Proof of the 3n+1 problem for n $\geq$ 1

by

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Abstract

I establish the existence of a unique binary pattern inherent to the 3n+1 step, and then use this binary pattern to prove the 3n+1 problem for all positive integers.

Introduction

Observe that every positive odd integer n, defined as $n = \sum_{i=0}^{x} 4^i$, $x \in \mathbf{Z}^+$ requires one 3n+1 step and then 2(x+1) consecutive n/2 steps to be reduced to 1.

The truth of this statement will become apparent when the 3n+1 step with such an integer is observed in base 2.

Example 1: Let $n = \sum_{i=0}^{2} 4^i = 21 = 10101_2$, then

$$10101_2 * 10_2 \Rightarrow 101010_2 + 10101_2 \Rightarrow 111111_2 + 1_2 = 1000000_2 = 2^6, \text{ and}$$

$$1000000_2/10_2 \Rightarrow 100000_2/10_2 \Rightarrow 10000_2/10_2 \Rightarrow 1000_2/10_2 \Rightarrow 100_2/10_2 \Rightarrow 10_2/10_2 = 1.$$

Therefore, the base 2 representation of positive integers furnishes more insight into the 3n+1 problem than their base 10 representation.

Proof

Let $\mathbf{O}^+$ be the set of positive odd integers, then

$$\mathbf{O}^+ = \{x = \mathbf{Z} \mid x=2y+1,\ y \geq 0,\ y \in \mathbf{Z}\}.$$

Theorem 1:	Let **P** designate the 3n+1 problem. Then if **P** is true for all positive odd integers, it is true for all positive integers. $\forall a \in \mathbf{O}^+ : \mathbf{P}(a) \Rightarrow \forall b \in \mathbf{Z}^+ : \mathbf{P}(b)$

Proof: Case 1: Power of two

Let $n = 2^x$, $x \in \mathbf{Z}^+$. Then n requires x consecutive n/2 steps to be reduced to 1.

Case 2: Odd integer multiplied by a power of two

Let $y = 2^x n$, $n \in \mathbf{O}^+$ and $x \in \mathbf{Z}^+$. Then x consecutive n/2 steps are required to have y = n.

Since these cases are exhaustive, it shows that if the 3n+1 problem is true for all $a \in \mathbf{O}^+$ it hast to be true for all $b \in \mathbf{Z}^+$.

The iteration between the 3n+1 step and the n/2 step modifies every integer n , $n \in \mathbf{O}^+$ in such a way that, at some point the integer becomes 2^x, $x \in \mathbf{Z}^+$.

However, the process of this transformation is obscured by the n/2 step. In order to make the process apparent, the n/2 step is omitted and the addition of 1 in the 3n+1 step is modified to compensate for the omission of the n/2 step.

Example 2: Let $n = 9 = 1001_2$, then $3n+2^x$ produces this pattern:

$$\begin{aligned}
1001_2 * 11_2 &\Rightarrow 11011_2 + 1_2 = 11100_2 \\
11100_2 * 11_2 &\Rightarrow 1010100_2 + 100_2 = 1011000_2 \\
1011000_2 * 11_2 &\Rightarrow 100001000_2 + 1000_2 = 100010000_2 \\
100010000_2 * 11_2 &\Rightarrow 1100110000_2 + 10000_2 = 1101000000_2 \\
1101000000_2 * 11_2 &\Rightarrow 100111000000_2 + 1000000_2 = 101000000000_2 \\
101000000000_2 * 11_2 &\Rightarrow 1111000000000_2 + 1000000000_2 = 10000000000000_2 = 2^{13}.
\end{aligned}$$

In example 2, the least significant bit transcends the most significant bit after six $3n+2^x$ steps, transforming n into a power of two.

<u>Definition 1:</u> Let LSB be the least significant bit of $s \in \mathbf{Z}^+$, then

$$LSB = \{2^r, r \geq 0, r \in \mathbf{Z} \mid 2^r = s/t, t \in \mathbf{O}^+\}.$$

<u>Theorem 2:</u> The 3n+LSB step and the 3n+1 step are isomorphic.

Proof: Suppose $n_0 \in \mathbf{O}^+$. Let $n_1 = 3n_0+1$ and $n_2 = n_1/LSB$, then

$$\frac{3n_1+LSB}{3n_2+1} = \frac{3n_1+LSB}{3\left(\frac{n_1}{LSB}\right)+1} = \frac{3n_1+LSB}{\frac{3n_1+LSB}{LSB}} = LSB.$$

$$\therefore\ 3n+LSB \equiv 0 \pmod{3n+1}.$$

Because a modular congruence exists between the 3n+LSB step and the 3n+1 step, they are therefore isomorphic.

The pattern in example 2 is composed of two functions. The first function increases the most significant power of two or most significant bit of n, and the second function increases the least significant power of two or least significant bit of n.

Let m(x) be the function for repeated multiplication of n by 3 in terms of x, $x \in \mathbf{Z}^+$. Then $m(x) = 3^{x+\delta}n$.

Let lsb(x) be the function for repeated multiplication by 4 (3(LSB)+LSB) of the least significant bit of n in terms of x, $x \in \mathbf{Z}^+$. Then $lsb(x) = 4^{x+\delta}$.

<u>Definition 2:</u> Let f(x) be the function for the 3n+LSB step for $n \in \mathbf{O}^+$ in terms of x, $x \in \mathbf{Z}^+$. Then

$$f(x) = m(x) + lsb(x) = 3^{x+\delta}n + 4^{x+\delta}.$$

Suppose that Tlsb(x) is the function that gives the true position of the least significant bit of the 3n+LSB step for $n \in \mathbf{O}^{+}$ in terms of x, $x \in \mathbf{Z}^{+}$. Then

$$\delta = \sum_{1}^{x} \mathrm{Tlsb}(x) - \mathrm{lsb}(x).$$

Example 3: Tlsb(x) > lsb(x)

Assume that multiplying n_k by 3 produces $\cdots 001111100 \cdots$ somewhere in the binary representation of the result; and that the rightmost 1 is $\mathrm{LSB} = 2^{x}$. Let lsb(x) = Tlsb(x). Adding LSB to n_k yields $\cdots 010000000 \cdots$, then

$$\delta = \sum_{x}^{x} \mathrm{Tlsb}(x) - \mathrm{lsb}(x) = \sum_{x}^{x} 2^{x+5} - 2^{x+2} = \sum_{x}^{x} x+5-x-2 = \sum_{x}^{x} 3 = 3.$$

Example 4: Tlsb(x) < lsb(x)

Assume that multiplying n_k by 3 and adding LSB produces $\cdots 001111100 \cdots$ somewhere in the binary representation of the result; and that the rightmost 1 is $\mathrm{LSB} = 2^{x}$. Let lsb(x) = Tlsb(x). Then repeated multiplication by 3 and addition of LSB will produce this pattern:

$\cdots 001111100 \cdots$ times 3 plus 2^{x}
$\cdots 101111000 \cdots$ times 3 plus 2^{x+1}
$\cdots 001110000 \cdots$ times 3 plus 2^{x+2}
$\cdots 101100000 \cdots$ times 3 plus 2^{x+3}
$\cdots 001000000 \cdots$, then

$$\delta = \sum_{x}^{x+3} \mathrm{Tlsb}(x) - \mathrm{lsb}(x) = \sum_{x}^{x+3} 2^{x+1} - 2^{x+2} = \sum_{x}^{x+3} x+1-x-2 = \sum_{x}^{x+3} -1 = -4.$$

$$\therefore \quad (\delta < 0) \ \vee \ (\delta = 0) \ \vee \ (\delta > 0)$$

Assume $x \in \mathbf{Z}^{+}$, then m(x) < lsb(x) implies that a single power of two is larger than a sum of powers of two.

Using example 2 as an illustration:

$m(x) - lsb(x) = 9\left(3^{x+2}\right) - 4^{x+2} = 0$ for $x \approx 5.6377$.

The integer after the root necessitates that $m(x) < lsb(x)$. In other words, it requires six 3n+LSB steps for 9 to converge to 2^{13}.

Theorem 3:	For all positive odd integers n, there exists a positive integer x such that $m(x) < lsb(x)$. $\forall n\,(n \in \mathbf{O^+})\,\exists x \in \mathbf{Z^+}\,(m(x) < lsb(x))$

Proof: Case 1:

$\delta \le -1,\ \delta \in \mathbf{Z}$

Assume $n \in \mathbf{O^+}$ and let $m(x) - lsb(x) = 3^{x-\delta}n - 4^{x-\delta} = 0$.

Then $x = \dfrac{\log(1/n)}{\log(3/4)} + \delta$.

$$\therefore\ \exists! x \in \mathbf{R^+}\,(3^{x-\delta}n - 4^{x-\delta} = 0) \ \Rightarrow\ \exists x \in \mathbf{Z^+}\,(m(x) < lsb(x))$$

Case 2:

$\delta = 0$

Assume $n \in \mathbf{O^+}$ and let $m(x) - lsb(x) = 3^{x}n - 4^{x} = 0$.

Then $x = \dfrac{\log(1/n)}{\log(3/4)}$.

$$\therefore\ \exists! x \in \mathbf{R^+}\,(3^{x}n - 4^{x} = 0) \ \Rightarrow\ \exists x \in \mathbf{Z^+}\,(m(x) < lsb(x))$$

Case 3:

$\delta \ge 1,\ \delta \in \mathbf{Z}$

Assume $n \in \mathbf{O^+}$ and let $m(x) - lsb(x) = 3^{x+\delta}n - 4^{x+\delta} = 0$.

Then $x = \dfrac{\log(1/n)}{\log(3/4)} - \delta$.

$$\therefore\ \exists! x \in \mathbf{R^+}\,(3^{x+\delta}n - 4^{x+\delta}) \ \Rightarrow\ \exists x \in \mathbf{Z^+}\,(m(x) < lsb(x))$$

Because these cases are exhaustive, it shows that

$$\forall n\,(n \in \mathbf{O^+})\,\exists x \in \mathbf{Z^+}\,(m(x) < lsb(x)).$$

For all $n \in \mathbf{O}^+$ there exists an $x \in \mathbf{Z}^+$ such that $m(x) < \mathrm{lsb}(x)$ (Theorem 3), therefore $f(x)$ converges to 2^y, $y \in \mathbf{Z}^+$. And since the 3n+LSB step and the 3n+1 step are isomorphic (Theorem 2), it can be concluded that if $a_0 = n$, $n \in \mathbf{O}^+$, then

$$a_{i+1} = \begin{cases} a_i/2 & \text{for even } a_i \\ 3a_i+1 & \text{for odd } a_i \end{cases}, \text{ converges to 1.}$$

Because the 3n+1 problem is true for all positive odd integers, then by Theorem 1 the truth extends to all positive integers. Therefore, if $a_0 = n$, $n \in \mathbf{Z}^+$, then

$$a_{i+1} = \begin{cases} a_i/2 & \text{for even } a_i \\ 3a_i+1 & \text{for odd } a_i \end{cases}, \text{ converges to 1.}$$

Q.E.D