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TETRON MODEL BUILDING

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Abstract

Spin models are considered on a discretized inner symmetry space with tetrahedral symmetry as possible dynamical schemes for the tetron model. Parity violation, which corresponds to a change of sign for odd permutations, is shown to dictate the form of the Hamiltonian. It is further argued that such spin models can be obtained from more fundamental principles by considering a (6+1)- or (7+1)-dimensional spacetime with octonion multiplication.

1 Introduction

Particle physics phenomena can be described, for example, by the left-right symmetric Standard Model with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ [1] and 24 left-handed and 24 right-handed fermion fields which including antiparticles amounts to 96 degrees of freedom, i.e. this model has right handed neutrinos as well as righthanded weak interactions.

In recent papers [2, 3, 4] it was shown that there is a natural one-to-one correspondence between the quarks and leptons and the elements of the permutation group S_4, as made explicit in table 1 and natural in the sense that the color, isospin and family structure correspond to the K, Z_2 and Z_3 subgroups of S_4, [1] where Z_n is the cyclic group of n elements and K is the so-called Kleinsche Vierergruppe which consists of the 3 even permutations $\overline{2143}$, $\overline{3412}$, $\overline{4321}$, where 2 pairs of numbers are interchanged, plus the identity.

In other words, S_4 is a semi-direct product $S_4 = K \diamond Z_3 \diamond Z_2$ where the Z_3 factor is the family symmetry and Z_2 and K can be considered to be the 'germs' of weak isospin and color symmetry (cf. [3]). Furthermore it does not only describe quarks and leptons (table 1) but also leads to a new ordering scheme for the Standard Model gauge bosons, cf. ref. [2].

In refs. [2, 3, 4] a constituent picture was suggested where quarks and leptons are assumed to be built from 4 tetron 'flavors' a, b, c and d, whose interchanges generate the permutation group S_4. In the present paper I follow a somewhat different approach which relies on the fact that S_4 is also the symmetry group of a tetrahedral lattice or of a fluctuating S_4-permutation (quantum) lattice. In this approach the inner symmetry space is not continuous (with a continuous symmetry group) but has instead the discrete structure of a tetrahedral or S_4-permutation lattice, and the original dynamics is governed by some unknown lattice interaction instead of by four real tetron constituents.

The observed quarks and leptons can then be interpreted as excitations on

[1] S_4 is isomorphic to the rotational symmetry group of a regular tetrahedron and, up to a parity factor, the symmetry group of a 3-dimensional cube and octahedron.

	...1234... family 1	...1423... family 2	...1243... family 3
	τ, $b_{1,2,3}$	μ, $s_{1,2,3}$	e, $d_{1,2,3}$
ν	$\overline{1234}(id)$	$\overline{2314}$	$\overline{3124}$
u_1	$\overline{2143}(k_1)$	$\overline{3241}$	$\overline{1342}$
u_2	$\overline{3412}(k_2)$	$\overline{1423}$	$\overline{2431}$
u_3	$\overline{4321}(k_3)$	$\overline{4132}$	$\overline{4213}$
	ν_τ, $t_{1,2,3}$	ν_μ, $c_{1,2,3}$	ν_e, $u_{1,2,3}$
l	$\overline{3214}(1 \leftrightarrow 3)$	$\overline{1324}(2 \leftrightarrow 3)$	$\overline{2134}(1 \leftrightarrow 2)$
d_1	$\overline{2341}$	$\overline{3142}$	$\overline{1243}(3 \leftrightarrow 4)$
d_2	$\overline{1432}(2 \leftrightarrow 4)$	$\overline{2413}$	$\overline{3421}$
d_3	$\overline{4123}$	$\overline{4231}(1 \leftrightarrow 4)$	$\overline{4312}$

Table 1: List of elements of S_4 ordered in 3 fermion families. k_i denote the elements of K and $(a \leftrightarrow b)$ a simple permutation where a and b are interchanged. Permutations with a 4 at the last position form a S_3 subgroup of S_4 and may be thought of giving the set of lepton states. It should be noted that this is only a heuristic assignment. Actually one has to consider linear combinations of permutation states as discussed in section 2.

this lattice and characterized by representations of the lattice symmetry group S_4, i.e. by $A_1 + A_2 + 2E + 3T_1 + 3T_2$ or $2G_1 + 2G_2 + 4H$, just as in the 'classical' tetron model [2, 3, 4], and the original dynamics is governed by some unknown lattice interaction instead of by four real tetron constituents.

The lattice ansatz also naturally explains the selection rule mentioned in ref. [2] that all physical states must be permutation states: just because the lattice excitations must transform under representations of S_4.

In the following I will make the additional assumption that not only the inner symmetry is discrete but that physical space is a lattice, too. The main reason for this assumption is that although theories with a discrete inner symmetry over a continuous base manifold have been examined [9] they seem to me rather artificial because they usually lead to domain walls and other discontinuities. In addition, this line of thought takes up an old dream that field theoretical UV-infinities and renormalization problems can eventually be avoided by considering a fundamental theory living on a discretized instead of a continuous spacetime, with the average lattice spacing typically of the order of the Planck scale.

To distinguish the inner S_4-symmetry from the symmetries of the spatial lattice I will denote it by S_4^{in} in the following.

Quantum theory dictates that there is an uncertainty in the position of the lattice points. Therefore instead of a fixed spatial lattice one should allow the lattice points to fluctuate, with the fluctuations following some (quantum) stochastic process [10]. Working in a semiclassical approximation one may neglect these fluctuations to first order and consider a fixed lattice with tetrahedral symmetry.

There is some relation of this idea to other models which involve a fundamental length scale, like quantum foam models, which however assume gravity to play the central role in producing the new length scale, while in the tetron model gravitational interactions and cosmological phenomena appear only as byproducts of the spin lattice interactions.

In the present paper, dynamical models based on such lattices will be considered. They are typically spin models or fermionic lattice models. Variants of such models will be presented in the next sections: in section 2 a simple spin model on a 3-dimensional lattice will be discussed which is assigned to describe the tetron phenomenology. Such a naive approach, however, may not lead to a well described continuous field theory like the Standard Model. Therefore in section 3 I start with continuous spin vectors on a face centered cubic (fcc) lattice and follow the idea that the S_4 inner symmetry is generated by the interplay within the two fcc sublattices. A more fundamental alternative is presented in section 4 by going to higher dimensions, i.e. assuming that the spatial and inner symmetry lattices can be unified to a higher dimensional lattice. One intriguing possibility is a 7-dimensional spatial lattice involving octonions.

2 Single-S Model with a discrete inner symmetry space

Starting with a spatial lattice, the most straightforward idea is to consider spin models. Spin models have been considered in statistical and solid state physics for a long time, and they have been used to describe magnetism and magnetic excitations as well as many other phenomena.

Spin models work as follows: since one cannot put a fermion field on the lattice, because this leads to non-local interactions, one considers 'spin vectors' $\vec{S}_i$ sitting on each lattice site i together with a Hamiltonian which in simple cases is just given by

$$H = g_{SS} \sum_{i,j} \vec{S}_i \vec{S}_j \tag{1}$$

where g_{SS} is the coupling strength and the sum runs over all neigbouring lattice sites i and j. One may distinguish the Ising model and its generalizations where the spin vector can take values only in a discrete set from the Heisenberg model which works with a continuous spin vector.

Fermionic excitations can arise in spin models [11] when one decomposes the spin vectors in more fundamental fermion degrees of freedom. In fact one may write

$$\vec{S} = f^\dagger \vec{\sigma} f \tag{2}$$

where f is a Pauli spinor and $\vec{\sigma}$ is the triplet of Pauli matrices. [2]

In the present case the main challenge is to obtain the excitation spectrum of the discrete inner S_4^{in} symmetry group. To this end we start with a 3-dimensional spatial lattice as discussed at the end of the last section. In order to obtain the tetron spectrum, the most straightforward approach (followed in this section) is to assume that the spin vectors take values in an at least 3-dimensional inner symmetry lattice with symmetry group S_4^{in}. The dimension d_{in} of the inner symmetry lattice is restricted to be ≥ 3 because it is required to have a tetrahedral symmetry. The most straightforward choice is $d_{in} = 3$ but we shall also consider an example with $d_{in} = 4$.

What kind of Hamiltonian to choose? This is a very delicate question because this Hamiltonian eventually has to generate the full Standard Model phenomenology.

One essential requirement is that parity violation of the weak interactions should be described correctly. Usually a natural explanation of weak parity violation in subquark models is a real challenge. In the framework of the tetron idea the situation is somewhat simpler. The point is that in the tetron framework, as can be seen in table 1, weak isospin transformations are related to odd permutations. Therefore the Hamiltonian should transform non-trivially (i.e. antisymmetric) under odd permutations (of inner symmetry points as well the base points). Furthermore, odd permutations of the inner symmetry group should be related to helicity/parity transformations in the sense that left handed transitions are energetically favoured as compared to righthanded ones.

A simple Heisenberg like spin model Hamiltonian eq. (1) fails to fulfil this requirement. However, there are two reasonable alternatives:

[2] For the moment I work in the nonrelativistic limit and postpone the discussion of the relativistic case and antiparticles to the last section.

- on an inner 3-dimensional tetrahedral lattice there is the triple product

$$H = g_{SSS} \sum_t (\vec{S}_4 - \vec{S}_1)[(\vec{S}_3 - \vec{S}_1) \times (\vec{S}_2 - \vec{S}_1)] \tag{3}$$

  where g_{SSS} is the coupling strength and the sum runs over all tetrahedral plaquettes $t = 1, 2, 3, 4$ of the lattice and $\vec{S}_i$ is the value of spin vector on site i.

- on an inner 4-dimensional tetrahedral lattice there is the antisymmetric combination

$$H = g_{SSSS} \sum_t \epsilon_{abcd} S_1^a S_2^b S_3^c S_4^d \tag{4}$$

  where S_i^a are the four components of the vector $\vec{S}_i$ sitting on site i.

In both cases the interaction is antisymmetric under odd permutations in the base lattice as well as in the inner symmetry lattice. This is precisely what is needed to describe the parity violation in the tetron view on particle physics. The point is that all suggested Hamiltonians give a negative energy contribution to the partition function, if an odd transformation is applied to the points of the base tetrahedron. This odd permutation is a reflection (proper rotation times parity transformation) both in the inner and in the spatial lattice. A combination of an odd and an even fermion corresponds to a parity violating weak vector boson (even×odd=odd).

There are several disadvantages of the approach presented in this section. First of all in terms of the fundamental fermion f (eq. (2)), eqs. (3) and (4) correspond to effective 6- or 8-fermion interactions, which does not look very fundamental. Secondly, it is neither straigtforward nor natural to obtain a field theory for fermions from discrete (Ising-like) inner symmetry spaces. In the next two sections I will consider alternatives which overcome this dificulty.

3 Tetrons on a FCC Lattice

Here I want to suggest a spin model with continuous spin vectors which works in 3 dimensions and leads to an S_4^{in} spectrum of excitations.

I will assume that space has the structure of an fcc cubic lattice (like the NaCl crystal), or a fluctuating quantum version thereof, i.e. it consists of 2 tetrahedral sublattices with 2 types of continuous inner spin vectors $\vec{S}$ and $\vec{T}$ sitting on the Na and Cl sites of the crystal.

The lattice allows for ordinary spatial symmetry transformations, but in addition there is a discrete inner S_4^{in} symmetry arising from relative rotations of the S and T sublattices.

Such a model is more likely to produce excitations effectively described by field theories than the models with a discrete inner symmetry lattice considered in the last section.

One can also consider it as being inspired or even induced by the 7 dimensional models presented below, because, as we shall see, it relates the inner S_4 tetron symmetry to spatial transformations.

A Heisenberg-like Hamiltonian for such a system is given by

$$H = g_{SS} \sum_{i,j} \vec{S}_i \vec{S}_j + g_{TT} \sum_{i,j} \vec{T}_i \vec{T}_j + g_{ST} \sum_{i,j} \vec{S}_i \vec{T}_j \tag{5}$$

where in the first 2 terms the sums run over 12 face diagonal next neighbours and in the last term it runs over 8 body diagonal next neighbours.

As stated, quark and lepton degrees of freedom arise from the relative rotations of the S and T sublattices, and in particular the appearance of weak isospin is related to odd permutations in this inner S_4 symmetry. [3] In contrast, spatial parity transformations in this lattice correspond to the exchange of $\vec{S}$ and $\vec{T}$ sublattices. Therefore the model will show parity violation as soon as the couplings g_{SS} and g_{TT} are different.

Similar to eq. (2) the spin vectors $\vec{S}$ and $\vec{T}$ may be decomposed in 2 fundamental fermions f and g: $\vec{S} = f^\dagger \vec{\sigma} f$ and $\vec{T} = g^\dagger \vec{\sigma} g$.

[3] More precisely, and if one includes antiparticles in the discussion, fermions and antifermions sit on the S and T sublattices, respectively. When an odd fermion and an even antifermion (each with inner symmetry S_4^{in}) approach each other to interact, they form a parity violating state. For a more extensive discussion of antiparticle, see the next section.

4 Higher dimensional Models

In the foregoing sections I have tried to set up spin models in 3 spatial dimensions in an attempt to obtain the phenomenology of the tetron model. In this section we follow a more universal idea, namely, that spatial and inner symmetry lattices can be united to one big lattice with a universal lattice constant in some n-dimensional space $\mathbb{R}^n$.

In contrast to ref. [12] I will ignore gravity and consider all spaces to be flat. Furthermore, at this point I still restrict to the nonrelativistic limit, i.e. work without antiparticles. Antiparticles will be included later by going from $\mathbb{R}^n$ to $\mathbb{R}^{(n,1)}$.

It is an old dream that inner symmetries may be obtained by extending ordinary 3-dimensional space to higher dimensions, and in particular to 7 dimensions, because a division algebra with a corresponding spinor structure can be defined there, namely the nonassociative algebra of octonions [5, 6, 7, 8]. [4] It is related to SO(7) just as the algebra of quaternions is related to SO(3). More precisely, the group of unit octonions can be identified with the covering group Spin(7) of SO(7) just as the group of unit quaternions defines the covering group SU(2) of SO(3).

Considering $\mathbb{R}^7$ one is automatically led to SO(4) as inner symmetry group. In fact, any evidence for an SO(4) inner symmetry particle spectrum would long since have lead to physical models based on octonions with a 'compactification' $SO(7,1) \rightarrow SO(4) \times SO(3,1)$ to a (3+1)-dimensional spacetime $\mathbb{R}_{sp}^{(3,1)}$ corresponding to a trivial fibration $\mathbb{R}^{(7,1)} \rightarrow \mathbb{R}_{in}^4 \oplus \mathbb{R}_{sp}^{(3,1)}$ with fibers $\mathbb{R}_{in}^4$ (where *sp* and *in* stands for spatial and inner, respectively).

In the tetron model we do not consider continuous SO(4) but the discrete inner symmetry group S_4. This may be considered a subgroup of SO(4), and this fact will now be used to suggest a tetron model based on a discretized 7-dimensional space, i.e. a lattice in $\mathbb{R}^7$. Concerning the 'compactification' of such a lattice we encounter a situation which is depicted graphically in

[4] While in continuous field theories there are many restrictions for such an approach (from the Coleman to the Weinberg-Witten theorem), this is not the case for the discrete spin models considered here.

figure 1 for the corresponding fibration $\mathbb{R}^3 \to \mathbb{R}^2_{in} \oplus \mathbb{R}^1_{sp}$ of a 3-dimensional space.

To be concrete, we consider a 7-dimensional lattice with S_8 rotational symmetry. Such a lattice is spanned by the 7-simplex in $\mathbb{R}^7$ just as a 3-dimensional lattice with S_4 tetrahedral symmetry is spanned by the 3-simplex (=tetrahedron) in $\mathbb{R}^3$ (fig. 1).

Next, the S_8 lattice is assumed to 'compactify' by some unknown mechanism into two $\mathbb{R}^3$ lattices, one inner symmetry part and one one spatial part with symmetry S_4^{in} and S_4^{sp}, respectively. These lattices can be explicitly constructed in the following way: we ignore quantum fluctuations of lattice points and make the semiclassical approximation of a fixed 7-dimensional lattice with a symmetry group which contains $S_4^{in} \times S_4^{sp}$. It is straightforward to define this lattice as the span of the regular 7-simplex in $\mathbb{R}^7$, i.e. to have it spanned by 8 linear independent unit vectors P_{1-8} regularly distributed on the 6-sphere in $\mathbb{R}^7$. The first four of these points P_{1-4} are assumed to span a regular tetrahedral lattice with symmetry group S_4^{sp} in what is assumed to be ordinary space. They can therefore be given in terms of quaternions I, J and $K = IJ$.

$$
\begin{aligned}
P_1 &= (-1,-1,-1) = -I - J - K \\
P_2 &= (-1,+1,+1) = -I + J + K \\
P_3 &= (+1,-1,+1) = I - J + K \\
P_4 &= (+1,+1,-1) = I + J - K
\end{aligned}
\tag{6}
$$

where as usual in quaternion constructions I, J and K are taken to form an orthogonal basis of $\mathbb{R}^3$ (just as the octionion basis I,J,K,L,IL,JL and KL will be used form an orthogonal basis of $\mathbb{R}^7$). The rest of the points P_{5-8} span another tetrahedral lattice with symmetry S_4^{in} in another $\mathbb{R}^3$ within $\mathbb{R}^7$, which forms a tower of tetrahedral lattices in the 7th dimension. The situation is depicted graphically in figure 1 where instead of the reduction $\mathbb{R}^7 \to \mathbb{R}^3_{in} \oplus \mathbb{R}^3_{sp}$ I have drawn $\mathbb{R}^3 \to \mathbb{R}^1_{in} \oplus \mathbb{R}^1_{sp}$ and instead of symmetry group $S_4^{in} \times S_4^{sp}$ one has $S_2^{in} \times S_2^{sp}$ (the reflections of a line at the origin).

Having set the geometrical framework one can now fix the physical objects to be spin vectors in 7 dimensions. These can be decomposed as

$$\vec{S} = F^{\dagger}\vec{e}F \tag{7}$$

where F are spinors under SO(7) and $\vec{e} = (e_1, ..., e_7)$ are the generalizations of the Pauli matrices in 7 dimensions, i.e. they span the space of the 7-dimensional representation $\underline{\mathbf{7}}$ of SO(7) just as the ordinary Pauli matrices span the 3-dimensional representation of SO(3). Furthermore, the e_i, i.e. the $\underline{\mathbf{7}}$ of SO(7) can be constructed can be obtained from the spinor representation $\underline{8}$ of SO(7) in the same way as the $\underline{\mathbf{3}}$ of SO(3) can be obtained from $\underline{\mathbf{2}} \otimes \underline{\mathbf{2}} = \underline{\mathbf{3}} + \underline{\mathbf{1}}$, namely

$$\underline{\mathbf{8}} \otimes \underline{\mathbf{8}} = \underline{\mathbf{1}} + \underline{\mathbf{7}} + \underline{\mathbf{21}} + \underline{\mathbf{35}} \tag{8}$$

The e_i are closely related to the octonion algebra, just as the Pauli matrices are to the quaternion. While the Pauli matrices can be identified more or less directly with the quaternion units, the situation in 7 dimensions is somewhat more subtle because the octonion algebra is not associative, i.e. cannot be represented by the matrices e_i. An explicit representation of the marices e_i can be found, for example, in the book by Dixon [8].

From the tetron point of view the big advantage to consider spin models on higher dimensional lattices is that one can have *continuous* spin vectors and at the same time an energy spectrum of the *discrete* S_4 inner symmetry. The point is that half of the spatial lattice serves as inner S_4^{in} symmetry while the field values can remain continuous.

The totality of excitations on the $S_4^{in} \times S_4^{sp}$ lattice can be classified according to [12]

$$(A_1 + A_2 + 2E + 3T_1 + 3T_2)^{in} \otimes (G_1 + G_2 + 2H)^{sp} \tag{9}$$

where the first factor contains the 24 inner symmetry d.o.f of quarks and leptons, and G_1^{sp} describes their (spin 1/2) spatial transformation behavior.

If one takes serious the assumption that $S_4^{in} \times S_4^{sp}$ originally stems from a higher (S_8) symmetry, eq. (9) should better be replaced by

$$(G_1 + G_2 + 2H)_{in} \otimes (G_1 + G_2 + 2H)_{sp} \tag{10}$$

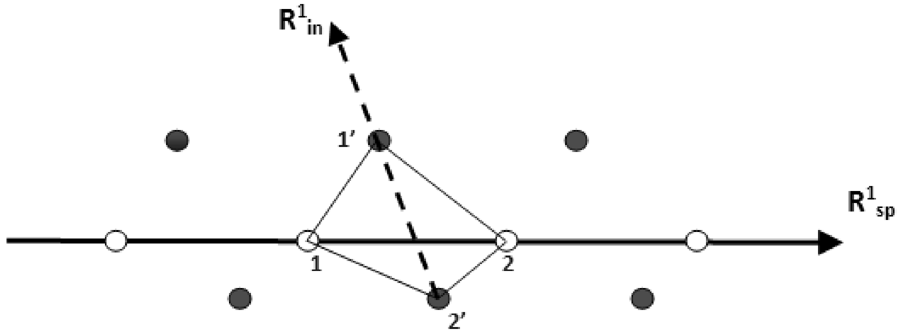

Figure 1: One dimensional spin chain from a three dimensional tetrahedral lattice in $\mathbb{R}^3 \to \mathbb{R}^1_{in} \oplus \mathbb{R}^1_{sp}$ as visualization for $\mathbb{R}^7 \to \mathbb{R}^3_{in} \oplus \mathbb{R}^3_{sp}$. The depicted lattice is assumed to possess a tetrahedral (S_4) symmetry with $S_4 \to S_2^{in} \times S_2^{sp}$. Transitions $1 \leftrightarrow 2$ correspond to spatial S_2 transformations, transitions $1' \leftrightarrow 2'$ to inner S_2 transformations.

i.e. one should work with projective representations of the tetrahedral group both in the inner symmetry and in the spatial sector, simply because there are no mixed bosonic and fermionic S_8 representations.

5 Conclusions

In conclusion, in the present paper lattice spin models have been discussed as possible dynamical schemes for the implementation of the tetron idea. Possible Hamiltonians have been presented, while the calculation of partition functions and expectation values for the excitation states is postponed to a forthcoming publication.

It is probable that the presented models are only effective descriptions of a more fundamental theory yet to be developed. However, presently there is no indication that this fundamental theory will have anything to do with the nowadays popular string or brane like structures. On the contrary, the appearance of S_4 symmetric states points to discrete structures at small distances and that the superstring ansatz is not opportune to describe natural phenomena.

The tetron idea is not only in opposition to string theories. It also reduces the celebrated gauge theories and the $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ Standard model gauge symmetry to what they are: a nice and logical theoretical framework which however holds true only on a certain level of matter (the TeV energy range). Tetrons are an idea that go beyond this level (just as quarks go beyond nuclear physics) and also offer explanations for outstanding cosmological problems [12]. Why did I suggest several dynamical schemes? Because it is difficult und ambiguous to formulate dynamics for tetrons, because one does not have many informations apart from the low energy behavior (the Standard Model).

In fact, such a situation is not unusual in the development of science. It is well known from the macroscopic world as well as from nano physics, molecular and atomic physics that when going to a lower level of matter one has to give up the full understanding of some emergent phenomena known from the higher levels. In the present case we have given up continuus Lorentz invariance (which is restaurated at low energies) in favor of a fluctuating quantum lattice picture. Furthermore we consider the appearance of gauge symmetries as collective emergent effects.

It is possible that eventually the underlying structure (e.g. involving octonions as in section 3) turns out to be in some sense supersymmetric. However at the present stage I consider this option far from being compelling.

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