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A CONJECTURE ABOUT THE RIEMANN XI-FUNCTION $\xi\left(\frac{1}{2}+iz\right)$ AND FUNCTIONAL DETERMINANTS

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- ABSTRACT: We give a possible interpretation of the Xi-function of Riemann as the Functional determinant $\det(E-H)$ for a certain Hamiltonian quantum operator in one dimension $-\frac{d^2}{dx^2}+V(x)$ for a real-valued function V(x) , this potential V is related to the half-integral of the logarithmic derivative for the Riemann Xi-function, through the paper we will assume that the reduced Planck constant is defined in units where $\hbar=1$ and that the mass is $2m=1$
- *Keywords: =* Riemann Hypothesis, Functional determinant, WKB semiclassical Approximation , Trace formula , Quantum chaos.

RIEMANN FUNCTION AND SPECTRAL DETERMINANTS

The Riemann Hypothesis is one of the most important open problems in mathematics, Hilbert and Polya [4] gave the conjecture that would exists an operator $\frac{1}{2}+iL$ with $L=L^{\dagger}$ so the eigenvalues of this operator would yield to the non-trivial zeros for the Riemann zeta function, for the physicists one of the best candidates would be a Hamiltonian operator in one dimension $-\frac{d^2}{dx^2}+V(x)$, so when we apply the quantization rules the Eigenvalues (energies) of this operator would appear as the solution of the spectral determinant $\det(E-H)$, if we define the Xi-function by

$\xi(s)=\frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2}$, then RH (Riemann Hypothesis) is equivalent to the fact that the function $\xi\left(\frac{1}{2}+iE\right)$ has REAL roots only , and then from the Hadamard product expansion [1] for the Xi-function , then $\frac{\xi\left(\frac{1}{2}+iE\right)}{\xi(1/2}\approx \det(E-H)$ is an spectral (Functional) determinant of the Hamiltonian operator, if we could give an expression for the potential V(x) so the eigenvalues are the non-trivial zeros of the zeta function, then RH would follow, we will try to use the semiclassical WKB analysis [8] to obtain an approximate expression for the inverse of the potential.

Trough this paper we will use the definition of the half-derivative $D_x^{1/2}f$ and the half integral $D_x^{-1/2}f$, this can be defined in terms of integrals and derivatives as

$$\frac{d^{1/2}f(x)}{dx^{1/2}}=\frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x\frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-1/2}f(x)}{dx^{1/2}}=\frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (1)$$

The case $D_x^{3/2}f$ we can simply use the identity $D_x^{3/2}f=\frac{d}{dx}\left(D_x^{1/2}f\right)$, these half-integral and derivative will be used further in the paper in order to relate the inverse of the potential V(x) to the density of states g(E) that 'counts' the energy levels of a one dimensional (x,t) quantum system.

- *Semiclassical evaluation of the potential V(x) :*

Unfortunately the potential V can not be exactly evaluated, a calculation of the potential can be made using the semiclassical WKB quantization of the Energy

$$\left(n(E)+\frac{1}{2}\right)2\pi\approx 2\int_0^{a=a(E)}\sqrt{E-V(x)}dx\rightarrow 2\int_0^E\sqrt{E-V}\frac{dx}{dV}=\sqrt{\pi}D_x^{3/2}\left(\frac{dV^{-1}(x)}{dx}\right) \quad (2)$$

Here we have introduced the fractional integral of order 3/2 , for a review about fractional Calculus we recommend the text by Oldham [10] , a solution to equation (2) can be obtained by applying the inverse operator $D_x^{-1/2}$ on the left side to get

$$V^{-1}(x)\approx 2\sqrt{\pi}\frac{d^{1/2}n(x)}{dx^{1/2}} \qquad V^{-1}(x)\approx 2\sqrt{\pi}\frac{d^{-1/2}g(x)}{dx^{-1/2}} \qquad \frac{dn}{dx}=g(x) \quad (3)$$

Here n(E) or N(E) is the function that counts how many energy levels are below the energy E , and g(E) is the density of states $g(E)=\sum_{n=0}^{\infty}\delta(E-E_n)$, for the case of Harmonic oscillator $N(E)=\frac{E}{\omega}$ so using formula (2) and taking the inverse function we recover the potential $V(x)=\frac{\omega^2 x^2}{4}$,which is the usual Harmonic potential for a mass $2m=1$ a similar calculation can be made for the infinite potential well of length 'L' with boundary conditions on $[0,L]$ to check that our formula (3) can give coherent results

In general g(E) is difficult to calculate and we can only give semiclassical approximations to it via the Gutzwiller Trace formula [8] , for the case of the Riemann Zeta function , N(E) can be defined by the equation

$$N(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right) \qquad \xi(s)=\frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} \qquad (4)$$

So, in this case the Potential V(x) inside the one dimensional Hamiltonian operator whose energies are precisely the imaginary part of the Riemann zeros is given implicitly by the functional equation

$$V^{-1}(x)\approx\frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\xi'}{\xi}\left(\frac{1}{2}+ix\right)\right)+\frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\xi'}{\xi}\left(\frac{1}{2}-ix\right)\right) \qquad (5)$$

$V^{-1}(x)$ is the inverse of V(x) , taking the inverse function of formula (5) we could recover the potential at least numerically.

Using the asymptotic calculation of the smooth density of states , we could separate formula (4) into an oscillating part defined by the logarithmic derivative of the Riemann zeta function and a smooth part whose behaviour is well-known for big 'x'

$$\frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+ix\right)\right)+\frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ix\right)\right)+\frac{1}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\left(x\log x-x+c\right) \quad (6)$$

$c=\frac{7\pi}{4}$, Using Zeta regularization , as we did in our previous paper [6] we can expand the oscillating part of formula (6) into the divergent series

$$\frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+ix\right)\right)+\frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ix\right)\right)=2\sum_{n=2}^{\infty}\frac{\Lambda(n)\cos(x\log n+\pi/4)}{\sqrt{n\pi\log n}} \quad (7)$$

$\Lambda(n)$ is the Von-Mangoldt function that takes the value log(p) if $n=p^m$ for some positive integer 'm' and a prime p and 0 otherwise , so the last sum inside (7) involves a sum over primes and prime powers.

Then , using (3) we have found a relationship between a classical quantity , the potential V(x), and the density of states $g(E)=\sum_{n=0}^{\infty}\delta(E-E_n)$ of a one dimensional dynamical system, the problem here is that g(E) can not be determined exactly unless for trivial Hamiltonians (Harmonic oscillator, potential well) the best evaluation for g(E) would come from the Gutzwiller Trace [8] , some people believe [4] that a possible proof for the Riemann Hypothesis would follow from the quantization of an hypothetical dynamical system whose dynamical zeta function is proportional to $\zeta\left(\frac{1}{2}+iE\right)$ to the spectral determinant of this dynamical system is $\det\left(E-H\right)\approx e^{-iN(E)}\zeta\left(\frac{1}{2}-iE\right)$, in this simple case the periodic orbits of the dynamical system are proportional to $\log\left(p^{m}\right)$ for m positive integer and 'p' a prime number, in this case the Quantization of the Hamiltonian 'H' would yield to the imaginary part of the non-trivial zeros, these zeros then would appear to be eigenvalues (energies) of H , since H is self-adjoint /Hermitean this energies would be all REAL and all the non-trivial zeros would be of the form $\frac{1}{2}+it \quad t\in R$, in this case the approximate Gutzwiller Trace would be of the form

$$g(E)\approx g_{smooth}(E)+\frac{1}{\pi}\Im m\left(\frac{\partial}{\partial E}\log\zeta\left(\frac{1}{2}+iE\right)\right) \quad (8)$$

Here $g_{smooth}(E)\approx\frac{1}{2\pi}Arg\Gamma\left(\frac{s}{2}\right)s(s-1)\pi^{-s/2}, \quad s=\frac{1}{2}+iE$ this contribution is well-known , for big energies E this is the main contribution to the density of states g(E) , the part involving the logarithmic derivative inside (8) is the oscillating part of the potential giving the zeros, if we combine (5) and (8) we can obtain an expression for the inverse of the potential V(x) , then solving the Hamiltonian $H=-\frac{d^{2}}{dx^{2}}+V(x)$ with the potential given by formulae (5) and (6) we could obtain approximately the imaginary parts of the non-trivial zeros.

- *Numerical calculations of functional determinants using the Gelfand-Yaglom formula :*

In the semiclassical approach to Quantum mechanics we must calculate path integrals of the form $\int_{V} D\left[\phi\right]e^{-\langle\phi|H|\phi\rangle}\approx\frac{1}{\sqrt{\det H}}$ and hence compute a Functional determinant, one of the fastest and easiest way is the approach by Gelfand and Yaglom [2] , this technique is valid for one dimensional potential and allows you calculate the functional determinant of a certain operator 'H' without needing to compute any eigenvalue, for example if we assume Dirichlet boundary conditions on the interval [0,L]

$$\frac{\det\left(H+z^{2}\right)}{\det(H)}=\frac{\prod_{n=0}^{\infty}(\lambda_{n}-z^{2})}{\prod_{n=0}^{\infty}\lambda_{n}}=\prod_{n=0}^{\infty}\left(1-\frac{z^{2}}{\lambda_{n}}\right)=\frac{\Psi^{(z)}(L)}{\Psi^{(0)}(L)} \qquad (9)$$

Here the function $\Psi^{(z)}(L)$ is the solution of the Cauchy initial value problem

$$\left(-\frac{d^2}{dx^2}+V(x)+z^2\right)\Psi^{(z)}(x)=0 \qquad \Psi^{(z)}(0)=0 \qquad \frac{d\Psi^{(z)}(0)}{dx}=1 \qquad (10)$$

For our Hilbert-Polya Hamiltonian , the imaginary part of the non-trivial zeros would appear as the solution of the eigenvalue problem $H\phi = E_n\phi$ with the conditions

$$\phi(0)=\phi(L)=0 \quad L\to\infty \qquad V_{RH}^{-1}(x)\approx\frac{2}{\sqrt{\pi}}\Re e\left(\frac{1}{\sqrt{i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\xi'}{\xi}\left(\frac{1}{2}+ix\right)\right)\right) \qquad (11)$$

Since $N(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$ then $N(0)=0$, also the Riemann Xi-function is an even function $\xi(s)=\xi(1-s)$ if 's' lies on the critical line, $s=\frac{1}{2}+iz$, another possible Dirichlet boundary condition is $\phi(-L)=\phi(L)=0$ as $L\to\infty$, this is equivalent to the assertion that $\phi\in L^2(R)$, in QM the eigenfunctions must be square-integrable $\int_{-\infty}^{\infty}dx|\phi(x)|^2<\infty$, then using the Gelfand-Yaglom theorem to evaluate the spectral determinant we find $\frac{\det\left(H+z^2\right)}{\det(H)}=\frac{\xi\left(\frac{1}{2}+iz\right)}{\xi\left(1/2\right)}=\frac{\phi^{(z)}(L)}{\phi^{(0)}(L)} \quad L\to\infty$, the eigenfunctions are the solution to the problem exposed in formula (11) , and they satisfy the initial value problem

$$(H+z^2)\phi^{(z)}(x)=\left(-\frac{d^2}{dx^2}+V(x)+z^2\right)\phi^{(z)}(x)=0 \qquad \phi^{(z)}(0)=0 \qquad \frac{d\phi^{(z)}(0)}{dx}=1 \qquad (12)$$

If we take the logarithm inside the Gelfand-Yaglom expression for the functional determinants [2] we can also get an expression for the spectral zeta function of eigenvalues for integer values of 's' $\log\frac{\det\left(H+z^2\right)}{\det(H)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\zeta_H(n)z^{2n}$ with $\sum_{n=0}^{\infty}\frac{1}{\lambda_n^{\,s}}=\zeta_H(s)$, for the case of the Riemann Xi-function , if RH is true then we should have that the Taylor expansion of $\log\xi\left(\frac{1}{2}+iz\right)-\log\xi\left(\frac{1}{2}\right)$ can be used to extract information about the sums $\sum_{n=0}^{\infty}\frac{1}{\gamma_n^{\,k}}$ involving the imaginary parts of the Riemann zeros , in general these sums $\sum_{n=0}^{\infty}\frac{1}{\gamma_n^{\,k}}$ can be evaluated by numerical methods so we can compare the Taylor series of the logarithm of Xi function near x=0 and these sums to check the validity (at least numerically) of Riemann Hypothesis.

CONCLUSIONS AND FINAL REMARK

Using the semiclassical analysis and the WKB quantization of energies, we have managed to prove that for one dimensional systems the inverse of the potential inside the Hamiltonian $H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$ (for simplicity through this paper we have used the notation $\hbar=1=2m$ to simplify calculation) is related to the half-derivative of the energy counting function N(E) or the half-integral of the density of states $g(E)=\sum_{n=0}^{\infty}\delta(E-E_n)$ in the (approximate) form $V^{-1}(x)\approx\sqrt{\frac{2\pi\hbar^2}{m}}\frac{d^{-1/2}g(x)}{dx^{-1/2}}$. From the definition of the counting function N(T) for the nontrivial zeros of the Riemann Zeta function lying on the critical strip $N(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$ and using our formula (2) with 2m=1 we have obtained the semiclassical approximation for the inverse of the potential V(x) (4) and have given some Numerical test that could be done in order to check that our definition for the inverse of the potential $V^{-1}(x)$ can be used to obtain a Hilbert-Polya differential operator whose eigenvalues (Energies) are precisely the imaginary part of the non-trivial zeros of the Riemann Zeta function, in case our formula is correct and (4) is the potential of a Hilbert-Polya operator satisfying Riemann Hypothesis then imposing Dirichlet boundary conditions $[0,L]$ with $L\to\infty$ then the Gelfand-Yaglom formula should give $\frac{\xi\left(\frac{1}{2}+iz\right)}{\xi(1/2)}=\frac{\det\left(H+z^2\right)}{\det(H)}=\frac{\phi^{(z)}(L)}{\phi^{(0)}(L)}\quad L\to\infty$ so the functional determinant of $H=-\frac{d^2}{dx^2}+V(x)$ is proportional to the Riemann Xi-function , this potential V(x) is given implicitly in (5) and (6) . A possible better improvement of our formula (3) would be to write down $V^{-1}(x)\approx A\frac{d^{1/2}n(x)}{dx^{1/2}}$ for some constant ‘A’ to be fixed in order to obtain (if possible) the correct energies of our Hamiltonian at least when $n\to\infty$ E_n

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