Generalization of a Remarkable Theorem

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In [1] Professor Claudiu Coandă proves the following theorem using the barycentric coordinates.

Theorem 1 (C. Coandă):

Let ABC a triangle with $m(\measuredangle A) \neq 90^o$ and Q_1, Q_2, Q_3 three points on its circumscribed circle. We note $BQ_i \cap AC = \{B_i\}$, $i = \overline{1,3}$, $CQ_i \cap AB = \{C_i\}$, $i = \overline{1,3}$. Then B_1C_1, B_2C_2, B_3C_3 are concurrent.

We will generalize this theorem and we'll prove it using several results from projective geometry relative to the notions of pole and polar

Theorem 2 (Generalization of C. Coandă Theorem):

Let ABC a triangle with $m(\measuredangle A) \neq 90^o$ and $Q_1, Q_2, ..., Q_n$ points on its circumscribed circle $(n \in N,\ n \geq 3)$. We note $BQ_i \cap AC = \{B_i\}$, $i = \overline{1,n}$; $CQ_i \cap AB = \{C_i\}$, $i = \overline{1,n}$. Then the lines B_1C_1, B_2C_2, ..., B_nC_n are concurrent in a fixed point.

To prove this theorem we'll use the following lemmas:

Lemma 1:

If $ABCD$ is a quadrilateral inscribed in a circle and $\{P\} = AB \cap CD$, then the polar of P in rapport to the circle is the line EF, where $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$.

Lemma 2: The pole of a line is the intersection of the corresponding polar to any two points of the line.

The poles of concurrent lines in rapport to a given circle are collinear points and reciprocally: the polar of some collinear points, in rapport to a given circle, are concurrent lines.

Lemma 3:

If $ABCD$ is a quadrilateral inscribed in a circle and $\{P\} = AB \cap CD$, $\{E\} = AC \cap BD$, and $\{F\} = BC \cap AD$, then the polar of the point E in rapport to the circle is the line PF.

The poof of the lemmas 1 – 3 and other properties regarding the notions of pole and polar in rapport to a circle can be found in [2] or in [3].

Proof of Theorem 2:

Let's consider $Q_1, Q_2, ..., Q_n$ points on the circumscribed circle to the triangle ABC. See the fig 1.

We consider the inscribed quadrilaterals $ABCQ_i$, $i = \overline{1,n}$ and we note $\{T_i\} = AQ_i \cap BC$.

Taking into consideration lemma 1 and lemma 3 the lines B_iC_i are respectively the polar (in rapport with the circle circumscribed to the triangle ABC) of the points T_i.

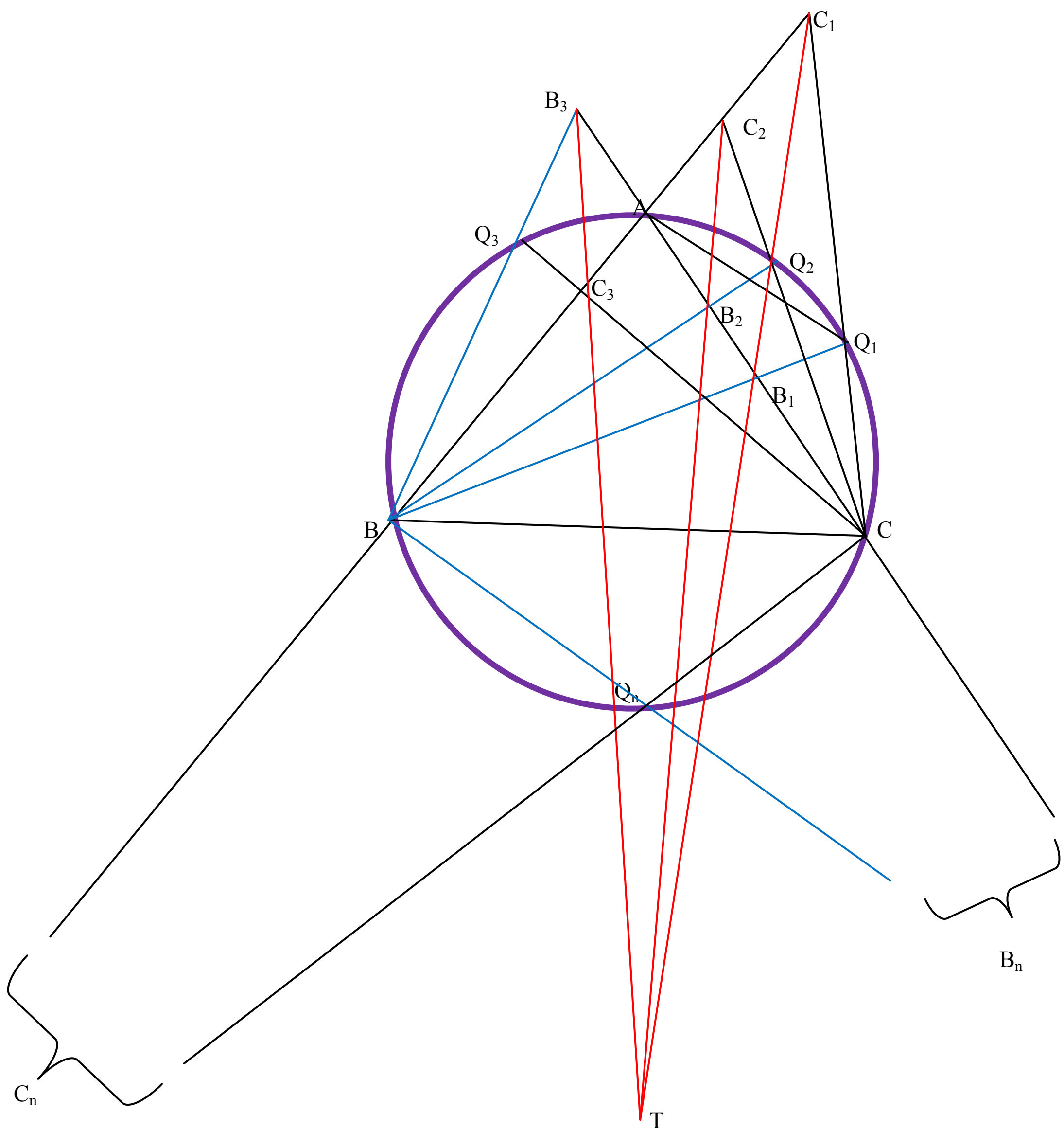

Fig. 1

Because the points T_i are collinear (belong to the line BC) from lemma 2 it results that their polar, that is, the lines B_iC_i, are concurrent in a point T.

Remark:

The point of concurrency T of the lines B_iC_i, $i=\overline{1,n}$ is fixed because we can consider $Q_k = C$ and $Q_j = B$, in which case B_jC_j and B_kC_k are the tangents in B and C to the circumscribed circle to the triangle ABC, and these tangents intersect in a fixed point on the median constructed from the point A of the triangle ABC.

It can be shown that the point T is the harmonic conjugate in rapport with the circle of the simedian center K of the triangle.

Open Questions:

1) Generalize the above two theorems taking instead of a triangle inscribed in a circle a polygon inscribed in a circle.
2) Let's consider instead of a triangle *ABC* inscribed in a circle a polygon $A_1A_2...A_n$ inscribed in a circle in the previous two theorems. Then we split this polygon into many triangles $A_iA_jA_k$, with $1 \le i, j, k \le n$, and $i \neq j \neq k \neq i$.

2.1. Let's have three points Q_1, Q_2, Q_3 on the circumscribed circle of the polygon. Apply C. Coanda's Theorem 1 for each triangle $A_iA_jA_k$ and the given points Q_1, Q_2, Q_3 and get a point of concurrence C_{ijk}. What is the locus of all these concurrence points?

2.2. Let's have *n* points $Q_1, Q_2, ..., Q_n$ on the circumscribed circle of the polygon. Now, apply Theorem 2 for each triangle $A_iA_jA_k$ and the given points $Q_1, Q_2, ..., Q_n$ and get a point of concurrence C_{ijk}. What is the locus of all these concurrence points?

Some Remarks:

If the triangle ABC is inscribed in the circle of center (O) and the vertices B and C are fixed while the vertex A is mobile on the circumscribed circle of center (O), then the lines B_iC_i pass through a fixed point which is at the intersection of the tangents drawn through the points B and C to the circle of center (O).

The points B_i and C_i are at the intersection of the lines BQ_i and CQ_i with AC and respectively AB, where Q_i is on the circle.

In the case when the angle A = 90° the lines B_iC_i from the previous Theorems 1 and 2 intersect at infinity.

References

[1] Claudiu Coandă, "Geometrie analitică în coordinate baricentrice", Editura Reprograph, Craiova, 2005.

[2] Ion Pătraşcu, "O aplicație practică a unei teoreme de geometrie proiectivă", in <Sfera matematicii> journal, No. 1b (2/2009-2010), Editura Reprograph, Craiova.

[3] Roger A. Johnson, "Advanced Euclidean Geometry", Dover Publications, Inc., Mineola, New York, 2007.