On The 5D Extra-Force according to Basini-Capozziello-Ponce De Leon Formalism and three important features: Chung-Freese Superluminal BraneWorld,Strong Gravitational Fields and the Pioneer Anomaly.

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April 30, 2010

Abstract

We use the $5D$ Extra Dimensional Force according to Basini-Capozziello-Ponce De Leon,Overduin-Wesson and Mashoon-Wesson-Liu Formalisms to study the behaviour of the Chung-Freese Superluminal BraneWorld compared to the Alcubierre Warp Drive and we arrive at some interesting results from the point of view of the Alcubierre ansatz although we used two different Shape Functions one continuous $g(rs)$ as an alternative to the original Alcubierre $f(rs)$ and a Piecewise Shape Function $f_{pc}(rs)$ with a behaviour similar to the Natario Warp Drive. We introduce here the Casimir Warp Drive.We also demonstrate that in flat $5D$ Minkowsky Spacetime or weak Gravitational Fields we cannot tell if we live in a $5D$ or a $4D$ Universe according to Basini-Capozziello-Ponce De Leon,Overduin-Wesson and Mashoon-Wesson-Liu Dimensional Reduction but in the extreme conditions of Strong Gravitational Fields we demonstrate that the effects of the $5D$ Extra Dimension becomes visible and perhaps the study of the extreme conditions in Black Holes can tell if we live in a Higher Dimensional Universe.We use a $5D$ Maartens-Clarkson Schwarzschild Cosmic Black String centered in the Sun coupled to the $5D$ Extra Force from Ponce De Leon together with Mashoon-Wesson-Liu and the definitions of Basini-Capozziello and Bertolami-Paramos for the Warp Fields in order to demonstrate that the Anomalous Effect disturbing two American space probes known as the Pioneer Anomaly is a force of $5D$ Extra Dimensional Nature.As a matter of fact the Pioneer Anomaly is the first experimental evidence of the "Fifth Force" predicted years ago by Mashoon-Wesson-Liu and we also demonstrate that this Extra Force is coming from the Sun.

1 The Beginning

The Space Era started on 4 October 1957 with the Aerospace Engineer Serguei Pavlovitch Koroliev and the rocket RS-7 Semyorka that launched into Outer Space the first Human Made artificial satellite:the Sputnik I of the former Union of Soviet Socialist Republics.Only fifty years have passed since the dawn of the Human Adventure in the Skies and its too early to intend (or pretend) that our science knows everything about the Outer Space.The Universe Out There will remain forever a mysterious place filled with secrets we cannot barely imagine and in the remote future at millions of years from now we can take for granted that strange and unknown phenomena left by Mother Nature will be Up There In The Skies waiting for ourselves in order to be discovered.

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This works deals with Extra Dimensions in an attempt to explain one of these strange phenomena:A unknown anomaly recently discovered in Outer Space affecting two American space probes.This Phenomena is simply known as:The Pioneer Anomaly.

This work also deals with the original and wonderful idea of the Alcubierre Warp Drive.[1]

2 Introduction

Much has been said about the so-called Extra Dimentional nature of the Universe.It was first proposed by Theodore Kaluza and Oskar Klein in 1918[2] in an attempt to unify Gravity and Electromagnetism. However the physical nature of the Extra Dimension was not well defined in a clear way. They used a so-called Compactification Mechanism to explain why we cannot see the $5D$ Extra Dimension but this mechanism was not clearly understood. Later on and with more advanced scientific knowledge other authors appeared with the same idea under the exotic concept of the so-called BraneWorld. In the BraneWorld concept our visible Universe is a $3+1$ Dimensional sheet of Spacetime :a Brane involved by a Spacetime of Higher Dimensional nature. This idea came mainly from Strings Theory where Gravitational Forces are being represented by an Elementary Particle called Graviton while other interactions are represented by other sets of Elementary Particles:Electromagnetic Interaction is being represented by an Elementary Particle called Photon. According to Strings Theory Gravitons are Closed Loops and can leave easily our $3+1$ Dimensional Spacetime and escape into the Extra Dimensions while Photons are Open Strings and are "trapped" in our $3+1$ Spacetime. To resume: In 1918 Kaluza-Klein tried to unify Gravity and Electromagnetism and Einstein among other scientists were interested in the same thing. But however inside the framework of the so-called Strings Theory how can a Closed Loop be unified with a Open String ??? How can an Interaction that with some degrees of freedom is allowed to probe the Extra Dimensional Spacetime be unified with Interactions confined to our $3+1$ Spacetime??? A puzzle to solve.So the so-called Strings Theory is trying to unify Gravity with Electromagnetism and other Interactions but the framework is not completed or not well understood. On the other hand the Compactification Mechanism in the original Kaluza-Klein theory explains why we cannot see beyond the $3+1$ Spacetime because the Extra Dimensions are Compactified or Curled Up but it does not explains why we have $3+1$ Uncurled or Uncompactified Dimensions while the remaining Extra Dimensions are Curled and what generates this Compactification Mechanism in the first place??? We adopt in this work the so-called Basini-Capozziello Ponce De Leon formalism coupled to the formalisms of Mashoon-Wesson-Liu and Overduin-Wesson in which Extra Dimensions are not compactified but opened like the $3+1$ Spacetime Dimensions we can see.There are small differences between these formalisms but Basini-Capozziello Ponce De Leon admits a non-null ${}^5R_{AB}$ Ricci Tensor while the others make the Ricci Tensor ${}^5R_{AB} = 0$[3] but essentially these formalisms are mathematically equivalent.In the Basini-Capozziello Ponce de Leon the ordinary Spacetime of $3+1$ Dimensions is embedded into a large Higher Dimensional Spacetime,however in a flat or Minkowsly Spacetime the Spacetime Curvature eg Ricci and Einstein Tensors of the Higher Dimensional Spacetime reduces to the same Ricci and Einstein Tensors of a $3+1$ Spacetime. This explains without Compactification Mechanisms why we cannot see beyond the $3+1$ Spacetime:our everyday Spacetime is

[1]readers interested only in Alcubierre Warp Drive may skip to the end of the Introduction section

[2]see [21] for an excellent account on Kaluza-Klein History

[3]see [21] pg 31 after eq 48 and see [2] eq 20 [11] eq 21 and [20] eq 8.We prefer to assume that exists matter in the $5D$ due to the last section of [20] about the particle Z

essentialy Minkowskian or flat [4]and a $5D$ Ricci Tensor reduces to a Ricci Tensor of a $3+1$ Spacetime. On the other hand in this formalism all masses,electric charges and spins of all the Elementary Particles seen in $4D$ are function of a $5D$ rest-mass coupled with Spacetime Geometry. We can observe in $4D$ Spacetime a multitude of Elementary Particles with different masses,electric charges or spins but according to the Basini-Capozziello Ponce De Leon formalism eg the $5D$ to $4D$ Dimensional Reduction all these different Elementary Particles with all these $4D$ rest-masses,electric charges or spins are as a matter of fact a small group of $5D$ Elementary Particles with a $5D$ rest-mass and is the geometry of the $5D$ Spacetime coupled with the Dimensional Reduction that generates these apparent differences. Hence two particles with the same $5D$ rest-mass M_5 can be seen in $4D$ with two different rest-masses m_0 making ourselves think that the particles are different but the difference is apparent and is generated by the Dimensional Reduction from $5D$ to $4D$.Look to the set of equations below:We will explore in this work these equations with details but two particles with the same $5D$ rest mass M_5 can be seen in the $4D$ with two different rest masses m_0 if the Dimensional Reduction from $5D$ to $4D$ or the Spacetime Geometric Coupling $\sqrt{1-\Phi^2(\frac{dy}{ds})^2}$ is different for each particle. All the particles in the Table of Elementary Particles given below[5] with non-zero rest-mass m_0 seen in $4D$ can as a matter of fact have the same rest-mass M_5 in $5D$ and the Dimensional Reduction term $\sqrt{1-\Phi^2(\frac{dy}{ds})^2}$ generates the apparent different $4D$ rest-masses.This is very attractive from the point of view of a Unified Physics theory.There exists a small set of particles in $5D$ and all the huge number of Elementary Particles in $4D$ is a geometric projection from the $5D$ Spacetime into a $4D$ one([2] eq 20,[11] eq 21 and [20] eq 8)([2] eq 14,[20] eq 1 and eq 2).

$$m_0 = \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}} \tag{1}$$

$$dS^2 = g_{uv}dx^u dx^v - \Phi^2 dy^2 \tag{2}$$

$$dS^2 = ds^2 - \Phi^2 dy^2 \tag{3}$$

[4]we consider our Spacetime as a Schwarzschild Spacetime however at a large distance from the Gravitational Source it reduces to a Minkowsky SR Spacetime due to a large R and the ratio $\frac{M}{R}$ tends to zero

[5]extracted from the Formulary Of Physics by J.C.A. Wevers available on Internet

Particle	spin ($\hbar$)	B	L	T	T_3	S	C	B^*	charge (e)	m_0 (MeV)	antipart.
u	1/2	1/3	0	1/2	1/2	0	0	0	+2/3	5	$\overline{\mathrm{u}}$
d	1/2	1/3	0	1/2	−1/2	0	0	0	−1/3	9	$\overline{\mathrm{d}}$
s	1/2	1/3	0	0	0	−1	0	0	−1/3	175	$\overline{\mathrm{s}}$
c	1/2	1/3	0	0	0	0	1	0	+2/3	1350	$\overline{\mathrm{c}}$
b	1/2	1/3	0	0	0	0	0	−1	−1/3	4500	$\overline{\mathrm{b}}$
t	1/2	1/3	0	0	0	0	0	0	+2/3	173000	$\overline{\mathrm{t}}$
e^-	1/2	0	1	0	0	0	0	0	−1	0.511	e^+
μ^-	1/2	0	1	0	0	0	0	0	−1	105.658	μ^+
τ^-	1/2	0	1	0	0	0	0	0	−1	1777.1	τ^+
ν_e	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_e$
ν_μ	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_\mu$
ν_τ	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_\tau$
γ	1	0	0	0	0	0	0	0	0	0	γ
gluon	1	0	0	0	0	0	0	0	0	0	$\overline{\text{gluon}}$
W^+	1	0	0	0	0	0	0	0	+1	80220	W^-
Z	1	0	0	0	0	0	0	0	0	91187	Z
graviton	2	0	0	0	0	0	0	0	0	0	graviton

We employ in this work the Basini-Capozziello Ponce De Leon Formalism to study the behavior of the so-called Superluminal Chung-Freese BraneWorld compared to the Alcubierre Warp Drive with some interesting results for the Alcubierre metric but with a diferent Shape Function $f(rs)$.More on Alcubierre Warp Drive in the end of this section.We also demonstrate that while in flat or Minkowsky Spacetime the curvature in $5D$ reduces to a one in $4D$ due to the Dimensional Reduction suffered by the Ricci and Einstein Tensors and we cannot tell if we live in a $5D$ or in a $4D$ Universe due to the absence of Strong Gravitational Fields but in an environment of Strong Gravity the $5D$ Ricci and Einstein Tensors cannot be reduced to similar $4D$ ones and the Curvature of a $5D$ Spacetime is different than the one of a $4D$ because the $5D$ extra terms in the Ricci and Einstein Tensors have the terms of the Strong Gravitational Field and cannot be reduced to $4D$.Perhaps the study of the conditions of extreme Gravitational Fields in large Black Holes will tell if we live in a $5D$ Universe or in a $4D$ one.Also we employ in this work a Maartens-Clarkson $5D$ Schwarzschild Black String centered on the Sun and we apply the Basini-Capozziello Ponce De Leon $5D$ Extra Force coupled with the $5D$ Extra Force definition from Mashoon-Wesson-Liu and the Definitions of Bertolami-Paramos[6] to demonstrate that the anomalous phenomena known as the Pioneer Anomaly is being generated by a $5D$ Extra Force that comes from the Sun.The Pioneer Anomaly is the first experimental evidence of the "Fifth Force" predicted by Mashoon-Wesson-Liu in [9].Basically we place the Sun as being the center of a Maartens-Clarkson Cosmic Black String defined as([7] eq 1,[20] eq 380):

$$dS^2 = [(1-\frac{2GM}{R})dt^2 - \frac{dR^2}{(1-\frac{2GM}{R})} - R^2 d\eta^2] - \Phi dy^2 \tag{4}$$

where M is the $4D$ rest-mass of the Sun defined in function of the $5D$ rest-mass of the Sun M_5 as follows([2] eq 20,[11] eq 21 and [20] eq 8):

$$M = \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}} \tag{5}$$

[6] better explained in the Pioneer Anomaly section

and R is the distance between the Sun and the Pioneer spacecrafts Φ is the so-called Warp Field written in function of both the $4D$ observable Spacetime Dimensions and the $5D$ Extra Dimension according to Basini-Capozziello Ponce De Leon definition([1] eq 76,[5] eq 70 and [20] eq 132)[7]

$$\Phi = \phi(t,x)\chi(y) \tag{6}$$

coupled with the definitions from Bertolami-Paramos[8] and y is the $5D$ Extra Dimension.

Applying the Basini-Capozziello Ponce De Leon $5D$ Extra Force defined as follows([2] eq 25,[20] eq 15 and [9] eq 24 and 38):

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} - \Phi(x^{3+1},y)u^u \frac{\partial \Phi(x^{3+1},y)}{\partial x^u}(\frac{dy}{ds})^2 \tag{7}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} - \phi(t,x)\chi(y)u^u \frac{\partial \phi(t,x)\chi(y)}{\partial x^u}(\frac{dy}{ds})^2 \tag{8}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} - \phi(t,x)\chi(y)^2 u^u \frac{\partial \phi(t,x)}{\partial x^u}(\frac{dy}{ds})^2 \tag{9}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} \tag{10}$$

where g_{uv} are the Spacetime Metric Tensor components of the Cosmic Black String we will show that this $5D$ Force is being generated by variations in the $4D$ rest-mass of the Sun in function of the $5D$ one plus variations in the Spacetime Coupling from $5D$ to $4D$ namely variations in the term $\sqrt{1-\Phi^2(\frac{dy}{ds})^2}$.Note that if the Warp Field $\Phi = 1$ which means to say that we have a $5D$ Extra Dimension without a Warp Field and we will analyze some cases of this Geometry or if the Warp Field is defined only in function of the $5D$ Extra Dimensions as $\Phi = \chi(y)$ eqs 7,8 and 9 reduces to 10(see [9] eq 24 and 38). The term $\frac{1}{M}\frac{dM}{ds}$ the $5D$ Extra Force according to Mashoon-Wesson-Liu is what is causing the de-acceleration of the Pioneers and in last case the reported Anomaly[9].The Pioneer Anomaly is the first experimental evidence of the Fifth Force predicted by Mashoon-Wesson-Liu in [9] and is also a geometrical property of the Spacetime.Any ship placed at the same distance the Pioneers are now from the Sun would suffer from the same Anomaly and this perhaps can be measured by precision Spacecrafts as suggested by Bertolami-Paramos in abstract of [27].This Anomaly is very small and in our Planetary System could be masked by the gravity of massive planets like Jupiter or Saturn disturbing the trajectory of Spaceships but far away from the Sun where gravity becomes weak this effect becomes noticeable.Although the Anomaly due to the $5D$ Extra Force is very small,too small to affect a massive body like a planet and even a small planet would be massive enough to absorb the effects of the Anomaly and remain undisturbed making the Anomaly undetectable,for small objects eg Space Probes the Anomaly becomes noticeable.This is at last the final explanation for the so-called Pioneer Anomaly:The first experimental evidence of the Fifth Force predicted years ago by Mashoon-Wesson-Liu in [9].This work is divided in Sections.Section 3 presents the Basini-Capozziello-Ponce

[7] the term "Warp" appears in pg 1340 in [2]

[8] better explained in the Pioneer Anomaly section

[9] Mashoon-Wesson-Liu mentions in [9] pg 565 that the $5D$ Extra Force is of Cosmological Nature and generates in local physics an anomalous effect of de-acceleration and this is analogous to what is happening with the Pioneers.We have the same effect of de-acceleration generated by the $5D$ Maartens-Clarkson Cosmic Black String in the Center of the Sun if we assume the Sun generating the $5D$ Extra Force.See also the definition of $\beta = \frac{1}{M}\frac{dM}{ds}$ in [9] pg 566 after eq 41 and compare [9] eq 41,39 with [9] eqs 24 and 38.See also the definition of β in [9] pg 565 when Mashoon-Wesson-Liu mentions the de-acceleration

De Leon Formalism and comparisons between this and the Mashoon-Wesson-Liu or Overduin-Wesson Formalisms,Section 4 presents the Dimensional Reduction from $5D$ to $4D$ in flat Minkowsky Spacetimes and in Spacetimes of Strong Gravitational Fields.Section 5 the most important of this work:It presents the Pioneer Anomaly explained by the $5D$ Extra Force from Basini-Capozziello-Ponce De Leon coupled to the definitions of Mashoon-Wesson-Liu and the Warp Fields from Basini-Capozziello-Ponce De Leon coupled to the Definitions of Bertolami-Paramos and we demonstrate that the Pioneer Anomaly is the first experimental evidence of the Fifth Force predicted by Mashoon-Wesson-Liu years ago in [9].

Finally Section 6 presents the Superluminal Chung-Freese BraneWorld compared to the Alcubierre Warp Drive[10].Our results are interesting to the Alcubierre metric but we must outline that our Shape Function $f(rs)$ is not the original from Alcubierre and have a behaviour similar to the Natario Warp Drive.As far as we know Miguel Alcubierre idea of the Warp Drive remained right after all these years.We will demonstrate this in this paper.The Alcubierre Warp Drive is still a valid Superluminal ansatz.Horizons will exists but we present a way that might be useful to overcome this problem[11].The front part of the Warp Bubble cannot be signalized and will always remains causally disconnected from the spaceship but the rear part of the Warp Bubble can be signalized.We also lower the energy density to low and affordable levels due to our Shape Function $f(rs)$.Also we demonstrate that the Doppler Blueshift of photons impacting the Warp Bubble never existed as a serious problem because if the Alcubierre Warp Drive possesses negative energy then it possesses negative mass and this would mean in a Gravitational repulsion and a Spacetime curvature that would repeal objects instead of attract.Of course this is also due to our Shape Function $f(rs)$ that allow a different distribution of the negative energy density T^{00}.While in original Alcubierre Shape Function the negative energy is concentrated in a region toroidal perpendicular to the direction of the motion of the spaceship(see pg 6 and fig 3 pg 7 in [28] and pg 70 and fig 5.3 pg 71 in [48]) while the spaceship remains on empty space vulnerable to Doppler Blueshifted photons[12] our Warp Bubble involves the spaceship with a sphere of negative energy protecting it from impacts.We can also overcome the restrictions of Lobo and Visser.Alcubierre Warp Drive possesses a negative Gravitational force and a negative Gravitational Bending Of Light.Doppler Blueshifts would be hazardous for Superluminal ansatz with positive energy but fortunately this is not the case.While other authors consider the negative energy a pathology we consider it a bless because it can deflect objects from the Warp Bubble.The only two problems that remains are what would happen if a large Black Hole appears in front of a Superluminal Warp Bubble and the fact that we don't know how to assemble effectively an Alcubierre Warp Drive. We compare the Alcubierre Warp Drive with the Chung-Freese Superluminal Braneworld and we arrive at the conclusion that the Alcubierre Warp Drive is suitable to subluminal travel or Superluminal travel but at "short" distances(eg some light-years) while the Chung-Freese Superluminal Braneworld is suitable for larger distances.Also we demonstrate that the Chung-Freese Superluminal Braneworld can avoid impacts with large Black Holes(we hope so at least in theory otherwise the Superluminal Chung-Freese Braneworld would be in deep trouble) but we also reckognize that we dont know how to "engineer" neither the Alcubierre Warp Drive nor the Superluminal Chung-Freese Brameworld.

[10]the beginning of section 6 explains why the Alcubierre Warp Drive appears in a work dedicated primarily to the Pioneer Anomaly and Extra Dimensions

[11]see the comment on pg 3 of [45] about the actions to create or change the Warp Bubble trajectory or speed being taken by an external observer whose light cone contains all the trajectory of the Warp Bubble

[12]the ship remains in the center of the Warp Bubble and this means the black regions on fig 3 or fig 5.3 leaving the ship "unprotected" while the negative energy regions are the white toroidal regions

3 The Basini-Capozziello Ponce De Leon Formalism and resemblances with Mashoon-Wesson-Liu and Overduin-Wesson Formalisms

Basini-Capozziello Ponce de Leon argues that our $3+1$ Dimensional Spacetime we can see is a Dimensional Reduction from a larger $5D$ one and according to a given Spacetime Geometry we can see(or not) the $5D$ Extra Dimension .This is also advocated in the almost similar Formalisms of Mashoon-Wesson-Liu and Overduin-Wesson. A $5D$ Spacetime metric is defined as([1] eq 32,[5] eq 18,[20] eq 62 and [9] pg 556 Section 2) and contains all the $3+1$ Spacetime Dimensions of our observable Universe plus the $5D$ Extra Dimension.Then $A, B = 0, 1, 2, 3, 4$ where $0, 1, 2, 3$ are the Dimensons of the $4D$ Spacetime and 4 is the script of the $5D$ Extra Dimension(see [5] pg 2225 after eq 18 and again [9] pg 556).

$$dS^2 = g_{AB}dx^A dx^B \tag{11}$$

Note that this equation is common not only to Basini-Capozziello-Ponce De Leon but also to Mashoon-Wesson-Liu and Overduin-Wesson Formalisms.These formalisms advocates the Dimensional Reduction from $5D$ to $4D$[13] and we need to separate in this Spacetime Metric both the $3+1$ Components of our visible Universe and the components of the $5D$ Extra Dimension.The resulting equation would then be([1] eq 56,[2] eq 12 and 14,[5] eq 42,[9] eq 32 and 33 without vector potential,[11] eq 10, [12] pg 308,[20] eq 109 and [13] pg 1346)[14][15][16]

$$dS^2 = g_{AB}dx^A dx^B = g_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{12}$$

Note that when the Warp Field $\Phi = 1$ the Spacetime Metric becomes:

$$dS^2 = g_{AB}dx^A dx^B = g_{\alpha\beta}dx^\alpha dx^\beta - dy^2 \tag{13}$$

Writing the ${}^5R_{\alpha\beta}$ Ricci Tensor and the ${}^5R$ Ricci Scalar according to Basini-Capozziello using these equations:($\alpha, \beta = 0, 1, 2, 3$)([1] eq 58, [5] eq 44 and [20] eq 111.See also [21] eq 48 for ${}^5R_{\alpha\beta}$)[17]

$${}^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} - \frac{1}{2\Phi^2}\left(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}\right) \tag{14}$$

$${}^5R = R - \frac{\Phi_{,a;b}}{\Phi}g^{\alpha\beta} - \frac{1}{2\Phi^2}g^{\alpha\beta}\left(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}\right) \tag{15}$$

Simplifying for diagonalized metrics we should expect for:

$${}^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} - \frac{1}{2\Phi^2}\left(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}\right) \tag{16}$$

$${}^5R = R - \frac{\Phi_{,a;b}}{\Phi}g^{\alpha\beta} - \frac{1}{2\Phi^2}g^{\alpha\beta}\left(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}\right) \tag{17}$$

[13] We will skip a tedious definition and concentrate on the Dimensional Reduction.A unfamiliar reader must study first [1] pg 122 Section 2.2 to pg 127,[5] pg 2225 Section 3 to pg 2229 and [20] pg 1434 Section 4 to pg 1441.see also [21] pg 29 Section 6 to pg 31

[14] [12] with spacelike signature

[15] see [13] pg 1341 the Campbell-Magaard Theorem

[16] see [1] eq 57,[5] eq 43 and [21] eq 47.

[17] Working with diagonalized metrics the terms α,λ,μ,β and ν are all equal

Note that the term ${}^4\Box\Phi = \nabla_\alpha \Phi^\alpha = g^{\alpha\beta}(\Phi_\alpha)_{;\beta} = g^{\alpha\beta}[(\Phi_\alpha)_\beta - \Gamma^K_{\beta\alpha}\Phi_K]$ corresponds to the D'Alembertian in $4D$[18][19] so we can write for the Ricci Scalar the following expression:

$$ {}^5R = R - \frac{{}^4\Box\Phi}{\Phi} - \frac{1}{2\Phi^2} g^{\alpha\beta}\left(\frac{\Phi_{,4} g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2}\right) \tag{18} $$

If according to Basini-Capozziello the terms $g_{\alpha\beta}$ have no dependance with respect to to the Extra Coordinate y after the Reduction from $5D$ to $4D$ then all the derivatives with respect to y vanish and we are left out with the following expression for the Ricci Scalar:([1] eq 59,[5] eq 45 and [20] eq 116)).We will analyze this in details when studying the $5D$ to $4D$ Dimensional Reduction.

$$ {}^5R = R - \frac{{}^4\Box\Phi}{\Phi} \tag{19} $$

Writing the remaining Ricci Tensors we should expect for([21] eq 48)[20]:

$$
\begin{aligned}
\hat{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{\nabla_\beta(\partial_\alpha\Phi)}{\Phi} - \frac{1}{2\Phi^2}\left(\frac{\partial_4\Phi\,\partial_4 g_{\alpha\beta}}{\Phi} - \partial_4 g_{\alpha\beta}\right. \\
&\quad \left. + g^{\gamma\delta}\,\partial_4 g_{\alpha\gamma}\,\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\,\partial_4 g_{\gamma\delta}\,\partial_4 g_{\alpha\beta}}{2}\right) \quad , \\
\hat{R}_{\alpha 4} &= \frac{g^{44} g^{\beta\gamma}}{4}(\partial_4 g_{\beta\gamma}\,\partial_\alpha g_{44} - \partial_\gamma g_{44}\,\partial_4 g_{\alpha\beta}) + \frac{\partial_\beta g^{\beta\gamma}\,\partial_4 g_{\gamma\alpha}}{2} \\
&\quad + \frac{g^{\beta\gamma}\,\partial_4(\partial_\beta g_{\gamma\alpha})}{2} - \frac{\partial_\alpha g^{\beta\gamma}\,\partial_4 g_{\beta\gamma}}{2} - \frac{g^{\beta\gamma}\,\partial_4(\partial_\alpha g_{\beta\gamma})}{2} \\
&\quad + \frac{g^{\beta\gamma} g^{\delta\epsilon}\,\partial_4 g_{\gamma\alpha}\,\partial_\beta g_{\delta\epsilon}}{4} + \frac{\partial_4 g^{\beta\gamma}\,\partial_\alpha g_{\beta\gamma}}{4} \quad , \\
\hat{R}_{44} &= \Phi\Box\Phi - \frac{\partial_4 g^{\alpha\beta}\,\partial_4 g_{\alpha\beta}}{2} - \frac{g^{\alpha\beta}\,\partial_4(\partial_4 g_{\alpha\beta})}{2} \\
&\quad + \frac{\partial_4\Phi\, g^{\alpha\beta}\,\partial_4 g_{\alpha\beta}}{2\Phi} - \frac{g^{\alpha\beta} g^{\gamma\delta}\,\partial_4 g_{\gamma\beta}\,\partial_4 g_{\alpha\delta}}{4} \quad ,
\end{aligned}
\tag{20}
$$

where "$\Box$" is defined as usual (in four dimensions) by $\Box\Phi \equiv g^{\alpha\beta}\nabla_\beta(\partial_\alpha\Phi)$.

Note that the Overduin-Wesson definition is exactly equal to the one presented by Basini-Capozziello Ponce De Leon.Both Formalisms are equivalent except that Basini-Capozziello-Ponce De Leon admits a ${}^5R_{AB}$ not null.

Working with diagonalized Spacetime Metrics of signature (+,-,-,-,-)[21] the Ricci Tensors would be written as:

$$
\begin{aligned}
\hat{R}_{\alpha\alpha} &= R_{\alpha\alpha} - \frac{\nabla_\alpha(\partial_\alpha\Phi)}{\Phi} - \frac{1}{2\Phi^2}\left(\frac{\partial_4\Phi\,\partial_4 g_{\alpha\alpha}}{\Phi} - \partial_4 g_{\alpha\alpha}\right. \\
&\quad \left. + g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}\,\partial_4 g_{\alpha\alpha} - \frac{g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}}{2}\right) \quad ,
\end{aligned}
$$

[18] see pg 129 in [1] and pg 2230 in [5]

[19] see also pg 311 in [12]

[20] adapted from the arXiv.org LaTeX file of [21] eq 48.Note the difference between the first term $\hat{R}_{\alpha\beta}$ between [21] eq 48 ,[1] eq 58,[5] eq 44 and [20] eq 111

[21] $\alpha = \beta = \gamma = \delta = \epsilon$

$$
\begin{aligned}
\hat{R}_{\alpha 4} &= \frac{g^{44}g^{\alpha\alpha}}{4}(\partial_4 g_{\alpha\alpha}\,\partial_\alpha g_{44} - \partial_\alpha g_{44}\,\partial_4 g_{\alpha\alpha}) + \frac{\partial_\alpha g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}}{2} \\
&+ \frac{g^{\alpha\alpha}\,\partial_4(\partial_\alpha g_{\alpha\alpha})}{2} - \frac{\partial_\alpha g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}}{2} - \frac{g^{\alpha\alpha}\,\partial_4(\partial_\alpha g_{\alpha\alpha})}{2} \\
&+ \frac{g^{\alpha\alpha}g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}\,\partial_\alpha g_{\alpha\alpha}}{4} + \frac{\partial_4 g^{\alpha\alpha}\,\partial_\alpha g_{\alpha\alpha}}{4} \quad , \\
\hat{R}_{44} &= \Phi\Box\Phi - \frac{\partial_4 g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}}{2} - \frac{g^{\alpha\alpha}\,\partial_4(\partial_4 g_{\alpha\alpha})}{2} \\
&+ \frac{\partial_4\Phi\, g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}}{2\Phi} - \frac{g^{\alpha\alpha}g^{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}\,\partial_4 g_{\alpha\alpha}}{4} \quad ,
\end{aligned}
\tag{21}
$$

Compare the first of the Ricci Tensors above with [20] eq 113. Note that the Mashoon-Wesson-Liu Formalism is exactly equal to the Basini-Capozziello-Ponce De Leon and Overduin-Wesson Formalisms. Look to the equations [9] eq 32 and 33 without vector potential.Compare with

$$dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{22}$$

One can see that we already presented this equation proving without shadows of doubt that the three formalisms are equivalent.Mashoon-Wesson-Liu in [9] pg 557 makes $g_{44} = -\Phi^2$. They also makes $g_{44} = -1$ (see pg 558)giving the equation below:

$$dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta - dy^2 \tag{23}$$

We already presented this equation:is the $5D$ Spacetime Geometry without the Warp Field.

One thing advocated by Mashoon-Wesson-Liu and Basini-Capozziello Ponce De Leon is the fact that the $5D$ Extra Dimension generate a $5D$ Extra Force that can be detected in $4D$.(see [9] abstract and pgs 556,562 look to eq 24,pg 563 look to eq 31 and the comment below this equation,pg 565 eq 38,39 and the comments on the de-acceleration,pg 566 definition of β,pg 567 look to the comment of a small force but detectable).(see also [2] abstract and pgs 1336,1337,1341 eq 20, pg 1342 eq 25,pg 1343 eq 30).We will now prove that the $5D$ Extra Force in both formalisms is equivalent.

If we have a Spacetime Geometry defined as:

$$dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{24}$$

$$dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta - [\phi(t,x)\chi(y)]^2 dy^2 \tag{25}$$

where we defined the Warp Field Φ according to Basini-Capoziello([1] eq 76,[5] eq 70 and [20] eq 132) we have two choices:

- M_5 the $5D$ Mass is not zero and we have matter in the $5D$ Extra Dimension according to one of the Ponce De Leon Options making also ${}^5R_{AB}$ the Ricci Tensor in $5D$ not null.
- M_5 The $5D$ Mass is zero and we have no matter in the $5D$ Extra Dimension according to another of the Ponce De Leon Options making also ${}^5R_{AB}$ the Ricci Tensor in $5D$ null.

Overduin-Wesson and Mashoon-Wesson-Liu formalisms agree with the second option of Ponce De Leon.(see [21] pg 31 after eq 48 and [9] pg 557 eq 2)

According to Ponce De Leon in option 1 if we have a rest-mass in $5D$ M_5 this rest-mass will be seen in $4D$ as a rest-mass m_0 as follows([2] eq 20,[11] eq 21 and [20] eq 8):

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{26}$$

$$m_0 = \frac{M_5}{\sqrt{1 - [\phi(t,x)\chi(y)]^2(\frac{dy}{ds})^2}} \tag{27}$$

We have Quantum Chromodynamics for Quarks and a Quantum Electrodynamics for Leptons like Electron but as a matter of fact two particles with the same rest-mass in $5D$ M_5 can appear in our $4D$ Spacetime with different rest masses m_0 making one appear as a Quark and the other as a Lepton depending on the Dimensional Reduction from $5D$ to $4D$ or the Spacetime Coupling $\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}$,$\sqrt{1 - [\phi(t,x)\chi(y)]^2(\frac{dy}{ds})^2}$ although in $5D$ both particles are the same.

This is a very interesting perspective of Modern Physics.Why Quantum Electrodynamics and Quantum Chromodynamics in $4D$ while as a matter of fact in $5D$ both are the same???.Look again to the table below[22] :

Particle	spin ($\hbar$)	B	L	T	T_3	S	C	B^*	charge (e)	m_0 (MeV)	antipart.
u	1/2	1/3	0	1/2	1/2	0	0	0	+2/3	5	$\bar{\mathrm{u}}$
d	1/2	1/3	0	1/2	−1/2	0	0	0	−1/3	9	$\bar{\mathrm{d}}$
s	1/2	1/3	0	0	0	−1	0	0	−1/3	175	$\bar{\mathrm{s}}$
c	1/2	1/3	0	0	0	0	1	0	+2/3	1350	$\bar{\mathrm{c}}$
b	1/2	1/3	0	0	0	0	0	−1	−1/3	4500	$\bar{\mathrm{b}}$
t	1/2	1/3	0	0	0	0	0	0	+2/3	173000	$\bar{\mathrm{t}}$
e^-	1/2	0	1	0	0	0	0	0	−1	0.511	e^+
μ^-	1/2	0	1	0	0	0	0	0	−1	105.658	μ^+
τ^-	1/2	0	1	0	0	0	0	0	−1	1777.1	τ^+
ν_e	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_e$
ν_μ	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_\mu$
ν_τ	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_\tau$
γ	1	0	0	0	0	0	0	0	0	0	γ
gluon	1	0	0	0	0	0	0	0	0	0	$\overline{\text{gluon}}$
W^+	1	0	0	0	0	0	0	0	+1	80220	W^-
Z	1	0	0	0	0	0	0	0	0	91187	Z
graviton	2	0	0	0	0	0	0	0	0	0	graviton

The Extra Force generated by the $5D$ seen in $4D$ for a massive $5D$ particle M_5 seen in $4D$ as m_0 according to Ponce De Leon is defined as follows([2] eq 25 and [20] eq 15):

$$\frac{1}{m_0}\frac{dm_0}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} - \Phi u^u \frac{\partial \Phi}{\partial x^u}(\frac{dy}{ds})^2 \tag{28}$$

We have here two choices:

- The Warp Field $\Phi = [\phi(t,x)\chi(y)]$ is not null and we have a Warp Field coupled to the $5D$ Extra Dimension.

[22] extracted from the Formulary Of Physics by J.C.A. Wevers available on Internet

- The Warp Field $\Phi = 1$ and we have no Warp Field at all.

For a $5D$ Extra Dimension coupled with a Warp Field according to Basini-Capozziello([1] eq 76,[5] eq 70 and [20] eq 132) the Extra Force is given by:

$$\frac{1}{m_0}\frac{dm_0}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} - \phi(t,x)\chi(y)u^u \frac{\partial \phi(t,x)\chi(y)}{\partial x^u}(\frac{dy}{ds})^2 \tag{29}$$

$$\frac{1}{m_0}\frac{dm_0}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} - \phi(t,x)\chi(y)^2 u^u \frac{\partial \phi(t,x)}{\partial x^u}(\frac{dy}{ds})^2 \tag{30}$$

If we have no Warp Field at all $\Phi = 1$ the equation is simply:

$$\frac{1}{m_0}\frac{dm_0}{ds} = -\frac{1}{2}u^u u^v \frac{\partial g_{uv}}{\partial y}\frac{dy}{ds} \tag{31}$$

Note that this equation is exactly equal to the $5D$ Extra Force equation as defined by Mashoon-Wesson-Liu([9] eq 24 and 38 because they used $g_{44} = -\Phi^2 = -1$ see [9] pg 558).Of course we expected this result because Basini-Capozziello Ponce De Leon and Mashoon-Wesson-Liu formalisms are equivalent.

For the case of a null $5D$ rest-mass M_5 the option 2 of Ponce De Leon the equation of the $5D$ Extra Force seen in $4D$ is given by([2] eq 30 and [20] eq 19):

$$\frac{1}{m_0}\frac{dm_0}{ds} =\mp \frac{1}{2\Phi}\frac{\partial g_{uv}}{\partial y}u^u u^v - \frac{u^u}{\Phi}\frac{\partial \Phi}{\partial x^u} \tag{32}$$

$$\frac{1}{m_0}\frac{dm_0}{ds} =\mp \frac{1}{2\phi(t,x)\chi(y)}\frac{\partial g_{uv}}{\partial y}u^u u^v - \frac{u^u}{\phi(t,x)}\frac{\partial \phi(t,x)}{\partial x^u} \tag{33}$$

If we have a no Warp Field at all the equation becomes:

$$\frac{1}{m_0}\frac{dm_0}{ds} =\mp \frac{1}{2}\frac{\partial g_{uv}}{\partial y}u^u u^v \tag{34}$$

This is equal to ([9] eq 24 and 38 with $dy = ds$ a Null-Like $5D$ Spacetime Geometry)

According to the following Spacetime Geometry as defined by Basini-Capozziello Ponce De Leon,Mashoon-Wesson-Liu and Overduin-Wesson formalisms([1] eq 56,[2] eq 12 and 14,[5] eq 42 and [20] eq 1)

$$dS^2 = g_{uv}dx^u dx^v - \Phi^2 dy^2 \tag{35}$$

$$dS^2 = ds^2 - \Phi^2 dy^2 \tag{36}$$

$$ds^2 = g_{uv}dx^u dx^v \tag{37}$$

We have three different types of Spacetime Geometries:

- Timelike $5D$ Geometry

$$dS^2 > 0 \dashrightarrow ds^2 - \Phi^2 dy^2 > 0 \dashrightarrow ds^2 > \Phi^2 dy^2 \dashrightarrow \frac{1}{\Phi^2} > (\frac{dy}{ds})^2 \dashrightarrow Timelike5D \tag{38}$$

- Null-Like $5D$ Geometry

$$dS^2 = 0 \dashrightarrow ds^2 - \Phi^2 dy^2 = 0 \dashrightarrow ds^2 = \Phi^2 dy^2 \dashrightarrow \frac{1}{\Phi^2} = (\frac{dy}{ds})^2 \dashrightarrow Nulllike5D \tag{39}$$

- Spacelike $5D$ Geometry

$$dS^2 < 0 \dashrightarrow ds^2 - \Phi^2 dy^2 < 0 \dashrightarrow ds^2 < \Phi^2 dy^2 \dashrightarrow \frac{1}{\Phi^2} < (\frac{dy}{ds})^2 \dashrightarrow Spacelike5D \tag{40}$$

Note that for a Null-Like $5D$ Geometry the equation of the $4D$ rest-mass m_0 in function of the $5D$ rest-mass M_5 is not valid.([2] eq 20,[11] eq 21 and [20] eq 8).

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2 (\frac{dy}{ds})^2}} \tag{41}$$

Hence we suppose that for a Null-Like $5D$ Geodesics the Extra Dimension have no mass at all or all matter in the $5D$ Extra Dimension obeys Timelike $5D$ Geometries.

Then we can say that the Basini-Capozziello Ponce De Leon $5D$ formalism is for Timelike $5D$ Geometries bacause they admit a non-null $5D$ rest-mass while the formalisms of Mashoon-Wesson-Liu and Overduin-Wesson are valid for a Null-Like $5D$ Geometry where we have a $M_5 = 0$ a null $5D$ Ricci Tensor or a flat $5D$ Spacetime.

- $$\frac{1}{\Phi^2} > (\frac{dy}{ds})^2 \dashrightarrow 1 > \Phi^2 (\frac{dy}{ds})^2 \tag{42}$$

- $$\frac{1}{\Phi^2} = (\frac{dy}{ds})^2 \dashrightarrow 1 = \Phi^2 (\frac{dy}{ds})^2 \tag{43}$$

- $$\frac{1}{\Phi^2} < (\frac{dy}{ds})^2 \dashrightarrow 1 < \Phi^2 (\frac{dy}{ds})^2 \tag{44}$$

Note that a small Warp Field $0 < \Phi^2 < 1$ will generate a large $\frac{1}{\Phi^2}$ ideal for a $5D$ Timelike Geodesics.A small Warp Field will appear in the Pioneer Section due to the work of Bertolami-Paramos[23].

Although we can have a Null $5D$ rest-mass M_5 the Warp Field in the $5D$ Extra Dimension can still account for the generation of rest-masses in $4D$.

See these Ponce De Leon Equations for the $4D$ rest-mass m_0 ([2] eq 27 and 28,[20] eq 16,17 and 18)

$$m_0 = ^+_- \Phi \frac{dy}{d\lambda} \tag{45}$$

$$d\lambda = \frac{1}{m_0} ds \tag{46}$$

Combining eqs 45 and 46 we can clearly see that:[24]

[23] the Yukawa Potential defined by Bertolami-Paramos eq 7 in [27] uses a rest-mass M in $4D$ that can be defined in function of the $5D$ rest-mass M_5 according to eq 20 in [2] where the Warp Field appears

[24] see pg 1343 in [2] but the result is obvious from [20] eq 10

$$\frac{dy}{ds} = \frac{1}{\Phi} \tag{47}$$

The $5D$ Extra Force seen in $4D$ for massless particles in $5D$ is given by:([2] eq 30,[20] eq 19)[25][26]

$$\frac{1}{m_0}\frac{dm_0}{ds} =^{-}_{+} \frac{1}{2\Phi}\frac{\partial g_{uv}}{\partial y}u^u u^v - \frac{u^u}{\Phi}\frac{\partial \Phi}{\partial x^u} \tag{48}$$

This equation although for massless $5D$ particles have many resemblance with its similar for massive $5D$ particles as pointed out by Ponce De Leon and can easily be obtained combining eqs 15 and 18 of [20](see pg 1343 in [2]).

According to the Table of Elementary Particles already presented in this work(two times and we think its enough) Photons or Gravitons have a $4D$ rest-mass $m_0 = 0$ corresponding to a $5D$ Null-Like Spacetime Geometry or in hence a stationary particle a particle that is at the rest in the $5D$ Spacetime,a particle with a $m_0 =^{+}_{-} \Phi\frac{dy}{d\lambda} \dashrightarrow m_0 = 0 \dashrightarrow \frac{dy}{d\lambda} = 0 \dashrightarrow \frac{dy}{ds} = 0$.

But of course we can have a $5D$ rest-mass $M_5 = 0$ giving a non-null $4D$ rest-mass $m_0 \neq 0$ even with a Warp Field $\Phi = 1$ if $\frac{dy}{d\lambda} \neq 0$ according to the following equations although we believe that non-null rest-masses m_0 in $4D$ comes from non-null rest-masses M_5 in $5D$(see sections 8 and 9 about particle Z in [20]) :

$$m_0 =^{+}_{-} \frac{dy}{d\lambda} \tag{49}$$

$$d\lambda = \frac{1}{m_0}ds \tag{50}$$

$$\frac{dy}{ds} = 1 \tag{51}$$

$$\frac{1}{m_0}\frac{dm_0}{ds} =^{-}_{+} \frac{1}{2}\frac{\partial g_{uv}}{\partial y}u^u u^v \tag{52}$$

Note that if the Warp Field $\Phi = 1$ with $\frac{dy}{ds} = 1$ and $dS^2 = 0$ the equation of the $5D$ Extra Force for a massless particle in $5D$ $M_5 = 0$ becomes equivalent to [9] eq 24 proving that the Ponce De Leon equations are equivalent to the Mashoon-Wesson-Liu ones.

$$\frac{1}{m_0}\frac{dm_0}{ds} =^{-}_{+} \frac{1}{2}\frac{\partial g_{uv}}{\partial y}u^u u^v \tag{53}$$

[25] note that like for its analogous $5D$ massive counterpart the Warp Field function only of the Extra Coordinate makes the second term vanish(examine eqs 50 and 52 in [1])

[26] compare this equation with [9] eq 24 and look for the + signal in this equation while [9] eq 24 only have the - sign

4 Dimensional Reduction from $5D$ to $4D$ according to Basini-Capozziello Ponce De Leon, Mashoon-Wesson-Liu and Overduin-Wesson.Possible detection of Extra Dimensions in Strong Gravitational Fields and Gravitational Bending Of Light in Extra Dimensions according to Kar-Sinha and perhaps a possible connection with the Pioneer Anomaly and the Bertolami-Paramos Definitions

The most important thing to keep in mind when we study models of BraneWorlds or Extra Dimensions is to explain why we cannot "see" directly the presence of the Extra Dimension although we can "feel" its effects in the $4D$ everyday Physics.We avoid here the models with compactification or "curling-up" of the Extra Dimension because these models don't explain why we have $3+1$ Large Dimensions while the remaining ones are small and "unseen" Extra Dimensions and also these models dont explain what generates the "Compactification" or "Curling" mechanism in the first place.Also some of these models develop "Unphysical" features.An excellent account of the difference between compactified and uncompactified models of Extra Dimensions is given by [21](see pgs 2 to 31).We prefer to adopt the fact that like the $3+1$ ordinary Large Spacetime Dimensions the Extra Dimensions are Large and uncompactified but due to a Dimensional Reduction from $5D$ to $4D$ we cannot "see" these Extra Dimensions although we can "feel" some of its effects. (see abs and pg 123 of [1] when Basini-Capozziello mentions the fact that we cannot perceive Time as the fourth Dimension and hence we cannot perceive the Spacelike Nature of the $5D$ Extra Dimension).(see also pg 1424 and pg 1434 begining of section 4 in [20]).(see also abs pg 2218 and 2219 of [5].Note the comment on Dimensional reduction and a $4D$ Spacetime embedded into a larger $5D$ one).We will now demonstrate how the Dimensional Reduction from $5D$ to $4D$ work and why in ordinary conditions we cannot "see" the $5D$ Extra Dimensions but we can "feel" some of its effects.Also we will see that changing the Spacetime Geometry and the shape of the Warp Field the $5D$ Extra Dimension will become visible.(Dimensional Reductions from $5D$ to $4D$ appears also in pg 2040 of [4] and pg 4 eq 3 in [34]). We know that in ordinary $3+1$ Spacetime the curvature of the Einstein Tensor is neglectable and Spacetime can be considered as Minkowskian or flat where Special Relativity holds.A Minkowskian $5D$ Spacetime with a Warp Field can be given by(see eq 325 in [20]):

$$dS^2 = dt^2 - dX^2 - \Phi^2 dy^2 \tag{54}$$

The Warp Field considered here have small values between 0 and 1 nearly close to 0 and we recover the ordinary Special Relativity Ansatz.More on this in the section of the Pioneer Anomaly.A Minkowskian $5D$ Spacetime with no Warp Field at all would be given by(see eq 326 in [20]):

$$dS^2 = dt^2 - dX^2 - dy^2 \tag{55}$$

The Ricci Tensors and Scalars for the Basini-Capozziello $5D$ Spacetime Fomalism and Ansatz given by $dS^2 = g_{\mu\nu}dx^\mu dx^\nu - \Phi^2 dy^2$ are shown below:(see pg 128 eq 58 in [1],pg 2230 eq 44 in [5] and pg 1442 eqs 111 to 115 in [20])

$$^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{56}$$

$$^5R = R - \frac{\Phi_{,a;b}}{\Phi}g^{\alpha\beta} - \frac{1}{2\Phi^2}g^{\alpha\beta}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{57}$$

$$
{}^5R = R - \frac{{}^4\Box\Phi}{\Phi} - \frac{1}{2\Phi^2} g^{\alpha\beta}\left(\frac{\Phi_{,4} g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2}\right) \tag{58}
$$

For a $5D$ Spacetime Metric without Warp Field defined as $dS^2 = g_{\mu\nu}dx^\mu dx^\nu - dy^2$ the Ricci Tensor and Scalar would then be(see eqs 330 and 331 pg 1477 in [20]):

$$
{}^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\left(-g_{\alpha\beta,44} + \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2}\right) \tag{59}
$$

$$
{}^5R = R - \frac{1}{2} g^{\alpha\beta}\left(-g_{\alpha\beta,44} + \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2}\right) \tag{60}
$$

But remember that a Minkowskian $5D$ Spacetime in which Special Relativity holds have all the $3+1$ Spacetime Metric Tensor Components defined by $g_{\mu\nu} = (+1, -1, -1, -1)$(see pg 1476 and 1477 in [20]) and the derivatives of the Metric Tensor vanishes and hence we are left with the following results(see eqs 332 and 333 in [20])

$$
{}^5R_{\alpha\beta} = R_{\alpha\beta} \tag{61}
$$

$$
{}^5R = R \tag{62}
$$

From the results above in a flat Minkowsky $5D$ Spacetime the Ricci Tensor in $5D$ is equal to its counterpart in $4D$ and since the Spacetime is flat then both are equal to zero.Then its impossible to tell if we live in a $4D$ Spacetime or in a larger $5D$ Extra Dimensional one.(see pg 1477 after eq 333 in [20]).If the Geometry of a flat $5D$ Extra Dimensional Spacetime is equivalent to the Geometry of a $3+1$ Spacetime we cannot distinguish if we live in a $4D$ or a $5D$ Universe.This is one of the most important things in the Dimensional Reduction from $5D$ to $4D$ as proposed by Basini-Capozziello.The $5D$ Extra Dimension is Large and Uncompactified but the physical reality we see is a Dimensional Reduction from $5D$ to $4D$ because we live in a nearly flat Minkowsky Spacetime where Special Relativity holds and the $5D$ Ricci Tensor is equal to the $3+1$ counterpart.No Compactification mechanisms needed.The $5D$ Extra Dimension have a real physical meaning(see pg 2226 in [5] and pg 127 in [1]).(see also pg 2230 in [5] the part of the reduction of the Ricci Tensor from $5D$ to $4D$ eqs 44 and 45.if the Warp Field $\Phi = 1$ both $5D$ and $4D$ Ricci Tensors from eq 45 are equal.the same can be seen in pg 128 to 129 eqs 58 to 59 in [1].see also pg 1442 eqs 115 to 116 in [20]). If in a flat $5D$ Minkowsky Spacetime we cannot "see" the Extra Dimension then we have three choices in order to tell if we live in a $5D$ Extra Dimensional Spacetime or a $3+1$ Ordinary Dimensional one.The choices are:

- Making the Warp Field $\Phi \neq 1$ in order to generate a difference between the $5D$ Extra Dimensional Ricci Tensor and the $3+1$ Spacetime counterpart according to eq 45 in [5],eq 59 in [1] and eq 116 in [20].This difference can tell the difference between a $5D$ Universe and a $3+1$ one.(see pg 1477 in [20])

- Making the $3+1$ Spacetime Metric Tensor components be a function of the $5D$ Extra Dimension in order to do not vanish the derivatives of the Metric Tensor with respect to the Extra Dimension generating a difference between the $5D$ Ricci Tensor and the $3+1$ counterpart according to eqs 330 and 331 pg 1477 in [20]. A Strong Gravitational Field of a Large Maartens-Clarkson $5D$ Schwarzschild Black String have the Spacetime Metric Tensor Components defined in function of the $4D$ rest-mass M but the $4D$ rest-mass is function of the $5D$ Extra Dimensional Spacetime Geometry according to eq 20 in [2].

- Making both conditions above hold true

We will examine all of the items above in this section. We live in a region of Spacetime where the Warp Field $\Phi = 1$ then we cannot see the $5D$ Extra Dimension.Or we can live in a region of Spacetime where the Warp Field $\Phi = 0$ and this cancels out the term $\Phi^2 dy^2$ in the $5D$ Spacetime Ansatz making the Extra Dimension invisible.Or perhaps we can live in a region of spacetime where $0 \leq \Phi \leq 1$ but near to 0 or 1 so its very difficult to detect the presence of the $5D$ Extra Dimension although we can "feel" some of its effects.More on this in the Pioneer Anomaly section.Considering now a Warp Field $\Phi \neq 1$ the Minkowsky $5D$ Spacetime Ansatz would still have the terms of the $3+1$ Spacetime Metric Tensor given by $g_{\mu\nu} = (+1, -1, -1, -1)$.Hence the $5D$ Spacetime Ansatz would then be:

$$dS^2 = dt^2 - dX^2 - \Phi^2 dy^2 \tag{63}$$

The derivatives of the $3+1$ components of the Spacetime Metric Tensor vanishes but note that the $5D$ component do not vanish.The Ricci Tensor and Scalar would be given by the following expressions(see eqs 335,336 and 337 in [20]):

$$^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} \tag{64}$$

$$^5R = R - \frac{\Phi_{,a;b}}{\Phi} g^{\alpha\beta} \tag{65}$$

$$^5R = R - \frac{^4\Box\Phi}{\Phi} \tag{66}$$

Note that now the scenario is different:while with the Warp Field $\Phi = 1$ the $5D$ Ricci Tensor is equal to its $3+1$ counterpart and we cannot tell if we live in a $5D$ or in a $3+1$ Universe but when the Warp Field $\Phi \neq 1$ there exists a difference between the Ricci Tensor in $5D$ and the $3+1$ one.The Geometrical Properties of Spacetime of the $5D$ Spacetime are now different than the $3+1$ equivalent one and this makes the $5D$ Extra Dimension visible.More on this in the Pioneer Anomaly Section.(see also pg 1478 after eq 337 in [20])

According to Basini-Capozziello the Warp Field can be decomposed in two parts:one in $3+1$ ordinary Spacetime and another in the $5D$ Extra Dimension given by the following equation:([1] eq 76,[5] eq 70 and [20] eq 132 and 338)

$$\Phi = \phi(t, x)\chi(y) \tag{67}$$

Note that when we compute the covariant derivative of the Warp Field with respect to the $3+1$ Spacetime the terms of the $5D$ Extra Dimension are cancelled out and we are left with derivatives of the $3+1$ components of the Warp Field(see eq 339 in [20])

$$\Phi_{,a;b} = \chi(y)[\phi_{,a;b}] = \chi(y)[(\phi_\alpha)_\beta] - \Gamma^K_{\beta\alpha}\phi_K] \dashrightarrow \frac{\Phi_{,a;b}}{\Phi} = \frac{\chi(y)[\phi_{,a;b}]}{\phi(t,x)\chi(y)} = \frac{[\phi_{,a;b}]}{\phi(t,x)} = \frac{[(\phi_\alpha)_\beta - \Gamma^K_{\beta\alpha}\phi_K]}{\phi(t,x)} \tag{68}$$

The result shown below is very important.It demonstrates that only the $3+1$ component of the Warp Field $\phi(t, x)$ fortunately the component that lies in "our side of the wall" and its derivatives with(again fortunately) respect to our $3+1$ Spacetime coordinates can make the $5D$ Ricci Tensor be different than its $3+1$ counterpart and since we are considering in this case a flat Minkowsky Spacetime the $4D$ Ricci Tensor reduces to zero and this means to say that ${}^5R_{\alpha\beta} = -\frac{\phi_{,a;b}}{\phi}$ or better ${}^5R_{\alpha\beta} = -\frac{[(\phi_\alpha)_\beta - \Gamma^K_{\beta\alpha}\phi_K]}{\phi(t,x)}$

$$ {}^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{\phi_{,a;b}}{\phi} \tag{69} $$

$$ {}^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{[(\phi_\alpha)_\beta - \Gamma^K_{\beta\alpha}\phi_K]}{\phi(t,x)} \tag{70} $$

Note that if the $4D$ Ricci Tensor vanishes due to a flat Minkowsky Spacetime and we are left with derivatives of the $3+1$ Spacetime components of the Warp Field with respect to (again fortunately for the second time) $3+1$ Spacetime coordinates and we are left with a result in which the $5D$ Ricci Tensor and our capability to detect the existence of the $5D$ Extra Dimension depends on the shape of the $3+1$ component of the Warp Field.Note that the $3+1$ component can be defined like a Warp Field coupled to Gravity(in [23] pg 6 Bertolami-Paramos mentions a Warp Field coupled to Gravity in a resemblance of what Basini-Capozziello mentions in [1] pg 119 and [5] pg 2235).If we can detect the derivatives of the Warp Field we can detect the existence of the $5D$.Note that this is similar the results of the Pioneer Anomaly outside Solar System where Gravity of the Sun vanishes and we are left with $3+1$ components of the Warp Field and also its derivatives.More on this on the Pioneer Anomaly section.

The other way to make the $5D$ Extra Dimension visible is to make the derivatives of the $3+1$ Spacetime Metric Tensor components $g_{\mu\nu} = (g_{00}, g_{11}, g_{22}, g_{33})$non-null with respect to the $5D$ Extra Dimension.

$$ dS^2 = dt^2 - g_{\mu\mu}d(X^\mu)^2 - \Phi^2 dy^2 \tag{71} $$

For our special case of diagonalized metric:

$$ dS^2 = dt^2 - g_{\mu\mu}d(X^\mu)^2 - \Phi^2 dy^2 \tag{72} $$

Considering the $3+1$ Spacetime Metric Tensor Components g_{00} and g_{11}(see eqs 353 and 354 in [20]).

$$ {}^5R_{00} = R_{00} - \frac{\Phi_{,0;0}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4}g_{00,4}}{\Phi} - g_{00,44} + \frac{g^{00}g_{00,4}g_{00,4}}{2}) \tag{73} $$

$$ {}^5R_{11} = R_{11} - \frac{\Phi_{,1;1}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4}g_{11,4}}{\Phi} - g_{11,44} + \frac{g^{11}g_{11,4}g_{11,4}}{2}) \tag{74} $$

Now we can see that if the derivatives of g_{00} and g_{11} do not vanish with respect to the Extra Coordinate then the terms $\frac{\Phi_{,0;0}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4}g_{00,4}}{\Phi} - g_{00,44} + \frac{g^{00}g_{00,4}g_{00,4}}{2})$ and $\frac{\Phi_{,1;1}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4}g_{11,4}}{\Phi} - g_{11,44} + \frac{g^{11}g_{11,4}g_{11,4}}{2})$ will generates a difference between the $5D$ Ricci Tensor and its $3+1$ Ordinary Spapcetime Dimensional counterpart.Remember also that g_{00} and g_{11} can be defined as the Spacetime Metric Tensor Components of the Maartens-Clarkson $5D$ Schwarzschild Black String centerd on a large Black Hole for example in which M is the $4D$ rest-mass of the Black Hole but M can be defined in function of the $5D$ Extra Dimensional rest-mass M_5 and also defined in function of the $5D$ Spacetime Geometry according to Ponce De Leon eq 20 in [2].This can make the $5D$ Extra Dimension becomes visible. Writing the Maartens-Clarkson $5D$ Schwarzschild Cosmic Black String as follows:([7] eq 1,[20] eq 380):

$$ dS^2 = [(1 - \frac{2GM}{R})dt^2 - \frac{dR^2}{(1-\frac{2GM}{R})} - R^2 d\eta^2] - \Phi dy^2 \tag{75} $$

Where the Spacetime Metric Tensor Components of the Black String are given by:$g_{00} = (1 - \frac{2GM}{R})$ and $g_{11} = -(1 - \frac{2GM}{R})^{-1}$.The derivatives with respect to the Extra Coordinate are then[27][28]:

$$\frac{\partial g_{00}}{\partial y} = \frac{\partial(1 - \frac{2GM}{R})}{\partial y} = -2G\frac{\partial \frac{M}{R}}{\partial y} = -2G[\frac{\partial M}{\partial y} \times R^{-1} + \frac{\partial R^{-1}}{\partial y} \times M] \tag{76}$$

We know that the $4D$ rest-mass M of the Maartens-Clarkson $5D$ Schwarzschild Cosmic Black String can be defined in function of the Ponce De Leon $5D$ rest-mass M_5 eq 20 in [2].The final result would then be:

$$\frac{\partial g_{00}}{\partial y} = -2G[\frac{1}{R}\frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3}[\Phi^2\frac{dy}{ds}\frac{\partial\frac{dy}{ds}}{\partial y} + (\frac{dy}{ds})^2\Phi\frac{\partial\Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y})\frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] \tag{77}$$

$$\frac{\partial g_{11}}{\partial y} = \frac{\frac{\partial g_{00}}{\partial y}}{g_{00}^2} = \frac{-2G}{g_{00}^2}[\frac{1}{R}\frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3}[\Phi^2\frac{dy}{ds}\frac{\partial\frac{dy}{ds}}{\partial y} + (\frac{dy}{ds})^2\Phi\frac{\partial\Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y})\frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] \tag{78}$$

Note that in a Strong Gravitational Field these derivatives will have high values and this will make the $5D$ Ricci Tensor be highly different than its $3+1$ counterpart making the $5D$ Extra Dimension be visible but far away from the center of the Black String $\frac{M_5}{R} \simeq 0$ and the derivatives will vanish however the term corresponding to the Warp Field will remain as shown below:

$$^5R_{00} = R_{00} - \frac{\Phi_{,0;0}}{\Phi} \tag{79}$$

$$^5R_{11} = R_{11} - \frac{\Phi_{,1;1}}{\Phi} \tag{80}$$

Then in a weak or null Gravitational Field is the Warp Field that can make the $5D$ Ricci Tensor be different than its $3+1$ counterpart and can tell if we live in a $5D$ Extra Dimensional Spacetime or in a ordinary $3+1$ one. This is similar to the Pioneer Anomaly situation:outside Solar System Gravity vanishes and the terms that can account for the Pioneer Anomaly are functions only of the Warp Field.Remember that at faraway distances from Gravitational Field the Spacetime is flat or Minkowskian and the $3+1$ Ricci Tensor is zero or nearly zero.Then in the case of the Pioneers we could rewrite the two equations above as follows:

$$^5R_{00} = -\frac{\Phi_{,0;0}}{\Phi} \tag{81}$$

$$^5R_{11} = -\frac{\Phi_{,1;1}}{\Phi} \tag{82}$$

We already know that when computing derivatives of the Warp Field with respect to $3+1$ Coordinates the $5D$ Extra Dimensional terms are cancelled out and we will get these results:

$$^5R_{00} = -\frac{[(\phi_0)_0 - \Gamma^K_{00}\phi_K]}{\phi(t,x)} \tag{83}$$

[27]see also Pioneer Anomaly section for derivatives of the Spacetime Metric Tensor components of the Maartens-Clarkson $5D$ Schwarzschild Cosmic Black String

[28]only time and radial components are considered here.see also Pioneer Anomaly Section

$$^5R_{11} = -\frac{[(\phi_1)_1 - \Gamma^K_{11}\phi_K]}{\phi(t,x)} \tag{84}$$

Again remarkably similar to the Pioneer Anomaly situation in which we are left with derivatives of the $3+1$ component of the Warp Field.Then if outside Solar System the $3+1$ Ricci Tensor is null[29][30] and we detected the Pioneer Anomaly the results above are perhaps the $5D$ Ricci Tensors behind the Pioneer Anomaly and the first proof that the $5D$ Extra Dimension really exists.

Writing the Ricci Tensors with the derivatives of the $3+1$ Spacetime Metric Tensor Components of the Warp Field explicitly written we have:

$$^5R_{00} = -\frac{[\frac{\partial^2\phi(t,x)}{\partial t^2} - \Gamma^K_{00}\frac{\partial\phi(t,x)}{\partial x^K}]}{\phi(t,x)} \tag{85}$$

$$^5R_{11} = -\frac{[\frac{\partial^2\phi(t,x)}{\partial R^2} - \Gamma^K_{11}\frac{\partial\phi(t,x)}{\partial x^K}]}{\phi(t,x)} \tag{86}$$

$$\Gamma^K_{00}\frac{\partial\phi(t,x)}{\partial x^K} = \Gamma^0_{00}\frac{\partial\phi(t,x)}{\partial t} + \Gamma^1_{00}\frac{\partial\phi(t,x)}{\partial R} \tag{87}$$

$$\Gamma^K_{11}\frac{\partial\phi(t,x)}{\partial x^K} = \Gamma^0_{11}\frac{\partial\phi(t,x)}{\partial t} + \Gamma^1_{11}\frac{\partial\phi(t,x)}{\partial R} \tag{88}$$

We still don't know the shape of the Warp Field but remember that the $3+1$ component of the Warp Field can be coupled to Gravity as defined by [23] pg 6 from Bertolami-Paramos and Basini-Capozziello in [1] pg 119 and [5] pg 2235.

Considering only valid Christoffel Symbols we have[31]:

$$\Gamma^K_{00}\frac{\partial\phi(t,x)}{\partial x^K} = \Gamma^0_{00}\frac{\partial\phi(t,x)}{\partial t} = \frac{1}{2}\frac{1}{g_{00}}\frac{\partial g_{00}}{\partial t}\frac{\partial\phi(t,x)}{\partial t} = \frac{1}{2}\frac{1}{g_{00}}\frac{\partial g_{00}}{\partial y}\frac{\partial\phi(t,x)}{\partial t}\frac{\partial y}{\partial t} \tag{89}$$

$$\Gamma^K_{11}\frac{\partial\phi(t,x)}{\partial x^K} = \Gamma^1_{11}\frac{\partial\phi(t,x)}{\partial R} = \frac{1}{2}\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial R}\frac{\partial\phi(t,x)}{\partial R} = \frac{1}{2}\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial y}\frac{\partial\phi(t,x)}{\partial R}\frac{\partial y}{\partial R} \tag{90}$$

These expressions are valid for Strong or Weak Gravitational Fields which means to say inside or outside Solar System.But we are considering here the case of the Pioneers outside Solar System where Gravitational Force vanishes and the derivatives of the Spacetime Metric Tensor Components of the Maartens-Clarkson $5D$ Schwarzschild Cosmic Black String vanishes due to the term $\frac{M_5}{R}$[32] and the final expression for the $5D$ Ricci Tensors can be given by:

$$^5R_{00} = -\frac{[\frac{\partial^2\phi(t,x)}{\partial t^2}]}{\phi(t,x)} \tag{91}$$

$$^5R_{11} = -\frac{[\frac{\partial^2\phi(t,x)}{\partial R^2}]}{\phi(t,x)} \tag{92}$$

[29] we consider the Sun a $5D$ Maartens-Clarkson Schwarzschild Black String but at large distances outside Solar System makes the $3+1$ component be a Minkowsky Spacetime

[30] only time and radial components are considered here.see also Pioneer Anomaly Section

[31] diagonalized metrics

[32] outside Solar System making $g_{00} = 1$ and $g_{11} = 1$

The $5D$ Ricci Scalar would be given by:

$$^5R =^5 R_{00} +^5 R_{11} = -\frac{[\frac{\partial^2\phi(t,x)}{\partial t^2}]}{\phi(t,x)} + -\frac{[\frac{\partial^2\phi(t,x)}{\partial R^2}]}{\phi(t,x)} = -\frac{1}{\phi(t,x)}{}^4\Box\phi(t,x)^2 \tag{93}$$

Note the remarkable resemblances with this expression for the Yukawa Potential of Bertolami-Paramos that will be introduced in the Pioneer Anomaly Section

$$\frac{1}{\phi(t,x)}{}^4\Box\phi(t,x)^2 + \frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y) = \frac{G}{R}M_5\frac{1}{[\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}]^3}(\frac{dy}{ds})^2 \tag{94}$$

We are dealing here with Ricci Tensors and Scalars and outside Solar System where Gravitational Force vanishes and where the Pioneers are located right now.Then only the Warp Field components remains so the expression can be written as follows[33]

$$\frac{1}{\phi(t,x)}{}^4\Box\phi(t,x)^2 + \frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y) = 0 \tag{95}$$

Remarkably we can extract the Ricci Scalar from the $5D$ to $4D$ Dimensional Reduction of Basini-Capozziello from the Bertolami-Paramos Yukawa Potential.Then outside Solar System both are similar in many ways and perhaps this is not a coincidence.Mother Nature is trying to tell us something.We will see in the Pioneer Anomaly Section that the Pioneer Anomaly is the final proof and the first confirmation that the $5D$ Extra Force predicted years ago by Mashoon-Wesson-Liu in [9] really exists.More of this in the Pioneer Anomaly Section.

If the Warp Field Coupled to Gravity is defined by Basini-Capozziello and Bertolami-Paramos even in $4D$ do not vanish and neither its partial derivatives then the equation can be written as([23] pg 6 from Bertolami-Paramos and Basini-Capozziello in [1] pg 119 and [5] pg 2235):

$$\frac{1}{\phi(t,x)}{}^4\Box\phi(t,x)^2 + \frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y) =^5 R - \frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y) = 0 \tag{96}$$

$$^5R = \frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y) \tag{97}$$

And this can be regarded as a final proof that the $5D$ Extra Dimension really exists

Considering now the case of the $5D$ Spacetime Metric with no Warp Field at all $\Phi = 1$ the difference between the $5D$ and the $4D$ Ricci Tensors will depend on the derivatives of the Spacetime Metric Tensor Components with respect to the $5D$ Extra Dimension that will vanish far away from the center of the Maartens Clarkson $5D$ Schwarzschild Cosmic Black String making the $5D$ be invisible but in the regions of intense Gravitational Field the $5D$ Ricci Tensor will be different than its $4D$ counterpart making the $5D$ Extra Dimension becomes visible

$$^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}(-g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{98}$$

$$^5R = R - \frac{1}{2}g^{\alpha\beta}(-g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{99}$$

[33] $\frac{M_5}{R} \simeq 0$

Writing the Ricci Tensor for the time and radial components we have(see eqs 343 and 344 pg 1479 in [20]):

$$^5R_{00} = R_{00} - \frac{1}{2}(-g_{00,44} + \frac{g^{00}g_{00,4}g_{00,4}}{2}) \tag{100}$$

$$^5R_{11} = R_{11} - \frac{1}{2}(-g_{11,44} + \frac{g^{11}g_{11,4}g_{11,4}}{2}) \tag{101}$$

From the Pioneer Anomaly Section we know that the derivatives of the Spacetime Metric Tensor for the $5D$ Maartens-Clarskon Schwarzschild Cosmic Black String are given by:

$$\frac{\partial g_{00}}{\partial y} = -2G[\frac{1}{R}\frac{M_5}{[\sqrt{1-(\frac{dy}{ds})^2}]^3}\frac{dy}{ds}\frac{\partial(\frac{dy}{ds})}{\partial y} - \frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}}\frac{1}{R^2}\frac{\partial R}{\partial y}] \tag{102}$$

$$\frac{\partial g_{11}}{\partial y} = \frac{\frac{\partial g_{00}}{\partial y}}{g_{00}^2} = -\frac{1}{g_{00}^2}2G[\frac{1}{R}\frac{M_5}{[\sqrt{1-(\frac{dy}{ds})^2}]^3}\frac{dy}{ds}\frac{\partial(\frac{dy}{ds})}{\partial y} - \frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}}\frac{1}{R^2}\frac{\partial R}{\partial y}] \tag{103}$$

Note that far away from the Black String the ratio $\frac{M_5}{R} \simeq 0$ and the derivatives of the Spacetime Metric Tensor Components will vanish making the $5D$ Ricci Tensor equal to its $3+1$ counterpart and the $5D$ Extra Dimension will become invisible.But in the neighborhoods of the Black String center Gravity becomes so high that the extra terms will make the $5D$ Ricci Tensor be different than its $3+1$ counterpart. Another way to measure the presence of the $5D$ Extra Dimension is to measure how the Extra Dimension affects the Gravitational Bending of Light in the vicinity of the Black String according to Kar-Sinha.In one of our works([20] pg 1495 section 8) we proposed the use of the International Space Station ISS[34] to measure the Kar-Sinha Gravitational Bending of Light of the Sun to find out if it can be affected by the presence of the $5D$ Extra Dimension(see also pg 1467 before eq 290 in [20])[35].[36] While the Sun have a "weak" Gravitational Field a Black String is a Black Hole in $5D$ and in the vicinity of the Black String perhaps the Gravitational Bending Of Light affected by the presence of the $5D$ Extra Dimension according to Kar-Sinha would be better noticeable. Writing the Kar-Sinha Gravitational Bending Of Light affected by the presence of the $5D$ Extra Dimension in the neighborhoods of a Black String with a non-null Warp Field as follows(see pg 1467 eq 288 to 291 and pg 1468 eq 295 to 296 in [20])(see also pg 1781 eq 18 in [3])[37][38]:

$$\triangle\omega = \frac{2GM}{R}(2 + [\Phi\frac{dy}{dt}]^2) \tag{104}$$

[34] more on General Relativity and ISS in [6],[8],[16],[17] and [18]

[35] we contacted ESA European Space Agency to see if experiments like these are feasible but they told us that while our research is very interesting the ISS International Space Station have more priority tasks such as the monitorization of the Antarctica Polar Ice Cap Meltdown that could affect Earth Climate .We agree with them and accepted their decision.more on this in this section

[36] This is the reason why we shifted ourselves to large Black Holes(Black Strings)

[37] equations written without Warp Factors and with the Gravitational Constant

[38] see also eqs 156 to 158 pg 70 section 8.7 in [21].see also in the same reference the comment on the velocity along the $5D$ Extra Dimension in pg 71 after eq 159 $\frac{d\psi}{dt}$ similar to our $\frac{dy}{dt}$.see also between page 70 and 71 the comment that the shift is physically measurable.we will examine photon paths in the $5D$ Maartens-Clarkson Schwarzschild Cosmic Black String in this section also but we will use the Ponce De Leon point of view of pg 1343 after eq 30 in [2].look to the Ponce De Leon comment of genuine manifestation of the $5D$ Extra Dimension before section 4

$$\triangle\omega = \frac{2GM}{R}(2 + [\phi(t,x)\chi(y)\frac{dy}{dt}]^2) \quad (105)$$

$$\triangle\omega = \frac{2G}{R}\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}(2 + [\Phi\frac{dy}{dt}]^2) \quad (106)$$

$$\triangle\omega = \frac{2G}{R}\frac{M_5}{\sqrt{1-\phi(t,x)^2\chi(y)^2(\frac{dy}{ds})^2}}(2 + [\phi(t,x)\chi(y)\frac{dy}{dt}]^2) \quad (107)$$

The same expression for a null Warp Field would be given by:

$$\triangle\omega = \frac{2GM}{R}(2 + [\frac{dy}{dt}]^2) \quad (108)$$

$$\triangle\omega = \frac{2G}{R}\frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}}(2 + [\frac{dy}{dt}]^2) \quad (109)$$

We know that the Warp Field must have values between 0 and 1 so the shift in the Gravitational Bending Of Light must be very small making the value of the expression in $5D$ be close to its $4D$ counterpart.The presence of the Gravitational Constant $G = 6,67 \times 10^{-11}\frac{Newton\ timesm^2}{kg^2}$ would make the things even worst.This is the reason why we need a Black String of large rest mass M or M_5 to make the shift noticeable.Perhaps in the Sun we would never be able to measure the shift.[39] Note also the comment on [21] pg 71 that the derivative $\frac{d\psi}{dt}$ analogous to our $\frac{dy}{dt}$ is null for photons and we know from Ponce De Leon that the $5D$ Spacetime Metric $dS^5 = ds^2 - \Phi^2 dy^2$ is null for photons making $ds^2 = \Phi^2 dy^2$ $M_5 = 0$ and $m_0 = 0$.Remember that in $4D$ SR $ds^2 = 0$ for photons and $\frac{dy}{ds} = 0$ making the shift in $5D$ be equal to its $4D$ counterpart.Kar-Sinha mentions in pg 1783 [3] the fact that if the photon propagates in $5D$ the value of $\frac{dy}{ds} < 2,8 \times 10^{-4}$ and the shift $\triangle\omega$ affected by the $5D$ Extra Dimension must lie between the error margins of the observed values of pgs 39 to 41 in [37][40].The Gravitational Bending can be observed for other particles with a non-null M_5 and a non-null m_0 and perhaps the study of the motion of high-speed relativistic particles from accretion disks of large Black Holes can tell the difference between the $5D$ $\triangle\omega$ and its $4D$ counterpart.For a non-null $\frac{dy}{ds}$ particle the Gravitational Bending formulas could be given by:

$$\triangle\omega = \frac{2GM}{R}(2 + [\Phi\frac{dy}{ds}\frac{ds}{dt}]^2) \quad (110)$$

$$\triangle\omega = \frac{2GM}{R}(2 + [\phi(t,x)\chi(y)\frac{dy}{ds}\frac{ds}{dt}]^2) \quad (111)$$

$$\triangle\omega = \frac{2G}{R}\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}(2 + [\Phi\frac{dy}{ds}\frac{ds}{dt}]^2) \quad (112)$$

$$\triangle\omega = \frac{2G}{R}\frac{M_5}{\sqrt{1-\phi(t,x)^2\chi(y)^2(\frac{dy}{ds})^2}}(2 + [\phi(t,x)\chi(y)\frac{dy}{ds}\frac{ds}{dt}]^2) \quad (113)$$

[39] this is is fact the motive why we accepted the ESA European Space Agency decision to do not use the ISS International Space Station for Gravitational Bending of Light experiments

[40] this reference contains one of the best explanations for the Gravitational Bending Of Light Geometry and describes even the 30 percent margin of error in the 1919 measurements

5 The Pioneer Anomaly:Explanation using the $5D$ Extra-Force of Mashoon-Wesson-Liu and Ponce De Leon equations and the Basini-Capozziello Warp Field Φ coupled to the Definitions of Bertolami-Paramos and the Maartens-Clarkson $5D$ Schwarzschild Black String centered on the Sun

The Pioneer spacecrafts were the first Man-Made objects to reach the neighborhoods of Jupiter and Saturn and also the first ones to leave our Solar System to enter Interstellar Space.Pioneer 10 was launched to Jupiter on March 2 1972(see pg 1 of [25],see pg 2 of [38]),and Pioneer 11 was launched to Saturn on April 5 1973(see pg 1 of [25]).The last transmission received from the Pioneers was on January 23 2003(see pg 2 of [24]).For a detailed account on the Pioneer Mission History see [24],[25],[26],[36] and [38]. The Pioneer Anomaly was discovered in the $1980's$.Excluding all the known gravitational and non-gravitational forces an anomalous sunward acceleration exists with a value of $a_p = (8.74 \pm 1.33) \times 10^{-8}\ cm/s^2$(see pg 1 of [23],pg 1 of [24],pg 3 of [24],abstract of [25],eq 2 pg 4 of [26],pg 4 of [38],pg 1 of [27] and pg 1 and 2 of [36]). This acceleration is one of the greatest mysteries of modern science.At the present time there are no acceptable explanations for this Anomaly.(see abstract of [27],pg 1 of [36], abstract of [26],pg 4 of [26],abstract of [25],pg 4 of [25],abstract of [24],pg 2 of [24], and abstract of [38]).In this work we avoid the conventional explanations of gas leak,oil leak or plutonium reactor leak and we adopt the point of view of [27] partially supported by [23]. We agree entirely with the point of view of Bertolami-Paramos that advocates the fact that the Pioneer Anomaly requires major modifications in General Relativity and Einstein Field Equations and using the Mashoon-Wesson-Liu and Ponce De Leon $5D$ Extra Force Formalism in addiction to Basini-Capozziello formalism we will demonstrate that the force behind the Pioneer Anomaly can be correctly and entirely explained by the Definitions of Bertolami-Paramos as a Geometrical Property of Spacetime and is being generated by the Sun.The Pioneer Anomaly is the first experimental evidence of the Fifth Force predicted by Mashoon-Wesson-Liu years ago in [9].The Anomaly was powerful enough to shift the Pioneers from its original position by more than $400.000km$ until its final loss in 2003[41].Writing now the Definitions of Bertolami-Paramos for the correct explanation for the Pioneer Anomaly we should expect for:

- 1)-The Anomaly is Centered on the Sun(see pg 6 of [27] before eq 7)
- 2)-Its a Yukawa-like Potential Function as defined by eq 7 of [27].However we adopt the fact that the coupling strength is zero.Bertolami-Paramos mentions a coupling strength close to zero.
- 3)-Bertolami-Paramos defines in pg 6 and 7 of [23] a Potential Function $V \simeq -\frac{1}{\Phi^a}$.We will demonstrate that the solution of the equation of motion for the Warp Field Φ see eq 13 in [23] can also be proportional to $-\frac{1}{\Phi^a}$ specially for the regions outside our Solar System where the Pioneers are moving now.

The Bertolami-Paramos Definitions are in complete agreement with Mashoon-Wesson-Liu [9] pg 565 and $\beta = \frac{1}{M}\frac{dM}{ds}$ in [9] pg 566 and the Pioneer de-acceleration is exactly the one mentioned by Mashoon-Wesson-Liu in [9] pg 565.(see [9] abstract and pgs 556,562 look to eq 24,pg 563 look to eq 31 and the comment below this equation,pg 565 eq 38,39 and the comments on the de-acceleration,pg 566 definition of β,pg 567 look to the comment of a small force but detectable).Mashoon-Wesson-Liu mentions the fact that the $5D$ Extra

[41] interview of Orfeu Bertolami to the Portuguese newspaper "O Correio Da Manha" January 27 2007.we can provide for those interested the original Portuguese text of the interview in PDF Acrobat Reader with an English translation.

Force can be detected in $4D$ but is of Cosmological Nature and causes a local-de-acceleration.Assuming instead that the $5D$ Extra Force is generated by the Sun according to Bertolami-Paramos the same effect of de-acceleration appears as a Geometrical Property of Spacetime.This in fact is the cause of the Pioneer Anomaly:A de-acceleration generated by the $5D$ Extra Dimension centered in the Sun assuming that the Sun behaves like a Maartens-Clarkson $5D$ Schwarzschild Black String. We will develop now two examples of Black Strings one with a Warp Field $\Phi = 1$ which means to say no Warp Field at all and another with a Warp Field defined according to Basini-Capozziello Formalism.Although the solution with a Warp Field $\Phi = 1$ could be used to explain the Pioneer Anomaly we strongly believe that the Pioneers are in a region of Spacetime where the Warp Field behaves according to the Basini-Capozziello definition $\Phi = \phi(t, x)\chi(y)$ and the Warp Field have values between 0 and 1 to keep compatibility with the $5D$ to $4D$ Dimensional Reduction.We will see that the Pioneer Anomaly is a Geometrical Property of the Spacetime and according to Basini-Capozziello,Bertolami-Paramos and Mashoon-Wesson-Liu is the final proof that the $5D$ Extra Force really exists.The De-Acceleration of the Pioneers is exactly the one predicted by Mashoon-Wesson-Liu.As a Geometrical Property of Spacetime we would like to mention the fact that if a spacecraft is about to be launched from Earth[42] to investigate the Pioneer Anomaly (see abstract,bottom of pg 2,pg 5 bottom of section 3, and beginning of section 4 of [27]) this spacecraft will be affected also by the Pioneer Anomaly and this perhaps can be used to reveal once for all the existence of the $5D$ Extra Force.

- 1) Maartens-Clarkson 5 Schwarzschild Black String with no Warp Field at all:$\Phi = 1$

  A Maartens-Clarkson Black String as described above centered on the Sun can be given by the following equation:(see for example [9] pg 557 for the comment on $g_{44} = -\Phi^2 = -1$ and [20] eq 110)

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu - dy^2 \tag{114}$$

  The De-Acceleration of the Pioneer spacecrafts is a Geometrical Property of Spacetime and is being generated by the rest-mass of the Sun according to the following equation from the $5D$ Extra Dimensional Force of Mashoon-Wesson-Liu in complete agreement with the Bertolami-Paramos Definitions:(see [9] eqs 24,31,38,39,bottom of page 565 for the de-acceleration comment and the definition of $\beta = \frac{1}{M}\frac{dM}{ds}$ pg 566 after eq 41,see also eq 25 in [2] and eq 15 in [20])[43]

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y}\frac{dy}{ds} \tag{115}$$

  The Maartens-Clarkson $5D$ Schwarzschild Black String is written explicitly as given below([7] eq 1,[20] eq 380).Remember that in this case the Warp Field $g_{44} = -\Phi^2 = -1$)

$$dS^2 = [(1 - \frac{2GM}{R})dt^2 - \frac{dR^2}{(1 - \frac{2GM}{R})} - R^2 d\eta^2] - dy^2 \tag{116}$$

[42] There is one thing that the Pioneer Anomaly Group Meeting and the Pioneer Anomaly Science Team must take into account:Pioneers are travelling in Outer Space for almost 40 years.The distance between Earth and the Pioneers is of order of many billions of kilometers.A spaceship launched from Earth with the current propulsion systems would take many years to reach the region where the Pioneers are right now in order to feel the same Anomaly effects.We have two choices:Or we launch the ship with the current technology seat down and wait many years to get the answers or we develop faster and better propulsion systems.More on this in section 4 about Superluminal Branes and Warp Drives

[43] Equations without the Warp Field Φ because we are considering $\Phi = 1$

In the equation above R is the distance between the Sun and the Pioneers and M is the Sun rest-mass seen in $4D$.Assuming that the Pioneers are moving away from the Sun with the radial component R being affected due to Pioneers sheer velocity and the angular speed with respect to the Sun is neglectable[44] we can rewrite the Black String as:

$$dS^2 = (1 - \frac{2GM}{R})dt^2 - \frac{dR^2}{(1 - \frac{2GM}{R})} - dy^2 \tag{117}$$

According to Basini-Capozziello Ponce De Leon the Black String have two components(see eq 14 in [2],eq 54,56 in [1] and eq 2,6 in [20])[45] :

- A) A $3 + 1$ Spacetime Component ds^2
- B) a $5D$ Extra Dimensional Component

Solving the $3 + 1$ Schwarzschild component we have:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dx_0^2 + g_{11}dx_1^2 \tag{118}$$

A classical Schwarzschild Metric:

$$ds^2 = (1 - \frac{2GM}{R})dt^2 - \frac{dR^2}{(1 - \frac{2GM}{R})} \tag{119}$$

The Radial and Time dependent components of the Spacetime Metric Tensor are given by:

$$g_{00} = (1 - \frac{2GM}{R}) \tag{120}$$

$$g_{11} = -\frac{1}{(1 - \frac{2GM}{R})} \tag{121}$$

Inserting these these values in the Mashoon-Wesson-Liu $5D$ Extra Force and de-acceleration equation we should expect for:

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(u^\mu)^2\frac{\partial g_{\mu\mu}}{\partial y}\frac{dy}{ds} = -\frac{1}{2}[(u^0)^2\frac{\partial g_{00}}{\partial y} + (u^1)^2\frac{\partial g_{11}}{\partial y}]\frac{dy}{ds} \tag{122}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}[(\frac{dx^0}{ds})^2\frac{\partial g_{00}}{\partial y} + (\frac{dx^1}{ds})^2\frac{\partial g_{11}}{\partial y}]\frac{dy}{ds} \tag{123}$$

Note that the components of the Spacetime Metric tensor in the Schwarzschild case obeys this equation:A special case

[44] Even in the Inner Solar System it was Jupiter and Saturn that commanded the Pioneers Orbit and the angular component from the Sun was too small to be taken into account.Outside Solar System the angular component is irrelevant.The Pioneer Anomaly is too small and in the Solar System was masked by Gravity from Jupiter or Saturn.Outside Solar System where Gravity no longer can affect the Pioneers the Anomaly effects becames noticeable.We will address this in this section

[45] Without Warp Factors Ω and with a Warp Field $\Phi = 1$

$$\frac{\partial g_{11}}{\partial y} = \frac{\frac{\partial g_{00}}{\partial y}}{g_{00}^2} = \frac{1}{g_{00}^2}\frac{\partial g_{00}}{\partial y} \tag{124}$$

Applying the result above the Mashoon-Wesson-Liu $5D$ Extra Force and de-acceleration equation we will have the following results:

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}[(\frac{dx^0}{ds})^2\frac{\partial g_{00}}{\partial y} + (\frac{dx^1}{ds})^2\frac{1}{g_{00}^2}\frac{\partial g_{00}}{\partial y}]\frac{dy}{ds} = -\frac{1}{2}\frac{\partial g_{00}}{\partial y}[(\frac{dx^0}{ds})^2 + (\frac{dx^1}{ds})^2\frac{1}{g_{00}^2}]\frac{dy}{ds} \tag{125}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}\frac{\partial g_{00}}{\partial y}(\frac{dx^0}{ds})^2[1+(\frac{dx^1}{dx^0})^2\frac{1}{g_{00}^2}]\frac{dy}{ds} = -\frac{1}{2}\frac{\partial(1-\frac{2GM}{R})}{\partial y}(\frac{dt}{ds})^2[1+(\frac{dR}{dt})^2\frac{1}{(1-\frac{2GM}{R})^2}]\frac{dy}{ds} \tag{126}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}\frac{\partial(1-\frac{2GM}{R})}{\partial y}(\frac{dt}{ds})^3[1+(\frac{dR}{dt})^2\frac{1}{(1-\frac{2GM}{R})^2}]\frac{dy}{dt} \tag{127}$$

This equation below shows that a natural de-acceleration(or acceleration) appears as a Geometrical Property of Spacetime according to Mashoon-Wesson-Liu and Bertolami-Paramos for a Maartens-Clarkson $5D$ Schwarzschild Black String centered on the Sun.This is in fact the true Nature of the $5D$ Extra Dimensional Force and the true nature of the Pioneer Anomaly:The sign of this force depends on the parameter $\frac{\partial(\frac{M}{R})}{\partial y}$ and perhaps the term $\frac{dy}{dt}$.[46]

$$\frac{1}{M}\frac{dM}{ds} = G\frac{\partial(\frac{M}{R})}{\partial y}(\frac{dt}{ds})^3[1+(\frac{dR}{dt})^2\frac{1}{(1-\frac{2GM}{R})^2}]\frac{dy}{dt} \tag{128}$$

We must outline the fact that the rest-mass M of the Sun seen in $4D$ is function of the $5D$ rest-mass M_5 according to Ponce De Leon and is given by:([2] eq 20,[11] eq 21 and [20] eq 8)[47]

$$M = \frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}} \tag{129}$$

The Pioneer sheer velocity is given by:

$$0 < v_{Pioneer} = \frac{dR}{dt} < 1 \dashrightarrow c = 1 \tag{130}$$

$$v_{Pioneer} = \frac{dR}{dt} \simeq 0 \tag{131}$$

This velocity is very small compared to the light speed Solving now the term $\frac{\partial(\frac{M}{R})}{\partial y}$ with a constant $5D$ rest-mass of the Sun M_5 we should expect for:[48]

[46] We outline again the reader to see the bottom of page 565 for the de-acceleration comment and the definition of $\beta = \frac{1}{M}\frac{dM}{ds}$ pg 566 after eq 41 in see [9].Read again the Bertolami-Paramos Definitions in this section.References [9] and [27] were the main inspirations behind our work

[47] Equations without the Warp Field Φ because we are considering $\Phi = 1$

[48] to avoid complications and we still dont know the behavior of the $5D$ rest-mass

$$\frac{1}{M}\frac{dM}{ds}=G\frac{\partial(\frac{M}{R})}{\partial y}(\frac{dt}{ds})^3[1+\frac{v_{Pioneer}^2}{(1-\frac{2GM}{R})^2}]\frac{dy}{dt} \tag{132}$$

$$\frac{\partial(\frac{M}{R})}{\partial y}=\frac{\partial(\frac{1}{R}\frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}})}{\partial y} \tag{133}$$

$$\frac{\partial(\frac{M}{R})}{\partial y}=\frac{1}{R}\frac{M_5}{[\sqrt{1-(\frac{dy}{ds})^2}]^3}\frac{dy}{ds}\frac{\partial(\frac{dy}{ds})}{\partial y}-\frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}}\frac{1}{R^2}\frac{\partial R}{\partial y} \tag{134}$$

Note that in a Timelike $5D$ Spacetime Metric $dS^2>0$ $ds^2>dy^2$ according to Ponce de Leon(see [20] eq 3) and the term $\sqrt{1-(\frac{dy}{ds})^2}\simeq 1$ will be of small significance for the final result.

$$0<1-(\frac{dy}{ds})^2<1 \dashrightarrow 0<\frac{dy}{ds}<1 \dashrightarrow 0<(\frac{dy}{ds})^2<1 \dashrightarrow \frac{dy}{ds}>(\frac{dy}{ds})^2 \dashrightarrow [\sqrt{1-(\frac{dy}{ds})^2}]\simeq 1 \tag{135}$$

$$dS^2=ds^2-dy^2 \dashrightarrow dS^2>0 \dashrightarrow ds^2>>dy^2 \dashrightarrow \frac{dy}{ds}\simeq 0 \tag{136}$$

The Final Expression for the $5D$ Mashoon-Wesson-Liu Extra Force for a Maartens-Clarkson $5D$ Schwarzschild Black String centered on the Sun according to the Bertolami-Paramos Definitions that can explain the Pioneer Anomaly is then given by:

$$\frac{1}{M}\frac{dM}{ds}=G[\frac{1}{R}\frac{M_5}{[\sqrt{1-(\frac{dy}{ds})^2}]^3}\frac{dy}{ds}\frac{\partial(\frac{dy}{ds})}{\partial y}-\frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}}\frac{1}{R^2}\frac{\partial R}{\partial y}](\frac{dt}{ds})^3[1+\frac{v_{Pioneer}^2}{(1-\frac{2GM}{R})^2}]\frac{dy}{dt} \tag{137}$$

It can produces a de-acceleration as a Geometrical Property of the Spacetime like the one observed in the Pioneer Anomaly if these conditions given in the following equation below are satisfied.We still don't know the behavior of the derivatives of the $5D$ Extra Dimension.Note that inside Solar System $\frac{M_5}{R}$ does not vanish but outside Solar System where the Pioneers are right now $\frac{M_5}{R}\simeq 0$.This could explain the extremely small value of the Pioneer Anomaly however with a Spacetime Metric where the Warp Field $\Phi=1$ as far the Pioneers are moving away from the Sun the effect of de-acceleration is decreasing in function of the distance R between the Pioneers and the Sun itself and will vanish someday.Note also that we don't know the behavior of the terms $\frac{\partial R}{\partial y}$,$\frac{dy}{ds}<<\frac{1}{R}$ and $\frac{\partial(\frac{dy}{ds})}{\partial y}\simeq 0$

$$\frac{1}{M}\frac{dM}{ds}<0 \dashrightarrow \frac{1}{R}\frac{M_5}{[\sqrt{1-(\frac{dy}{ds})^2}]^3}\frac{dy}{ds}\frac{\partial(\frac{dy}{ds})}{\partial y}<\frac{M_5}{\sqrt{1-(\frac{dy}{ds})^2}}\frac{1}{R^2}\frac{\partial R}{\partial y} \dashrightarrow \frac{dy}{ds}<<\frac{1}{R} \dashrightarrow \frac{\partial(\frac{dy}{ds})}{\partial y}\simeq 0 \tag{138}$$

- 2) Maartens-Clarkson $5D$ Schwarzschild Black String with a Warp Field $\Phi = \phi(t,x)\chi(y)$ according to Basini-Capozzielo Ponce De Leon([1] eq 76,[5] eq 70 and [20] eq 132)

  We use here also the Bertolami-Paramos Definitions and we believe that the Warp Field may be possibly generated by the Sun[49].We strongly believe that this case is physically more real than the previous one and we are living in a region of Spacetime where the Warp Field $0 < \Phi < 1$.

  In [23] pg 6 Bertolami-Paramos mentions a Warp Field coupled to Gravity in a resemblance of what Basini-Capozziello mentions in [1] pg 119 and [5] pg 2235

  Bertolami-Paramos also defines in pg 6 and 7 of [23] a Potential Function $V \simeq -\frac{1}{\Phi^a}$

  This requires of course a non-null Warp Field $\Phi \neq 1 \dashrightarrow \Phi = \phi(t,x)\chi(y)$

  The $5D$ Timelike Spacetime Metric according to Basini-Capozziello and Ponce De Leon in this case is given by:([1] eq 56,[2] eq 12 and 14,[5] eq 42,[9] eq 32 and 33 without vector potential,[11] eq 10, [12] pg 308,[20] eq 109 and [13] pg 1346)[50][51][52]

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu - \Phi^2 dy^2 \tag{139}$$

  The equation of the $5D$ Extra Force that causes the Pioneer Anomaly as a de-acceleration and a Geometrical Property of Spacetime according to Mashoon-Wesson-Liu and Ponce De Leon is given by:([2] eq 25,[20] eq 15):

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y}\frac{dy}{ds} - \Phi u^\mu \frac{\partial \Phi}{\partial x^\mu}(\frac{dy}{ds})^2 \tag{140}$$

  The Maartens-Clarkson $5D$ Schwarzschild Cosmic Black String in this case is given by the following equations([7] eq 1,[20] eq 380)[53]:

$$dS^2 = g_{\mu\mu}(dx^\mu)^2 - \Phi^2 dy^2 \tag{141}$$

$$dS^2 = [(1 - \frac{2GM}{R})dt^2 - \frac{dR^2}{(1 - \frac{2GM}{R})} - R^2 d\eta^2] - \Phi^2 dy^2 \tag{142}$$

  As in the previous case we consider only the radial and time components and we neglect the angular ones.M is the $4D$ rest mass of the Sun and R is the distance between the Sun and the Pioneers.

$$dS^2 = (1 - \frac{2GM}{R})dt^2 - \frac{dR^2}{(1 - \frac{2GM}{R})} - \Phi^2 dy^2 \tag{143}$$

  Inserting now the Spacetime Metric Components into the equation of the $5D$ Extra Force we should expect for:

[49] We agree that although the mass of the Sun generates the Gravitational Field may not be entirely responsible for the generation of the Warp Field that can be of small value but scattered across a vast region of space.This could be explored in further studies.

[50] [12] with spacelike signature

[51] see [13] pg 1341 the Campbell-Magaard Theorem

[52] see [1] eq 57,[5] eq 43 and [21] eq 47.

[53] diagonalized metrics

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(u^{\mu})^2\frac{\partial g_{\mu\mu}}{\partial y}\frac{dy}{ds} - \Phi u^{\mu}\frac{\partial \Phi}{\partial x^{\mu}}(\frac{dy}{ds})^2 \tag{144}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}[(u^{0})^2\frac{\partial g_{00}}{\partial y} + (u^{1})^2\frac{\partial g_{11}}{\partial y}]\frac{dy}{ds} - \Phi[u^{0}\frac{\partial \Phi}{\partial x^{0}} + u^{1}\frac{\partial \Phi}{\partial x^{1}}](\frac{dy}{ds})^2 \tag{145}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}[(\frac{dt}{ds})^2\frac{\partial g_{00}}{\partial y} + (\frac{dR}{ds})^2\frac{\partial g_{11}}{\partial y}]\frac{dy}{ds} - \Phi[\frac{dt}{ds}\frac{\partial \Phi}{\partial t} + \frac{dR}{ds}\frac{\partial \Phi}{\partial R}](\frac{dy}{ds})^2 \tag{146}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(\frac{dt}{ds})^2[\frac{\partial g_{00}}{\partial y} + (\frac{dR}{dt})^2\frac{\partial g_{11}}{\partial y}]\frac{dy}{ds} - \Phi\frac{dt}{ds}[\frac{\partial \Phi}{\partial t} + \frac{dR}{dt}\frac{\partial \Phi}{\partial R}](\frac{dy}{ds})^2 \tag{147}$$

As in the previous case we apply the properties of the Schwarzschild Spacetime Metric Components g_{00} and g_{11}.

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(\frac{dt}{ds})^2[\frac{\partial g_{00}}{\partial y} + (\frac{dR}{dt})^2\frac{1}{g_{00}^2}\frac{\partial g_{00}}{\partial y}]\frac{dy}{ds} - \Phi\frac{dt}{ds}[\frac{\partial \Phi}{\partial t} + \frac{dR}{dt}\frac{\partial \Phi}{\partial R}](\frac{dy}{ds})^2 \tag{148}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(\frac{dt}{ds})^2\frac{\partial g_{00}}{\partial y}[1 + (\frac{dR}{dt})^2\frac{1}{g_{00}^2}]\frac{dy}{ds} - \Phi\frac{dt}{ds}[\frac{\partial \Phi}{\partial t} + \frac{dR}{dt}\frac{\partial \Phi}{\partial R}](\frac{dy}{ds})^2 \tag{149}$$

But we also know that $v_{Pioneer} = \frac{dR}{dt}$ is the sheer velocity of the Pioneers.Then we can write:

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(\frac{dt}{ds})^2\frac{\partial g_{00}}{\partial y}[1 + v_{Pioneer}^2\frac{1}{g_{00}^2}]\frac{dy}{ds} - \Phi\frac{dt}{ds}[\frac{\partial \Phi}{\partial t} + v_{Pioneer}\frac{\partial \Phi}{\partial R}](\frac{dy}{ds})^2 \tag{150}$$

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}(\frac{dt}{ds})^3\frac{\partial g_{00}}{\partial y}[1 + v_{Pioneer}^2\frac{1}{g_{00}^2}]\frac{dy}{dt} - \Phi(\frac{dt}{ds})^3[\frac{\partial \Phi}{\partial t} + v_{Pioneer}\frac{\partial \Phi}{\partial R}](\frac{dy}{dt})^2 \tag{151}$$

Again as we stated before:This equation below shows that a natural de-acceleration(or acceleration) appears as a Geometrical Property of Spacetime according to Mashoon-Wesson-Liu Ponce De Leon and Bertolami-Paramos for a Maartens-Clarkson $5D$ Schwarzschild Black String centered on the Sun.This is in fact the true Nature of the $5D$ Extra Dimensional Force and the true nature of the Pioneer Anomaly:The sign of this force depends on the parameter $\frac{\partial(\frac{M}{R})}{\partial y}$ and perhaps the term $\frac{dy}{dt}$.[54][55]

$$\frac{1}{M}\frac{dM}{ds} = -\frac{1}{2}\frac{\partial(1-\frac{2GM}{R})}{\partial y}(\frac{dt}{ds})^3[1+v_{Pioneer}^2\frac{1}{(1-\frac{2GM}{R})^2}]\frac{dy}{dt} - \Phi(\frac{dt}{ds})^3[\frac{\partial \Phi}{\partial t} + v_{Pioneer}\frac{\partial \Phi}{\partial R}](\frac{dy}{dt})^2 \tag{152}$$

We already know that the rest-mass M of the Sun seen in $4D$ is function of the $5D$ rest-mass M_5 according to Ponce De Leon and is given by:([2] eq 20,[11] eq 21 and [20] eq 8)

[54] We outline again the reader to see the bottom of page 565 for the de-acceleration comment and the definition of $\beta = \frac{1}{M}\frac{dM}{ds}$ pg 566 after eq 41 in see [9].Read again the Bertolami-Paramos Definitions in this section.References [9] and [27] were the main inspirations behind our work

[55] We know the we are repeating ourselves here but this is as a matter of fact the most important part of this work

$$M = \frac{M_5}{\sqrt{1 - \Phi^2 (\frac{dy}{ds})^2}} \tag{153}$$

Computing now the remaining terms of the equation we should expect for:

$$\frac{\partial (1 - \frac{2G}{R} \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}})}{\partial y} = -2G[\frac{1}{R} \frac{\partial \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}}{\partial y} - \frac{1}{R^2}(\frac{\partial R}{\partial y}) \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] = A \tag{154}$$

$$-\frac{1}{2} A = G[\frac{1}{R} \frac{\partial \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}}{\partial y} - \frac{1}{R^2}(\frac{\partial R}{\partial y}) \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] = B \tag{155}$$

$$\frac{1}{M} \frac{dM}{ds} = B(\frac{dt}{ds})^3 [1 + v_{Pioneer}^2 \frac{1}{(1 - \frac{2G \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}}{R})^2}] \frac{dy}{dt} - \Phi (\frac{dt}{ds})^3 [\frac{\partial \Phi}{\partial t} + v_{Pioneer} \frac{\partial \Phi}{\partial R}](\frac{dy}{dt})^2 \tag{156}$$

$$\frac{\partial \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}}{\partial y} = \frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3} [\Phi^2 \frac{dy}{ds} \frac{\partial \frac{dy}{ds}}{\partial y} + (\frac{dy}{ds})^2 \Phi \frac{\partial \Phi}{\partial y}] \tag{157}$$

$$-\frac{1}{2} A = G[\frac{1}{R} \frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3} [\Phi^2 \frac{dy}{ds} \frac{\partial \frac{dy}{ds}}{\partial y} + (\frac{dy}{ds})^2 \Phi \frac{\partial \Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y}) \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] = B \tag{158}$$

We know that in a Timelike $5D$ Spacetime Metric the following equations holds true:[56]

$$dS^2 = ds^2 - \Phi^2 dy^2 \dashrightarrow dS^2 > 0 \dashrightarrow ds^2 >> \Phi^2 dy^2 \dashrightarrow \Phi \frac{dy}{ds} \simeq 0 \dashrightarrow \Phi \frac{dy}{ds} < 1 \dashrightarrow \frac{dy}{ds} = \frac{1}{N\Phi} \tag{159}$$

$$-\frac{1}{2} A = G[\frac{1}{R} \frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3}][\frac{\Phi}{N} \frac{\partial \frac{1}{N\Phi}}{\partial y} + \frac{1}{N} \frac{\partial \Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y}) \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] = B \tag{160}$$

$$-\frac{1}{2} A = G[\frac{1}{R} \frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3}][-\frac{1}{N\Phi^2} \frac{\partial \Phi}{\partial y} + \frac{1}{N} \frac{\partial \Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y}) \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] = B \tag{161}$$

$$-\frac{1}{2} A = G[\frac{1}{R} \frac{M_5}{[\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}]^3}][\frac{1}{N} \frac{\partial \Phi}{\partial y} - \frac{1}{N\Phi^2} \frac{\partial \Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y}) \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}}] = B \tag{162}$$

[56]In the following equations we consider the term N a constant

$$\frac{1}{N}\frac{\partial\Phi}{\partial y} - \frac{1}{N\Phi^2}\frac{\partial\Phi}{\partial y} < 0 \dashrightarrow B < 0 \dashrightarrow 0 < \Phi < 1 \dashrightarrow \frac{1}{N\Phi^2} >> 0 \tag{163}$$

$$-\frac{1}{2}A = \frac{G}{N}[\frac{1}{R}\frac{M_5}{[\sqrt{1-\Phi^2(\frac{dy}{ds})^2}]^3}][\frac{\partial\Phi}{\partial y} - \frac{1}{\Phi^2}\frac{\partial\Phi}{\partial y}] - \frac{1}{R^2}(\frac{\partial R}{\partial y})\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}] = B \tag{164}$$

The Final Expression for the non null Warp Field $5D$ Mashoon-Wesson-Liu and Ponce De Leon Extra Force for a Maartens-Clarkson $5D$ Schwarzschild Black String centered on the Sun according to the Bertolami-Paramos Definitions that can explain the Pioneer Anomaly is then given by:

$$-\frac{1}{2}A = \frac{G}{N}\frac{\partial\Phi}{\partial y}[\frac{1}{R}\frac{M_5}{[\sqrt{1-\Phi^2(\frac{dy}{ds})^2}]^3}][1 - \frac{1}{\Phi^2}] - \frac{1}{R^2}(\frac{\partial R}{\partial y})\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}] = B \tag{165}$$

$$\frac{1}{M}\frac{dM}{ds} = B(\frac{dt}{ds})^3[1 + v_{Pioneer}^2\frac{1}{(1-\frac{2G\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}}{R})^2}]\frac{dy}{dt} - \Phi(\frac{dt}{ds})^3[\frac{\partial\Phi}{\partial t} + v_{Pioneer}\frac{\partial\Phi}{\partial R}](\frac{dy}{dt})^2 \tag{166}$$

It can produces a de-acceleration as a Geometrical Property of the Spacetime like the one observed in the Pioneer Anomaly Note that in these equations above the effect of the de-acceleration can be noted more explicitly due to the term $1 - \frac{1}{\Phi^2}$.If we live in a Spacetime where the Warp Field is too small $0 < \Phi < 1$ and $1 - \frac{1}{\Phi^2} < 0$ because $\frac{1}{\Phi^2} >> 0$.The previous case without Warp Field could produce an acceleration or a de-acceleration but this one will produce a de-acceleration always.This is the reason why we favor this second case.Remember also that Pioneers are already outside Solar System where the term $\frac{M_5}{R} \simeq 0$.The Warp Field terms can produce a de-acceleration far away from our Solar System if this field is of small value but scattered across a vast region of space due to the terms $-\Phi(\frac{dt}{ds})^3[\frac{\partial\Phi}{\partial t} + v_{Pioneer}\frac{\partial\Phi}{\partial R}](\frac{dy}{dt})^2$.If the Pioneers are now in a region of Spacetime where Gravity can be neglected and we note the sunward acceleration:a de-acceleration[57] then it can perfectly be the Basini-Capozziello coupled to Bertolami-Paramos Warp Field what is causing the Pioneer Anomaly as a Geometrical Property of Spacetime and we are observing the first physical evidence of the Warp Field and the $5D$ Extra Dimension.We still don't know the shape of the Warp Field.The expressions for $\frac{dt}{ds}$ are given below as:

$$\frac{dt}{ds} = \frac{1}{\sqrt{(1-\frac{2GM}{R}) - \frac{\frac{dR^2}{dt^2}}{(1-\frac{2GM}{R})}}} \tag{167}$$

$$\frac{dt}{ds} = \frac{1}{\sqrt{(1-\frac{2GM}{R}) - \frac{v_{Pioneer}^2}{(1-\frac{2GM}{R})}}} \tag{168}$$

According to Basini-Capozziello Ponce De Leon the Warp Field have a $3+1$ component and a $5D$ Extra Dimensional one defined by the following relation $\Phi = \phi(t,x)\chi(y)$:(see [1] eq 76,[5] eq 70 and [20] eq 132)

[57]from the last set of Pioneers available raw of data data

Writing again the Pioneer Anomaly equations in function of the Basini-Capozziello Ponce De Leon Warp Field with the components written explicitly we should expect for:

$$-\frac{1}{2}A = \frac{G\phi(t,x)}{N}\frac{\partial\chi(y)}{\partial y}[\frac{1}{R}\frac{M_5}{[\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}]^3}][1-\frac{1}{(\phi(t,x)\chi(y))^2}] - D = B \tag{169}$$

$$D = \frac{1}{R^2}(\frac{\partial R}{\partial y})[\frac{M_5}{\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}}] \tag{170}$$

$$\frac{1}{M}\frac{dM}{ds} = B(\frac{dt}{ds})^3[1+v_{Pioneer}^2\frac{1}{(1-\frac{2G\frac{M_5}{\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}}}{R})^2}]\frac{dy}{dt} - E \tag{171}$$

$$E = \phi(t,x)(\frac{dt}{ds})^3\chi(y)^2[\frac{\partial\phi(t,x)}{\partial t} + v_{Pioneer}\frac{\partial\phi(t,x)}{\partial R}](\frac{dy}{dt})^2 \tag{172}$$

Note that outside Solar System where the Pioneers are right now the Gravitational Force of the Sun vanishes and hence the ratio $\frac{M}{R} \simeq 0$ or $\frac{M_5}{R} \simeq 0$.The term responsible for the current account of the Pioneer Anomaly is given by the result above. Note also that $0 < v_{Pioneer} < 1$ and the ratio $\frac{dt}{ds}$ would then be given by:

$$\frac{dt}{ds} = \frac{1}{\sqrt{1-v_{Pioneer}^2}} \tag{173}$$

Hence the Pioneer Anomaly as a de-acceleration of the two spacecrafts we observed until 2003 can be written as:

$$\frac{1}{M}\frac{dM}{ds} = -\phi(t,x)(\frac{1}{\sqrt{1-v_{Pioneer}^2}})^3\chi(y)^2[\frac{\partial\phi(t,x)}{\partial t} + v_{Pioneer}\frac{\partial\phi(t,x)}{\partial R}](\frac{dy}{dt})^2 \tag{174}$$

Note that without the $5D$ Extra Dimension $\frac{dy}{dt} = 0$ and we would have no de-acceleration at all.The Pioneer Anomaly is the strongest proof of the existence of the $5D$ Extra Dimension.Again as we stated before:We don't know the shape of the Warp Field and the Geometry of the Extra Dimension although we believe that the Warp Field is scattered across large distances in Space and have a small value between 0 and 1.Note that the eq above used derivatives of the $3+1$ components of the Warp Field.See also our comment on derivatives of the Warp Field in the section of Dimensional Reduction from $5D$ to $4D$.[58][59]

The Bertolami-Paramos Definitions uses a Yukawa Potential defined by eq 7 of [27] given below:

$$V = -\frac{GM}{(1+\alpha)R}(1+\alpha\varepsilon^{\frac{-R}{\lambda}}) \tag{175}$$

[58] We outline again the reader to see the bottom of page 565 for the de-acceleration comment and the definition of $\beta = \frac{1}{M}\frac{dM}{ds}$ pg 566 after eq 41 in see [9]

[59] We know the we are repeating ourselves here but this is as a matter of fact the most important part of this work

Writing the Yukawa Potential using the $5D$ rest-mass M_5 the Warp Field appears explicitly in the Potential expression.Remember that Bertolami-Paramos says that the Potential obeys the following condition $V \cong -\frac{1}{\Phi^a}$ (see pg 6 and 7 of [23]):

$$V = -\frac{G\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}}}{(1+\alpha)R}(1+\alpha\varepsilon^{\frac{-R}{\lambda}}) \tag{176}$$

Following Bertolami-Paramos we consider the coupling strength $0 < \alpha < 1$(see pg 6 of [27]) and we rewrite the Potential as follows:

$$V = -G\frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}} \tag{177}$$

We will now demonstrate that the solution of the equation of motion for the Warp Field Φ see eq 13 in [23] can also be proportional to $-\frac{1}{\Phi^a}$ specially for the regions outside our Solar System where the Pioneers are moving now. Writing the Bertolami-Paramos equation of motion as:

$$^5\Box\Phi^2 + \frac{dV}{d\Phi} = 0 \tag{178}$$

We will now take the derivatives of the modified Yukawa Potential using the $5D$ rest-mass M_5 according to Ponce De Leon.The expression is given below:

$$\frac{dV}{d\Phi} = -\frac{G}{R}M_5\frac{1}{[\sqrt{1-\Phi^2(\frac{dy}{ds})^2}]^3}\Phi(\frac{dy}{ds})^2 \tag{179}$$

The Bertolami-Paramos equation of motion will then be:

$$^5\Box\Phi^2 = \frac{G}{R}M_5\frac{1}{[\sqrt{1-\Phi^2(\frac{dy}{ds})^2}]^3}\Phi(\frac{dy}{ds})^2 \tag{180}$$

But we also know that according to Basini-Capozziello the Warp Field Φ can be split in a $3+1$ Spacetime component and a $5D$ Extra Dimensional component.Inserting the components we would find the result given below:

$$^5\Box\Phi^2 = \chi(y)^4\Box\phi(t,x)^2 + \phi(t,x)\frac{\partial^2}{\partial y^2}\chi(y) = -\frac{dV}{d\Phi} \tag{181}$$

Equalizing with the derivative of the Yukawa Potential we would have:

$$^5\Box\Phi^2 = \chi(y)^4\Box\phi(t,x)^2 + \phi(t,x)\frac{\partial^2}{\partial y^2}\chi(y) = \frac{G}{R}M_5\frac{1}{[\sqrt{1-\Phi^2(\frac{dy}{ds})^2}]^3}\Phi(\frac{dy}{ds})^2 \tag{182}$$

$$^5\Box\Phi^2 = \chi(y)^4\Box\phi(t,x)^2 + \phi(t,x)\frac{\partial^2}{\partial y^2}\chi(y) = \frac{G}{R}M_5\frac{1}{[\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}]^3}\phi(t,x)\chi(y)(\frac{dy}{ds})^2 \tag{183}$$

$$\chi(y)^4\Box\phi(t,x)^2 + \phi(t,x)\frac{\partial^2}{\partial y^2}\chi(y) = \frac{G}{R}M_5\frac{1}{[\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}]^3}\phi(t,x)\chi(y)(\frac{dy}{ds})^2 \tag{184}$$

$$\frac{1}{\phi(t,x)}^4\Box\phi(t,x)^2+\frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y)=\frac{G}{R}M_5\frac{1}{[\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}]^3}(\frac{dy}{ds})^2 \tag{185}$$

Remember that Pioneers are now outside Solar System and the Gravitational Force vanishes as mentioned before.Then $\frac{M}{R}\simeq 0$ or $\frac{M_5}{R}\simeq 0$ and we are being helped by the Gravitational Constant $G = 6,67\times 10^{-11}\frac{Nm^2}{Kg^2}$ to make the result of the Bertolami-Paramos equation of motion even close to 0

$$\frac{1}{\phi(t,x)}^4\Box\phi(t,x)^2+\frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y)=0 \tag{186}$$

$$\frac{1}{\phi(t,x)}^4\Box\phi(t,x)^2=-\frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y) \tag{187}$$

If perhaps a future Space Mission like the one being planned by the Pioneer Anomaly International Science Rescue Team(see pg 1 in [27])[60] can detect variations in the $3+1$ component of the Warp Field then the variations in the $5D$ Extra Dimensional ones cannot be null.Otherwise there would be nothing to be detected.Again the Pioneer Anomaly is the strongest proof that we live in a Extra Dimensional Universe.For the variations in the Warp Field see the section on Dimensional Reduction from $5D$ to $4D$.

If we still consider the distance between the Sun and the Pioneers R as meaningful we can apply the Ponce De Leon definition of electrical charge from the $5D$ Extra Dimension rest-mass M_5 as follows:(see eq 19 in [11])

$$q=-\frac{M_5\Phi^2\frac{dy}{ds}}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}} \tag{188}$$

Inserting the Ponce de Leon charge into the Bertolami-Paramos equation we would get the following result:

$$\frac{1}{\phi(t,x)}^4\Box\phi(t,x)^2+\frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y)=-\frac{G}{R}q\frac{1}{\Phi^2[\sqrt{1-(\phi(t,x)\chi(y))^2(\frac{dy}{ds})^2}]^3}\frac{dy}{ds} \tag{189}$$

As stated before we don't know the Geometry of the Extra Dimension and the shape of the Warp Field but we know that the Warp Field must have values between 0 and 1 and we know that in a $5D$ Timelike Geodesics according to Ponce De Leon (see eq 3 in [20]) $1>\Phi\frac{dy}{ds}$ and $0<\frac{dy}{ds}<1$. Hence $\Phi>\Phi\frac{dy}{ds}$ due to the factor $\frac{dy}{ds}$.Then we can say that $[\sqrt{1-(\Phi)^2(\frac{dy}{ds})^2}]\simeq 1$ and the Bertolami-Paramos equation of motion would then be:

$$\frac{1}{\phi(t,x)}^4\Box\phi(t,x)^2+\frac{1}{\chi(y)}\frac{\partial^2}{\partial y^2}\chi(y)=-\frac{G}{R}q\frac{1}{\Phi^2}\frac{dy}{ds} \tag{190}$$

The result above satisfies the Bertolami-Paramos relation $V\cong-\frac{1}{\Phi^a}$ with $a=2$ (see pg 6 and 7 of [23]) although we consider here a equation of motion and not a Potential Function .

To terminate this section we would like to outline that the Pioneer Anomaly is not generated by gas leak,oil leak or plutonium reactor leak but instead is a Geometric Property of Spacetime and is the first observational evidence of the $5D$ Extra Dimensional Force predicted by Mashoon-Wesson-Liu years ago

[60] and pg 4 in the same reference the part of the probe encompassing a device to measure electric potential

in [9].We got the inspiration to use Extra Dimensions to explain the Pioneer Anomaly from the works of Bertolami-Paramos in [23] and [27] although we used the Basini-Capozziello Ponce De Leon Formalism(see [1],[2] and [20]).If the Pioneer Anomaly International Science Rescue Team is about to really send a spaceship to verify if the Anomaly remains then our answer is positive:any ship would suffer from the same perturbations generated by the $5D$ Extra Force(see pg 1 of [27]).We know that some of our results are assumptions from Theoretical Physics and there will be much more to say on this subject but we believe that these equations presented here can satisfy the point of view of Bertolami-Paramos because we agree entirely with both of them when they say that a new physics is needed to explain the observed anomalous de-acceleration.

6 The Superluminal Chung-Freese Braneworld under the Basini-Capozziello Ponce De Leon Formalism compared to the Alcubierre Warp Drive.A Shape Function $f(rs)$ to make an Alcubierre Warp Drive without expansion/contraction of the Spacetime behaving effectively like a Natario Warp Drive and the Introduction to the Casimir Warp Drive.

At last we arrived to the final section of this work. This is a work conceived to explain the Pioneer Anomaly but is also a work dedicated to the $50th$ Anniversary of the Human Adventure In The Skies and a Homage[61] to the day 4 October 1957. At this time we will raise two points:

- 1)The Pioneer Anomaly International Science Mission Rescue Team is planning to send a spaceship to the region where the Pioneers were lost([27],[24]) to see if the Anomaly remains.We know that this ship will also be affected by the Anomaly.However the Pioneers were launched almost 40 years ago and this would mean we should have to wait more 40 years in order to give time to the ship reach the same region considering of course the current propulsion systems. Unless we develop a faster and better propulsion system.
- 2)Since this is also a work to celebrate the Human Conquest Of Space we will raise here a fundamental question:Will Humans ever be able to leave our Solar System and reach the Stars???

Both point converges to a faster and better propulsion system.But for the second point another question arises:The nearest Star close to our Solar System is Proxima Centauri at $4,3$ light-years away.This means to say that the light speed takes $4,3$ years to travel from the Sun to Proxima Centauri by Earth clocks.In order to reach the nearest Star in a affordable amount of time for a manned space mission(not considering robotic space probes,Von Neumann machines,generational ships where the grandfather starts a journey to a distant star and the grandson finally arrives or other exotic concepts like cryogenic hibernation etcetera) we need to find out a way to go Faster Than Light(see item 2 section 3.1 in [49]).This is the main reason why we enclosed this section in this work:Faster Than Light Space Travel.

In this section we will examine some proposed solutions for Faster Than Light Space Travel:

- 1)-Alcubierre Warp Drive
- 2)-Chung-Freese Superluminal BraneWorld

The Challenge of Faster Than Light Space Travel started in 1905 when Einstein published his Special Theory of Relativity(SR).Einstein established a Universe speed limit:Light Speed c.Nothing can travel Faster Than the Speed Of Light(at least locally in SR Frames of Reference) or as is also known:"Thou Shall Not Travel Faster Than The Speed Of Light". This can be better pictured by the equations given below:

$$m = \frac{m_0}{\sqrt{1 - (\frac{v^2}{c^2})}} \tag{191}$$

$$E_0 = m_0 c^2 \tag{192}$$

[61] the word Homage is French:it means Celebration

$$E = mc^2 \tag{193}$$

$$K = E - E_0 \tag{194}$$

From the set of equations above if we accelerate a body we give it Kinetic energy K but since due to the equivalence between mass and energy this Kinetic energy also have mass..so a body in motion have Kinetic energy but have a mass m that is much heavier than the same body at the rest with mass m_0 because the Kinetic energy accounts for a mass increase.As faster the body moves the body possesses more Kinetic energy and more mass....it becomes more heavier...as it becomes more heavier it will require a stronger force to accelerate the body giving even more Kinetic energy which means to say even more mass and the body becomes even more heavier.....Ad Infinitum..... In order to reach the Speed Of Light an infinite amount of energy and an infinite force is needed.So its impossible to reach The Speed Of Light and if we cannot reach it we cannot surpass it.This is the reason why we cannot travel Faster Than Light.This line of reason started by Einstein was scientifically correct from 1905 to 1993. Until 1994 Faster Than Light Space Travel was regarded a Province of the Realm of Science Fiction but in that year something happened:one of the major shifts in the line of reason of Modern Science and perhaps the Greatest Of All.1994 was the year when the concept of Faster Than Light Space Travel changed radically from the "Complete Impossible And Solely In Science Fiction" to a "Maybe Possible But We Don't Have The Technology To Afford It".In the year of 1994 a Revolutionary paper appeared bringing by the first time to Modern Science the possibility of Faster Than Light Space Travel.Now we know that the idea behind the 1994 paper will not work at least in a $3+1$ Einstein Universe[62] and we will demonstrate in this section that we really need Extra Dimensions to achieve Faster Than Light Space Travel[63].But the 1994 paper will ever be considered a Historical paper and a Landmark because perhaps this paper changed forever the course of Modern Science.It is not exaggerated to say that the Humanity started is first steps in the Route To The Stars in the year of 1994[64].The Historical Revolutionary paper of 1994 was written by the Mexican mathematician Miguel Alcubierre from Universidad Nacional Autonona de Mexico(UNAM) and is entitled "The Warp Drive Hyper-Fast Travel Within General Relativity".This paper launched the foundations of the so-called Alcubierre Warp Drive.The main idea behind the Alcubierre Warp Drive is to create a local Spacetime Distortion surrounding a spaceship that will generates an expansion of Spacetime behind the spaceship and a contraction in front of the spaceship(see abstract and pg 3 and 6 of [31]).Since the spaceship is at the rest inside the local Spacetime distortion(known as Warp Bubble) and the Warp Bubble moves away from the spaceship the departure point (expansion) while at the same time brings to the front of the spaceship the destination point(contraction) the spaceship do not moves at all inside the Warp Bubble so while remaining at the rest inside the Warp Bubble the spaceship will not have the mass increase of Special Relativity(see pg 6 of [31],pg 2 in [29])[65] and can afford Faster Than Light Space Travel(see pg 8 of [31],pg 2 of [28],pg 3 of [30],pg 2 in [29],pg 1 in [42],pg 1 in [43],pg 2 in [45]).We will now examine in details some of the features of the Alcubierre Warp Drive. The Alcubierre Warp Drive can be defined by the following Spacetime ansatz in $3+1$ Spacetime Dimensions(see pg 4 and eqs 6 and 8 in [31],pg 3 and

[62]this is not entirely true and we will demonstrate that the Alcubierre Warp Drive is still a valid solution of the Einstein Field Equations of General Relativity that allows Faster Than Light Space Travel but of course an impact with a Large Black Hole would change the picture

[63]Extra Dimensions are a safer way to circumvent Large Black Holes

[64]interview of Miguel Alcubierre to Sergio de Regules of the magazine "Como Ves" of Universidad Nacional Autonoma de Mexico(UNAM).We can provide the original PDF Acrobat Reader of the interview in Spanish with an English translation for those interested

[65]if the spaceship don't suffer time dilatation then according to Lorentz Transformations it will not suffer mass increase

eqs 1 to 3 in [28],pg 2 and eqs 1 to 2 in [29],pg 4 to 5 and eqs 1 to 3 in [30],pg 2 to 3 eqs 4 to 6 in [42],pg 2 eqs 2.1,2.2 and pg 3 eq 2.11 in [43],pg 2 eqs 2 to 3 in [45]) [66][67]:

$$ds^2 = dt^2 - [dx - vsf(rs)dt]^2 - dy^2 - dz^2 \tag{195}$$

where

$$vs = \frac{dxs}{dt} \tag{196}$$

$$rs(t) = \sqrt{[x - xs(t)]^2 + y^2 + z^2} \tag{197}$$

$$f(rs) = \frac{tanh[\delta(rs + R)] - tanh[\delta(rs - R)]}{2tanh(\delta R)} \tag{198}$$

Now a little bit of Warp Drive Basics:xs is the center of the Warp Bubble where the spaceship remains,x is any position inside the Warp Bubble.For the spaceship $x = xs$(see pg 5 and 6 in [31],pg 3 in [28],pg 2 in [29])[68].$vs = \frac{dxs}{dt}$ is the speed of the center of the Warp Bubble where the spaceship resides with respect to a distant observer at the rest with respect to a SR Local Frame(see pg 3 in [28],pg 4 in [30]).To put it simply the ship is as the rest inside the Warp Bubble and feels no acceleration and no g-forces but the Warp Bubble can move itself with a arbitrary large speed $vs = \frac{dxs}{dt}$ with respect to a distant observer(see pg 2 in [29],see pg 2 in [42],see pg 2 in [45]).rs is the distance between a given point and the center of the Warp Bubble at $x = xs$(see pg 3 in [28]).$f(rs)$ is the so-called Shape Function that makes the Alcubierre Warp Drive works.The parameters of $f(rs)$ are these:δ is the thickness of the Warp Bubble and can be given arbitrarily,R is the radius of the Warp Bubble and rs was already described(see pg 3 in [28]).We have three possible values for the Shape Function:

- 1)Shape Function inside the Warp Bubble $rs \leq R$ (Flat Spacetime) $f(rs) = 1$ everywhere
- 2)Shape Function in the walls of the Warp Bubble(Warped Region) $0 < f(rs) < 1$
- 3)Shape Function outside the Warp Bubble $rs \gg R$ (Flat Spacetime) $f(rs) = 0$

According to pg 3 in [28],eq 7 pg 4 in [31],eq 7 pg 3 in [42],pg 2 in [43] and pg 2 in [45] the Shape Function $f(rs) = 1$ inside the Warp Bubble where $rs \leq R$ and $f(rs) = 0$ outside the Warp Bubble where $rs \gg R$.Assuming a continuous $f(rs)$ there must exists a region where $f(rs)$ decreases from 1 to 0 and this region is the Warped Region associated to the Warp Bubble Walls[69].

Considering that δ the thickness of the Warp Bubble Walls can have a significant value we will redefine the values of the Shape Function as follows[70]:

- 1)Shape Function inside the Warp Bubble $rs \leq R - \delta$ (Flat Spacetime) $f(rs) = 1$ everywhere

[66] Alcubierre,Ford,Pfenning,Broeck,Clark Hiscock and Larson used a signature (-,+,+,+),we use a signature (+,-,-,-)

[67] Alcubierre Warp Drive is not a diagonalized metric because it contains the Spacetime Metric Tensor components g_{01} and g_{10} and $g_{01} = g_{10}$.This can be easily seen from the factor $[dx - vsf(rs)dt]^2$.Alcubierre Warp Drive is a particular case of a family of Spacetime ansatz as we shall see in this section

[68] pg 5 in [31] the center of the Warp Bubble is defined by $[xs(t), 0, 0]$.look also to eq 6 pg 4 in [28] and consider the Spatial Components only.remember that inside the Warp Bubble $f(rs) = 1$ so Ford-Pfenning are taking time derivatives of the Alcubierre center

[69] δ defines the "size" of the Warped Region

[70] combining eqs 3 and 4 pg 3 in [28]

- 2)Shape Function in the walls of the Warp Bubble(Warped Region) $R-\delta < rs < R+\delta$ $0 < f(rs) < 1$
- 3)Shape Function outside the Warp Bubble $rs \geq R+\delta$ (Flat Spacetime) $f(rs) = 0$

We have also three possible locations to place observers:

- 1)Observer in the spaceship in the center of the Warp Bubble $f(rs) = 1$
- 2)Observer in the middle of the Warped Region $0 < f(rs) < 1$
- 3)Observer outside the Warped Bubble at a faraway distance $f(rs) = 0$

According to pg 6 in [31] the Warped Region is the worst location to place an observer because Spacetime is not flat and the Tidal Forces are very large.Then we are left with two locations only:

- 1)Observer in the spaceship in the center of the Warp Bubble $f(rs) = 1$
- 2)Observer outside the Warped Bubble at a faraway distance $f(rs) = 0$

We will now demonstrate how Alcubierre in 1994 bypassed the limitations posed by Einstein from 1905 to 1993. Imagine two observers A and B at the rest initially with clocks synchronized with respect to a SR Local Frame C.Both have with respect to C the same $ds^2 = dt^2$.

- 1)Observer A enters in a Warp Bubble that instantaneously achieves a large $vs >> 1$[71].Faster Than Light ($c = 1$).In the center of the Warp Bubble $x = xs$ and $f(rs) = 1$
- 2)Observer B remains outside the Warp Bubble $f(rs) = 0$

Although y and z cannot be null[72] the motion will occurs only in the $x-axis$ so we can write the Alcubierre Warp Drive as follows:

$$ds^2 = dt^2 - [dx - vsf(rs)dt]^2 \tag{199}$$

- 1)Situation seen by A:$x = xs$:$f(rs) = 1$:$vs = \frac{dxs}{dt}$:$dx = dxs$

$$ds^2 = dt^2 - [dxs - vsdt]^2 \tag{200}$$

$$ds^2 = dt^2 - [dxs - \frac{dxs}{dt}dt]^2 \tag{201}$$

$$ds^2 = dt^2 - [dxs - dxs]^2 \tag{202}$$

$$ds^2 = dt^2 \tag{203}$$

[71]the action is being taken by an external observer
[72]we will see why y and z cannot be null

- 2)Situation seen by B while observing A moving away:$f(rs) = 0$ for B but B can observe A moving away from him with speed vs:$x = xs$ for A:$dx = dxs$ for A

$$ds^2 = dt^2 - dxs^2 \tag{204}$$

Remember that observer B is still synchronized to the Frame C so with respect to C B possesses a $ds^2 = dt^2$ as a proper time.But A passed instantaneously from the rest to a $vs >> 1$ and A measuring its proper time measures also a $ds^2 = dt^2$ and remains in a Timelike Geodesics and at the rest with respect to a Local Frame inside the Warp Bubble.A felt no accelerations when passing from the rest to a $vs >> 1$.So A remains synchronized to C because A passed instantaneously from the rest to $vs >> 1$ while synchronized to C.On the other hand B measures A with a speed $vs >> 1$ and in a Spacelike Geodesics.Both have the same proper time with respect to C.No time dilatation.No Special Relativity and A is moving away from B Faster Than Light.

This is the Revolutionary concept of the Warp Drive as a Dynamical Spacetime introduced by Alcubierre in 1994

- 1)Motion Faster Than Light
- 2)No limitations from Special Relativity.No time dilatation.No mass increase
- 3)Both observers remains synchronized between themselves

However it was discovered that the Alcubierre Warp Drive have serious problems and drawbacks that from a realistic point of view can poses serious obstacles to its physical feasibility

- 1)Horizons-Causally Disconnected portions of Spacetime
- 2)Doppler Blueshifts and Impacts with hazardous objects
- 3)Enormous energy densities required to create it

We will now examine these in details.While negative energy problem can be ameliorated and the levels of energy can be lowered [29] and [41] the two firsts are serious obstacles [30],[32] and [45].If we keep the Alcubierre Warp Drive at subluminal speeds the Horizon problem disappears and perhaps with a unknown form of Quantum Gravity that encompasses non-local Quantum Entanglements[73] would be possible to solve Horizons at superluminal speeds but the Doppler Blueshifts and the collision with large objects remains the most serious problem whether subluminal or superluminal. It is easy to see that an Alcubierre Warp Drive even with the negative energy or Horizons problem solved by a still unknown spacetime metric would even face the Doppler Blueshifts of incoming photons from Cosmic Background Radiation with wavelengths shifted towards synchrotron radiation[74] impacting the Warp Bubble making the Alcubierre Warp Drive unstable.Plus impact with hazardous objects such as protons,electrons,Clouds of Space Dust or Debris,Asteroids,Meteors,Comets,Supernovas,Neutron Stars or Black Holes that would appear in front

[73]EPR-Einstein Podolsky Rosen paradox,Bell Inequalities etcetera

[74]one of the most lethal forms or radiation with even more penetrating capability than gamma radiation.Think about how many photons of COBE we have per cubit centimeter of space and think in how many cubic centimeters of space exists between Earth and Proxima Centauri at 4 light-years away to better understand how synchrotron radiation is hazardous for the Alcubierre Warp Drive.

of an Alcubierre Warp Drive in a realistic travel across Outer Space in our Galaxy[75] would disrupt and destroy any kind of Alcubierre Warp Drive.[76]

- 1)Horizons-Causally Disconnected portions of Spacetime

  Imagine a observer in a spaceship in the center of the Warp Bubble and a Warp Bubble speed vs.For a while we assume the Warp Bubble moves at subluminal speed $vs << 1$ and the motion occurs in the $x - axis$.The Alcubierre Warp Drive can be written in the following form(see pg 2 eq 1 in [45] and pg 2 eq 2.3 in [43][77] :

$$ds^2 = [1 - (vsf(rs))^2]dt^2 + 2vsf(rs)dxdt - dx^2 \tag{205}$$

  Lets imagine that the observer in the spaceship at the Warp Bubble center is sending photons to the front or the rear of the Warp Bubble in order to accelerate it or to drive its control.We know that photons moves in a Null-Like Geodesics $ds^2 = 0$.Then we have:

$$0 = [1 - (vsf(rs))^2]dt^2 + 2vsf(rs)dxdt - dx^2 \tag{206}$$

$$0 = [1 - (vsf(rs))^2] + 2vsf(rs)\frac{dx}{dt} - (\frac{dx}{dt})^2 \tag{207}$$

  but $v = \frac{dx}{dx}$ is the speed of the photon so we are left with[78]:

$$0 = [1 - (vsf(rs))^2] + 2vsf(rs)v - v^2 \tag{208}$$

$$A = vsf(rs) \tag{209}$$

$$0 = 1 - A^2 + 2Av - v^2 \tag{210}$$

  Pay attention in the multiplication by -1

$$0 = -1 + A^2 - 2Av + v^2 \tag{211}$$

$$v^2 - 2Av + A^2 - 1 = 0 \tag{212}$$

$$B = -2A \tag{213}$$

[75] Clark,Hiscock,Larson mentions in pg 4 of [30] "Warp Drive Starships Plying The Galaxy".However Space is not empty and a collision with a Black Hole at 1000 times the speed of light is not for fun.

[76] "Travelling in Hyperspace is not dusting crops boy.Without precise calculations we will impact a Neutron Star or a Supernova or even a Meteor Shower".Harrison Ford as Captain Han Solo talking to Mark Hamill as Luke Skywalker in the George Lucas movie Star Wars Episode IV A New Hope

[77] Everett,Roman and Gonzalez-Diaz used a signature (-,+,+,+) we use a signature (+,-,-,-)

[78] Alcubierre Warp Drive do not obey Lorentz Tansformations

$$C = A^2 - 1 \tag{214}$$

$$v^2 + Bv + C = 0 \tag{215}$$

$$v^2 + Bv + C = 0 \tag{216}$$

$$v = \frac{-B \pm \sqrt{B^2 - 4C}}{2} \tag{217}$$

$$v = \frac{2A \pm \sqrt{4A^2 - 4A^2 + 4}}{2} \tag{218}$$

$$v = \frac{2A \pm \sqrt{4}}{2} \dashrightarrow v = \frac{2A \pm 2}{2} \dashrightarrow v = A \pm 1 \dashrightarrow v = vsf(rs) \pm 1 \tag{219}$$

The expression above have two solutions:one is for the photon sent towards the rear of the Warp Bubble and the other is for the photon sent towards the front of the Warp Bubble

- 1.1)-photon sent towards the rear of the Warp Bubble

$$v2 = vsf(rs) + 1 \tag{220}$$

  Nothing special to say in this case except that the photon will reach the rear Warp Bubble Wall and outside the Warp Bubble the Shape Function $f(rs) = 0$ and the photon will leave the Warp Bubble with a speed $v2 = 1$(the speed is really -1)[79].

- 1.2)-photon sent towards the front of the Warp Bubble

$$v1 = vsf(rs) - 1 \tag{221}$$

  This case is more complicated because inside the Warp Bubble the Shape Function $f(rs) = 1$.The expression becomes

$$v1 = vs - 1 \tag{222}$$

  For a subluminal $vs < 1$ ok but assuming we are accelerating the Warp Bubble to achieve a Faster Than Light speed $vs > 1$ in a given moment we must pass by $vs = 1$ and $v1 = 0$.The photon stops.Any photon sent by the observer to the front of the Warp Bubble will never reach it.The observer looses the capability to signal the front of the Warp Bubble that becomes Causally Disconnected from the observer.Et Voila Le Horizon(see abs and pg 3 of [45],fig 1 and pg 7 of [32]).An observer inside the Warp Bubble cannot accelerate the Warp Bubble to a Faster Than Light Speed.It must be made from outside by an external observer (see the comment on

[79] actually the speed is -1.the photon moves in a direction opposite to the direction of the Warp Bubble motion.we multiplied by -1 between eqs 210 and 211

pg 3 of [45] about the actions to create or change the Warp Bubble trajectory or speed being taken by an external observer whose light cone contains all the trajectory of the Warp Bubble). Assuming that we have an Alcubierre Warp Drive on Earth Orbit with some crew members ready to go to the Messier-1 the Crab Nebula at 6000 light-years from Earth.Earth can accelerate the Alcubierre Warp Drive to a Faster Than Light speed and sent it.But when arriving at Crab Nebula the astronauts will want of course to de-accelerate the Warp Bubble and stop to explore the object.Passing from a $vs >> 1$ to a $vs = 0$ to explore the object will require a continuous de-acceleration and in a given time $vs = 1$ and $v1 = 0$ and the crew will loose contact with the front of the Warp Bubble.Unless is somebody out there in he Crab Nebula to help our astronauts they will never de-accelerate.The Horizon problem is one of the most serious faced by the Alcubierre Warp Drive in its present form. In pg 5 of [43] Gonzalez-Diaz propose the use of the Alcubierre Warp Drive as a Time-Machine to overcome the Horizons Problem.In our attempt to "save and rescue back" the Alcubierre Warp Drive we will propose another way to overcome the Horizons problem that seems to be more realistic than Time Travel.

- 1)-According to Everett-Roman an external observer contains all the Warp Bubble trajectory.[80] Hence our mothership to Crab Nebula would have many Space Probes eg Warp Drive Drones.The mothership would eject one of these Warp Drones and the Drone would "engineer" the Spacetime Metric around the motherhip creating the Alcubierre Warp Drive and accelerate it to a $vs >> 1$ and send it to Crab Nebula.The mothership do not create the Warp Bubble.It would be created by the Warp Drone that would performs as the Everett-Roman external observer whose light cone contains all the Warp Bubble . If NASA uses the gravitational field of Venus as a "slingshot" to accelerate Space Probes to Jupiter and NASA also used the Command Module,the Service Module and a Lunar Module(a spaceship divided in 3 parts) to land on the Moon and only the Command Module returned to Earth while the Lunar Module was abandoned on the Moon and the Service Module on Earth orbit so why not an external Warp Drone generating the Warp Bubble???.This is more reasonable and more affordable than Time Machines.
- 2)-Once at Crab Nebula the crew could perhaps de-accelerate from a $vs >> 1$ to a $vs = 1$. They would loose contact with the front part of the Warp Bubble but the crew members can still signal the rear part of the Warp Bubble.They could send a signal to destroy the rear part of the Warp Bubble from behind.The front part of the Warp Bubble would go on since it cannot be signalized.We still don't know the consequences of the destruction or disruption of the Warp Bubble from behind.The crew would have to de-accelerate by conventional propulsion eg Bussard Ramscoops or Bussard Ramjets.
- 3)-In order to go back to Earth the mothership would eject another Warp Drone to create another Alcubierre Warp Drive and accelerate it to a $vs >> 1$ and send the Warp Bubble back to Earth

This approach seems to be more reasonable than Time Machines[81] and would solve the Horizons Problem eliminating the first obstacle against the Alcubierre Warp Drive.However we can point also 4 drawbacks:

- 1)-The Warp Drone on Earth orbit could perhaps be re-utilized by another mothership but the one at Crab Nebula could be used only one time.After sending the Warp Bubble back

[80] see the comment on pg 3 of [45] about the actions to create or change the Warp Bubble trajectory or speed being taken by an external observer whose light cone contains all the trajectory of the Warp Bubble

[81] more of this on the negative energy section

to Earth it would remain abandoned[82].

  * 2)-Each Alcubierre Warp Drive could only be used for a "one-way" trip.
  * 3)-We don't know what would happens to the mothership if the Warp Bubble is destroyed from behind.
  * 4)-We don't know how to "engineer" the Spacetime to create the Alcubierre Warp Drive.

- 2)Doppler Blueshifts and Impacts with hazardous objects:This is the most terrible and formidable obstacle against the physical feasibility of the Alcubierre Warp Drive. While the mothership remains at the rest inside the Warp Bubble the Warp Bubble moves itself with a great velocity vs with respect to the rest of the Universe and will impact hazardous objects as we pointed out before.We will use the Clark-Hiscock-Larson-Natario approach to demonstrate how terrible are impacts against the Warp Bubble.Considering light particles eg photons of COBE and we have too many per cubic centimeter of space.Applying the non-relativistic Doppler-Fizeau expression [83]

$$f = f_0 \frac{c + va}{c - vb} \tag{223}$$

Where we have:

  - 1)-f is the photon frequency seen by an observer
  - 2)-f_0 is the original frequency of the emitted photon
  - 3)-c is the light speed.in our case $c = 1$
  - 4)-va is the speed of the light source approaching the observer.in our case is vs
  - 5)-vb in the speed of the light source moving away from the observer.in our case $vb = 0$

Rewriting the Doppler-Fizeau expression for an incoming photon approaching the Warp Bubble from the front we should expect for(see eq 26 pg 9 and pg 11 in [30],pg 8 in [32]):

$$f = f_0(1 + vs) \tag{224}$$

Energy E is Planck Constsnt $\hbar$ multiplied by frequency so for the energy we would have:

$$E = E_0(1 + vs) \tag{225}$$

Now we can see how bad is the Doppler Blueshift for the Alcubierre Warp Drive.The energy of the photon impacting the Warp Bubble E is much greater than the original photon energy $E_0.E >> E_0$ and as far as vs increases to Faster Than Light speeds the problem becomes worst.In order to achieve an affordable and reasonable time for an Interstellar Travel in our Galaxy from the point of view of the crew members the ship would perhaps needs to attain a $vs = 200$.Two hundred times Faster Than Light but see again pg 11 in [30].A photon of COBE would impact the Warp Bubble at two hundred times Faster Than Light with the energy of an entire Solar Photosphere!!!!!.And how many photons of COBE we have per cubic centimeter of space????.Each one impacting the Warp Bubble with the energy of a Solar Photosphere?????.Plus how many cubic centimeters of space we have between the

[82] we assume there are no "advanced civilizations" at Crab Nebula
[83] remember again the fact that the Alcubierre Warp Drive do not obey Lorentz Transformations

Sun and Proxima Centauri???.The approach of Clark-Hiscock-Larson and Natario is the most terrible and formidable obstacle against the Alcubierre Warp Drive. Perhaps the Doppler Blueshift problem can be overcomed and the second obstacle against the Alcubierre Warp Drive can be removed and the Alcubierre Warp Drive can still be "saved and rescued back" in this way:

- 1)-According to eq 19 pg 8 in [31],eq 8 pg 6 in [28] and eq 5.8 pg 70 in [48] the energy density of the Alcubierre Warp Drive is negative.It violates all the energy conditions(WEC,NEC,SEC).We are not concerned and not worried about with this due to [29],[46] and [41] where the energy density can be lowered to affordable levels and also due to abs of [44] where macroscopic amounts of negative energy densities can be created.We also know that the Casimir Effect can create the negative energy densities.(see pg 9 in [31]).Still according to eq 25 pg 9 in [28] if the energy density is negative then the total energy of the Alcubierre Warp Drive is negative too.But if the energy E of the Alcubierre Warp Drive is negative then the total mass M of the Alcubierre Warp Drive is also negative.
- 2)-A negative mass M would generate a negative Gravitational Bending Of Light and a negative Gravitational Field that would repeal objects instead of attract.Consider a Asteroid of mass M_A positive of course and an Alcubierre Warp Drive of mass M_W negative.The Gravitational Force would be negative and given by $F = G\frac{M_A M_W}{d^2}$.So the Asteroid would be naturally shifted from the Warp Bubble.Of course this is also due to our Shape Function $f(rs)$ that allow a different distribution of the negative energy density T^{00}.While in original Alcubierre Shape Function the negative energy is concentrated in a region toroidal perpendicular to the direction of the motion of the spaceship(see pg 6 and fig 3 pg 7 in [28] and pg 70 and fig 5.3 pg 71 in [48]) while the spaceship remains on empty space vulnerable to Doppler Blueshifted photons[84] our Warp Bubble involves the spaceship with a sphere of negative energy protecting it from impacts.Due to the negative Bending of Light the photons would be shifted too[85]. The Spacetime Curvature of an object of negative mass is opposite to the similar one of a positive mass so while the Sun bends photons in a inwards direction a negative mass objects would bend photons in a outwards direction. While other autors considers the negative energy a pathology we consider it a bless because a positive energy density would attract objects disrupting effectively the Warp Bubble.
- 3)-The Warp Bubble cannot be signalized from outside with photons.Our Warp Drone while "engineering" the Alcubierre Warp Drive of negative mass M_W would send packets of "Casimir matter" of negative mass M_C to the Warp Bubble by a still unknown process.The gravitational force between two negative masses would cancel the minus sign in both masses making the gravitational force attractive so the Warp Bubble would only be signalized by negative masses.Also the crew members while signalizing the rear part of the Warp Bubble must uses negative matter to disrupt the Warp Bubble from behind.Remember that the negative matter moves at subluminal speeds but at least we know that the Warp Bubble lies entirely in the light-cone of the Warp Drone and we also know that the crew members can signalize the rear part of the Warp Bubble.

This could perhaps solve the Doppler Blueshift and Impacts with Hazardous Objects Problem as the second obstacle against the Alcubierre Warp Drive but again we must point at least 3 drawbacks:

[84]the ship remains in the center of the Warp Bubble and this means the black regions on fig 3 or fig 5.3 leaving the ship "unprotected" while the negative energy regions are the white toroidal regions

[85]more of this on the negative energy section

- 1)-It could perhaps shift COBE photons or Asteroids but what if a Large Black Hole appears...???
- 2)-A very massive positive object would generate a strong gravitational repulsive force repealing the Warp Bubble.What the consequences???Could the Warp Bubble be disrupted or destroyed by a large repulsive gravitational force?????
- 3)-We don't know how to "engineer" the Spacetime to create the Alcubierre Warp Drive.

- 3)Enormous energy densities required to create it:This item is the most easily to be overcomed:See [29] and [41] but instead of other Warp Drive geometries [29],[32] more complicated [86] we will demonstrate how the negative energy can be "lowered" in the original Alcubierre Warp Drive.The solution will come from the work of Ford and Pfenning [28],[48]. All we have to do is to choose the "correct" Shape Function.The energy density in the Alcubierre Warp Drive is given by(see eq 19 pg 8 in [31],eq 8 pg 6 in [28] and eq 5.8 pg 70 in [48]):

$$T^{00} = \frac{1}{8\pi}G^{00} = -\frac{1}{8\pi}\frac{vs^2\rho^2}{4rs^2}[\frac{df(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df(rs)}{drs}]^2 \tag{226}$$

With [87] :

$$\rho^2 = y^2 + z^2 \tag{227}$$

As far as the Warp Bubble accelerates to $vs >> 1$[88] the energy density becomes bigger...and more negative.Assuming that the Alcubierre Warp Drive have motion only in the $x-axis$ then ρ^2 is a constant.The "trick" to lower the energy density lies in the term $\frac{df(rs)}{drs}$.As Ford and Pfenning pointed out correctly in pg 3 of [28] and in pg 68 of [48] we dont need to choose a particular form of $f(rs)$:Any function $f(rs) = 1$ inside the Warp Bubble where $rs < R$ or $rs < R - \delta$ in the original Alcubierre Warp Bubble thickness δ and $f(rs) = 0$ where $rs > R$ or $rs > R+\delta$ while decreasing from 1 to 0 in the Warped Region $R-\delta < rs < R+\delta$ $0 < f(rs) < 1$ is a valid Shape Function.So everything depends on the form of the Shape Function.A "good" Shape Function will perform better and lowers the energy density more than a "bad" or a "evil" Shape Function.The original Alcubierre Shape Function is complicated due to the hyperbolic terms so in order to simplify the energy density calculations Ford and Pfenning introduced the Piecewise Shape Function.By manipulating the Piecewise Shape Function we can lower the energy density requirements without strange topologies.The Piecewise Shape Function is defined by(eq 4 pg 3 in [28] and eq 5.4 pg 68 in [48]):

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

The parameter Δ is the thickness of the Warped Region in the Piecewise Shape Function and is related to the thickness of the Warped Region δ in the original Alcubierre Shape Function by the following expression(eq 5 pg 4 in [28] and eq 5.5 pg 68 in [48]):

[86]The Warp Drives of Broeck and Natario are more complicated than the original Alcubierre one. It will be difficult to generate the Alcubierre Warp Drive not to mention these ones

[87]now we know why y and z cannot be zero.the T^{00} would be null leading to a unphysical result

[88]action taken by an external observer eg Warp Drone

$$\Delta = \frac{[1 + tanh^2(\delta R)]^2}{2\delta tanh(\delta R)} \tag{228}$$

The Piecewise Shape Function gives results similar to the Alcubierre Shape Function(see comments before eq 8 pg 6 in [28] and before eq 5.8 pg 70 in [48])[89]

Ford and Pfenning computed the total energy for an Alcubierre Warp Drive with a constant Faster Than Light Speed vs in pg 9 eqs 25 and 26 of [28] and in pg 73 eqs 5.25 and 5.26 of [48] as follows

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 \tag{229}$$

$$E = \int [T^{00}]dV_{volume} \tag{230}$$

$$E = -\frac{vs^2}{32\pi}\int [\frac{\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2]dV_{volume} \tag{231}$$

passing to spherical coordinates

$$E = -\frac{vs^2}{12}\int [rs^2[\frac{df_{pf}(rs)}{drs}]^2]drs \tag{232}$$

Note again that the "trick" to lower the integral of the total energy is the term $[\frac{df_{pf}(rs)}{drs}]^2$

Ford and Pfenning uses the so-called Quantum Inequalities(QI) to place limits on the thickness of the Warp Bubble Walls.Although we do not worry too much about the QI due to Krasnikov in [46] and [41] we will comment how the QI can easily be overcomed and QI can also be used to demonstrate how to lower the energy density to affordable levels.

Inserting the QI eqs 9 and 10 of [28] and eqs 5.9 and 5.10 of [48][90]

$$\frac{\tau_0}{\pi}\int_{-\infty}^{+\infty}\frac{T_{\mu\nu}U^{\mu}U^{\nu}}{\tau^2+\tau_0^2}d\tau = \frac{\tau_0}{\pi}\int_{-\infty}^{+\infty}\frac{T^{\mu\nu}U_{\mu}U_{\nu}}{\tau^2+\tau_0^2}d\tau = \frac{\tau_0}{\pi}\int_{-\infty}^{+\infty}\frac{T^{00}}{\tau^2+\tau_0^2}d\tau \geq -\frac{3}{32\pi^2\tau_0^4} \tag{233}$$

Where τ is an inertial observer proper time and τ_0 is an arbitrary sampling time(see pg 7 of [28] and pg 70 of [48])

$$\frac{\tau_0}{\pi}\int_{-\infty}^{+\infty}\frac{-\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2}{\tau^2+\tau_0^2}d\tau \geq -\frac{3}{32\pi^2\tau_0^4} \tag{234}$$

$$-\frac{\tau_0}{\pi}\int_{-\infty}^{+\infty}\frac{vs^2\rho^2[\frac{df_{pf}(rs)}{drs}]^2}{32\pi rs^2(\tau^2+\tau_0^2)}d\tau \geq -\frac{3}{32\pi^2\tau_0^4} \tag{235}$$

[89] fortunately this is not entirely true.The Ford-Pfenning Piecewise Shape Function can give results different than the original Alcubierre Shape Function as Ford-Pfenning mentioned in the comments.We will demonstrate this during the calculations of the total energy

[90] see eqs 8 and 10 of [28] and eqs 5.8 and 5.10 of [48]

$$-\frac{\tau_0}{\pi}\int_{-\infty}^{+\infty}[\frac{df_{pf}(rs)}{drs}]^2\frac{vs^2\rho^2}{32\pi rs^2(\tau^2+\tau_0^2)}d\tau \geq -\frac{3}{32\pi^2\tau_0^4} \tag{236}$$

And we arrived at the eq 10 of [28] and eq 5.10 of [48]

$$\tau_0\int_{-\infty}^{+\infty}[\frac{df_{pf}(rs)}{drs}]^2\frac{vs^2}{rs^2(\tau^2+\tau_0^2)}d\tau \leq \frac{3}{\rho^2\tau_0^4} \tag{237}$$

$$t_0\int_{-\infty}^{+\infty}[\frac{df_{pf}(rs)}{drs}]^2\frac{vs^2}{rs^2(t^2+t_0^2)}dt \leq \frac{3}{\rho^2 t_0^4} \tag{238}$$

The sampling time $\tau_0 = t_0$ according to Ford and Pfenning is very small compared to the time t the Warp Bubble is changing the speed vs so they consider a Warp Bubble with constant speed vs.(see pg 7 of [28] and pg 71 of [48]).

$$(t_0)(vs^2)\int_{-\infty}^{+\infty}[\frac{df_{pf}(rs)}{drs}]^2\frac{1}{rs^2(t^2+t_0^2)}dt \leq \frac{3}{\rho^2 t_0^4} \tag{239}$$

see the right side of eq 14 pg 8 in [28] and eq 5.14 pg 71 in [48]

$$t_0\int_{-\infty}^{+\infty}[\frac{df_{pf}(rs)}{drs}]^2\frac{1}{rs^2(t^2+t_0^2)}dt \leq \frac{3}{vs^2\rho^2 t_0^4} \tag{240}$$

We know that inside the Warp Bubble $f_{pf} = 1$ and outside the Warp Bubble $f_{pf} = 0$ so for both cases f_{pf} is a constant and $[\frac{df_{pf}(rs)}{drs}]^2 = 0$.We are interested in the behavior of the QI only in the Warped Region.Hence we can write the QI as:

$$t_0\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[\frac{df_{pf}(rs)}{drs}]^2\frac{1}{rs^2(t^2+t_0^2)}dt \leq \frac{3}{vs^2\rho^2 t_0^4} \tag{241}$$

But we know from the definition of the Piecewise Shape Function that[91]:

- 1) $[\frac{df_{pf}(rs)}{drs}]^2 = 0 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta}]^2 \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $[\frac{df_{pf}(rs)}{drs}]^2 = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

$$t_0\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[-\frac{1}{\Delta}]^2\frac{1}{rs^2(t^2+t_0^2)}dt \leq \frac{3}{vs^2\rho^2 t_0^4} \tag{242}$$

$$t_0\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}\frac{1}{\Delta^2}\frac{1}{rs^2(t^2+t_0^2)}dt \leq \frac{3}{vs^2\rho^2 t_0^4} \tag{243}$$

[91] we consider a Warp Bubble of constant radius R and a constant thickness Δ

Our expression differs a little bit from eq 14 pg 8 in [28] and eq 5.14 pg 71 in [48] but the right side is the same.

$$t_0 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} \frac{1}{rs^2(t^2+t_0^2)} dt \leq \frac{3\Delta^2}{vs^2 \rho^2 t_0^4} \tag{244}$$

As Ford and Pfenning says the sampling time $\tau_0 = t_0$ is arbitrary and very small compared to the time t the Warp Bubble is changing the speed vs so we can easily see that:(see pg 7 of [28] and pg 71 of [48]).

$$t_0 << \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} \frac{1}{rs^2(t^2+t_0^2)} dt \tag{245}$$

Hence we can make the sampling time very very small for a Warp Bubble of almost constant speed vs since the time needed for change of the speed in the Warp Bubble would always be greater than the sampling time and for the energy calculations Ford and Pfenning considered a Warp Bubble of constant velocity vs although for a Warp Bubble of variable speed in a short period of time the sampling time would perhaps be not so small at all.We will address this later in this section.

$$\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} \frac{1}{rs^2(t^2+t_0^2)} dt \leq \frac{3\Delta^2}{vs^2 \rho^2 t_0^5} \tag{246}$$

A large Warp Bubble thickness Δ coupled to a very small sampling time t_0 would make the term $\frac{3\Delta^2}{vs^2 \rho^2 t_0^5}$ larger than the integral making the QI hold as shown in the expression above and not restraining the size of the Warp Bubble thickness.This is the reason why we agree with Krasnikov in [46] and [41].Note that for a large Warp Bubble speed vs the things becomes more difficult to make the QI hold because for a large speed vs the term $\frac{3\Delta^2}{vs^2 \rho^2 t_0^5}$ becomes smaller.We will see the same on the total energy integral calculations but we will present a way to overcome this.

Back to total energy integral of the Warp Bubble we should expect for(see eq 27 and 28 pg 10 in [28] and eq 5.27 pg 73 in [48]):

$$E = -\frac{1}{12} \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(vs^2)(rs^2)[\frac{df_{pf}(rs)}{drs}]^2] drs = -\frac{vs^2}{12} \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[\frac{df_{pf}(rs)}{drs}]^2] drs \tag{247}$$

$$E = -\frac{vs^2}{12} \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[-\frac{1}{\Delta}]^2] drs = -\frac{vs^2}{12} [-\frac{1}{\Delta}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} (rs^2) drs \tag{248}$$

$$E = -\frac{1}{12} [\frac{vs^2}{\Delta^2}] \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} (rs^2) drs = -\frac{1}{12} [\frac{vs}{\Delta}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} (rs^2) drs = -\frac{1}{12} (vs^2) (\frac{R^2}{\Delta} + \frac{\Delta}{12}) \tag{249}$$

From the expression above we can see that a large Warp Bubble speed vs will raise the amount of negative energy needed.Ford and Pfenning restricted the thickness Δ of the Warp Bubble to the same scale of the Planck size(see eq 23 pg 9 in [28] and eq 5.23 pg 72 in [48]) and the term $\frac{R^2}{\Delta}$ results in a huge number for the total energy integral.Dividing a Warp Bubble radius $R = 100m$

according to Ford-Pfenning by a number close to the Planck size results in a number roughly ten times the mass of the Universe of negative energy needed to sustain the Warp Bubble[92]. But note that Ford-Pfenning says also in pg 10 of [28] and in pg 73 of [48] that a Warp Bubble thickness $\Delta = 1$ and Ford-Pfenning mentions explicitly a violation of the QI integral would lower the magnitude of negative energy needed to sustain a Warp Bubble of radius $R = 100m$ from 10 times the mass of the Universe to a $\frac{1}{4}$ of a Solar Mass.An improvement without shadows of doubt but not too much

We have two choices to low the negative energy needed to sustain the Warp Bubble

- 1)-a Warp Bubble of a thickness $\Delta < R \dashrightarrow \Delta \simeq R$ would make the term $\frac{R^2}{\Delta} \simeq R$ but this would be similar to the situation $\Delta = 1$ due to the term $\frac{\Delta}{12} \simeq \frac{R}{12}$
- 2)-Introducing a new Shape Function

While the Alcubierre choice for a continuous Shape Function involved a toroidal geometry that would not protect the spaceship from impacts of Doppler Blueshifted photons(making valid the obstacle raised by Clark-Hiscock-Larson and Natario) and Ford-Pfenning introduced a Piecewise Shape Function that divides 1 by a Warp Bubble thickness Δ of the magnitude of the Planck Length and dividing 1 by a Planck Length of 10^{-35} would result in the huge number of 10 times the mass of the Universe in negative energy to sustain a Warp Bubble.Ford and Pfenning created a Piecewise Shape Function that really demands 10 times the mass of the Universe in negative energy to sustain a Warp Bubble.This don't means to say that the Warp Drive is impossible.Its impossible with the Shape Function choosed by Ford and Pfenning and impossible with the Shape Function choosed by Alcubierre.But as Ford-Pfenning pointed out any Shape Function that gives 1 inside the Warp Bubble 0 far from it and $0 < f(rs) < 1$ in the Warped Region is a equally valid Shape Function.We will now present three Shape Functions inspired on the Ford-Pfenning Piecewise Shape Function.Our Piecewise Shape Functions will low the energy density requirements of the Alcubierre Warp Drive to low and affordable levels and still have a Warp Bubble topology that will protect the ship against incoming Blueshifted photons or impacts with small objects eg Asteroids.[93]

The reason why Ford and Pfenning arrived at the huge number of 10 times the mass of the Universe or at least $\frac{1}{4}$ of a Solar mass of negative energy needed to sustain a Warp Bubble was due to the term they choosed for the Warped Region

$$f_{pf} = -\frac{1}{\Delta}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{250}$$

It is easy to see that if $\frac{1}{\Delta} = \frac{1}{10^{-35}}$ and the shape of the total energy integral is $E = -\frac{1}{12}[\frac{vs}{\Delta}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} (rs^2) drs$

then dividing 1 by 10^{-35} we would get a big number and to make the things even worst we are dividing $[\frac{vs}{\Delta}]^2$.We are dividing a high velocity vs by 10^{-35} and raise to a power of 2.Of course the result would then be a physical unattainable amount of negative energy.

Our Piecewise Shape Functions were designed to enter in the total energy integral

$E = -\frac{vs^2}{12} \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[\frac{df_{pf}(rs)}{drs}]^2] drs$

[92] see pg 10 after eq 31 of [28] and see pg 73 after eq 5.30 of [48]

[93] an impact with a Large Black Hole would perhaps destroy the Warp Bubble.more on this with the comparisons between Alcubierre and Chung-Freese

Lowering the energy density requirements due to our different factor $\frac{df_{pf}(rs)}{drs}$

Our Piecewise Shape Functions are in total agreement with the Ford-Pfenning definitions resembling the original Ford-Pfenning Piecewise Shape Function(eq 4 pg 3 in [28] and eq 5.4 pg 68 in [48]) and our Piecewise Shape Functions also gives 1 inside the Warp Bubble 0 far from it and $0 < f_{pf} < 1$ in the Warped Region.

We will introduce now a factor $h(rs)$ that is 1 in the region where the ship is located,starts to grow when entering in the Warped Region reaches its maximum value in the center of the Warped Region decreases when we approach the end of the Warped Region and is again 1 outside the Warped Region.

The continuous expression for $h(rs)$ is given by:

$$h(rs) = [\frac{1 + tanh[\delta(rs - R)]^2}{2}]^{-\frac{R}{\Delta}\frac{vs}{c}} \tag{251}$$

In the expression above $c = 1$ and we divide in the power factor the Radius R by the Thickness Δ and the ship speed vs by the light speed c.This enable ourselves to attain an enormous value for $h(rs)$ in the center of the Warped Region.[94]

Our continuous Shape Function would then be given by:

$$g(rs) = \frac{f(rs)}{h(rs)} \tag{252}$$

This is an alternative to the Alcubierre continuous Shape Function that provides $0 < f(rs) < 1$ in the Warped Region.Our factor $h(rs)$ gives us a $h(rs) >> 1$ in the Warped Region and dividing $\frac{f(rs)}{h(rs)}$ would enable ourselves to get a $0 < g(rs) < 1$ but closer to 0 than the original Alcubierre Shape Function.

Our Shape Function is somewhat more complicated than the Alcubierre one and we will use the Ford-Pfenning Piecewise behavior to study the factor $h(rs)$.

A Piecewise expression for our factor $h(rs)$ would be given by:

- 1) $hrs = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $h(rs) = |R|^2 or |R|^4 or |R|^6 \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $h_{rs} = 1 \dashrightarrow rs > R + \frac{\Delta}{2}$

To reproduce the enormous value attained by the continuous expressions for $h(rs)$ in the Warped Region[95] we use the modulus of the Radius.[96]

Our Piecewise Shape Functions resembles the original Ford-Pfenning Piecewise Shape Function but like our continuous Shape Function is also divided by the factor $h(rs)$.

Here are our three Piecewise Shape Functions:

[94] Consider for example a Radius $R = 20meters$,$\delta = 2meters$,Thickness $\Delta = 10meters$,speed $vs = 100timeslightspeed$,light speed $c = 1$ and a rs that varies from 0 to $40meters$.We can provide a Microsoft Excel simulation for those interested.

[95] Our Microsoft Excel simulation with the values provided by a previous footnote gives a value of $1,61 \times 10^{60}$

[96] in the previous version of this paper we got the idea of the modulus of the Radius however in the equations we placed the radius.This is a correction

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta|R|^2}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta|R|^4}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta|R|^6}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

It is easy to see that in the Warped Region our Shape Functions will also give the Ford-Pfenning result $0 < f_{pf} < 1$ due to the following expressions:[97]

$$f_{pf} = -\frac{1}{\Delta|R|^2}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{253}$$

$$f_{pf} = -\frac{1}{\Delta|R|^4}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{254}$$

$$f_{pf} = -\frac{1}{\Delta|R|^6}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{255}$$

Our Piecewise Shape Functions also use the Warp Bubble thickness Δ but we don't care about QI due to our previous QI calculations and note that Ford-Pfenning considered also a thickness $\Delta = 1$ in a clear QI violation.But we introduced the Warp Bubble modulus of the Radius R in the definitions of our Shape Functions.The Warp Bubble modulus of the Radius R is the key ingredient to lower the energy density requirements to low and affordable levels eliminating the third obstacle against the Alcubierre Warp Drive[98].We will see the calculations right now:

The Total Energy Integral is given by:(see eq 27 and 28 pg 10 in [28] and eq 5.27 pg 73 in [48])

$$E = -\frac{vs^2}{12} \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[\frac{df_{pf}(rs)}{drs}]^2] drs \tag{256}$$

Our factors $\frac{df_{pf}(rs)}{drs}$ when inserted in the Total Energy Integral will give the results shown below:

$$[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta|R|^2}]^2 = [\frac{1}{\Delta|R|^2}]^2 \tag{257}$$

[97] consider a Warp Bubble Radius $R = 100meters$ and a Warp Bubble Thickness $\Delta = 10meters$ and compute the values of the rs inside the Warped Region or example for a $rs = 100meters$.Anyone can see that our expressions also obeys $0 < f_{pf} < 1$

[98] if Ford and Pfenning can use the Warp Bubble thickness Δ in the definition of the Shape Function then why not use the Warp Bubble modulus of the Radius R?.it will low the energy density and will produce the desired result $0 < f_{pf} < 1$

$$[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta|R|^4}]^2 = [\frac{1}{\Delta|R|^4}]^2 \tag{258}$$

$$[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta|R|^6}]^2 = [\frac{1}{\Delta|R|^6}]^2 \tag{259}$$

$$E = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[-\frac{1}{\Delta|R|^2}]^2]drs = -\frac{vs^2}{12}[\frac{1}{\Delta|R|^2}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs \tag{260}$$

$$E = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[-\frac{1}{\Delta|R|^4}]^2]drs = -\frac{vs^2}{12}[\frac{1}{\Delta|R|^4}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs \tag{261}$$

$$E = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)[-\frac{1}{\Delta|R|^6}]^2]drs = -\frac{vs^2}{12}[\frac{1}{\Delta|R|^6}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs \tag{262}$$

Note that this negative energy would mean a negative mass and a negative gravitational force that would repeal objects instead of attract.This will be useful to protect the ship from incoming blueshifted photons and hazardous objects.We will show that our Piecewise Shape Functions don't have the toroidal geometry of the Alcubierre Shape Function and we can protect the ship against incoming blueshifted photons.This is the reason why we must have the negative energy in front of the ship and not in a toroidal distribution.

Remember that the Gravitational Force in units $G = c = 1$ for an Alcubierre Warp Drive of mass $M_w = -\frac{vs^2}{12}[\frac{1}{\Delta|R|^6}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs$ approaching a positive object(eg a small Asteroid) of mass M_a would be given by $F = \frac{M_w M_a}{d}$ being d the separation distance and as far as the object approaches the Alcubierre Warp Drive the distance d becomes smaller making the repulsive force bigger repealing the object.The negative force comes from the minus sign in $M_w = -\frac{vs^2}{12}[\frac{1}{\Delta|R|^6}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs$

From the expressions above it is easy to see that our Piecewise Shape Functions are able to low the energy density requirements in the original Alcubierre Warp Drive Geometry without recurring to other more complicated solutions(Broeck [29],Natario [32])[99].With the third obstacle solved we can say with confidence that the Alcubierre Warp Drive is "still alive"[100] as a fully functional Superluminal and Faster Than Light Spacetime Ansatz of General Relativity.Miguel Alcubierre was right after all these years[101].The Alcubierre Warp Drive is a Wonderful Idea.The Historical Paper of 1994 will ever be considered as a Revolutionary Paper for the Human Science because it launched the First Foundations of the Faster Than Light Space Travel.

[99]the funny thing is the fact that as large as the Warp Bubble modulus of the Radius R is greater as low the energy density becomes.This can be a math curiosity but anyone can see that our calculations are right due to the form of our Piecewise Shape Functions.

[100]if an Alcubierre Warp Drive collides with a Large Black Hole it would be destroyed.This don't means to say the Alcubierre Warp Drive is impossible because NASA Space Shuttle in the neighborhoods of the same Black Hole would be destroyd too and we know that the Space Shuttle is possible

[101]although the original Shape Function choosed by Alcubierre with the toroidal distribution of negative energy up and below the spaceship would not protect the ship against blueshifted photons making valid the objections raised by Clark-Hiscock-Larson and Natario

While in original Alcubierre Shape Function the negative energy is concentrated in a region toroidal perpendicular to the direction of the motion of the spaceship(see pg 6 and fig 3 pg 7 in [28] and pg 70 and fig 5.3 pg 71 in [48]) while the spaceship remains on empty space vulnerable to Doppler Blueshifted photons[102] our Warp Bubble involves the spaceship with a sphere of negative energy protecting it from impacts.Due to the negative Bending of Light the photons would be shifted too

The energy density T^{00} of the Stress-Energy-Momentum Tensor for our Piecewise Shape Functions is given by:

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{1}{\Delta|R|^2}]^2 \tag{263}$$

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{1}{\Delta|R|^4}]^2 \tag{264}$$

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{1}{\Delta|R|^6}]^2 \tag{265}$$

Note that in the region where the negative energy resides T^{00} cannot be zero and this region is the Warped Region where $0 < f_{pf} < 1$ and $\frac{df_{pf}(rs)}{drs} \neq 0$.This means to say the region where rs approaches the Warp Bubble Radius R or better the region :$R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$

$$T^{00} = -\frac{vs^2}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta|R|^2}]^2 \tag{266}$$

$$T^{00} = -\frac{vs^2}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta|R|^4}]^2 \tag{267}$$

$$T^{00} = -\frac{vs^2}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta|R|^6}]^2 \tag{268}$$

A Warp Bubble Sphere of Radius $R = x^2 + y^2 + z^2$ surrounding the spaceship is the ideal way.Note that if we keep fixed y and z and move ourselves in the $x-axis$ starting from the spaceship position xs towards the Warp Bubble Radius and towards the Warped Region in the front of the spaceship then according to the definition of rs when rs approaches R and we would have $R = [x - xs(t)]^2 + y^2 + z^2$

Note that in this case T^{00} do no vanish and we have negative energy in front of the spaceship protecting it from incoming objects.

Suppose now we are in the position of the spaceship $x = xs$ and we move ourselves in the plane $y - z$ to the Warp Bubble Radius R and the Warped Region upside(or downside the ship).The expressions for T^{00} would then be:

$$T^{00} = -\frac{vs^2}{32\pi}[\frac{1}{\Delta|R|^2}]^2 \tag{269}$$

[102]the ship remains in the center of the Warp Bubble and this means the black regions on fig 3 or fig 5.3 leaving the ship "unprotected" while the negative energy regions are the white toroidal regions

$$T^{00} = -\frac{vs^2}{32\pi}[\frac{1}{\Delta|R|^4}]^2 \tag{270}$$

$$T^{00} = -\frac{vs^2}{32\pi}[\frac{1}{\Delta|R|^6}]^2 \tag{271}$$

Note that we can move backwards or forwards or upstairs or downstairs but the ship will always have a non-null T^{00} involving the ship as a protective "cocoon" a real Warp Bubble.

While a toroidal distribution of the negative energy upside and downside with the front of the ship in empty space leaves the ship unprotected we closed the front of the ship in order to protect it.The original Ford-Pfenning Piecewise Shape Function could also be used to close the front of the ship.This is the main difference between the Alcubierre Shape Function and the Ford-Pfenning Piecewise Shape Function.

The analogous expressions for the Ford-Pfenning Piecewise would then be:

$$T^{00} = -\frac{vs^2}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta}]^2 \tag{272}$$

$$T^{00} = -\frac{vs^2}{32\pi}[\frac{1}{\Delta}]^2 \tag{273}$$

Note that Ford-Pfenning also describes a Warp Bubble as a "cocoon" surrounding and protecting the ship although this "cocoon" requires more energy than ours because they did not included the Warp Bubble modulus of the Radius R in the definition of the Piecewise Shape Function.

Another remarkable thing is the fact that the Alcubierre Warp Drive defines a expansion of the Spacetime behind the spaceship and a contraction in front of it due to the nature of the Alcubierre Shape Function.

Our Piecewise Shape Function and the one of Ford-Pfenning does not defines expansion or contraction.Our Warp Drive and Ford-Pfenning one involves the ship in a "cocoon" that is being carried out by the Spacetime "stream" just like a fish in the stream of a river while the ship remains at the rest with respect to the "cocoon" center.Imagine a fish a salmon for example being carried out by a river stream and imagine our "cocoon" being carried out by the Spacetime "stream".Our "cocoon" is the fish the river is the Spacetime "stream" and inside the "coccon" similar to the fish the ship remains at the rest free from g-forces.In a way our Warp Drive and Ford-Pfenning one are very similar to the Natario Warp Drive that also don't suffer expansion or contraction.(see abstract and pg 1 of [32])

We will demonstrate this right now:

Starting with the original Alcubierre expression for the expansion of the volume elements given as follows but using our Piecewise Shape Functions:(see eq 12 pg 5 in [31],eq 9 pg 3 in [42] and pg 4 of [32])

$$\theta = vs\frac{x-xs}{rs}\frac{df_{pf}(rs)}{drs} \tag{274}$$

$$\theta = vs\frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^2}] \tag{275}$$

$$\theta = vs \frac{x - xs}{\sqrt{[x - xs(t)]^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^4}] \tag{276}$$

$$\theta = vs \frac{x - xs}{\sqrt{[x - xs(t)]^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^6}] \tag{277}$$

Assuming that the ship lies at the rest in the center of the Warp Bubble and for a Frame placed in the ship with the ship in the origin then $xs = 0$ and $rs = \sqrt{x^2 + y^2 + z^2}$(see eq 24 pg 9 in [28] and eq 5.24 pg 73 in [48])[103]

$$\theta = vs \frac{x}{\sqrt{x^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^2}] \tag{278}$$

$$\theta = vs \frac{x}{\sqrt{x^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^4}] \tag{279}$$

$$\theta = vs \frac{x}{\sqrt{x^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^6}] \tag{280}$$

If we define the following coordinate(a comoving ship Frame)$x^l = x - xs$

$$\theta = vs \frac{x^l}{\sqrt{(x^l)^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^2}] \tag{281}$$

$$\theta = vs \frac{x^l}{\sqrt{(x^l)^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^4}] \tag{282}$$

$$\theta = vs \frac{x^l}{\sqrt{(x^l)^2 + y^2 + z^2}} [-\frac{1}{\Delta |R|^6}] \tag{283}$$

It is easy to see that for example the expansion when $x^l = -5$ is balanced by the contraction when $x^l = 5$ but remember that we have division by powers of the Warp Bubble Radius R[104] multiplied by the Warp Bubble Thickness Δ to reduce the effect to almost zero.

In order to reduce the expansion/contraction of the Spacetime volume elements would be desirable to remove the speed of the Warp Bubble vs from the expressions above.As a matter of fact would be desirable to removes vs from the Total Energy Integral and from the energy density to low even more the energy requirements.This can be done by redefining the Piecewise Shape Functions including also the Warp Bubble speed vs together with the Warp Bubble modulus of the Radius R and the Warp

[103] see Ford and Pfenning observation of a constant speed vs and observation taken in the initial time $t = 0$

[104] at least in the case R^6.Consider a Warp Bubble Radius of $R = 100 meters$ and a speed 100 times the light speed and we will have an expansion/contraction factor of 10^{-2}

Bubble Thickness Δ to allow the remotion of the term vs in the expansion/contraction expression as follows[105][106]:

Here are the new Piecewise Shape Functions:

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta|R|^2 vs}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta|R|^4 vs}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

- 1) $f_{pf} = 1 \dashrightarrow rs < R - \frac{\Delta}{2}$
- 2) $f_{pf} = -\frac{1}{\Delta|R|^6 vs}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2}$
- 3) $f_{pf} = 0 \dashrightarrow rs > R + \frac{\Delta}{2}$

We are interested in the terms inside the Warped Region which means to say:

$$f_{pf} = -\frac{1}{\Delta|R|^2 vs}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{284}$$

$$f_{pf} = -\frac{1}{\Delta|R|^4 vs}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{285}$$

$$f_{pf} = -\frac{1}{\Delta|R|^6 vs}(rs - R - \frac{\Delta}{2}) \dashrightarrow R - \frac{\Delta}{2} < rs < R + \frac{\Delta}{2} \tag{286}$$

Their derivatives:

$$[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta|R|^2 vs}]^2 = [\frac{1}{\Delta|R|^2 vs}]^2 \tag{287}$$

$$[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta|R|^4 vs}]^2 = [\frac{1}{\Delta|R|^4 vs}]^2 \tag{288}$$

$$[\frac{df_{pf}(rs)}{drs}]^2 = [-\frac{1}{\Delta|R|^6 vs}]^2 = [\frac{1}{\Delta|R|^6 vs}]^2 \tag{289}$$

Total Energy Integral:

[105] like Ford and Pfenning we consider a constant Faster Than Light Speed vs in pg 9 eqs 25 and 26 of [28] and in pg 73 eqs 5.25 and 5.26 of [48]

[106] note that inside the Warped Region any one of these new Piecewise Shape Functions defined in function of the Warp Bubble Radius R Thickness Δ and speed vs satisfies the Ford-Pfenning requirement of $0 < f_{pf} < 1$.Consider a Radius $R = 100meters$ a Thickness $\Delta = 10meters$ and a speed $vs = 100timeslightspeed$

$$E = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)[-\frac{1}{\Delta|R|^2vs}]^2]drs = -\frac{1}{12}[\frac{1}{\Delta|R|^2}]^2\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)]drs \tag{290}$$

$$E = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)[-\frac{1}{\Delta|R|^4vs}]^2]drs = -\frac{1}{12}[\frac{1}{\Delta|R|^4}]^2\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)]drs \tag{291}$$

$$E = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)[-\frac{1}{\Delta|R|^6vs}]^2]drs = -\frac{1}{12}[\frac{1}{\Delta|R|^6}]^2\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)]drs \tag{292}$$

Energy Density:

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{1}{\Delta|R|^2vs}]^2 = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^2}]^2 \tag{293}$$

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{1}{\Delta|R|^4vs}]^2 = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^4}]^2 \tag{294}$$

$$T^{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df_{pf}(rs)}{drs}]^2 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{1}{\Delta|R|^6vs}]^2 = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^6}]^2 \tag{295}$$

$$T^{00} = -\frac{1}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta|R|^2}]^2 \tag{296}$$

$$T^{00} = -\frac{1}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta|R|^4}]^2 \tag{297}$$

$$T^{00} = -\frac{1}{32\pi}\frac{y^2+z^2}{[x-xs(t)]^2+y^2+z^2}[\frac{1}{\Delta|R|^6}]^2 \tag{298}$$

Expansion of the volume elements:[107]

$$\theta = vs\frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^2vs}] = \frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^2}] \tag{299}$$

$$\theta = vs\frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^4vs}] = \frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^4}] \tag{300}$$

$$\theta = vs\frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^6vs}] = \frac{x-xs}{\sqrt{[x-xs(t)]^2+y^2+z^2}}[-\frac{1}{\Delta|R|^6}] \tag{301}$$

Note that the inclusion of the Warp Bubble speed vs in the definition of the Piecewise Shape Functions reduces the energy density even more and makes the expansion/contraction of the volume elements

[107] even for the first case R^2 consider a Warp Bubble Radius $R = 100meters$ and a Warp Bubble Thickness $\Delta = 10meters$.anyone can see that the expansion/contraction is almost close to zero

almost close to zero resembling the Natario Warp Drive.An Alcubierre Warp Drive practically without expansion or contraction just like the Natario one.Our Piecewise Shape Functions fits well in the Alcubierre Warp Drive Spacetime to restore its physical feasibility as a valid and fully functional Superluminal and Faster Than Light Spacetime Ansatz of General Relativity.All the pathologies can be solved after all.Again we want to say that Miguel Alcubierre had a Wonderful Idea.By removing the original Alcubierre Shape Function and by modifying the Ford-Pfenning Piecewise Shape Function to satisfy our needs we can overcome the unphysical features and retain the Geometrical Beauty of the original Alcubierre Warp Drive without recurring to more complicated topologies(Broeck [29],Natario [32]).

Our Piecewise Shape Function can even overcome the restrictions raised by Lobo-Visser([42]) and we will demonstrate this for the WEC but in order to do that we will compute the mass M_w of the Alcubierre Warp Drive as follows:

$$M_w = -\frac{1}{12}[\frac{1}{\Delta|R|^2}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs = -\frac{1}{12}[\frac{1}{\Delta|R|^2}]^2|_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[\frac{1}{3}rs^3] + C \tag{302}$$

$$M_w == -\frac{1}{12}[\frac{1}{\Delta|R|^4}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs = -\frac{1}{12}[\frac{1}{\Delta|R|^4}]^2|_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[\frac{1}{3}rs^3] + C \tag{303}$$

$$M_w = -\frac{1}{12}[\frac{1}{\Delta|R|^6}]^2 \int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}} [(rs^2)]drs = -\frac{1}{12}[\frac{1}{\Delta|R|^6}]^2|_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[\frac{1}{3}rs^3] + C \tag{304}$$

$$M_w = -\frac{1}{36}[\frac{1}{\Delta|R|^2}]^2[(R+\frac{\Delta}{2})^3 - (R-\frac{\Delta}{2})^3] + C = -\frac{1}{18}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] + C \tag{305}$$

$$M_w = -\frac{1}{36}[\frac{1}{\Delta|R|^4}]^2[(R+\frac{\Delta}{2})^3 - (R-\frac{\Delta}{2})^3] + C = -\frac{1}{18}[\frac{1}{\Delta|R|^4}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] + C \tag{306}$$

$$M_w = -\frac{1}{36}[\frac{1}{\Delta|R|^6}]^2[(R+\frac{\Delta}{2})^3 - (R-\frac{\Delta}{2})^3] + C = -\frac{1}{18}[\frac{1}{\Delta|R|^6}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] + C \tag{307}$$

In the last set of eqs above we consider a Warp Bubble Radius $R = 100meters$ and a Thickness $\Delta = 10meters$.

Our Piecewise Shape Function can satisfy the restrictions posed by Lobo and Visser([42]) concerning the relations between the mass of the spaceship and the negative mass of the Alcubierre Warp Drive. While Lobo and Visser considered subluminal velocities in order to use the linearized gravity applied to the weak-field limit(not possible at Superluminal velocities) applied to the Alcubierre original Shape Function where the Warp Bubble velocity vs appears in the energy density equation affecting the energy conditions(WEC in our case[108]) and the relations between the mass of the spaceship

[108] BEC,NEC,SEC,DEC,PEC,REC or whatever will appear in a future work.Our goal is to convince the maximum nmber of readers that the Alcubierre Warp Drive is "still alive" as a fully functional Superluminal and Faster Than Light ansatz of General Relativity and WEC is enough

and the effective Warp Drive mass(see abstract of [42] eq 14 and eqs 91 to 95 for the WEC)[109][110] constraining the Warp Bubble velocity to subluminal levels but this constriction is a particular consequence of the original Alcubierre Shape Function.We demonstrate that by eliminating the velocity from the energy density equation due to our Piecewise Shape Function we can also eliminate the Warp Bubble velocity from the energy conditions violations equations.We do not restrict the speed of the Alcubierre Warp Drive to subluminal speeds allowing our version of the Alcubierre Warp Drive to attain Superluminal and Faster Than Light velocities while maintaining the equilibrium of the WEC energy condition considering the relations between the mass of the spaceship and the effective mass of the Alcubierre Warp Drive.

We will demonstrate this right now:

Lobo and Visser defines the WEC equilubrium condition as $T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$ where V^{μ} is any Timelike Vector or $T_{\mu\nu}U^{\mu}U^{\nu} \geq 0$ where U^{μ} is the four-velocity of an Eulerian Observer.The physical interpretation of the WEC is the fact that the local energy density is always positive for a non-massless systems.(see pg 3 before eq 10 in [42])

The WEC energy condition violation is given by Lobo-Visser as follows:(see eqs 11 and 12 pg 4 of [42])

$$T_{\mu\nu}U^{\mu}U^{\nu} < 0 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df(rs)}{drs}]^2 < 0 = T_{00} < 0 \tag{308}$$

From above we can reckognize the expression for the energy density of the Alcubierre Warp Drive and this means to say that the energy density of the Alcubierre Warp Drive is always negative in a clear WEC violation.(see comments on pg 4 between eqs 12 and 13 of [42] and comment on page 8 after eq 19 in [31])

Lobo-Visser defines the mass of the Alcubierre Warp Drive as follows:(see eq 13 pg 4 of [42])

$$M_w = -\frac{vs^2}{12}\int_{R-\frac{\Delta}{2}}^{R+\frac{\Delta}{2}}[(rs^2)[\frac{df(rs)}{drs}]^2]drs \tag{309}$$

The integral above can be solved exactly but due to the hyperbolic nature of the original Alcubierre Shape Function it will result in difficult and useless polylog functions[111] as mentioned by Lobo-Visser.

The mass of the Alcubierre Warp Drive is computed by Lobo-Visser for the particular case of the Alcubierre Shape Function as the following estimative(see eq 14 pg 4 in [42]):

$$M_w = -vs^2R^2\delta = \frac{-vs^2R^2}{\Delta} \tag{310}$$

Note that the mass of the Alcubierre Warp Drive and hence the negative energy density requirements raises quadratically with the Warp Bubble Radius R and velocity vs and inversely to the Warp Bubble Thickness Δ.(see comment on pg 4 after eq 14 in [42])

[109] see also pg 3 between eqs 8 and 9 where Lobo-Visser says that the total ADM mass of the spaceship and Warp Field Generators must be exactly compensated by the ADM mass due to the Stress Energy Momentum Tensor of the Warp Field itself

[110] in our approach:what spaceship????the mothership or the Warp Drone????

[111] Mathematica,Mathlab,Maple etc.see comment on pg 4 after eq 14 in [42]

As fastest the Warp Bubble moves or as bigger the Warp Bubble becomes the negative energy density requirements and hence WEC violations becomes more worst.
This is a consequence of the original Alcubierre Shape Function.

Lobo and Visser in pg 6 section III of [42] discuss linearized gravity applied to a subluminal version of the Alcubierre Warp Drive $vs << 1$.We concern ourselves with section $III.A$ pg 6 where WEC is analyzed.According to the linearized gravity in weak-field limit the WEC is given by the following equations:(see eqs 33,38 and 39 pg 7 of [42])

$$T_{\mu\nu}U^{\mu}U^{\nu} = T_{00} = \frac{1}{8\pi}G_{00} \tag{311}$$

$$G_{00} = O(vs^2) \tag{312}$$

$$T_{\mu\nu}U^{\mu}U^{\nu} = T_{00} = O(vs^2) \tag{313}$$

$$\int T_{00}dx^3 = \int O(vs^2)dx^3 = M_w = -vs^2R^2\delta = \frac{-vs^2R^2}{\Delta} \tag{314}$$

Remember that the eqs above are exclusively for the mass,Stress Energy Momentum Tensor and energy density of the Alcubierre Warp Drive.Specially the last equation from the set above relates the Alcubierre Warp Drive energy density to its mass.This equation will be used later in this section.

Lobo and Visser applies linearized gravity not only to the Alcubierre Warp Drive as above but also to a static source (eg the spaceship) in section $III.D$ pg 8 of [42].

The metric is given by:(see eq 52 pg 8 in [42])[112][113]

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + \Phi(x,y,z)(dt^2 + dx^2 + dy^2 + dz^2) \tag{315}$$

This equation cames from the formalism of the linearized theory given below with $h_{\mu\nu} << 1$ and $|\Phi| << 1$.(see top and between eqs 55 and 56 of pg 9 in [42])

$$ds^2 = (n_{\mu\nu} + h_{\mu\nu})dx^{\mu}dx^{\nu} \tag{316}$$

From the Einstein Tensor Component G_{00} applied to $h_{\mu\nu}$ we arrive at the classical Poisson equation(see appendix C pg 16 and 17 and eq 55 pg 9 in [42])

$$\nabla^2\Phi = 4\pi\rho \tag{317}$$

In the equation above ρ is the mass density of the spaceship.Hence we can say that the "mass-energy" density (eg the Stress Energy Momentum Tensor) of the spaceship is:

[112] Lobo and Visser used the signature (-+,+,+).we will use the signature (+,-,-,-).by inverting the signature we invert the signs of the matrix elements of eq 53

[113] here Φ is the weak gravitational field unable to produce velocities close to that of light.see comment on pg 9 of [42]

$$T_{\mu\nu}U^{\mu}U^{\nu} = T_{00} = \rho > 0 \tag{318}$$

$$M_{ship} = \int \rho dx^3 > 0 \tag{319}$$

Note that the mass-energy density of the spaceship satisfies the WEC because the mass of the ship is positive.

Now the reader can understand the point of view of Lobo and Visser:we have one Stress Energy Momentum Tensor for the spaceship that do not violates the WEC due to the positive mass of the spaceship and another Stress Energy Momentum Tensor for the mass of the Alcubierre Warp Drive Bubble where the ship is immersed and this Stress Energy Momentum Tensor violates the WEC due to the negative mass of the Alcubierre Warp Drive.Both spaceship and Alcubierre Warp Drive forms a combined system by addition of both masses and why?:Because the ship is immersed inside the Warp Drive Bubble so both masses must be treated as a single one combined system and this combined system would be desirable to do not violate the WEC.If we want to keep the WEC valid for the combined system then the modulus of the mass of the ship must always be greater than the modulus of the mass of the Alcubierre Warp Drive where the ship is immersed because the ship mass is positive and the Alcubierre Warp Drive mass is negative.This is the main idea behind the work of Lobo and Visser.

$$M_{combined_system} = M_{ship} + M_w > 0 \tag{320}$$

$$|M_{ship}| > |M_w| \tag{321}$$

The equations above are the fundamental conditions for a combined system spaceship-Alcubierre Warp Bubble that do not violates the WEC according to Lobo-Visser

We don't need to analyze the rest of the calculations of Lobo and Visser to demonstrate that the mass of the combined system is what they call "ADM mass" because we already got the idea behind their work and this is the most important thing.Omitting the details of linearized gravity in the weak-field limit with first order or second order terms etcetera and keeping in mind that the ship when immersed inside an Alcubierre Warp Drive both forms a combined system spaceship-Alcubierre Warp Drive by adding both masses because both masses are integrated into a single piece interacting gravitationally (and negatively) between each other and must be treated together as a combined system.We will compare two combined systems:

- 1)-Combined system between a spaceship of mass M_{ship} immersed inside an Alcubierre Warp Drive Bubble of mass $M_{f(rs)}$ with the original Alcubierre Shape Function $f(rs)$
- 2)-Combined system between a spaceship of mass M_{ship} immersed inside an Alcubierre Warp Drive Bubble of mass $M_{f_{pc}(rs)}$ with our Piecewise Shape Functions $f_{pc}(rs)$

This is meant to demonstrate quickly to the reader that whether subluminal or Superluminal our Piecewise Shape Function satisfies the ADM requirements of Lobo-Visser for a combined system and don't suffer WEC violations.This eliminates the last constriction against the Original and Wonderful idea of the Alcubierre Warp Drive.Although the original Alcubierre Shape Function needs to be replaced by a continuous Function with the behavior of our Piecewise Shape Function to avoid all these pathologies in order to make the Alcubierre Warp Drive a valid and fully-functional Superluminal and Faster Than Light ansatz of General Relativity(and we are working on a continuous Shape Function with this behavior that will appear in a future work) we want to say that Miguel Alcubierre was Brilliant and the 1994 paper will ever be regarded as a Historical and Revolutionary paper.

- 1)-Combined system between a spaceship of mass M_{ship} immersed inside an Alcubierre Warp Drive Bubble of mass $M_{f(rs)}$ with the original Alcubierre Shape Function $f(rs)$ that satisfies the WEC

$$M_{combined_system} = M_{ship} + M_{f(rs)} > 0 \tag{322}$$

$$M_{combined_system} = M_{ship} - \frac{vs^2 R^2}{\Delta} > 0 \tag{323}$$

$$M_{ship} > \frac{vs^2 R^2}{\Delta} \tag{324}$$

  Compare the expression above with eqs 94 and 95 pg12 of [42] and see the comment between these expressions.The net energy of the Alcubierre Warp Drive cannot exceed the net energy of the spaceship itself if we want to satisfy the WEC.This places severe restrictions to the radius R and speed vs of the Alcubierre Warp Drive.

- 2)-Combined system between a spaceship of mass M_{ship} immersed inside an Alcubierre Warp Drive Bubble of mass $M_{f_{pc}(rs)}$ with our Piecewise Shape Functions $f_{pc}(rs)$ that satisfies the WEC

$$M_{combined_system} = M_{ship} + M_{f_{pc}(rs)} > 0 \tag{325}$$

$$M_{combined_system} = M_{ship} - \frac{1}{18}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] > 0 \tag{326}$$

$$M_{combined_system} = M_{ship} - \frac{1}{18}[\frac{1}{\Delta|R|^4}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] > 0 \tag{327}$$

$$M_{combined_system} = M_{ship} - \frac{1}{18}[\frac{1}{\Delta|R|^6}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] > 0 \tag{328}$$

$$M_{ship} > \frac{1}{18}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{329}$$

$$M_{ship} > \frac{1}{18}[\frac{1}{\Delta|R|^4}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{330}$$

$$M_{ship} > \frac{1}{18}[\frac{1}{\Delta|R|^6}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{331}$$

Our Piecewise Shape Function allows the possibility of an Alcubierre Warp Drive that whether subluminal or Superluminal will always satisfy the Lobo-Visser WEC energy conditions.Consider a Warp Bubble of Radius $R = 100meters$ a Thick of $\Delta = 10meters$ and a speed $vs = 100timesfasterthanlight$.And this is due to the fact that by including the speed of the Warp Bubble vs in the definition of the Piecewise Shape Function $f_{pc}(rs)$ we were able to remove the speed from the energy density and from the energy conditions.

While we lowered the energy density requirements of the Alcubierre Warp Drive to acceptable and physically reasonable levels we would like to comment the work of Ford-Roman.[44]

We know that the Alcubierre Warp Drive violates the WEC energy condition $T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$(cite pg 2 of [44]) but we also know that our Piecewise Shape Function allows the reduction of the energy density requirements to sustain a Warp Bubble.

In abstract,pages 2,3,5 after eq 15 and pg 18 in the beginning of section 5 of [44]) Ford-Roman mentions the possibility to produce large amounts of negative energy.This sounds good for the Warp Drive of course because if we have no restrictions of negative energy production then the physical feasibility of the Warp Drive is reinforced.But of course while we welcome the large outputs of negative energy density we want to use low and affordable energy density levels for the Warp Drive in order to sustain its credibility.

According to Ford-Roman and Basini-Capozziello the Stress Energy Momentum Tensor for a generically coupled scalar field would be given by(Basini-Capozziello see eq 4 in [1] and eq 85 in [20]):

$$T_{\mu\nu} = \nabla_{\mu}\Phi\nabla_{\nu}\Phi - \frac{1}{2}g_{\mu\nu}\nabla_{\mu}\Phi\nabla^{\mu}\Phi + g_{\mu\nu}V(\Phi) \tag{332}$$

$$T_{\mu\nu} = \nabla_{\mu}\Phi\nabla_{\nu}\Phi - \frac{1}{2}g_{\mu\nu}\nabla_{\mu}\Phi g^{\mu\mu}\nabla_{\mu}\Phi + g_{\mu\nu}V(\Phi) \tag{333}$$

The following expressions are taken from eq 3 pg 3 in [44] without the parameter ξ

$$T_{\mu\nu} = \nabla_{\mu}\Phi\nabla_{\nu}\Phi - \frac{1}{2}g_{\mu\nu}\nabla_{\mu}\Phi\nabla^{\mu}\Phi - g_{\mu\nu}V(\Phi) \tag{334}$$

$$T_{\mu\nu} = \nabla_{\mu}\Phi\nabla_{\nu}\Phi - \frac{1}{2}g_{\mu\nu}\nabla_{\mu}\Phi g^{\mu\mu}\nabla_{\mu}\Phi - g_{\mu\nu}V(\Phi) \tag{335}$$

The energy density for the Warp Drive is given by T_{00} from the following equation:(see eqs 11 and 12 pg 4 and eqs 33,38 and 39 pg 7 of [42]:

$$T_{\mu\nu}U^{\mu}U^{\nu} < 0 = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df(rs)}{drs}]^2 < 0 = T_{00} < 0 \tag{336}$$

Note that Ford-Roman mentions explicitly in pg 2 of [44] the solutions of Visser-Barcelo wormhole in which a large flux of negative energy is injected into a wormhole ”throat” in order to keep its stability.

The positive mass of of the Black Hole coupled together with the negative energy flux is also a combined system just like the spaceship and the Warp Drive Bubble.The WEC according to Ford-Roman can be violated(see pg 2 of [44] the mention to the works of Bekenstein,Deser,Flanagan and Wald).Note that the energy flux although great will not exceed the total rest mass M of the Black Hole.

But since our energy density for the Warp Drive is low and affordable we don’t need to violate the WEC

Writing the energy density for the Alcubierre Warp Drive in function of the scalar field for the Basini-Capozziello Stress Energy Momentum Tensor and using our Piecewise Shape Function we would have:

$$T_{00} = \nabla_0\Phi\nabla_0\Phi - \frac{1}{2}g_{00}\nabla_0\Phi g^{00}\nabla_0\Phi + g_{00}V(\Phi) \tag{337}$$

We know that $g_{00} \times g^{00} = 1$

$$T_{00} = \nabla_0\Phi\nabla_0\Phi - \frac{1}{2}\nabla_0\Phi\nabla_0\Phi + V(\Phi) \tag{338}$$

$$T_{00} = \frac{1}{2}\nabla_0\Phi\nabla_0\Phi + V(\Phi) \tag{339}$$

$$T_{00} = \frac{1}{2}\nabla_0^2\Phi + V(\Phi) \tag{340}$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 + V(\Phi) \tag{341}$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 + V(\Phi) = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^2}]^2 \tag{342}$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 + V(\Phi) = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^4}]^2 \tag{343}$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 + V(\Phi) = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^6}]^2 \tag{344}$$

Using the Ford-Roman expression:

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 - V(\Phi) \tag{345}$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 - V(\Phi) = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^2}]^2 \tag{346}$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 - V(\Phi) = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^4}]^2 \quad (347)$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 - V(\Phi) = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^6}]^2 \quad (348)$$

In pg 4 of [44] Ford-Roman makes $V(\Phi) = 0$.Then we would have:

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^2}]^2 \quad (349)$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^4}]^2 \quad (350)$$

$$T_{00} = \frac{1}{2}[\frac{\partial\Phi}{\partial t}]^2 = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^6}]^2 \quad (351)$$

Although we have a WEC violation for the negative mass-energy of the Alcubierre Warp Drive the partial derivative of the scalar field have low values.

The negative energy density needed to create the Alcubierre Warp Drive could perhaps be obtained by the Casimir Effect as mentioned by Miguel Alcubierre in pg 9 of [31].

Combining the negative energy density of the Casimir Effect with the negative energy of the Alcubierre Warp Drive we will obtain a new set of equations.Although these equations are only a theoreticsl analysis of a possible "mini Backreaction" between Alcubierre and Casimir we call it simply:The Casimir Warp Drive

A detailed explanation of the Casimir Effect can be found in pg 6 to 9 of [40],pg 2 to 9 in [47]

According to pg 4 of [47],pg 6 of [40] and pg 2 of [39] the negative energy of the Casimir Effect between parallel conducting and reflecting plates is given by:(see eq 14 pg 9 of [40])[114]

$$\frac{1}{A}\xi_0 = -\frac{\pi^2}{720}\frac{\hbar c}{L^3} \quad (352)$$

Where $A = L_z \times L_y$ is the area of each parallel plate in the plane yOz and L is the distance between the plates.(see fig 1 pg 6 in [40])

$$\xi_0 = -\frac{\pi^2}{720}\frac{\hbar c}{L^3}A \quad (353)$$

The energy density between planes inclined by an angle α is given by:(see eqs 4.2,4.3 and 4.4 pg 38 in [47]).

$$T^{00} = -\frac{f(\alpha)}{720\pi^2 L^4} \quad (354)$$

Conformal scalar with Dirichlet boundary conditions:

$$f(\alpha) = \frac{\pi^2}{2\alpha^2}(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2}) \quad (355)$$

[114] equations 353 and 354 written with the conventional values for G,c and $\hbar$.for the rest we use $G = c = \hbar = 1$.

Electromagnetism with perfect coundutor boundary conditions:

$$f(\alpha) = (\frac{\pi^2}{\alpha^2} + 11)(\frac{\pi^2}{\alpha^2} - 1) \tag{356}$$

Ford and Sopova in pg 2 of [39] presents the energy density as:

$$T_{00} = -\frac{\pi^2}{720}\frac{1}{L^4} \tag{357}$$

Ford and Sopova also mentions in pg 2 the WEC violation of the Casimir Effect and the need of exotic matter for transversable wormholes.

Equalizing the negative energy of the Casimir Effect with the one of the Alcubierre Warp Drive with our Piecewise Shape Function we should expect for:

$$T_{00} = -\frac{1}{32\pi}\frac{vs^2\rho^2}{rs^2}[\frac{df(rs)}{drs}]^2 = -\frac{\pi^2}{720}\frac{1}{L^4} \tag{358}$$

$$T_{00} = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^2}]^2 = -\frac{\pi^2}{720}\frac{1}{L^4} \tag{359}$$

$$T_{00} = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^4}]^2 = -\frac{\pi^2}{720}\frac{1}{L^4} \tag{360}$$

$$T_{00} = -\frac{1}{32\pi}\frac{\rho^2}{rs^2}[\frac{1}{\Delta|R|^6}]^2 = -\frac{\pi^2}{720}\frac{1}{L^4} \tag{361}$$

Now we can compute the distance between the parallel plates of our Casimir Warp Drive to generate the Alcubierre Spacetime Geometry(at least in theory)

$$L^4 = \frac{32\pi^3}{720}\frac{rs^2}{\rho^2}[\Delta|R|^2]^2 \tag{362}$$

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If the distance between the plates grows then the negative energy density becomes smaller.However we know that the plates must stay close to each other

Computing the negative energy E_w of the Alcubierre Warp Drive and relating it with the Casimir negative energy we should expect for:

$$E_w = -\frac{c^2}{18}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] = -\frac{\pi^2}{720}\frac{\hbar c}{L^3}A \tag{365}$$

$$E_w = -\frac{c^2}{18}[\frac{1}{\Delta|R|^4}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] = -\frac{\pi^2}{720}\frac{\hbar c}{L^3}A \tag{366}$$

$$E_w = -\frac{c^2}{18}[\frac{1}{\Delta|R|^6}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] = -\frac{\pi^2}{720}\frac{\hbar c}{L^3}A \tag{367}$$

Now the distance between the Casimir plates to generate the exotic matter for the Alcubierre Warp Drive is given by:

$$\frac{40c}{\hbar A\pi^2}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] = \frac{1}{L^3} \tag{368}$$

$$\frac{40c}{\hbar A\pi^2}[\frac{1}{\Delta|R|^4}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] = \frac{1}{L^3} \tag{369}$$

$$\frac{40c}{\hbar A\pi^2}[\frac{1}{\Delta|R|^6}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] = \frac{1}{L^3} \tag{370}$$

Note that the magnitude of the value of the Planck Constant 10^{-34} combined with the magnitude of the light speed value 10^8 reduces the distance between the plates.

We know that our approach of the Alcubierre Warp Drive is only theoretical but at least we tried to solve the unphysical pathologies.Only three obstacle remains:

- 1)- We don't know what would happen to the spaceship if the Warp Bubble is destroyed from behind.
- 2)- An impact with a large positive mass object would generate a strong gravitational repulsive force that could disrupt or destroy the front part of the Warp Buble.Again we don't know the consequences for the spaceship if the Warp Bubble is destroyed from the front.More on this when analyzing the Chung-Freese Superluminal BraneWorld.
- 3)-We don't know to generate the Shape Function in order to "engineer" the Alcubierre Warp Drive Spacetime.

In order to terminate the discussion about the subject of the Alcubierre Warp Drive we will compute the Gravitational Bending Of Light in the neighborhoods of the Warp Bubble negative mass.

The classical $4D$ formula of General Relativity for the Bending Of Light is given by(see pg 1781 eq 18 in [3] and pg 70 eq 157 in [21]):[115]

$$\Delta\omega = \frac{4M}{r_0} \tag{371}$$

Where M is the mass of the body that generates the Bending Of Light and r_0 is the distance between the body and the Light photons.The Bending Of Light is positive for a body of positive mass.

Computing for the Alcubierre Warp Drive using the results of our Piecewise Shape Function we should expect for:

$$\Delta\omega = -\frac{4}{18r_0}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{372}$$

[115] equations written with the $5D$ Extra Dimensional term removed.

$$\Delta\omega = -\frac{4}{18r_0}[\frac{1}{\Delta|R|^4}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{373}$$

$$\Delta\omega = -\frac{4}{18r_0}[\frac{1}{\Delta|R|^6}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{374}$$

We can clearly see from above that the Bending Of Light although very small is negative for the Alcubierre Warp Drive.

The negative Gravitational Force between an incoming particle of a given positive mass M_p approaching the Warp Bubble at a distance d considering our Piecewise Shape Function is given by:

$$F = -GM_p\frac{1}{18d^2}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{375}$$

$$F = -GM_p\frac{1}{18d^2}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{376}$$

$$F = -GM_p\frac{1}{18d^2}[\frac{1}{\Delta|R|^2}]^2[R^2\Delta + R^2\frac{\Delta}{2} + \frac{\Delta^3}{8}] \tag{377}$$

Note that if the distance d becomes small the negative force grows proportionally.Also note the fact that we mentioned the Warp Bubble being destroyed from behind and the front part going on this would mean that the pieces of the small negative energy density from the front part would be repealed by the negative gravitational force between these pieces and positive mass objects.The reminiscent of the front part would never be a threat for ships or planets.

This could deflect small incoming particles and protect the Warp Bubble from impacts although a giant Black Hole or a Neutron Star would pose a very serious threat.

In our example of Horizons we used the Crab Nebula but for distances of 6000 Light-Years the probability of an impact with a large positive mass object would be of a great concern.

Alcubierre Warp Drive would be suitable perhaps for interstellar travels of "short" distance eg Sun and Proxima Centauri or Sirius because we know that(we hope) there are no Neutron Stars or Black Holes between Earth and these Stars.For interstellar travels of long distance the Superluminal Chung-Freese Braneworld would probably be more adequate.This terminates our discussion on the Alcubierre Warp Drive.Lastly we would like to say that we introduced the Alcubierre Warp Drive in a work of $5D$ Extra Dimensions and the Pioneer Anomaly because even a subluminal version of the Alcubierre Warp Drive would be of an Immense value.A spaceship equipped with a subluminal version of the Alcubierre Warp Drive could reach in a matter of days(or hours) the region of Space where the Pioneers were lost avoiding the need to wait more 40 years in order to solve the mystery .Also we tried to demonstrate that Superluminal and Faster Than Light Space Travel is possible in $3 + 1$ Spacetime although with some limitations..We will now examine the Superluminal Chung-Freese Braneworld.

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. The Superluminal Chung-Freese BraneWorld was first proposed in [22] as a solution of the so called Cosmological Horizon Problem in the Past of the Universe(time of nucleosynthesis see pg 1 and causality violations at an Universe Early Time however hidden today in the beginning of pg 8 in [22]).Our approach will be different because we consider the Universe in our Present time.

Chung-Freese proposes that a "signal" leaves our $3 + 1$ Einstein Universe and enters in a $5D$ Extra Dimension.When returning to the Einstein Universe the "signal" would appear in a distant different place but in a shorter time than a "signal" confined to the Einstein Universe would take.According to Chung-Freese thsee $5D$ geodesics can connect distant points of our Universe that otherwise would be causally disconnected from each other.(see abs pg 2 and 9 of [22]).

. Imagine now that instead of a "signal" we have a real spaceship that can leave our $3+1$ Spacetime when on an Earth orbit travels into the $5D$ Extra Dimension and when returning to our Spacetime the ship will appear in the neighborhoods of the Crab Nebula.In this case and still according to Chung-Freese the $5D$ Extra Dimensional geodesics would connect Earth and Crab Nebula as two distant points in our Universe but causally connected due to the presence of the Extra Dimension.By replacing the Chung-Freese "signal" by a spaceship we transform the Superluminal Chung-Freese BraneWorld into a valid and fully functioning Superluminal and Faster than Light Spacetime ansatz[116].Another remarkable thing is the fact that when in the $5D$ Spacetime the spaceship would remain subluminal free from impacts with lethal objects[117] but when "seen" from our Universe the ship would appear as Superluminal to a distant observer.(see pg 1849,1850 in [10],pg 2036 in [4] and pg 1474 in [20]). We will diverge from [4] and [10] and examine the Superluminal Chung-Freese BraneWorld according to the Basini-Capozziello-Ponce De Leon Formalism.

. Examine carefully fig 1 pg 135 in [1] from Basini-Capozziello work:The white curved surface where the points A and B are located corresponds to the ordinary $3+1$ Spacetime while the shaded or shadowed area corresponds to a Spacetime Shortcut through the $5D$ Extra Dimension connecting the distant points A and B in the same way as proposed by Chung-Freese.(see also comments on this fig in pg 1474 immediately before eq 320 in [20]) If we choose to travel from A to B across the white surface this would mean a long trip through the $3+1$ Spacetime but however if we choose to use the shaded area of the $5D$ Extra Dimension the trip from A to B will be made in a shorter time.This fig from Basini-Capozziello illustrates without shadows of doubt the essence of the Chung-Freese idea.A similar presentation is given by Chung-Freese in fig 2 pg 5 in [22].The point 1 in Chung-Freese corresponds to the point A of Basini-Capozziello while the point 2 corresponds to the point B.The point 3 lies between the points 1 and 2 at a shorter distance from 2 with respect to 1.Chung-Freese illustrates the fact that using a $5D$ Extra Dimensional Shortcut a signal can arrive at the point 2 and achieve a farther distance from point 1 while point 3 confined to our $3+1$ Spacetime cannot reach the distance of point 2.Note that according to Basini-Capozziello fig the trip from A to B is made almost instantaneously.This agrees with a work similar to the work of Chung-Freese:the Kalbermann-Halevi work [33] where in the abs and pg 3 we can see all the matter in our Universe connected by Superluminal "signals" in a $5D$ Extra Dimensional Spacetime and a "signal" entering in the Extra Dimension would leap huge distances in a small amount of time.Examine pg 8 of [33] after eq 8.Kalbermann-Halevi demonstrates that a distance of 100 megaparsecs could be transversed in the small amount of time of $t = 2,5 \times 10^{-10}$ seconds using the $5D$ Extra Dimension while a photon confined to the

[116]As a matter of fact the ansatz of the Superluminal Chung-Freese BraneWorld resembles or is a reminiscent of the Star Wars Hyperdrive. Examine the abstract of http://www.arxiv.org/abs/hep-ph/9910235 where it can be seen the "signal" leaving our Universe entering into the Extra Dimension and returning back to a distant point in our Universe faster than a "signal" confined in our Univere would need and examine the Star Wars definition of the Hyperdrive available at http://www.starwars.com/databank/technology/hyperdrive/ where a spaceship leaves the Einstein Universe enters in Hyperspace and travels interstellar distances in minutes returning then to our Universe.We will retain in this work the scientific vocabulary and avoid the mention to the term "Hyperdrive" because the term Warp Drive is already well accepted by the scientific community and appears in many peer-review published works while as far as we know there are no published works with the term "Hyperdrive". Although we agree with the fact that both Chung-Freese and Hyperdrive have many resemblances between each other our vocabulary will be strictly scientific.

[117]remember that according to eq 20 pg 1341 in [2] any object of $4D$ rest-mass M posseses a $5D$ rest-mass M_5.Then when the ship "leaves" our Universe it do not "leaves" a large Black Hole behind.The Black Hole exists also in $5D$.The motion in the $5D$ Spacetime starts when $\frac{dy}{ds} \neq 0$.However we conjecture that due to the subluminality of the ship in the $5D$ Spacetime the ship can perhaps manoeuvre to avoid the collision between its $5D$ rest-mass with the $5D$ rest-mass of the Black Hole.This would be of course an advantage over the Alcubierre Warp Drive in long distance interstellar travel

$3+1$ Spacetime needs 326 millon light-years[118].Without shadows of doubt this is an example of an almost instantaneous $a-luminal$ trip according to the fig of Basini-Capozziello.Note that Basini-Capozziello also proposes the $5D$ for Superluminal travel.(see pg 127 in [1] and pg 2227 in [5])

The ansatz of Chung-Freese is given by(see eq 3 pg 3 in [22],eq 1 pg 2035 in [4],eq 1 pg 1850 in [10] and eq 320 pg 1474 in [20]) :

$$dS^2 = dt^2 - e^{-2ky}a^2dX^2 - dy^2 \tag{378}$$

In the equation above and according to ([22] pg 3,[4] pg 2035,[10] pg 1850 and [20] pg 1474) X corresponds to our $3D$ spacetime while y is the $5D$ Extra Dimension.Note that according to Chung-Freese pg 3 in [22] our $3+1$ Universe lies in the Brane $y=0$.We cannot "see" the Extra Dimension when the Chung-Freese ansatz reduces to our Universe due to the reasons presented at section 4 of this work.

Solving the Null-Like Geodesics in $5D$ $dS^2 = 0$ we get(see eq 2 pg 2036 in [4],eq 4 pg 1850 in [10] and eq 321 pg 1475 in [20]):

$$\frac{dX}{dt} = \frac{e^{ky}}{a}\sqrt{1-(\frac{dy}{dt})^2} \tag{379}$$

According to original Chung-Freese point of view[119] a particle subluminal in the Extra Dimension $\frac{dy}{dt} < 1$ would be seen in our Universe(Brane) as Superluminal and Faster Than Light $\frac{dX}{dt} > 1$ due to the factor $\frac{e^{ky}}{a}$.(see pg 2036 in [4],pg 1850 in [10] and pg 1475 in [20]).The k here is the Chung-Freese Brane Lensing coefficient(see pg 2035 in [4],pg 1850 in [10] and pg 1475 in [20]).

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. How or why does the Superluminal Chung-Freese Braneworld fits perfectly well in the Basini-Cappzziello-Ponce De Leon????

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. Look again to fig 1 pg 135 in [1].Consider the point A in the white region the Earth and the point B also in the white region the Crab Nebula at 6000 light-years of distance. Look also to the arrows departing from the points A and B.In the point of each arrow we can draw two points:C in the arrow of A and D in the arrow of B.Suppose that C and D are two large Black Holes between Earth and Crab Nebula. According to Basini-Capozziello-Ponce De Leon in section 3 of this work our $3+1$ Universe is a Dimensional Reduction from a Universe of $5D$. Then Earth,Crab Nebula and the Black Holes C and D we can see in $3+1$ Spacetime are Dimensional Reductions from its $5D$ counterparts.Now draw straight lines from A,B,C and D that will intersect the shaded area that corresponds to the Extra Dimension.The points where the lines intersect the Extra Dimension are the $5D$ counterparts of Earth,Crab Nebula and Black Holes each one with a respective $5D$ rest-mass M_5 inside the shaded area and all the $4D$ rest-masses m_0 inside the white area we can see for each one of these bodies are Dimensional Reductions according to Ponce De Leon.

We would have the following equations for the relation between the $5D$ rest-masses placed in the shaded area and the $4D$ ones placed in the white surface.([2] eq 20,[11] eq 21 and [20] eq 8)[120]

[118] We don't need to be so fast.A "speed" of 100.000 light-years in 7 weeks would be more than enough to explore our Galaxy.However we would become satisfied with a modest speed of some light-years to reach our neighbor stars.Examine again the fig of Basini-Capoziello:A small white surface of some light-years would mean that the points A and B would remain "close" to each other allowing an Alcubierre Warp Drive to travel in the white region with a reduced propability to impact a Black Hole while a large white region of thousands of light years would make the probability grows dangerously.

[119] also Basini-Capozziello in fig 1 pg 135 in [1]

[120] Equations without the Warp Field Φ and Warp Factor Ω

$$M(Earth) = \frac{M_5(Earth)}{\sqrt{1 - (\frac{dy}{ds}(Earth))^2}} \tag{380}$$

$$M(Crab - Nebula) = \frac{M_5(Crab - Nebula)}{\sqrt{1 - (\frac{dy}{ds}(Crab - Nebula))^2}} \tag{381}$$

$$M(Black - Hole - C) = \frac{M_5(Black - Hole - C)}{\sqrt{1 - (\frac{dy}{ds}(Black - Hole - C))^2}} \tag{382}$$

$$M(Black - Hole - D) = \frac{M_5(Black - Hole - D)}{\sqrt{1 - (\frac{dy}{ds}(Black - Hole - D))^2}} \tag{383}$$

We assume that at each one of these bodies have at least a respective $\frac{dy}{ds} \neq 0$ in order to generate the $5D$ Extra Force according to Basini-Capozziello-Ponce de Leon and Mashoon-Wesson-Liu.

Considering a spaceship of very low mass neglecting the ship own Gravitational Force in order to avoid the complications between a merge of the Chung-Freese and Schwarzschild Spacetime metrics and considering the spacetime around the spaceship as a SR spacetime with an Extra Dimension(see eq 55 section 3 in this work).

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. Imagine now that we find out a way to "engineer" the Spacetime around the spaceship to create the Superluminal Chung-Freese BraneWorld.The ship "leaves" our Universe at point A the Earth and enter into the Extra Dimension.This happens when the ship starts the motion defined by $\frac{dy}{dt}$ in the shaded area of the fig 1 in the Basini-Capozziello work.Actually the ship never "leaves" our Universe because the Dimensional Reduction from $5D$ to $4D$ is what makes the subluminal ship in $5D$ being seen as Superluminal in $4D$.Just draw a straight line between each point of the ship path in $5D$ the shaded area that will intersect the white surface in $4D$.The line obtained in the white surface is the Dimensional Reduction from the ship path in $5D$ eg the ship path "seen" in $4D$ from its $5D$ counterpart. The ship moves slowly in the $5D$ manoeuvering and avoiding the rest-masses of the Black-Holes C and D arriving in safe to the Crab Nebula at point B.This is one of the most important features of the Chung-Freese Superluminal BraneWorld.The ship can manoeuvre avoiding hazardous objects.

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. However this scenario have two serious drawbacks:

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- 1)- We don't know how to navigate in the shaded area in order to avoid the rest-masses of the two Black Holes.We cannot signalize it with photons or gravitons because for these particles $\frac{dy}{ds} = 0$ according to Ponce De Leon at least in our approach[121] .This means to say that these rest-masses do not emit photons or gravitons.How can we detect these masses???? Our hope is the fact that perhaps the Table of Elementary Particles presented here may be incomplete and a new unknown particle will allow the spaceship to detect the presence of the $5D$ rest-masses of potential hazardous objects.Otherwise the Superluminal Chung-Freese Braneworld would be in deep trouble.[122] [123]

[121] We consider only non-zero $5D$ M_5 rest-masses here. Zero $5D$ rest-masses will appear in a future work.It is important to have this fact in mind when reading this work

[122] We provide a possible way before the end of this section.

[123] If God created this Universe of billion,billions and billions of light-years with only 4 Fundamental Forces and the small

- 2)-We don't know how to "engineer" the Superluminal Chung-Freese Braneworld

The energy density is positive and low and is given by:(see eq 37 pg 8 in [22],eq 3 pg 2036 in [4],eq 5 pg 1850 in [10] and eq 322 pg 1475 in [20])

$$T^{00} = -6k^2 + 3[(\frac{1}{a})(\frac{da}{dt})]^2 \tag{384}$$

This is a remarkable feature of the Chung-Freese Superluminal Braneworld:energy density in a Superluminal environment without violating the energy conditions WEC,NEC,SEC,BEC,GEC etcetera.We will provide an example(see eqs 345 to 347 in [20]):

Consider $a = e^{-y}$(pg 2036 in [4],pg 1852 to 1853 in [10] and pg 1479 in [20]).The ansatz of Chung-Freese would then be:

$$dS^2 = dt^2 - e^{-2y(k+1)}dX^2 - dy^2 \tag{385}$$

The Null-Like geodesics and energy density would be respectively:

$$\frac{dX}{dt} = e^{y(k+1)}\sqrt{1 - (\frac{dy}{dt})^2} \tag{386}$$

$$T^{00} = -6k^2 + 3[(\frac{dy}{dt})]^2 \tag{387}$$

Note that this ansatz retains the Superluminal behaviour $\frac{dy}{dt} < 1$ $\frac{dX}{dt} >> 1 \dashrightarrow e^{y(k+1)} >> 1$ and the positive (and low) energy density features if $y > \sqrt{2}kt \dashrightarrow y \simeq \sqrt{2}kt \dashrightarrow T^{00} > 0 \dashrightarrow T^{00} \simeq 0$.

. For the ansatz described above we need a Large and Not Compactified Extra Dimension in agreement with Basini-Capozziello-Ponce De Leon at section 4 of this work.

. In order to "see" the $5D$ Extra Dimension we must make the Warp Field $\Phi \neq 1$.Then the complete ansatz of Chung-Freese with Warp Field in a Basini-Capozziello reduction from $5D$ to $4D$ would be given by:(see eqs 350 and 351 pg 1480 in [20])

$$dS^2 = dt^2 - e^{-2ky}a^2dX^2 - \Phi^2dy^2 \tag{388}$$

For our special case:

$$dS^2 = dt^2 - e^{-2y(k+1)}dX^2 - \Phi^2dy^2 \tag{389}$$

The Null-Like geodesics in the complete Chung-Freese BraneWorld(with Warp Field Φ) is given by:(see eqs 363 and 364 pg 1481 in [20])

$$\frac{dX}{dt} = \frac{e^{ky}}{a}\sqrt{1 - (\Phi\frac{dy}{dt})^2} \tag{390}$$

or for our special case $a = e^{-y}$

$$\frac{dX}{dt} = e^{y(k+1)}\sqrt{1 - (\Phi\frac{dy}{dt})^2} \tag{391}$$

group of Elementary Particles presented in a table in this work then we hope God will forgive ourselves when we think God have lacks of immagination.But in God we Believe and what we know about the Universe is Just The Beginning.We believe that God had left in the Universe many pleasant scientific surprises.More on this in the Conclusion of the Pioneer Anomaly

This ansatz have the Superluminal capability due to the factor $\frac{e^{ky}}{a}$ or $e^{y(k+1)}$ if $\Phi\frac{dy}{dt} < 1$

Our scenario according to Basini-Capozziello-Ponce De Leon with again the fig 1 pg 135 of [1] of the relation between the $5D$ and the $4D$ rest-masses of the "trip" between Earth and Crab Nebula would be given by:([2] eq 20,[11] eq 21 and [20] eq 8)[124]

$$M(Earth) = \frac{M_5(Earth)}{\sqrt{1 - (\Phi\frac{dy}{ds}(Earth))^2}} \tag{392}$$

$$M(Crab - Nebula) = \frac{M_5(Crab - Nebula)}{\sqrt{1 - (\Phi\frac{dy}{ds}(Crab - Nebula))^2}} \tag{393}$$

$$M(Black - Hole - C) = \frac{M_5(Black - Hole - C)}{\sqrt{1 - (\Phi\frac{dy}{ds}(Black - Hole - C))^2}} \tag{394}$$

$$M(Black - Hole - D) = \frac{M_5(Black - Hole - D)}{\sqrt{1 - (\Phi\frac{dy}{ds}(Black - Hole - D))^2}} \tag{395}$$

$$M(Black - Hole - D) = \frac{M_5(Black - Hole - D)}{\sqrt{1 - (\Phi\frac{dy}{ds}(Black - Hole - D))^2}} \tag{396}$$

We will include the mass of the Sun in this scenario due to our conclusions in the section 5 of this work:The Pioneer Anomaly.

$$M(Sun) = \frac{M_5(Sun)}{\sqrt{1 - (\Phi\frac{dy}{ds}(Sun))^2}} \tag{397}$$

Note that in this scenario the mass of the Sun would generate the de-acceleration observed in the Pioneer Anomaly according to the $5D$ Extra Force of Ponce De Leon and Mashoon-Wesson-Liu as shown in our conclusions however any ship placed near the Black Holes would experience the same $5D$ Extra Force with more intensity.The Pioneer Anomaly is not a phenomena between the Pioneers and the Sun:It is a Universal phenomena.

Note that each body have its own motion in the $5D$ Spacetime $\frac{dy}{ds}$ and its own Warp Field Φ.

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. What happens if a body remains at the rest with respect to the $5D$ Extra Dimension which means to say $\frac{dy}{ds} = 0$?

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. In this case each M_5 $5D$ rest mass would be equal to its own counterpart m_0 rest-mass in $4D$.

$$M(Earth) = M_5(Earth) \tag{398}$$

$$M(Crab - Nebula) = M_5(Crab - Nebula) \tag{399}$$

$$M(Black - Hole - C) = M_5(Black - Hole - C) \tag{400}$$

[124] Equations with the Warp Field Φ and without Warp Factor Ω

$$M(Black - Hole - D) = M_5(Black - Hole - D) \quad (401)$$

This will lead ourselves to two important conclusions[125]:

- 1)-When a body is at the rest in the $4D$ spacetime then the same body is at the rest in the $5D$ Extra Dimension and both rest masses are the same.
- 2)-When a body starts to move in the $5D$ Extra Dimension such body will have a $\frac{dy}{dt} \neq 0$ and this means to say a $\frac{dy}{ds}\frac{ds}{dt}$.Then any motion inside the $5D$ Extra Dimension will produce a projection into the $4D$ ordinary spacetime and vice versa.Examine again fig 1 pg 135 of [1].Plot a motion in $4D$ the white area and draw some points E and F in the white region.Now draw straight lines from E and F that will intersect the shaded area the $5D$ Extra Dimension.The intersection points will be the points G and H.Then E and F are $5D$ to $4D$ Dimensional Reductions from G and H.

Then why we cannot "see" the Extra Dimension?????

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. In Section 4 we gave many explanations.We will apply these ones to the Superluminal Chung-Freese BraneWorld.

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Applying the Basini-Capozziello $5D$ to $4D$ reduction to the original Chung-Freese ansatz (eq 1 pg 2035 in [4],eq 1 pg 1850 in [10] and eq 342 pg 1478 in [20]) :

$$dS^2 = dt^2 - e^{-2ky}a^2dX^2 - dy^2 \quad (402)$$

Note that in this case we have a Warp Field $\Phi = 1$ and the spacetime metric components are given by $(g_{ab} = +1, -e^{-2ky}a^2, -1)$ and the Ricci Tensors are given by R_{00} and R_{11}.Since the derivatives of the Warp Field will vanish we are left with the following expressions(eqs 343 and 344 pg 1479 in [20]):

$$^5R_{00} = R_{00} - \frac{1}{2}(-g_{00,44} + \frac{g^{00}g_{00,4}g_{00,4}}{2}) \quad (403)$$

$$^5R_{11} = R_{11} - \frac{1}{2}(-g_{11,44} + \frac{g^{11}g_{11,4}g_{11,4}}{2}) \quad (404)$$

Note also that the derivatives of $g_{00} = +1$ with respect to the Extra Coordinate vanishes leaving us with a reduction from $5D$ to $4D$ $^5R_{00} = R_{00}$ and the $5D$ Extra Dimension only affects the $^5R_{11}$.Note that if $e^{2ky} >> a$ then the derivatives of g_{11} with respect to the Extra Coordinate will vanish too and we will have the approximate result $^5R_{11} \simeq R_{11}$.We will provide an example:

Consider the case of $a = e^{-y}$(pg 2036 in [4],pg 1852 to 1853 in [10],pg 1479 eq 345 in [20]).The ansatz of Chung-Freese would then be:

$$dS^2 = dt^2 - e^{-2y(k+1)}dX^2 - dy^2 \quad (405)$$

Note that in this case the derivatives of the g_{11} spacetime metric component with respect to the Extra Coordinate do not vanish and are given by:(eqs 348 to 350 pg 1479 in [20])

[125]The conclusions are valid for only non-zero $5D$ M_5 rest-masses . Zero $5D$ rest-masses will appear in a future work

$$g_{11,4} = \frac{2[a(\frac{da}{dy}) - ka^2]}{e^{2ky}} = -\frac{2(k+1)}{e^{2y(k+1)}} \tag{406}$$

$$g_{11,4}^2 = \frac{4[a(\frac{da}{dy}) - ka^2]^2}{e^{4ky}} = \frac{4(k+1)^2}{e^{4y(k+1)}} \tag{407}$$

$$g_{11,44} = \frac{2[(\frac{da}{dy})^2 + a(\frac{d^2a}{dy^2}) - 2ka(\frac{da}{dy})] - 2k[a(\frac{da}{dy}) - ka^2]}{e^{2ky}} = -\frac{4k(y+1)}{e^{2y(k+1)}} \tag{408}$$

If we have a value for the Chung-Freese Brane Lensing coefficient $k = 100$ for example then $e^k = e^{100} = 2,6 \times 10^{43}$ and $\frac{k^2}{e^k} \simeq \frac{10^4}{e^{100}} \simeq 3,7 \times 10^{-40}$ making this a value practically negligible and also making valid the relation ${}^5R_{11} \simeq R_{11}$. This is effectively and approximately a reduction from $5D$ to $4D$ as Basini-Capozziello stated.Even consider a Extra Dimension $y \neq 0$ the Ricci Tensor in $5D$ is approximately equal to a one in $4D$ and we cannot "see" the $5D$ Extra Dimension due to the reasons presented here in section 4 although a particle in the $5D$ Extra Dimension would "appear" as Superluminal to a potential observer in $4D$.A new line of reason of course would be to consider the apparent non-local effects of Quantum Mechanics as the "Superluminality" of the Einstein-Podolsky-Rosen(EPR) Paradox as matter moving inside the $5D$ but apparently invisible to us.

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In order to "see" the $5D$ Extra Dimension according to section 4 in this work we must make the Warp Field $\Phi \neq 1$.Then the complete ansatz of Chung-Freese with Warp Field in a Basini-Capozziello reduction from $5D$ to $4D$ for this case would be given by:

$$dS^2 = dt^2 - e^{-2ky}a^2 dX^2 - \Phi^2 dy^2 \tag{409}$$

For our special case:

$$dS^2 = dt^2 - e^{-2y(k+1)} dX^2 - \Phi^2 dy^2 \tag{410}$$

$${}^5R_{00} = R_{00} - \frac{\Phi_{,0;0}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4} g_{00,4}}{\Phi} - g_{00,44} + \frac{g^{00} g_{00,4} g_{00,4}}{2}) \tag{411}$$

$${}^5R_{11} = R_{11} - \frac{\Phi_{,1;1}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4} g_{11,4}}{\Phi} - g_{11,44} + \frac{g^{11} g_{11,4} g_{11,4}}{2}) \tag{412}$$

Note again that the derivatives of $g_{00} = +1$ vanishes leaving us with the following result:

$${}^5R_{00} = R_{00} - \frac{\Phi_{,0;0}}{\Phi} \tag{413}$$

And from our previous calculations we know that the derivatives of $g_{11} = -e^{-2ky}a^2$ or $g_{11} = -e^{-2y(k+1)}$ can be neglected leaving us with the expression:

$${}^5R_{11} \simeq R_{11} - \frac{\Phi_{,1;1}}{\Phi} \tag{414}$$

The equation above agrees with our results from section 4 for a Warp Field $\Phi \neq 1$.

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. We mentioned here the fact that the "classical" $4D$ General Relativity photon or graviton with zero $4D$ rest-mass m_0 travels at Light-Speed and do not propagates into the $5D$ Extra Dimension[126]so a spacecraft travelling inside the $5D$ Chung-Freese Superluminal Braneworld could not detect the presence of a large M_5 rest mass of a Black Hole acting as a Black String and the impact in the shaded area between the $5D$ rest-masses $M_5(spacecraft)$ and the $M_5(BlackString)$ would be a disaster. In order to prove that Chung-Freese Superluminal Braneworld can manoeuvre inside the $5D$ Extra Dimension avoiding the rest-masses of potential hazardous objects we will turn our attention to the Maartens-Clarkson massive modes of Kaluza-Klein gravitons propagating in the $5D$ Extra Dimension. According to the classical General Relativity and Quantum Mechanics photons and gravitons are particles of zero rest-mass as seen in the Table Of Elementary Particles presented in sections 2 and 3 of this work.According to Ponce De Leon in eq 20 of [2] the $5D$ rest-mass M_5 for both particles is zero.However in pg 1681 of [7] Maartens-Clarkson mentions that gravitons can propagates in the $5D$ Extra Dimension leaving a "smoking gun" that could be used to demonstrate the existence of the $5D$ Extra Dimension.In pg 1683 of [7] Maartens-Clarkson mentions the fact that the $5D$ Extra Dimensional polarized modes of the graviton would be "seen" in $4D$ as massive Kaluza-Klein (KK) modes.This is very important because these massive $4D$ (KK) graviton modes of non-zero rest-mass m_0 would correspond to a non-zero $5D$ rest-mass M_5 propagating in the $5D$ Extra Dimension at least in agreement with Ponce De Leon.At the top of pg 1684 of [7] Maartens-Clarkson mentions the possibility of the use of he $LIGO$ and $LISA$ Space Telescopes to detect these massive modes while at the bottom they mention the fact that the gravity signal of a $5D$ Black String is a combination between the classical $4D$ General Relativity standard graviton of $m_0 = 0$ and $5D$ $M_5 = 0$ coupled with these massive modes of $5D$ $M_5 \neq 0$ and $4D$ $m_0 \neq 0$.The difference between the classical General Relativity Black Hole gravity signal and the $5D$ Black String signal is given by Maartens-Clarkson in fig 3 pg 1685 of [7] where the massive modes can be seen.Also note the comment of Maartens-Clarkson about the gravitational events mainly taking place inside the $5D$ Extra Dimension.At pg 1686 of [7] Maartens-Clarkson mentions the fact that these massive graviton modes travels below the Light-Speed and this is a behavior in complete agreement of what would be expected for a non null $5D$ rest-mass M_5 according to Ponce De Leon. Then a $5D$ Black String in the shaded area of the Basini-Capozziello fig 1 pg 135 of [1] emits these non-null M_5 graviton modes at subluminal speeds and these modes propagating in the $5D$ Extra Dimension the Basini-Capozziello shaded area could perhaps be detected by the Superluminal Chung-Freese spacecraft also in the shaded area and also at subluminal speed.Perhaps the spacecraft could manoeuvre avoiding a lethal collision and this is a remarkable property of the Chung-Freese spacetime.Remember that the massive graviton mode at subluminal speed in the shaded area would be seen a subluminal speed also in the $4D$ spacetime the Basini-Capozziello white area.The capacity to travel subluminally in the $5D$ Extra Dimension but being seen as Superluminal and Faster Than Light in the $4D$ spacetime is a particular feature of the geometry of the Superluminal Chung-Freese Braneworld.

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. We will terminate now the part of the Superluminal Chung-Freese Braneworld mentioning that only in $4D$ Spacetime without the Extra Dimension the Alcubierre Warp Drive in this "trip" to Crab Nebula would need "luck" in order to avoid the Black Holes while the Superluminal Chung-Freese BraneWorld could perhaps avoid a collision.Between Sun and Proxima Centauri it seems that there are no large Black Holes so the Alcubierre Warp Drive would be suitable to "short" distance travel like Sun-Proxima Centauri but for long distance travel the Superluminal Chung-Freese Braneworld seems to be better.But at least we demonstrated that the Alcubierre Warp Drive as a Superluminal and Faster Than Light ansatz in $4D$ without recurring to Extra Dimensions is "still alive".However we don't know how to engineer the

[126]Our approach is different than the approach of Strings Theory

7 Conclusion

This section is divided by the following topics:

- 1)-Conclusion on Basini-Capozziello-Ponce De Leon Formalism
- 2)-Conclusion on the Pioneer Anomaly
- 3)-Conclusion on the Superluminal Chung-Freese BraneWorld
- 4)-Conclusion on the Alcubierre Warp Drive

- 1)-Conclusion on Basini-Capozziello-Ponce De Leon Formalism

  Our approach to the study of Extra Dimensions was centered on the Basini-Capozziello-Ponce de Leon Formalism.While other formalisms of Extra Dimensions uses $3+1$ uncompactified ordinary spacetime dimensions while the Extra Dimensions are compactified bringing the question of why $3+1$ large ordinary dimensions and the rest of the Extra Dimensions "curled-up" over themselves and what causes or generates the "compactification mechanism"????.In the Basini-Capozziello-Ponce de Leon Formalism the Extra Dimensions are large the same size of the $3+1$ ordinary dimensions avoiding the need of "exotic" compactification mechanisms but we cannot "see" these dimensions in normal conditions due to the reasons presented in this work. Also it can explain the multitude of particles seen in $4D$ as Dimensional Reductions from a small group of particles in $5D$ allowing perhaps the "unification" of Physics from the point of view of the Extra Dimensional Spacetime. This is very attractive from the point of view of a Unified Physics theory.There exists a small set of particles in $5D$ and all the huge number of Elementary Particles in $4D$ is a geometric projection from the $5D$ Spacetime into a $4D$ one([2] eq 20,[11] eq 21 and [20] eq 8).[127]

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{415}$$

[127] We know that we are repeating the table for the third time but the table coupled with the Ponce De Leon equation illustrates the beauty of this point of view

Particle	spin ($\hbar$)	B	L	T	T_3	S	C	B^*	charge (e)	m_0 (MeV)	antipart.
u	1/2	1/3	0	1/2	1/2	0	0	0	+2/3	5	$\overline{\mathrm{u}}$
d	1/2	1/3	0	1/2	−1/2	0	0	0	−1/3	9	$\overline{\mathrm{d}}$
s	1/2	1/3	0	0	0	−1	0	0	−1/3	175	$\overline{\mathrm{s}}$
c	1/2	1/3	0	0	0	0	1	0	+2/3	1350	$\overline{\mathrm{c}}$
b	1/2	1/3	0	0	0	0	0	−1	−1/3	4500	$\overline{\mathrm{b}}$
t	1/2	1/3	0	0	0	0	0	0	+2/3	173000	$\overline{\mathrm{t}}$
e^-	1/2	0	1	0	0	0	0	0	−1	0.511	e^+
μ^-	1/2	0	1	0	0	0	0	0	−1	105.658	μ^+
τ^-	1/2	0	1	0	0	0	0	0	−1	1777.1	τ^+
ν_{e}	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_{\mathrm{e}}$
ν_μ	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_\mu$
ν_τ	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_\tau$
γ	1	0	0	0	0	0	0	0	0	0	γ
gluon	1	0	0	0	0	0	0	0	0	0	$\overline{\mathrm{gluon}}$
W^+	1	0	0	0	0	0	0	0	+1	80220	W^-
Z	1	0	0	0	0	0	0	0	0	91187	Z
graviton	2	0	0	0	0	0	0	0	0	0	graviton

Look to the Elementary Particles Table above:we have a multitude of Quarks,Leptons,Muons and Heavy particles with apparently different rest-masses m_0 in $4D$ but these particles can have the same rest-mass M_5 in the $5D$ Extra Dimension and the differences are being generated by the Dimensional Reduction from $5D$ to $4D$ according to the Basini-Capozziello-Ponce De Leon formalism.This can bring new perspectives for the desired dream of the Unification of Physics

- 2)-Conclusion on the Pioneer Anomaly

  Some years ago 3 physicists Bahran Mashoon from Missouri State University United States,Paul Wesson from University of Waterloo Ontario Canada and Hongya Liu from Dalian University The People Republic of China predicted the existence of fifth force in the Universe beyond the four known in the scientific paper "Dynamics Of Kaluza-Klein Gravity and a Fifth Force". According to them this force could be detected and observed.. The Pioneer Anomaly according to our work is the first experimental evidence of the Mashoon-Wesson-Liu fifth force and the first experimental confirmation that we live in a Universe that contains more than 4 dimensions. The Sun is a Maartens-Clarkson 5D Schwarzschild Black String and the anomalous de-acceleration centered in the Sun and observed in the spaceships Pioneer X and XI is explained by this extra force coupled with the work of Orfeu Bertolami and Jorge Paramos from Instituto Superior Tecnico (IST) Lisboa Portugal. The "sunshine gleaming" in $5D$ Extra Dimensions generates the fifth force that de-accelerates the ships. This is not a Sun-Pioneer phenomena:The Anomaly is scattered across the entire Universe. The Space Era started on 4 October 1957 50 years ago and the discovery of the Pioneer Anomaly is a consequence of this Human Adventure in the Skies.It could never be spotted from the Earth. As far the Human Exploration of Space grows our way to see the Universe and the laws of physics will change in a way that could never happen solely on Earth.We believe that God had plenty of immagination when created the Universe and what we know is just the beginning. Axions,Magnetars,MACHOS,Dirac Magnetic Monopoles and many other unknown wonders will appear as a consequence of our Epic Odyssey. The Universe have many secrets Up There In The Skies waiting for ourselves in order to be discovered and many of these discoveries will force ourselves to change our comprehension of the physical laws and the way we see the Universe. It happened before with the Pioneer Anomaly.The Anomaly is not due to gas leak,oil leak or plutonium reactor leak but instead forced ourselves to re-examine the equations of Einstein General Relativity. Like we said it already happened before but we can take it for granted that as far as we go further sailing in the Oceans of the Frozen Immensity of Space it will happen again and again and again and again.

- 3)-Conclusion on the Superluminal Chung-Freese BraneWorld

  In this work we studied the physical features of the Superluminal Chung-Freese BraneWorld. According to the abstract of Chung-Freese original paper a particle,a signal or a spaceship can "leave" our 3+1 Universe enter into the Extra Dimensional spacetime and when returning back to our Universe the particle,signal or spaceship would appear in a distant place far away from the original departure point and in a shorter time than a signal confined to our 3+1 Spacetime would need to travel the same distance.We kept ourselves in this work with the scientific vocabulary and avoided science fiction terms(the term "Warp Drive" was well accepted by the scientific community and appears in many peer-review published papers) but the behavior of the Superluminal Chung-Freese Braneworld is a remnant or a reminiscent of the "Hyperdrive" of science fiction as depicted in the science-fiction novel "Star Wars". The "Hyperdrive" dont suffer the pathology of negative energy density and while subluminal in the Extra Dimensional spacetime the "Hyperdrive" would be "seen" as Superluminal by an observer in our Universe. The "Hyperdrive" would be adequated to "long" distance interstellar travel eg Sun-Crab Nebula because it can manoeuvre between hazardous objects eg large Black Holes. However we still don't know how to generate the "Hyperdrive" spacetime. Lastly we will comment the prediction of "Hyperdrive Starships Plying the Galaxy"[128]. We are confident that this prediction will one day become true and the Space Pilots of the Distant Future will look backwards to their Remote Past to Salute the year of 1999 when the American Physicists Daniel Chung and Katherine Freese from Michigan University Ann Arbor Michigan United States appeared with a paper called "Can Geodesics In Extra Dimensions Solve the Cosmological Horizon Problem??". This paper will ever be regarded as the paper that started it all:If this Legend of Space Pioneers will ever become true then Daniel Chung and Katherine Freese already have a place in the Human History.They will always be remembered from Here to the Eternity and with the most Profound Feeling of Gratitude by the Future Generations as the Father and the Mother of the "Hyperdrive".

- 4)-Conclusion on the Alcubierre Warp Drive

  In this work we tried to restore the physical feasibility of the original and wonderful idea of the Alcubierre Warp Drive.Our new Piecewise Shape Functions solved the pathologies of negative energy and Doppler Blueshifts using the original Alcubierre geometry but without expansion or contraction imitating the behavior of the Natario Warp Drive.The expansion/contraction is a consequence of the Alcubierre choice for the Shape Function. For the problem of the Horizons we took inspiration from the NASA Apollo spacecrafts eg the Command Module and the Service Module.The Alcubierre Warp Drive spaceship(the mothership) is like the Command Module:the mothership must eject the Warp Drone that contains the entire light cone of the mothership as the external observer that will create the Warp Bubble following the work of the American physicists Allen Everett and Thomas Roman from Tufts University Medford Massachussets United States and the Warp Drone will create the Warp Bubble with the mothership inside and will trigger the motion[129]. The Warp Drone is like

[128]adapted from our conclusions with respect to the Alcubierre Warp Drive

[129]see the comment on pg 3 of [45] about the actions to create or change the Warp Bubble trajectory or speed being taken by an external observer whose light cone contains all the trajectory of the Warp Bubble

the Service Module. When arriving at the destination point the mothership will destroy the rear part of the Warp Bubble.The front will go on but since we use low negative energy and the gravitational force is repulsive the front will not be harmful to planets for example. Once triggered the motion the Warp Drone will be abandoned in Space just like the Apollo Service Module. Each Alcubierre Warp Drive can only be used for a "one-way" trip. Impact with large Black Holes are serious threats but considering a Warp Bubble of large radius the impact would destroy the front of the Bubble and the ship would perhaps stop but at a safe distance from the Events Horizon allowing the ship to escape.The ship would then eject another Warp Drone to continue the journey however this is an expensive way to travel.The Alcubierre Warp Drive would be suitable to interstellar travel of "short" distances eg Sun-Sirius,Sun-Tau Ceti,Sun-Proxima Centauri where no Black Holes are expected to be found.For "long" interstellar travel the Chung-Freese Superluminal Braneworld would be better. We still don't know how to generate the Alcubierre Warp Drive but once generated it would be of Immense value. Even a subluminal version of the Alcubierre Warp Drive would be able to reach the region of Space where the Pioneers were lost in a matter of days(or hours) avoiding the need to wait more 40 years. Lastly we will comment the prediction of 3 American physicists Chad Clark,William Hiscock and Shane Larson from Montana State University Bozeman Montana United States in the paper "Null Geodesics In The Alcubierre Warp Drive:The View From The Bridge" about the "Warp Drive Starships Plying the Galaxy"[130].We are confident that this prediction will one day become true and the Space Pilots of the Distant Future will look backwards to their Remote Past to Salute the year of 1994 when the Mexican Mathematician Miguel Alcubierre from Universidad Nacional Autonoma de Mexico(UNAM) appeared with a paper called "The Warp Drive:Hyper Fast Travel Within General Relativity".This paper will ever be regarded as the paper that started it all.Miguel Alcubierre was the first to overcome the limitations of Einstein Special Relativity. If this Legend of Space Pioneers will ever become true then Miguel Alcubierre already have a place in the Human History:He will always be remembered from Here to the Eternity and with the most Profound Feeling of Gratitude by the Future Generations as the Father of the Warp Drive

[130] Clark,Hiscock,Larson mentions in pg 4 of [30] "Warp Drive Starships Plying The Galaxy".

8 The End

The Pioneer Anomaly is perhaps the strongest proof of a Universe of Higher Dimensions affecting our everyday Universe of $4D$ Dimensions.We will terminate this work exactly by the same way we started the beginning: The Space Era started on 4 October 1957 with the Aerospace Engineer Serguei Pavlovitch Koroliev and the rocket RS-7 Semyorka that launched into Outer Space the first Human Made artificial satellite:the Sputnik I of the former Union of Soviet Socialist Republics.Only fifty years have passed since the dawn of the Human Adventure in the Skies and its too early to intend (or pretend) that our science knows everything about the Outer Space. The Universe Out There will remain forever a mysterious place filled with secrets we cannot barely imagine and in the remote future at millions of years from now we can take for granted that strange and unknown phenomena left by Mother Nature will be Up There In The Skies waiting for ourselves in order to be discovered. Young Jedi Knight Padawan:Stay Away From The Dark Side And May The Force Be With You[131]

[131]slightly modified from Frank Oz as Master Yoda in the George Lucas Movie:Star Wars Episode II The Attack Of The Clones

9 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[132]
- "We need someone to shake our views. Otherwise we are doomed to live in the past."-Edward Halerewicz Jr.[133]
- "Only once we leave our tiny precious homeworld and venture out in the void of space will we truly find our destiny out among the stars. Its up to us each and every one of us to fight hard and work to make it happen"-Simon Jenks[134]
- "Earth was the cradle of Humanity but Humans will not stay on the cradle forever"-Konstantin Edvardovitch Tsiolkovsky[135]
- "The Victory Belongs to the ones that Believe it the Most.And Believe it the Longest:We gonna make Believe"-Alec Baldwin[136]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[137][138]

[132] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[133] American Physicist from Sprinngfield Illinois and author of one of the most beautiful webpages on the Internet about the Physics of the Alcubierre Warp Drive

[134] British Web designer living in San Antonio Texas United States of America and also author of one of the most beautiful webpages on the Internet about the Physics of the Alcubierre Warp Drive

[135] The Father of the Russia Space Program. Mentor of Serguei Pavlovitch Koroliev and Yuri Alekseyevitch Gagarin

[136] Alec Baldwin as Lieutenant-Colonel James B.Doolitle-April 1942. From the Jerry Bruckheimer and Michael Bay movie:Pearl Harbor

[137] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[138] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

10 Acknowledgements

We would like to express the most profound and sincere gratitude towards Professor Doctor Martin Tajmar of University Of Viena, Austria – ESA (European Space Agency) – ESTEC-SV (European Space Technology and Engineering Center - Space Vehicles Division) and Seibesdorf Austria Aerospace Corporation GmBH-ASPS (Advanced Space Propulsion Systems), for his kindness and goodwill for being our arXiv.org sponsor. We would like also to acknowledge Paulo Alexandre Santos and Dorabella Martins da Silva Santos from University of Aveiro Portugal for the access to the scientific publication General Relativity and Gravitation(GRG). We would like also to express our sincere gratitude towards the arXiv.org readers that spotted the division by R in our Piecewise Shape Functions giving a dimensional coefficient while the original Alcubierre continuous function and the Ford-Pfenning Piecewise Shape Function are both dimensionless specially naming T.Desiato. We also would like to express our immense gratitude to Professor Doctor Marcelo Ribeiro from the Department of Physics of Universidade Federal do Rio de Janeiro(UFRJ)-Brazil for his kindness to provide access to the original Classical and Quantum Gravity(CQG) and Physical Review D(PRD) published papers on the Alcubierre Warp Drive(Alcubierre,Ford-Pfenning,Broeck,Clark-Hiscock-Larson,Natario,Everett-Roman,Diaz Gonzalez). In closing we would also like to acknowledge the Administrators and Moderators of arXiv.org at the Cornell University, United States of America for their agreement in accepting this document.

11 Remarks

The bulk of the bibliographic sources used in our research came from the refereed scientific publication General Relativity and Gravitation (GRG) from Springer-Verlag GmBH (formerly Kluver/Plenum Academic Publishing Corp)(ISSN:0001-7701 paper)(ISSN:1572-9532 electronic) under the auspices of the International Comitee on General Relativity and Gravitation and quoted by Deutsche Zentralblatt Math of EMS(European Mathematical Society). The Volume 36 Issue 03 March 2004 under the title:"Fundamental Physics on the ISS" was totally dedicated to test experimentally General Relativity,Quantum Gravity,Extra Dimensions and other physics theories in Outer Space on-board International Space Station(ISS) under the auspices of ESA(European Space Agency) and NASA(National Aeronautics and Space Administration).Seven references came from the prestige refereed scientific publication Classical and Quantum Gravity(CQG) from the excellent well-reputation IOP Institute of Physics Publishing(ISSN:0264-931 paper)(ISSN:1361-6382 electronic) and one reference came from Journal Of Physics A(JPhysA) also from IOP(ISSN:1751-8121).Another eight references came from the prestige refereed scientific publication Physical Review D(PRD) from American Physical Society(ISSN: 1550-7998 paper)(ISSN: 1550-2368 Online).Another reference came from the published papers of European Space Agency(ESA) and another from the prestige refereed scientific publication Physics Reports(PhysRpt) from the well prestige publisher Elsevier-North Holland Publishing(ISSN: 0370-1573).To terminate the origin of the published papers we use two from International Journal of Modern Physics A and D(IJMPA)(IJMPD) from the also well reputation World Scientific Publishing Corporation(ISSN: 0218-2718) one from the Canadian Journal of Physics(Can J Phys)(ISSN:0008-4204 paper)(ISSN: 1208-6045 online),another from the prestigious scientific publisher Living Reviews in Relativity(Liv Rev Rel)from the well reputated Max Planck Institut Fur Physik Und Astrophysik-Albert Einstein Institut(AEI) Golm Germany(ISSN:1433-8351),another from Europhysics News(EuPhysNew) from European Physical Society(ISSN:1432-1092) and another from Gravitation and Cosmology (GravCosmol)(ISSN:0202-2893) from the Ministry of Industry, Science and Technologies of Russia and another published by Stanford University California United States.All the mention to pages of the references in the main text and in the footnotes of this work are for GRG and Liv Rev Rel references originally from the published version since we have access to GRG and Liv Rel Rel although we provide the number of the arXiv.org available GRG and Liv Rev Rel papers but for mentions to CQG[139],JPhysA,PRD,IMJPA,IJMPD,PhysRpt,EuPhysNew,GravCosmol and Can J Phys references the page numbers are originally from the arXiv.org version since we cannot access these journals and sometimes exists differences in page numbers between the arXiv.org version and the published version due to different editorial styles preferred by scientific journals[140].One reference[141] is a Post-Doctoral Dissertation written by one of the World's best Warp Drive scientists from one of the best Universities of the World:Tufts University Medford Massachussets United States Of America.Another reference [142] cames from one of the best NASA Aerospace Engineeers working at John Glenn Space Center Lewis Field Cleveland Ohio United States of America.The most important reference[143] of this work is an unpublished paper from arXiv.org but we took in mind the immense scientific prestige of Instituto Superior Tecnico(IST) Lisboa Portugal from where this paper came from, and also the prestige of the authors of this paper specially because they outline the fact that the Pioneer Anomaly is not due to a gas leak or an oil leak or a plutonium reactor

[139] the original CQG and PRD papers on the Alcubierre Warp Drive arrived when we terminated the Warp Drive section but indeed we are extremely grateful to Professor Doctor Marcelo Ribeiro from the Physics Department-Universidade Federal do Rio de Janeiro UFRJ-Brazil

[140] readers that can access GRG can compare for example gr-qc/0310078 with [2] or gr-qc/0603106 with [20]

[141] see [48]

[142] see [49]

[143] see [27]

leak but instead it requires a major revision of the equations of Einstein General Relativity[144] and we agree entirely with their point of view.This reference was the source of the Bertolami-Paramos definitions that coupled with the $5D$ Extra Dimensional Force predicted years ago by Mashoon-Weson-Liu in [145] can perhaps tell us what happened with the Pioneer spaccrafts.We choose to adopt in our research mainly refereed published papers of excellent quality from these publications/institutions not only due to its/their prestige and reputation among the scientific community but also because we are advocating new points of view in this work but based on the solid ground of certifiable and credible references

[144] read first page of [27].Note the fact that the authors outline a needed modification of the Einstein Field Equations and also the fact that how can two space probes moving in different directions having the same anomaly.The same gas leak in both probes?:Highly Unlikely

[145] see [9]

12 Legacy

This work is dedicated to the Memory of the Aerospace Engineer Serguei Pavlovitch Koroliev[146] Father of the Mighty Powerful Rocket RS-7 Semyorka that launched into Outer Space the Sputnik I on 4 October 1957 50 years ago opening to the Human Race the Doors of the Astronautic Era.This work is also dedicated to the 50th Anniversary of the Human Adventure In the Skies and Specially Dedicated to the Memory of the Admirable Brave Souls of these Cosmonauts and Astronauts that died in a Supreme Sacrifice of this Epic Odyssey[147] For All Mankind and in a Noble Call of Duty:Vladimir Mikhailovich Komarov(Soviet Union)(He died during the re-entrance in the atmosphere of the spaceship Soyuz I April 24,1967),Vladislav Volkov(Soviet Union),Georgi Dobrovolski(Soviet Union),Viktor Patsayev(Soviet Union)(They died by asphyxia due to a failure in the Life Support System during the re-entrance in the atmosphere of the spaceship Soyuz 11 June 30,1971), Edward Higgins White(United States),Virgil Ivan "Gus" Grissom(United States),Roger Bruce Chaffee(United States)(They died in the explosion during the launch of the spaceship Apollo I January 27 1967),Sharon Christa Corrigan McAuliffe(United States),Michael "Mike" John Smith(United States),Francis Richard "Dick" Scobee(United States),Ronald Erwin McNair(United States),Ellison Shoji Onizuka(Japan-United States),Gregory Bruce Jarvis(United States),Judith Arlene Resnik(United States)(They died in the explosion during the launch of the Space Shuttle Challenger January 28,1986),Kalpana Chawla(India),Richard Douglas Husband(United States),William Cameron "Willie" McCool(United States),David McDowell Brown (United States),Michael Phillip Anderson(United States),Laurel Blair Salton Clark (United

[146]Serguei Pavlovitch Koroliev will figure among the best Aerospace Engineers of all Mankind.He was born in the small town of Zhitomir Ukrania:Due to his Ukranian Nationality he suffered the prosecution of the Soviet Dictator Josif Vissarionovitch Vassilievitch(Stalin).He was a victim of Stalin rage against Ukranian intellectuals.He was sent as a prisioner to a Gulag in Central Siberia and condemned to slave labor in the gold mine fields of Kolyma.Soviet Union at late '50s' needed a Ballistic Missile able to launch nuclear weapons over US territory because while Americans had more than 600 Long Range Strategic Bombers (Boeing B-52) able to reach at ease any part of Soviet Union,Soviets had less than 80 Medium Range Bombers(Myasishev Mya-4 "Bison") that could barely reach American airspace.Soviets unable to compete against the American economic power and edge in Bombers shifted their attention to a fast,cheap but effective way to launch weapons over the United States.They choosed the Intercontinental Ballistic Missile,however all the Soviet attempts to launch an ICBM failed and it seemed that the Soviet effort was doomed to failure:Red Army General Mikhail Tukaschenko released Koroliev and despite of his nationality gave him the Prime Leadership in the ICBM project and took him to the Red Army Missile Test Facility of Kapustin Yar.While others failed Koroliev accomplished success.He becames the Father of the first Intercontinental Ballistic Missile in the World:The ICBM SS-6 Sapwood.Some of these ICBM's were placed in Strategic Alert in the top secret Missile Complex of Plesetsk.But Koroliev passion was really Space Science.He knew that ICBM's with some small modifications would be perfect to perform as Space Rockets.The ICBM SS-6 Sapwood was modified to become the RS-7 Semyorka.Koroliev worked actively not only in Sputnik I but also in the Sputnik II that carried the first living creature on Earth to go to Outer Space:the female dog Laika on 3 November 1957.He worked also on strengthened versions of the RS-7 able to carry large payloads to Outer Space and one of these new versions launched in 12 April 1961 the first Human Astronaut to Outer Space:the Major Yuri Alekseyevitch Gagarin of the Red Air Force in the Spaceship Vostok I(Orient I).It is now known that by 1966 year of his death Koroliev completed the design of a enormous Manned Moon Rocket by a special request from the Soviet Premier Leonid Brezhnev and Soviets could perfectly be the first ones in the Human History to attempt a Manned Lunar Landing had Soviet Union not closed the project due to economic reasons.When Koroliev died even the Soviet Premier Leonid Brezhnev made question to join the funeral ceremonies and was among the ones that helped to carry his coffin to the cemetery.He was incinerated and his ashes were buried in the Patriotic Wall at the Kremlin.Serguei Pavlovitch Koroliev was the Father of the World's First Space Rocket and will ever be considered one of the greatest Aerospace Engineers of all Mankind.Due to the Stalin rage against Ukranians he was sent as a prisoner to a Gulag in Central Siberia to work as a slave in a gold mine to the death.But in the end the Soviet Union needed his knowledge,expertise and skills and he served his Mother Country with Devotion,Patriotism and Commitment.A street as a Homage to him exists in Moscow.He was regarded as a Hero of the Soviet Union.After the fall of the communism Russia President Boris Yeltsin changed the name of the city of Kalinigrad to city of Koroliev

[147]a Homage to Jim Lovell,Fred Haise and Jack Swigert to their Bravery during the disaster of Apollo 13 flight.The Command Module was named "Odyssey".Fortunately they are not on the list

States),Ilan Ramon(Israel)(They died during the re-entrance in the atmosphere of the Space Shuttle Columbia February 1,2003). Space is the Final Frontier and the Ultimate Destiny of the Humans as a Specie but unfortunately the Conquest of a New Frontier sometimes demands a price to be paid and sometimes the price is too expensive too painful or too high.This was a work destined to propose an explanation for the mystery behind the so-called Pioneer Anomaly and a work destined to rescue back the physical feasibility of the original and wonderful idea of the Alcubierre Warp Drive but it was above everything else a work dedicated to the 50th Anniversary of the Human Adventure In The Skies and a Homage[148] to the Date 4 October 1957 [149] and we would like to mention here that these Brave Men and Women paid the price of the Space Conquest in the Name Of All The Humanity.They paid the price of the Future of All Mankind Up There Among The Stars In The Middle of the Frozen Immensity Of Space.They paid the price with the cost of their own deaths.They died For All Of Us.They died like Heroes.They died like Pioneers.May God Bless All The Brave Souls Of These Cosmonauts and Astronauts and Give To Each One Of Them The Sacred Rest For All The Eternity.They Made To Deserve It and They Will Never Be Forgotten.....So Be It.

[148] the word "Homage" is French:it means Celebration

[149] as a Homage to the 4 October 1957 we sent this work to arXiv.org on 4 October 2007 exactly 50 years later

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