Physics as Infinite-dimensional Geometry and Generalized Number Theory: Basic Visions

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Abstract

There are two basic approaches to the construction of quantum TGD. The first approach relies on the vision of quantum physics as infinite-dimensional Kähler geometry for the "world of classical worlds" identified as the space of 3-surfaces in in certain 8-dimensional space. Essentially a generalization of the Einstein's geometrization of physics program is in question. The second vision is the identification of physics as a generalized number theory. This program involves three threads: various p-adic physics and their fusion together with real number based physics to a larger structure, the attempt to understand basic physics in terms of classical number fields (in particular, identifying associativity condition as the basic dynamical principle), and infinite primes whose construction is formally analogous to a repeated second quantization of an arithmetic quantum field theory. In this article brief summaries of physics as infinite-dimensional geometry and generalized number theory are given to be followed by more detailed articles.

Keywords: Kähler geometry, infinite-dimensional geometry, p-adic number fields, classical number fields, infinite primes.

Contents

1 Introduction

There are two basic approaches to the construction of quantum TGD. The first approach relies on the vision of quantum physics as infinite-dimensional Kähler geometry [30] for the "world of classical worlds" identified as the space of 3-surfaces in in certain 8-dimensional space. Essentially a generalization of the Einstein's geometrization of physics program is in question. The second vision is the identification of physics as a generalized number theory.

1.1 Physics as infinite-dimensional geometry

It is good to start with an attempt to give overall view about what the dream about physics as infinite-dimensional geometry is. The basic vision is generalization of the Einstein's program for the geometrization of classical physics so that entire quantum physics would be geometrized. Finite-dimensional geometry is certainly not enough for this purposed but physics as infinite-dimensional geometry of what might be called world of classical worlds (WCW) -or more neutrally configuration space of 3-surfaces of some higher-dimensional imbeddign space- might make sense. The requirement that the Hermitian conjugation of quantum theories has a geometric realization forces Kähler geometry for WCW. WCW defines the fixed arena of quantum physics and physical states are identified as spinor fields in WCW. These spinor fields are classical and no second quantization is needed at this level. The justification comes from the observation that infinite-dimensional Clifford algebra [34] generated by gamma matrices allows a natural identification as fermionic oscillator algebra.

The basic challenges are following.

1. Identify WCW.
2. Provide WCW with Kähler metric and spinor structure
3. Define what spinors and spinor fields in WCW are.

There is huge variety of finite-dimensional geometries and one might think that in infinite-dimensional case one might be drowned with the multitude of possibilities. The situation is however exactly opposite. The loop spaces associated with groups have a unique Kähler geometry due to the simple condition that Riemann connection exists mathematically [49]. This condition requires that the metric possesses maximal symmetries. Thus raises the vision that infinite-dimensional Kähler geometric existence is unique once one poses the additional condition that the resulting geometry satisfies some basic constraints forced by physical considerations.

The observation about the uniqueness of loop geometries leads also to a concrete vision about what this geometry could be. Perhaps WCW could be refarded as a union of symmetric spaces [28] for which every point is equivalent with any other. This would simplify the construction of the geometry immensely and would mean a generalization of cosmological principle to infinite-D context [8].

This still requires an answer to the question why $M^4 \times CP_2$ is so unique. Something in the structure of this space must distinguish it in a unique manner from any other candidate. The uniqueness of M^4 factor can be understood from the miraculous conformal symmetries of the light-cone boundary but in the case of CP_2 there is no obvious mathematical argument of this kind although physically CP_2 is unique [?]. The observation that $M^4 \times CP_2$ has dimension 8, the space-time surfaces have dimension 4, and partonic 2-surfaces, which are the fundamental objects by holography have dimension 2, suggests that classical number fields [38, 39, 40] are involved and one can indeed end up to the choice $M^4 \times CP_2$ from physics as generalized number theory vision by simple arguments [23]. In particular, the choices M^8 -a subspace of complexified octonions (for octonions see [40]), which I have used to call hyper-octonions- and $M^4 \times CP_2$ can be regarded as physically equivalent: this "number theoretical compactification" is analogous to spontaneous compactification in M-theory. No dynamical compactification takes place so that $M^8 - H$ duality is a more appropriate term.

1.2 Physics as generalized number theory

Physics as a generalized number theory (for an overview about number theory see [37]) program consists of three separate threads: various p-adic physics and their fusion together with real number based physics to a larger structure [22], the attempt to understand basic physics in terms of classical number fields [23] (in particular, identifying associativity condition as the basic dynamical principle), and infinite primes [24], whose construction is formally analogous to a repeated second quantization of an arithmetic quantum field theory. In this article a summary of the philosophical ideas behind this dream and a summary of the technical challenges and proposed means to meet them are discussed.

The construction of p-adic physics and real physics poses formidable looking technical challenges: p-adic physics should make sense both at the level of the imbedding space, the "world of classical worlds" (WCW), and space-time and these physics should allow a fusion to a larger coherent whole. This forces to generalize the notion of number by fusing reals and p-adics along rationals and common algebraic numbers. The basic problem that one encounters is definition of the definite integrals and harmonic analysis [29] in the p-adic context [25]. It turns out that the representability of WCW as a union of symmetric spaces [28] provides a universal group theoretic solution not only to the construction of the Kähler geometry of WCW but also to this problem. The p-adic counterpart of a symmetric space is obtained from its discrete invariant by replacing discrete points with p-adic variants of the continuous symmetric space. Fourier analysis [29] reduces integration to summation. If one wants to define also integrals at space-time level, one must pose additional strong constraints which effectively reduce the partonic 2-surfaces and perhaps even space-time surfaces to finite geometries and allow assign to a given partonic 2-surface a unique power of a unique p-adic prime characterizing the measurement resolution in angle variables. These integrals might make sense in the intersection of real and p-adic worlds defined by algebraic surfaces.

The dimensions of partonic 2-surface, space-time surface, and imbedding space suggest that classical number fields might be highly relevant for quantum TGD. The recent view about the connection is based on hyper-octonionic representation of the imbedding space gamma matrices, and the notions of associative and co-associative space-time regions defined as regions for which the modified gamma matrices span quaternionic or co-quaternionic plane at each point of the region. A further condition is that the tangent space at each point of space-time surface contains a preferred hyper-complex (and thus commutative) plane identifiable as the plane of non-physical polarizations so that gauge invariance has a purely number theoretic interpretation. WCW can be regarded as the space of sub-algebras of the local octonionic Clifford algebra [34] of the imbedding space defined by space-time surfaces with the property that the local sub-Clifford algebra spanned by Clifford algebra valued functions restricted at them is associative or co-associative in a given region.

The recipe for constructing infinite primes is structurally equivalent with a repeated second quantization of an arithmetic super-symmetric quantum field theory. At the lowest level one has fermionic and bosonic states labeled by finite primes and infinite primes correspond to many particle states of this theory. Also infinite primes analogous to bound states are predicted. This hierarchy of quantizations can be continued indefinitely by taking the many particle states of the previous level as elementary particles at the next level. Construction could make sense also for hyper-quaternionic and hyper-octonionic primes although non-commutativity and non-associativity pose technical challenges. One can also construct infinite number of real units as ratios of infinite integers with a precise number theoretic anatomy. The fascinating finding is that the quantum states labeled by standard model quantum numbers allow a representation as wave fuctions in the discrete space of these units. Space-time point becomes infinitely richly structured in the sense that one can associate to it a wave function in the space of real (or octonionic) units allowing to represent the WCW spinor fields. One can speak about algebraic holography or number theoretic Brahman=Atman identity and one can also say that the points of imbedding space and space-time surface are subject to a number theoretic evolution.

2 Questions

The experience has shown repeatedly that a correct question and identification of some weakness of existing vision is what can only lead to a genuine progress. In the following I discuss the basic questions, which have stimulated progress in the challenge of constructing WCW geometry.

2.1 What is WCW?

Concerning the identification of WCW I have made several guesses and the progress has been basically due to the gradual realization of various physical constraints and the fact that standard physics ontology is not enough in TGD framework.

1. The first guess was that WCW corresponds to all possible space-like 3-surfaces in $H = M^4 \times CP_2$, where M^4 denotes Minkowski space and CP_2 denotes complex projective space of two complex dimensions having also representation as coset space $SU(3)/U(2)$ (see the separate article summarizing the basic facts about CP_2 and how it codes for standard model symmetries [27]). What led to the this particular choice H was the observation that the geometry of H codes for standard model quantum numbers and that the generalization of particle from point like particle to 3-surface allows to understand also remaining quantum numbers having no obvious explanation in standard model (family replication phenomenon). What is important to notice is that Poincare symmetries act as exact symmetries of M^4 rather than space-time surface itself: this realizes the basic vision about Poincare invariant theory of gravitation. This lifting of symmetries to the level of imbedding space and the new dynamical degrees of freedom brought by the sub-manifold geometry of space-time surface are absolutely essential for entire quantum TGD and distinguish it from general relativity and string models. There is however a problem: it is not obvious how to get cosmology.

2. The second guess was that WCW consists of space-like 3-surfaces in $H_+ = M^4_+ \times CP2$, where M^4_+ future light-cone having interpretation as Big Bang cosmology at the limit of vanishing mass density with light-cone property time identified as the cosmic time. One obtains cosmology but loses exact Poincare invariance in cosmological scales since translations lead out of future light-cone. This as such has no practical significance but due to the metric 2-dimensionality of light-cone boundary δM^4_+ the conformal symmetries of string model assignable to finite-dimensional Lie group generalize to conformal symmetries assignable to an infinite-dimensional symplectic group of $S^2 \times CP_2$ and also localized with respect to the coordinates of 3-surface. These symmetries are simply too beautiful to be important only at the moment of Big Bang and must be present also in elementary particle length scales. Note that these symmetries are present only for 4-D Minkowski space so that a partial resolution of the old conundrum about why space-time dimension is just four emerges.

3. The third guess was that the light-like 3-surfaces in H or H_+ are more attractive than space-like 3-surfaces. The reason is that the infinite-D conformal symmetries characterize also light-like 3-surfaces because they are metrically 2-dimensional. This leads to a generalization of Kac-Moody symmetries [46] of super string models with finite-dimensional Lie group replaced with the group of isometries of H. The natural identification of light-like 3-surfaces is as 3-D surfaces defining the regions at which the signature of the induced metric changes from Minkowskian $(1, -1, -1, -1)$ to Euclidian $(-1 - 1 - 1 - 1)$- I will refer these surfaces as throats or wormhole throats in the sequel. Light-like 3-surfaces are analogous to blackhole horizons and are static because strong gravity makes them light-like. Therefore also the dimension 4 for the space-time surface is unique.

This identification leads also to a rather unexpected physical interpretation. Single lightlike wormhole throat carriers elementary particle quantum numbers. Fermions and their superpartners are obtained by glueing Euclidian regions (deformations of so called CP_2 type vacuum extremals of Kähhler action) to the background with Minkowskian signature. Bosons are identified as wormhole contacts with two throats carrying fermion *resp.* antifermionic quantum numbers. These can be identified as deformations of CP_2 vacuum extremals between between two parallel Minkowskian space-time sheets. One can say that bosons and their superpartners emerge. This has dramatic implications for quantum TGD [13] and QFT limit of TGD [15].

The question is whether one obtains also a generalization of Feynman diagrams. The answer is affirmative. Light-like 3-surfaces or corresponding Euclidian regions of space-time are analogous to the lines of Feynman diagram and vertices are replaced by 2-D surface at which these surfaces glued together. One can speak about Feynman diagrams with lines thicknened to light-like 3-surfaces and vertices to 2-surfaces. The generalized Feynman diagrams are singular as 3-manifolds but the vertices are non-singular as 2-manifolds. Same applies to the corresponding space-time surfaces and space-like 3-surfaces. Therefore one can say that WCW consists of generalized Feynman diagrams- something rather different from the original identification as space-like 3-surfaces and one can wonder whether these identification could be equvalent.

4. The fourth guess was a generalization of the WCW combining the nice aspects of the identifications $H = M^4 \times CP_2$ (exact Poincare invariance) and $H = M^4_+ \times CP_2$ (Big Bang cosmology). The idea was to generalize WCW to a union of basic building bricks -causal diamonds (CDs) - which themselves are analogous to Big Bang-Big Crunch cosmologies breaking Poincare invariance, which is however regained by the allowance of union of Poincare trnsforms of the causal diamonds.

   The starting point is General Coordinate Invariance (GCI). It does not matter, which 3-D slice of the space-time surface one choose to represent physical data as long as slices are related by a diffeomorphism of the space-time surface. This condition implies holography in the sense that 3-D slices define holograms about 4-D reality.

   The question is whether one could generalize GCI in the sense that the descriptions using space-like and light-like 3-surfaces would be equivalent physically. This requires that finite-sized space-like 3-surfaces are somehow equivalent with light-like 3-surfaces. This suggests that the light-like 3-surfaces must have ends. Same must be true for the space-time surfaces and must define preferred space-like 3-surfaces just like wormhole throats do. This makes sense only if the 2-D intersections of these two kinds of 3-surfaces -call them partonic 2-surfaces- and their 4-D tangent spaces carry the information about quantum physics. A strenghening of holopraphy principle would be the outcome. The challenge is to understand, where the intersections defining the partonic 2-surfaces are located.

   Zero energy ontology (ZEO) allows to meet this challenge.

   (a) Assume that WCW is union of sub-WCWs identified as the space of light-like 3-surfaces assignable to $CD \times CP_2$ with given CD defined as an intersection of future and past directed lightcones of M^4. The tips of CDs have localization in M^4 and one can perform for CD both translations and Lorentz boost for CDs. Space-time surfaces inside CD define the basic building brick of WCW. Also unions of CDs allowed and the CDs belonging to the union can intersect. One can of course consider the possibility of intersections and analogy with the set theoretic realization of topology.

   (b) ZEO property means that the light-like boundaries of these objects carry positive and negative energy states, whose quantum numbers are opposite. Everything can be created from vacuum and can be regarded as quantum fluctuations in the standard vocabulary of quantum field theories.

   (c) Space-time surfaces inside CDs begin from the lower boundary and end to the upper boundary and in ZEO it is natural to identify space-like 3-surfaces as pairs of space-like 3-surfaces at these boundaries. Light-like 3-surfaces connect these boundaries.

   (d) The generalization of GCI states that the descriptions based on space-like 3-surfaces must be equivalent with that based on light-like 3-surfaces. Therefore only the 2-D intersections of light-like and space-like 3-surfaces - partonic 2-surfaces- and their 4-D tangent spaces (4-surface is there!) matter. Effective 2-dimensionality means a strengthened form of holography but does not imply exact 2-dimensionality, which would reduce the theory to a mere string model like theory. Once these data are given, the 4-D space-time surface is fixed and is analogous to a generalization of Bohr orbit to infinite-D context. This is the first guess. The situation is actually more delicate due to the non-determinism of Kähler action motivating the interaction of the hierarchy of CDs within CDs.

In this framework one obtains cosmology: CDs represent a fractal hierarchy of big bang-big crunch cosmologies. One obtains also Poincare invariance. One can also interpret the non-conservation of gravitational energy in cosmology which is an empirical fact but in conflict with exact Poincare invariance as it is realized in positive energy ontology [20, 21]. The reason is that energy and four-momentum in zero energy ontology correspond to those assignable to the positive energy part of the zero energy state of a particular CD. The density of energy as cosmologist defines it is the statistical average for given CD: this includes the contibutions of sub-CDs. This average density is expected to depend on the size scale of CD density is should therefore change as quantum dispersion in the moduli space of CDs takes place and leads to large time scale for any fixed sub-CD.

Even more, one obtains actually quantum cosmology! There is large variety of CDs since they have position in M^4 and Lorentz transformations change their shape. The first question is whether the M^4 positions of both tips of CD can be free so that one could assign to both tips of CD momentum eigenstates with opposite signs of four-momentum. The proposal, which might look somewhat strange, is that this not the case and that the proper time distance between the tips is quantized in octaves of a fundamental time scale $T = R/c$ defined by CP_2 size R. This would explains p-adic length scale hypothesis which is behind most quantitative predictions of TGD. That the time scales assignable to the CD of elementary particles correspond to biologically important time scales [26] forces to take this hypothesis very seriously.

The interpretation for T could be as a cosmic time quantized in powers of two. Even more general quantization is proposed to take place. The relative position of the second tip with respect to the first defines a point of the proper time constant hyperboloid of the future light cone. The hypothesis is that one must replace this hyperboloid with a lattice like structure. This implies very powerful cosmological predictions finding experimental support from the quantization of redshifts for instance [21]. For quite recent further empirical support see [60].

One should not take this argument without a grain of salt. Can one really realize zero energy ontology in this framework? The geometric picture is that translations correspond to translations of CDs. Translations should be done independently for the upper and lower tip of CD if one wants to speak about zero energy states but this is not possible if the proper time distance is quantized. If the relative M^4_+ coordinate is discrete, this pessimistic conclusion is strengthened further.

The manner to get rid of problem is to assume that translations are represented by quantum operators acting on states at the light-like boundaries. This is just what standard quantum theory assumes. An alternative- purely geometric- way out of difficulty is the Kac-Moody symmetry associated with light-like 3-surfaces meaning that local M^4 translations depending on the point of partonic 2-surface are gauge symmetries. For a given translation leading out of CD this gauge symmetry allows to make a compensating transformation which allows to satisfy the constraint.

This picture is roughly the recent view about WCW. What deserves to be emphasized is that a very concrete connection with basic structures of quantum field theory emerges already at the level of basic objects of the theory and GCI implies a strong form of holography and almost stringy picture.

2.2 Some Why's

In the following I try to summarize the basic motivations behind quantum TGD in form of various Why's.

1. Why WCW?

   Einstein's program has been extremely successful at the level of classical physics. Fusion of general relativity and quantum theory has however failed. The generalization of Einstein's geometrization program of physics from classical physics to quantum physics gives excellent hopes about the success in this project. Infinite-dimensional geometries are highly unique and this gives hopes about fixing the physics completetely from the uniqueness of the infinite-dimensional Kähler geometric existence.

2. Why spinor structure in WCW?

   Gamma matrices defining the Clifford algebra [34] of WCW are expressible in terms of fermionic oscillator operators. This is obviously something new as compared to the view about gamma matrices as bosonic objects. There is however no deep reason denying this kind of identification. As a consequence, a geometrization of fermionic oscillator operator algebra and fermionic statistics follows as also geometrization of super-conformal symmetries [45, 46] since gamma matrices define super-generators of the algebra of WCW isometries extended to a super-algebra.

3. Why Kähler geometry?

   Geometrization of the bosonic oscillator operators in terms of WCW vector fields and fermionic oscillator operators in terms of gamma matrices spanning Clifford algebra. Gamma matrices span hyper-finite factor of type II_1 and the extremely beautiful properties of these von Neuman algebras [50] (one of the three von Neuman algebras that von Neumann suggests as possible mathematical frameworks behind quantum theory) lead to a direct connection with the basic structures of modern physics (quantum groups, non-commutative geometries,.. [54]).

   A further reason why is the finiteness of the theory.

   (a) In standard QFTs there are two kinds of divergences. Action is a local functional of fields in 4-D sense and one performs path integral over **all** 4-surfaces to construct S-matrix. Mathematically path integration is a poorly defined procedure and one obtains diverging Gaussian determinants and divergences due to the local interaction vertices. Regularization provides the manner to get rid of the infinities but makes the theory very ugly.

   (b) Kähler function defining the Kähler geometry is a expected to be non-local functional of the partonic 2-surface (Kähler action for a preferred extremal having as its ends the positive and negative energy 3-surfaces). Path integral is replaced with a functional integral which is mathematically well-defined procedure and one perfoms functional integral only over the partonic 2-surfaces rather than all 4-surfaces (holography). The exponent of Kähler function defines a unique vacuum functional. The local divergences of local quantum field theories of local quantum field theories since there are no local interaction vertices. Also the divergences associated with the Gaussian determinant and metric determinant cancel since these two determinants cancel each other in the integration over WCW. As a matter fact, symmetric space property suggest a much more elegant manner to perform the functional integral by reducing it to harmonic analysis in infinite-dimensional symmetric space [11].

   (c) One can imagine also the possibility of divergences in fermionic degrees of freedom but it has turned out that the generalized Feynman diagrams in ZEO are manifestly finite. Even more: it is quite possible that only finite number of these diagrams give non-vanishing contributions to the scattering amplitude. This is essentially due to the new view about virtual particles, which are identified as bound states of on mass shell states assigned with the throats of wormhole contacts so that the integration over loop momenta of virtual particles is extremely restricted [11].

4. Why infinite-dimensional symmetries?

   WCW must be a union of symmetric spaces in order that the Riemann connection exists (this generalizes the finding of Freed for loop groups [49]). Since the points of symmetric spaces are metrically equivalent, the geometrization becomes tractable although the dimension is infinite. A union of symmetric spaces is required because 3-surfaces with a size of galaxy and electron cannot be metrically equivalent. Zero modes distinguish these surfaces and can be regarded as purely classical degrees of freedom whereas the degrees of freedom contributing to the WCW line element are quantum fluctuating degrees of freedom.

One immediate implication of the symmetric space property is constant curvature space property meaning that the Ricci tensor proportional to metric tensor. Infinite-dimensionality means that Ricci scalar either vanishes or is infinite. This implies vanishing of Ricci tensor and vacuum Einstein equations for WCW.

5. Why $M^4 \times CP_2$?

   This choice provides an explanation for standard model quantum numbers. The conjecture is that infinite-D geometry of 3-surfaces exists only for this choice. As noticed, the dimension of space-time surfaces and M^4 fixed by the requirement of generalized conformal invariance [44] making possible to achieve symmetric space property. If $M^4 \times CP_2$ is so special, there must be a good reason for this. Number theoretical vision [23] indeed leads to the identification of this reason. One can assign the hierarchy of dimensions associated with partonic 2-surfaces, space-time surfaces and imbedding space to classical number fields and can assign to imbedding space what might be called hyper-octonionic structure. "Hyper" comes from the fact that the tangent space of H corresponds to the subspaces of complexified octonions with octonionic imaginary units multiplied by a commuting imaginary unit. The space-time reions would be either hyper-quaternionic or co-hyper-quaternionic so that associativity/co-associativity would become the basic dynamical principle at the level of space-time dynamics. Whether this dynamical principle is equivalent with the preferred extremal property of Kähler action remains an open conjecture.

6. Why zero energy ontology and why causal diamonds?

   The consistency between Poincare invariance and GRT requires ZEO. In positive energy ontology only one of the infinite number of classical solutions is realized and partially fixed by the values of conserved quantum numbers so that the theory becomes obsolote. Even in quantum theory conservation laws mean that only those solutions of field equations with the quantum numbers of the initial state of the Universe are interesting and one faces the problem of understanding what the the initial state of the universe was. In ZEO these problems disappear. Everything is creatable from vacuum: if the physical state is mathematically realizable it is in principle reachable by a sequence of quantum jumps. There are no physically non-reachable entities in the theory. Zero energy ontology leads also to a fusion of thermodynamics with quantum theory. Zero energy states ae defined as entangled states of positive and negative energy states and entanglement coefficients define what I call M-matrix identified as "complex square root" of density matrix expressible as a product of diagonal real and positive density matrix and unitary S-matrix [13].

   There are several good reasons why for causal diamonds. ZEO requires CDs, the generalized form of GCI and strong form of holography (light-like and space-like 3-surfaces are physically equivalent representations) require CDs, and also the view about light-like 3-surfaces as generalized Feynman diagrams requires CDs. Also the classical non-determinism of Kähler action can be understood using the hierarchy CDs and the addition of CDs inside CDs to obtain a fractal hierarchy of them provides an elegant manner to undersand radiative corrections and coupling constant evolution in TGD framework.

   A strong physical argument in favor of CDs is the finding that the quantized proper time distance between the tips of CD fixed to be an octave of a fundamental time scale defined by CP_2 happens to define fundamental biological time scale for electron, u quark and d quark [26]: there would be a deep connection between elementary particle physics and living matter leading to testable predictions.

3 What could be the symmetries of WCW geometry?

Symmetry principles play key role in the construction of WCW geometry have become and deserve a separate explicit treament even at the risk of repetitions.

3.1 General Coordinate Invariance

General coordinate invariance is certainly of the most important guidelines and is much more powerful in TGD framework thanin GRT context.

1. General coordinate transformations as a gauge symmetry so that the diffeomorphic slices of space-time surface equivalent physically. 3-D light-like 3-surfaces defined by wormhole throats define preferred slices and allows to fix the gauge partially apart from the remaining 3-D variant of general coordinate invariance and possible gauge degeneracy related to the choice of the light-like 3-surface due to the Kac-Moody invariance. This would mean that the random light-likeness represents gauge degree of freedome except at the ends of the light-like 3-surfaces.

2. GCI can be strengthed so that the pairs of space-like ends of space-like 3-surfaces at CDs are equivalent with light-like 3-surfaces connecting them. The outcome is effective 2-dimensionality because their intersections at the boundaries of CDs must carry the physically relevant information.

3.2 Generalized conformal symmetries

One can assign Kac-Moody type conformal symmetries to light-like 3-surfaces as isometries of H localized with respect to light-like 3-surfaces. Kac Moody algebra essentially the Lie algebra of gauge group with central extension meaning that projective representation in which representation matrices are defined only modulo a phase factor. Kac-Moody symmetry is not quite a pure gauge symmetry.

One can assign a generalization of Kac-Moody symmetries to the boundaries of CD by replacing Lie-group of Kac Moody algebra with the group of symplectic (contact-) tranformations [32] of H_+ provided with a degenerate Kähler structure made possible by the effective 2-dimensionality of δM^4_+. The light-like radial coordinate of δM^4_+ plays the role of the complex coordinate of conformal transformations or their hyper-complex analogs. These symmetries are also localized with respect to the internal coordinates of the partonic 2-surface so that rather huge symmetry group is in question. The basic hypothesis is that these transformations with possible some restrictions on the depedence on the coordinates of X^2 define the isometries of WCW.

A further physically well-motivated hypothesis inspired by holography and extended GCI is that these symmetries extend so that they apply at the entire space-time sheet. This requires the slicing of space-time surface by partonic 2- surfaces and by stringy world sheets such that each point of stringy world sheet defines a partonic 2-surface and vice versa. This slicing has deep physical motivations since it realizes geometrically standard facts about gauge invariance (partonic 2-surface defines the space of physical polarizations and stringy space-time sheet corresponds to non-physical polarizations) and its existence is a hypothesis about the properties of the preferred extremals of Kähler action. There is a similar decomposition also at the level of CD and so called Hamilton-Jacobi coordinates for M^4_+ [19] define this kind of slicings. This slicing can induced the slicing of the space-time sheet. The number theoretic vision gives a further justification for this hypothesis and also strengthens it by postulating the presence of the preferred time direction having interpretation in terms of real unit of octonions. In ZEO this time direction corresponds to the time-like vector connecting the tips of CD.

Conformal symmetries would provide the realization of WCW as a union of symmetric spaces. Symmetric spaces are coset spaces of form G/H. The natural identification of G and H is as groups of X^2-local symplectic transformations and local Kac-Moody group of X^2-local H isometries. Quantum fluctuating (metrically non-trivial) degrees of freedom would correspond to symplectic transformations of H_+ and induced Kähler form at X^2 would define a local representation for zero modes: not necessarily all of them.

3.3 Equivalence Principle and super-conformal symmetries

Equivalence Principle (EP) is a second corner stone of General Relativity and together with GCI leads to Einstein's equations. What EP states is that inertial and gravitational masses are identical. In this form it is not well-defined even in GRT since the definition of gravitational and inertial four-momenta is highly problematic because Noether theorem is not avaible. The realization is in terms of local equations identifying energy momentum tensor with Einstein tensor.

Whether EP is realized in TGD has been a longstanding open question[20]. The problem has been that at the classical level EP in its GRT form can hold true only in long enough length scales and it took long to time to realize that only the stringy form of this principle is required. The first question is how to identify the gravitational and inertial four-momenta. This is indeed possible. One can assign to the two types super-conformal symmetries assigned with light-like 3-surfaces and space-like 3-surfaces four-momenta to both. EP states that these four momenta are identical and is equivalent with the generalization of GCI and effective 2-dimensionality. The condition generalizes so that it applies to the generators of super-conformal algebras associated with the two super-conformal symmetries. This leads to a generalization of a standard mathematical construction of super-conformal theories known as coset representation [47]. What the construction states is that the differences of super-conformal generators defined by super-symmetric algebra and Kac-Moody algebra annihilate physical states.

3.4 Extension to super-conformal symmetries

The original idea behind the extension of conformal symmetries to super-conformal symmetries was the observation that isometry currents defining infinitesimal isometries of WCW have natural super-counterparts obtained by contracting the Killing vector fields with the complexified gamma matrices of the imbedding space.

This vision has generalized considerably as the construction of WCW spinor structure in terms of modified Dirac action has developed. The basic philosophy behind this idea is that configuration space spinor structure must relate directly to the fermionic sector of quantum physics. In particular, modified gamma matrices should be expressible in terms of the fermionic oscillator operators associated with the second quantized induced spinor fields. The explicit realization of this program leads to an identification of rich spectrum of super-conformal symmetries and generalization of the ordinary notion of space-time supersymmetry. What happens that all fermionic oscillator operator generate broken super-symmetries whereas in SUSYs there is only finite number of them. One can however identify sub-algebra of super-conformal symmetries associated with right handed neutrino and this gives $\mathcal{N}=1$ super-symmetry [56] of SUSYs [14].

3.5 Further symmetries

Besides super-conformal symmetries one can imagine candidates for other symmetries. Electric-magnetic duality is the Mother of all dualities and the question however it might be present in TGD popped up from beginning and its precise formulation has quite recently led to a dramatic progress in the understanding of both physics and mathematics of TGD [8]. This duality suggests even a modification of the classical dynamics of TGD by replacing Kähler form J of CP_2 with the sum $J+J_1$, where J_1 is the Kähler form of $r_M = constant$ sphere defining a magnetic monopole field in CD. This option extends dramatically the space of vacuum extremals which is highly desired outcome since it makes possible to imbed much larger set of solutions of Einstein's equations as vacuum extremals. Hyper-Kähler structure [35] or at least quaternionic structure [36] is an additional attractive candidate for a symmetry. So called M^8-H duality was inspired by number theoretical vision before the realization that imbedding space gamma matrices allow an octonionic generalization reducing the dynamics of the space-time time surfaces to associativity or co-associativity.

3.5.1 Could a weak form of electric-magnetic duality hold true?

Holography means that the initial data at the partonic 2-surfaces should fix the configuration space metric. A weak form of this condition allows only the partonic 2-surfaces defined by the wormhole throats at which the signature of the induced metric changes. Number theoretical vision suggests that hyper-quaternionicity *resp.* co-hyperquaternionicity constraint could be enough to fix the initial values of time derivatives of the imbedding space coordinates in the space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. This is a condition on modified gamma matrices and hyper-quaternionicity states that they span a hyper-quaternionic sub-space.

One can also consider alternative conditions possibly equivalent with this condition. The argument goes as follows.

1. The expression of the matrix elements of the metric and Kähler form of WCW in terms of the Kähler fluxes weighted by Hamiltonians of $\delta M^4_{\pm}$ at the partonic 2-surface X^2 looks very

attractive. These expressions however carry no information about the 4-D tangent space of the partonic 2-surfaces so that the theory would reduce to a genuinely 2-dimensional theory, which cannot hold true. One would like to code to the WCW metric also information about the electric part of the induced Kähler form assignable to the complement of the tangent space of $X^2 \subset X^4$.

2. Electric-magnetic duality [59] of the theory looks a highly attractive symmetry. The trivial manner to get electric magnetic duality at the level of the full theory would be via the identification of the flux Hamiltonians as sums of of the magnetic and elecric fluxes. The presence of the induced metric is however troublesome since the presence of the induced metric means that the simple transformation properties of flux Hamiltonians under symplectic transformations -in particular color rotations- are lost.

3. A less trivial formulation of electric-magnetic duality would be as an initial condition which eliminates the induced metric from the electric flux. In the Euclidian version of 4-D YM theory this duality allows to solve field equations exactly in terms of instantons. This approach involves also quaternions. These arguments suggest that the duality in some form might work. The full electric magnetic duality is certainly too strong and implies that space-time surface at the partonic 2-surface corresponds to piece of CP_2 type vacuum extremal and can hold only in the deep interior of the region with Euclidian signature. In the region surrounding wormhole throat at both sides the condition must be replaced with a weaker condition.

4. To formulate a weaker form of the condition let us introduce coordinates (x^0, x^3, x^1, x^2) such (x^1, x^2) define coordinates for the partonic 2-surface and (x^0, x^3) define coordinates labeling partonic 2-surfaces in the slicing of the space-time surface by partonic 2-surfaces and string world sheets making sense in the regions of space-time sheet with Minkowskian signature. The assumption about the slicing allows to preserve general coordinate invariance. The weakest condition is that the generalized Kähler electric fluxes are apart from constant proportional to Kähler magnetic fluxes. This requires the condition

$$J^{03}\sqrt{g_4} = KJ_{12} \ . \tag{3.1}$$

5. Information about the tangent space of the space-time surface can be coded to the configuration space metric with loosing the nice transformation properties of the magnetic flux Hamiltonians if Kähler electric fluxes or sum of magnetic flux and electric flux satisfying this condition are used and K is symplectic invariant. Using the sum

$$J_e + J_m = (1 + K)J_{12} \tag{3.2}$$

makes it possible to have a non-trivial configuration space metric even for $K = 0$, which could correspond to the ends of a cosmic string like solution carrying only Kähler magnetic fields. This condition suggests that it can depend only on Kähler magnetic flux and other symplectic invariants. Whether local symplectic coordinate invariants are possible at all is far from obvious, If the slicing itself is symplectic invariant then K could be a non-constant function of X^2 depending on string world sheet coordinates. The light-like radial coordinate of the light-cone boundary indeed defines a symplectically invariant slicing and this slicing could be shifted along the time axis defined by the tips of CD.

6. A more general form of this duality is suggested by the considerations of [8] reducing Kähler function for preferred extremals to Chern-Simons terms at the boundaries of CD and at light-like wormhole throats. This form is following

$$J^{n\beta}\sqrt{g_4} = K\epsilon \times \epsilon^{n\beta\gamma\delta}J_{\gamma\delta}\sqrt{g_4} \ . \tag{3.3}$$

Here the index n refers to a normal coordinate for the space-like 3-surface at either boundary of CD or for light-like wormhole throat. ϵ is a sign factor which is opposite for the two ends of CD.

It could be also opposite of opposite at the opposite sides of the wormhole throat. Note that the dependence on induced metric disappears at the right hand side and this condition eliminates the potentials singularity due to the reduction of the rank of the induced metric at wormhole throat.

7. Physical arguments force to consider also a more general variant of the weak self-duality in which appears also the symplectic form of $r_M = constant$ sphere of light-cone boundary -call it J^1 - defining a magnetic monopole field of a monopole at the line connecting the tips of CD. This form reads as

$$J^{n\beta}\sqrt{g_4} = K\epsilon \times \epsilon^{n\beta\gamma\delta}(J_{\gamma\delta} + \epsilon J^1_{\gamma\delta})\sqrt{g_4} \ . \tag{3.4}$$

Here ϵ is a pure number $\epsilon = \pm 1$ is favored and should be opposite at the opposite boindaries of CD. This leads to ask whether one should replace J in Kähler action with $J + J_1$ [8]. This would break Lorentz invariance for a given CD to $SO(3)$ but this is caused in any case by the geometry of CD and is compensated by the fact that the Lorentz boots of CDs are allowed.

They physical implications of the weak form of electric-magnetic duality are surprisingly far reaching [8].

1. The basic point is that long range magnetic monopole fields have not been observed and the only possibility is that the wormhole throat carrying magnetic charge is accompanied by a second one carrying opposite magnetic charge. The other throat can carry also weak isospin compensating that of the throat and this leads to a precise understanding of how the finite range of weak interactions emerges by the screening caused by this charge. One could speak of weak confinement realized in terms of magnetic monopole confinement. By generalizing a little bit also color confiement can be interpreted in terms of magnetic confinement. Note that the magnetic fluxes connecting the monopoles make particles string like objects with length of order intetermediate gauge boson Compton length.

2. A further implication is the reduction of the Kähler action to generalized Chern-Simons action [58] if the Coulomb interaction energy vanishes in some gauge. This gauge exists if the integrability condition $j_K \wedge dj_K = 0$ holds true. This condition reduces to a similar condition for instanton current using the general solution ansatz for the extremals and the solution ansatz actually requires this condition so that there is consistency with "classical" TGD [8].

3.5.2 Does WCW possess quaternion or even Hyper-Kähler structure?

CP_2 has quaternion structure [31] with quaternionic units defined by metric and three components of Weyl tensor one of which is covariantly constant Kähler form so that a symmetry breaking occurs. Something like this expected also in infinite-D context. Hyper-Kähler structure [35] would be stronger symmetry and mean that the Kähler structures are parametrized by the points of sphere S^2. In the case of CP_2 there is unique Kähler structure and the physical interpretation is in terms of electroweak symmetry breaking selecting a unique isospin direction so that superpositions of quantum states with different electromagnetic charges are not possible. It is a general theorem that Hyper-Kähler structure implies vacuum Einstein equations (vanishing Ricci tensor) and this must hold true for WCW. Perhaps Hyper-Kähler property is too much to hope and it might be indeed inconsistent with electroweak symmetry breaking.

3.5.3 Does $M^8 - H$ duality hold true?

The hopes of giving $M^4 \times CP_2$ hyper-octonionic structure as something analogous to complex structure are meager since this would require generalization of holomorphic functions to hyper-octonionic context. Hyper-octonionic structure however allows a beautiful realization in terms of the hyper-octonionic representation of the imbedding space gamma matrices as I learned later. The modified gamma matrices realized in terms of hyper-octonionic gamma matrices are required to span hyper-quaternionic gamma matrix algebra and this condition generalizes the notion of hyper-quaternionic (and thus associative manifold).

One can however still argue that CP_2 must be something very special. This circumstance forced to ask whether four-surfaces $X^4 \subset M^8$ could under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly so that the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. The following arguments suggest that this is indeed the case.

The hard mathematical fact behind the "number theoretical compactification" (not real as in M-theory) is that the hyper-quaternionic sub-spaces of octonions containing a fixed hyper-complex sub-space defining preferred real and imaginary units are parameterized by CP_2 just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. The preferred imaginary and real units span the subs-space of non-physical polarizations and also quantization direction for energy (preferred rest system) and quantization axis of angular momentum.

1. The space of complex structures of the octonion space is parameterized by S^6. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Hyper-quaternions can be identified as $U(2)$ Lie-algebra but it is obvious that hyper-octonions do not allow an identification as $SU(3)$ Lie algebra. Rather, octonions decompose as $1 \oplus 1 \oplus 3 \oplus \overline{3}$ to the irreducible representations of $SU(3)$.

2. Geometrically the choice of a preferred complex (quaternionic) structure means fixing of complex (quaternionic) sub-space of octonions. The fixing of a hyper-quaternionic structure of hyper-octonionic M^8 means a selection of a fixed hyper-quaternionic sub-space $M^4 \subset M^8$ implying the decomposition $M^8 = M^4 \times E^4$. If M^8 is identified as the tangent space of $H = M^4 \times CP_2$, this decomposition results naturally. It is also possible to select a fixed hyper-complex structure, which means a further decomposition $M^4 = M^2 \times E^2$.

3. The basic result behind number theoretic compactification and $M^8 - H$ duality is that hyper-quaternionic sub-spaces $M^4 \subset M^8$ containing a fixed hyper-complex sub-space $M^2 \subset M^4$ or its light-like line $M_\pm$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

4. Space-time surface $X^4 \subset M^8$ is by stardard definition hyper-quaternionic if the tangent spaces of X^4 are hyper-quaternionic planes. Co-hyper-quaternionictity means the same for normal spaces. The presence of fixed hyper-complex structure means at space-time level that the tangent space of X^4 contains fixed M^2 at each point. Under this assumption one can map the points $(m, e) \in M^8$ to points $(m, s) \in H$ by assigning to the point (m, e) of X^4 the point (m, s), where $s \in CP_2$ characterize $T(X^4)$ as hyper-quaternionic plane. This definition is not the only one and even the appropriate one in TGD context the replacement of the tangent plane with the 4-D plane spanned by modified gamma matrices defined by Kähler action is a more natural choice. This plane is not parallel to tangent plane in general. In the sequel $T(X^4)$ denotes the preferred 4-plane which co-incides with tangent plane of X^4 only if the action defining modified gamma matrices is 4-volume.

5. The choice of M^2 can be made also local in the sense that one has $T(X^4) \supset M^2(x) \subset M^4 \subset H$. It turns out that strong form of number theoretic compactification requires this kind of generalization. In this case one must be able to fix the convention how the point of CP_2 is assigned to a hyper-quaternionic plane so that it applies to all possible choices of $M^2 \subset M^4$. Since $SO(3)$ hyper-quaternionic rotation relates the hyper-quaternionic planes to each other, the natural assumption is hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Under this assumption it is possible to map hyper-quaternionic surfaces of M^8 for which $M^2 \subset M^4$ depends on point of X^4 to H.

4 Various approaches to the construction of WCW Kähler geometry

One can imagine several approaches to the construction of Kähler geometry. One can try to guess the Kähler function using quantum classical correspondence and general coordinate invariance as guidelines [8]. One can try to construct the matrix elements of Kähler metric and form directly using the huge super-conformal symmetries [8]. One can also try to reduce the construction to that of WCW spinor structure in terms of second quantized induced spinor fields by using the dynamics of the modified Dirac action super-symmetrically related to that of Kähler action [10, 11].

4.1 Guess the Kähler function

The idea is to guess the Kähler function from quantum classical correspondence and general coordinate invariance (GCI). The basic objects are 3-surface but 4-D general coordinte invariance is required. This means that the definition of Kähler function must assign to a given 3-surface X^2 a more or less unique 4-surface $X^4(X^3)$. The interpretation is in terms of holography and quantum classical correspondence meaning that classical physics becomes part of WCW geometry.

4.1.1 Kähler function as Kähler action for its preferred extremal

Quantum classical correspondence requires that space-time surfaces are preferred extremals of some classical variational principle. So called Kähler action -a Maxwell action associated with the Maxwell field defined by the projection of the CP_2 Kähler form is in respect a completely unique choice and is characterized by enormous vacuum degeneracy corresponding to induced Kähler forms which vanish. Any space-time surface for which CP_2 projection is at most 2-D Lagrangian manifold [33] is vacuum extremal. Also the non-vacuum deformations of the vacuum extremals fail to be determistic in the standard sense of the word and one must generalize the notion of 3-surface by allowing unions of disjoint 3-surfaces with time-like separations. This allows to extend quantum classical correspondence so that also quantum jump sequences have classical space-time correlates.

Kähler function for given 3-surfaces as Kähler action for a preferred extremal $X^4(X^3)$ of Kähler action. This raises however questions. First of all: what are these preferred extremals?

1. The first guess was that they correspond to absolute minima of Kähler action. This guess turned out to be non-realistic.

2. TGD Universe is quantum critical in the sense that the only parameter of the theory- Kähler coupling strength- is analogous to critical temperature mathematically. The natural proposal is that criticality property selects preferred extremals. Critical extremals are analogous to a minimum of potential function for which the matrix defined by second derivatives has vanishing determinant and thus non-maximal rank. The catastrophe theory of Thom relies on the classification of catastrophes in terms of the manifolds at which the rank is not minimal.

3. The construction of WCW spinor structure in terms of the modified Dirac action forces criticality as a prerequisite for conservation laws associated with the critical deformations of space-time surface and also suggests that the algebra of critical deformations is always infinite-dimensional and one has inclusion hierarchies of this kind of algebras directly related to the inclusion hierarchies of hyper-finite factors of type II_1 about which configuration space Clifford algebra is an example of.

4. Number theoretic vision requires that space-time surfaces decompose to associative and co-associative regions -or equivalently- to quaternionic and co-quaternionic regions. Co-property - means that normal space has the property in the sense that tangent space gamma matrices in octonionic representation multiplied by a fixed octonionic imaginary unit are quaternionic. This would be essentially algebraic characterization of space-time surfaces. An open conjecture is whether this characterization is consistent with extremal property.

5. Number theoretic vision- in particular $M^8 - H$ lead to strong constraints on the space-time surfaces. In particular the tangent space of the quaternionic tangent space must contain a

preferred plane hyper-complex plane $M^2 \subset M^4$ having interpretation as plane of preferred polarizations. M^2 could also depend on point of M^4 and does so in so called Hamilton-Jacobi coordinates defining slicings of M^4 to string world sheets and partonic 2-surfaces. The properties of known extremals of Kähler action [19] are consistent with these constraints.

6. Holography implies that everything should reduce to 4-D boundary conditions at partonic 2-surfaces giving constraints on 4-D tangent space of X^4 at X^2. What are these conditions? Is quaternionicity/co-quaternionicity enough? Could one pose weak form of electric-magnetic duality and does it imply the hyper-quaternionicity? Note than in 4-D Euclidian YM theories strong form of electric-magnetic duality is assume and quaternionic structure is involved in an essential manner.

Quantum classical correspondence and quantum meaurement theory require that space-time geometry carries information about quantum number of zero energy states. In the formulation based on modified Dirac action this is achieved by adding to the modified Dirac action a measurement interaction term linear in conserved charges- in particular momenta. This implies that the preferred extremal depends on the momenta through boundary conditions at the ends of space-time surface.

Kähler function determined up to the addition of a real part of holomorphic function depending also on zero modes since this modification does not affect metric and Kähler form. This means an additional U(1) gauge invariance. There are several intepretations. First of all it could corresponds to the possibility to Kac-Moody invariance of light-like 3-surfaces leaving the partonic 2-surfaces and corresponding tangent space distributions invariant. Also it could relate to the possibility to choose any partonic 2-surface in the slicing as a representative of space-time surface. Also the modification of the measurement interaction terms characterizing particular measurement would affect the preferred extremal via boundary conditions but leave Kähler metri cnvariant.

4.1.2 Hierarchy of Planck constants and Kähler function

The hypothesis about hierarchy of Planck constants and generalization of the imbedding space H or rather $CD \times CP_2$ to a book like structure with pages corresponding to products of singular coverings (and perhaps also singular factor spaces) of CD and CD [16] has become a central part of quantum TGD.

Induced Kähler form defines a Maxwell field and it is important to characterize precisely its relationship to the gauge fields as they are defined in gauge theories. Kähler form J is related to the corresponding Maxwell field F via the formula

$$J = \frac{g_K}{\hbar} F \ . \tag{4.1}$$

Similar relationship holds true also for the other induced gauge fields. The inverse proportionality of J to $\hbar$ does not matter in the ordinary gauge theory context where one routinely choses units by putting $\hbar = 1$ but becomes very important when one considers a hierachy of Planck constants [16]. By $\alpha_K = g_K^2/4\pi\hbar$ the large Planck constant means weaker interactions and convergence of the functional integral defined by the exponent of Kähler function and one can argue that the convergence of the functional integral is what forces the hierarchy of Planck constants. This is in accordance with the vision that Mother Nature likes theoreticians and takes care that the perturbation theory works by making a phase transition increasing the value of the Planck constant in the situation when perturbation theory fails. This leads to a replacement of the M^4 (or more precisely, causal diamond CD) and CP_2 factors of the imbedding space ($CD \times CP_2$) with its $r = \hbar/\hbar_0$-fold singular covering (one can consider also singular factor spaces). If the components of the space-time surfaces at the sheets of the covering are identical, one can interpret r-fold value of Kähler action as a sum of r identical contributions from the sheets of the covering with ordinary value of Planck constant and forget the presence of the covering. Physical states are however different even in the case that one assumes that sheets carry identical quantum states and anyonic phase could correspond to this kind of phase [18].

The hierarchy of Planck constants was originally introduced on basis of purely physical motivations. It has however turned out that there might be no need to introduce it as a separate hypothesis in TGD framework. The realization that the hierarchy of Planck constant realized in terms of coverings of the imbedding space follows from basic quantum TGD leads to a further understanding. The extreme non-linearity of the canonical momentum densities as functions of time derivatives of the imbedding space coordinates implies that the correspondence between these two variables is not 1-1 so that there is natural to introduce coverings of $CD \times CP_2$. This leads also to a precise geometric characterization of the criticality of the preferred extremals [8].

4.1.3 Can one guess Kähler function directly?

The basic challenge is the explicit identification of WCW Kähler function K. Two assumptions lead to the identification of K as a sum of Chern-Simons type terms associated with the ends of causal diamond and with the light-like wormhole throats at which the signature of the induced metric changes [8]. The first assumption is the weak form of electric magnetic duality. Second assumption is that the Kähler current for the preferred extremals satisfies the condition $j_K \wedge dj_K = 0$ implying that the flow parameter of the flow lines of j_K defines a global space-time coordinate.

The integrability conditions follow also from the construction of the extremals of Kähler action [19]. The conjecture was that for the extremals the 4-dimensional Lorentz force vanishes (no dissipation): this requires $j_K \wedge J = 0$. One manner to guarantee this is the topologization of the Kähler current meaning that it is proportional to the instanton current: $j_K = \phi j_I$, where $j_I = {}^*(J \wedge A)$ is the instanton current, which is not conserved for 4-D CP_2 projection. The conservation of j_K implies the condition $j_I^\alpha \partial_\alpha \phi = \partial_\alpha j^\alpha \phi$ and from this ϕ can be integrated if the integrability condition $j_I \wedge dj_I = 0$ holds true implying the same condition for j_K. By introducing at least 3 or CP_2 coordinates as space-time coordinates, one finds that the contravariant form of j_I is purely topological so that the integrability condition fixes the dependence on M^4 coordinates and this selection is coded into the scalar function ϕ. These functions define families of conserved currents $j_K^\alpha \phi$ and $j_I^\alpha \phi$ and could be also interpreted as conserved currents associated with the critical deformations of the space-time surface.

This result would mean that the vision about the reduction to almost topological QFT [55] is realized. The number of the consistency conditions is four and the same as the number of field equations so that it might be possible to find solutions to the conditions by some kind of imbedding procedure. An enormous boost at conceptual level as well as the level of practical calculations would be implied since one would be able to directly calculate Kähler function and Kähler metric, and also to deduce directly the symmetries of the Kähler function. Chern-Simons action however allows non-trivial WCW Kähler metric in M^4 degrees of freedom only if one replaces J with $J + J_1$ in the Kähler action [8]. As found in [8], this has highly desirable consequences as far as the description of classical gravitation is considered.

4.2 Construct Kähler metric from symmetries

One can also try to construct the matrix elements of the metric and Kähler form using infinite-dimensional super-conformal symmetries [8]. It is enough to construct them at single point for given values of zero modes. The basic technical idea is to consider matrix elements (J_a, J_b) of the metric between isometry currents and formulate explicitly the conditions for invariance under infinitesimal symmetries. The simplest possible example in finite-dimensional context is the construction of the matrix elements of the Kähler form and metric between the Killing vector fields of rotations around 3-axes at -say- North Pole of S^2. The matrix elements of 3 Killing vector fields J_i, $i = 1, 2, 3$ define a diagonal matrix $(J_i, J_j) = P_{ij} = diag(1, 1, 0)$, where the third element is the rotation leaving poles invariant. In complexified basis J_3, J_+, J_- one obtains matrix elements of Kähler metric as Poisson brackets of corresponding complexified Hamiltonians.

In the recent case the natural physical counterpart for the North Pole is the space-time surface (and corresponding partonic 2-surface) corresponding to a maximum of Kähler function. In perturbative approach the construction of WCW metric only at this point might be all that is needed.

Second hypothesis is that WCW isometries which are symplectic transformations [32] are induced from the symplectic transformations of $\delta CD \times CP_2$ so that the matrix elements to be conside with be associated with the conformal variant of the symplectic algebra of $\delta CD \times CP_2$. Super-symmetry implies that the matrix element of the WCW Kähler form and metric correspond to anticommucators of the Clifford algebra elements defined as contractions of the symplectic Killing vector fields with gamma matrices [10]. The crucial observation is that the components of Kähler form between Killing vector fields correspond to the Poisson brackets of the corresponding Hamiltonians. From the fact, that the elements of Kähler metric in complexified basis differ from those of Kähler form only by a multiplication with imaginary unit one can conclude that this approach gives also the matrix elements of the metric.

4.3 Construct spinor structure first

The most practical approach is based on the idea of constructing the WCW spinor structure in terms of the modified Dirac action and identifying the exponent of Kähler function as Dirac determinant defined as the product of generalized eigen values of the Dirac operator in question [10, 11]. The key observation is that for the induced spinor structure the counterpart of ordinary Dirac equation is internally consistent only if the space-time surface is a minimal surface. This principle generalizes and one can assign to any general coordinate invariant action principle a unique Dirac operator, which I call modified Dirac operator. This unique correspondence between the dynamics in space-time and spinorial degrees of freedom has interpretation in terms of super-symmetry. In the case of Kähler action this operator is denoted by D_K. The general expression for the modified Dirac operator is

$$\begin{aligned} D_K &= \hat{\Gamma}^\alpha D_\alpha \ , \\ \hat{\Gamma}^\alpha &= \Pi^\alpha_k \Gamma^k \ , \ \Pi^\alpha_k = \frac{\partial L_K}{\partial(\partial_\alpha h^k)} \ . \end{aligned} \tag{4.2}$$

D_α denote covariant derivatives in induced spinor structure defined by the projections of imbedding space spinor connection and modified gamma matrices are contractions of canonical momentum currents Π^α_k with imbedding space gamma matrices. Modified gamma matrices define currents which must be conserved in order that modified Dirac equation for Ψ and its conjugate are equivalent and extremal property guarantees the conservation.

The skeptic reader can pose two questions.

1. Does the Dirac determinant really define something having interpretation as Kähler function? How this determinant is defined?

2. Does this Kähler function correspond to Kähler action plus real part of holomorphic function of WCW coordinates?

Despite the lack of clearcut answers to these questions the approach is very attractive since it gives a direct connection with the second quanization of induced spinor fields and leads to a detailed understanding of the realization of super-conformal symmetries in terms of fermionic oscillator operators. The extremal property of Kähler action emerges as a consistency condition and if one requires that there exists deformations of space-time surface acting as symmetries one finds that they correspond to critical deformations for which the second variation of Kähler action vanishes. Thus preferred extremals could be critical and therefore analogous to minima of a potential function when the rank of the determinant defined by the second derivatives of potential function is not maximal. Thom's catastrophe theory treats just this kind of situation. Criticality can interpreted as the space-time correlate of quantum criticality. The conjecture is that due to vacuum degeneracy there always exists infinite number of deformations for which the second variation vanishes and that corresponding super-conformal symmety algebras from infinite inclusion hierarchies identifiable in terms of inclusion hierarchies of hyper-finite factors of type II_1.

Quantum classical correspondence requires that one must add to the modified Dirac equation coupling to conserved charges- in particular isometry charges in order to obtain correlation of space-time geometry with quantum numbers. The condition that modified Dirac operator allows symmetries as deformations of the space-time surface requires that each deformation of space-time surface defining symmetry corresponds to a critical deformation of space-time surface. Criticality condition follows from the condition that symmetries exist.

IIf the exponent of Kähler action for the preferred extremal is equal to the Dirac determinant defined as product of generalized eigenvalues of 3-D Chern-Simons Dirac operator (this option turned out to be the only possible option) and effective 2-dimensionality holds true, there is no need to know the entire $X^4(X^3)$ to calculate the Kähler function. Only the data at partonic 2-surfaces is needed (4-D tangent space data is needed).

5 Physics as a generalized number theory

Physics as a generalized number theory program involves three threads: various p-adic physics and their fusion together with real number based physics to a larger structure [22], the attempt to understand basic physics in terms of classical number fields [23], and infinite primes [24] whose construction is formally analogous to a repeated second quantization of an arithmetic quantum field theory. A common denominator of these approaches is a precise mathematical formulation for the notion of finite measurement resolution, which could be taken as one of the basic guiding principles of quantum TGD and is at quantum level realized in terms of inclusions of hyper-finite factors about which configuration space spinor fields provide an example [17]. In the following these threads are described briefly. More detailed summaries will be given in separate articles.

5.1 p-Adic physics and unification of real and p-adic physics

p-Adic numbers [42] became a part of TGD through the successes of p-adic thermodynamics in the description of elementary particle massivations [6]. The p-adicization program attempts to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in an essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals.

5.1.1 Real and p-adic regions of the space-time as geometric correlates of matter and mind

One could end up with p-adic space-time sheets via field equations. The solutions of the equations determining space-time surfaces are restricted by the requirement that the coordinates are real. When this is not the case, one might apply instead of a real completion with some p-adic completion. It however seems that p-adicity is present at deeper level and automatically present via the generalization of the number concept obtained by fusing reals and p-adics along rationals and common algebraics.

p-Adic non-determinism due to the presence of non-constant functions with a vanishing derivative implies extreme flexibility and therefore suggests the identification of the p-adic regions as seats of cognitive representations. Unlike the completion of reals to complex numbers, the completions of p-adic numbers preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

5.1.2 The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

5.1.3 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts zero energy ontology [12, 13].

1. Zero energy ontology classically

In TGD inspired cosmology [21] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [20] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

2. Zero energy ontology at quantum level

Also the construction of S-matrix [13] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state have interpretation as M-matrix identifiable as a "complex square root" of density matrix expressible as a product of positive diagonal square root of the density matrix and of a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

The collection of M-matrices defines an orthonormal state basis for zero energy states and together they define unitary U-matrix charactering transition amplitudes between zero energy states. This matrix would not be however the counterpart of the usual S-matrix. Rather the unitary matrix phase of a given M-matrix would define the S-matrix measured in laboratory. U-matrix would also characterize the transitions between different number fields possible in the intersection of rel and p-adic worlds and having interpretation in terms of intention and cognition.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by the time scale of the causal diamond (CD) and the rational (perhaps integer) characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also. CDs are indeed important also in TGD inspired cosmology [21].

3. Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [17]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

4. The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [17]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

5. The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [13]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

5.1.4 What number theoretical universality might mean?

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough. It is however clear that number theoretical criticality could provide important insights to quantum TGD. p-Adic thermodynamics [6] is an excellent example of this. In consciousness theory the transitions transforming intentions to actions and actions to cognitions would be key applications. Needless to say, zero energy ontology is absolutely essential: otherwise this kind of transitions would not make sense.

5.1.5 p-Adicization by algebraic continuation

The basic challenges of the p-adicization program are following.

1. The first problem -the conceptual one- is the identification of preferred coordinates in which functions are algebraic and for which algebraic values of coordinates are in preferred position. This problem is encountered both at the level of space-time, imbedding space, and configuration space. Here the group theoretical considerations play decisive role and the selection of preferred coordinates relates closely to the selection of quantization axes. This selection has direct physical correlates at the level of imbedding space and the hierarchy of Planck constants has interpretation as a correlate for the selection of quantization axes [16].

   Algebraization does not necessarily mean discretization at space-time level: for instance, the coordinates characterizing partonic 2-surface can be algebraic so that algebraic point of the configuration space results and surface is not discretized. If this kind of function spaces are finite-dimensional, it is possible to fix X^2 completely data for a finite number of points only.

2. Local physics generalizes as such to p-adic context (field equations, etc...). The basic stumbling block of this program is integration already at space-time (Kähler action, flux Hamiltonians, etc..). The problem becomes really horrible looking at configuration space level (functional integral). Algebraic continuation could allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics. Also the existence of the Kähler geometry does this and the solution to the constraint is that WCW is a union of symmetric spaces.

   In the case of symmetric spaces Fourier analysis generalizes to harmonics analysis and one can reduce integration to summation for functions allowing Fourier decomposition. In p-adic context the existence of plane waves requires an algebraic extension allowing roots of unity characterizing the measurement accuracy for angle like variables. This leads in the case of symmetric spaces to a general p-adicization recipe. One starts from a discrete variant of the symmetric space for which points correspond to roots of unity and replaces each discrete point with is p-adic completion representing the p-adic variant of the symmetric space so that kind of fractal variant of the symmetric space is obtained. There is an infinite hierarchy of p-adicizations corresponding to measurement resolutions and to the choice of preferred coordinates and the interpretation is in terms of cognitive representations. This requires a refined view about General Coordinate Invariance taking into account the fact that cognition is also part of the quantum state.

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane.

1. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

2. For instance, residue calculus essential in the construction of N-point functions of conformal field theory might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole

common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

3. Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality.

4. For instance, the idea that number theoretically critical partonic 2-surfaces are expressible in terms of rational functions with rational or algebraic coefficients so that also p-adic variants of these surfaces make sense, is very attractive.

5. Finite sums and products respect algebraic number property and the condition of finiteness is coded naturally by the notion of finite measurement resolution in terms of the notion of (number theoretic) braid. This simplifies dramatically the algebraic continuation since configuration space reduces to a finite-dimensional space and the space of configuration space spinor fields reduces to finite-dimensional function space.

The real configuration space can well contain sectors for which p-adicization does not make sense. For instance, if the exponent of Kähler function and Kähler are not expressible in terms of algebraic functions with rational or at most algebraic functions or more general functions making sense p-adically, the continuation is not possible. p-Adic non-determinism in p-adic sectors makes also impossible the continuation to real sector. All this is consistent with vision about rational and algebraic physics as as analog of rational and algebraic numbers allowing completion to various continuous number fields.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. For instance, most points p-adic space-time sheets reside at infinity in real sense and p-adically infinitesimal is infinite in real sense. Two types of cutoffs are predictedp-adic length scale cutoff and a cutoff due to phase resolution related to the hierarchy of Planck constants. Zero energy ontology provides natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

5.2 TGD and classical number fields

This chapter is second one in a multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme is the role of classical number fields in quantum TGD. A central notion is M^8-H duality which might be also called number theoretic compactification. This duality allows to identify imbedding space equivalently either as M^8 or $M^4 \times CP_2$ and explains the symmetries of standard model number theoretically. These number theoretical symmetries induce also the symmetries dictaging the geometry of the "world of classical worlds" (WCW) as a union of symmetric spaces. This infinite-dimensional Kähler geometry is expected to be highly unique from the mere requirement of its existence requiring infinite-dimensional symmetries provided by the generalized conformal symmetries of the light-cone boundary $\delta M^4_+ \times S$ and of light-like 3-surfaces and the answer to the question what makes 8-D imbedding space and $S = CP_2$ so unique would be the reduction of these symmetries to number theory.

Zero energy ontology has become the corner stone of both quantum TGD and number theoretical vision. In zero energy ontology either light-like or space-like 3-surfaces can be identified as the fundamental dynamical objects, and the extension of general coordinate invariance leads to effective 2-dimensionality (strong form of holography) in the sense that the data associated with partonic 2-surfaces and the distribution of 4-D tangent spaces at them located at the light-like boundaries of causal diamonds (CDs) defined as intersections of future and past directed light-cones code for quantum physics and the geometry of WCW.

The basic number theoretical structures are complex numbers, quaternions and octonions, and their complexifications obtained by introducing additional commuting imaginary unit $\sqrt{-1}$. Hyper-octonionic (-quaternionic,-complex) sub-spaces for which octonionic imaginary units are multiplied by commuting $\sqrt{-1}$ have naturally Minkowskian signature of metric. The question is whether and how the hyper-structures could allow to understand quantum TGD in terms of classical number fields. The answer which looks the most convincing one relies on the existence of octonionic representation of 8-D gamma matrix algebra.

1. The first guess is that associativity condition for the sub-algebras of the local Clifford algebra defined in this manner could select 4-D surfaces as associative (hyper-quaternionic) sub-spaces of this algebra and define WCW purely number theoretically. The associative sub-spaces in question would be spanned by the modified gamma matrices defined by the modified Dirac action fixed by the variational principle (Kähler action) selecting space-time surfaces as preferred extremals [11].

2. This condition is quite not enough: one must strengthen it with the condition that a preferred commutative and thus hyper-complex sub-space is contained in the tangent space of the space-time surface. This condition actually generalizes somewhat since one can introduce a family of so called Hamilton-Jacobi coordinates for M^4 allowing an integrable distribution of decompositions of tangent space to the space of non-physical and physical polarizations [19]. The physical interpretation is as a number theoretic realization of gauge invariance selecting a preferred local commutative plane of non-physical polarizations.

3. Even this is not yet the whole story: one can define also the notions of co-associatitivy and co-commutativity applying in the regions of space-time surface with Euclidian signature of the induced metric. The basic unproven conjecture is that the decomposition of space-time surfaces to associative and co-associative regions containing preferred commutative *resp.* co-commutative 2-plane in the 4-D tangent plane is equivalent with the preferred extremal property of Kähler action and the hypothesis that space-time surface allows a slicing by string world sheets and by partonic 2-surfaces [11].

5.2.1 Hyper-octonions and hyper-quaternions

The discussions for years ago with Tony Smith [41] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space (for octonions see [40]. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity. The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

Note that hyper-variants of number fields make also sense p-adically unlike the notions of number fields themselves unless restricted to be algebraic extensions of rational variants of number fields. What deserves separate emphasis is that the basic structure of the standard model would reduce to number theory.

5.2.2 Number theoretical compactification and $M^8 - H$ duality

The notion of hyper-quaternionic and octonionic manifold makes sense but it not plausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Situation changes if H is replaced with hyper-octonionic M^8. Suppose that $X^4 \subset M^8$ consists of hyper-quaternionic and co-hyper-quaternionic regions. The basic observation is that the hyper-quaternionic sub-spaces of M^8 with a fixed hyper-complex structure (containing in their tangent space a fixed hyper-complex subspace M^2 or at least one of the light-like lines of M^2) are labeled by points of CP_2. Hence each hyper-quaternionic and co-hyper-quaternionic four-surface of M^8 defines a 4-surface of $M^4 \times CP_2$. One can loosely say that the number-theoretic analog of spontaneous compactification occurs: this of course has nothing to do with dynamics.

This picture was still too naive and it became clear that not all known extremals of Kähler action contain fixed $M^2 \subset M^4$ or light-like line of M^2 in their tangent space.

1. The first option represents the minimal form of number theoretical compactification. M^8 is interpreted as the tangent space of H. Only the 4-D tangent spaces of light-like 3-surfaces X_l^3 (wormhole throats or boundaries) are assumed to be hyper-quaternionic or co-hyper-quaternionic and contain fixed M^2 or its light-like line in their tangent space. Hyper-quaternionic regions would naturally correspond to space-time regions with Minkowskian signature of the induced metric and their co-counterparts to the regions for which the signature is Euclidian. What is of special importance is that this assumption solves the problem of identifying the boundary conditions fixing the preferred extremals of Kähler action since in the generic case the intersection of M^2 with the 3-D tangent space of X_l^3 is 1-dimensional. The surfaces $X^4(X_l^3) \subset M^8$ would be hyper-quaternionic or co-hyper-quaternionic but would not allow a local mapping between the 4-surfaces of M^8 and H.

2. One can also consider a more local map of $X^4(X_l^3) \subset H$ to $X^4(X_l^3) \subset M^8$. The idea is to allow $M^2 \subset M^4 \subset M^8$ to vary from point to point so that $S^2 = SO(3)/SO(2)$ characterizes the local choice of M^2 in the interior of X^4. This leads to a quite nice view about strong geometric form of $M^8 - H$ duality in which M^8 is interpreted as tangent space of H and $X^4(X_l^3) \subset M^8$ has interpretation as tangent for a curve defined by light-like 3-surfaces at X_l^3 and represented by $X^4(X_l^3) \subset H$. Space-time surfaces $X^4(X_l^3) \subset M^8$ consisting of hyper-quaternionic and co-hyper-quaternionic regions would naturally represent a preferred extremal of E^4 Kähler action. The value of the action would be same as CP_2 Kähler action. $M^8 - H$ duality would apply also at the induced spinor field and at the level of configuration space.

3. Strong form of $M^8 - H$ duality satisfies all the needed constraints if it represents Kähler isometry between $X^4(X_l^3) \subset M^8$ and $X^4(X_l^3) \subset H$. This implies that light-like 3-surface is mapped to light-like 3-surface and induced metrics and Kähler forms are identical so that also Kähler action and field equations are identical. The only differences appear at the level of induced spinor fields at the light-like boundaries since due to the fact that gauge potentials are not identical.

4. The map of $X_l^3 \subset H \rightarrow X_l^3 \subset M^8$ would be crucial for the realization of the number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as those preferred coordinates in which the points of imbedding space are rational/algebraic. Thus the point of $X^4 \subset H$ is algebraic if it is mapped to algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactication could thus be motivated by the number theoretical universality.

5. The possibility to use either M^8 or H picture might be extremely useful for calculational purposes. In particular, M^8 picture based on $SO(4)$ gluons rather than $SU(3)$ gluons could perturbative description of low energy hadron physics. The strong $SO(4)$ symmetry of low energy hadron physics can be indeed seen direct experimental support for the $M^8 - H$ duality.

5.3 Infinite primes

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors. Many interpretations for infinite primes have been competing for survival but it seems that the recent state of TGD allows to exclude some of them from consideration.

5.3.1 The notion of infinite prime

Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense in the sequence of quantum jumps: the reason is simply that the size of primes is bounded below. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct infinite primes by repeating a procedure analogous to a quantization of a super symmetric arithmetic quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

This and other observations suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [43] providing rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Somewhat surprisingly, infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively and this raises the question whether the tangent space for the configuration space of 3-surfaces could be regarded as the space of generalized 8-D hyper-octonionic numbers.

5.3.2 Infinite primes and physics in TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1. Infinite primes, cognition, and intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point -or more generally wave functions in the space of the units associated with the point- might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

3. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions

of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

2. Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. It turns out that this idea makes sense when one considers discrete wave functions in the space of infinite primes and that one can indeed represent standard model quantum numbers in this manner.

4. The number theoretical supersymmetry suggests also space-time supersymmetry TGD framework. Space-time super-symmetry in its standard form is not possible in TGD Universe and this cheated me to believe that this supersymmetry is completely absent in TGD Universe. The progress in the understanding of the properties of the modified Dirac action however led to a generalization of the space-time super-symmetry as a dynamical and broken symmetry of quantum TGD [14].

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [17] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [12].

3. Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

G_2 acts as automorphisms of hyper-octonions and $SU(3)$ as its subgroup respecting the choice of a preferred imaginary unit. The discrete subgroups of $SU(3)$ permuting to each other hyper-octonionic primes are analogous to Galois group and turned out to play a crucial role in the understanding of the correspondence between infinite hyper-octonionic primes and quantum states predicted by quantum TGD.

4. The notion of finite measurement resolution as the key concept

TGD predicts several hierarchies: the hierarchy of space-time sheets, the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution[17], the dark matter hierarchy characterized by increasing values of $\hbar$ [16], the hierarchy of extensions of a given p-adic number field. TGD inspired theory of consciousness predictes the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. These hierarchies should be closely related.

The notion of finite measurement resolution turns out to be the key concept: the p-adic norm of the rational defined by the infinite prime characterizes the angle measurement resolution for given p-adic prime p. It is essential that one has what might be called a state function reduction selecting a fixed p-adic prime which could be also infinite. This gives direct connections with cognition and with the p-adicization program relying also on angle measurement resolution. Also the value the integers characterizing the singular coverings of CD and CP_2 defining as their product Planck constant characterize the measurement resolution for a given p-adic prime in CD and CP_2 degrees of freedom. This conforms with the fact that elementary particles are characterized by two infinite primes. Hence finite measurement resolution ties tightly together the three threads of the number theoretic vision. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory so that the characterization of the finite measurement resolution, which has been the ugly ducling of theoretical physics, transforms to a beatiful swan.

5. Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. Note that this conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

The representation of space-time surfaces as algebraic surfaces in M^8 is however too naive idea and the attempt to map hyper-octonionic infinite primes to algebraic surfaces has not led to any concrete progress.

The solution came from quantum classical correspondence, which requires the map of the quantum numbers of configuration space spinor fields to space-time geometry. The modified Dirac equation with measurement interaction term realizes this requirement. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries.

5.3.3 Generalization of ordinary number fields: infinite primes and cognition

Both fermions and p-adic space-time sheets are identified as correlates of cognition in TGD Universe. The attempt to relate these two identifications leads to a rather concrete model for how bosonic generators of super-algebras correspond to either real or p-adic space-time sheets (actions and intentions) and fermionic generators to pairs of real space-time sheets and their p-adic variants obtained by algebraic continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads also to a generalization of classical number fields since an infinite algebra of real (complex, etc...) units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real and also more general units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is invisible and is relevant only for the physics of cognition. On the other hand, one can consider the possibility of mapping the configuration space and configuration space spinor fields to the number theoretical anatomies of a single point of imbedding space so that the structure of this point would code for the world of classical worlds and for the quantum states of the Universe. Quantum jumps would induce changes of configuration space spinor fields interpreted as wave functions in the set of number theoretical anatomies of single point of imbedding space in the ordinary sense of the word, and evolution would reduce to the evolution of the structure of a typical space-time point in the system. Physics would reduce to space-time level but in a generalized sense. Universe would be an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

Infinite rationals are in one-one correspondence with quantum states and in zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

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