Quark lepton braids and heterotic supersymmetry

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Abstract. A unique matrix is easily assigned to each Bilson-Thompson braid diagram. The quark and lepton matrices are then related to bosons via a twisted quantum Fourier transform, for which fermion and boson multiplets fit the dimension structure of heterotic strings.

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1. Introduction

In a category theory approach to the Standard Model one expects to characterise the quarks and leptons using simple diagrammatic elements. One existing braid diagram scheme [1] characterises the electroweak interactions through matrix multiplication, and there exists a full matrix scheme [2] based on measurement algebra. This paper assigns matrices to the braid diagrams [1] in order to (a) analyse certain quantum information aspects of this scheme and (b) consider them a useful reduction of the full scheme [2] in the case that spin information is suppressed.

A projection of braid diagrams onto the underlying permutations loses the crossing information. However, the permutations and the charge assignments for each braid strand are enough to classify Standard Model particles [1]. Here we employ permutation like matrices with phased non zero entries representing charge.

These matrices have only one non zero entry in each row and column. It will be useful to consider matrices whose entries are complex roots of unity, in which case all such matrices fall into a class of operators associated to the so called field with one element [3]. In this abstract arithmetic context, although given by ordinary matrix multiplication, all relations between particles are in principle independent of ordinary linear algebra, since the matrices are viewed as operators on certain sets, rather than vector spaces. Moreover, in this context there exists a natural homomorphism from braid groups to permutations, acting somewhat like the classical limit $q \to 1$ of deformation theory.

The permutation scheme will thus be seen as classical physical information arising from a measurement algebra which is deeply arithmetic in nature. The finite set of complex numbers appearing here are typically characters for finite fields, which themselves play an abstract role in the theory of the field with one element. Although remarkably simple, this permutation structure appears to contain valuable information about Standard Model phenomenology.

The next section defines the fermions in terms of permutation operators, and in the following section we introduce the twisted Fourier transform and the boson matrices. Supersymmetry and crossing information are discussed in sections 4 and 5. In the future, more sophisticated versions of these 3×3 operators will be used to study the structure of Koide mass triplets [4] and the CKM and neutrino mixing matrices.

2. The Fermion Zoo

Bilson-Thompson [1] characterises the quarks and leptons with braid diagrams, using three ribbon strands for each particle. These braids all represent the permutations (231) or (312), up to two possible choices for the braid crossings. The allowed crossing choices are such that a product of three identical braid elements results in mutually unlinked, though braided, strands. This limitation on braid elements from B_3 suggests seeking an algebraic characterisation for the braids, beginning with the underlying permutations, introducing the crossing information later on.

To each braid diagram we can assign a unique 3×3 matrix based on the permutation matrices. If a ribbon has a positive twist, representing a charge of $+1/3$, the matrix entry is the symbol ω. Similarly, negative twists are denoted by $\overline{\omega}$, and no twists by 1. In this paper, ω will be viewed as a complex number, but the motivating mathematics does allow for more abstract interpretations of matrix entries.

Using this scheme, Figure 1 lists all the Bilson-Thompson leptons and half of the quarks. Up quarks are left handed, whereas down quarks are right handed. Note that charge conjugation sends ω to $\overline{\omega}$. Left handed particles correspond to the permutation (231), while right handed ones correspond to the other 1-circulant permutation (312).

$$
\begin{array}{cccccc}
e_L^- & e_L^+ & e_R^- & e_R^+ & \nu_L & \overline{\nu}_R \\
\begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ \omega & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \\ 0 & \overline{\omega} & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \omega \\ \omega & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\overline{u}_L(1) & \overline{u}_L(2) & \overline{u}_L(3) & d_R(1) & d_R(2) & \overline{d}_R(1) \\
\begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & 1 \\ \overline{\omega} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 1 \\ \overline{\omega} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

Figure 1. quark and lepton braids

The ribbon rule $\omega \cdot \overline{\omega} = 1$ is satisfied by any complex phase ω. If ω is a primitive nth root of unity, n twists must return the identity, so build ups of charge require either large values of n or irrational phases. However, in order to study the basic interaction rules one never requires more than one twist, so it is enough to set ω to a cubed root of unity. With this reduction, the permutation rule $(231)^3 = (123)$ is obeyed.

3. Twisted Fourier Transform and Bosons

The quantum Fourier transform [5] plays an important role in quantum arithmetic [3]. In particular, complementary observables in a p dimensional space, for p a prime power, are characterised by a set of $p+1$ operators, each defining a basis. The p dimensional Fourier operator F_p is one of this set, while the other members are powers of a given circulant matrix, and circulants are diagonalised by F_p.

A 3×3 1-circulant matrix is a discrete Fourier series, with basis the odd permutations, (123), (231) and (312). Given the 1-circulants of Figure 1, we would like to know to which diagonal operators they correspond. Since the fermion zoo is naturally associated to a three point noncommutative momentum space, any quantum transform of Fourier type should shift the zoo to a dual space that is also of physical relevance.

In this section $\phi = e^{2\pi i/3}$ is the primitive cubed root of unity, and ω may be a general complex phase. Under the 3×3 Fourier transform any phased odd permutation goes to the product of a diagonal D and a 1-circulant, as in

$$\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \phi & \overline{\phi} \\ 1 & \overline{\phi} & \phi \end{pmatrix}\begin{pmatrix} 0 & \omega_1 & 0 \\ 0 & 0 & \omega_2 \\ \omega_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\phi} & \phi \\ 1 & \phi & \overline{\phi} \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\phi} & 0 \\ 0 & 0 & \phi \end{pmatrix}\begin{pmatrix} \omega_1+\omega_2+\omega_3 & \overline{\phi}\omega_1+\phi\omega_2+\omega_3 & \phi\omega_1+\overline{\phi}\omega_2+\omega_3 \\ \phi\omega_1+\overline{\phi}\omega_2+\omega_3 & \omega_1+\omega_2+\omega_3 & \overline{\phi}\omega_1+\phi\omega_2+\omega_3 \\ \overline{\phi}\omega_1+\phi\omega_2+\omega_3 & \phi\omega_1+\overline{\phi}\omega_2+\omega_3 & \omega_1+\omega_2+\omega_3 \end{pmatrix}$$

This suggests taking a look at the twisted Fourier transform of a matrix M, defined by

$$T(M) = D^{-1}FMF^{\dagger} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ \phi & \overline{\phi} & 1 \\ \overline{\phi} & \phi & 1 \end{pmatrix} M \begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{\phi} & \phi \\ 1 & \phi & \overline{\phi} \end{pmatrix} \tag{2}$$

Figure 2 shows the T transformed matrices for the quark and leptons. The quark phases are expressed in terms of integer powers of $e^{\pi i/6}$, using the value $\omega = \phi$. That is, the matrix entries represent 12th roots of unity expressed in terms of integers mod 12. Note that the triple values $(1,5,9)$ represent evenly spaced 12th roots, summing to zero. Similarly, the triple $(11,3,7) = (-1,3,7)$ also represents evenly spaced 12th roots. These phases are multiplicative characters for $\mathbb{F}_{13}$, the field with 13 elements. Note that only for the value $\omega = \phi$ do the transformed quark entries all have the same norm.

For more general ω, the transformed quark operators are simple combinations of two diagonally phased idempotent circulants, as in

$$T(\overline{u}_L(1)) = \overline{\omega}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3}$$

$$T(d_R(1)) = \begin{pmatrix} 2 & -\phi & -\overline{\phi} \\ -\phi & 2\overline{\phi} & -1 \\ -\overline{\phi} & -1 & 2\phi \end{pmatrix} + \overline{\omega}\begin{pmatrix} 1 & \phi & \overline{\phi} \\ \phi & \overline{\phi} & 1 \\ \overline{\phi} & 1 & \phi \end{pmatrix}$$

$$\begin{array}{cccccc} T(e_L^-) & T(e_L^+) & T(e_R^-) & T(e_R^+) & T(\nu_L) & T(\overline{\nu}_R) \\ \begin{pmatrix} \varpi & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & \varpi \end{pmatrix} & \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} & \begin{pmatrix} \varpi & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & \omega \end{pmatrix} \end{array}$$

$$\begin{array}{cccccc} T(\overline{u}_L(1)) & T(\overline{u}_L(2)) & T(\overline{u}_L(3)) & T(d_R(1)) & T(d_R(2)) & T(\overline{d}_R(1)) \\ \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix} & \begin{pmatrix} 9 & 9 & 5 \\ 5 & 9 & 9 \\ 9 & 5 & 9 \end{pmatrix} & \begin{pmatrix} 9 & 5 & 9 \\ 9 & 9 & 5 \\ 5 & 9 & 9 \end{pmatrix} & \begin{pmatrix} 11 & 11 & 3 \\ 11 & 7 & 7 \\ 3 & 7 & 3 \end{pmatrix} & \begin{pmatrix} 11 & 7 & 7 \\ 3 & 7 & 3 \\ 11 & 11 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 9 & 1 \\ 9 & 9 & 5 \\ 1 & 5 & 5 \end{pmatrix} \end{array}$$

Figure 2. transformed quarks and leptons

where each pair (A, B) of phased circulants is annihilating, the up quark case satisfying $AB = 0$ and the down transform satisfying $(AB)^2 = 0$. Whether phased or not, the circulants $(2, -1, -1)$ and $(1, 1, 1)$ have eigenvalue sets $(0, 1, 1)$ and $(0, 0, 1)$ respectively. We observe that these are two elementary Koide mass matrices [4] for particle triplets. For example, the mass triplet $(0, 1, 1)$ describes the relative masses of the boson triplet (γ, W^+, W^-).

Diagonal matrices with strands of like charge represent bosons, as in

$$T(e_L^{\pm}) = W^{\pm} \qquad T(\nu_L) = \gamma \tag{4}$$

The Z boson is also characterised by Bilson-Thompson [1], as a product of the three transformed right handed leptons in Figure 2. The photon γ, as the identity, is also expressed as the product of the three right handed leptons from Figure 1. In order to distinguish the Z boson from a photon, the three factors must remain distinct. Note that each diagonal factor breaks the symmetry of the strands in much the same way as for up quarks. In other words, the Z boson appears here as a hadron like colour neutral particle, composed of three transformed fermions. The Z factors are also responsible for mapping anti up quarks to down quarks, through left multiplication.

The inverse twisted transform takes bosons to fermions, as in

$$e_L^- = \frac{1}{3} \begin{pmatrix} 1 & \overline{\phi} & \phi \\ 1 & \phi & \overline{\phi} \\ 1 & 1 & 1 \end{pmatrix} W^- \begin{pmatrix} 1 & 1 & 1 \\ 1 & \phi & \overline{\phi} \\ 1 & \overline{\phi} & \phi \end{pmatrix} = T^{-1}(W^-) \tag{5}$$

where the two transform matrices are mirrored vertically now, rather than horizontally. When $\omega = \phi$, the application of T to the bosons (γ, W^+, W^-) gives the Z boson triplet, and T^3 acts as the identity on these multiples of the identity. On a Z boson factor, T returns a circulant of Pauli type, which[1] is associated to a quark transform pair in the next section.

Note that T does not satisfy $T(M_1)T(M_2) = T(M_1 M_2)$, since it uses two matrices that are not inverses. However, it does satisfy the twisted rule

$$T(M_1)T(M_2) = T(M_1 \overline{\nu}_R M_2) \tag{6}$$

and similarly for other twisted transforms.

4. Fourier Supersymmetry

The transformed quarks are not obviously representable as braids. However, products of these matrices do form braid like ones. For example, at $\omega = \phi$,

$$T(\overline{u}_L(1))T(d_R(1)) = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix} \begin{pmatrix} 11 & 11 & 3 \\ 11 & 7 & 7 \\ 3 & 7 & 3 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \quad (7)$$

These matrices are distinct from the ones listed so far, and are known as generalised Pauli operators [3]. As in dimension 2, the eigenvector sets of such a Pauli operator defines complementary observables in dimension 3, here associated with 6 space like directions associated to 6 quark pairs.

There is a natural set of eight basic 3×3 Pauli operators (gluons) that generalises the three σ_k in dimension 2. These may be used to generate the Lie algebra $su(3)$, just as $su(2)$ is generated by the σ_k.

The application of T to a Pauli operator, which is fermion like, results in a circulant matrix. Since $\omega = \phi$, this circulant is a basic permutation, namely $\overline{\nu}_R$. Thus a double application of T to the quark boson pair results again in a Z boson factor. In this way, the full Z boson triplet is directly associated to the quark colour index.

Thus a full chiral particle scheme for the first generation includes 16 fermions (left handed charged leptons, neutrinos and 12 quarks) and 10 apparent bosons (four electroweak and 6 paired quark transforms). Such a list brings to mind the dimensions of heterotic strings, for which the 16 fermionic dimensions are associated to an $E_8 \times E_8$ symmetry, and 10 spacetime dimensions involve an observable 4 dimensional component. From the information theory point of view, however, background independent braid observables form a full list of observables and there is no requirement of, for instance, extra supersymmetric partners, which arise only from an ad hoc implementation of a continuum supersymmetry.

There exist many relations between the simple matrices on both sides of a twisted transform, and these match the usual rules [1], albeit without the braid crossing information. For example, $e_L^- e_R^+ = \gamma$ describes pair annihilation.

Note that the naming of quarks and anti-quarks is mere convention, so by swapping their names we can arbitrarily state that protons are made of anti-quarks and anti-protons of quarks. Given this convention, that ordinary baryons are made of anti-quarks rather than quarks, there would be no mysterious asymmetry between matter and antimatter, which are equally represented in the braid scheme.

Ignoring quark color, let a fundamental set of four quarks and four leptons represent one information theoretic spatial direction [4]. Quaternion numbers are expected to appear when spin information is introduced to the particle scheme [2]. Similarly, additional nonassociative structure will introduce octonions. Note that nonassociative braids are those for which strand groupings are recorded. In M Theory, the 3×3 Jordan algebra of octonion Hermitian matrices plays an important role.

The tripling of quarks due to flavor, and of leptons due to generation, would result in a total of 24 fundamental fermions, eight for each spatial direction, giving the off diagonal factors of the 27 dimensional Jordan algebra. These 24 dimensions should be associated to the 24 dimensions of the Leech lattice, which appears in 26 dimensional bosonic string theory.

Note that from a categorical perspective, vertex operator algebras are described by the theory of higher dimensional operads, where nonassociative braids appear in the definition of operad polytopes. The simpler case of the associahedra polytopes provides the combinatorics of BCFW rules in massless particle twistor scattering theory, and the associahedra also appear in cohomological descriptions of Veneziano n point functions. In other words, there are many reasons to expect that higher dimensional operads play an important role in the definition of mass observables.

Further arithmetic aspects of the 3×3 scheme are expected to clarify appearances of special functions in stringy physics. For example, the j invariant has a triality involving theta functions, and such a triality should correspond to the generation tripling.

5. Crossing Information

The Burau representation of B_3, in terms of a parameter t, assigns 2×2 matrices to braids. For the e_L^- braid, constructed from two generators, we have

$$\tau_1^{-1}\tau_2 = \begin{pmatrix} -t^{-1} & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix} = \begin{pmatrix} -t^{-1} & 1 \\ -t^{-1} & 1-t \end{pmatrix} \tag{8}$$

and electron positron annihilation is expressed by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -t^{-1} & 1 \\ -t^{-1} & 1-t \end{pmatrix} \begin{pmatrix} 1-t & -1 \\ t^{-1} & -t^{-1} \end{pmatrix} \tag{9}$$

As a 1×1 matrix, the parameter $-t$ corresponds to the twist symbol ω for the two strand case, B_2. The B_3 1-circulant permutations also have a 2×2 representation using cubed roots of unity, given by

$$(231) = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \qquad (312) = \begin{pmatrix} \overline{\omega} & 0 \\ 0 & \overline{\omega} \end{pmatrix} \tag{10}$$

So when $-t = \omega$, the 2×2 form of e_R^- is expressed as the product

$$\begin{pmatrix} 1+\overline{\omega} & 1+\overline{\omega} \\ 1 & \overline{\omega} \end{pmatrix} = \begin{pmatrix} \overline{\omega} & 0 \\ 0 & \overline{\omega} \end{pmatrix} \begin{pmatrix} 1+\omega & 1+\omega \\ \omega & 1 \end{pmatrix} \tag{11}$$

Observe that the right hand matrix here is a conjugated mirror image of the one on the left hand side. The permutation (312) represents $\overline{\nu}_R$, which was used to twist matrix products under the transform T. It follows that the action of T on a braid pair, multiplying here by another braid matrix on the left, obeys the same twist rule. In other words, the crossing information does not destroy the Fourier supersymmetry of the particle scheme.

6. Discussion

An elementary means of augmenting the 3×3 permutation matrices to account for extra binary information is to add minus signs to each fermion matrix. For example, place a minus sign at the entry corresponding to the strand that passes another strand only once, going under it, as in

$$e_L^+ = \begin{pmatrix} 0 & -\omega & 0 \\ 0 & 0 & \omega \\ \omega & 0 & 0 \end{pmatrix} \qquad e_L^- = \begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & -\overline{\omega} \\ \overline{\omega} & 0 & 0 \end{pmatrix} \tag{12}$$

When $\omega = \phi$ this introduces sixth roots of unity. On the leptons in this example, T gives a circulant Hermitian, as in

$$T(e_L^+) = \frac{\omega}{3} \begin{pmatrix} 1 & -2\overline{\phi} & -2\phi \\ -2\phi & 1 & -2\overline{\phi} \\ -2\overline{\phi} & -2\phi & 1 \end{pmatrix} \tag{13}$$

The eigenvalues of such a Hermitian matrix are $(1, 1, -1)$, and substituting 1 or $\overline{\phi}$ for ϕ gives three matrices that provide a convenient basis for all Koide mass matrices [4]. Note the similarity between these operators and the transformed quark basis.

In this general scheme, the electroweak interaction is now described by two triplets: the Z boson triplet and (γ, W^+, W^-). The latter triplet may be viewed as an index of spatial directions. We interpret the Z boson triplet as an index for three imaginary time directions, which average out to one observable time direction in the domain of colour confinement. From Machian principles, a true derivation of physical mass matrices should incorporate a cosmological element, with regard to which the Z boson triplet presumably appears.

The multiplicity of Z boson factors in Figure 2 corresponds to three possible choices of the diagonal D in defining a twisted Fourier transform. The alternatives may be marked by a choice of neutral reference strand, and the same reference selects a set of four fundamental quarks, out of twelve. Marking the neutral strand with a 0 is a means of reducing 3×3 matrices to two dimensional ones. Arithmetically, this may be achieved by reducing the cubed roots, which are characters for $\mathbb{F}_4$, to square roots of unity, as characters for $\mathbb{F}_3$. A Z boson matrix would then reduce to a two dimensional operator, namely the Pauli matrix σ_Z. In the full matrix particle scheme [2], this Pauli operator selects one spatial direction. However, all three Z boson factors reduce to the same Pauli operator, indicating that the Z boson triplet is indeed associated to colour and not to generation.

The idempotent circulants underlying the transformed quark matrices are also basic mass operators, since they have eigenvalue sets like $(0, 1, 1)$. That is, a Fourier transform sends such circulants to another basic 2×2 operator. All these 2×2 matrices are naturally acted upon by the two dimensional Fourier transform,

$$F_2 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{14}$$

Observe that the tribimaximal form of the neutrino mixing matrix may be expressed as $F_3 F_2$, the product of two Fourier operators, where F_2 is thought here to give a choice of reference spatial direction, while F_3 acts on the electroweak operators of the particle scheme. This suggests a study of the CKM mixing matrix using transform operators for all three directions.

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