

Funcoids and Reloids*

a generalization of proximities and uniformities

BY VICTOR PORTON

September 25, 2010

Abstract

It is a part of my Algebraic General Topology research.

In this article I introduce the concepts of *funcoids* which generalize proximity spaces and *reloids* which generalize uniform spaces. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of analysis and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula are generalized continuity, proximity continuity, and uniform continuity.

Keywords: algebraic general topology, quasi-uniform spaces, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

A.M.S. subject classification: 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99

Table of contents

1 Common	2
1.1 Draft status	2
1.2 Earlier works	2
1.3 Used concepts, notation and statements	3
1.3.1 Filters	3
2 Partially ordered dagger categories	4
2.1 Partially ordered categories	4
2.2 Dagger categories	4
2.2.1 Monovalued and entirely defined morphisms	5
3 Funcoids	5
3.1 Informal introduction into funcoids	5
3.2 Basic definitions	6
3.2.1 Composition of funcoids	7
3.3 Funcoid as continuation	8
3.4 Lattice of funcoids	9
3.5 More on composition of funcoids	10
3.6 Domain and range of a funcoid	11
3.7 Category of funcoids	12
3.8 Specifying funcoids by functions or relations on atomic filter objects	12
3.9 Direct product of filter objects	14
3.10 Atomic funcoids	16
3.11 Complete funcoids	18
3.12 Completion of funcoids	19
3.13 Monovalued funcoids	21

*. This document has been written using the GNU $\text{T}_{\text{E}}\text{X}_{\text{M}}\text{A}^{\text{C}}\text{S}$ text editor (see www.texmacs.org).

3.14	T_0 -, T_1 - and T_2 -separable functors	22
3.15	Filter objects closed regarding a functor	22
4	Reloids	23
4.1	Composition of reloids	23
4.2	Direct product of filter objects	24
4.3	Restricting reloid to a filter object. Domain and image	25
4.4	Category of reloids	26
4.4.1	Monovalued reloids	26
4.5	Complete reloids and completion of reloids	26
5	Relationships of functors and reloids	27
5.1	Functor induced by a reloid	27
5.2	Reloids induced by functor	29
6	Galois connections of functors and reloids	31
7	Continuous morphisms	31
7.1	Traditional definitions of continuity	31
7.1.1	Pre-topology	31
7.1.2	Proximity spaces	32
7.1.3	Uniform spaces	32
7.2	Our three definitions of continuity	32
7.3	Continuousness of a restricted morphism	33
8	Connectedness regarding functors and reloids	34
8.1	Some lemmas	34
8.2	Endomorphism series	35
8.3	Connectedness regarding binary relations	35
8.4	Connectedness regarding functors and reloids	36
8.5	Algebraic properties of S and S^*	38
9	Postface	38
9.1	Misc	38
9.2	Pointfree functors and reloids	39
Appendix A Some counter-examples		39
Bibliography		41

1 Common

1.1 Draft status

This article is a draft.

This text refers to a preprint edition of [13]. Theorem number clashes may appear due editing both of these manuscripts.

1.2 Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [4].

Some references to predecessors:

- In [5], [6], [11], [2], [16] are studied generalized uniformities and proximities.
- Proximities and uniformities are also studied in [9], [10], [15], [17], [18].
- [7] and [8] contains recent progress in quasi-uniform spaces. [8] has a very long list of related literature.

Some works ([14]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work is not done the attempt to define or research their generalization, compactness of funcoids or reloids. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate article.

[3] studies mappings between proximity structures. (In this work no attempt to research mappings between funcoids is done, as this seems a particularly difficult topic.) [12] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

1.3 Used concepts, notation and statements

The set of functions from a set A to a set B is denoted as B^A .

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x .

I will denote $\langle f \rangle X = \{f\alpha \mid \alpha \in X\}$ for a set X .

For simplicity I will assume that all sets in consideration are subsets of universal set \mathcal{U} .

1.3.1 Filters

In this work the word *filter* will refer to a filter on a set \mathcal{U} (in contrast to [13] where are considered filters on arbitrary posets). Note that I do not require filters to be proper.

I will call the set of filters ordered reverse to set-theoretic inclusion of filters *the set of filter objects* \mathfrak{F} and its element *filter objects* (f.o. for short). I will denote $\text{up } \mathcal{F}$ the filter corresponding to a filter object \mathcal{F} . So we have $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \text{up } \mathcal{A} \supseteq \text{up } \mathcal{B}$ for every filter objects \mathcal{A} and \mathcal{B} . We also will equate filter objects corresponding to principal filters with corresponding sets. (Thus we have $\mathcal{P}\mathcal{U} \subseteq \mathfrak{F}$.) See [13] for formal definition of filter objects in the framework of ZF. Filters (and filter objects) are studied in the work [13].

Prior reading of [13] is needed to understand this work.

Filter objects corresponding to ultrafilters are atoms of the lattice \mathfrak{F} and will be called *atomic filter objects*.

Also we will need to introduce the concept of *generalized filter base*.

Definition 1. *Generalized filter base* is a set $S \in \mathcal{P}\mathfrak{F} \setminus \{\emptyset\}$ such that

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S: \mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Proposition 2. Let S is a generalized filter base. If $A_1, \dots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists \mathcal{C} \in S: \mathcal{C} \subseteq A_1 \cap \dots \cap A_n.$$

Proof. Can be easily proved by induction. □

Theorem 3. If S is a generalized filter base, then $\text{up } \bigcap^{\mathfrak{F}} S = \bigcup \langle \text{up} \rangle S$.

Proof. Obviously $\text{up } \bigcap^{\mathfrak{F}} S \supseteq \bigcup \langle \text{up} \rangle S$. Reversely, let $K \in \text{up } \bigcap^{\mathfrak{F}} S$; then $K = A_1 \cap \dots \cap A_n$ where $A_i \in \text{up } \mathcal{A}_i$ where $\mathcal{A}_i \in S$, $i = 1, \dots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \dots \cap \mathcal{A}_n \subseteq A_1 \cap \dots \cap A_n = K$, $K \in \text{up } \mathcal{C}$, $K \in \bigcup \langle \text{up} \rangle S$. □

Corollary 4. If S is a generalized filter base, then $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in S$.

Proof. $\bigcap^{\mathfrak{S}} S = \emptyset \Leftrightarrow \emptyset \in \text{up} \bigcap^{\mathfrak{S}} S \Leftrightarrow \emptyset \in \bigcup \langle \text{up} \rangle S \Leftrightarrow \exists \mathcal{X} \in S: \emptyset \in \text{up} \mathcal{X} \Leftrightarrow \emptyset \in S.$ \square

2 Partially ordered dagger categories

2.1 Partially ordered categories

Definition 5. I will call a *partially ordered (pre)category* a (pre)category together with partial order \subseteq on each of its Hom-sets with the additional requirement that

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\text{Src } f_1 = \text{Src } f_2 \wedge \text{Dst } f_1 = \text{Dst } f_2 = \text{Src } g_1 = \text{Src } g_2 \wedge \text{Dst } g_1 = \text{Dst } g_2.$

2.2 Dagger categories

Definition 6. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger.$

In other words, a *dagger precategory* is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g :

1. $f^{\dagger\dagger} = f;$
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger.$

Definition 7. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger.$

In other words, a *dagger category* is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A :

1. $f^{\dagger\dagger} = f;$
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger;$
3. $(1_A)^\dagger = 1_A.$

Theorem 8. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A.$ Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A. \quad \square$$

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g)

$$f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}.$

Definition 9. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}.$

Definition 10. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger.$

Definition 11. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f.$

Theorem 12. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^\dagger \circ f$.

Proof.

(1) \Rightarrow (2). If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

(2) \Rightarrow (1). $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive. \square

2.2.1 Monovalued and entirely defined morphisms

Definition 13. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \subseteq 1_{\text{Dst } f}$.

Definition 14. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \supseteq 1_{\text{Src } f}$.

Remark 15. Easy to show that this is a generalization of monovalued and entirely defined binary relations as morphisms of the category **Rel**.

Definition 16. For a given partially ordered dagger category C the *category of monovalued (entirely defined) morphisms* of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined) morphisms of C with the composition of morphisms the same as in C .

We need to prove that these are really categories, that is that composition of monovalued (entirely defined) morphisms is monovalued (entirely defined) and that identity morphisms are monovalued and entirely defined.

Proof.

Monovalued. Let f and g are monovalued morphisms, $\text{Dst } f = \text{Src } g$. $(g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \subseteq g \circ 1_{\text{Dst } f} \circ g^\dagger = g \circ 1_{\text{Src } g} \circ g^\dagger = g \circ g^\dagger \subseteq 1_{\text{Dst } g} = 1_{\text{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\text{Dst } 1_A} \subseteq 1_{\text{Dst } 1_A}$.

Entirely defined. Let f and g are entirely defined morphisms, $\text{Dst } f = \text{Src } g$. $(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \supseteq f^\dagger \circ 1_{\text{Src } g} \circ f = f^\dagger \circ 1_{\text{Dst } f} \circ f = f^\dagger \circ f \supseteq 1_{\text{Src } f} = 1_{\text{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\text{Src } 1_A} \subseteq 1_{\text{Src } 1_A}$. \square

3 Functors

3.1 Informal introduction into functors

Functors are a generalization of proximity spaces and a generalization of pretopological spaces. Also functors are a generalization of binary relations.

That functors are a common generalization of “spaces” (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement “ f is a continuous function from a space μ to a space ν ” can be described in terms of functors as the formula $f \circ \mu \subseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A \delta B$ is defined for every sets A and B . We will extend it from sets to filter objects by the formula:

$$\mathcal{A} \delta' \mathcal{B} \Leftrightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}: A \delta B.$$

Then (as will be proved below) exist two functions $\alpha, \beta \in \mathfrak{F}^{\delta}$ such that

$$\mathcal{A} \delta' \mathcal{B} \Leftrightarrow \mathcal{B} \cap^{\delta} \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap^{\delta} \beta \mathcal{B} \neq \emptyset.$$

The pair $(\alpha; \beta)$ is called *funcoïd* when $\mathcal{B} \cap^{\delta} \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap^{\delta} \beta \mathcal{B} \neq \emptyset$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β . The first component of a funcoïd f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of discrete funcoids (see below) these coincide.)

One of the most important properties of a funcoïd is that it is uniquely determined by just one of its components. That is a funcoïd f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoïd f is uniquely determined by $\langle f \rangle|_{\mathcal{P}U}$ that is by values of function $\langle f \rangle$ on sets.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^U$. Then we define $\alpha' X \stackrel{\text{def}}{=} \bigcup^{\delta} \{\alpha X \mid x \in X\}$ for every set X . We will prove that there exists a unique funcoïd f such that $\alpha' = \langle f \rangle|_{\mathcal{P}U}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p exists unique funcoïd such that $\langle f \rangle|_{\mathcal{P}U} = p$; in this case $\langle f \rangle|_{\mathcal{P}U} \in \mathcal{P}U^{\mathcal{P}U}$.

For every binary relation p exists unique funcoïd f such that $\forall X \in \mathcal{P}U: \langle f \rangle X = \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoïd is uniquely determined by the values of its first component on sets. I will call such funcoids *discrete*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of discrete funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle.$$

By the same formulas we can define composition of every two funcoids.

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below will be defined domain and image of a funcoïd (the domain and the image of a funcoïd are filter objects).

3.2 Basic definitions

Definition 17. Let's call a *funcoïd* a pair $(\alpha; \beta)$ where $\alpha, \beta \in \mathfrak{F}^{\delta}$ such that

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{Y} \cap^{\delta} \alpha \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\delta} \beta \mathcal{Y} \neq \emptyset).$$

Definition 18. $\langle (\alpha; \beta) \rangle \stackrel{\text{def}}{=} \alpha$ for a funcoïd $(\alpha; \beta)$.

Definition 19. $(\alpha; \beta)^{-1} = (\beta; \alpha)$ for a funcoïd $(\alpha; \beta)$.

Proposition 20. If f is a funcoïd then f^{-1} is also a funcoïd.

Proof. Follows from symmetry in the definition of funcoïd. □

Obvious 21. $(f^{-1})^{-1} = f$ for a funcoïd f .

Definition 22. The relation $[f] \in \mathcal{P}\mathfrak{F}^2$ is defined by the formula (for every filter objects \mathcal{X}, \mathcal{Y} and functor f)

$$\mathcal{X}[f]\mathcal{Y} \stackrel{\text{def}}{=} \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset.$$

Obvious 23. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y}$ for every filter objects \mathcal{X}, \mathcal{Y} and functor f .

Obvious 24. $[f^{-1}] = [f]^{-1}$ for a functor f .

Theorem 25.

1. For given value of $\langle f \rangle$ exists no more than one functor f .
2. For given value of $[f]$ exists no more than one functor f .

Proof. Let f and g are functors.

Obviously $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\langle f^{-1} \rangle = \langle g^{-1} \rangle \Rightarrow [f] = [g]$. So enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{X}[g]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset$ and consequently $\langle f \rangle \mathcal{X} = \langle g \rangle \mathcal{X}$ for every f.o. \mathcal{X} and \mathcal{Y} because the set of filter objects is separable [13], thus $\langle f \rangle = \langle g \rangle$. \square

Proposition 26. $\langle f \rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J}$ for every functor f and $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$.

Proof.

$$\begin{aligned} \star \langle f \rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) &= \\ \{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) \neq \emptyset\} &= \\ \{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset\} &= \text{(by corollary 10 in [13])} \\ \{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y}) \cup^{\mathfrak{F}} (\mathcal{J} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y}) \neq \emptyset\} &= \\ \{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \vee \mathcal{J} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset\} &= \\ \{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{I} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{J} \neq \emptyset\} &= \\ \{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{I}) \cup^{\mathfrak{F}} (\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{J}) \neq \emptyset\} &= \text{(by corollary 10 in [13])} \\ \{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} (\langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J}) \neq \emptyset\} &= \\ \star (\langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J}). & \end{aligned}$$

Thus $\langle f \rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J}$ because \mathfrak{F} is separable. \square

3.2.1 Composition of functors

Definition 27. *Composition* of functors is defined by the formula

$$(\alpha_2; \beta_2) \circ (\alpha_1; \beta_1) = (\alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Proposition 28. If f, g are functors then $g \circ f$ is functor.

Proof. Let $f = (\alpha_1; \beta_1)$, $g = (\alpha_2; \beta_2)$. For every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$ we have

$$\mathcal{Y} \cap^{\mathfrak{F}} (\alpha_2 \circ \alpha_1) \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \alpha_2 \alpha_1 \mathcal{X} \neq \emptyset \Leftrightarrow \alpha_1 \mathcal{X} \cap^{\mathfrak{F}} \beta_2 \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta_1 \beta_2 \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} (\beta_1 \circ \beta_2) \mathcal{Y} \neq \emptyset.$$

So $(\alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a functor. \square

Obvious 29. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every functors f and g .

Proposition 30. $(h \circ g) \circ f = h \circ (g \circ f)$ for every functors f, g, h .

Proof.

$$\langle (h \circ g) \circ f \rangle = \langle h \circ g \rangle \circ \langle f \rangle = (\langle h \rangle \circ \langle g \rangle) \circ \langle f \rangle = \langle h \rangle \circ (\langle g \rangle \circ \langle f \rangle) = \langle h \rangle \circ \langle g \circ f \rangle = \langle h \circ (g \circ f) \rangle. \quad \square$$

Theorem 31. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every functors f and g .

Proof. $\langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle = \langle f^{-1} \circ g^{-1} \rangle$. \square

3.3 Functor as continuation

Theorem 32. For every functor f and filter objects \mathcal{X} and \mathcal{Y}

1. $\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \text{up } \mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X[f]Y$.

Proof. 2. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \forall Y \in \text{up } \mathcal{Y}: Y \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \forall Y \in \text{up } \mathcal{Y}: \mathcal{X}[f]Y$.

Analogously $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}: X[f]\mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X[f]Y.$$

1. $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \forall X \in \text{up } \mathcal{X}: \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle X \neq \emptyset$.

Let's denote $W = \{\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle X \mid X \in \text{up } \mathcal{X}\}$. We will prove that W is a generalized filter base.

To prove this enough to show that $V = \{\langle f \rangle X \mid X \in \text{up } \mathcal{X}\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f \rangle A$, $\mathcal{Q} = \langle f \rangle B$ where $A, B \in \text{up } \mathcal{X}$; $A \cap B \in \text{up } \mathcal{X}$ and $\mathcal{R} \subseteq \mathcal{P} \cap^{\mathfrak{F}} \mathcal{Q}$ for $\mathcal{R} = \langle f \rangle (A \cap B) \in V$. So V is a generalized filter base and thus W is a generalized filter base.

$\emptyset \notin W \Leftrightarrow \bigcap^{\mathfrak{F}} W \neq \emptyset$ by the corollary 4 of the theorem 3. That is

$$\forall X \in \text{up } \mathcal{X}: \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle X \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \text{up } \mathcal{X} \neq \emptyset.$$

Comparing with the above, $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \text{up } \mathcal{X} \neq \emptyset$. So $\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \text{up } \mathcal{X}$ because the lattice of filter objects is separable. \square

Theorem 33.

1. A function $\alpha \in \mathfrak{F}^{\mathcal{P}\mathcal{U}}$ conforming to the formulas (for every $I, J \in \mathcal{P}\mathcal{U}$)

$$\alpha \emptyset = \emptyset, \quad \alpha(I \cup J) = \alpha I \cup^{\mathfrak{F}} \alpha J$$

can be continued to the function $\langle f \rangle$ for a unique functor f ;

$$\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha \rangle \text{up } \mathcal{X} \tag{1}$$

for every filter object \mathcal{X} .

2. A relation $\delta \in \mathcal{P}(\mathcal{P}\mathcal{U})^2$ conforming to the formulas (for every $I, J, K \in \mathcal{P}\mathcal{U}$)

$$\begin{aligned} \neg(\emptyset \delta I), \quad I \cup J \delta K &\Leftrightarrow I \delta K \vee J \delta K, \\ \neg(I \delta \emptyset), \quad K \delta I \cup J &\Leftrightarrow K \delta I \vee K \delta J \end{aligned} \tag{2}$$

can be continued to the relation $[f]$ for a unique functor f ;

$$\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X \delta Y \tag{3}$$

for every filter objects \mathcal{X}, \mathcal{Y} .

Proof. Existence of no more than one such functors and formulas (1) and (3) follow from the previous theorem.

2. Let define $\alpha \in \mathfrak{F}^{\mathcal{P}\mathcal{U}}$ by the formula $\partial(\alpha X) = \{Y \in \mathcal{P}\mathcal{U} \mid X \delta Y\}$ for every $X \in \mathcal{P}\mathcal{U}$. (It is obvious that $\{Y \in \mathcal{P}\mathcal{U} \mid X \delta Y\}$ is a free star.) Analogously can be defined $\beta \in \mathfrak{F}^{\mathcal{P}\mathcal{U}}$ by the formula $\partial(\beta X) = \{X \in \mathcal{P}\mathcal{U} \mid X \delta Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}^{\mathfrak{F}}$ and $\beta' \in \mathfrak{F}^{\mathfrak{F}}$ by the formulas

$$\alpha' \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha \rangle \text{up } \mathcal{X} \quad \text{and} \quad \beta' \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \beta \rangle \text{up } \mathcal{X}$$

and δ to $\delta' \in \mathcal{P}\mathfrak{F}^2$ by the formula

$$\mathcal{X} \delta' \mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X \delta Y.$$

$\mathcal{Y} \cap^{\mathfrak{s}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{s}} \bigcap^{\mathfrak{s}} \langle \alpha \rangle \text{up } \mathcal{X} \neq \emptyset \Leftrightarrow \bigcap^{\mathfrak{s}} \langle \mathcal{Y} \cap^{\mathfrak{s}} \rangle \langle \alpha \rangle \text{up } \mathcal{X} \neq \emptyset$. Let's prove that

$$W = \langle \mathcal{Y} \cap^{\mathfrak{s}} \rangle \langle \alpha \rangle \text{up } \mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle \alpha \rangle \text{up } \mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle \alpha \rangle \text{up } \mathcal{X}$ then exist $X_1, X_2 \in \text{up } \mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle \alpha \rangle \text{up } \mathcal{X}$. So $\langle \alpha \rangle \text{up } \mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

Accordingly the corollary 4 of the theorem 3, $\bigcap^{\mathfrak{s}} \langle \mathcal{Y} \cap^{\mathfrak{s}} \rangle \langle \alpha \rangle \text{up } \mathcal{X} \neq \emptyset$ is equivalent to

$$\forall X \in \text{up } \mathcal{X}: \mathcal{Y} \cap^{\mathfrak{s}} \alpha X \neq \emptyset,$$

what is equivalent to $\forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: Y \cap^{\mathfrak{s}} \alpha X \neq \emptyset \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X \delta Y$. Combining the equivalencies we get $\mathcal{Y} \cap^{\mathfrak{s}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow X \delta' Y$. Analogously $\mathcal{X} \cap^{\mathfrak{s}} \beta' \mathcal{Y} \neq \emptyset \Leftrightarrow X \delta' Y$. So $\mathcal{Y} \cap^{\mathfrak{s}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{s}} \beta' \mathcal{Y} \neq \emptyset$, that is $(\alpha'; \beta')$ is a funcooid. From the formula $\mathcal{Y} \cap^{\mathfrak{s}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow X \delta' Y$ follows that $[(\alpha'; \beta')]$ is a continuation of δ .

1. Let define the relation $\delta \in \mathcal{P}(\mathcal{P}\mathcal{U})^2$ by the formula $X \delta Y \Leftrightarrow Y \cap^{\mathfrak{s}} \alpha X \neq \emptyset$.

That $\neg(\emptyset \delta I)$ and $\neg(I \delta \emptyset)$ is obvious. We have $I \cup J \delta K \Leftrightarrow (I \cup J) \cap^{\mathfrak{s}} \alpha K \neq \emptyset \Leftrightarrow (I \cup^{\mathfrak{s}} J) \cap^{\mathfrak{s}} \alpha K \neq \emptyset \Leftrightarrow (I \cap^{\mathfrak{s}} \alpha K) \cup^{\mathfrak{s}} (J \cup^{\mathfrak{s}} \alpha K) \neq \emptyset \Leftrightarrow I \cap^{\mathfrak{s}} \alpha K \neq \emptyset \vee J \cup^{\mathfrak{s}} \alpha K \neq \emptyset \Leftrightarrow I \delta K \vee J \delta K$ and $K \delta I \cup J \Leftrightarrow K \cap^{\mathfrak{s}} \alpha(I \cup J) \neq \emptyset \Leftrightarrow K \cap^{\mathfrak{s}} \alpha(I \cup J) \neq \emptyset \Leftrightarrow K \cap^{\mathfrak{s}} (\alpha I \cup^{\mathfrak{s}} \alpha J) \neq \emptyset \Leftrightarrow (K \cap^{\mathfrak{s}} \alpha I) \cup^{\mathfrak{s}} (K \cap^{\mathfrak{s}} \alpha J) \neq \emptyset \Leftrightarrow K \cap^{\mathfrak{s}} \alpha I \neq \emptyset \vee K \cap^{\mathfrak{s}} \alpha J \neq \emptyset \Leftrightarrow K \delta I \vee K \delta J$.

That is the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some funcooid f .

$\forall X, Y \in \mathcal{P}\mathcal{U}: (Y \cap^{\mathfrak{s}} \langle f \rangle X \neq \emptyset \Leftrightarrow X[f]Y \Leftrightarrow Y \cap^{\mathfrak{s}} \alpha X \neq \emptyset)$, consequently $\forall X \in \mathcal{P}\mathcal{U}: \alpha X = \langle f \rangle X$. So $\langle f \rangle$ is a continuation of α . \square

Note that by the last theorem to every proximity δ corresponds a unique funcooid. So funcooids are a generalization of proximity structures.

Definition 34. Any (multivalued) function f will be considered as a funcooid, where by definition $\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{s}} \langle \langle f \rangle \rangle \text{up } \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff.

Definition 35. Funcooids corresponding to a binary relation are called *discrete funcooids*.

We may equate discrete funcooids with corresponding binary relations by the method of appendix B in [13]. This is useful for describing relationships of funcooids and binary relations, such as for the formulas of continuous functions and continuous funcooids (see below). For simplicity I will not dive here into formal definition of equating discrete funcooids with binary relations (by the method shown in appendix B in [13]) but we simply will (informally) assume that discrete funcooids can be equated with binary relations.

I will denote FCD the set of funcooids or the category of funcooids (see below) dependently on context.

3.4 Lattice of funcooids

Definition 36. $f \subseteq g \stackrel{\text{def}}{=} [f] \subseteq [g]$ for $f, g \in \text{FCD}$.

Thus FCD is a poset.

Definition 37. I will call the *filtrator of funcooids* (see [13] for the definition of filtrators) the filtrator (FCD; $\mathcal{P}\mathcal{U}^2$).

Conjecture 38. The filtrator of funcooids is:

1. with separable core;

2. with co-separable core.

Theorem 39. The set of funcoids is a complete lattice. For every $R \in \mathcal{P}\text{FCD}$ and $X, Y \in \mathcal{P}\mathcal{U}$

1. $X[\bigcup^{\text{FCD}} R]Y \Leftrightarrow \exists f \in R: X[f]Y$;
2. $\langle \bigcup^{\text{FCD}} R \rangle X = \bigcup^{\mathfrak{F}} \{ \langle f \rangle X \mid f \in R \}$.

Proof.

2. $\alpha X \stackrel{\text{def}}{=} \bigcup^{\mathfrak{F}} \{ \langle f \rangle X \mid f \in R \}$. We have $\alpha \emptyset = \emptyset$;

$$\begin{aligned} \alpha(I \cup J) &= \bigcup^{\mathfrak{F}} \{ \langle f \rangle (I \cup J) \mid f \in R \} \\ &= \bigcup^{\mathfrak{F}} \{ \langle f \rangle (I \cup^{\mathfrak{F}} J) \mid f \in R \} \\ &= \bigcup^{\mathfrak{F}} \{ \langle f \rangle I \cup^{\mathfrak{F}} \langle f \rangle J \mid f \in R \} \\ &= \bigcup^{\mathfrak{F}} \{ \langle f \rangle I \mid f \in R \} \cup^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \{ \langle f \rangle J \mid f \in R \} \\ &= \alpha I \cup^{\mathfrak{F}} \alpha J. \end{aligned}$$

So α can be continued to $\langle h \rangle$ for a funcoid h . Obviously

$$\forall f \in R: h \supseteq f. \quad (4)$$

And h is the least funcoid for which holds the condition (4). So $h = \bigcup^{\text{FCD}} R$.

1. $X[\bigcup^{\text{FCD}} R]Y \Leftrightarrow Y \cap^{\mathfrak{F}} \langle \bigcup^{\text{FCD}} R \rangle X \neq \emptyset \Leftrightarrow Y \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \{ \langle f \rangle X \mid f \in R \} \neq \emptyset \Leftrightarrow \exists f \in R: Y \cap^{\mathfrak{F}} \langle f \rangle X \neq \emptyset \Leftrightarrow \exists f \in R: X[f]Y$ (used the theorem 52 in [13]). \square

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 40. For every funcoids f and g and a filter object \mathcal{X}

1. $\langle f \cup^{\text{FCD}} g \rangle \mathcal{X} = \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$;
2. $[f \cup^{\mathfrak{F}} g] = [f] \cup [g]$.

Proof.

1. Let $\alpha \mathcal{X} \stackrel{\text{def}}{=} \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$; $\beta \mathcal{Y} \stackrel{\text{def}}{=} \langle f^{-1} \rangle \mathcal{Y} \cup^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. Then

$$\begin{aligned} \mathcal{Y} \cap^{\mathfrak{F}} \alpha \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \vee \mathcal{X} \cap^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta \mathcal{Y} \neq \emptyset. \end{aligned}$$

So $h = (\alpha; \beta)$ is a funcoid. Obviously $h \supseteq f$ and $h \supseteq g$. If $p \supseteq f$ and $p \supseteq g$ for some funcoid p then $\langle p \rangle \mathcal{X} \supseteq \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X} = \langle h \rangle \mathcal{X}$ that is $p \supseteq h$. So $f \cup^{\text{FCD}} g = h$.

2. $\mathcal{X}[f \cup^{\text{FCD}} g] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \cup^{\text{FCD}} g \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} (\langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}) \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}[f] \mathcal{Y} \vee \mathcal{X}[g] \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. \square

3.5 More on composition of funcoids

Proposition 41. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for $f, g \in \text{FCD}$.

Proof. $\mathcal{X}[g \circ f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \circ f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X}[g] \mathcal{Y} \Leftrightarrow \mathcal{X}([g] \circ \langle f \rangle) \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. $[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} = \langle g^{-1} \rangle^{-1} \circ [f]$. \square

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x(g \circ f)z \Leftrightarrow \exists y(xfy \wedge ygz)$ for every x and z and every binary relations f and g .

Theorem 42. For every $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$ and $f, g \in \text{FCD}$

$$\mathcal{X}[g \circ f]\mathcal{Z} \Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{X}[f]y \wedge y[g]\mathcal{Z}).$$

Proof.

$$\begin{aligned} \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{X}[f]y \wedge y[g]\mathcal{Z}) &\Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle y \neq \emptyset \wedge y \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset) \\ &\Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle y \neq \emptyset \wedge y \subseteq \langle f \rangle \mathcal{X}) \\ &\Rightarrow \mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle \langle f \rangle \mathcal{X} \neq \emptyset \\ &\Leftrightarrow \mathcal{X}[g \circ f]\mathcal{Z}. \end{aligned}$$

Reversely, if $\mathcal{X}[g \circ f]\mathcal{Z}$ then $\langle f \rangle \mathcal{X}[g]\mathcal{Z}$, consequently exists $y \in \text{atoms}^{\mathfrak{F}}\langle f \rangle \mathcal{X}$ such that $y[g]\mathcal{Z}$; we have $\mathcal{X}[f]y$. \square

Theorem 43. If f, g, h are funcoids then

1. $f \circ (g \cup^{\text{FCD}} h) = f \circ g \cup^{\text{FCD}} f \circ h$;
2. $(g \cup^{\text{FCD}} h) \circ f = g \circ f \cup^{\text{FCD}} h \circ f$.

Proof. I will prove only the first equality because the other is analogous.

For every $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$

$$\begin{aligned} \mathcal{X}[f \circ (g \cup^{\text{FCD}} h)]\mathcal{Z} &\Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{X}[g \cup^{\text{FCD}} h]y \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: ((\mathcal{X}[g]y \vee \mathcal{X}[h]y) \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{X}[g]y \wedge y[f]\mathcal{Z} \vee \mathcal{X}[h]y \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{X}[g]y \wedge y[f]\mathcal{Z}) \vee \exists y \in \text{atoms}^{\mathfrak{F}}\mathcal{U}: (\mathcal{X}[h]y \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \mathcal{X}[f \circ g]\mathcal{Z} \vee \mathcal{X}[f \circ h]\mathcal{Z} \\ &\Leftrightarrow \mathcal{X}[f \circ g \cup^{\text{FCD}} f \circ h]\mathcal{Z}. \end{aligned}$$

\square

3.6 Domain and range of a funcoid

Definition 44. Let $\mathcal{A} \in \mathfrak{F}$. The *identity funcoid* $I_{\mathcal{A}}^{\text{FCD}} = (\mathcal{A} \cap^{\mathfrak{F}}; \mathcal{A} \cap^{\mathfrak{F}})$.

Proposition 45. The identity funcoid is a funcoid.

Proof. We need to prove that $(\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) \cap^{\mathfrak{F}} \mathcal{Y} \neq \emptyset \Leftrightarrow (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{Y}) \cap^{\mathfrak{F}} \mathcal{X} \neq \emptyset$ what is obvious. \square

Obvious 46. $(I_{\mathcal{A}})^{-1} = I_{\mathcal{A}}$.

Obvious 47. $\mathcal{X}[I_{\mathcal{A}}]\mathcal{Y} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} \neq \emptyset$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$.

Definition 48. I will define *restricting* of a funcoid f to a filter object \mathcal{A} by the formula $f|_{\mathcal{A}}^{\text{FCD}} \stackrel{\text{def}}{=} f \circ I_{\mathcal{A}}$.

Obviously the last definition does not contradict to the previous.

Definition 49. *Image* of a funcoid f will be defined by the formula $\text{im } f = \langle f \rangle \mathcal{U}$.

Domain of a funcoid f is defined by the formula $\text{dom } f = \text{im } f^{-1}$.

Proposition 50. $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap^{\mathfrak{F}} \text{dom } f)$ for every $f \in \text{FCD}$, $\mathcal{X} \in \mathfrak{F}$.

Proof. For every filter object \mathcal{Y} we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle (\mathcal{X} \cap^{\mathfrak{F}} \text{dom } f) \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \text{dom } f \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \text{im } f^{-1} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset$. Thus $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap^{\mathfrak{F}} \text{dom } f)$ because the lattice of filter objects is separable. \square

Proposition 51. $\mathcal{X} \cap^{\mathfrak{F}} \text{dom } f \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X} \neq \emptyset$ for every $f \in \text{FCD}$, $\mathcal{X} \in \mathfrak{F}$.

Proof. $\mathcal{X} \cap^{\mathfrak{F}} \text{dom } f \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{U} \neq \emptyset \Leftrightarrow \mathcal{U} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X} \neq \emptyset.$ \square

Corollary 52. $\text{dom } f = \bigcup^{\mathfrak{F}} \{a \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{U}, \langle f \rangle a \neq \emptyset\}.$

Proof. This follows from that \mathfrak{F} is an atomistic lattice. \square

3.7 Category of funcoids

I will define the category FCD of funcoids:

- The set of objects is \mathfrak{F} .
- The set of morphisms from a filter object \mathcal{A} to a filter object \mathcal{B} is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a funcoid such that $\text{dom } f \subseteq \mathcal{A}$, $\text{im } f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object \mathcal{A} is $(I_{\mathcal{A}}^{\text{FCD}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 53. For every funcoid f and filter objects \mathcal{X} and \mathcal{Y}

1. $\langle f \rangle \mathcal{X} = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle \text{atoms}^{\mathfrak{F}} \mathcal{X};$
2. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} \mathcal{X}, y \in \text{atoms}^{\mathfrak{F}} \mathcal{Y}: x[f]y.$

Proof. 1.

$$\begin{aligned} \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \\ &\Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} \mathcal{X}: x \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \\ &\Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} \mathcal{X}: \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle x \neq \emptyset. \end{aligned}$$

$$\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle \langle \langle f \rangle \rangle \text{atoms}^{\mathfrak{F}} \mathcal{X} = \partial \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle \text{atoms}^{\mathfrak{F}} \mathcal{X}.$$

2. If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset$, consequently exists $y \in \text{atoms}^{\mathfrak{F}} \mathcal{Y}$ such that $y \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset$, $\mathcal{X}[f]y$. Repeating this second time we get that there exist $x \in \text{atoms}^{\mathfrak{F}} \mathcal{X}$ such that $x[f]y$. From this follows

$$\exists x \in \text{atoms}^{\mathfrak{F}} \mathcal{X}, y \in \text{atoms}^{\mathfrak{F}} \mathcal{Y}: x[f]y.$$

The reverse is obvious. \square

Theorem 54.

1. A function $\alpha \in \mathfrak{F}^{\text{atoms}^{\mathfrak{F}} \mathcal{U}}$ such that (for every $a \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$)

$$\alpha a \subseteq \bigcap^{\mathfrak{F}} \left\langle \bigcup^{\mathfrak{F}} \circ \langle \alpha \rangle \circ \text{atoms}^{\mathfrak{F}} \right\rangle \text{up } a \quad (5)$$

can be continued to the function $\langle f \rangle$ for a unique funcoid f ;

$$\langle f \rangle \mathcal{X} = \bigcup^{\mathfrak{F}} \langle \alpha \rangle \text{atoms}^{\mathfrak{F}} \mathcal{X} \quad (6)$$

for every filter object \mathcal{X} .

2. A relation $\delta \in \mathcal{P}(\text{atoms}^{\mathfrak{F}} \mathcal{U})^2$ such that (for every $a, b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$)

$$\forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms}^{\mathfrak{F}} X, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \Rightarrow a \delta b \quad (7)$$

can be continued to the relation $[f]$ for a unique funcoid f ;

$$\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} \mathcal{X}, y \in \text{atoms}^{\mathfrak{F}} \mathcal{Y}: x \delta y \quad (8)$$

for every filter objects \mathcal{X}, \mathcal{Y} .

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the previous theorem.

1. Consider the function $\alpha' \in \mathfrak{F}^{\mathcal{U}}$ defined by the formula (for every $X \in \mathcal{P}\mathcal{U}$)

$$\alpha'X = \bigcup^{\mathfrak{F}} \langle \alpha \rangle \text{atoms}^{\mathfrak{F}} X.$$

Obviously $\alpha' \emptyset = \emptyset$. For every $I, J \in \mathcal{P}\mathcal{U}$

$$\begin{aligned} \alpha'(I \cup J) &= \bigcup^{\mathfrak{F}} \langle \alpha' \rangle \text{atoms}^{\mathfrak{F}}(I \cup J) \\ &= \bigcup^{\mathfrak{F}} \langle \alpha' \rangle (\text{atoms}^{\mathfrak{F}} I \cup \text{atoms}^{\mathfrak{F}} J) \\ &= \bigcup^{\mathfrak{F}} (\langle \alpha' \rangle \text{atoms}^{\mathfrak{F}} I \cup \langle \alpha' \rangle \text{atoms}^{\mathfrak{F}} J) \\ &= \bigcup^{\mathfrak{F}} \langle \alpha' \rangle \text{atoms}^{\mathfrak{F}} I \cup \bigcup^{\mathfrak{F}} \langle \alpha' \rangle \text{atoms}^{\mathfrak{F}} J. \\ &= \alpha' I \cup \alpha' J. \end{aligned}$$

Let continue α' till a funcoid f (by the theorem 25): $\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha' \rangle \text{up } \mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned} \bigcap^{\mathfrak{F}} \langle \bigcup^{\mathfrak{F}} \langle \alpha \rangle \circ \text{atoms}^{\mathfrak{F}} \rangle \text{up } a &= \bigcap^{\mathfrak{F}} \langle \bigcup^{\mathfrak{F}} \langle \alpha \rangle \rangle \langle \text{atoms}^{\mathfrak{F}} \rangle \text{up } a \\ &\subseteq \bigcap^{\mathfrak{F}} \langle \bigcup^{\mathfrak{F}} \langle \alpha \rangle \rangle \{ \{a\} \} \\ &= \bigcap^{\mathfrak{F}} \{ (\bigcup^{\mathfrak{F}} \langle \alpha \rangle) \{a\} \} \\ &= \bigcap^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \langle \alpha \rangle \{a\} \} \\ &= \bigcap^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \{ \alpha a \} \} = \bigcap^{\mathfrak{F}} \{ \alpha a \} = \alpha a. \end{aligned}$$

Finally,

$$\alpha a = \bigcap^{\mathfrak{F}} \langle \bigcup^{\mathfrak{F}} \langle \alpha \rangle \circ \text{atoms}^{\mathfrak{F}} \rangle \text{up } a = \bigcap^{\mathfrak{F}} \langle \alpha' \rangle \text{up } a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α .

2. Consider the relation $\delta' \in \mathcal{P}(\mathcal{P}\mathcal{U})^2$ defined by the formula (for every $X, Y \in \mathcal{P}\mathcal{U}$)

$$X \delta' Y \Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} X, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y.$$

Obviously $\neg(X \delta' \emptyset)$ and $\neg(\emptyset \delta' Y)$.

$$\begin{aligned} (I \cup J) \delta' Y &\Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}}(I \cup J), y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \\ &\Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} I \cup \text{atoms}^{\mathfrak{F}} J, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \\ &\Leftrightarrow \exists x \in \text{atoms}^{\mathfrak{F}} I, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \vee \exists x \in \text{atoms}^{\mathfrak{F}} J, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \\ &\Leftrightarrow I \delta' Y \vee J \delta' Y; \end{aligned}$$

analogously $X \delta' (I \cup J) \Leftrightarrow X \delta' I \vee X \delta' J$. Let's continue δ' till a funcoid f (by the theorem 25):

$$\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \forall X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X \delta' Y$$

The reverse of (7) implication is trivial, so

$$\forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms}^{\mathfrak{F}} X, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \Leftrightarrow a \delta b.$$

$$\forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms}^{\mathfrak{F}} X, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \Leftrightarrow \forall X \in \text{up } a, Y \in \text{up } b: X \delta' Y \Leftrightarrow a[f]b.$$

So $a \delta b \Leftrightarrow a[f]b$, that is $[f]$ is a continuation of δ . \square

One of uses of the previous theorem is proof of the following theorem:

Theorem 55. If R is a set of funcoids, $x, y \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$, then

1. $\langle \bigcap^{\text{FCD}} R \rangle x = \bigcap^{\mathfrak{F}} \{ \langle f \rangle x \mid f \in R \}$;
2. $x [\bigcap^{\text{FCD}} R] y \Leftrightarrow \forall f \in R: x [f] y$.

Proof. 2. Let denote $x \delta y \Leftrightarrow \forall f \in R: x[f]y$.

$$\begin{aligned} & \forall X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms}^{\mathfrak{F}} X, y \in \text{atoms}^{\mathfrak{F}} Y: x \delta y \Leftrightarrow \\ & \forall f \in R, X \in \text{up } a, Y \in \text{up } b \exists x \in \text{atoms}^{\mathfrak{F}} X, y \in \text{atoms}^{\mathfrak{F}} Y: x[f]y \Rightarrow \\ & \quad \forall f \in R, X \in \text{up } a, Y \in \text{up } b: X[f]Y \Rightarrow \\ & \quad \quad \forall f \in R: a[f]b \Leftrightarrow \\ & \quad \quad \quad a \delta b. \end{aligned}$$

So, by the theorem 54, δ can be continued till $[p]$ for some funcoind p .

For every funcoind q such that $\forall f \in R: q \subseteq f$ we have $x[q]y \Rightarrow \forall f \in R: x[f]y \Leftrightarrow x \delta y \Leftrightarrow x[p]y$, so $q \subseteq p$. Consequently $p = \bigcap^{\text{FCD}} R$.

From this $x[\bigcap^{\text{FCD}} R]y \Leftrightarrow \forall f \in R: x[f]y$.

1. From the former $y \in \text{atoms}^{\mathfrak{F}} \langle \bigcap^{\text{FCD}} R \rangle x \Leftrightarrow y \cap^{\mathfrak{F}} \langle \bigcap^{\text{FCD}} R \rangle x \neq \emptyset \Leftrightarrow \forall f \in R: y \cap^{\mathfrak{F}} \langle f \rangle x \neq \emptyset \Leftrightarrow y \in \bigcap \langle \text{atoms}^{\mathfrak{F}} \rangle \{ \langle f \rangle x \mid f \in R \} \Leftrightarrow y \in \text{atoms}^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{ \langle f \rangle x \mid f \in R \}$ for every $y \in \text{atoms}^{\mathfrak{F}} \mathfrak{U}$. From this follows $\langle \bigcap^{\text{FCD}} R \rangle x = \bigcap^{\mathfrak{F}} \{ \langle f \rangle x \mid f \in R \}$. \square

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is direct product of two filter objects as defined in the theory of funcoids:

Definition 56. *Direct product* of filter objects \mathcal{A} and \mathcal{B} is such a funcoind $\mathcal{A} \times^{\text{FCD}} \mathcal{B}$ that

$$\mathcal{X}[\mathcal{A} \times^{\text{FCD}} \mathcal{B}] \mathcal{Y} \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset.$$

Proposition 57. $\mathcal{A} \times^{\text{FCD}} \mathcal{B}$ is really a funcoind and

$$\langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset; \\ \emptyset & \text{if } \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} = \emptyset. \end{cases}$$

Proof. Obvious. \square

Obvious 58. $A \times B = A \times^{\text{FCD}} B$ for sets A and B .

Proposition 59. $f \subseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B} \Leftrightarrow \text{dom } f \subseteq \mathcal{A} \wedge \text{im } f \subseteq \mathcal{B}$ for every $f \in \text{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. If $f \subseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ then $\text{dom } f \subseteq \text{dom}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\text{im } f \subseteq \text{im}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\text{dom } f \subseteq \mathcal{A} \wedge \text{im } f \subseteq \mathcal{B}$ then

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{X}[f] \mathcal{Y} \Rightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset);$$

consequently $f \subseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$. \square

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 60. $f \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = I_B^{\text{FCD}} \circ f \circ I_A^{\text{FCD}}$ for every $f \in \text{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $h \stackrel{\text{def}}{=} I_B^{\text{FCD}} \circ f \circ I_A^{\text{FCD}}$. For every $\mathcal{X} \in \mathfrak{F}$

$$\langle h \rangle \mathcal{X} = \langle I_B^{\text{FCD}} \rangle \langle f \rangle \langle I_A^{\text{FCD}} \rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ for a funcoind g then $\text{dom } g \subseteq \mathcal{A}$, $\text{im } g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{F}} \langle g \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) \subseteq \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) = \langle I_B^{\text{FCD}} \rangle \langle f \rangle \langle I_A^{\text{FCD}} \rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{B})$. \square

Corollary 61. $f|_{\mathcal{A}}^{\text{FCD}} = f \cap (\mathcal{A} \times^{\text{FCD}} \mathfrak{U})$ for every $f \in \text{FCD}$ and $\mathcal{A} \in \mathfrak{F}$.

Proof. $f \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{U}) = I_{\mathcal{U}}^{\text{FCD}} \circ f \circ I_{\mathcal{A}}^{\text{FCD}} = f \circ I_{\mathcal{A}}^{\text{FCD}} = f|_{\mathcal{A}}^{\text{FCD}}$. \square

Corollary 62. $f \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \neq \emptyset \Leftrightarrow \mathcal{A}[f]\mathcal{B}$ for every $f \in \text{FCD}$, $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $f \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \neq \emptyset \Leftrightarrow \langle f \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \rangle \mathcal{U} \neq \emptyset \Leftrightarrow \langle I_{\mathcal{B}}^{\text{FCD}} \circ f \circ I_{\mathcal{A}}^{\text{FCD}} \rangle \mathcal{U} \neq \emptyset \Leftrightarrow \langle I_{\mathcal{B}}^{\text{FCD}} \rangle \langle f \rangle \langle I_{\mathcal{A}}^{\text{FCD}} \rangle \mathcal{U} \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\text{FCD}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{U}) \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A}[f]\mathcal{B}$. \square

Corollary 63. The filtrator of funcoids is star-separable.

Proof. The set of direct products of sets is a separation subset of the lattice of funcoids. \square

Theorem 64. If $S \in \mathcal{P}\mathfrak{F}^2$ then

$$\bigcap^{\text{FCD}} \{ \mathcal{A} \times^{\text{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} = \bigcap^{\mathfrak{F}} \text{dom } S \times^{\text{FCD}} \bigcap^{\mathfrak{F}} \text{im } S.$$

Proof. If $x \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$ then by the theorem 55

$$\left\langle \bigcap^{\text{FCD}} \{ \mathcal{A} \times^{\text{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \bigcap^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \}.$$

If $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \text{dom } S \neq \emptyset$ then

$$\begin{aligned} \forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x = \mathcal{B}); \\ \{ \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} = \text{im } S; \end{aligned}$$

if $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \text{dom } S = \emptyset$ then

$$\begin{aligned} \exists (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \wedge \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x = \emptyset); \\ \{ \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} \ni \emptyset. \end{aligned}$$

So

$$\left\langle \bigcap^{\text{FCD}} \{ \mathcal{A} \times^{\text{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \begin{cases} \bigcap^{\mathfrak{F}} \text{im } S & \text{if } x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \text{dom } S \neq \emptyset; \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \text{dom } S = \emptyset. \end{cases}$$

From this follows the statement of the theorem. \square

Corollary 65. $(\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \cap^{\text{FCD}} (\mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\text{FCD}} \mathcal{A}_1) \times^{\text{FCD}} (\mathcal{B}_0 \cap^{\text{FCD}} \mathcal{B}_1)$ for every $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof. $(\mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0) \cap^{\text{FCD}} (\mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1) = \bigcap^{\mathfrak{F}} \{ \mathcal{A}_0 \times^{\text{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\text{FCD}} \mathcal{B}_1 \}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap^{\text{FCD}} \mathcal{A}_1) \times^{\text{FCD}} (\mathcal{B}_0 \cap^{\text{FCD}} \mathcal{B}_1)$. \square

Theorem 66. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\text{FCD}}$ is a complete homomorphism of the lattice \mathfrak{F} to a complete sublattice of the lattice FCD , if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Proof. Let $S \in \mathcal{P}\mathfrak{F}$, $X \in \mathcal{P}\mathcal{U}$, $x \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$.

$$\begin{aligned} \left\langle \bigcup^{\text{FCD}} \langle \mathcal{A} \times^{\text{FCD}} \rangle S \right\rangle X &= \bigcup^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle X \mid \mathcal{B} \in S \} \\ &= \begin{cases} \bigcup^{\mathfrak{F}} S & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\ &= \langle \mathcal{A} \times^{\text{FCD}} \bigcup^{\mathfrak{F}} S \rangle X; \\ \left\langle \bigcap^{\text{FCD}} \langle \mathcal{A} \times^{\text{FCD}} \rangle S \right\rangle x &= \bigcap^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle x \mid \mathcal{B} \in S \} \\ &= \begin{cases} \bigcap^{\mathfrak{F}} S & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\ &= \langle \mathcal{A} \times^{\text{FCD}} \bigcap^{\mathfrak{F}} S \rangle x. \end{aligned}$$

If $\mathcal{A} \neq \emptyset$ then obviously the function $\mathcal{A} \times^{\text{FCD}}$ is injective. \square

The following proposition states that cutting a rectangle of atomic width from a funcoïd always produces a rectangular (representable as a direct product of filter objects) funcoïd (of atomic width).

Proposition 67. If a is an atomic filter object, $f \in \text{FCD}$ then $f|_a^{\text{FCD}} = a \times^{\text{FCD}} \langle f \rangle a$.

Proof. Let $\mathcal{X} \in \mathfrak{F}$.

$$\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f|_a^{\text{FCD}} \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \langle f|_a^{\text{FCD}} \rangle \mathcal{X} = \emptyset. \quad \square$$

3.10 Atomic funcoïds

Theorem 68. A funcoïd is an atom of the lattice of funcoïds iff it is direct product of two atomic filter objects.

Proof.

\Rightarrow . Let f is an atomic funcoïd. Let's get elements $a \in \text{atoms}^{\mathfrak{F}} \text{dom } f$ and $b \in \text{atoms}^{\mathfrak{F}} \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{F}$

$$\mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \langle a \times^{\text{FCD}} b \rangle \mathcal{X} = \emptyset \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle a \times^{\text{FCD}} b \rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

So $a \times^{\text{FCD}} b \subseteq f$; because f is an atomic funcoïd $f = a \times^{\text{FCD}} b$.

\Leftarrow . Let $a, b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$, $f \in \text{FCD}$. If $b \cap^{\mathfrak{F}} \langle f \rangle a = \emptyset$ then $\neg(a[f]b)$, $f \cap^{\mathfrak{F}} (a \times^{\text{FCD}} b) = \emptyset$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}$: $(\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\text{FCD}} b$. Consequently $f \cap^{\text{FCD}} (a \times^{\text{FCD}} b) = \emptyset \vee f \supseteq a \times^{\text{FCD}} b$; that is $a \times^{\text{FCD}} b$ is an atomic filter object. \square

Theorem 69. The lattice of funcoïds is atomic.

Proof. Let f is a non-empty funcoïd. Then $\text{dom } f \neq \emptyset$, thus by the theorem 46 in [13] exists $a \in \text{atoms}^{\mathfrak{F}} \text{dom } f$. So $\langle f \rangle a \neq \emptyset$ thus exists $b \in \text{atoms} \langle f \rangle a$. Finally the atomic funcoïd $a \times^{\text{FCD}} b \subseteq f$. \square

Theorem 70. The lattice of funcoïds is separable.

Proof. Let $f, g \in \text{FCD}$, $f \subset g$. Then exists $a \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$ such that $\langle f \rangle a \subset \langle g \rangle a$. So because the lattice \mathfrak{F} is atomically separable then exists $b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$ such that $\langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset$ and $b \subseteq \langle g \rangle a$. For every $x \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$

$$\begin{aligned} \langle f \rangle a \cap^{\mathfrak{F}} \langle a \times^{\text{FCD}} b \rangle a &= \langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset, \\ x \neq a &\Rightarrow \langle f \rangle x \cap^{\mathfrak{F}} \langle a \times^{\text{FCD}} b \rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \emptyset = \emptyset \end{aligned}$$

Thus $\langle f \rangle x \cap^{\mathfrak{F}} \langle a \times b \rangle x = \emptyset$ and consequently $f \cap^{\text{FCD}} (a \times^{\text{FCD}} b) = \emptyset$.

$$\begin{aligned} \langle a \times^{\text{FCD}} b \rangle a &= b \subseteq \langle g \rangle a, \\ x \neq a &\Rightarrow \langle a \times^{\text{FCD}} b \rangle x = \emptyset \subseteq \langle g \rangle a. \end{aligned}$$

Thus $\langle a \times^{\text{FCD}} b \rangle x \subseteq \langle g \rangle x$ and consequently $a \times^{\text{FCD}} b \subseteq g$.

So the lattice of funcoïds is separable by the theorem 19 in [13]. \square

Corollary 71. The lattice of funcoïds is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [13]. \square

Remark 72. For more ways to characterize (atomic) separability of the lattice of funcoids see [13], subsections “Separation subsets and full stars” and “Atomically separable lattices”.

Corollary 73. The lattice of funcoids is an atomistic lattice.

Proof. Let f is a funcoid. Suppose contrary to the statement to be proved that $\bigcup^{\mathfrak{S}} \text{atoms}^{\text{FCD}} f \subset f$. Then exists $a \in \text{atoms}^{\text{FCD}} f$ such that $a \cap^{\mathfrak{S}} \bigcup^{\mathfrak{S}} \text{atoms}^{\text{FCD}} f = \emptyset$ what is impossible. \square

Proposition 74. $\text{atoms}^{\text{FCD}}(f \cup^{\mathfrak{S}} g) = \text{atoms}^{\text{FCD}} f \cup \text{atoms}^{\text{FCD}} g$ for every funcoids f and g .

Proof. $(a \times^{\text{FCD}} b) \cap^{\text{FCD}} (f \cup^{\text{FCD}} g) \neq \emptyset \Leftrightarrow a[f \cup^{\text{FCD}} g]b \Leftrightarrow a[f]b \vee a[g]b \Leftrightarrow (a \times^{\text{FCD}} b) \cap^{\text{FCD}} f \neq \emptyset \vee (a \times^{\text{FCD}} b) \cap^{\text{FCD}} g \neq \emptyset$ for every atomic filter objects a and b . \square

Corollary 75. For every $f, g, h \in \text{FCD}$, $R \in \mathcal{P}\text{FCD}$

1. $f \cap^{\text{FCD}} (g \cup^{\text{FCD}} h) = (f \cap^{\text{FCD}} g) \cup^{\text{FCD}} (f \cap^{\text{FCD}} h)$;
2. $f \cup^{\text{FCD}} \bigcap^{\text{FCD}} R = \bigcap^{\text{FCD}} \langle f \cup^{\text{FCD}} \rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice. To be concise I will write atoms instead of $\text{atoms}^{\text{FCD}}$ and \cap and \cup instead of \cap^{FCD} and \cup^{FCD} .

1. $\text{atoms}(f \cap (g \cup h)) = \text{atoms } f \cap \text{atoms}(g \cup h) = \text{atoms } f \cap (\text{atoms } g \cup \text{atoms } h) = (\text{atoms } f \cap \text{atoms } g) \cup (\text{atoms } f \cap \text{atoms } h) = \text{atoms}(f \cap g) \cup \text{atoms}(f \cap h) = \text{atoms}((f \cap g) \cup (f \cap h))$.
2. $\text{atoms}(f \cup \bigcap^{\text{FCD}} R) = \text{atoms } f \cup \text{atoms } \bigcap^{\text{FCD}} R = \text{atoms } f \cup \bigcap^{\text{FCD}} \langle \text{atoms} \rangle R = \bigcap^{\text{FCD}} \langle (\text{atoms } f) \cup \rangle \langle \text{atoms} \rangle R = \bigcap^{\text{FCD}} \langle \text{atoms} \rangle \langle f \cup \rangle R = \text{atoms } \bigcap^{\text{FCD}} \langle f \cup \rangle R$. (Used the following equality.)

$$\begin{aligned}
& \langle (\text{atoms } f) \cup \rangle \langle \text{atoms} \rangle R = \\
& \{ (\text{atoms } f) \cup A \mid A \in \langle \text{atoms} \rangle R \} = \\
& \{ (\text{atoms } f) \cup A \mid \exists C \in R: A = \text{atoms } C \} = \\
& \{ (\text{atoms } f) \cup (\text{atoms } C) \mid C \in R \} = \\
& \{ \text{atoms}(f \cup C) \mid C \in R \} = \\
& \{ \text{atoms } B \mid \exists C \in R: B = f \cup C \} = \\
& \{ \text{atoms } B \mid B \in \langle f \cup \rangle R \} = \\
& \langle \text{atoms} \rangle \langle f \cup \rangle.
\end{aligned}$$

\square

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

Corollary 76. The lattice of funcoids is co-brouwerian.

The next proposition is one more (among the theorem 42) generalization for funcoids of composition of relations.

Proposition 77. For every $f, g \in \text{FCD}$

$\text{atoms}^{\text{FCD}}(g \circ f) = \{ x \times^{\text{FCD}} z \mid x, z \in \text{atoms}^{\mathfrak{S}} \mathfrak{U}, \exists y \in \text{atoms}^{\mathfrak{S}} \mathfrak{U}: (x \times^{\text{FCD}} y \in \text{atoms}^{\text{FCD}} f \wedge y \times^{\text{FCD}} z \in \text{atoms}^{\text{FCD}} g) \}$.

Proof. $(x \times^{\text{FCD}} z) \cap^{\text{FCD}} (g \circ f) \neq \emptyset \Leftrightarrow x[g \circ f]z \Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{S}} \mathfrak{U}: (x[f]y \wedge y[g]z) \Leftrightarrow \exists y \in \text{atoms}^{\mathfrak{S}} \mathfrak{U}: ((x \times^{\text{FCD}} y) \cap^{\text{FCD}} f \neq \emptyset \wedge (y \times^{\text{FCD}} z) \cap^{\text{FCD}} g \neq \emptyset)$ (were used the theorem 42). \square

Conjecture 78. The set of discrete funcoids is the center of the lattice of funcoids.

3.11 Complete functors

Definition 79. I will call *co-complete* such a functor f that $\forall X \in \mathcal{P}\mathcal{U}: \langle f \rangle X \in \mathcal{P}\mathcal{U}$.

Remark 80. I will call *generalized closure* such a function $\alpha \in \mathcal{P}\mathcal{U}^{\mathcal{P}\mathcal{U}}$ that

1. $\alpha \emptyset = \emptyset$;
2. $\forall I, J \in \mathcal{P}\mathcal{U}: \alpha(I \cup J) = \alpha I \cup \alpha J$.

Obvious 81. A functor f is co-complete iff $\langle f \rangle|_{\mathcal{P}\mathcal{U}}$ is a generalized closure.

Remark 82. Thus functors can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of functors.

Definition 83. I will call a *complete functor* a functor whose reverse is co-complete.

Theorem 84. The following conditions are equivalent for every functor f :

1. functor f is complete;
2. $\forall S \in \mathcal{P}\mathcal{F}, J \in \mathcal{P}\mathcal{U}: (\bigcup^{\mathfrak{F}} S[f]J \Leftrightarrow \exists I \in S: I[f]J)$;
3. $\forall S \in \mathcal{P}\mathcal{P}\mathcal{U}, J \in \mathcal{P}\mathcal{U}: (\bigcup S[f]J \Leftrightarrow \exists I \in S: I[f]J)$;
4. $\forall S \in \mathcal{P}\mathcal{F}: \langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
5. $\forall S \in \mathcal{P}\mathcal{P}\mathcal{U}: \langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
6. $\forall A \in \mathcal{P}\mathcal{U}: \langle f \rangle A = \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in A \}$.

Proof.

(3) \Rightarrow (1). For every $S \in \mathcal{P}\mathcal{P}\mathcal{U}, J \in \mathcal{P}\mathcal{U}$

$$\bigcup S \cap^{\mathfrak{F}} \langle f^{-1} \rangle J \neq \emptyset \Leftrightarrow \exists I \in S: I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J \neq \emptyset, \quad (9)$$

consequently by the theorem 52 in [13] we have $\langle f^{-1} \rangle J \in \mathcal{P}\mathcal{U}$.

(1) \Rightarrow (2). For every $S \in \mathcal{P}\mathcal{F}, J \in \mathcal{P}\mathcal{U}$ we have $\langle f^{-1} \rangle J \in \mathcal{P}\mathcal{U}$, consequently the formula (9) is true. From this follows (2).

(6) \Rightarrow (5). $\langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in \bigcup S \} = \bigcup^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in A \} \mid A \in S \} = \bigcup^{\mathfrak{F}} \{ \langle f \rangle A \mid A \in S \} = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$.

(2) \Rightarrow (4). $J \cap^{\mathfrak{F}} \langle f \rangle \bigcup^{\mathfrak{F}} S \neq \emptyset \Leftrightarrow \bigcup^{\mathfrak{F}} S[f]J \Leftrightarrow \exists I \in S: I[f]J \Leftrightarrow \exists I \in S: J \cap^{\mathfrak{F}} \langle f \rangle I \neq \emptyset \Leftrightarrow J \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S \neq \emptyset$ (used the theorem 52 in [13]).

(2) \Rightarrow (3), (4) \Rightarrow (5), (5) \Rightarrow (3), (5) \Rightarrow (6). Obvious. \square

The following proposition shows that complete functors are a direct generalization of pre-topological spaces.

Proposition 85. To specify a complete functor f it is enough to specify $\langle f \rangle$ on one-element sets, values of $\langle f \rangle$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f \rangle$ on one-element sets $\langle f \rangle$ can be found on every set and then its value can be inferred for every filter objects.

Choosing arbitrarily the values of $\langle f \rangle$ on one-element sets we can define a complete functor the following way: $\langle f \rangle X \stackrel{\text{def}}{=} \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{ \alpha \} \mid \alpha \in X \}$ for every $X \in \mathcal{P}\mathcal{U}$. Obviously it is really a complete functor. \square

Theorem 86. A functor is discrete iff it is both complete and co-complete.

Proof.

\Rightarrow . Obvious.

\Leftarrow . Let f is both a complete and co-complete funcoid. Consider the relation g defined by that $\langle g \rangle \{ \alpha \} = \langle f \rangle \{ \alpha \}$ (g is correctly defined because f is a generalized closure). Because f is a complete funcoid $f = g$. \square

Theorem 87. If R is a set of (co-)complete funcoids then $\bigcup^{\text{FCD}} R$ is a (co-)complete funcoid.

Proof. It is enough to prove only for co-complete funcoids. Let R is a set of co-complete funcoids. Then for every $X \in \mathcal{P}\mathcal{U}$

$$\left\langle \bigcup^{\text{FCD}} R \right\rangle X = \bigcup \{ \langle f \rangle X \mid f \in R \} \in \mathcal{P}\mathcal{U}$$

(used the theorem 39). \square

Corollary 88. If R is a set of binary relations then $\bigcup^{\text{FCD}} R = \bigcup R$.

Proof. From two last theorems. \square

Theorem 89. The filtrator of funcoids is filtered.

Proof. It's enough to prove that every funcoid is representable as (infinite) meet (on the lattice of funcoids) of some set of discrete funcoids.

Let $f \in \text{FCD}$, $A \in \mathcal{P}\mathcal{U}$, $B \in \text{up}\langle f \rangle A$, $g(A; B) \stackrel{\text{def}}{=} A \times^{\text{FCD}} B \cup^{\text{FCD}} \bar{A} \times^{\text{FCD}} \bar{U}$. For every $X \in \mathcal{P}\mathcal{U}$

$$\langle g(A; B) \rangle X = \langle A \times^{\text{FCD}} B \rangle X \cup \langle \bar{A} \times^{\text{FCD}} \bar{U} \rangle X = \left(\begin{array}{l} \emptyset \text{ if } X = \emptyset \\ B \text{ if } \emptyset \neq X \subseteq A \\ \bar{U} \text{ if } X \not\subseteq A \end{array} \right) \supseteq \langle f \rangle X;$$

so $g(A; B) \supseteq f$. For every $A \in \mathcal{P}\mathcal{U}$

$$\bigcap^{\mathfrak{S}} \{ \langle g(A; B) \rangle A \mid B \in \text{up}\langle f \rangle A \} = \bigcap^{\mathfrak{S}} \{ B \mid B \in \text{up}\langle f \rangle A \} = \langle f \rangle A;$$

consequently

$$\bigcap^{\text{FCD}} \{ g(A; B) \mid A \in \mathcal{P}\mathcal{U}, B \in \text{up}\langle f \rangle A \} = f. \quad \square$$

Conjecture 90. If f is a complete funcoid and R is a set of funcoids then $f \circ \bigcup^{\text{FCD}} R = \bigcup^{\text{FCD}} \langle f \circ \rangle R$.

This conjecture can be weakened:

Conjecture 91. If f is a discrete funcoid and R is a set of funcoids then $f \circ \bigcup^{\text{FCD}} R = \bigcup^{\text{FCD}} \langle f \circ \rangle R$.

I will denote ComplFCD and CoComplFCD the sets of complete and co-complete funcoids correspondingly.

Obvious 92. ComplFCD and CoComplFCD are closed regarding composition of funcoids.

Proposition 93. ComplFCD and CoComplFCD (with induced order) are complete lattices.

Proof. Follows from the corollary 87. \square

3.12 Completion of funcoids

Theorem 94. $\text{Cor } f = \text{Cor}' f$ for an element f of the filtrator of funcoids. (Core part is taken for the filtrator of funcoids.)

Proof. From the theorem 26 in [13] and the corollary 88 and theorem 89. \square

Definition 95. *Completion* of a funcooid f is the complete funcooid $\text{Compl } f$ defined by the formula $\langle \text{Compl } f \rangle \{\alpha\} = \langle f \rangle \{\alpha\}$ for $\alpha \in \mathcal{U}$.

Definition 96. *Co-completion* of a funcooid f is defined by the formula

$$\text{CoCompl } f = (\text{Compl } f^{-1})^{-1}.$$

Obvious 97. $\text{Compl } f \subseteq f$ and $\text{CoCompl } f \subseteq f$ for every funcooid f .

Proposition 98. The filtrator $(\text{FCD}; \text{ComplFCD})$ is filtered.

Proof. Because the filtrator $(\text{FCD}; \mathcal{P}\mathcal{U}^2)$ is filtered. \square

Theorem 99. $\text{Compl } f = \text{Cor}^{(\text{FCD}; \text{ComplFCD})} f = \text{Cor}'^{(\text{FCD}; \text{ComplFCD})} f$.

Proof. $\text{Cor}^{(\text{FCD}; \text{ComplFCD})} f = \text{Cor}'^{(\text{FCD}; \text{ComplFCD})} f$ since (the theorem 26 in [13]) the filtrator $(\text{FCD}; \text{ComplFCD})$ is filtered and with join closed core (the theorem 87).

Let $g \in \text{up}^{(\text{FCD}; \text{ComplFCD})} f$. Then $g \in \text{ComplFCD}$ and $g \supseteq f$. Thus $g = \text{Compl } g \supseteq \text{Compl } f$.

Thus $\forall g \in \text{up}^{(\text{FCD}; \text{ComplFCD})} f: g \supseteq \text{Compl } f$.

Let $\forall g \in \text{up}^{(\text{FCD}; \text{ComplFCD})} f: h \subseteq g$ for some $h \in \text{ComplFCD}$.

Then $h \subseteq \bigcap \text{up}^{(\text{FCD}; \text{ComplFCD})} f = f$ and consequently $h = \text{Compl } h \subseteq \text{Compl } f$.

Thus $\text{Compl } f = \bigcap \text{ComplFCD}_{\text{up}}^{(\text{FCD}; \text{ComplFCD})} f = \text{Cor}^{(\text{FCD}; \text{ComplFCD})} f$. \square

Theorem 100. Atoms of the lattice ComplFCD are exactly direct products of the form $\{\alpha\} \times^{\text{FCD}} b$ where $\alpha \in \mathcal{U}$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\text{FCD}} b$ are elements of ComplFCD . Also \emptyset is an element of ComplFCD .

$\{\alpha\} \times^{\text{FCD}} b$ are atoms of ComplFCD because these are atoms of FCD .

Remain to prove that if f is an atom of ComplFCD then $f = \{\alpha\} \times^{\text{FCD}} b$ for some $\alpha \in \mathcal{U}$ and an atomic f.o. b .

Suppose f is a non-empty complete funcooid. Then exists $\alpha \in \mathcal{U}$ such that $\langle f \rangle \{\alpha\} \neq \emptyset$. Thus $\{\alpha\} \times^{\text{FCD}} b \subseteq f$ for some atomic f.o. b . If f is an atom then $f = \{\alpha\} \times^{\text{FCD}} b$. \square

Theorem 101. $\langle \text{CoCompl } f \rangle X = \text{Cor} \langle f \rangle X$ for every funcooid f and set X .

Proof. $\text{CoCompl } f \subseteq f$ thus $\langle \text{CoCompl } f \rangle X \subseteq \langle f \rangle X$, but $\langle \text{CoCompl } f \rangle X \in \mathcal{P}\mathcal{U}$ thus $\langle \text{CoCompl } f \rangle X \subseteq \text{Cor} \langle f \rangle X$.

Let $\alpha X = \text{Cor} \langle f \rangle X$. Then $\alpha \emptyset = \emptyset$ and

$$\alpha(X \cup Y) = \text{Cor} \langle f \rangle (X \cup Y) = \text{Cor}(\langle f \rangle X \cup \langle f \rangle Y) = \text{Cor} \langle f \rangle X \cup \text{Cor} \langle f \rangle Y = \alpha X \cup \alpha Y.$$

(used the theorem 64 from [13]). Thus α can be continued till $\langle g \rangle$ for some funcooid g . This funcooid is co-complete.

Evidently g is the greatest co-complete funcooid which is lower than f .

Thus $g = \text{CoCompl } f$ and so $\text{Cor} \langle f \rangle X = \alpha X = \langle g \rangle X = \langle \text{CoCompl } f \rangle X$. \square

Theorem 102. ComplFCD is an atomistic lattice.

Proof. Let $f \in \text{ComplFCD}$. $\langle f \rangle X = \bigcup^{\mathfrak{F}} \{\langle f \rangle \{x\} \mid x \in X\} = \bigcup^{\mathfrak{F}} \left\{ \left\langle f \Big|_{\{x\}}^{\text{FCD}} \right\rangle \{x\} \mid x \in X \right\} = \bigcup^{\mathfrak{F}} \left\{ \left\langle f \Big|_{\{x\}}^{\text{FCD}} \right\rangle X \mid x \in X \right\}$, thus $f = \bigcup^{\text{FCD}} \left\{ f \Big|_{\{x\}}^{\text{FCD}} \mid x \in X \right\}$. It is trivial that every $f \Big|_{\{x\}}^{\text{FCD}}$ is a union of atoms of ComplFCD . \square

Theorem 103. A funcooid is complete iff it is a join (on the lattice FCD) of atomic complete funcooids.

Proof. Follows from the theorem 87 and the previous theorem. \square

Corollary 104. ComplFCD is join-closed.

Theorem 105. $\text{Compl}(\bigcup^{\text{FCD}} R) = \bigcup^{\text{FCD}} \langle \text{Compl} \rangle R$ for every set R of funcoids.

Proof. $\langle \text{Compl}(\bigcup^{\text{FCD}} R) \rangle X = \bigcup^{\mathfrak{F}} \{ \langle \bigcup^{\text{FCD}} R \rangle \{ \alpha \} \mid \alpha \in X \} = \bigcup^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{ \alpha \} \mid f \in R \} \mid \alpha \in X \} = \bigcup^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{ \alpha \} \mid \alpha \in X \} \mid f \in R \} = \bigcup^{\mathfrak{F}} \{ \langle \text{Compl} f \rangle X \mid f \in R \} = \langle \bigcup^{\text{FCD}} \langle \text{Compl} \rangle R \rangle X$ for every set X . \square

Corollary 106. Compl is an lower adjoint.

Conjecture 107. Compl is not an upper adjoint (in general).

Conjecture 108. $\text{Compl} f = f \setminus *^{\text{FCD}} (\Omega \times^{\text{FCD}} \mathcal{U})$ for every funcoid f .

This conjecture may be proved by considerations similar to these in the section “Fréchet filter” in [13].

Lemma 109. Co-completion of a complete funcoid is complete.

Proof. Let f is a complete funcoid.

$\langle \text{CoCompl} f \rangle X = \text{Cor} \langle f \rangle X = \text{Cor} \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{ x \} \mid x \in X \} = \bigcup \{ \text{Cor} \langle f \rangle \{ x \} \mid x \in X \} = \bigcup \{ \langle \text{CoCompl} f \rangle \{ x \} \mid x \in X \}$ for every set X . Thus $\text{CoCompl} f$ is complete. \square

Theorem 110. $\text{Compl} \text{CoCompl} f = \text{CoCompl} \text{Compl} f = \text{Cor} f$ for every funcoid f .

Proof. $\text{Compl} \text{CoCompl} f$ is co-complete since (used the lemma) $\text{CoCompl} f$ is co-complete. Thus $\text{Compl} \text{CoCompl} f$ is a discrete funcoid. $\text{CoCompl} f$ is the the greatest co-complete funcoid under f and $\text{Compl} \text{CoCompl} f$ is the greatest complete funcoid under $\text{CoCompl} f$. So $\text{Compl} \text{CoCompl} f$ is greater than any discrete funcoid under $\text{CoCompl} f$ which is greater than any discrete funcoid under f . Thus $\text{Compl} \text{CoCompl} f$ it is the greatest discrete funcoid under f . Thus $\text{Compl} \text{CoCompl} f = \text{Cor} f$. Similarly $\text{CoCompl} \text{Compl} f = \text{Cor} f$. \square

Question 111. Is ComplFCD a co-brouwerian lattice?

3.13 Monovalued funcoids

Following the idea of definition of monovalued morphism let's call *monovalued* such a funcoid f that $f \circ f^{-1} \subseteq I_{\text{im}}^{\text{FCD}} f$.

Obvious 112. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of funcoids is monovalued iff the funcoid f is monovalued.

Theorem 113. The following statements are equivalent for a funcoid f :

1. f is monovalued.
2. $\forall a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}: \langle f \rangle a \in \text{atoms}^{\mathfrak{F}} \mathcal{U} \cup \{ \emptyset \}$.
3. $\forall \mathcal{I}, \mathcal{J} \in \mathfrak{F}: \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J}$.
4. $\forall I, J \in \mathcal{P} \mathcal{U}: \langle f^{-1} \rangle (I \cap J) = \langle f^{-1} \rangle I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J$.

Proof.

(2) \Rightarrow (3). Let $a \in \text{atoms}^{\mathfrak{F}} \mathcal{U}$, $\langle f \rangle a = b$. Then because $b \in \text{atoms}^{\mathfrak{F}} \mathcal{U} \cup \{ \emptyset \}$

$$\begin{aligned} (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} b \neq \emptyset &\Leftrightarrow \mathcal{I} \cap^{\mathfrak{F}} b \neq \emptyset \wedge \mathcal{J} \cap^{\mathfrak{F}} b \neq \emptyset; \\ a[f](\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) &\Leftrightarrow a[f] \mathcal{I} \wedge a[f] \mathcal{J}; \\ (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J})[f^{-1}] a &\Leftrightarrow \mathcal{I}[f^{-1}] a \wedge \mathcal{J}[f^{-1}] a; \\ a \cap^{\mathfrak{F}} \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \neq \emptyset &\Leftrightarrow a \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{I} \neq \emptyset \wedge a \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J} \neq \emptyset; \\ \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) &= \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J}. \end{aligned}$$

(4) \Rightarrow (1). $\langle f^{-1} \rangle a \cap^{\mathfrak{F}} \langle f^{-1} \rangle b = \emptyset$ for every two distinct atomic filter objects a and b . This is equivalent to $\neg(\langle f^{-1} \rangle a[f]b)$; $b \cap^{\mathfrak{F}} \langle f \rangle \langle f^{-1} \rangle a = \emptyset$; $b \cap^{\mathfrak{F}} \langle f \circ f^{-1} \rangle a = \emptyset$; $\neg(a[f \circ f^{-1}]b)$. So $a[f \circ f^{-1}]b \Rightarrow a = b$ for every atomic filter objects a and b . This is possible only when $f \circ f^{-1} \subseteq I_{\text{Dst}}^{\text{FCD}} f$.

(3) \Rightarrow (4). Obvious.

\neg (2) \Rightarrow \neg (1). Suppose $\langle f \rangle a \notin \text{atoms}^{\mathfrak{F}} \mathcal{B} \cup \{\emptyset\}$ for some $a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}$. Then there exist two atomic filter objects $p \neq q$ such that $\langle f \rangle a \supseteq p \wedge \langle f \rangle a \supseteq q$. Consequently $p \cap^{\mathfrak{F}} \langle f \rangle a \neq \emptyset$; $a \cap^{\mathfrak{F}} \langle f^{-1} \rangle p \neq \emptyset$; $a \subseteq \langle f^{-1} \rangle p$; $\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \supseteq \langle f \rangle a \supseteq q$; $\langle f \circ f^{-1} \rangle p \not\subseteq p$. So it cannot be $f \circ f^{-1} \subseteq I_{\text{Dst}}^{\text{FCD}} f$. \square

Corollary 114. A binary relation is a monovalued funcoid iff it is a function.

Proof. Because $\forall I, J \in \mathcal{P}\mathcal{U}$: $\langle f^{-1} \rangle (I \cap J) = \langle f^{-1} \rangle I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J$ is true for a binary relation f if and only if it is a function. \square

Remark 115. This corollary can be reformulated as follows: For binary relations the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

3.14 T_0 -, T_1 - and T_2 -separable funcoids

For funcoids can be generalized T_0 -, T_1 - and T_2 - separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 116. Let call T_1 -separable such funcoid f that for every $\alpha, \beta \in \mathcal{U}$ is true

$$\alpha \neq \beta \Rightarrow \neg(\{\alpha\}[f]\{\beta\})$$

Definition 117. Let call T_0 -separable such funcoid f that $f \cap^{\text{FCD}} f^{-1}$ is T_1 -separable.

Definition 118. Let call T_2 -separable such funcoid f that the funcoid $f^{-1} \circ f$ is T_1 -separable.

For symmetric transitive funcoids T_1 - and T_2 -separability are the same (see theorem 12).

Obvious 119. A funcoid f is T_2 -separable iff $\alpha \neq \beta \Rightarrow \langle f \rangle \{\alpha\} \cap^{\mathfrak{F}} \langle f \rangle \{\beta\} = \emptyset$ for every $\alpha, \beta \in \mathcal{U}$.

3.15 Filter objects closed regarding a funcoid

Definition 120. Let's call *closed* regarding a funcoid f such filter object \mathcal{A} that $\langle f \rangle \mathcal{A} \subseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 121. If \mathcal{I} and \mathcal{J} are closed (regarding some funcoid), S is a set of closed filter objects, then

1. $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ is a closed filter object;
2. $\bigcap^{\mathfrak{F}} S$ is a closed filter object.

Proof. Let denote the given funcoid as f . $\langle f \rangle (\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J} \subseteq \mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$, $\langle f \rangle \bigcap^{\mathfrak{F}} S \subseteq \bigcap^{\mathfrak{F}} \langle f \rangle S \subseteq \bigcap^{\mathfrak{F}} S$. Consequently the filter objects $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ and $\bigcap^{\mathfrak{F}} S$ are closed. \square

Proposition 122. If S is a set of closed regarding a complete funcoid filter objects, then the filter object $\bigcup^{\mathfrak{F}} S$ is also closed regarding our funcoid.

Proof. $\langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle f \rangle S \subseteq \bigcup^{\mathfrak{F}} S$ where f is the given funcoid. \square

4 Reloids

Definition 123. I will call a *reloid* a filter object on the set of binary relations.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations (the set of binary relations is a subset of the set of reloids, I will call *discrete* these reloids which are binary relations).

Definition 124. The *reverse* reloid of a reloid f is defined by the formula

$$\text{up } f^{-1} = \{F^{-1} \mid F \in \text{up } f^{-1}\}.$$

Reverse reloid is a generalization of conjugate quasi-uniformity.

I will denote RLD either the set of reloids or the category of reloids (defined below), dependently on context.

4.1 Composition of reloids

Definition 125. Composition of reloids is defined by the formula

$$g \circ f = \bigcap^{\text{RLD}} \{G \circ F \mid F \in \text{up } f, G \in \text{up } g\}.$$

Composition of reloids is a reloid.

Theorem 126. $(h \circ g) \circ f = h \circ (g \circ f)$ for every reloids f, g, h .

Proof. For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow (\forall K \in A \exists L \in B: L \subseteq K) \wedge (\forall K \in B \exists L \in A: L \subseteq K).$$

It is easy to see that \sim is a transitive relation.

I will denote $B \circ A = \{L \circ K \mid K \in A, L \in B\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B: K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ exist $P' \in B \circ C$ such that $P' \subseteq P$; vice versa is analogous. So $A \circ C \sim B \circ C$.

$\text{up}((h \circ g) \circ f) \sim \text{up}(h \circ g) \circ \text{up } f$, $\text{up}(h \circ g) \sim (\text{up } h) \circ (\text{up } g)$. By proven above $\text{up}((h \circ g) \circ f) \sim (\text{up } h) \circ (\text{up } g) \circ (\text{up } f)$.

Analogously $\text{up}(h \circ (g \circ f)) \sim (\text{up } h) \circ (\text{up } g) \circ (\text{up } f)$.

So $\text{up}((h \circ g) \circ f) \sim \text{up}(h \circ (g \circ f))$ what is possible only if $\text{up}((h \circ g) \circ f) = \text{up}(h \circ (g \circ f))$. \square

Theorem 127.

1. $f \circ f = \bigcap^{\text{RLD}} \{F \circ F \mid F \in \text{up } f\}$;
2. $f^{-1} \circ f = \bigcap^{\text{RLD}} \{F^{-1} \circ F \mid F \in \text{up } f\}$;
3. $f \circ f^{-1} = \bigcap^{\text{RLD}} \{F \circ F^{-1} \mid F \in \text{up } f\}$.

Proof. I will prove only (1) and (2) because (3) is analogous to (2).

1. Enough to show that $\forall F, G \in \text{up } f \exists H \in \text{up } f: H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.
2. Enough to show that $\forall F, G \in \text{up } f \exists H \in \text{up } f: H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq G^{-1} \circ F$. \square

Conjecture 128. If f, g, h are reloids then

1. $f \circ (g \cup^{\text{RLD}} h) = f \circ g \cup^{\text{RLD}} f \circ h$;

$$2. (g \cup^{\text{RLD}} h) \circ f = g \circ f \cup^{\text{RLD}} h \circ f.$$

4.2 Direct product of filter objects

In theory of reloids direct product of filter objects \mathcal{A} and \mathcal{B} is defined by the formula

$$\mathcal{A} \times^{\text{RLD}} \mathcal{B} \stackrel{\text{def}}{=} \bigcap^{\mathfrak{F}} \{A \times B \mid A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}\}.$$

Theorem 129. $\mathcal{A} \times^{\text{RLD}} \mathcal{B} = \bigcup^{\mathfrak{F}} \{a \times^{\text{RLD}} b \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \text{atoms}^{\mathfrak{F}} \mathcal{B}\}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. Obviously

$$\mathcal{A} \times^{\text{RLD}} \mathcal{B} \supseteq \bigcup^{\mathfrak{F}} \{a \times^{\text{RLD}} b \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \text{atoms}^{\mathfrak{F}} \mathcal{B}\}$$

Reversely, let $K \in \text{up } \bigcup^{\mathfrak{F}} \{a \times^{\text{RLD}} b \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \text{atoms}^{\mathfrak{F}} \mathcal{B}\}$. Then $K \in \text{up}(a \times^{\text{RLD}} b)$ for every $a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \text{atoms}^{\mathfrak{F}} \mathcal{B}$; $K \supseteq X_a \times^{\text{RLD}} Y_b$ for some $X_a \in \text{up } a, Y_b \in \text{up } b$; $K \supseteq \bigcup \{X_a \times Y_b \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \text{atoms}^{\mathfrak{F}} \mathcal{B}\} = \bigcup \{X_a \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{A}\} \times \bigcup \{Y_b \mid b \in \text{atoms}^{\mathfrak{F}} \mathcal{B}\} \supseteq A \times B$ where $A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}$; $K \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$. \square

Theorem 130. $(\mathcal{A}_0 \times^{\text{RLD}} \mathcal{B}_0) \cap^{\text{RLD}} (\mathcal{A}_1 \times^{\text{RLD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\text{RLD}} \mathcal{A}_1) \times^{\text{RLD}} (\mathcal{B}_0 \cap^{\text{RLD}} \mathcal{B}_1)$ for every $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof.

$$\begin{aligned} (\mathcal{A}_0 \times^{\text{RLD}} \mathcal{B}_0) \cap^{\text{RLD}} (\mathcal{A}_1 \times^{\text{RLD}} \mathcal{B}_1) &= \bigcap^{\text{RLD}} \{P \cap Q \mid P \in \text{up}(\mathcal{A}_0 \times^{\text{RLD}} \mathcal{B}_0), Q \in \text{up}(\mathcal{A}_1 \times^{\text{RLD}} \mathcal{B}_1)\} \\ &= \bigcap^{\text{RLD}} \{(A_0 \times B_0) \cap (A_1 \times B_1) \mid A_0 \in \text{up } \mathcal{A}_0, B_0 \in \text{up } \mathcal{B}_0, A_1 \in \text{up } \mathcal{A}_1, B_1 \in \text{up } \mathcal{B}_1\} \\ &= \bigcap^{\text{RLD}} \{(A_0 \cap A_1) \times (B_0 \cap B_1) \mid A_0 \in \text{up } \mathcal{A}_0, B_0 \in \text{up } \mathcal{B}_0, A_1 \in \text{up } \mathcal{A}_1, B_1 \in \text{up } \mathcal{B}_1\} \\ &= \bigcap^{\text{RLD}} \{K \times L \mid K \in \text{up}(\mathcal{A}_0 \cap \mathcal{A}_1), L \in \text{up}(\mathcal{B}_0 \cap \mathcal{B}_1)\} \\ &= (\mathcal{A}_0 \cap^{\text{RLD}} \mathcal{A}_1) \times^{\text{RLD}} (\mathcal{B}_0 \cap^{\text{RLD}} \mathcal{B}_1). \end{aligned}$$

\square

Theorem 131. If $S \in \mathcal{P}\mathfrak{F}^2$ then

$$\bigcap^{\text{RLD}} \{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} = \bigcap^{\mathfrak{F}} \text{dom } S \times^{\text{RLD}} \bigcap^{\mathfrak{F}} \text{im } S.$$

Proof. Let $\mathcal{P} = \bigcap^{\mathfrak{F}} \text{dom } S, \mathcal{Q} = \bigcap^{\mathfrak{F}} \text{im } S; l = \bigcap^{\text{RLD}} \{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\}$.

$\mathcal{P} \times^{\text{RLD}} \mathcal{Q} \subseteq l$ is obvious.

Let $F \in \text{up}(\mathcal{P} \times^{\text{RLD}} \mathcal{Q})$. Then exist $P \in \text{up } \mathcal{P}$ and $Q \in \text{up } \mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \cap \dots \cap P_n$ where $P_i \in \langle \text{up} \rangle \text{dom } S$ and $Q = Q_1 \cap \dots \cap Q_m$ where $Q_i \in \langle \text{up} \rangle \text{im } S$.

$P \times Q = \bigcap_{i,j} (P_i \times Q_j)$.

$P_i \times Q_j \supseteq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for some $(\mathcal{A}; \mathcal{B}) \in S$. $P \times Q = \bigcap_{i,j} (P_i \times Q_j) \supseteq l$. $F \in \text{up } l$. \square

Conjecture 132. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\text{RLD}}$ is a complete homomorphism of the lattice \mathfrak{F} to a complete sublattice of the lattice RLD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Definition 133. I will call a reloid *convex* iff it is a union of direct products.

Example 134. Non-convex reloids exist.

Proof. Let a is a non-trivial atomic f.o. Then $(=)|_a$ is non-convex. This follows from the fact that only direct products which are below $(=)$ are direct products of atomic f.o. and $(=)|_a$ is not their join. \square

I will call two filter objects *isomorphic* when the corresponding filters are isomorphic (in the sense defined in [13]).

Theorem 135. The reloid $\{a\} \times^{\text{RLD}} \mathcal{F}$ is isomorphic to the filter object \mathcal{F} for every $a \in \mathcal{U}$.

Proof. Consider $B = \{a\} \times \mathcal{U}$ and $f = \{(x; (a; x)) \mid x \in \mathcal{U}\}$. Then f is a bijection from \mathcal{U} to B .

If $X \in \text{up } \mathcal{F}$ then $\langle f \rangle X \subseteq B$ and $\langle f \rangle X = \{a\} \times X \in \text{up}(\{a\} \times^{\text{RLD}} \mathcal{F})$.

For every $Y \in \text{up}(\{a\} \times^{\text{RLD}} \mathcal{F}) \cap \mathcal{P}B$ we have $Y = \{a\} \times X$ for some $X \in \text{up } \mathcal{F}$ and thus $Y = \langle f \rangle X$.

So $\langle f \rangle|_{\text{up } \mathcal{F} \cap \mathcal{P}\mathcal{U}} = \langle f \rangle|_{\text{up } \mathcal{F}}$ is a bijection from $\text{up } \mathcal{F} \cap \mathcal{P}\mathcal{U}$ to $\text{up}(\{a\} \times^{\text{RLD}} \mathcal{F}) \cap \mathcal{P}B$.

We have $\text{up } \mathcal{F} \cap \mathcal{P}\mathcal{U}$ and $\text{up}(\{a\} \times^{\text{RLD}} \mathcal{F}) \cap \mathcal{P}B$ directly isomorphic and thus $\text{up } \mathcal{F}$ is isomorphic to $\text{up}(\{a\} \times^{\text{RLD}} \mathcal{F})$. \square

4.3 Restricting reloid to a filter object. Domain and image

Definition 136. I call *restricting* a reloid f to a filter object \mathcal{A} as $f|_{\mathcal{A}}^{\text{RLD}} = f \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{U})$.

Definition 137. *Domain* and *image* of a reloid f are defined as follows:

$$\text{dom } f = \bigcap^{\mathfrak{F}} \langle \text{dom} \rangle \text{up } f; \quad \text{im } f = \bigcap^{\mathfrak{F}} \langle \text{im} \rangle \text{up } f.$$

Proposition 138. $f \subseteq \mathcal{A} \times^{\text{RLD}} \mathcal{B} \Leftrightarrow \text{dom } f \subseteq \mathcal{A} \wedge \text{im } f \subseteq \mathcal{B}$.

Proof.

\Rightarrow . Follows from $\text{dom}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \text{im}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.

\Leftarrow . $\text{dom } f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \text{up } \mathcal{A} \exists F \in \text{up } f: \text{dom } F \subseteq A$. Analogously

$$\text{im } f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \text{up } \mathcal{B} \exists G \in \text{up } f: \text{im } G \subseteq B.$$

Let $\text{dom } f \subseteq \mathcal{A} \wedge \text{im } f \subseteq \mathcal{B}$, $A \in \text{up } \mathcal{A}$, $B \in \text{up } \mathcal{B}$. Then exist $F \in \text{up } f$, $G \in \text{up } f$ such that $\text{dom } F \subseteq A \wedge \text{im } G \subseteq B$. Consequently $F \cap G \in \text{up } f$, $\text{dom}(F \cap G) \subseteq A$, $\text{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. So exists $H \in \text{up } f$ such that $H \subseteq A \times B$ for every $A \in \text{up } \mathcal{A}$, $B \in \text{up } \mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\text{RLD}} \mathcal{B}$. \square

Definition 139. I call *identity reloid* for a filter object \mathcal{A} the reloid $I_{\mathcal{A}}^{\text{RLD}} \stackrel{\text{def}}{=} (=)|_{\mathcal{A}}^{\text{RLD}}$.

Theorem 140. $I_{\mathcal{A}}^{\text{RLD}} = \bigcap^{\mathfrak{F}} \{I_A \mid A \in \text{up } \mathcal{A}\}$ where I_A is the identity relation on a set A .

Proof. Let $K \in \text{up } \bigcap^{\mathfrak{F}} \{I_A \mid A \in \text{up } \mathcal{A}\}$, then exists $A \in \text{up } \mathcal{A}$ such that $K \supseteq I_A$. Then $I_A = (=)|_{\mathcal{A}}^{\text{RLD}} = (=) \cap^{\text{RLD}} (\mathcal{A} \times \mathcal{U}) \subseteq (=) \cap (A \times \mathcal{U}) = I_A \subseteq K$; $K \in \text{up } I_A$.

Reversely let $K \in \text{up } I_{\mathcal{A}}^{\text{RLD}} = \text{up}((=) \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{U}))$, then exists $A \in \text{up } \mathcal{A}$ such that $K \in \text{up}((=) \cap (A \times \mathcal{U})) = \text{up } I_A \subseteq \text{up } \bigcap^{\mathfrak{F}} \{I_A \mid A \in \text{up } \mathcal{A}\}$. \square

Proposition 141. $(I_{\mathcal{A}}^{\text{RLD}})^{-1} = I_{\mathcal{A}}^{\text{RLD}}$.

Proof. Follows from the previous theorem. \square

Theorem 142. $f|_{\mathcal{A}}^{\text{RLD}} = f \circ I_{\mathcal{A}}^{\text{RLD}}$ for every reloid f and filter object \mathcal{A} .

Proof. We need to prove that $f \cap^{\text{RLD}} (\mathcal{A} \times \mathcal{U}) = f \circ \bigcap^{\text{RLD}} \{I_A \mid A \in \text{up } \mathcal{A}\}$. $f \circ \bigcap^{\text{RLD}} \{I_A \mid A \in \text{up } \mathcal{A}\} = \bigcap^{\text{RLD}} \{F \circ I_A \mid F \in \text{up } f, A \in \text{up } \mathcal{A}\} = \bigcap^{\text{RLD}} \{F|_A \mid F \in \text{up } f, A \in \text{up } \mathcal{A}\} = \bigcap^{\text{RLD}} \{F \cap (A \times \mathcal{U}) \mid F \in \text{up } f, A \in \text{up } \mathcal{A}\} = \bigcap^{\text{RLD}} \{F \mid F \in \text{up } f\} \cap \bigcap^{\text{RLD}} \{A \times \mathcal{U} \mid A \in \text{up } \mathcal{A}\} = f \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{U})$. \square

Theorem 143. $(g \circ f)|_{\mathcal{A}}^{\text{RLD}} = g \circ (f|_{\mathcal{A}}^{\text{RLD}})$ for every reloids f and g and filter object \mathcal{A} .

Proof. $(g \circ f)|_{\mathcal{A}}^{\text{RLD}} = (g \circ f) \circ I_{\mathcal{A}}^{\text{RLD}} = g \circ (f \circ I_{\mathcal{A}}^{\text{RLD}}) = g \circ (f|_{\mathcal{A}}^{\text{RLD}})$. \square

Theorem 144. $f \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = I_{\mathcal{B}}^{\text{RLD}} \circ f \circ I_{\mathcal{A}}^{\text{RLD}}$ for every reloid f and filter objects \mathcal{A} and \mathcal{B} .

Proof. $f \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = f \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{U}) \cap^{\text{RLD}} (\mathcal{U} \times^{\text{RLD}} \mathcal{B}) = f|_{\mathcal{A}}^{\text{RLD}} \cap^{\text{RLD}} (\mathcal{U} \times \mathcal{B}) = f \circ I_{\mathcal{A}}^{\text{RLD}} \cap^{\text{RLD}} (\mathcal{U} \times \mathcal{B}) = ((f \circ I_{\mathcal{A}}^{\text{RLD}})^{-1} \cap^{\text{RLD}} (\mathcal{U} \times^{\text{RLD}} \mathcal{B})^{-1})^{-1} = ((I_{\mathcal{A}}^{\text{RLD}} \circ f^{-1}) \cap^{\text{RLD}} (\mathcal{B} \times^{\text{RLD}} \mathcal{U}))^{-1} = (I_{\mathcal{A}}^{\text{RLD}} \circ f^{-1} \circ I_{\mathcal{B}}^{\text{RLD}})^{-1} = I_{\mathcal{B}}^{\text{RLD}} \circ f \circ I_{\mathcal{A}}^{\text{RLD}}.$ \square

4.4 Category of reloids

I will define the category RLD of reloids:

- The set of objects is \mathfrak{F} .
- The set of morphisms from a filter object \mathcal{A} to a filter object \mathcal{B} is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a reloid such that $\text{dom } f \subseteq \mathcal{A}$, $\text{im } f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object \mathcal{A} is $(I_{\mathcal{A}}^{\text{RLD}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

4.4.1 Monovalued reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f \circ f^{-1} \subseteq I_{\text{im } f}^{\text{RLD}}$.

Obvious 145. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloids is monovalued iff the reloid f is monovalued.

Conjecture 146. If a reloid is monovalued then it is a monovalued function restricted to some filter object.

Conjecture 147. A reloid f is monovalued iff $\forall g \in \text{RLD}: (g \subseteq f \Rightarrow \exists \mathcal{A} \in \mathfrak{F}: g = f|_{\mathcal{A}}^{\text{RLD}})$.

Conjecture 148. A monovalued reloid restricted to an atomic filter object is atomic or empty.

A weaker conjecture:

Conjecture 149. A (monovalued) function restricted to an atomic filter object is atomic or empty.

4.5 Complete reloids and completion of reloids

Definition 150. A *complete* reloid is a reloid representable as join of direct products $\{\alpha\} \times^{\text{RLD}} b$ where $\alpha \in \mathcal{U}$ and b is an atomic f.o.

Definition 151. A *co-complete* reloid is a reloid representable as join of direct products $a \times^{\text{RLD}} \{\beta\}$ where $\beta \in \mathcal{U}$ and a is an atomic f.o.

I will denote the sets of complete and co-complete reloids correspondingly as CompRLD and CoCompRLD .

Obvious 152. Complete and co-complete are dual.

Obvious 153. Complete and co-complete reloids are convex.

Obvious 154. Discrete reloids are complete and co-complete.

Conjecture 155. If a reloid is both complete and co-complete then it is discrete.

Conjecture 156. Composition of complete reloids is complete.

Obvious 157. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 158. ComplRLD (with the induced order) is a complete lattice.

Definition 159. *Completion* and *co-completion* of a reloid f are defined by the formulas:

$$\text{Compl } f = \text{Cor}^{(\text{RLD}; \text{ComplRLD})} f \quad \text{and} \quad \text{CoCompl } f = \text{Cor}^{(\text{RLD}; \text{CoComplRLD})} f.$$

Theorem 160. Atoms of the lattice ComplRLD are exactly direct products of the form $\{\alpha\} \times^{\text{RLD}} b$ where $\alpha \in \mathcal{U}$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\text{FCD}} b$ are elements of ComplRLD . Also \emptyset is an element of ComplRLD .

$\{\alpha\} \times^{\text{RLD}} b$ are atoms of ComplFCD because these are atoms of RLD .

Remain to prove that if f is an atom of ComplRLD then $f = \{\alpha\} \times^{\text{RLD}} b$ for some $\alpha \in \mathcal{U}$ and an atomic f.o. b .

Suppose f is a non-empty complete reloid. Then $\{\alpha\} \times^{\text{RLD}} b \subseteq f$ for some $\alpha \in \mathcal{U}$ and atomic f.o. b . If f is an atom then $f = \{\alpha\} \times^{\text{FCD}} b$. \square

Obvious 161. ComplRLD is an atomistic lattice.

Conjecture 162. $\text{Compl } f \cap^{\text{RLD}} \text{Compl } g = \text{Compl}(f \cap^{\text{RLD}} g)$ for every reloids f and g .

Conjecture 163. $\text{Compl}(\bigcup^{\text{RLD}} R) = \bigcup^{\text{RLD}} \langle \text{Compl} \rangle R$ for every set R of reloids.

Conjecture 164. $\text{Compl } \text{CoCompl } f = \text{CoCompl } \text{Compl } f = \text{Cor } f$ for every reloid f .

Question 165. Is ComplRLD a distributive lattice? Is ComplRLD a co-brouwerian lattice?

Conjecture 166. If f is a complete reloid and R is a set of reloids then

$$f \circ \bigcup^{\text{RLD}} R = \bigcup^{\text{RLD}} \langle f \circ \rangle R.$$

This conjecture can be weakened:

Conjecture 167. If f is a discrete reloid and R is a set of reloids then

$$f \circ \bigcup^{\text{RLD}} R = \bigcup^{\text{RLD}} \langle f \circ \rangle R.$$

Conjecture 168. $\text{Compl } f = f \setminus^{*\text{RLD}} (\Omega \times^{\text{RLD}} \mathcal{U})$ for every reloid f .

5 Relationships of funcoids and reloids

5.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\text{FCD})f$ by the following formulas (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$):

$$\begin{aligned} \mathcal{X}[(\text{FCD})f]\mathcal{Y} &\Leftrightarrow \forall F \in \text{up } f: \mathcal{X}[F]\mathcal{Y} \\ \langle (\text{FCD})f \rangle \mathcal{X} &= \bigcap^{\mathfrak{F}} \{ \langle F \rangle \mathcal{X} \mid F \in \text{up } f \}. \end{aligned}$$

We should prove that $(\text{FCD})f$ is really a funcoid.

Proof. We need to prove that

$$\mathcal{X}[(\text{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle (\text{FCD})f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle (\text{FCD})f^{-1} \rangle \mathcal{Y} \neq \emptyset.$$

The above formula is equivalent to:

$$\forall F \in \text{up } f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{ \langle F \rangle \mathcal{X} \mid F \in \text{up } f \} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{ \langle F^{-1} \rangle \mathcal{Y} \mid F \in \text{up } f \} \neq \emptyset.$$

We have $\mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{\langle F \rangle \mathcal{X} \mid F \in \text{up } f\} = \bigcap^{\mathfrak{F}} \{\mathcal{Y} \cap^{\mathfrak{F}} \langle F \rangle \mathcal{X} \mid F \in \text{up } f\}$.

Let's denote $W = \{\mathcal{Y} \cap^{\mathfrak{F}} \langle F \rangle \mathcal{X} \mid F \in \text{up } f\}$.

$\forall F \in \text{up } f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \forall F \in \text{up } f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \emptyset \notin W$.

We need to prove that $\emptyset \notin W \Leftrightarrow \bigcap^{\mathfrak{F}} W \neq \emptyset$. (The rest follows from symmetry.)

This follows from the fact that W is a generalized filter base.

Let's prove that W is a generalized filter base. For this enough to prove that $V = \{\langle F \rangle \mathcal{X} \mid F \in \text{up } f\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \langle P \rangle \mathcal{X}$, $\mathcal{B} = \langle Q \rangle \mathcal{X}$ where $P, Q \in \text{up } f$. Then for $\mathcal{C} = \langle P \cap Q \rangle \mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base. \square

Proposition 169. $(\text{FCD})f = f$ for every binary relation f .

Proof. $\mathcal{X}[(\text{FCD})f]\mathcal{Y} \Leftrightarrow \forall F \in \text{up } f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{X}[f]\mathcal{Y}$ (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$). \square

Theorem 170. $\mathcal{X}[(\text{FCD})f]\mathcal{Y} \Leftrightarrow (\mathcal{X} \times^{\text{RLD}} \mathcal{Y}) \cap^{\text{RLD}} f \neq \emptyset$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$ and $f \in \text{RLD}$.

Proof.

$$\begin{aligned} (\mathcal{X} \times^{\text{RLD}} \mathcal{Y}) \cap^{\text{RLD}} f \neq \emptyset &\Leftrightarrow \forall F \in \text{up } f, P \in \text{up}(\mathcal{X} \times^{\text{RLD}} \mathcal{Y}): P \cap F \neq \emptyset \\ &\Leftrightarrow \forall F \in \text{up } f, X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: (X \times^{\text{RLD}} Y) \cap^{\text{RLD}} F \neq \emptyset \\ &\Leftrightarrow \forall F \in \text{up } f, X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: (X \times Y) \cap F \neq \emptyset \\ &\Leftrightarrow \forall F \in \text{up } f, X \in \text{up } \mathcal{X}, Y \in \text{up } \mathcal{Y}: X[F]Y \\ &\Leftrightarrow \forall F \in \text{up } f: \mathcal{X}[F]\mathcal{Y} \\ &\Leftrightarrow \mathcal{X}[(\text{FCD})f]\mathcal{Y}. \end{aligned}$$

\square

Theorem 171. $(\text{FCD})f = \bigcap^{\text{FCD}} \text{up } f$ for every reloid f .

Proof. Let a is an atomic filter object.

$((\text{FCD})f)a = \bigcap^{\mathfrak{F}} \{\langle F \rangle a \mid F \in \text{up } f\}$ by the definition of (FCD).

$\langle \bigcap^{\text{FCD}} \text{up } f \rangle a = \bigcap^{\mathfrak{F}} \{\langle F \rangle a \mid F \in \text{up } f\}$ by the theorem 55.

So $\langle (\text{FCD})f \rangle a = \langle \bigcap^{\text{FCD}} \text{up } f \rangle a$ for every atomic filter object a . \square

Lemma 172. $\langle g \rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle S$ if g is a funcoid and S is a filter base.

Proof. $\text{up} \bigcap^{\mathfrak{F}} S = \bigcup \langle \text{up} \rangle S$ by the theorem 3.

$\langle g \rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle \text{up} \bigcap^{\mathfrak{F}} S$ by the theorem 32.

$\bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle \text{up} \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle \bigcup \langle \text{up} \rangle S$.

Easy to see that $\bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle \bigcup \langle \text{up} \rangle S = \bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle S$ because $S \subseteq \bigcup \langle \text{up} \rangle S$.

Combining these equalities we produce $\langle g \rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g \rangle \rangle S$. \square

Lemma 173. For every two filter bases S and T of binary relations and every set A

$$\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T \Rightarrow \bigcap^{\mathfrak{F}} \{\langle F \rangle A \mid F \in S\} = \bigcap^{\mathfrak{F}} \{\langle G \rangle A \mid G \in T\}$$

Proof. Let $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T$.

First let prove that $\{\langle F \rangle A \mid F \in S\}$ is a filter base. Let $X, Y \in \{\langle F \rangle A \mid F \in S\}$. Then $X = \langle F_X \rangle A$ and $Y = \langle F_Y \rangle A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \subseteq F_X \cap F_Y$. So $\langle F_Z \rangle A \subseteq X \cap Y$ and $\langle F_Z \rangle A \in \{\langle F \rangle A \mid F \in S\}$. So $\{\langle F \rangle A \mid F \in S\}$ is a filter base.

Suppose $X \in \text{up} \bigcap^{\mathfrak{F}} \{\langle F \rangle A \mid F \in S\}$. Then exists $X' \in \{\langle F \rangle A \mid F \in S\}$ where $X \supseteq X'$ because $\{\langle F \rangle A \mid F \in S\}$ is a filter base. That is $X' = \langle F \rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$ because T is a filter base. Let $Y' = \langle G \rangle A$. We have $Y' \subseteq X' \subseteq X$; $Y' \in \{\langle G \rangle A \mid G \in T\}$; $Y' \in \text{up} \bigcap^{\mathfrak{F}} \{\langle G \rangle A \mid G \in T\}$; $X \in \text{up} \bigcap^{\mathfrak{F}} \{\langle G \rangle A \mid G \in T\}$. The reverse is symmetric. \square

Lemma 174. $\{G \circ F \mid F \in \text{up } f, G \in \text{up } g\}$ is a filter base for every reloids f and g .

Proof. Let denote $D = \{G \circ F \mid F \in \text{up } f, G \in \text{up } g\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \text{up } f$ and $G_A, G_B \in \text{up } g$. So $A \cap B \supseteq (G_A \cap G_B) \circ (F_A \cap F_B) \in D$ because $F_A \cap F_B \in \text{up } f$ and $G_A \cap G_B \in \text{up } g$. \square

Theorem 175. $(\text{FCD})(g \circ f) = ((\text{FCD})g) \circ ((\text{FCD})f)$ for every reloids f and g .

Proof.

$$\begin{aligned} \langle (\text{FCD})(g \circ f) \rangle X &= \bigcap^{\mathfrak{F}} \{ \langle H \rangle X \mid H \in \text{up}(g \circ f) \} \\ &= \bigcap^{\mathfrak{F}} \{ \langle H \rangle X \mid H \in \text{up} \bigcap^{\text{RLD}} \{ G \circ F \mid F \in \text{up } f, G \in \text{up } g \} \}. \end{aligned}$$

Obviously

$$\bigcap^{\text{RLD}} \{ G \circ F \mid F \in \text{up } f, G \in \text{up } g \} = \bigcap^{\text{RLD}} \text{up} \bigcap^{\text{RLD}} \{ G \circ F \mid F \in \text{up } f, G \in \text{up } g \};$$

from this by the lemma 173 (taking in account that $\{G \circ F \mid F \in \text{up } f, G \in \text{up } g\}$ and $\bigcap^{\text{RLD}} \{G \circ F \mid F \in \text{up } f, G \in \text{up } g\}$ are filter bases)

$$\bigcap^{\mathfrak{F}} \{ \langle H \rangle X \mid H \in \text{up} \bigcap^{\text{RLD}} \{ G \circ F \mid F \in \text{up } f, G \in \text{up } g \} \} = \bigcap^{\mathfrak{F}} \{ \langle G \circ F \rangle X \mid F \in \text{up } f, G \in \text{up } g \}.$$

On the other side

$$\begin{aligned} \langle ((\text{FCD})g) \circ ((\text{FCD})f) \rangle X &= \langle (\text{FCD})g \rangle \langle (\text{FCD})f \rangle X \\ &= \langle (\text{FCD})g \rangle \bigcap^{\mathfrak{F}} \{ \langle F \rangle X \mid F \in \text{up } f \} \\ &= \bigcap^{\mathfrak{F}} \{ \langle G \rangle \bigcap^{\mathfrak{F}} \{ \langle F \rangle X \mid F \in \text{up } f \} \mid G \in \text{up } g \}. \end{aligned}$$

Let's prove that $\{ \langle F \rangle X \mid F \in \text{up } f \}$ is a filter base. If $A, B \in \{ \langle F \rangle X \mid F \in \text{up } f \}$ then $A = \langle F_1 \rangle X$ and $B = \langle F_2 \rangle X$ where $F_1, F_2 \in \text{up } f$. $A \cap B \supseteq \langle F_1 \cap F_2 \rangle X \in \{ \langle F \rangle X \mid F \in \text{up } f \}$. So $\{ \langle F \rangle X \mid F \in \text{up } f \}$ is really a filter base.

By the lemma 172 $\langle G \rangle \bigcap^{\mathfrak{F}} \{ \langle F \rangle X \mid F \in \text{up } f \} = \bigcap^{\mathfrak{F}} \{ \langle G \rangle \langle F \rangle X \mid F \in \text{up } f \}$. So continuing the above equalities,

$$\begin{aligned} \langle ((\text{FCD})g) \circ ((\text{FCD})f) \rangle X &= \bigcap^{\mathfrak{F}} \{ \bigcap^{\mathfrak{F}} \{ \langle G \rangle \langle F \rangle X \mid F \in \text{up } f \} \mid G \in \text{up } g \} \\ &= \bigcap^{\mathfrak{F}} \{ \langle G \rangle \langle F \rangle X \mid F \in \text{up } f, G \in \text{up } g \} \\ &= \bigcap^{\mathfrak{F}} \{ \langle G \circ F \rangle X \mid F \in \text{up } f, G \in \text{up } g \}. \end{aligned}$$

Combining these equalities we get $\langle (\text{FCD})(g \circ f) \rangle X = \langle ((\text{FCD})g) \circ ((\text{FCD})f) \rangle X$ for every set X . \square

Proposition 176. $(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B}) = \mathcal{A} \times^{\text{FCD}} \mathcal{B}$ for every f.o. \mathcal{A} and \mathcal{B} .

Proof. $\mathcal{X}[(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B})] \mathcal{Y} \Leftrightarrow \forall F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}): \mathcal{X}[F] \mathcal{Y}$ (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$).

Evidently $\forall F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}): \mathcal{X}[F] \mathcal{Y} \Rightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}: \mathcal{X}[A \times B] \mathcal{Y}$.

Let $\forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}: \mathcal{X}[A \times B] \mathcal{Y}$. Then if $F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B})$ then there are $A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}$ such that $F \supseteq A \times B$. So $\mathcal{X}[F] \mathcal{Y}$.

We proved $\forall F \in \text{up}(\mathcal{A} \times^{\text{RLD}} \mathcal{B}): \mathcal{X}[F] \mathcal{Y} \Leftrightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}: \mathcal{X}[A \times B] \mathcal{Y}$.

Further $\forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}: \mathcal{X}[A \times B] \mathcal{Y} \Leftrightarrow \forall A \in \text{up } \mathcal{A}, B \in \text{up } \mathcal{B}: \mathcal{X} \cap^{\mathfrak{F}} A \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} B \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset \Leftrightarrow \mathcal{X}[\mathcal{A} \times^{\text{FCD}} \mathcal{B}] \mathcal{Y}$.

Thus $\mathcal{X}[(\text{FCD})(\mathcal{A} \times^{\text{RLD}} \mathcal{B})] \mathcal{Y} \Leftrightarrow \mathcal{X}[\mathcal{A} \times^{\text{FCD}} \mathcal{B}] \mathcal{Y}$. \square

5.2 Reloids induced by funcoid

Every funcoid f induces a reloid in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$\begin{aligned} (\text{RLD})_{\text{out}} f &\stackrel{\text{def}}{=} \bigcap^{\text{RLD}} \text{up } f; \\ (\text{RLD})_{\text{in}} f &\stackrel{\text{def}}{=} \bigcup^{\text{RLD}} \{ \mathcal{A} \times^{\text{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\text{FCD}} \mathcal{B} \subseteq f \} \end{aligned}$$

Theorem 177. $(\text{RLD})_{\text{in}}f = \bigcup^{\text{RLD}} \{a \times^{\text{RLD}} b \mid a, b \in \text{atoms}^{\mathfrak{U}}, a \times^{\text{FCD}} b \subseteq f\}$.

Proof. Follows from the theorem 129. \square

Remark 178. It seems that $(\text{RLD})_{\text{in}}$ has smoother properties and is more important than $(\text{RLD})_{\text{out}}$. (However see also the exercise below for $(\text{RLD})_{\text{in}}$ not preserving identities.)

Proposition 179. $(\text{RLD})_{\text{out}}f = f$ for every binary relation f .

Proof. $(\text{RLD})_{\text{out}}f = \bigcap^{\text{RLD}} \text{up } f = \min \text{up } f = f$. \square

Lemma 180. $F \in \text{up } (\text{RLD})_{\text{in}}f \Leftrightarrow \forall a, b \in \text{atoms}^{\mathfrak{U}}: (a[f]b \Rightarrow F \supseteq a \times^{\text{RLD}} b)$ for a funcoïd f .

Proof.

$$\begin{aligned} F \in \text{up } (\text{RLD})_{\text{in}}f &\Leftrightarrow F \in \text{up } \bigcup^{\mathfrak{U}} \{a \times^{\text{RLD}} b \mid a, b \in \text{atoms}^{\mathfrak{U}}, a \times^{\text{FCD}} b \subseteq f\} \\ &\Leftrightarrow \forall a, b \in \text{atoms}^{\mathfrak{U}}: (a \times^{\text{FCD}} b \subseteq f \Rightarrow F \in \text{up}(a \times^{\text{RLD}} b)) \\ &\Leftrightarrow \forall a, b \in \text{atoms}^{\mathfrak{U}}: ((a \times^{\text{FCD}} b) \cap^{\text{FCD}} f \neq \emptyset \Rightarrow F \supseteq a \times^{\text{RLD}} b) \\ &\Leftrightarrow \forall a, b \in \text{atoms}^{\mathfrak{U}}: (a[f]b \Rightarrow F \supseteq a \times^{\text{RLD}} b). \end{aligned}$$

\square

Surprisingly a funcoïd is greater inward than outward:

Theorem 181. $(\text{RLD})_{\text{out}}f \subseteq (\text{RLD})_{\text{in}}f$ for a funcoïd f .

Proof. We need to prove

$$\bigcap^{\text{RLD}} \text{up } f \subseteq \bigcup^{\text{RLD}} \{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\text{FCD}} \mathcal{B} \subseteq f\}.$$

Let

$$K \in \text{up } \bigcup^{\mathfrak{U}} \{\mathcal{A} \times^{\text{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\text{FCD}} \mathcal{B} \subseteq f\}.$$

Then

$$\begin{aligned} K &= \bigcup \{X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\text{FCD}} \mathcal{B} \subseteq f\} \\ &= \bigcup^{\text{RLD}} \{X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\text{FCD}} \mathcal{B} \subseteq f\} \\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}} \in \text{up } \mathcal{A}$, $Y_{\mathcal{B}} \in \text{up } \mathcal{B}$. So $K \in \text{up } f$; $K \supseteq \bigcap^{\text{RLD}} \text{up } f$; $K \in \text{up } \bigcap^{\text{RLD}} \text{up } f$. \square

Theorem 182. $(\text{FCD})(\text{RLD})_{\text{in}}f = f$ for every funcoïd f .

Proof. For every sets X and Y

$$\begin{aligned} X[(\text{FCD})(\text{RLD})_{\text{in}}f]Y &\Leftrightarrow \\ (X \times^{\text{RLD}} Y) \cap^{\text{RLD}} (\text{RLD})_{\text{in}}f \neq \emptyset &\Leftrightarrow \\ (X \times Y) \cap^{\text{RLD}} \bigcup^{\text{RLD}} \{a \times^{\text{RLD}} b \mid a, b \in \text{atoms}^{\mathfrak{U}}, a \times^{\text{FCD}} b \subseteq f\} &\Leftrightarrow \text{(theorem 52 in [13])} \\ \exists a, b \in \text{atoms}^{\mathfrak{U}}: (a \times^{\text{FCD}} b \subseteq f \wedge (X \times Y) \cap^{\text{RLD}} (a \times^{\text{RLD}} b) \neq \emptyset) &\Leftrightarrow \\ \exists a, b \in \text{atoms}^{\mathfrak{U}}: (a[f]b \subseteq f \wedge a \subseteq X \wedge b \subseteq Y) &\Leftrightarrow \\ \exists a \in \text{atoms}^{\mathfrak{U}} X, b \in \text{atoms}^{\mathfrak{U}} Y: a[f]b &\Leftrightarrow \\ X[f]Y. & \end{aligned}$$

Thus $(\text{FCD})(\text{RLD})_{\text{in}}f = f$. \square

Obvious 183. $(\text{RLD})_{\text{in}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for every f.o. \mathcal{A}, \mathcal{B} .

Question 184. $(\text{RLD})_{\text{out}}(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathcal{A} \times^{\text{RLD}} \mathcal{B}$ for every f.o. \mathcal{A}, \mathcal{B} ?

Conjecture 185. $(\text{RLD})_{\text{out}} I_{\mathcal{A}}^{\text{FCD}} = I_{\mathcal{A}}^{\text{RLD}}$ for every f.o. \mathcal{A} .

Exercise 1. Prove that generally $(\text{RLD})_{\text{in}} I_{\mathcal{A}}^{\text{FCD}} \neq I_{\mathcal{A}}^{\text{RLD}}$.

Remark 186. The above theorem allows to represent funcoids as reloids.

Conjecture 187. For a convex reloid f

1. $(\text{RLD})_{\text{out}}(\text{FCD})f = f$;
2. $(\text{RLD})_{\text{in}}(\text{FCD})f = f$.

6 Galois connections of funcoids and reloids

Theorem 188. (FCD) is the lower adjoint of $(\text{RLD})_{\text{in}}$.

Proof. Because (FCD) and $(\text{RLD})_{\text{in}}$ are trivially monotone, it's enough to prove

$$f \subseteq (\text{RLD})_{\text{in}}(\text{FCD})f \text{ and } (\text{FCD})(\text{RLD})_{\text{in}}g \subseteq g.$$

The second formula follows from the fact that $(\text{FCD})(\text{RLD})_{\text{in}}g = g$.

$$\begin{aligned} & (\text{RLD})_{\text{in}}(\text{FCD})f = \\ & \bigcup^{\text{RLD}} \{a \times^{\text{RLD}} b \mid a, b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}, a \times^{\text{FCD}} b \subseteq (\text{FCD})f\} = \\ & \bigcup^{\text{RLD}} \{a \times^{\text{RLD}} b \mid a, b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}, a[(\text{FCD})f]b\} = \\ & \bigcup^{\text{RLD}} \{a \times^{\text{RLD}} b \mid a, b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}, (a \times^{\text{RLD}} b) \cap^{\text{RLD}} f \neq \emptyset\} \supseteq \\ & \bigcup^{\text{RLD}} \{p \mid a, b \in \text{atoms}^{\mathfrak{F}} \mathcal{U}, p \in \text{atoms}^{\text{RLD}}(a \times^{\text{RLD}} b), p \cap^{\text{RLD}} f \neq \emptyset\} = \\ & \bigcup^{\text{RLD}} \{p \mid p \in \text{atoms}^{\text{RLD}}(\mathcal{U} \times \mathcal{U}), p \cap^{\text{RLD}} f \neq \emptyset\} = \\ & \bigcup^{\text{RLD}} \{p \mid p \in \text{atoms}^{\text{RLD}} f\} = f. \end{aligned}$$

□

Corollary 189.

1. $(\text{FCD}) \bigcup^{\text{RLD}} S = \bigcup^{\text{FCD}} \langle (\text{FCD}) \rangle S$ if S is a set of reloids.
2. $(\text{RLD})_{\text{in}} \bigcap^{\text{FCD}} S = \bigcap^{\text{RLD}} \langle (\text{RLD})_{\text{in}} \rangle S$ if S is a set of funcoids.

Conjecture 190. $(\text{RLD})_{\text{in}}$ is not a lower adjoint (in general).

Conjecture 191. $(\text{RLD})_{\text{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

See also corollary 251 below.

7 Continuous morphisms

This section will use the apparatus from the section “Partially ordered dagger categories”.

7.1 Traditional definitions of continuity

7.1.1 Pre-topology

Let μ and ν are funcoids representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall \epsilon \in \text{up} \langle \nu \rangle f a \exists \delta \in \text{up} \langle \mu \rangle \{a\}: \langle f \rangle \delta \subseteq \epsilon.$$

Equivalently transforming this formula we get:

$$\begin{aligned} \forall \epsilon \in \text{up} \langle \nu \rangle f a: \langle f \rangle \langle \mu \rangle \{a\} \subseteq \epsilon; \\ \langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle f a; \\ \langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle \langle f \rangle \{a\}; \\ \langle f \circ \mu \rangle \{a\} \subseteq \langle \nu \circ f \rangle \{a\}. \end{aligned}$$

So f is a continuous map from μ to ν in every point of its domain iff $f \circ \mu \subseteq \nu \circ f$.

7.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a nearness-continuous map from μ to ν iff

$$\forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow (\langle f \rangle X)[\nu](\langle f \rangle Y)).$$

Equivalently transforming this formula we get:

$$\begin{aligned} \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y \cap \langle \nu \rangle \langle f \rangle X \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y \cap \langle \nu \circ f \rangle X \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X[\nu \circ f] \langle f \rangle Y); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y[(\nu \circ f)^{-1}]X); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y[f^{-1} \circ \nu^{-1}]X); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X \cap \langle f^{-1} \circ \nu^{-1} \rangle \langle f \rangle Y \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X \cap \langle f^{-1} \circ \nu^{-1} \circ f \rangle Y \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow Y[f^{-1} \circ \nu^{-1} \circ f]X); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X[f^{-1} \circ \nu \circ f]Y); \\ \mu \subseteq f^{-1} \circ \nu \circ f. \end{aligned}$$

So a function f is nearness-continuous iff $\mu \subseteq f^{-1} \circ \nu \circ f$.

7.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu \forall (x; y) \in \delta: (fx; fy) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\begin{aligned} \forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu \forall (x; y) \in \delta: \{(fx; fy)\} \subseteq \epsilon \\ \forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu \forall (x; y) \in \delta: f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon \\ \forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu: f \circ \delta \circ f^{-1} \subseteq \epsilon \\ \forall \epsilon \in \text{up} \nu: f \circ \mu \circ f^{-1} \subseteq \epsilon \\ f \circ \mu \circ f^{-1} \subseteq \nu. \end{aligned}$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \subseteq \nu$.

7.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in C(\mu; \nu) \Leftrightarrow f \in \text{Mor}(\text{Ob } \mu; \text{Ob } \nu) \wedge f \circ \mu \subseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$\begin{aligned} f \in C'(\mu; \nu) &\Leftrightarrow f \in \text{Mor}(\text{Ob } \mu; \text{Ob } \nu) \wedge \mu \subseteq f^\dagger \circ \nu \circ f; \\ f \in C''(\mu; \nu) &\Leftrightarrow f \in \text{Mor}(\text{Ob } \mu; \text{Ob } \nu) \wedge f \circ \mu \circ f^\dagger \subseteq \nu. \end{aligned}$$

Remark 192. In the examples about funcoids and reloids the “dagger functor” is the inverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

Proposition 193. Every of these three definitions of continuity forms a sub-precategory (sub-category if the original precategory is a category).

Proof.

C. Let $f \in C(\mu; \nu)$, $g \in C(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in C(\mu; \pi)$. $1_{\text{Ob } \mu} \in C(\mu; \mu)$ is obvious.

C'. Let $f \in C'(\mu; \nu)$, $g \in C'(\nu; \pi)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$, $\nu \subseteq g^\dagger \circ \pi \circ g$;

$$\mu \subseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in C'(\mu; \pi)$. $1_{\text{Ob } \mu} \in C'(\mu; \mu)$ is obvious.

C''. Let $f \in C''(\mu; \nu)$, $g \in C''(\nu; \pi)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$, $g \circ \nu \circ g^\dagger \subseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \subseteq \pi.$$

So $g \circ f \in C''(\mu; \pi)$. $1_{\text{Ob } \mu} \in C''(\mu; \mu)$ is obvious. \square

Proposition 194. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in C'(\mu; \nu) \Rightarrow f \in C(\mu; \nu) \Rightarrow f \in C''(\mu; \nu).$$

Proof. Let $f \in C'(\mu; \nu)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$; $f \circ \mu \subseteq f \circ f^\dagger \circ \nu \circ f \subseteq 1_{\text{Dst } f} \circ \nu \circ f = \nu \circ f$; $f \in C(\mu; \nu)$.

Let $f \in C(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f \circ \mu \circ f^\dagger \subseteq \nu \circ f \circ f^\dagger \subseteq \nu \circ 1_{\text{Dst } f} = \nu$; $f \in C''(\mu; \nu)$. \square

Proposition 195. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in C''(\mu; \nu) \Rightarrow f \in C(\mu; \nu) \Rightarrow f \in C'(\mu; \nu).$$

Proof. Let $f \in C''(\mu; \nu)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$; $f \circ \mu \circ f^\dagger \circ f \subseteq \nu \circ f$; $f \circ \mu \circ 1_{\text{Src } f} \subseteq \nu \circ f$; $f \circ \mu \subseteq \nu \circ f$; $f \in C(\mu; \nu)$.

Let $f \in C(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f^\dagger \circ f \circ \mu \subseteq f^\dagger \circ \nu \circ f$; $1_{\text{Src } f} \circ \mu \subseteq f^\dagger \circ \nu \circ f$; $\mu \subseteq f^\dagger \circ \nu \circ f$; $f \in C'(\mu; \nu)$. \square

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 196. If f is a monovalued and entirely defined morphism then

$$f \in C'(\mu; \nu) \Leftrightarrow f \in C(\mu; \nu) \Leftrightarrow f \in C''(\mu; \nu).$$

Proof. From two previous propositions. \square

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 197. If an entirely defined morphism of the category of reloids $f \in C''(\mu; \nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\text{FCD})f \in C'((\text{FCD})\mu; (\text{FCD})\nu)$.

Exercise 2. I leave a simple exercise for the reader to prove the last theorem.

7.3 Continuousness of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of functors or semigroup of relocks regarding the composition.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to *identity element* I_A of the semigroup (for example identity functor or identity relock). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is “dagger” (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define *restricting* an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for functors and for relocks):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 198. For every elements f, μ, ν of our semigroup and an object A

1. $f \in C(\mu; \nu) \Rightarrow f|_A \in C(I_A \circ \mu \circ I_A; \nu)$;
2. $f \in C'(\mu; \nu) \Rightarrow f|_A \in C'(I_A \circ \mu \circ I_A; \nu)$;
3. $f \in C''(\mu; \nu) \Rightarrow f|_A \in C''(I_A \circ \mu \circ I_A; \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof.

1. $f|_A \in C(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftrightarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftrightarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in C(\mu; \nu)$.
2. $f|_A \in C'(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^\dagger \circ \nu \circ f|_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow \mu \subseteq f^\dagger \circ \nu \circ f \Leftrightarrow f \in C'(\mu; \nu)$.
3. $f|_A \in C''(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ \mu \circ f^\dagger \subseteq \nu \Leftrightarrow f \in C''(\mu; \nu)$. \square

8 Connectedness regarding functors and relocks

8.1 Some lemmas

Lemma 199. If $\neg(A[f]B) \wedge A \cup B \supseteq \text{dom } f \cup \text{im } f$ then f is closed on A for a functor f and sets A and B .

Proof. $\neg(A[f]B) \Leftrightarrow B \cap \langle f \rangle A = \emptyset \Leftrightarrow (\text{dom } f \cup \text{im } f) \cap B \cap \langle f \rangle A = \emptyset \Rightarrow ((\text{dom } f \cup \text{im } f) \setminus A) \cap \langle f \rangle A = \emptyset \Leftrightarrow \langle f \rangle A \subseteq A$. \square

Corollary 200. If $\neg(A[f]B) \wedge A \cup B \supseteq \text{dom } f \cup \text{im } f$ then f is closed on $A \setminus B$ for a functor f and sets A and B .

Proof. Let $\neg(A[f]B) \wedge A \cup B \supseteq \text{dom } f \cup \text{im } f$. Then $\neg((A \setminus B)[f]B) \wedge (A \setminus B) \cup B \supseteq \text{dom } f \cup \text{im } f$. \square

Lemma 201. If $\neg(A[f]B) \wedge A \cup B \supseteq \text{dom } f \cup \text{im } f$ then $\neg(A[f^n]B)$ for every whole positive n .

Proof. Let $\neg(A[f]B) \wedge A \cup B \supseteq \text{dom } f \cup \text{im } f$. From the above proposition $\langle f \rangle A \subseteq A$. $B \cap \langle f \rangle A = \emptyset$, consequently $\langle f \rangle A \subseteq A \setminus B$. Because (by the above corollary) f is closed on $A \setminus B$, then $\langle f \rangle \langle f \rangle A \subseteq A \setminus B$, $\langle f \rangle \langle f \rangle \langle f \rangle A \subseteq A \setminus B$, etc. So $\langle f^n \rangle A \subseteq A \setminus B$, $B \cap \langle f^n \rangle A = \emptyset$, $\neg(A[f^n]B)$. \square

8.2 Endomorphism series

Definition 202. $S_1(\mu) \stackrel{\text{def}}{=} \mu \cup \mu^2 \cup \mu^3 \cup \dots$ for an endomorphism μ of a precategory with countable union of morphisms.

Definition 203. $S(\mu) \stackrel{\text{def}}{=} \mu^0 \cup S_1(\mu)$ where $\mu^0 \stackrel{\text{def}}{=} I_{\text{Ob } \mu}$ (identity morphism for the object $\text{Ob } \mu$) where $\text{Ob } \mu$ is the object of endomorphism μ for an endomorphism μ of a category with countable union of morphisms.

I call S_1 and S *endomorphism series*.

We will consider the collection of all binary relations (on a set \mathcal{U}), as well as the collection of all functors and the collection of all relops, as categories with single object \mathcal{U} and the identity morphism $(=)$ or $(=)|_{\mathcal{U}}$.

So if μ is a binary relation or a functor or a reloid we have

$$S_1(\mu) = \mu \cup \mu^2 \cup \mu^3 \cup \dots \text{ and } S(\mu) = (=) \cup \mu \cup \mu^2 \cup \mu^3 \cup \dots$$

Proposition 204. $S(\mu)$ is transitive for the category of binary relations.

Proof.

$$\begin{aligned} S(\mu) \circ S(\mu) &= \mu^0 \circ S(\mu) \cup \mu \circ S(\mu) \cup \mu^2 \circ S(\mu) \cup \dots \\ &= (\mu^0 \cup \mu^1 \cup \mu^2 \cup \dots) \cup (\mu^1 \cup \mu^2 \cup \mu^3 \cup \dots) \cup (\mu^2 \cup \mu^3 \cup \mu^4 \cup \dots) \\ &= \mu^0 \cup \mu^1 \cup \mu^2 \cup \dots \\ &= S(\mu). \end{aligned}$$

\square

8.3 Connectedness regarding binary relations

Before going to research connectedness for functors and relops we will excuse into the basic special case of connectedness regarding binary relations.

Definition 205. A set A is called (*strongly*) *connected* regarding a binary relation μ when

$$\forall X, Y \in \mathcal{P}\mathcal{U} \setminus \{\emptyset\}: (X \cup Y = A \Rightarrow X[\mu]Y).$$

Definition 206. *Path* between two elements $a, b \in \mathcal{U}$ in a set A through binary relation μ is the finite sequence $x_0 \dots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i(\mu \cap A \times A)x_{i+1}$ for every $i = 0, \dots, n - 1$. n is called *path length*.

Proposition 207. There exists path between every element $a \in \mathcal{U}$ and that element itself.

Proof. It is the path consisting of one vertex (of length 0). \square

Proposition 208. There is a path from element a to element b in a set A through a binary relation μ iff $a(S(\mu \cap A \times A))b$ (that is $(a, b) \in S(\mu \cap A \times A)$).

Proof.

- \Rightarrow . If exists a path from a to b , then $\{b\} \subseteq \langle (\mu \cap A \times A)^n \rangle \{a\}$ where n is the path length. Consequently $\{b\} \subseteq \langle S(\mu \cap A \times A) \rangle \{a\}$; $a (S(\mu \cap A \times A)) b$.
- \Leftarrow . If $a (S(\mu \cap A \times A)) b$ then exists $n \in \mathbb{N}$ such that $a (\mu \cap A \times A)^n b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \dots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i (\mu \cap A \times A) x_{i+1}$ for every $i = 0, \dots, n - 1$. That is there is path from a to b . \square

Theorem 209. The following statements are equivalent for a relation μ and a set A :

1. For every $a, b \in A$ there is a path between a and b in A through μ .
2. $S(\mu \cap A \times A) \supseteq A \times A$.
3. $S(\mu \cap A \times A) = A \times A$.
4. A is connected regarding μ .

Proof.

- (1) \Rightarrow (2). Let for every $a, b \in A$ there is a path between a and b in A through μ . Then $a (S(\mu \cap A \times A)) b$ for every $a, b \in A$. It is possible only when $S(\mu \cap A \times A) \supseteq A \times A$.
- (3) \Rightarrow (1). For every two vertices a and b we have $a (S(\mu \cap A \times A)) b$. So (by the previous theorem) for every two vertices a and b exist path from a to b .
- (3) \Rightarrow (4). Suppose that $\neg(X[\mu \cap A \times A]Y)$ for some $X, Y \in \mathcal{P}U \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg(X[(\mu \cap A \times A)^n]Y)$ for every $n \in \mathbb{N}$. Consequently $\neg(X[S(\mu \cap A \times A)]Y)$. So $S(\mu \cap A \times A) \neq A \times A$.
- (4) \Rightarrow (3). If $\langle S(\mu \cap A \times A) \rangle \{v\} = A$ for every vertex v then $S(\mu \cap A \times A) = A \times A$. Consider the remaining case when $V \stackrel{\text{def}}{=} \langle S(\mu \cap A \times A) \rangle \{v\} \subset A$ for some vertex v . Let $W = A \setminus V$. If $\text{card } A = 1$ then $S(\mu \cap A \times A) \supseteq (=) = A \times A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V[\mu]W$ what is equivalent to $V[\mu \cap A \times A]W$ that is $\langle \mu \cap A \times A \rangle V \cap W \neq \emptyset$. This is impossible because $\langle \mu \cap A \times A \rangle V = \langle \mu \cap A \times A \rangle \langle S(\mu \cap A \times A) \rangle V = \langle S_1(\mu \cap A \times A) \rangle V \subseteq \langle S(\mu \cap A \times A) \rangle V = V$.
- (2) \Rightarrow (3). Because $S(\mu \cap A \times A) \subseteq A \times A$. \square

Corollary 210. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap A \times A$.

Definition 211. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A .

Theorem 212. The set A is partitioned into connected components (regarding every binary relation F).

Proof. Consider the binary relation $a \sim b \Leftrightarrow a (S(F)) b \wedge b (S(F)) a$. \sim is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q . Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. \square

Proposition 213. A set is connected (regarding a binary relation) iff it has one connected component.

Proof. Direct implication is obvious. Reverse is proved by contradiction. \square

8.4 Connectedness regarding funcoids and reloids

Definition 214. $S_1^*(\mu) = \bigcap^{\mathfrak{F}} \{S_1(M) \mid M \in \text{up } \mu\}$ for a reloid μ .

Definition 215. *Connectivity reloid* $S^*(\mu)$ for a reloid μ is defined as follows:

$$S^*(\mu) = \bigcap^{\mathfrak{F}} \{S(M) \mid M \in \text{up } \mu\}.$$

Remark 216. Do not mess the word *connectivity* with the word *connectedness* which means being connected.¹

Proposition 217. $S^*(\mu) = (=) \cup^{\text{RLD}} S_1^*(\mu)$ for every reloid μ .

Proof. Follows from the theorem about distributivity of \cup regarding $\bigcap^{\mathfrak{F}}$ (see [13]). \square

Proposition 218. $S^*(\mu) = S(\mu)$ if μ is a binary relation.

Proof. $S^*(\mu) = \bigcap^{\mathfrak{F}} \{S(\mu)\} = S(\mu)$. \square

Definition 219. A filter \mathcal{A} is called *connected* regarding a reloid μ when $S^*(\mu \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{A})) \supseteq \mathcal{A} \times^{\text{RLD}} \mathcal{A}$.

Obvious 220. A filter object \mathcal{A} is connected regarding a reloid μ when $S^*(\mu \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{A})) = \mathcal{A} \times^{\text{RLD}} \mathcal{A}$.

Definition 221. A filter object \mathcal{A} is called *connected* regarding a funoid μ when

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\}: (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = \mathcal{A} \Rightarrow \mathcal{X}[\mu]\mathcal{Y}).$$

Proposition 222. A set A is connected regarding a funoid μ iff

$$\forall \mathcal{X}, \mathcal{Y} \in \mathcal{P}\mathcal{U} \setminus \{\emptyset\}: (X \cup Y = A \Rightarrow X[\mu]Y).$$

Proof.

\Rightarrow . Obvious.

\Leftarrow . Follows from co-separability of filter objects. \square

Theorem 223. The following are equivalent for every set A and binary relation μ :

1. A is connected regarding binary relation μ .
2. A is connected regarding μ considered as a reloid.
3. A is connected regarding μ considered as a funoid.

Proof.

(1) \Leftrightarrow (2). $S^*(\mu \cap^{\text{RLD}} (A \times^{\text{RLD}} A)) = S^*(\mu \cap A \times A) = S(\mu \cap A \times A)$. So $S^*(\mu \cap^{\text{RLD}} A \times^{\text{RLD}} A) \supseteq A \times^{\text{RLD}} A \Leftrightarrow S(\mu \cap A \times A) \supseteq A \times A$.

(1) \Leftrightarrow (3). Follows from the previous proposition. \square

Next is conjectured a statement more strong than the above theorem:

Conjecture 224. A filter object is connected regarding a binary relation considered as a funoid iff it is connected regarding this binary relation considered as a reloid.

Obvious 225. A filter object is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \cap^{\text{RLD}} (\mathcal{A} \times^{\text{RLD}} \mathcal{A})$.

Obvious 226. A filter object is connected regarding a funoid μ iff it is connected regarding the funoid $\mu \cap^{\text{FCD}} (\mathcal{A} \times^{\text{FCD}} \mathcal{A})$.

¹. In some math literature these two words are used interchangeably.

Theorem 227. A filter object \mathcal{A} is connected regarding a reloid f iff it is connected regarding every $F \in \text{up } f$ (considered as reloid).

Proof.

\Rightarrow . Obvious.

\Leftarrow . F is connected iff $S(F) = F^0 \cup F^1 \cup F^2 \cup \dots \supseteq \mathcal{A} \times^{\text{RLD}} \mathcal{A}$.

$$S^*(f) = \bigcap^{\mathfrak{F}} \{S(F) \mid F \in \text{up } f\} \supseteq \bigcap^{\mathfrak{F}} \{\mathcal{A} \times^{\text{RLD}} \mathcal{A} \mid F \in \text{up } f\} = \mathcal{A} \times^{\text{RLD}} \mathcal{A}. \quad \square$$

Conjecture 228. A filter object \mathcal{A} is connected regarding a funcooid μ iff \mathcal{A} is connected for every $F \in \text{up } \mu$ (considered as funcooid).

The above conjecture is open even for the case when \mathcal{A} is a set.

Conjecture 229. A filter object \mathcal{A} is connected regarding a reloid f iff it is connected regarding the funcooid (FCD) f .

The above conjecture is true in the special case of principal filters:

Proposition 230. A set A is connected regarding a reloid f iff it is connected regarding the funcooid (FCD) f .

Proof. The set A is connected regarding a reloid f iff it is connected regarding every $F \in \text{up } f$ that is when (taken in account that connectedness for F regarded as a reloid is the same as connectedness for F regarded as a funcooid)

$$\begin{aligned} \forall F \in \text{up } f \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\}: (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \mathcal{X}[F]\mathcal{Y}) &\Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} \forall F \in \text{up } f: (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \mathcal{X}[F]\mathcal{Y}) &\Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \forall F \in \text{up } f: \mathcal{X}[F]\mathcal{Y}) &\Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = A \Rightarrow \mathcal{X}[(\text{FCD})f]\mathcal{Y}) & \end{aligned}$$

that is when the set A is connected regarding the funcooid (FCD) f . \square

8.5 Algebraic properties of S and S^*

Theorem 231. $S^*(S^*(f)) = S^*(f)$ for every reloid f .

Proof. $S^*(S^*(f)) = \bigcap^{\mathfrak{F}} \{S(R) \mid R \in \text{up } S^*(f)\} \subseteq \bigcap^{\mathfrak{F}} \{S(R) \mid R \in \{S(F) \mid F \in \text{up } f\}\} = \bigcap^{\mathfrak{F}} \{S(S(F)) \mid F \in \text{up } f\} = \bigcap^{\mathfrak{F}} \{S(F) \mid F \in \text{up } f\} = S^*(f)$.

So $S^*(S^*(f)) \subseteq S^*(f)$. That $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. \square

Corollary 232. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for any reloid f .

Proof. Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.

But $S^*(S(f)) \subseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \subseteq S^*(S^*(f)) = S^*(f)$. \square

Conjecture 233. $S(S(f)) = S(f)$ for

1. every reloid f ;
2. every funcooid f .

Conjecture 234. For every reloid f

1. $S(f) \circ S(f) = S(f)$;
2. $S^*(f) \circ S^*(f) = S^*(f)$;
3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 235. $S(f) \circ S(f) = S(f)$ for every funcooid f .

9 Postface

9.1 Misc

See this Web page for my research plans: <http://www.mathematics21.org/agt-plans.html>

I deem that now two most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids.

Also a future research topic are n -ary (where n is an ordinal, or more generally an index set) funcoids and reloids (plain funcoids and reloids are binary by analogy with binary relations).

We should also research relationships between complete funcoids and complete reloids.

9.2 Pointfree funcoids and reloids

I have set wiki site <http://funcoids.wikidot.com> to write on that site the pointfree variant of the theory of funcoids and reloids (that is generalized funcoids on arbitrary lattices rather than funcoids on a lattice of sets as in this work).

However I consider for me research of pointfree funcoids and pointfree reloids a low priority project. (There are yet enough research topics in the point-set topology and I don't want to meddle into pointfree topology in foreseeable future.)

The work about pointfree funcoids and reloids seems being largely technical and boring. Pointfree theory of funcoids and reloids seems being a trivial generalization of the theory of point-set funcoids and reloids. It is not similar to the traditional pointfree topology which is not an obvious generalization of point-set topology.

But if someone indeed wishes to treat pointfree funcoids, please use the above mentioned wiki.

Appendix A Some counter-examples

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap^{\mathfrak{F}} \{(-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}.$$

Example 236. There exist a funcoid f and a set S of funcoids such that $f \cap^{\text{FCD}} \bigcup^{\text{FCD}} S \neq \bigcup^{\text{FCD}} \langle f \cap^{\text{FCD}} \rangle S$.

Proof. Let $f = \Delta \times^{\text{FCD}} \{0\}$ and $S = \{(\varepsilon; +\infty) \times^{\text{FCD}} \{0\} \mid \varepsilon > 0\}$. Then $f \cap^{\text{FCD}} \bigcup^{\text{FCD}} S = (\Delta \times^{\text{FCD}} \{0\}) \cap^{\text{FCD}} ((0; +\infty) \times^{\text{FCD}} \{0\}) = (\Delta \cap^{\text{FCD}} (0; +\infty)) \times^{\text{FCD}} \{0\} \neq \emptyset$ while $\bigcup^{\text{FCD}} \langle f \cap^{\text{FCD}} \rangle S = \bigcup^{\text{FCD}} \{\emptyset\} = \emptyset$. \square

Conjecture 237. There exist a set R of funcoids and a funcoid f such that $f \circ \bigcup^{\text{FCD}} R \neq \bigcup^{\text{FCD}} \langle f \circ \rangle R$.

Example 238. There exist a set R of funcoids and f.o. \mathcal{X} and \mathcal{Y} such that

1. $\mathcal{X}[\bigcup^{\text{FCD}} R]\mathcal{Y} \wedge \nexists f \in R: \mathcal{X}[f]\mathcal{Y}$;
2. $\langle \bigcup^{\text{FCD}} R \rangle \mathcal{X} \supset \bigcup^{\mathfrak{F}} \{\langle f \rangle \mathcal{X} \mid f \in R\}$.

Proof.

1. Let $\mathcal{X} = \Delta$ and $\mathcal{Y} = \mathbb{R}$. Let $R = \{(\varepsilon; +\infty) \times^{\text{FCD}} \mathbb{R} \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}$. Then $\bigcup^{\text{FCD}} R = (0; +\infty) \times^{\text{FCD}} \mathbb{R}$. So $\mathcal{X}[\bigcup^{\text{FCD}} R]\mathcal{Y}$ and $\forall f \in R: \neg(\mathcal{X}[f]\mathcal{Y})$.
2. With the same \mathcal{X} and R we have $\langle \bigcup^{\text{FCD}} R \rangle \mathcal{X} = \mathbb{R}$ and $\langle f \rangle \mathcal{X} = \emptyset$ for every $f \in R$, thus $\bigcup^{\mathfrak{F}} \{\langle f \rangle \mathcal{X} \mid f \in R\} = \emptyset$. \square

Theorem 239. For a f.o. a we have $a \times^{\text{RLD}} a \subseteq (=)|_{\mathcal{U}}$ only in the case if $a = \emptyset$ or a is a trivial atomic f.o. (that is an one-element set).

Proof. If $a \times^{\text{RLD}} a \subseteq (=)|_{\mathcal{U}}$ then exists $m \in \text{up}(a \times^{\text{RLD}} a)$ such that $m \subseteq (=)|_{\mathcal{U}}$. Consequently exist $A, B \in \text{up } a$ such that $A \times B \subseteq (=)|_{\mathcal{U}}$ what is possible only in the case when $A = B = a$ is an one-element set or empty set. \square

Corollary 240. Direct product (in the sense of reلودs) of non-trivial atomic filter objects is non-atomic.

Proof. Obviously $(a \times^{\text{RLD}} a) \cap^{\text{RLD}} (=)|_{\mathcal{U}} \neq \emptyset$ and $(a \times^{\text{RLD}} a) \cap^{\text{RLD}} (=)|_{\mathcal{U}} \subset a \times^{\text{RLD}} a$. \square

Example 241. There exist two atomic reلودs whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object and $x \in \mathcal{U}$. Then

$$(a \times \{x\}) \circ (\{x\} \times a) = \bigcap^{\mathfrak{F}} \{(A \times \{x\}) \circ (\{x\} \times A) \mid A \in \text{up } a\} = \bigcap^{\mathfrak{F}} \{A \times A \mid A \in \text{up } a\} = a \times a$$

is non-atomic despite of $a \times \{x\}$ and $\{x\} \times a$ are atomic. \square

Example 242. There exists non-monovalued atomic reلود.

Proof. From the previous example follows that the atomic reلود $\{x\} \times a$ is not monovalued. \square

Example 243. $(\text{RLD})_{\text{in}} f \neq (\text{RLD})_{\text{out}} f$ for a funcooid f .

Proof. Let $f = (=)|_{\mathcal{U}}$. Then $(\text{RLD})_{\text{in}} f = \bigcup^{\text{RLD}} \{a \times^{\text{RLD}} a \mid a \in \text{atoms}^{\mathfrak{F}} \mathcal{U}\}$ and $(\text{RLD})_{\text{out}} f = (=)|_{\mathcal{U}}$. But as we shown above $a \times^{\text{RLD}} a \not\subseteq (=)|_{\mathcal{U}}$ for non-trivial f.o. a , and so $(\text{RLD})_{\text{in}} f \not\subseteq (\text{RLD})_{\text{out}} f$. \square

Proposition 244. $(=)|_{\mathcal{U}} \cap^{\text{FCD}} (\mathcal{U} \times \mathcal{U} \setminus f) = (=)|_{\Omega} \neq \emptyset$ (where Ω is the Fréchet filter object).

Proof. Note that $\langle (=)|_{\Omega} \rangle \mathcal{X} = \mathcal{X} \cap^{\mathfrak{F}} \Omega$.

Let x is a non-trivial atomic f.o. If $X \in \text{up } x$ then $\text{card } X \geq 2$ (In fact, X is infinite but we don't need this.) and consequently $\langle g \rangle X = \mathcal{U}$. Thus $\langle g \rangle x = \mathcal{U}$. Consequently

$$\langle f \cap^{\text{FCD}} g \rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \langle g \rangle x = x \cap^{\mathfrak{F}} \mathcal{U} = x.$$

Also $\langle (=)|_{\Omega} \rangle x = x \cap^{\mathfrak{F}} \Omega = x$.

Let now x is a trivial f.o. Then $\langle f \rangle x = x$ and $\langle g \rangle x = \mathcal{U} \setminus x$. So

$$\langle f \cap^{\text{FCD}} g \rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \langle g \rangle x = x \cap^{\mathfrak{F}} (\mathcal{U} \setminus x) = x \cap (\mathcal{U} \setminus x) = \emptyset.$$

Also $\langle (=)|_{\Omega} \rangle x = x \cap^{\mathfrak{F}} \Omega = \emptyset$.

So $\langle f \cap^{\text{FCD}} g \rangle x = \langle (=)|_{\Omega} \rangle x$ for every atomic f.o. x . Thus $f \cap^{\text{FCD}} g = (=)|_{\Omega}$. \square

Example 245. There exist discrete funcooids f and g such that $f \cap^{\text{FCD}} g \neq f \cap g$.

Proof. From the proposition above. \square

Example 246. There exists funcooid h such that $\text{up } h$ is not a filter.

Proof. Consider the funcooid $h = (=)|_{\Omega}$. We have (from the proposition) that $f \in \text{up } h$ and $g \in \text{up } f$, but $f \cap g = \emptyset \notin \text{up } h$. \square

Example 247. There exists a funcooid $h \neq \emptyset$ such that $(\text{RLD})_{\text{out}} h = \emptyset$.

Proof. Consider $h = (=)|_{\Omega}$. By proved above $h = f \cap^{\text{FCD}} g$ where $f = (=)|_{\mathcal{U}}$ and $g = \mathcal{U} \times \mathcal{U} \setminus f$.

We have $f, g \in \text{up } h$.

So $(\text{RLD})_{\text{out}}h = \bigcap^{\text{RLD}} \text{up } h \subseteq f \cap^{\text{RLD}} g = f \cap g = \emptyset$; and thus $(\text{RLD})_{\text{out}}h = \emptyset$. \square

Example 248. There exists a funcoid h such that $(\text{FCD})(\text{RLD})_{\text{out}}h \neq h$.

Proof. Follows from the previous example. \square

Example 249. There exist funcoids f and g such that

$$(\text{RLD})_{\text{out}}(g \circ f) \neq (\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f.$$

Proof. Take $f = (=)|_{\Omega}$ and $g = \mathcal{U} \times^{\text{FCD}} \{\alpha\}$ for some $\alpha \in \mathcal{U}$. Then $(\text{RLD})_{\text{out}}f = \emptyset$ and thus $(\text{RLD})_{\text{out}}g \circ (\text{RLD})_{\text{out}}f = \emptyset$.

We have $g \circ f = \Omega \times^{\text{FCD}} \{\alpha\}$.

Let's prove $(\text{RLD})_{\text{out}}(\Omega \times^{\text{FCD}} \{\alpha\}) = \Omega \times^{\text{RLD}} \{\alpha\}$.

Really: $(\text{RLD})_{\text{out}}(\Omega \times^{\text{FCD}} \{\alpha\}) = \bigcap^{\text{RLD}} \text{up}(\Omega \times^{\text{FCD}} \{\alpha\}) = \bigcap^{\text{RLD}} \{K \times \{\alpha\} \mid K \in \text{up } \Omega\} = \bigcap^{\mathfrak{F}} \{K \mid K \in \text{up } \Omega\} \times^{\text{RLD}} \{\alpha\} = \bigcap^{\mathfrak{F}} \{K \mid K \in \text{up } \Omega\} \times^{\text{RLD}} \{\alpha\} = \Omega \times^{\text{RLD}} \{\alpha\}$. [TODO: Detailed proof of $\bigcap^{\text{RLD}} \{K \times \{\alpha\} \mid K \in \text{up } \Omega\} = \bigcap^{\mathfrak{F}} \{K \mid K \in \text{up } \Omega\} \times^{\text{RLD}} \{\alpha\}$.]

Thus $(\text{RLD})_{\text{out}}(g \circ f) = \Omega \times^{\text{RLD}} \{\alpha\} \neq \emptyset$. \square

Example 250. (FCD) does not preserve finite meets.

Proof. $(\text{FCD})((=)|_{\mathcal{U}} \cap^{\text{RLD}} (\mathcal{U} \times \mathcal{U} \setminus f)) = (\text{FCD})\emptyset = \emptyset$.

On the other hand

$$(\text{FCD})((=)|_{\mathcal{U}} \cap^{\text{FCD}} (\text{FCD})(\mathcal{U} \times \mathcal{U} \setminus f)) = (=)|_{\mathcal{U}} \cap^{\text{FCD}} (\mathcal{U} \times \mathcal{U} \setminus f) = (=)|_{\Omega} \neq \emptyset$$

(used the proposition 169). \square

Corollary 251. (FCD) is not an upper adjoint (in general).

Bibliography

- [1] E. M. Alfsen and J. E. Fenstad. On the equivalence between proximity structures and totally bounded uniform structures. *Mathematica Scandinavica*, 7:353–360, 1959.
- [2] E.M. Alfsen and Olav Njåstad. Proximity and generalized uniformity. *Fund. Math.*, 52:235–252, 1963.
- [3] C. H. Dowker. Mappings of proximity structures. *General Topology and its Relations to Modern Analysis and Algebra*, Proceedings of the symposium held in Prague in September 1961:139–141, 1961.
- [4] V.A. Efremovich. The geometry of proximity i. *Mat. Sb. (N.S.)*, 31(73):189–200, 1952.
- [5] Miroslav Hušek. Generalized proximity and uniform spaces. i. *Commentationes Mathematicae Universitatis Carolinae*, 5(4):247–266, 1964.
- [6] Miroslav Hušek. Generalized proximity and uniform spaces. ii. *Commentationes Mathematicae Universitatis Carolinae*, 6(1):119–139, 1965.
- [7] Hans-Peter A. Kunzi. Quasi-uniform spaces, 2001.
- [8] Hans-Peter A. Kunzi. Quasi-uniform spaces in the year 2001, 2001.
- [9] Mrówka. On complete proximity spaces. *Doklady Akad. Nauk. SSSR*, N.S. 108:587–590, 1956.
- [10] Mrówka. On the notion of completeness in proximity spaces. *Bull. Acad. Polon. Sci.*, III, 4:201–216, 1956.
- [11] Olav Njåstad. Some properties of proximity and generalized uniformity. *Mathematica Scandinavica*, 12:47–56, 1963.
- [12] William J. Pervin. Quasi-uniformization of topological spaces. *Mathematische Annalen*, 147:316–317, 1962. 10.1007/BF01440953.
- [13] Victor Porton. Filters on posets and generalizations. At <http://www.mathematics21.org/binaries/filters.pdf>.
- [14] Yu. M. Smirnov. On proximity spaces. *Mat. Sb. (N.S.)*, 31(73):543–574, 1952.
- [15] Yu. M. Smirnov. On completeness of proximity spaces. *Doklady Acad. Nauk SSSR*, 88:761–764, 1953.

- [16] Yu. M. Smirnov. On completeness of uniform spaces and proximity spaces. *Doklady Acad. Nauk SSSR*, 91:1281–1284, 1953.
- [17] Yu. M. Smirnov. On completeness of proximity spaces i. *Trudy Moscov. Mat. Obsc.*, 3:271–306, 1954.
- [18] Yu. M. Smirnov. On completeness of proximity spaces ii. *Trudy Moscov. Mat. Obsc.*, 4:421–438, 1955.