A Clifford algebra realization of Supersymmetry and its Polyvector extension in Clifford Spaces

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Abstract

It is shown explicitly how to construct a novel (to our knowledge) realization of the Poincare superalgebra in $2D$. These results can be extended to other dimensions and to (extended) superconformal and (anti) de Sitter superalgebras. There is a fundamental difference between the findings of this work with the other approaches to Supersymmetry (over the past four decades) using Grassmannian calculus and which is based on anti-commuting numbers. We provide an algebraic realization of the anti-commutators and commutators of the $2D$ super-Poincare algebra in terms of the generators of the tensor product $Cl_{1,1}(R) \otimes \mathcal{A}$ of a two-dim Clifford algebra and an internal algebra $\mathcal{A}$ whose generators can be represented in terms of powers of a 3×3 matrix $\mathcal{Q}$, such that $\mathcal{Q}^3 = 0$. Our realization $differs$ from the standard realization of superalgebras in terms of $differential$ operators in Superspace involving Grassmannian (anti-commuting) coordinates θ^α and bosonic coordinates x^μ. We conclude in the final section with an analysis of how to construct Polyvector-valued extensions of supersymmetry in Clifford Spaces involving spinor-tensorial supercharge generators $\mathcal{Q}_\alpha^{\mu_1\mu_2....\mu_n}$ and momentum polyvectors $P_{\mu_1\mu_2....\mu_n}$. Clifford-Superspace is an extension of Clifford-space and whose symmetry transformations are generalized polyvector-valued supersymmetries.

KEYWORDS : Clifford algebras; Supersymmetry; Polyvector-supersymmetry; M, F theory superalgebras.

1 Clifford algebra realization of Supersymmetry

Clifford algebras have been a very useful tool for a description of geometry and physics [4], [5]. In [5],[3],[6] it was proposed that every physical quantity is in fact a *polyvector* where the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford-algebra-valued objects : *polyvectors*. Such a manifold has been called a pan-dimensional continuum [3] or C-space [1], [2]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also lines, surfaces, volumes,.., called 0-loops (points), 1-loops (closed strings) 2-loops (closed membranes), 3-loops, etc.. It is a sort of a *dimension* category, where the role of functorial maps is played by C-space transformations which reshuffles a p-brane history for a p'-brane history or a mixture of all of them, for example.

The above geometric objects may correspond to the well-known physical objects, namely closed p-branes. Technically those transformations in C-space, that reshuffle objects of different dimensions, are generalizations of ordinary Lorentz transformations of spacetime events to C-space. In that sense the C-space is roughly speaking a sort of generalized Penrose-Twistor space from which the ordinary spacetime is a *derived* concept. Penrose's twistor theory has been generalized to Clifford algebras by [7] where the basic geometric forms and their relationships are expressed algebraically. In addition, by means of an inner automorphism of this algebra, it is possible to regard these forms and relationships as *emerging* from an underlying pre-space.

In this section we show explicitly how to construct a novel (to our knowledge) algebraic realization of the Poincare superalgebra in $2D$. These results can be extended to other dimensions and to (extended) superconformal and (anti) de Sitter superalgebras. There is a fundamental difference between the findings of this work with the other approaches to Supersymmetry (over the past four decades) using Grassmannian calculus, and which is based on anticommuting numbers. We provide an algebraic explicit realization of the anticommutators and commutators of the super-Poincare algebra in terms of a two-dim Clifford algebra generators and an internal algebra $\mathcal{A}$, whose generators can be represented in terms of powers of a 3×3 matrix $\mathcal{Q}$, such that $\mathcal{Q}^3 = 0$. The realization $differs$ from the standard realization of superalgebras in terms of $differential$ operators in Superspace involving Grassmannian (anti-commuting) coordinates θ^α and bosonic coordinates x^μ.

It is well known that the particle content of supersymmetric theories fall under irreducible representations of Clifford algebras. The N extended supersymmetry algebra in $4D$ Minkowski spacetime is based mainly on the anticommutators $\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij}(\mathcal{C}\gamma^\mu)_{\alpha\beta}P_\mu$, for $i, j = 1, 2, 3, ...N$; and $\mathcal{C}$ is the charge conjugation matrix. In the rest frame for massive particles $m \neq 0$, the anticommutator takes the form of an algebra of $2n$ fermionic creation and annihilation operators isomorphic to the Clifford algebra $Cl(4n)$. Its unique *irreducible* representation is 2^{2n} dimensional and contains both boson and fermions as required by supersymmetry. In the massless case, there is no rest frame and there are only 2^n states that are classified according to helicity, rather than spin. In ordinary Poincare supersymmetry, the anti-commutator is

$$\{Q_\alpha, Q_\beta\ \} = \frac{1}{2} C\gamma^\mu P_\mu \tag{1.1}$$

In $D = 4$, with signatures $-,+,+,+$ one can find a charge conjugation matrix C and its transpose C^T obeying the properties

$$(C\gamma^\mu)^T = (C\gamma^\mu). \quad (C\gamma^{\mu\nu})^T = (C\gamma^{\mu\nu}) \tag{1.2}$$

$$C^T = -C, \quad C\gamma_\mu C^{-1} = -\gamma_\mu^T. \quad C^\dagger C = CC^\dagger = 1, \quad C^{-1}\gamma_{\mu\nu}C = -\gamma_{\mu\nu}^T. \tag{1.3}$$

The crux of our findings is that in $D = 2$, for example, one may find a realization of the $2D$ Poincare superalgebra in terms of two-dim Clifford algebra generators, and an additional algebra $\mathcal{A}$, by expressing the supercharges $Q^\alpha, Q^\beta, \ \alpha,\beta = 1,2$ as

$$\mathbf{Q}^1 \ = \ i \ \Gamma^1 \otimes \mathcal{Q}, \quad \mathbf{Q}^2 \ = \ \Gamma^2 \otimes \mathcal{Q} \tag{1.4}$$

and such that the anti-commutators are

$$\{ \ \mathbf{Q}^\alpha, \ \mathbf{Q}^\beta \ \} \ = \ \{ \ \Gamma^\alpha \otimes \mathcal{Q}, \ \Gamma^\beta \otimes \mathcal{Q} \ \} = \ 2 \ \delta^{\alpha\beta} \ (\ \mathbf{1} \otimes \mathcal{Q}^2 \ ) \tag{1.5}$$

where $\mathcal{Q}$, $(\mathcal{Q})^2 \equiv \mathcal{Q}^2 = \mathcal{P}$ are two 3×3 matrix generators associated with the second factor algebra $\mathcal{A}$ defined below in eqs-(1.18, 1.19). The representation of the gamma matrices in $2D$ is a Majorana one given by purely imaginary matrices Γ_1, Γ_2 (as opposed to real matrices)

$$\Gamma_1 \ = \ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \Gamma_2 \ = \ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \Gamma_1\Gamma_2 = \Gamma_3 \ = \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.6}$$

The momentum operators can be represented as the tensor products

$$\mathbf{P}_\mu = P_\mu \otimes \mathcal{P} \ = \ \frac{1}{2}\Gamma_\mu(1-\Gamma_3) \otimes \mathcal{P}, \quad \mu,\nu = 1,2 \tag{1.7}$$

where $\mathcal{P} = \mathcal{Q}^2$ is one of 3×3 matrix generators of the algebra $\mathcal{A}$ described in eqs-(1.18, 1.19) below. Using the relations in (1.7)

$$\Gamma^1\Gamma_1 = g^{11}\Gamma_1\Gamma_1 = g^{11}g_{11} \ \mathbf{1} = \mathbf{1}, \quad \Gamma^2\Gamma_2 = g^{22}\Gamma_2\Gamma_2 = g^{22}g_{22} \ \mathbf{1} = \mathbf{1} \tag{1.8}$$

and $\{\Gamma_\mu, \Gamma_3\} = 0$, allows to evaluate

$$\Gamma^\mu \ P_\mu \ + \ P_\mu \ \Gamma^\mu = \ (1-\Gamma_3) \ + \ (1+\Gamma_3) = \ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{1.9}$$

One should remark that because $P_\mu = \frac{1}{2}\Gamma_\mu(1-\Gamma_3)$ are now represented as matrices in terms of a $Cl_{1,1}(R)$ algebra generators, one then has that $\Gamma^\mu P_\mu \neq P_\mu \Gamma^\mu$, and it is for this reason that one may choose the following ordering procedure

$$2 \ \Gamma^\mu \ P_\mu \ \leftrightarrow \ \Gamma^\mu \ P_\mu \ + \ P_\mu \ \Gamma^\mu \tag{1.10}$$

such that one can write the anti-commutators as

$$\{ \ \mathbf{Q}^\alpha, \ \mathbf{Q}^\beta \ \} \ = \ 2 \ \delta^{\alpha\beta} \ (\mathbf{1} \otimes \mathcal{Q}^2) \ = \ (\Gamma^\mu P_\mu \ + \ P_\mu \ \Gamma^\mu)^{\alpha\beta} \ (\mathbf{1} \otimes \mathcal{P}) \tag{1.11}$$

since $(\Gamma^\mu P_\mu + P_\mu\ \Gamma^\mu)^{\alpha\beta} = 2\ \delta^{\alpha\beta}$, and $\mathcal{Q}^2 = \mathcal{P}$, as displayed in the Appendix. The most salient feature of the above equation is that eq-(1.11) has a one-to-one correspondence with an ordinary anti-commutator relation of the form $\{Q^\alpha, Q^\beta\} = 2(\Gamma^\mu P_\mu)^{\alpha\beta}$.

The explicit derivation of the other commutators

$$[\ \mathbf{P}_\mu,\ \mathbf{Q}^\alpha\ ]\ =\ 0,\ \ [\ \mathcal{M}_{\mu\nu},\ \mathbf{Q}^\alpha\ ]\ =\ (\Gamma_{\mu\nu}\ \mathbf{Q})^\alpha. \tag{1.12}$$

$$[\ \mathbf{P}_\mu,\ \mathbf{P}_\nu\ ]\ =\ 0;\quad [\mathcal{M}_{\mu\nu},\ \mathbf{P}_\rho\ ]\ =\ -\ g_{\mu\rho}\ \mathbf{P}_\nu\ +\ g_{\nu\rho}\ \mathbf{P}_\mu. \tag{1.13}$$

$$[\ \mathcal{M}_{\mu\nu},\ \mathcal{M}_{\rho\tau}\ ]\ =\ g_{\nu\rho}\ \mathcal{M}_{\mu\tau}\ -\ \ g_{\mu\rho}\ \mathcal{M}_{\nu\tau}\ +\ g_{\mu\tau}\ \mathcal{M}_{\nu\rho}\ -\ g_{\nu\tau}\ \mathcal{M}_{\mu\rho}. \tag{1.14}$$

and the verification of the graded super Jacobi identities ensuring the closure of the superalgebra will be presented in the Appendix. We shall display in full detail all the calculations showing how the $2D$ Poincare superalgebra can be realized as the tensor product of two algebras $Cl_{1,1}(R)\otimes\mathcal{A}$. In a way this is not surprising since the Poincare superalgebra entails both bosonic and fermionic generators and this explains the need to extend the Clifford algebra $Cl_{1,1}(R)$ (with four generators) to the tensor product algebra $Cl_{1,1}(R)\otimes\mathcal{A}$ involving more generators.

This procedure differs from the constructions based on Grassmannian variables. For example, in $4D$ one has the expression for chiral (anti-chiral) supercharges, written in two-component spinor notation

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\ \bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu}.\quad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\ \frac{\partial}{\partial x^\mu} \tag{1.15}$$

in terms of the matrices $\sigma^\mu_{\alpha\dot\alpha}$, $\mu = 0,1,2,3$, given by the unit and three Pauli spin matrices, respectively; the Grassmannian (anti-commuting) coordinates $\theta^\alpha, \bar\theta^{\dot\alpha}$ and the bosonic coordinates x^μ.

To conclude, with the inclusion of the Appendix, we have found explicitly an algebraic realization of the (anti) commutators of a $2D$ super-Poincare algebra given entirely in terms of the generators of $Cl_{1,1}(R)\otimes\mathcal{A}$ and that $differs$ from the standard realization in terms of $differential$ operators in Superspace involving the Grassmannian (anti-commuting) coordinates θ^α and the bosonic coordinates x^μ.

To finalize this section, we should add that a construction of the so-called "super Clifford algebras" as extensions of superconformal algebras $SU(2,2|N)$ based on Z_4 gradings of Clifford algebras has been provided by [13]. A Clifford analysis of superspace based on Weyl-symplectic-Clifford and orthogonal-Clifford algebras can be found in [14]. However, the approaches and results of [13], [14] to superalgebras and supercalculus are very different from ours.

There are 8 generators in total, the double in the number of generators of $Cl_{1,1}(R)$, given by

$$P_1 = \Gamma_1(\mathbf{1} - \Gamma_3),\quad P_2 = \Gamma_2(\mathbf{1} - \Gamma_3),\quad \mathcal{M}_{12} = \frac{1}{2}\ \Gamma_3 \tag{1.16}$$

corresponding to the two momentum (translation) operators and the $2D$ Lorentz generator, respectively; and the additional generators

$$\mathbf{Q}^1 = i\ \Gamma^1 \otimes \mathcal{Q}, \quad \mathbf{Q}^2 = \Gamma^2 \otimes \mathcal{Q}, \quad \mathcal{Q}, \quad \mathcal{P}, \quad \mathbf{1} \tag{1.17}$$

such that

$$\mathcal{Q}^2 = \mathcal{P}, \quad \mathcal{Q}\ \mathcal{P}\ =\ \mathcal{P}\ \mathcal{Q}\ =\ \mathcal{Q}^3\ =\ 0. \tag{1.18}$$

The generators $\mathcal{Q}, \mathcal{P}$ admit a representation in terms of the 3×3 matrices

$$\mathcal{Q}\ =\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Q}^2\ =\ \mathcal{P}\ =\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Q}^3\ =\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.19}$$

The unit operator $\mathbf{1} = (\mathcal{Q})^0$ of the algebra $\mathcal{A}$ can be represented as the unit 3×3 matrix, whereas the unit operator $\mathbf{1}$ of the $Cl_{1,1}(R)$ algebra is the unit 2×2 matrix.

The authors [17] have studied a Z_3 graded extension of supersymmetry, what has been called *hypersymmetry* involving *ternary* algebraic extensions of Clifford algebras and the Grassmanian where now $\theta^3 = 0$ (instead of $\theta^2 = 0$) and the exterior differentials instead of obeying $\mathbf{d}^2 = 0$ obey now $\mathbf{d}^3 = 0$. A generalized cohomology complex $\mathcal{Q}^2_{BRST} = 0 \rightarrow \mathcal{Q}^3_{BRST} = 0$ and $\mathcal{Q}^n_{BRST} = 0, n \geq 2$ has been analyzed by [18]. It is associated with a higher spin fields cohomology. W_∞ algebras are the higher conformal spins $s = 2, 3, 4, ..., \infty$ extensions of the Virasoro algebra in $2D$ and which have been extensively studied over the past decades.

Our results in $2D$ can be extended to other dimensions. In particular, one could try to find realizations of the $\mathcal{N} = 1$ superconformal algebra in $D = 4$, and its extensions, like the celebrated $\mathcal{N} = 4$ superconformal algebra due to the finiteness of the quantum theory resulting from the fact that the β function for $\mathcal{N} = 4$ super Yang-Mills theory vanishes. The connection among Clifford algebras, conformal algebras and Twistors [7] deserves to be explored further, and its supersymmetric extensions, within the algebraic formalism presented here involving tensor products of a Clifford algebra with a judicious internal $\mathcal{A}$ algebra. Speaking of Twistors, one should say that an intrinsic massless-like structure is already operating in the choice of momentum operators (1.7) in $2D$ Minkowski spacetime. It is not difficult to see that the operators P_μ are nilpotent $P_1 P_1 = P_2 P_2 = P_1 P_2 = 0$ and that $P_1 = P_2$. The last relation is reminiscent of the on-shell momentum condition for massless particles in $2D$ Minkowski spacetime : $-(p_1)^2 + (p_2)^2 = 0 \Rightarrow p_1 = p_2$.

To finalize, we should add that the choice for the supercharges $\mathbf{Q}^1 = i(\Gamma^1)^\alpha_\beta \otimes \mathcal{Q}^a_b$ and $\mathbf{Q}^2 = (\Gamma^2)^\alpha_\beta \otimes \mathcal{Q}^a_b$, after inserting the spinorial indices in the 2×2 Gamma matrices and the internal indices ($a, b = 1, 2, 3$) of the 3×3 matrix generator $\mathcal{Q}$, is also reminiscent of having quaternionic-valued spinors which carry an additional internal $SU(3)$ color-like index structure; i.e. $2D$ spinors whose two entries are themselves quaternionic valued and which carry additional internal color-like labels represented by the indices $a, b = 1, 2, 3$ of the 3×3 matrix generator $\mathcal{Q}$. A theory of algebraic quark confinement involving a ternary and non-associative structure (like in octonions) can be found in [17] and references therein.

2 Polyvector Supersymmetry in Clifford Spaces

Clifford-Superspace is an extension of Clifford-space and whose symmetry transformations are generalized polyvector-valued supersymmetries. Polyvector super-Poincare algebras have been studied within a different context from the point of view of M, F theory superalgebras in $11D, 12D$, superconformal symmetries and supertwistor dynamics by [8], [9], [15]. For this reason we believe we ought to explore how to build the novel Polyvector Supersymmetic structures in Clifford Spaces.

To begin, we shall revert to standard realizations of the superalgebra in terms of $differential$ operators. As a reminder [30] we recall that a $4D$ Majorana spinor Ψ_M can be written in a Weyl basis in terms of two Weyl spinors $\chi_\alpha, \bar{\chi}^{\dot{\alpha}}$ where $\alpha = 1, 2$ and $\dot{\alpha} = \dot{1}, \dot{2}$. Spinor indices are raised and lower by the $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}, ...$ antisymmetric 2×2 matrices. This decomposition of a $4D$ Majorana spinor into two-component Weyl spinors and the 4×4 γ matrices in terms of blocks consisting of σ^μ Pauli 2×2 matrices (where σ^0 is the unit matrix, up to a sign) is very convenient. The chiral (antichiral) covariant differential operators in $N = 1$ superspace are

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\ \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^\mu}. \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\ \frac{\partial}{\partial x^\mu}. \tag{2.1}$$

and the chiral (antichiral) supersymmetry generators are

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\ \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^\mu}. \quad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\ \frac{\partial}{\partial x^\mu}. \tag{2.2}$$

$$\pi_\alpha \ \equiv\ \frac{\partial}{\partial\theta^\alpha}, \ \ \{\pi_\alpha,\ \theta^\beta\} \ =\ \{\frac{\partial}{\partial\theta^\alpha},\ \theta^\beta\} \ =\ \delta^\beta_\alpha, \tag{2.3}$$

$$\bar{\pi}_{\dot{\alpha}} \ \equiv\ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \ \ \{\ \bar{\pi}_{\dot{\alpha}},\ \bar{\theta}^{\dot{\beta}}\ \} \ =\ \{\ \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}},\ \bar{\theta}^{\dot{\beta}}\ \} \ =\ \delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{2.4}$$

$$\{\frac{\partial}{\partial\theta^\alpha},\ \frac{\partial}{\partial\theta^\beta}\} \ =\ 0, \ \ \{\theta^\alpha, \theta^\beta\} \ =\ 0, \ \ etc.... \tag{2.5}$$

The Q's and the D's anticommute among themselves $\{Q, D\} = 0$ and the only nonzero anticommutators are

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} \ =\ -2i\ \sigma^\mu_{\alpha\dot{\alpha}}\ \frac{\partial}{\partial x^\mu}, \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \ =\ 2i\ \sigma^\mu_{\alpha\dot{\alpha}}\ \frac{\partial}{\partial x^\mu} \tag{2.6}$$

Superfields form linear representations of supersymetry algebras. In general these representations are highly reducible. The extra components can be eliminated by imposing covariant constraints like $\bar{D}_{\dot{\alpha}}\Phi = 0,\ D_\alpha\Phi = 0$ leading to chiral (anti-chiral) superfields, respectively.

The anti-commutator associated with a Polyvector version of Poincare supersymmetry in *ordinary* spacetime (not in Clifford space) was provided by [9]

$$\{S_\alpha, S_\beta\} = \sum_k \ (\mathcal{C}\Gamma^{\mu_1\mu_2....\mu_k})_{\alpha\beta}\ W^{(k)}_{0\ \mu_1\mu_2\mu_2....\mu_k} \tag{2.7}$$

where α, β denote spinor indices and the summation over k must obey certain crucial restrictions to match degrees of freedom with the terms in the left hand side. These algebras [9] have the form $g = g_0 + g_1$, with $g_0 = so(V) + W_0$ and $g_1 = W_1$, where the algebra of generalized translations $W = W_0 + W_1$ is the maximal solvable ideal of g; W_0 is generated by W_1 and commutes with $W_0 + W_1$. The matrix $\mathcal{C}$ is the charge conjugation matrix. Depending on the given spacetime and its signature there are at most two charge conjugation matrices $\mathcal{C}_S, \mathcal{C}_A$ given by the product of all symmetric and all antisymmetric gamma matrices, respectively. In special spacetime signatures they collapse into a single matrix [8], [9] . These charge conjugation matrix $\mathcal{C}$ are essential in order to satisfy the nontrivial graded super Jacobi identities.

For example, the M-theory superalgebra in $D = 11$ is

$$\{Q_\alpha, Q_\beta\} = (\mathcal{A}\Gamma^\mu)_{\alpha\beta}P_\mu + (\mathcal{A}\Gamma^{\mu_1\mu_2})_{\alpha\beta}Z_{\mu_1\mu_2} + (\mathcal{A}\Gamma^{\mu_1\mu_2....\mu_5})_{\alpha\beta}Z_{\mu_1\mu_2....\mu_5}. \tag{2.8}$$

P_μ is the usual momentum operator; the antisymmetric tensorial central charges $Z_{\mu_1\mu_2},\ Z_{\mu_1\mu_2....\mu_5}$ are of ranks $2, 5$ respectively. The matrix $\mathcal{A}$ plays the role of the timelike γ^0 matrix in Minkowskian spacetimes and is used to introduced barred-spinors. In spacetimes of signature (s, t) $\mathcal{A}$ is given by the products of all the timelike gammas, up to an overall sign [8], [9] . Notice that the summation over the k indices in the r.h.s is very restricted since the $k = 1, 2, 5$ sectors of the r.h.s yield in $D = 11$ a total number of $11 + 55 + 462 = 528$ components which precisely match the number of independent components of a 32×32 symmetric real matrix in the l.h.s given by $(32 \times 33)/2 = 528$.

The 12-dim Euclidean generalized supersymmetric F algebra was

$$\{Q_\alpha, Q_\beta\} = (\mathcal{C}\Gamma^\mu)_{\alpha\beta}P_\mu + (\mathcal{C}\Gamma^{\mu_1\mu_2})_{\alpha\beta}Z_{\mu_1\mu_2} + (\mathcal{C}\Gamma^{\mu_1\mu_2....\mu_5})_{\alpha\beta}Z_{\mu_1\mu_2....\mu_5}. \tag{2.9}$$

together with its complex conjugation [8] . Other Hermitian versus holomorphic complex and quaternionic generalized supertranslations (" supersymmetries") of M-theory were classified by [8]. The classification of the family of symmetric matrices $(C\gamma^{\mu_1\mu_2....\mu_n})_{\alpha\beta}$ is what restricts the type of terms that appear in the $\{Q_\alpha, Q_\beta\}$ anticommutator and depends on the number of space time dimensions D, the signatures (s, t) and the rank n. A table of the allowed values of D, s, t, n can be found in [15] .

In particular, when $D = 4 = 3 + 1$, the $\{Q_\alpha, Q_\beta \}$ is a symmetric matrix in α, β with 10 independent components and which matches the degrees of freedom of the Clifford-space momentum vector P^μ and the momentum bivector $P^{\mu\nu}$ given by $4 + 6 = 10$, respectively. Therefore, in $D = 4$ one can postulate the anticommutators in Clifford space of the form

$$\{Q_\alpha,\ Q_\beta \} \ = \ \frac{1}{2}\ (C\ \Gamma^\mu\ P_\mu\ + C\ \Gamma^{\mu\nu}\ P_{\mu\nu}\ )_{\alpha\beta}. \tag{2.10}$$

since there is an exact match in the number of degrees of freedom in the left and right hand side and the matrix products yield a symmetric matrix in α, β.

The graded Jacobi identities are satisfied ensuring the closure of the superalgebra

$$[\mathcal{J}_{\mu_1\mu_2},\ P_{\rho_1\rho_2}] = -\eta_{\mu_1\rho_1} P_{\mu_2\rho_2} \pm ... \ ; [\mathcal{J}_{\mu_1\mu_2}, Q_\alpha] = -(\gamma_{\mu_1\mu_2})_\alpha^{\ \delta} Q_\delta \tag{2.11}$$

$$\{\ Q_\alpha,\ Q_\beta\ \} = \frac{1}{2} C\gamma^\nu P_\nu + \frac{1}{2} C\gamma^{\nu_1\nu_2} P_{\nu_1\nu_2} \tag{2.12}$$

$$[\ \mathcal{J}_{\mu\nu},\ \mathcal{J}_{\rho\tau}\ ] \ = \ g_{\nu\rho}\ \mathcal{J}_{\mu\tau}\ \ -i\ \ g_{\mu\rho}\ \mathcal{J}_{\nu\tau}\ \ +\ \ g_{\mu\tau}\ \mathcal{J}_{\nu\rho}\ \ -\ \ g_{\nu\tau}\ \mathcal{J}_{\mu\rho}. \tag{2.13}$$

and involves terms containing P_μ and $P_{\mu\nu}$. If one works with anti-Hermitian generators there is no need to introduce i factors in the right hand side of the above equations because the commutators of two anti-Hermitain operators are anti-Hermitian. The closure of the superalgebra (2.10-2.13) was shown in [16].

A naive polyvector valued extension of a supercharge operator $Q_\alpha =$

$$\frac{\partial}{\partial\theta^\alpha} - C\ \ (\Gamma^\mu P_\mu + \Gamma^{\mu_1\mu_2} P_{\mu_1\mu_2} + \Gamma^{\mu_1\mu_2\mu_3} P_{\mu_1\mu_2\mu_3} + \Gamma^{\mu_1\mu_2\mu_3\mu_4} P_{\mu_1\mu_2\mu_3\mu_4}\ )_{\alpha\beta}\ \ \theta^\beta. \tag{2.14}$$

furnishing the putative anti-commutator

$$\{Q_\alpha,\ Q_\beta\} \ = \ -2\, C\, (\Gamma^\mu P_\mu + \Gamma^{\mu_1\mu_2} P_{\mu_1\mu_2} + \Gamma^{\mu_1\mu_2\mu_3} P_{\mu_1\mu_2\mu_3} + \Gamma^{\mu_1\mu_2\mu_3\mu_4} P_{\mu_1\mu_2\mu_3\mu_4}\ )_{\alpha\beta}\,. \tag{2.15}$$

will *not* work since there is no exact match in the number of degrees of freedom in the left and right hand side of (2.15), and *not* all of the matrix products in (2.15) are going to yield a *symmetric* matrix in α, β, after evaluating the anti-commutator.

The momentum polyvectors in natural units of $\hbar = c = 1$ can be realized in terms of differential operators as

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad P_{\mu_1\mu_2} = \frac{\partial}{\partial x^{\mu_1\mu_2}} \tag{2.16}$$

$$P_{\mu_1\mu_2\mu_3} \ = \ \frac{\partial}{\partial x^{\mu_1\mu_2\mu_3}},\ P_{\mu_1\mu_2\mu_3\mu_4} \ = \ \frac{\partial}{\partial x^{\mu_1\mu_2\mu_3\mu_4}} \tag{2.17}$$

The Clifford polyvector extension of a chiral superfield in $2D$ has the form $\Phi(s, x^1, x^2, x^{12}; \theta^1, \theta^2)$ where s, x^1, x^2, x^{12} are the scalar, vector and bivector coordinates, respectively, associated with a polyvector in $2D$. The chiral superfield field can be expanded in powers of θ as

$$\Phi(s, x^1, x^2, x^{12}; \theta^1, \theta^2) = \phi(s, x^1, x^2, x^{12}) + \theta^\alpha \, \Psi_\alpha(s, x^1, x^2, x^{12}) + \theta^1\theta^2 \, F(s, x^1, x^2, x^{12}). \tag{2.18}$$

the powers of θ's terminates due to the Grassmanian nature of the coordinates $\theta^1\theta^1 = \theta^2\theta^2 = 0; \{\theta^1, \theta^2\} = 0$; θ^α is a $2D$ Majorana spinor with real Grassmanian-valued (anticommuting) entries θ^1, θ^2 . After some straightforward algebra and using the Grassmanian integration rules in chiral superspace , the action becomes

$$S = -\frac{i}{4\pi} \int \, [ds dx^1 dx^2 dx^{12}] \, [d^2\theta] \, \bar{D}_\alpha \Phi(s, x^\mu, x^{\mu\nu}; \theta^1, \theta^2) \, D^\alpha \Phi(s, x^\mu, x^{\mu\nu}; \theta^1, \theta^2) =$$

$$-\frac{1}{2\pi} \int \, [ds dx^1 dx^2 dx^{12}] \, [\, (\partial_\mu \phi) \, (\partial^\mu \phi) - i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - FF + ... \,]. \tag{2.19}$$

where the ellipsis ..... terms involve the additional contribution due to the scalar and bivector derivatives

$$(\frac{\partial \phi}{\partial x^{\mu\nu}})^2, \quad (\frac{\partial \phi}{\partial s})^2, \quad \bar{\Psi} \, \gamma^{\mu\nu} \, \frac{\partial \Psi}{\partial x^{\mu\nu}}, \quad \bar{\Psi} \frac{\partial \Psi}{\partial s}. \tag{2.20}$$

terms.

The natural supersymmetric extension of the Clifford space polyvector coordinates involves the introduction of spinor-polyvectors coordinates

$$X^\mu \ \leftrightarrow \ \Psi^\mu_\alpha, \quad X^{[\mu_1 \mu_2]} \ \leftrightarrow \ \Psi^{[\mu_1 \mu_2]}_\alpha, \quad ...., \quad X^{[\mu_1 \mu_2 .... \mu_n]} \ \leftrightarrow \ \Psi^{[\mu_1 \mu_2 .... \mu_n]}_\alpha \tag{2.21}$$

and such that the generalization of the anti-commutators to Clifford spaces (when $D \geq 1$) is given by the anti-commutators involving spinor-vectorial (spinor-tensorial) charges

$$\{Q^\alpha_\mu, \ Q^\beta_\nu\} \ = \ \{Q^\alpha \otimes \Gamma_\mu, \ Q^\beta \otimes \Gamma_\nu\} \ = \ \frac{1}{2} \, \{Q^\alpha, \ Q^\beta\} \otimes \{\Gamma_\mu, \ \Gamma_\nu\} \ +$$

$$\frac{1}{2} \, [Q^\alpha, \ Q^\beta] \otimes [\Gamma_\mu, \ \Gamma_\nu] \ = \ \{Q^\alpha, \ Q^\beta\} \otimes g_{\mu\nu} \, \mathbf{1} \ + \ [Q^\alpha, \ Q^\beta] \otimes \Gamma_{\mu\nu} \tag{2.22}$$

where $\{Q^\alpha, Q^\beta\}$ is given by an expression like eq-(2.7) symmetric in the α, β indices, and $[Q^\alpha, Q^\beta]$ must be given by a combination of matrix products which furnish an overall *antisymmetric* matrix in the α, β indices.

In addition, for arbitrary dimensions one has the anti-commutators of spinor-tensorial charges of arbitrary rank $r = 1, 2, ..., D$ given as

$$\{Q^\alpha_{\mu_1\mu_2....\mu_n},\ Q^\beta_{\nu_1\nu_2....\nu_m}\} \ = \ \{Q^\alpha \otimes \Gamma_{\mu_1\mu_2.....\mu_n},\ Q^\beta \otimes \Gamma_{\nu_1\nu_2....\nu_m}\} \ =$$

$$\frac{1}{2}\ \{Q^\alpha,\ Q^\beta\} \otimes \{\Gamma_{\mu_1\mu_2....\mu_n}, \Gamma_{\nu_1\nu_2....\nu_m}\} \ + \ \frac{1}{2}\ [Q^\alpha,\ Q^\beta] \otimes [\Gamma_{\mu_1\mu_2....\mu_n}, \Gamma_{\nu_1\nu_2....\nu_m}]. \tag{2.23}$$

The (anti) commutators of the antisymmetric products of the Gammas are well known. The relevant terms are once again $\{Q^\alpha, Q^\beta\}, [Q^\alpha, Q^\beta]$ and which are given in terms of suitable products of gamma matrices which furnish an overall symmetric and anti-symmetric matrix in the α, β indices, respectively. The classification of the family of symmetric and antisymmetric matrices $(C\gamma^{i_1 i_2....i_m})_{\alpha\beta}$, $(C\gamma^{j_1 j_2....j_n})_{\alpha\beta}$ depends on the number of space time dimensions D, the signatures (s, t) and the ranks m, n. A table of the allowed values of D, s, t, m furnishing a symmetric matrix can be found in [15] . One just needs to look at the table for the allowed values of D, s, t, n furnishing an *antisymmetric* matrix, in addition to the symmetric cases. Once this is done one can write the expressions for $\{Q^\alpha, Q^\beta\}, [Q^\alpha, Q^\beta]$ in terms of momentum polyvectors as

$$\{Q^\alpha, Q^\beta\} \ = \ \sum_k \ (C\gamma^{i_1 i_2....i_m})^{\alpha\beta}\ P_{i_1 i_2....i_m},$$

$$[Q^\alpha, Q^\beta] \ = \ \sum_k \ (C\gamma^{j_1 j_2....j_n})^{\alpha\beta}\ P_{j_1 j_2....j_n} \tag{2.24}$$

and where the choice of the allowed values of k in the respective summations is obtained from the tables. To conclude this section, eqs-(2.21-2.24) encode the proper procedure to generalize ordinary supersymmetry in ordinary spacetime (superspace) to polyvector-valued supersymmetry in $Clifford$ (super) spaces involving both (antisymmetric) tensorial coordinates and spinor-tensorial ones.

APPENDIX : CLOSURE OF THE POINCARE SUPERALGEBRA

To show the closure of the $2D$ superalgebra provided by eqs-(1.11-1.14) one needs to find the proper representation of the operators via the addition of an extra "internal" $\mathcal{A}$ algebra whose generators $\mathbf{1}, \mathcal{Q}, \mathcal{P}$ obey the defining relations

$$\mathcal{Q}^2 \ = \ \mathcal{P}, \ \ \mathcal{P}\ \mathcal{Q} \ = \ \mathcal{Q}\ \mathcal{P} \ = \ \mathcal{Q}^3 \ = \ 0 \Rightarrow [\ \mathcal{P},\ \mathcal{Q}\ ] \ = \ \{\ \mathcal{P},\ \mathcal{Q}\ \} \ = \ \mathbf{0} \quad (A.1)$$

An explicit 3×3 matrix realization of the algebra (A.1) was given by eq-(1.19). In this fashion we will provide a realization of the operators of the $2D$ Poincare superalgebra in terms of the tensor product algebra $Cl_{1,1}(R) \otimes \mathcal{A}$. The momentum operators are represented as

$$\mathbf{P}_\mu = \frac{1}{2}\Gamma_\mu(1 - \Gamma_3) \otimes \mathcal{P}, \quad \mu, \nu = 1, 2 \qquad (A.2)$$

where the generator $\mathcal{P}$ belongs to the second algebra $\mathcal{A}$ factor.

The $2D$ Lorentz generator is represented as

$$\mathcal{M}_{\mu\nu} \ = \ \frac{1}{2} \ \Gamma_{\mu\nu} \otimes \mathbf{1} \qquad (A.3)$$

where $\mathbf{1}$ is the unit operator of the algebra $\mathcal{A}$. The two supercharges $Q^\alpha = Q^1, Q^2$ are represented as

$$\mathbf{Q}^\alpha \ = \ Q^\alpha \ \otimes \mathcal{Q}, \ \alpha = 1,2 \ \Rightarrow \ \mathbf{Q}^1 \ = \ i \ \Gamma^1 \otimes \mathcal{Q}, \quad \mathbf{Q}^2 \ = \ \Gamma^2 \otimes \mathcal{Q} \qquad (A.4a)$$

leading to the anti-commutators

$$\{ \ \mathbf{Q}^\alpha, \ \mathbf{Q}^\beta \ \} \ = \ \{ \ \Gamma^\alpha \otimes \mathcal{Q}, \ \Gamma^\beta \otimes \mathcal{Q} \ \} = \ 2 \ \delta^{\alpha\beta} \ (\mathbf{1} \otimes \mathcal{Q}^2) \ = \ 2 \ \delta^{\alpha\beta} \ (\mathbf{1} \otimes \mathcal{P}) \ =$$

$$(\ \Gamma^\mu P_\mu \ + \ P_\mu \Gamma^\mu \ )^{\alpha\beta} \ (\mathbf{1} \otimes \mathcal{P}) \qquad (A.4b)$$

after using $(\Gamma_3)^2 = \Gamma_1\Gamma_2\Gamma_1\Gamma_2 = -(\Gamma_1)^2(\Gamma_2)^2 = \mathbf{1}$; $(\Gamma_1)^2 = g_{11}\mathbf{1} = -\mathbf{1}$; $(\Gamma_2)^2 = g_{22}\mathbf{1} = \mathbf{1}$. The evaluation of the anti-commutators (A.4b) is explicitly derived from the above definitions (A.4a) of the supercharges and the following relations involving the Kronecker tensor products $\otimes$ of operators (matrices, for example)

$$(A \otimes C) \ (B \otimes D) \ = \ (AB) \otimes (CD) \qquad (A.5)$$

$$[\ A \otimes C, \ B \otimes D \ ] \ = \ \frac{1}{2} \ [\ A, B \ ] \otimes \{ \ C, D \ \} \ + \ \frac{1}{2} \ \{ \ A, B \ \} \otimes [\ C, D \ ] \qquad (A.6)$$

$$\{ \ A \otimes C, \ B \otimes D \ \} \ = \ \frac{1}{2} \ \{ \ A, B \ \} \otimes \{ \ C, D \ \} \ + \ \frac{1}{2} \ [\ A, B \ ] \otimes [\ C, D \ ] \qquad (A.7)$$

Using (A.5-A.7) one arrives at

$$[\ \mathbf{P}_\mu, \ \mathbf{Q}^1 \ ] \ = \ \frac{i}{2} [\ \Gamma_\mu (1 - \Gamma_3) \otimes \mathcal{P}, \ \Gamma^1 \otimes \mathcal{Q} \ ] \ =$$

$$\frac{i}{4} [\ (\Gamma_\mu (1 - \Gamma_3), \ \Gamma^1 \ ] \otimes \{\mathcal{P}, \mathcal{Q}\} \ + \ \frac{i}{4} \{ \ \Gamma_\mu (1 - \Gamma_3), \ \Gamma^1 \ \} \otimes [\mathcal{P}, \mathcal{Q}] \ = \ 0 \qquad (A.8)$$

$$[\ \mathbf{P}_\mu, \ \mathbf{Q}^2 \ ] \ = \ \frac{1}{2} [\ \Gamma_\mu (1 - \Gamma_3) \otimes \mathcal{P}, \ \Gamma^2 \otimes \mathcal{Q} \ ] \ =$$

$$\frac{1}{4} [\ (\Gamma_\mu (1 - \Gamma_3), \ \Gamma^2 \ ] \otimes \{\mathcal{P}, \mathcal{Q}\} \ + \ \frac{i}{4} \{ \ \Gamma_\mu (1 - \Gamma_3), \ \Gamma^2 \ \} \otimes [\mathcal{P}, \mathcal{Q}] \ = \ 0 \qquad (A.9)$$

as a direct result of the algebraic relations of eqs-(1.18, A.1) associated with the algebra $\mathcal{A}$: $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \mathcal{Q}^3 = 0 \Rightarrow [\mathcal{P}, \mathcal{Q}] = \{\mathcal{P}, \mathcal{Q}\} = 0$. It is important to emphasize that *without* the presence of the terms involving the second algebra factors in (A.8, A.9) one would *not* have been able to obtain a *zero* commutator for $[\mathbf{P}_\mu, \mathbf{Q}^\alpha]$.

The commutator $[\ \mathcal{M}_{\mu\nu}, \ \mathbf{Q}^\alpha \ ]$ for $\alpha = 1, 2$, after using the definitions $\mathbf{Q}^1 = \Gamma^1 \otimes \mathcal{Q}$ and $\mathbf{Q}^2 = \Gamma^2 \otimes \mathcal{Q}$, is

$$\frac{i}{2} \ [\ \Gamma_3 \otimes \mathbf{1}, \ \Gamma^1 \otimes \mathcal{Q} \ ] \ = \ \frac{i}{4} \ [\Gamma_3, \ \Gamma^1] \ \otimes \ \{ \ \mathbf{1}, \ \mathcal{Q} \ \} \ =$$

$$i\ (\Gamma_3\ \Gamma^1)\ \otimes\ \mathcal{Q} = (\Gamma_3 \otimes \mathbf{1})\ (i\ \Gamma^1 \otimes \mathcal{Q})\ =\ (\Gamma_3 \otimes \mathbf{1})\ \mathbf{Q}^1 \qquad (A.10)$$

and

$$\frac{1}{2}\ [\ \Gamma_3 \otimes \mathbf{1},\ \Gamma^2 \otimes \mathcal{Q}\ ]\ =\ \frac{1}{4}\ [\Gamma_3,\ \Gamma^2]\ \otimes\ \{\ \mathbf{1},\ \mathcal{Q}\ \}\ =$$
$$(\Gamma_3\ \Gamma^2)\ \otimes\ \mathcal{Q} = (\Gamma_3 \otimes \mathbf{1})\ (\Gamma^2 \otimes \mathcal{Q})\ =\ (\Gamma_3 \otimes \mathbf{1})\ \mathbf{Q}^2 \qquad (A.11)$$

respectively, after using eqs-(A.6,A.7,A.8) and $[\mathbf{1}, \mathcal{Q}] = 0$. Hence, as expected, we arrive in $2D$

$$[\mathcal{M}_{12}, \mathbf{Q}^\alpha]\ =\ (\Gamma_{12} \otimes \mathbf{1})\ \mathbf{Q}^\alpha\ =\ (\Gamma_3 \otimes \mathbf{1})\ \mathbf{Q}^\alpha \qquad (A.12)$$

after using the definition $\mathcal{M}_{12} = \frac{1}{4}[\Gamma_1, \Gamma_2] = \frac{1}{2}\Gamma_{12} = \frac{1}{2}\Gamma_3$.

The momentum operators commute

$$[\ \mathbf{P}_\mu,\ \mathbf{P}_\nu\ ]\ =\ \frac{1}{4}\ [\ \Gamma_\mu(1-\Gamma_3) \otimes \mathcal{P},\ \Gamma_\nu(1-\Gamma_3) \otimes \mathcal{P}\ ]\ =$$
$$\frac{1}{8}\ [\ \Gamma_\mu(1-\Gamma_3),\ \Gamma_\nu(1-\Gamma_3)\ ] \otimes \{\mathcal{P}, \mathcal{P}\}\ =\ 0 \qquad (A.13)$$

since

$$[\mathcal{P}, \mathcal{P}]\ =\ 0, \quad \Gamma_\mu\ (1-\Gamma_3)\ \Gamma_\nu(1-\Gamma_3)\ =\ \Gamma_\mu\ \Gamma_\nu\ (1+\Gamma_3)\ (1-\Gamma_3)\ =\ \mathbf{0}$$
$$\Gamma_\nu\ (1-\Gamma_3)\ \Gamma_\mu(1-\Gamma_3)\ =\ \Gamma_\nu\ \Gamma_\mu\ (1+\Gamma_3)\ (1-\Gamma_3)\ =\ \mathbf{0} \qquad (A.14)$$

because $\{\Gamma_3, \Gamma_\mu\} = 0$ and $(\Gamma_3)^2 = \mathbf{1}$ for Minkowskian signature in $2D$.

The commutator $[\mathcal{M}_{\mu\nu},\ \mathbf{P}_\rho\ ]$ is

$$\frac{1}{4}\ [\ \Gamma_{\mu\nu} \otimes \mathbf{1},\ \Gamma_\rho(1-\Gamma_3) \otimes \mathcal{P}\ ]\ =\ \frac{1}{8}\ [\ \Gamma_{\mu\nu},\ \Gamma_\rho(1-\Gamma_3)] \otimes \{\mathbf{1},\ \mathcal{P}\}\ =$$
$$\frac{1}{4}\ (-2\ g_{\mu\rho}\ \Gamma_\nu\ +2\ g_{\nu\rho}\ \Gamma_\mu)\ (1-\Gamma_3)\ \otimes\ \mathcal{P}\ =\ -g_{\mu\rho}\ \mathbf{P}_\nu\ +\ g_{\nu\rho}\ \mathbf{P}_\mu \qquad (A.15)$$

after recurring to the relations

$$[\mathbf{1}, \mathcal{P}]\ =\ 0, \quad [\Gamma_{\mu\nu}, \Gamma_3]\ =\ 0, \quad [\Gamma_{\mu\nu}, \Gamma_\rho]\ =\ 2\ (\ -g_{\mu\rho}\Gamma_\nu + g_{\nu\rho}\Gamma_\mu\ ) \qquad (A.16)$$

Finally, the generators $\mathcal{M}_{\mu\nu}$ given by $\frac{1}{2}\Gamma_{\mu\nu} \otimes \mathbf{1}$ obey the commutators in eq-(18) associated with the Lorentz generators in any dimension because the bivectors $\frac{1}{2}\Gamma_{\mu\nu}$ obey the Lorentz algebra commutation relations. In $2D$ the commutators are trivially zero since there is only *one* generator $\mathcal{M}_{12} = -\mathcal{M}_{21} = \frac{1}{2}\Gamma_3 \otimes \mathbf{1}$, which behaves as a dilatation.

Therefore, once we have shown that the (anti) commutators indeed obey the $2D$ Poincare superalgebra given by eqs-(1.11-1.14), the graded super Jacobi identities are satisfied because it is well known that the Poincare superalgebra closes. For example one can show that

$$-[\ \mathcal{M}_{\mu\nu}, \{\mathbf{Q}^\alpha,\ \mathbf{Q}^\beta\}\ ]\ +\ \{\ \mathbf{Q}^\beta,\ [\mathcal{M}_{\mu\nu}, \mathbf{Q}^\alpha]\ \}\ -\ \{\mathbf{Q}^\alpha, [\mathbf{Q}^\beta, \mathcal{M}_{\mu\nu}]\ \}\ =\ 0 \quad (A.18)$$

is satisfied by studying all the cases when $\alpha, \beta = 1, 2$:

$$-[\ \frac{1}{2}\Gamma_3 \otimes \mathbf{1},\ \{i\Gamma^1 \otimes \mathcal{Q}, i\Gamma^1 \otimes \mathcal{Q}\}\ ]\ +\ \{\ i\Gamma^1 \otimes \mathcal{Q},\ [\ \frac{1}{2}\Gamma_3 \otimes \mathbf{1}, i\Gamma^1 \otimes \mathcal{Q}\ ]\ \}\ -$$

$$\{\ i\Gamma^1 \otimes \mathcal{Q},\ [\ i\Gamma^1 \otimes \mathcal{Q}, \frac{1}{2}\Gamma_3 \otimes \mathbf{1}\ ]\ \}\ =\ 0 \qquad (A.19)$$

The first term yields $[\Gamma_3 \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{Q}^2] = 0$, after using eqs-(A.6, A7), since all the Gamma matrices commute with the *unit* matrix and $[\mathbf{1}, \mathcal{Q}^2] = 0$. The second and third terms in (A.19) yield expressions of the form $\{\Gamma^1, \Gamma_3\Gamma^1\} \otimes \mathcal{Q}^2 = 0$, because

$$\{\Gamma^1,\ \Gamma_3\ \Gamma^1\}\ =\ \Gamma^1\ \Gamma_3\ \Gamma^1\ +\ \Gamma_3\ \Gamma^1\ \Gamma^1\ =\ -\ \Gamma_3\ \Gamma^1\ \Gamma^1\ +\ \Gamma_3\ \Gamma^1\ \Gamma^1\ =\ 0 \quad (A.20)$$

after using $\{\Gamma_3, \Gamma_1\} = 0$. Similar results are found in the other cases

$$[\ \frac{1}{2}\Gamma_3 \otimes \mathbf{1},\ \{\Gamma^2 \otimes \mathcal{Q}, \Gamma^2 \otimes \mathcal{Q}\}\ ]\ +\ \{\ \Gamma^2 \otimes \mathcal{Q},\ [\ \frac{1}{2}\Gamma_3 \otimes \mathbf{1}, \Gamma^2 \otimes \mathcal{Q}\ ]\ \}\ -$$

$$\{\ \Gamma^2 \otimes \mathcal{Q},\ [\ \Gamma^2 \otimes \mathcal{Q}, \frac{1}{2}\Gamma_3 \otimes \mathbf{1}\ ]\ \}\ =\ 0 \qquad (A.21)$$

after using $\{\Gamma^2, \Gamma_3\Gamma^2\} \otimes \mathcal{Q}^2 = 0$ and $[\Gamma_3 \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{Q}^2] = 0$. And finally,

$$-[\ \frac{1}{2}\Gamma_3 \otimes \mathbf{1},\ \{i\Gamma^1 \otimes \mathcal{Q},\ \Gamma^2 \otimes \mathcal{Q}\}\ ]\ +\ \{\ \Gamma^2 \otimes \mathcal{Q},\ [\frac{1}{2}\Gamma_3 \otimes \mathbf{1}, i\Gamma^1 \otimes \mathcal{Q}]\ \}\ -$$

$$\{i\Gamma^1 \otimes \mathcal{Q},\ [\Gamma^2 \otimes \mathcal{Q}, \frac{1}{2}\Gamma_3 \otimes \mathbf{1}]\ \}\ =\ 0 \qquad (A.22)$$

because

$$\{\ \Gamma^1,\ \Gamma_3\Gamma^2\ \} + \{\ \Gamma^2,\ \Gamma_3\Gamma^1\ \}\ =\ (\Gamma^1\Gamma_3\Gamma^2 + \Gamma_3\Gamma^1\Gamma^2)\ +\ (\Gamma_3\Gamma^2\Gamma^1 + \Gamma^2\Gamma_3\Gamma^1)\ =\ 0 \qquad (A.23)$$

after using $\{\Gamma_3, \Gamma^1\} = \{\Gamma_3, \Gamma^2\} = 0$.

The Jacobi identity

$$[\ \mathbf{Q}^\gamma, \{\mathbf{Q}^\alpha,\ \mathbf{Q}^\beta\}\ ]\ +\ \textit{cyclic permutation}\ =\ 0 \qquad (A.23)$$

is satisfied because the term

$$[\ \mathbf{Q}^\gamma,\ \{\mathbf{Q}^\alpha,\ \mathbf{Q}^\beta\}\ ]\ = 2\ \delta^{\alpha\beta}\ [\ \Gamma^\gamma \otimes \mathcal{Q},\ \mathbf{1} \otimes \mathcal{Q}^2\ ]\ =\ 0 \qquad (A.24)$$

after recurring to eqs-(A.6, A.7) since the Γ's 2×2 matrices commute with the 2×2 unit matrix operator $\mathbf{1}$ and $[\mathcal{Q}, \mathcal{Q}^2] = 0$. Similar results are found with the other permutation of indices in eq-(A.23). Concluding, the graded super Jacobi identities are satisfied and the superalgebra closes.

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References

[1] C. Castro , *Foundations of Physics.* **8**, 1301 (2000). *Jour. Chaos, Solitons and Fractals* **11** no.11, 1663 (2000). *Foundations of Physics* **35**, no. 6, 971 (2005).

[2] C. Castro, M. Pavsic, *Progress in Physics* **1** (2005) 31; *Phys. Letts B* **559** (2003) 74; *Int. J. Theor. Phys* **42** (2003) 1693.

[3] W. Pezzaglia, "Classification of Multivector Theories and the Modification of the Postulates of Physics" gr-qc/9306006; "Dimensionally Democratic Calculus and Principles of Polydimensional Physics" gr-qc/9912025; "Polydimensional Relativity, a Classical Generalization of the Automorphism Invariance Principle" gr-qc/9608052.

[4] J. Chisholm : J. Phys **A 35** (2002) 7359 J. Chisholm and R. Farwell , "Spin Gauge Theories, Principles and Predictions" in *Clifford Algebras and their Application sin Mathematical Physics* eds. F. Brack, R. Delanghe and H. Serras (Dordrecht, Kluwer 1993) p. 367. G. Trayling and W. Baylis : J. Phys. **A 34** (2001) 3309. J. Chisholm and R. Farwell, J. Phys. **A 32** (1999) 2805. J. Chisholm and R. Farwell, Foundations of Physics **25** (1999) 1511.

[5] D. Hestenes, *'Spacetime Algebra '* Gordon and Breach, New York, 1996. D. Hestenes and G. Sobcyk, *Clifford Algebra to Geometric Calculus* D. Reidel Publishing Company, Dordrecht, 1984. Clifford Algebras and their applications in Mathematical Physics" Vol 1: Algebras and Physics. eds by R. Ablamowicz, B. Fauser. Vol 2: Clifford analysis. eds by J. Ryan, W. Sprosig Birkhauser, Boston 2000. P. Lounesto: "Clifford Algebras and Spinors". Cambridge University Press. 1997.

[6] M.Pavsic, *The Landscape of Theoretical Physics: A Global View, From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle* (Kluwer Academic Publishers, Dordrecht-Boston-London, 2001).

[7] D. Bohm and B. Hiley, "Generalisation of the twistor to Clifford algebras as a basis for geometry" Revista Brasileira de Fisica, Volume Especial, Os 70 anos de Mario Schonberg, (1984) pp. 1-26.

[8] F. Toppan, " Hermitian versus holomorphic complex and quaternionic generalized supersymmetries of the M-theory : A classification" hep-th/0406022. " Exceptional Structures in Mathematics and Physics and the Role of the Octonions"

[9] D. Alekseevsky, V. Cortes, C. Devchand and A. Van Proeyen " Polyvector Super-Poincare Algebras " hep-th/0311107.

[10] R. Mosna, D. Miralles and J. Vaz, " Z_2 gradings of Clifford algebras and multivector structures" math-ph/0212020.

[11] A. Lasenby, C. Doran and S. Gull, Jour. Math. Phys **34** (8) (1993) 3683. Jour. Math. Phys **39** (1998) 3303.

[12] P. Freund, *Supersymmetry* (Cambridge University Press, 1986). J. Wess, J. Bagger, *Supersymmetry and Supergarvity* (Princeton Univ. Press, 1983). P. West, *Introduction to Supersymmetry and Supergravity* (World Scientific, Singapore, 1986). B. de Witt, *Supermanifolds* (Cambridge University Press 1984).

[13] G. Dixon, J. Math. Phys **19**, no. 10 (1976) 2103. Letts in Math. Phys **5** (1981) 411. Jour. Math Phys **vol 45** (10) (2004) 3878. *Divison Algebras, Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics* (Kluwer, Dordrecht, 1994).

[14] H. De Bie and F. Sommen, "A Clifford analysis approach to superspace" arXiv : 0707.2859.

[15] I. Rudychev, E. Sezgin, " Superparticles, p-form Coordinates and the BPS Condition" hep-th/9711128. I. Bars, C. Kounnas, "A new supersymmetry" hep-th/9612119. I. Bandos, J. Lukierski, "Generalized Superconformal Symmetries and Supertwistor Dynamics" hep-th/9912264.

[16] C. Castro, Int. J. Mod. Phys **A 21**, no. 10 (2006) 2149.

[17] V. Abramov, R. Kerner and B. Le Roy, J. Math. Phys. **38** (1997) 1650.

[18] M. Henneaux, Int. J. Geom. Meth. Mod. Phys. **5** (2008). M. Dubois-Violette, Czech. J. Phys. **46** (1997), 1227. M. Dubois-Violette, K-Theory **14** (1998), 371. M. Dubois-Violette, Contemporary Mathematics **219** (1998), 69. P.J. Olver, Differential hyperforms I, Univ. of Minnesota report 82-101.