Smarandache's Pedal Polygon Theorem in The Poincaré Disc Model of Hyperbolic Geometry

Cătălin Barbu

"Vasile Alecsandri" High School - Romania
kafka_mate@yahoo.com

Abstract. In this note, we present a proof of the hyperbolic a Smarandache's pedal polygon theorem in the Poincaré disc model of hyperbolic geometry.

2000 Mathematical Subject Classification: 30F45, 20N99, 51B10, 51M10
Keywords and phrases: hyperbolic geometry, hyperbolic triangle, gyro-vector, hyperbolic Pytagorean theorem

1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. In this note we choose the Poincaré disc model in order to present the hyperbolic version of the Smarandache's pedal polygon theorem. The Euclidean version of this well-known theorem states that if the points $M_i, i = \overline{1,n}$ are the projections of a point M on the sides $A_iA_{i+1}, i = \overline{1,n}$, where $A_{n+1} = A_1$, of the polygon $A_1A_2...A_n$, then $M_1A_1^2 + M_2A_2^2 + ... + M_nA_n^2 = M_1A_2^2 + M_2A_3^2 + ... + M_{n-1}A_n^2 + M_nA_1^2$ [1]. This result has a simple statement but it is of great interest.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a,b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\ gyr[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ for all points $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u},\mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u},\mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a},\mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1. *The hyperbolic distance function in D is defined by the equation*

$$d(a,b) = |a \ominus b| = \left|\frac{a-b}{1-\overline{a}b}\right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

Theorem 2. ***(The Möbius Hyperbolic Pythagorean Theorem)*** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$, with vertices $A, B, C \in V_s$, sides $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{V_s}$ and side gyrolenghts $a, b, c \in (-s, s)$, $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A, \mathbf{c} = -A \oplus B, a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$ and with gyroangles α, β, and γ at the vertices A, B, and C. If $\gamma = \pi/2$, then*

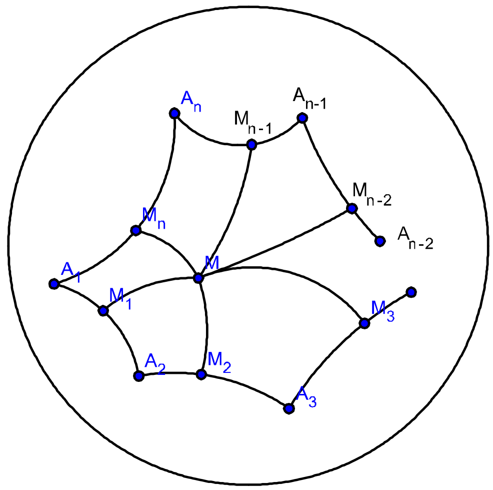
$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s}$$

(see [2, p 290])

For further details we refer to the recent book of A.Ungar [2].

Main result

In this sections, we present a proof of the hyperbolic a Smarandache's pedal polygon theorem in the Poincaré disc model of hyperbolic geometry.



Figure

Theorem 3. *Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré disc, whose vertices are the points $A_1, A_2, ..., A_n$ of the disc and whose sides (directed counterclockwise) are $\mathbf{a}_1 = -A_1 \oplus A_2$, $\mathbf{a}_2 = -A_2 \oplus A_3, ..., \mathbf{a}_n = -A_n \oplus A_1$. Let the points $M_i, i = \overline{1,n}$ be located on the sides $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ...,$ and M_n are concurrent, then the following holds:*

$$|-A_1 \oplus M_1|^2 \ominus |-M_1 \oplus A_2|^2 \oplus |-A_2 \oplus M_2|^2 \ominus |-M_2 \oplus A_3|^2 \oplus ... \oplus |-A_n \oplus M_n|^2 \ominus |-M_n \oplus A_1|^2 = 0$$

Proof. We assume that perpendiculars meet at a point of $A_1A_2...A_n$ and let denote this point by M (see Figure). The geodesic segments $-A_1 \oplus M$, $-A_2 \oplus M, ..., -A_n \oplus M, -M_1 \oplus M, -M_2 \oplus M, ..., -M_n \oplus M$ split the hyperbolic polygon into $2n$ right-angled hyperbolic triangles. We apply the theorem 2 to these $2n$ right-angled hyperbolic triangles one by one, and we easily obtain:

$$|-M \oplus A_k|^2 = |-A_k \oplus M_k|^2 \oplus |-M_k \oplus M|^2,$$

for all k from 1 to n, and $M_0 = M_n$.Using equalities

$$|-M \oplus A_k|^2 = |-A_k \oplus M|^2, k = \overline{1,n},$$

we have

$$\alpha_k = |-A_k \oplus M_k|^2 \oplus |-M_k \oplus M|^2 = |-M \oplus M_{k-1}|^2 \oplus |-M_{k-1} \oplus A_k|^2 = \alpha_k'$$

for all k from 1 to n, and $M_0 = M_n$. This implies

$$\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n = \alpha_1' \oplus \alpha_2' \oplus ... \oplus \alpha_n'.$$

Since $((-1,1), \oplus)$ is a commutative group, we immediately obtain

$$|-A_1 \oplus M_1|^2 \oplus |-A_2 \oplus M_2|^2 \oplus ... \oplus |-A_n \oplus M_n|^2 = |-M_1 \oplus A_2|^2 \oplus |-M_2 \oplus A_3|^2 \oplus ... \oplus |-M_n \oplus A_1|^2,$$

i.e.

$$|-A_1 \oplus M_1|^2 \ominus |-M_1 \oplus A_2|^2 \oplus |-A_2 \oplus M_2|^2 \ominus |-M_2 \oplus A_3|^2 \oplus ... \oplus |-A_n \oplus M_n|^2 \ominus |-M_n \oplus A_1|^2 = 0.$$

■

REFERENCES

[1] F. Smarandache, *Problémes avec et sans... probléms*!, pp. 49 & 54-60, Somipress, Fés, Morocoo, 1983.

[2] Ungar, A.A.,*Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, Hackensack, NJ:World Scientific Publishing Co.Pte. Ltd., 2008.