show how this equation is consistent with the quantum mechanics of the particle.

First notice that identifying the net constant η with Planck's constant h provides the de Broglie relation $\lambda\, \mathrm{p} = \mathrm{h}$ [16] with $\Delta \mathrm{x} = \lambda/2$, and a Heisenberg like relation $\Delta \mathrm{p} \Delta \mathrm{x} \sim \mathrm{h}/2$ [17,18].

Now a general way of forming the probabilities $\mathcal{P}_{i,j}$ for the branch (i,j) is through a vector dot product $\mathcal{P}_{i,j} = \boldsymbol{\phi}_{\mathrm{i,j}}.\ \boldsymbol{\phi}_{\mathrm{i,j}}$ with $\boldsymbol{\phi}_{\mathrm{i,j}} = \sqrt{\mathcal{P}_{i,j}}\,(\mathrm{a}(t'), b(t'))$ with real components that depend on the proper time t' at the net vertices. The probability is invariant in the rest frame of the particle and equivalent to a gauge relationship that conserves probability in S' so using the relativistic invariance $-mc^2\Delta t' = p\Delta x - E\Delta t$ we can write the vector as

$$\boldsymbol{\phi}_{\mathrm{i,j}} = \sqrt{\mathcal{P}_{i,j}} \begin{pmatrix} \mathrm{a}(t') \\ \mathrm{b}(t') \end{pmatrix} = \sqrt{\mathcal{P}_{i,j}} \begin{pmatrix} \cos\,(mc^2 t'/\hbar) \\ \sin\,(mc^2 t'/\hbar) \end{pmatrix} = \sqrt{\mathcal{P}_{i,j}} \begin{pmatrix} \cos\,((px - Et)/\hbar) \\ \sin\,((px - Et)/\hbar) \end{pmatrix}$$

(6)

Where x and t are defined at the discrete net vertices. We can rewrite equation (5) as

$$\boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{\phi}_{1,1} \\ \boldsymbol{\phi}_{2,1} \end{pmatrix} = \left[\begin{pmatrix} 0 & \sqrt{\Gamma} \\ 1/\sqrt{\Gamma} & 0 \end{pmatrix} \otimes \boldsymbol{I} \right] \begin{pmatrix} \boldsymbol{\phi}_{1,1} \\ \boldsymbol{\phi}_{2,1} \end{pmatrix}$$

(7)

which can be alternatively expressed in terms of a unique transformation matrix $\boldsymbol{M}$

$$\boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{\phi}_{1,1} \\ \boldsymbol{\phi}_{2,1} \end{pmatrix} = [\boldsymbol{M} \otimes \boldsymbol{I}] \begin{pmatrix} \boldsymbol{\phi}_{1,1} \\ \boldsymbol{\phi}_{2,1} \end{pmatrix}$$

(8)

where

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1/\gamma & v/c \\ v/c & -1/\gamma \end{pmatrix} = \frac{1}{E}\begin{pmatrix} mc^2 & pc \\ pc & -mc^2 \end{pmatrix} = \frac{H_D}{E}$$

(9)

Here we recognize H_D as the Dirac Hamiltonian for a free particle[19] with defined momentum p.

We see that the relational concept of time gives rise to many of the features of relativistic motion both at the classical and quantum mechanical level. To connect with the complete quantum mechanics we note that equation (9) can be put in the conventional form [10,19] by assuming that space-time is locally differentiable at the vertex allowing us to use the usual matrix operator $\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}}$ to replace the momentum eigenvalues p writing

$$\begin{pmatrix} mc^2 & c\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} \\ c\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} & -mc^2 \end{pmatrix}\Psi = \mathrm{E}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

(10)

where we have replaced the real vector $\boldsymbol{\phi}$ with the complex Dirac spinor $\boldsymbol{\Psi}$ for the free particle.

The net is consistent with the Foldy-Wouthuysen representation [20] where the positive and negative energy states are decoupled through a rotation of θ of the Dirac Hamiltonian. In the above discussion we have considered only the positive energy states and the net effectively propagates a pair of 2 component spinors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(px-Et)/\hbar} \qquad \widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} = p \qquad S = 1/2$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(px-Et)/\hbar} \qquad \widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} = -p \qquad S = -1/2$$

(11)

corresponding to $\pm 1/2$ spin and these two states are weighted by the branching probabilities on the net. From this representation we can see how the 3+1 dimensional problem reduces to a 1+1 dimensional space-time, since the causal net is constructed parallel to the momentum direction $\boldsymbol{p}/|\boldsymbol{p}|$ in space. The Foldy-Wouthuysen states correspond with our framework where the Hamiltonian and velocity operator satisfy their classical analogues. Importantly, in this representation, establishing an exact particle position is impossible (there is only a mean position operator) and a particle is viewed as spread out over a finite region of about a wavelength which is consistent with our net picture since we cannot localize a particle between two adjacent net vertices without an averaging over net vertices being performed in a measurement.

Note that the matrix $\boldsymbol{M}$ above is a unitary, orthogonal matrix which provides an SU(2) group transformation corresponding to an improper rotation – that is a rotation preceded by an inversion. It is not the same as the transfer matrix derived in [6] for ice-type models, and relates to proper time rather time in the observer frame as considered there. The matrix provides the transformations for the probabilities $\boldsymbol{\mathcal{P}} = \mathbf{M}^2\boldsymbol{\mathcal{P}} = \mathbf{I}\boldsymbol{\mathcal{P}}$ and probability amplitudes $\boldsymbol{\Psi} = \boldsymbol{M}\boldsymbol{\Psi}$. Importantly because $\mathbf{M}^3 = \mathbf{M}$ there automatically exists only two levels of symmetry at the vertex and the causal net provides a dual existence to both the probabilities and the underlying probability amplitudes. Since it is an improper rotation the symmetry determines a preferred axis which provides helicity along the axis of movement. This provides the property of spin when projected onto different spatial axis through the operator $\hat{\boldsymbol{\sigma}}.\hat{\boldsymbol{p}}$.

It is worth emphasizing that the full Dirac equation arises only when we impose a continuous space-time and complex probability amplitude on the causal net. In contrast the quantization is automatically provided by the net and does not rely on the formalism of quantum mechanics. To illustrate this consider the relativistic particle in the (1 dimensional) box problem. For a potential well of depth V and width L bound states exist for a “forbidden zone” given by imaginary momentum states in the potential region. Now from the causal net model we can consider there to be an integer number n of net vertices to be contained in the well. This provides the quantization condition $\mathrm{n} = \mathrm{L}\Delta x$ giving quantized momentum states $\mathrm{p} = 2\mathrm{nL}/\mathrm{h}$. This corresponds to the solution of the 1+1 dimensional Dirac equation [21] and reduces to the Schrödinger particle in the box problem in the non relativistic limit. Thus for the particle in say its n=2 state then there are 2 space events on the net that can occur at the same time with equal probability.

Although the causal net appears to support the Dirac equation for free particle motion the net concept itself does not naturally imply a principle of superposition. We briefly consider how a principle of superposition might be achieved. The projection of a causal net of events onto space-time in our model is qualitatively similar to the projection of a Dirac state vector onto space-time to give a wavefunction [22]. Since we can add an arbitrary phase in equation (6) for $\boldsymbol{\phi}$ whilst preserving the probabilities (a gauge invariance) we can define a complete set of possible causal nets for a specified momentum state. These nets are adjacent, with non overlapping vertices, and map out a wavefunction defined in continuous space-time $\boldsymbol{\Psi}_{\mathrm{p}}(\mathrm{x}, \mathrm{t})$ for a macroscopic observer (who views space-time as virtually continuous). An ensemble of causal nets with different momentum states corresponds to superposition of the wavefunction in quantum mechanics, and thus, by summing probability amplitudes of overlapping vertices from nets is equivalent to superposing different probability amplitudes. This analysis can be extended to an EPR type experiment with parallel spin settings [12] and recent similar experiments [23]. For two particles there exists a set of possible pairs of nets corresponding to different experimental setups and if a particular spin measurement direction is chosen at one detector then a unique pair of nets is selected which is mutually exclusive to all the other pairs of nets existing in the initial set.

We can perhaps consider a common physical analogy for the causal net. Combining the vectors $\boldsymbol{\phi}$ or probability amplitudes in a linear manner is analogous to combining AC currents in an electrical circuit linearly to preserve phase before calculating power emitted. For the case of a free particle we previously noted that $\mathcal{P}_{1,1} = \mathcal{P}_{2,2}$ and $\mathcal{P}_{1,2} = \mathcal{P}_{2,1}$ so the causal net is analogous to electrical currents 'crossing' in wires at each vertex and the probability corresponding to the total instantaneous power.

The case we have considered is that of a free particle but we could include a potential V on the causal net since E can be replaced by E-V in the construction of the net and the branching ratios. Between two media with different potentials the net is compressed or stretched in space in the potential region with a form similar to Snell's Law $cos\theta_2/cos\theta_1 = E/(E-V)$. We can write (9) as

$$\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{pmatrix} = \frac{1}{E-V}\begin{pmatrix} mc^2 & pc \\ pc & -mc^2 \end{pmatrix} = \frac{H_D}{E-V} \tag{12}$$

The probability current is conserved at a potential barrier if we consider the relativistic change in probability across the barrier arising from Lorentz contraction/expansion.

It is interesting to consider the transition from quantum to classical behaviour**.** For massive objects since $\Delta \mathrm{x} = h/2p$ from the uncertainty relations when p is large – true for high mass or virtually any velocity - then Δx is small and uncertainty in x is small relative to the size of the object. The object can be well localized or resolved on the net and is effectively non quantum although it can be relativistic.

In summary we have explored the possibility that a Riechenbach-like causal net can generate a form of relativistic quantum mechanics that includes emergent Dirac particles and many of the usual observed quantum phenomena. The causal net accommodates a transition from classical to quantum behavior and the quantum measurement process reduces to Bayesian statistics when net vertex states are irreversibly measured, providing an order to the flow of time. The casual net model provides a relational space time similar to that envisaged by Liebniz [11], though differing in the sense it considers ordering of possible, as well as real, events. Since causal net vertices represent possible events, it is impossible to say whether a particle itself has local reality between events when it is observed and only when an actual measurement occurs can we definitely say that the region of space-time has reality and an existence. Interestingly this provides a model for the universe in which the next time 'slice' of possible reality is simply computed [24] from the probabilities held in the previous time slice. Obviously this is an essentially realist interpretation of space-time – that both space and time exist outside the human mind. We feel that the value of our causal net approach lies mainly in the clear, conceptual outline it suggests for combining special relativity and quantum mechanics and the possible description it provides for many quantum phenomena on the fundamental basis of causally linked events.

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References

1. Bramwell S.T., Giblin S.R., Calder S., Aldus R., Prabhakaran D., Fennell T., Measurement of the Charge and Current of Magnetic Monopoles in Spin Ice Nature 461 956 (2009)
2. R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, Mcgraw-Hill (1965)
3. Gaveau B., Jacobson T., Kac M., Schulman L.S. Relativistic Extension of the Analogy between Quantum Mechanics and Brownian Motion, Phys. Rev. Lett. 53, 419 (1984)
4. Bialynicki-Birula I., Weyl, Dirac, and Maxwell equations on a lattice as unitary cellular automata, 49 12 (1994)
5. Gerrit Coddens, A Remark on the Dirac Equation, Found. Physics Lett., 8 3 (1995)
6. Schotte K.D., Iwabuchi S. and Truong T.T., Ice Models and a Lattice Version of the Dirac Equation, Z.Phys.B 60, 255 (1985)
7. Mecklenburg M. and Regan B.C., Spin and the Honeycomb Lattice: Lessons from Graphene, arXiv:1003.3715v3, to be published (2010)
8. Wang R.F. et al, Artificial Spin Ice in a Geometrically Frustrated Lattice of Nanoscale Ferromagnetic Islands, Nature 439, 303 (2006)
9. Reichenbach H. The Direction of Time, University California Press (1956), Dover (1999)
10. Dirac P.A.M. The Quantum Theory of the Electron, Proc. Royal Soc. A117 (1928)
11. Clarke S. A Collection of Papers which passed between the late learned Mr Leibniz and Dr Clarke in the years 1715 and 1716. Relating to the Principle of Natural Philosophy and Religion (1717).
12. Grasshoff G., Portmann S. and Wuthrich A., Minimal Assumption Derivation of a Bell-type Inequality, Brit. J. Phil. Sci. 56, 663 (2005)
13. Rindler W., Introduction to Special Relativity, Oxford (1982)
14. Feynman R.P. Space –Time Approach to Non Relativistic Quantum Mechanics, Rev. Mod. Physics 20, 2 367 (1948)
15. Malament D., Causal Theories of Time and the Conventionality of Simultaneity, Nous 11, 293 (1977)
16. De Broglie L. Ann. Phys. (Paris) 3, 22 (1925) and Nobel lecture “The wave nature of the electron” (1929)
17. Heisenberg W.Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Zeitschrift fur Physik, 43,172-198 (1927)
18. Schurmann T. and Hoffmann I., A Closer Look at the Uncertainty Relation of Position and Momentum, Found. Phys. 39, 958, (2009)
19. Davydov A.S. Quantum Mechanics, Pergamon Press (1965)
20. Foldy L. and Wouthuysen S., On the Dirac Theory of Spin ½ Particles and Its Non-Relativistic Limit, Phys . Rev. 78 1 (1950)
21. Coulter B., Adler C. The Relativistic One Dimensional Square Potential, AJP Vol 39, 305 (1971)
22. Dirac P.A.M., The Principles of Quantum Mechanics, 4^{TH} Edition Oxford (1974)
23. Jian-Wei Pan, Bouwmeester D., Daniell M.,Weinfurter H. and Zeilinger A., Experimental Test of Quantum Nonlocality in Three-Photon Greenberger-Horne-Zeilinger entanglement, 403, 515, Nature (2000)
24. Wolfram S., A New Kind of Science, Wolfram Media (2002)