Funcoids and Reloids*

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Abstract

It is a part of my Algebraic General Topology research.

In this article I introduce the concepts of *funcoids* which generalize proximity spaces and *reloids* which generalize uniform spaces. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of analysis and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula are generalized continuity, proximity continuity, and uniform continuity.

Keywords: algebraic general topology, quasi-uniform spaces, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

A.M.S. subject classification: 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99

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1 Common

1.1 Draft status

This article is a draft.

This text refers to a preprint edition of [5]. Theorem number clashes may appear due editing both of these manuscripts.

1.2 Used concepts, notation and statements

The set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will denote $\langle f\rangle X = \{f\alpha \mid \alpha\in X\}$ for a set X.

For simplicity I will assume that all sets in consideration are subsets of universal set $\mho$.

1.2.1 Filters

In this work the word *filter* will refer to a filter on a set $\mho$ (in contrast to [5] where are considered filters on arbitrary posets).

I will call the set of filters ordered reverse to set-theoretic inclusion of filters *the set of filter objects* $\mathfrak{F}$ and its element *filter objects* (f.o. for short). I will denote $\mathrm{up}\,\mathcal{F}$ the filter corresponding to a filter object $\mathcal{F}$. So we have $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \mathrm{up}\mathcal{A} \supseteq \mathrm{up}\mathcal{B}$ for every filter objects $\mathcal{A}$ and $\mathcal{B}$. We also will equate filter objects corresponding to principal filters with corresponding sets. (Thus we have $\mathscr{P}\mho \subseteq \mathfrak{F}$.) See [5] for formal definition of filter objects in the framework of ZF. Filters (and filter objects) are studied in the work [5].

Filter objects corresponding to ultrafilters are atoms of the lattice $\mathfrak{F}$ and will be called *atomic filter objects*.

Also we will need to introduce the concept of *generalized filter base*.

Definition 1. *Generalized filter base* is a set $S \in \mathscr{P}\mathfrak{F} \setminus \{\emptyset\}$ such that

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S : \mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Proposition 2. Let S is a generalized filter base. If $A_1, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists \mathcal{C} \in S : \mathcal{C} \subseteq A_1 \cap \ldots \cap A_n.$$

Proof. Can be easily proved by induction. □

Theorem 3. If S is a generalized filter base, then $\mathrm{up} \bigcap^{\mathfrak{F}} S = \bigcup \langle \mathrm{up} \rangle S$.

Proof. Obviously $\mathrm{up} \bigcap^{\mathfrak{F}} S \supseteq \bigcup \langle \mathrm{up} \rangle S$. Reversely, let $K \in \bigcap^{\mathfrak{F}} S$; then $K = A_1 \cap \ldots \cap A_n$ where $A_i \in \mathrm{up}\mathcal{A}_i \in \mathcal{S}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n \subseteq A_1 \cap \ldots \cap A_n = K$, $K \in \mathrm{up}\,\mathcal{C}$, $K \in \bigcup \langle \mathrm{up} \rangle S$. □

Corollary 4. If S is a generalized filter base, then $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in S$.

Proof. $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in \mathrm{up} \bigcap^{\mathfrak{F}} S \Leftrightarrow \emptyset \in \bigcup \langle \mathrm{up} \rangle S \Leftrightarrow \exists \mathcal{X} \in S : \emptyset \in \mathrm{up}\mathcal{X} \Leftrightarrow \emptyset \in S$. □

1.3 Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Some references to predecessors:

- In [1] and [2] are studied semi-uniformities and proximities.
- [3] and [4] contains recent progress in quasi-uniform spaces.

2 Partially ordered dagger categories

2.1 Partially ordered categories

Definition 5. I will call a *partially ordered (pre)category* a (pre)category together with partial order $\subseteq$ on each of its Hom-sets with the additional requirement that

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\mathrm{Src}\, f_1 = \mathrm{Src}\, f_2 \wedge \mathrm{Dst}\, f_1 = \mathrm{Dst}\, f_2 = \mathrm{Src}\, g_1 = \mathrm{Src}\, g_2 \wedge \mathrm{Dst}\, g_1 = \mathrm{Dst}\, g_2$.

2.2 Dagger categories

Definition 6. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^{\dagger}$.

In other words, a *dagger precategory* is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 7. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a *dagger category* is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Theorem 8. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A. \qquad \square$$

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g)

$$f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 9. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

Definition 10. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.

Definition 11. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

Theorem 12. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^\dagger \circ f$.

Proof.

(1) ⇒ (2). If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

(2) ⇒ (1). $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive. □

2.2.1 Monovalued and entirely defined morphisms

Definition 13. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \subseteq 1_{\mathrm{Dst}\, f}$.

Definition 14. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \supseteq 1_{\mathrm{Src}\, f}$.

Remark 15. Easy to show that this is a generalization of monovalued and entirely defined binary relations as morphisms of the category **Rel**.

Definition 16. For a given partially ordered dagger category C the *category of monovalued (entirely defined) morphisms* of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined) morphisms is monovalued (entirely defined) and that identity morphisms are monovalued and entirely defined.

Proof.

Monovalued. Let f and g are monovalued morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f) \circ (g \circ f)^{\dagger} = g \circ f \circ f^{\dagger} \circ g^{\dagger} \subseteq g \circ 1_{\mathrm{Dst}\, f} \circ g^{\dagger} = g \circ 1_{\mathrm{Src}\, g} \circ g^{\dagger} = g \circ g^{\dagger} \subseteq 1_{\mathrm{Dst}\, g} = 1_{\mathrm{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^{\dagger} = 1_A \circ 1_A = 1_A = 1_{\mathrm{Dst}\, 1_A} \subseteq 1_{\mathrm{Dst}\, 1_A}$.

Entirely defined. Let f and g are entirely defined morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f)^{\dagger} \circ (g \circ f) = f^{\dagger} \circ g^{\dagger} \circ g \circ f \supseteq f^{\dagger} \circ 1_{\mathrm{Src}\, g} \circ f = f^{\dagger} \circ 1_{\mathrm{Dst}\, f} \circ f = f^{\dagger} \circ f \supseteq 1_{\mathrm{Src}\, f} = 1_{\mathrm{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^{\dagger} \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\mathrm{Src}\, 1_A} \subseteq 1_{\mathrm{Src}\, 1_A}$. □

3 Funcoids

3.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of "spaces" (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement "f is a continuous function from a space μ to a space ν" can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A\, \delta\, B$ is defined for every sets A and B. We will extend it from sets to filter objects by the formula:

$$\mathcal{A}\, \delta'\, \mathcal{B} \Leftrightarrow \forall A \in \mathrm{up}\, \mathcal{A}, B \in \mathrm{up}\, \mathcal{B}: A\, \delta\, B.$$

Then (as will be proved below) exist two functions $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A}\, \delta'\, \mathcal{B} \Leftrightarrow \mathcal{B} \cap \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap \beta \mathcal{B} \neq \emptyset.$$

The pair $(\alpha; \beta)$ is called *funcoid* when $\mathcal{B} \cap \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap \beta \mathcal{B} \neq \emptyset$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of discrete funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by $\langle f \rangle|_{\mathscr{P}\mho}$ that is by values of function $\langle f \rangle$ on sets.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^{\mho}$. Then we define $\alpha' X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\alpha X \mid x \in X\}$ for every set X. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathscr{P}\mho}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p exists unique funcoid such that $\langle f \rangle|_{\mathscr{P}\mho} = p$; in this case $\langle f \rangle|_{\mathscr{P}\mho} \in \mathscr{P}\mho^{\mathscr{P}\mho}$.

For every binary relation p exists unique funcoid f such that $\forall X \in \mathscr{P}\mho: \langle f \rangle X = \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *discrete*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of discrete funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \left\langle (g \circ f)^{-1} \right\rangle = \left\langle f^{-1} \right\rangle \circ \left\langle g^{-1} \right\rangle.$$

By the same formulas we can define composition of every two funcoids.

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below will be defined domain and image of a funcoid (the domain and the image of a funcoid are filter objects).

3.2 Basic definitions

Definition 17. Let's call *a funcoid* a pair $(\alpha; \beta)$ where $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{Y} \cap^{\mathfrak{F}} \alpha \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta \mathcal{Y} \neq \emptyset).$$

Definition 18. $\langle (\alpha; \beta) \rangle \overset{\text{def}}{=} \alpha$ for a funcoid $(\alpha; \beta)$.

Definition 19. $(\alpha; \beta)^{-1} = (\beta; \alpha)$ for a funcoid $(\alpha; \beta)$.

Proposition 20. If f is a funcoid then f^{-1} is also a funcoid.

Proof. Follows from symmetry in the definition of funcoid. □

Obvious 21. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 22. The relation $[f] \in \mathscr{P}\mathfrak{F}^2$ is defined by the formula (for every filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f)

$$\mathcal{X} [f] \mathcal{Y} \overset{\text{def}}{=} \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset.$$

Obvious 23. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle \mathcal{Y}$ for every filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f.

Obvious 24. $\left[f^{-1} \right] = [f]^{-1}$ for a funcoid f.

Theorem 25.

1. For given value of $\langle f \rangle$ exists no more than one funcoid f.
2. For given value of $[f]$ exists no more than one funcoid f.

Proof. Let f and g are funcoids.

Obviously $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\left\langle f^{-1} \right\rangle = \left\langle g^{-1} \right\rangle \Rightarrow [f] = [g]$. So enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{X} [g] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset$ and consequently $\langle f \rangle \mathcal{X} = \langle g \rangle \mathcal{X}$ for every f.o. $\mathcal{X}$ and $\mathcal{Y}$ because the set of filter objects is separable [5], thus $\langle f \rangle = \langle g \rangle$. □

Proposition 26. $\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})=\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}$ for every funcoid f and $\mathcal{I},\mathcal{J}\in\mathfrak{F}$.

Proof.

$$\begin{aligned}
\star\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J}) &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\right\} &= \text{(by corollary 10 in [5])}\\
\left\{\mathcal{Y}\in\mathfrak{F}\mid(\mathcal{I}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y})\cup^{\mathfrak{F}}(\mathcal{J}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y})\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{I}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\vee\mathcal{J}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{I}\neq\emptyset\vee\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{J}\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid(\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{I})\cup^{\mathfrak{F}}(\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{J})\neq\emptyset\right\} &= \text{(by corollary 10 in [5])}\\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{Y}\cap^{\mathfrak{F}}(\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J})\neq\emptyset\right\} &= \\
\star(\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}). &
\end{aligned}$$

Thus $\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})=\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}$ because $\mathfrak{F}$ is separable. □

3.2.1 Composition of funcoids

Definition 27. *Composition* of funcoids is defined by the formula

$$(\alpha_2;\beta_2)\circ(\alpha_1;\beta_1)=(\alpha_2\circ\alpha_1;\beta_1\circ\beta_2).$$

Proposition 28. If f, g are funcoids then $g\circ f$ is funcoid.

Proof. Let $f=(\alpha_1;\beta_1)$, $g=(\alpha_2;\beta_2)$.

$\mathcal{Y}\cap^{\mathfrak{F}}(\alpha_2\circ\alpha_1)\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\alpha_2\alpha_1\mathcal{X}\neq\emptyset\Leftrightarrow\alpha_1\mathcal{X}\cap^{\mathfrak{F}}\beta_2\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\beta_1\beta_2\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}(\beta_1\circ\beta_2)\mathcal{Y}\neq\emptyset$.

So $(\alpha_2\circ\alpha_1;\beta_1\circ\beta_2)$ is a funcoid. □

Obvious 29. $\langle g\circ f\rangle=\langle g\rangle\circ\langle f\rangle$ for every funcoids f and g.

Proposition 30. $(h\circ g)\circ f=h\circ(g\circ f)$ for every funcoids f, g, h.

Proof.
$\langle(h\circ g)\circ f\rangle=\langle h\circ g\rangle\circ\langle f\rangle=(\langle h\rangle\circ\langle g\rangle)\circ\langle f\rangle=\langle h\rangle\circ(\langle g\rangle\circ\langle f\rangle)=\langle h\rangle\circ\langle g\circ f\rangle=\langle h\circ(g\circ f)\rangle$. □

Theorem 31. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every funcoids f and g.

Proof. $\left\langle(g\circ f)^{-1}\right\rangle=\left\langle f^{-1}\right\rangle\circ\left\langle g^{-1}\right\rangle=\left\langle f^{-1}\circ g^{-1}\right\rangle$. □

3.3 Funcoid as continuation

Theorem 32. For every funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}:X[f]Y$.

Proof. 2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\mathrm{up}\,\mathcal{Y}:Y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\mathrm{up}\,\mathcal{Y}:\mathcal{X}[f]Y$.
Analogously $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}:X[f]\mathcal{Y}$. Combining these two equalities we get

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}:X[f]Y.$$

1. $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset$

Let's denote $W=\left\{\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\mid X\in\mathrm{up}\,\mathcal{X}\right\}$. We will prove that W is a generalized filter base. To prove this enough to show that $V=\{\langle f\rangle X\mid X\in\mathrm{up}\,\mathcal{X}\}$ is a generalized filter base.

Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle A$, $\mathcal{Q}=\langle f\rangle B$ where $A,B\in\mathrm{up}\,\mathcal{X}$; $A\cap B\in\mathrm{up}\,\mathcal{X}$ and $\mathcal{R}\subseteq\mathcal{P}\cap^{\mathfrak{F}}\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle(A\cap B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$\emptyset \notin W \Leftrightarrow \bigcap^{\mathfrak{F}} W \not\Rightarrow \emptyset$ by the corollary 4 of the theorem 3. That is

$$\forall X \in \operatorname{up} \mathcal{X}: \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle X \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \operatorname{up} \mathcal{X} \neq \emptyset.$$

Comparing with the above, $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \operatorname{up} \mathcal{X} \neq \emptyset$. So $\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \operatorname{up} \mathcal{X}$ because the lattice of filter objects is separable. □

Theorem 33.

1. A function $\alpha \in \mathfrak{F}^{\mathscr{P}\mho}$ conforming to the formulas (for every $I, J \in \mathscr{P}\mho$)

   $$\alpha\emptyset = \emptyset, \quad \alpha(I \cup J) = \alpha I \cup \alpha J$$

   can be continued to the function $\langle f \rangle$ for a unique funcoid f;

   $$\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha \rangle \operatorname{up} \mathcal{X} \tag{1}$$

   for every filter object $\mathcal{X}$.

2. A relation $\delta \in \mathscr{P}(\mathscr{P}\mho)^2$ conforming to the formulas (for every $I, J, K \in \mathscr{P}\mho$)

   $$\begin{array}{ll} \neg(\emptyset \,\delta\, I), & I \cup J \,\delta\, K \Leftrightarrow I \,\delta\, K \vee J \,\delta\, K, \\ \neg(I \,\delta\, \emptyset), & K \,\delta\, I \cup J \Leftrightarrow K \,\delta\, I \vee K \,\delta\, J \end{array} \tag{2}$$

   can be continued to the relation $[f]$ for unique funcoid f;

   $$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: X \,\delta\, Y \tag{3}$$

   for every filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (1) and (3) follow from the previous theorem.

2. Let define $\alpha \in \mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\alpha X) = \{Y \in \mathscr{P}\mho \mid X \,\delta\, Y\}$ for every $X \in \mathscr{P}\mho$. Analogously can be defined $\beta \in \mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\beta X) = \{X \in \mathscr{P}\mho \mid X \,\delta\, Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}^{\mathfrak{F}}$ and $\beta' \in \mathfrak{F}^{\mathfrak{F}}$ by the formulas

$$\alpha' \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha \rangle \operatorname{up} \mathcal{X} \quad \text{and} \quad \beta' \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \beta \rangle \operatorname{up} \mathcal{X}.$$

and δ to $\delta' \in \mathscr{P}\mathfrak{F}^2$ by the formula

$$\mathcal{X} \,\delta'\, \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: X \,\delta\, Y.$$

$\mathcal{Y} \cap^{\mathfrak{F}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \alpha \rangle \operatorname{up} \mathcal{X} \neq \emptyset \Leftrightarrow \bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle \alpha \rangle \operatorname{up} \mathcal{X} \neq \emptyset$. Let's prove that

$$W = \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle \alpha \rangle \operatorname{up} \mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle \alpha \rangle \operatorname{up} \mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle \alpha \rangle \operatorname{up} \mathcal{X}$ then exist $X_1, X_2 \in \operatorname{up} \mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{A} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle \alpha \rangle \operatorname{up} \mathcal{X}$. So $\langle \alpha \rangle \operatorname{up} \mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

Accordingly the corollary 4 of the theorem 3, $\bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle \alpha \rangle \operatorname{up} \mathcal{X} \neq \emptyset$ is equivalent to

$$\forall X \in \operatorname{up} \mathcal{X}: \mathcal{Y} \cap^{\mathfrak{F}} \alpha X \neq \emptyset,$$

what is equivalent to $\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y}: X \,\delta\, Y$. Combining the equivalencies we get $\mathcal{Y} \cap^{\mathfrak{F}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \,\delta'\, \mathcal{Y}$. Analogously $\mathcal{X} \cap^{\mathfrak{F}} \beta' \mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \,\delta'\, \mathcal{Y}$. So $\mathcal{Y} \cap^{\mathfrak{F}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta' \mathcal{Y} \neq \emptyset$, that is $(\alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap^{\mathfrak{F}} \alpha' \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \,\delta'\, \mathcal{Y}$ follows that $[(\alpha'; \beta')]$ is a continuation of δ.

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}\mho)^2$ by the formula $X \,\delta\, Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$. Then the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some funcoid f.

$\forall X, Y \in \mathscr{P}\mho: X [f] Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$, consequently $\forall X \in \mathscr{P}\mho: \alpha X = \langle f \rangle X$. So $\langle f \rangle$ is a continuation of α. □

Note that by the last theorem to every proximity δ corresponds a unique funcoid. So funcoids are a generalization of proximity structures.

Definition 34. Any (multivalued) function f will be considered as a funcoid, where by definition $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\mathcal{X}$ for every $\mathcal{X}\in\mathfrak{F}$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff.

Definition 35. Funcoids corresponding to binary relation are called *discrete funcoids*.

We may equate discrete funcoids with corresponding binary relations by the method of appendix B in [5]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below). For simplicity I will not dive here into formal definition of equating discrete funcoids with binary relations (by the method shown in appendix B in [5]) but we simply will (informally) assume that discrete funcoids can be equated with binary relations.

I will denote FCD the set of funcoids or the category of funcoids (see below) dependently on context.

3.4 Lattice of funcoids

Definition 36. $f\subseteq g\overset{\mathrm{def}}{=}[f]\subseteq[g]$ for $f,g\in\mathsf{FCD}$.

Thus FCD is a poset.

Definition 37. I will call the *filtrator of funcoids* (see [5] for the definition of filtrators) the filtrator $(\mathsf{FCD};\mathscr{P}\mho^2)$.

Conjecture 38. The filtrator of funcoids is:

1. with separable core;
2. with co-separable core.

Theorem 39. The set of funcoids is a complete lattice. For every $R\in\mathscr{P}\mathsf{FCD}$ and $X,Y\in\mathscr{P}\mho$

1. $X\left[\bigcup^{\mathsf{FCD}}R\right]Y\Leftrightarrow\exists f\in R: X[f]Y$;
2. $\left\langle\bigcup^{\mathsf{FCD}}R\right\rangle X=\bigcup^{\mathfrak{F}}\{\langle f\rangle X\mid f\in R\}$.

Proof.

2. $\alpha X\overset{\mathrm{def}}{=}\bigcup^{\mathfrak{F}}\{\langle f\rangle X\mid f\in R\}$. $\langle h\rangle\emptyset=\emptyset$;

$$\begin{aligned}\alpha(I\cup J) &= \bigcup^{\mathfrak{F}}\{\langle f\rangle(I\cup J)\mid f\in R\}\\ &= \bigcup^{\mathfrak{F}}\{\langle f\rangle(I\cup^{\mathfrak{F}}J)\mid f\in R\}\\ &= \bigcup^{\mathfrak{F}}\{\langle f\rangle I\cup^{\mathfrak{F}}\langle f\rangle J\mid f\in R\}\\ &= \bigcup^{\mathfrak{F}}\{\langle f\rangle I\mid f\in R\}\cup^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\{\langle f\rangle J\mid f\in R\}\\ &= \alpha I\cup^{\mathfrak{F}}\alpha J.\end{aligned}$$

So α can be continued to $\langle h\rangle$ for a funcoid h. Obviously

$$\forall f\in R: h\supseteq f. \tag{4}$$

And h is the least funcoid for which holds the condition (4). So $h=\bigcup^{\mathsf{FCD}}R$.

1. $X\left[\bigcup^{\mathsf{FCD}}R\right]Y\Leftrightarrow Y\cap^{\mathfrak{F}}\left\langle\bigcup^{\mathsf{FCD}}R\right\rangle X\neq\emptyset\Leftrightarrow Y\cap^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\{\langle f\rangle X\mid f\in R\}\neq\emptyset\Leftrightarrow\exists f\in R: Y\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset\Leftrightarrow\exists f\in R: X[f]Y$ (used the theorem 52 in [5]. □

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 40. For every funcoids f and g and a filter object $\mathcal{X}$

1. $\langle f \cup^{\mathsf{FCD}} g \rangle \mathcal{X} = \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$;
2. $[f \cup^{\mathfrak{F}} g] = [f] \cup [g]$.

Proof.

1. Let $\alpha \mathcal{X} \overset{\text{def}}{=} \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$; $\beta \mathcal{Y} \overset{\text{def}}{=} \langle f^{-1} \rangle \mathcal{Y} \cup^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. Then
$$\begin{aligned} \mathcal{Y} \cap^{\mathfrak{F}} \alpha \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \vee \mathcal{X} \cap^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta \mathcal{Y} \neq \emptyset. \end{aligned}$$
So $h = (\alpha; \beta)$ is a funcoid. Consequently $f \cup^{\mathsf{FCD}} g = h$.
2. $\mathcal{X} [f \cup^{\mathsf{FCD}} g] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \cup^{\mathsf{FCD}} g \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} (\langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}) \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} [f] \mathcal{Y} \vee \mathcal{X} [g] \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. □

3.5 More on composition of funcoids

Proposition 41. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for $f, g \in \mathsf{FCD}$.

Proof. $\mathcal{X} [g \circ f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \circ f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X} [g] \mathcal{Y} \Leftrightarrow \mathcal{X} ([g] \circ \langle f \rangle) \mathcal{Y}$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. $[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} = \langle g^{-1} \rangle^{-1} \circ [f]$. □

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x (g \circ f) z \Leftrightarrow \exists y (x f y \wedge y g z)$ for every x and z and every binary relations f and g.

Theorem 42. For every $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$ and $f, g \in \mathsf{FCD}$
$$\mathcal{X} [g \circ f] \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}).$$

Proof.
$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}) &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle y \neq \emptyset \wedge y \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle y \neq \emptyset \wedge y \subseteq \langle f \rangle \mathcal{X}) \\ &\Rightarrow \mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle \langle f \rangle \mathcal{X} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} [g \circ f] \mathcal{Z}. \end{aligned}$$
Reversely, if $\mathcal{X} [g \circ f] \mathcal{Z}$ then $\langle f \rangle \mathcal{X} [g] \mathcal{Z}$, consequently exists $y \in \operatorname{atoms}^{\mathfrak{F}} \langle f \rangle \mathcal{X}$ such that $y [g] \mathcal{Z}$; we have $\mathcal{X} [f] y$. □

Theorem 43. If f, g, h are funcoids then

1. $f \circ (g \cup^{\mathsf{FCD}} h) = f \circ g \cup^{\mathsf{FCD}} f \circ h$;
2. $(g \cup^{\mathsf{FCD}} h) \circ f = g \circ f \cup^{\mathsf{FCD}} h \circ f$.

Proof. I will prove only the first equality because the other is analogous.

For every $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$
$$\begin{aligned} \mathcal{X} [f \circ (g \cup^{\mathsf{FCD}} h)] \mathcal{Z} &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X} [g \cup^{\mathsf{FCD}} h] y \wedge y [f] \mathcal{Z}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : ((\mathcal{X} [g] y \vee \mathcal{X} [h] y) \wedge y [f] \mathcal{Z}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X} [g] y \wedge y [f] \mathcal{Z} \vee \mathcal{X} [h] y \wedge y [f] \mathcal{Z}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X} [h] y \wedge y [f] \mathcal{Z}) \\ &\Leftrightarrow \mathcal{X} [f \circ g] \mathcal{Z} \vee \mathcal{X} [f \circ h] \mathcal{Z} \\ &\Leftrightarrow \mathcal{X} [f \circ g \cup^{\mathsf{FCD}} f \circ h] \mathcal{Z}. \end{aligned}$$
□

3.6 Domain and range of a funcoid

Definition 44. Let $\mathcal{A}\in\mathfrak{F}$. The *identity funcoid* $I_{\mathcal{A}}=(\mathcal{A}\cap^{\mathfrak{F}};\mathcal{A}\cap^{\mathfrak{F}})$.

Proposition 45. The identity funcoid is a funcoid.

Proof. We need to prove that $(\mathcal{A}\cap^{\mathfrak{F}}\mathcal{X})\cap^{\mathfrak{F}}\mathcal{Y}\neq\emptyset\Leftrightarrow(\mathcal{A}\cap^{\mathfrak{F}}\mathcal{Y})\cap^{\mathfrak{F}}\mathcal{X}\neq\emptyset$ what is obvious. □

Obvious 46. $(I_{\mathcal{A}})^{-1}=I_{\mathcal{A}}$.

Obvious 47. $\mathcal{X}[I_{\mathcal{A}}]\mathcal{Y}\Leftrightarrow\mathcal{A}\cap^{\mathfrak{F}}\mathcal{X}\cap^{\mathfrak{F}}\mathcal{Y}\neq\emptyset$ for any $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$.

Definition 48. I will define *restricting* of a funcoid f to a filter object $\mathcal{A}$ by the formula $f|_{\mathcal{A}}\stackrel{\text{def}}{=}f\circ I_{\mathcal{A}}$.

Obviously the last definition does not contradict to the previous.

Definition 49. *Image* of a funcoid f will be defined by the formula $\operatorname{im} f=\langle f\rangle\mho$.
Domain of a funcoid f is defined by the formula $\operatorname{dom} f=\operatorname{im} f^{-1}$.

Proposition 50. $\langle f\rangle\mathcal{X}=\langle f\rangle(\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f)$ for every $f\in\mathsf{FCD}$, $\mathcal{X}\in\mathfrak{F}$.

Proof. For every filter object $\mathcal{Y}$ we have $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle(\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f)\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\operatorname{im} f^{-1}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$. Thus $\langle f\rangle\mathcal{X}=\langle f\rangle(\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f)$ because the lattice of filter objects is separable. □

Proposition 51. $\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f\neq\emptyset\Leftrightarrow\langle f\rangle\mathcal{X}\neq\emptyset$ for every $f\in\mathsf{FCD}$, $\mathcal{X}\in\mathfrak{F}$.

Proof. $\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mho\neq\emptyset\Leftrightarrow\mho\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\langle f\rangle\mathcal{X}\neq\emptyset$. □

Corollary 52. $\operatorname{dom} f=\bigcup^{\mathfrak{F}}\{a\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mho,\langle f\rangle a\neq\emptyset\}$.

Proof. This follows from that $\mathfrak{F}$ is an atomistic lattice. □

3.7 Category of funcoids

I will define the category FCD of funcoids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f;\mathcal{A};\mathcal{B})$ where f is a funcoid such that $\operatorname{dom} f\subseteq\mathcal{A}$, $\operatorname{im} f\subseteq\mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}};\mathcal{A};\mathcal{A})$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 53. For every funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcup^{\mathfrak{F}}\langle\langle f\rangle\rangle\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{X},y\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{Y}:x[f]y$.

Proof. 1.

$$\begin{aligned}\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}:x\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset.\end{aligned}$$

$\partial\langle f\rangle\mathcal{X}=\bigcup\langle\partial\rangle\langle\langle f\rangle\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}=\partial\bigcup\langle\langle f\rangle\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}$.

2. If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$, consequently exists $y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}$ such that $y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$, $\mathcal{X}[f]y$. Repeating this second time we get that there exist $x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}$ such that $x[f]y$. From this follows

$$\exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X},y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}:x[f]y.$$

The reverse is obvious. □

Theorem 54.

1. A function $\alpha\in\mathfrak{F}^{\mathrm{atoms}^{\mathfrak{F}}\mho}$ such that (for every $a\in\mathrm{atoms}^{\mathfrak{F}}\mho$)

$$\alpha a\supseteq\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a\tag{5}$$

can be continued to the function $\langle f\rangle$ for a unique funcoid f;

$$\langle f\rangle\mathcal{X}=\bigcup^{\mathfrak{F}}\langle\alpha\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}\tag{6}$$

for every filter object $\mathcal{X}$.

2. A relation $\delta\in\mathscr{P}(\mathrm{atoms}^{\mathfrak{F}}\mho)^2$ such that (for every $a,b\in\mathrm{atoms}^{\mathfrak{F}}\mho$)

$$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y:x\,\delta\,y\Rightarrow a\,\delta\,b\tag{7}$$

can be continued to the relation $[f]$ for a unique funcoid f;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X},y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}:x\,\delta\,y\tag{8}$$

for every filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the previous theorem.

1. Consider the function $\alpha'\in\mathfrak{F}^{\mho}$ defined by the formula (for every $X\in\mathscr{P}\mho$)

$$\alpha'X=\bigcup^{\mathfrak{F}}\langle\alpha\rangle\mathrm{atoms}^{\mathfrak{F}}X.$$

Obviously $\alpha'\emptyset=\emptyset$. For every $I,J\in\mathscr{P}\mho$

$$\begin{aligned}\alpha'(I\cup J)&=\bigcup^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}(I\cup J)\\&=\bigcup^{\mathfrak{F}}\langle\alpha'\rangle(\mathrm{atoms}^{\mathfrak{F}}I\cup\mathrm{atoms}^{\mathfrak{F}}J)\\&=\bigcup^{\mathfrak{F}}\left(\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}I\cup\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}J\right)\\&=\bigcup^{\mathfrak{F}}\langle\alpha'\rangle\,\mathrm{atoms}^{\mathfrak{F}}I\cup^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}.\\&=\alpha'I\cup^{\mathfrak{F}}\alpha'J.\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 25): $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{up}\,\mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a&=\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right\rangle\left\langle\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a\\&\supseteq\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\&=\bigcap^{\mathfrak{F}}\left\{\left(\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right)\{a\}\right\}\\&=\bigcap^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\langle\alpha\rangle\{a\}\right\}\\&=\bigcap^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\{\alpha a\}\right\}=\bigcap^{\mathfrak{F}}\{\alpha a\}=\alpha a.\end{aligned}$$

Finally,

$$\alpha a=\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a=\bigcap^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{up}\,a=\langle f\rangle a,$$

so $\langle f\rangle$ is a continuation of α.

2. Consider the relation $\delta'\in\mathscr{P}(\mathscr{P}\mho)^2$ defined by the formula (for every $X,Y\in\mathscr{P}\mho$)

$$X\,\delta'\,Y\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y:x\,\delta\,y.$$

Obviously $\neg(X\,\delta'\,\emptyset)$ and $\neg(\emptyset\,\delta'\,Y)$.

$$\begin{aligned}(I\cup J)\,\delta'\,Y &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}(I\cup J), y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}I\cup\operatorname{atoms}^{\mathfrak{F}}J, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}I, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\vee\exists x\in\operatorname{atoms}^{\mathfrak{F}}J, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow I\,\delta'\,Y\vee J\,\delta'\,Y;\end{aligned}$$

analogously $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$. Let's continue δ' till a funcoid f (by the theorem 25):

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}, Y\in\operatorname{up}\mathcal{Y}: X\,\delta'\,Y$$

The reverse of (7) implication is trivial, so

$$\forall X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow\forall X\in\operatorname{up}a, Y\in\operatorname{up}b: X\,\delta'\,Y\Leftrightarrow a[f]b$.

So $a\,\delta\,b\Leftrightarrow a[f]b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is proof of the following theorem:

Theorem 55. If R is a set of funcoids, $x, y\in\operatorname{atoms}^{\mathfrak{F}}\mho$, then

1. $\langle\bigcap^{\mathsf{FCD}}R\rangle x=\bigcap^{\mathfrak{F}}\{\langle f\rangle x\mid f\in R\}$;
2. $x\left[\bigcap^{\mathsf{FCD}}R\right]y\Leftrightarrow\forall f\in R: x[f]y$.

Proof. 2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R: x[f]y$.

$$\begin{gathered}\forall X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow\\ \forall f\in R, X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x[f]y\Rightarrow\\ \forall f\in R, X\in\operatorname{up}a, Y\in\operatorname{up}b: X[f]Y\Rightarrow\\ \forall f\in R: a[f]b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 54, δ can be continued till $[p]$ for some funcoid p.

For every funcoid q such that $\forall f\in R: q\subseteq f$ we have $x[q]y\Rightarrow\forall f\in R: x[f]y\Leftrightarrow x\,\delta\,y\Leftrightarrow x[p]y$, so $q\subseteq f$. Consequently $p=\bigcap^{\mathsf{FCD}}R$.

From this $x\left[\bigcap^{\mathsf{FCD}}R\right]y\Leftrightarrow\forall f\in R: x[f]y$.

1. From the former $y\cap^{\mathfrak{F}}\langle\bigcap^{\mathsf{FCD}}R\rangle x\neq\emptyset\Leftrightarrow\forall f\in R: y\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset$ for every $y\in\operatorname{atoms}^{\mathfrak{F}}\mho$. From this follows what we need to prove. □

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is direct product of two filter objects as defined in the theory of funcoids:

Definition 56. *Direct product* of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ that

$$\mathcal{X}\left[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\wedge\mathcal{Y}\cap^{\mathfrak{F}}\mathcal{B}\neq\emptyset.$$

Proposition 57. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a funcoid and

$$\left\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if } \mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset;\\ \emptyset & \text{if } \mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}=\emptyset.\end{cases}$$

Proof. Obvious. □

Obvious 58. $A\times B=A\times^{\mathsf{FCD}}B$ for sets A and B.

Proposition 59. $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ for every $f \in \mathsf{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. If $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \subseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ then

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{X} [f] \mathcal{Y} \Rightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset);$$

consequently $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 60. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$ for every $f \in \mathsf{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $h \overset{\text{def}}{=} I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$. For every $\mathcal{X} \in \mathfrak{F}$

$$\langle h \rangle \mathcal{X} = \langle I_{\mathcal{B}} \rangle \langle f \rangle \langle I_{\mathcal{A}} \rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a funcoid g then $\operatorname{dom} g \subseteq \mathcal{A}$, $\operatorname{im} g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{F}} \langle g \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) \subseteq \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) = \langle I_{\mathcal{B}} \rangle \langle f \rangle \langle I_{\mathcal{A}} \rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 61. $f|_{\mathcal{A}} = f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mho)$ for every $f \in \mathsf{FCD}$ and $\mathcal{A} \in \mathfrak{F}$.

Proof. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mho) = I_{\mho} \circ f \circ I_{\mathcal{A}} = f \circ I_{\mathcal{A}} = f|_{\mathcal{A}}$. □

Corollary 62. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq \emptyset \Leftrightarrow \mathcal{A} [f] \mathcal{B}$ for every $f \in \mathsf{FCD}$, $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq \emptyset \Leftrightarrow \langle f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \rangle \mho \neq \emptyset \Leftrightarrow \langle I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}} \rangle \mho \neq \emptyset \Leftrightarrow \langle I_{\mathcal{B}} \rangle \langle f \rangle \langle I_{\mathcal{A}} \rangle \mho \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathsf{FCD}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mho) \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} [f] \mathcal{B}$. □

Corollary 63. The filtrator of funcoids is star-separable.

Proof. The set of direct products of sets is a separation subset of the lattice of funcoids. □

Theorem 64. If $S \in \mathscr{P} \mathfrak{F}^2$ then

$$\bigcap^{\mathsf{FCD}} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} = \bigcap^{\mathfrak{F}} \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap^{\mathfrak{F}} \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$ then by the theorem 55

$$\left\langle \bigcap^{\mathsf{FCD}} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \bigcap^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \}.$$

If $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S \neq \emptyset$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} = \operatorname{im} S;$$

if $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S = \emptyset$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \emptyset);$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} \ni \emptyset.$$

So

$$\left\langle \bigcap^{\mathsf{FCD}} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \begin{cases} \bigcap^{\mathfrak{F}} \operatorname{im} S & \text{if } x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S \neq \emptyset; \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S = \emptyset. \end{cases}$$

From this follows the statement of the theorem. □

Corollary 65. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap^{\mathsf{FCD}} (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\mathsf{FCD}} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathsf{FCD}} \mathcal{B}_1)$ for every $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap^{\mathsf{FCD}} (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigcap^{\mathfrak{F}} \{\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap^{\mathsf{FCD}} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathsf{FCD}} \mathcal{B}_1)$. □

Theorem 66. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice FCD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Proof. Let $S \in \mathscr{P}\mathfrak{F}$, $X \in \mathscr{P}\mho$, $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$.

$$\begin{aligned}
\left\langle \bigcup^{\mathsf{FCD}} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle X &= \bigcup^{\mathfrak{F}} \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle X \mid \mathcal{B} \in S\} \\
&= \begin{cases} \bigcup^{\mathfrak{F}} S & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\
&= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcup^{\mathfrak{F}} S \right\rangle X; \\
\left\langle \bigcap^{\mathsf{FCD}} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle x &= \bigcap^{\mathfrak{F}} \{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid \mathcal{B} \in S\} \\
&= \begin{cases} \bigcap^{\mathfrak{F}} S & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\
&= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcap^{\mathfrak{F}} S \right\rangle x.
\end{aligned}$$

If $\mathcal{A} \neq \emptyset$ then obviously the function $\mathcal{A} \times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a direct product of filter objects) funcoid (of atomic width).

Proposition 67. If a is an atomic filter object, $f \in \mathsf{FCD}$ then $f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a$.

Proof. Let $\mathcal{X} \in \mathfrak{F}$.

$$\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \langle f|_a \rangle \mathcal{X} = \emptyset.$$ □

3.10 Atomic funcoids

Theorem 68. A funcoid is an atom of the lattice of funcoids iff it is direct product of two atomic filter objects.

Proof.

⇒. Let f is an atomic funcoid. Let's get elements $a \in \operatorname{atoms}^{\mathfrak{F}} \operatorname{dom} f$ and $b \in \operatorname{atoms}^{\mathfrak{F}} \langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{F}$

$$\mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = \emptyset \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is an atomic funcoid $f = a \times^{\mathsf{FCD}} b$.

⇐. Let $a, b \in \operatorname{atoms}^{\mathfrak{F}} \mho$, $f \in \mathsf{FCD}$. If $b \cap^{\mathfrak{F}} \langle f \rangle a = \emptyset$ then $\neg(a [f] b)$, $f \cap^{\mathfrak{F}} a \times^{\mathsf{FCD}} b = \emptyset$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}: (\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \cap^{\mathsf{FCD}} a \times^{\mathsf{FCD}} b = \emptyset \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic filter object. □

Theorem 69. The lattice of funcoids is atomic.

Proof. Let f is a non-empty funcoid. Then $\operatorname{dom} f \neq \emptyset$, thus by the theorem 46 in [5] exists $a \in \operatorname{atoms}^{\mathfrak{F}} \operatorname{dom} f$. So $\langle f \rangle a \neq \emptyset$ thus exists $b \in \operatorname{atoms} \langle f \rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 70. The lattice of funcoids is separable.

Proof. Let $f,g\in\mathsf{FCD}$, $f\subset g$. Then exists $a\in\operatorname{atoms}^{\mathfrak{F}}\mho$ such that $\langle f\rangle a\subset\langle g\rangle a$. So because the lattice $\mathfrak{F}$ is atomically separable then exists $b\in\operatorname{atoms}^{\mathfrak{F}}\mho$ such that $\langle f\rangle a\cap^{\mathfrak{F}} b=\emptyset$ and $b\subseteq\langle g\rangle a$. For every $x\in\operatorname{atoms}^{\mathfrak{F}}\mho$

$$\langle f\rangle a\cap^{\mathfrak{F}}\left\langle a\times^{\mathsf{FCD}} b\right\rangle a=\langle f\rangle a\cap^{\mathfrak{F}} b=\emptyset,$$
$$x\neq a\Rightarrow\langle f\rangle x\cap^{\mathfrak{F}}\left\langle a\times^{\mathsf{FCD}} b\right\rangle x=\langle f\rangle x\cap^{\mathfrak{F}}\emptyset=\emptyset$$

Thus $\langle f\rangle x\cap^{\mathfrak{F}}\langle a\times b\rangle x=\emptyset$ and consequently $f\cap^{\mathsf{FCD}} a\times^{\mathsf{FCD}} b=\emptyset$.

$$\left\langle a\times^{\mathsf{FCD}} b\right\rangle a=b\subseteq\langle g\rangle a,$$
$$x\neq a\Rightarrow\left\langle a\times^{\mathsf{FCD}} b\right\rangle x=\emptyset\subseteq\langle g\rangle a.$$

Thus $\left\langle a\times^{\mathsf{FCD}} b\right\rangle x=b\subseteq\langle g\rangle x$ and consequently $a\times^{\mathsf{FCD}} b\subseteq g$.

So the lattice of funcoids is separable by the theorem 19 in [5]. □

Corollary 71. The lattice of funcoids is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [5]. □

Remark 72. For more ways to characterize (atomic) separability of the lattice of funcoids see [5], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 73. The lattice of funcoids is an atomistic lattice.

Proof. Let f is a funcoid. Suppose contrary to the statement to be proved that $\bigcup^{\mathfrak{F}}\operatorname{atoms}^{\mathsf{FCD}} f\subset f$. Then exists $a\in\operatorname{atoms}^{\mathsf{FCD}} f$ such that $a\cap^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\operatorname{atoms}^{\mathsf{FCD}} f=\emptyset$ what is impossible. □

Proposition 74. $\operatorname{atoms}^{\mathsf{FCD}}(f\cup^{\mathfrak{F}} g)=\operatorname{atoms}^{\mathsf{FCD}} f\cup\operatorname{atoms}^{\mathsf{FCD}} g$ for every funcoids f and g.

Proof. $(a\times^{\mathsf{FCD}} b)\cap^{\mathsf{FCD}}(f\cup^{\mathsf{FCD}} g)\neq\emptyset\Leftrightarrow a\left[f\cup^{\mathsf{FCD}} g\right]b\Leftrightarrow a[f]b\vee a[g]b\Leftrightarrow(a\times^{\mathsf{FCD}} b)\cap^{\mathsf{FCD}} f\neq\emptyset\vee(a\times^{\mathsf{FCD}} b)\cap^{\mathsf{FCD}} g\neq\emptyset$ for every atomic filter objects a and b. □

Corollary 75. For every $f,g,h\in\mathsf{FCD}$, $R\in\mathscr{P}\mathsf{FCD}$

1. $f\cap^{\mathsf{FCD}}(g\cup^{\mathsf{FCD}} h)=(f\cap^{\mathsf{FCD}} g)\cup^{\mathsf{FCD}}(f\cap^{\mathsf{FCD}} h)$;
2. $f\cup^{\mathsf{FCD}}\bigcap^{\mathsf{FCD}} R=\bigcap^{\mathsf{FCD}}\left\langle f\cup^{\mathsf{FCD}}\right\rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice. To be concise I will write atoms instead of $\operatorname{atoms}^{\mathsf{FCD}}$ and $\cap$ and $\cup$ instead of $\cap^{\mathsf{FCD}}$ and $\cup^{\mathsf{FCD}}$.

1. $\operatorname{atoms}(f\cap(g\cup h))=\operatorname{atoms} f\cap\operatorname{atoms}(g\cup h)=\operatorname{atoms} f\cap(\operatorname{atoms} g\cup\operatorname{atoms} h)=(\operatorname{atoms} f\cap\operatorname{atoms} g)\cup(\operatorname{atoms} f\cap\operatorname{atoms} h)=\operatorname{atoms}(f\cap g)\cup\operatorname{atoms}(f\cap h)=\operatorname{atoms}((f\cap g)\cup(f\cap h))$.
2. $\operatorname{atoms}\left(f\cup\bigcap^{\mathsf{FCD}} R\right)=\operatorname{atoms} f\cup\operatorname{atoms}\bigcap^{\mathsf{FCD}} R=\operatorname{atoms} f\cup\bigcap^{\mathsf{FCD}}\langle\operatorname{atoms}\rangle R=\bigcap^{\mathsf{FCD}}\langle(\operatorname{atoms} f)\cup\rangle\langle\operatorname{atoms}\rangle R=\bigcap^{\mathsf{FCD}}\langle\operatorname{atoms}\rangle\langle f\cup\rangle R=\operatorname{atoms}\bigcap^{\mathsf{FCD}}\langle f\cup\rangle R$. □

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

Corollary 76. The lattice of funcoids is co-brouwerian.

The next proposition is one more (among the theorem 42) generalization for funcoids of composition of relations.

Proposition 77. For every $f, g \in \mathsf{FCD}$

$\operatorname{atoms}^{\mathsf{FCD}}(g \circ f) = \left\{ x \times^{\mathsf{FCD}} z \mid x, z \in \operatorname{atoms}^{\mathfrak{F}} \mho, \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms}^{\mathsf{FCD}} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms}^{\mathsf{FCD}} g) \right\}$.

Proof. $(x \times^{\mathsf{FCD}} z) \cap^{\mathsf{FCD}} g \circ f \neq \emptyset \Leftrightarrow x [g \circ f] z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x [f] y \wedge y [g] z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : ((x \times^{\mathsf{FCD}} y) \cap^{\mathsf{FCD}} f \neq \emptyset \wedge (y \times^{\mathsf{FCD}} z) \cap^{\mathsf{FCD}} g \neq \emptyset)$ (were used the theorem 42). □

Conjecture 78. The set of discrete funcoids is the center of the lattice of funcoids.

3.11 Complete funcoids

Definition 79. I will call *co-complete* such a funcoid f that $\forall X \in \mathscr{P} \mho : \langle f \rangle X \in \mathscr{P} \mho$.

Remark 80. I will call *generalized closure* such a function $\alpha \in \mathscr{P} \mho^{\mathscr{P} \mho}$ that

1. $\alpha \emptyset = \emptyset$;
2. $\forall I, J \in \mathscr{P} \mho : \alpha (I \cup J) = \alpha I \cup \alpha J$.

Obvious 81. A funcoid f is co-complete iff $\langle f \rangle|_{\mathscr{P} \mho}$ is a generalized closure.

Remark 82. Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 83. I will call a *complete funcoid* a funcoid whose reverse is co-complete.

Theorem 84. The following conditions are equivalent for every funcoid f:

1. funcoid f is complete;
2. $\forall S \in \mathscr{P} \mathfrak{F}, J \in \mathscr{P} \mho : (\bigcup^{\mathfrak{F}} S [f] J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} [f] J)$;
3. $\forall S \in \mathscr{P} \mathscr{P} \mho, J \in \mathscr{P} \mho : (\bigcup S [f] J \Leftrightarrow \exists I \in S : I [f] J)$;
4. $\forall S \in \mathscr{P} \mathfrak{F} : \langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
5. $\forall S \in \mathscr{P} \mathscr{P} \mho : \langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
6. $\forall A \in \mathscr{P} \mho : \langle f \rangle A = \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in A \}$.

Proof.

(3) ⇒ (1). For every $S \in \mathscr{P} \mathscr{P} \mho$, $J \in \mathscr{P} \mho$

$$\bigcup S \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle J \neq \emptyset \Leftrightarrow \exists I \in S : I \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle J \neq \emptyset, \tag{9}$$

consequently by the theorem 52 in [5] we have $\left\langle f^{-1} \right\rangle J \in \mathscr{P} \mho$.

(1) ⇒ (2). For every $S \in \mathscr{P} \mathfrak{F}$, $J \in \mathscr{P} \mho$ we have $\left\langle f^{-1} \right\rangle J \in \mathscr{P} \mho$, consequently the formula (9) is true. From this follows (2).

(6) ⇒ (5). $\langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in \bigcup S \} = \bigcup^{\mathfrak{F}} \left\{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in A \} \mid A \in S \right\} = \bigcup^{\mathfrak{F}} \{ \langle f \rangle A \mid A \in S \} = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$.

(2) ⇒ (3), (4) ⇒ (5), (5) ⇒ (3), (2) ⇒ (4), (5) ⇒ (6). Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pre-topological spaces.

Proposition 85. To specify a complete funcoid f it is enough to specify $\langle f \rangle$ on one-element sets, values of $\langle f \rangle$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f \rangle$ on one-element sets $\langle f \rangle$ can be found on every sets and then its value can be inferred for every filter objects.

Choosing arbitrarily the values of $\langle f \rangle$ on one-element sets we can define a complete funcoid the following way: $\langle f \rangle X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{ \alpha \} \mid \alpha \in X \}$ for every $X \in \mathscr{P}\mho$. Obviously it is really a complete funcoid. □

Theorem 86. A funcoid is discrete iff it is both complete and co-complete.

Proof.

Direct implication. Obvious.

Reverse implication. Let f is both a complete and co-complete funcoid. Consider the relation g defined by that $\langle g \rangle \{ \alpha \} = \langle f \rangle \{ \alpha \}$ (g is correctly defined because f is a generalized closure). Because f is a complete funcoid $f = g$. □

Theorem 87. If R is a set of (co-)complete funcoids then $\bigcup^{\mathsf{FCD}} R$ is a (co-)complete funcoid.

Proof. It is enough to prove only for co-complete funcoids. Let R is a set of co-complete funcoids. Then for every $X \in \mathscr{P}\mho$

$$\left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X = \bigcup \{ \langle f \rangle X \mid f \in R \} \in \mathscr{P}\mho$$

(used the theorem 39). □

Corollary 88. If R is a set of binary relations then $\bigcup^{\mathsf{FCD}} R = \bigcup R$.

Proof. From two last theorems. □

Theorem 89. The filtrator of funcoids is filtered.

Proof. It's enough to prove that every funcoid is representable as (infinite) intersection (on the lattice of funcoids) of some set of discrete funcoids.

Let $f \in \mathsf{FCD}$, $A \in \mathscr{P}\mho$, $B \in \operatorname{up}\langle f \rangle A$, $g(A; B) \overset{\text{def}}{=} A \times B \cup^{\mathsf{FCD}} \overline{A} \times \mho$. For every $X \in \mathscr{P}\mho$

$$\langle g(A; B) \rangle X = \left\langle A \times^{\mathsf{FCD}} B \right\rangle X \cup \left\langle \overline{A} \times^{\mathsf{FCD}} \mho \right\rangle X = \left(\begin{cases} \emptyset & \text{if } X = \emptyset \\ B & \text{if } \emptyset \neq X \subseteq A \\ \mho & \text{if } X \nsubseteq A \end{cases} \right) \supseteq \langle f \rangle X;$$

so $g(A; B) \supseteq f$. For every $A \in \mathscr{P}\mho$

$$\bigcap^{\mathfrak{F}} \{ \langle g(A; B) \rangle A \mid B \in \operatorname{up}\langle f \rangle A \} = \bigcap^{\mathfrak{F}} \{ B \mid B \in \operatorname{up}\langle f \rangle A \} = \langle f \rangle A;$$

consequently

$$\bigcap^{\mathsf{FCD}} \{ g(A; B) \mid A \in \mathscr{P}\mho, B \in \operatorname{up}\langle f \rangle A \} = f.$$ □

In certain cases the theorem 43 can be generalized for infinite unions.

Conjecture 90. If f is a complete funcoid and R is a set of funcoids then $f \circ \bigcup^{\mathsf{FCD}} R = \bigcup^{\mathsf{FCD}} \langle f \circ \rangle R$.

This conjecture can be weakened:

Conjecture 91. If f is a discrete funcoid and R is a set of funcoids then $f \circ \bigcup^{\mathsf{FCD}} R = \bigcup^{\mathsf{FCD}} \langle f \circ \rangle R$.

3.12 Completion of funcoids

I will denote ComplFCD and CoComplFCD the sets of complete and co-complete funcoids correspondingly.

Obvious 92. ComplFCD and CoComplFCD are closed regarding composition of funcoids.

Proposition 93. ComplFCD and CoComplFCD (with induced order) are complete lattices.

Proof. Follows from the corollary 87. □

Theorem 94. $\mathrm{Cor}\, f = \mathrm{Cor}'\, f$ for an element f of the filtrator of funcoids.

Proof. From the theorem 26 in [5] and the corollary 88 and theorem 89. □

Definition 95. *Completion* of a funcoid f is the complete funcoid $\mathrm{Compl}\, f$ defined by the formula $\langle \mathrm{Compl}\, f\rangle\{\alpha\} = \langle f\rangle\{\alpha\}$ for $\alpha \in \mho$.

Definition 96. *Co-completion* of a funcoid f is defined by the formula

$$\mathrm{CoCompl}\, f = (\mathrm{Compl}\, f^{-1})^{-1}.$$

Obvious 97. $\mathrm{Compl}\, f \subseteq f$ and $\mathrm{CoCompl}\, f \subseteq f$ for every funcoid f.

Proposition 98. The filtrator $(\mathsf{FCD}; \mathrm{Compl}\mathsf{FCD})$ is filtered.

Proof. Because the filtrator $(\mathsf{FCD}; \mathscr{P}\mho^2)$ is filtered. □

Theorem 99. $\mathrm{Compl}\, f = \mathrm{Cor}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = \mathrm{Cor}'^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$.

Proof. $\mathrm{Cor}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = \mathrm{Cor}'^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$ since (the theorem 26 in [5]) the filtrator $(\mathsf{FCD}; \mathrm{Compl}\mathsf{FCD})$ is filtered (as a consequence of the theorem 89) and with join closed core (the theorem 87).

Let $g \in \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$. Then $g \in \mathrm{Compl}\mathsf{FCD}$ and $g \supseteq f$. Thus $g = \mathrm{Compl}\, g \supseteq \mathrm{Compl}\, f$.

Thus $\forall g \in \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f: g \supseteq \mathrm{Compl}\, f$.

Let $\forall g \in \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f: h \subseteq g$ for some $h \in \mathrm{Compl}\mathsf{FCD}$.

Then $h \subseteq \bigcap^{\mathsf{FCD}} \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = f$ and consequently $h = \mathrm{Compl}\, h \subseteq \mathrm{Compl}\, f$.

Thus $\mathrm{Compl}\, f = \bigcap^{\mathrm{Compl}\mathsf{FCD}} \mathrm{up}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f = \mathrm{Cor}^{(\mathsf{FCD};\mathrm{Compl}\mathsf{FCD})} f$. □

Theorem 100. Atoms of the lattice ComplFCD are exactly direct products of the form $\{\alpha\} \times^{\mathsf{FCD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of ComplFCD. Also $\emptyset$ is an element of ComplFCD.

$\{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of ComplFCD because these are atoms of FCD.

Remain to prove that if f is an atom of ComplFCD then $f = \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in \mho$ and an atomic f.o. b.

Suppose f is a non-empty complete funcoid. Then exists $\alpha \in \mho$ such that $\langle f\rangle\alpha \neq \emptyset$. Thus $\{\alpha\} \times^{\mathsf{FCD}} b \subseteq f$ for some atomic f.o. b. If f is an atom then $f = \{\alpha\} \times^{\mathsf{FCD}} b$. □

Theorem 101. $\langle \mathrm{CoCompl}\, f\rangle X = \mathrm{Cor}\, \langle f\rangle X$ for every funcoid f and set X.

Proof. $\mathrm{CoCompl}\, f \subseteq f$ thus $\langle \mathrm{CoCompl}\, f\rangle X \subseteq \langle f\rangle X$, but $\langle \mathrm{CoCompl}\, f\rangle X \in \mathscr{P}\mho$ thus $\langle \mathrm{CoCompl}\, f\rangle X \subseteq \mathrm{Cor}\, \langle f\rangle X$.

Let $\alpha X = \mathrm{Cor}\, \langle f\rangle X$. Then $\alpha\emptyset = \emptyset$ and

$$\alpha(X \cup Y) = \mathrm{Cor}\, \langle f\rangle(X \cup Y) = \mathrm{Cor}(\langle f\rangle X \cup \langle f\rangle Y) = \mathrm{Cor}\, \langle f\rangle X \cup \mathrm{Cor}\, \langle f\rangle Y = \alpha X \cup \alpha Y.$$

(used the theorem 64 from [5]). Thus α can be continued till $\langle g\rangle$ for some funcoid g. This funcoid is co-complete.

Evidently g is the greatest co-complete funcoid which is lower than f.

Thus $g = \mathrm{CoCompl}\, f$ and so $\mathrm{Cor}\, \langle f\rangle X = \alpha X = \langle g\rangle X = \langle \mathrm{CoCompl}\, f\rangle X$. □

Conjecture 102. If f, g are (co-)complete funcoids then $f \cap^{\mathsf{FCD}} g$ is a (co-)complete funcoid.

Its consequence:

Conjecture 103. If f, g are discrete funcoids then $f \cap^{\mathsf{FCD}} g$ is a discrete funcoid.

An equivalent conjecture:

Conjecture 104. If f, g are discrete funcoids then $f \cap^{\mathsf{FCD}} g = f \cap g$.

Theorem 105. ComplFCD is an atomistic lattice.

Proof. Let $f \in \mathrm{Compl}\mathsf{FCD}$. $\langle f\rangle X = \bigcup^{\mathfrak{F}}\{\langle f\rangle x \mid x \in X\} = \bigcup^{\mathfrak{F}}\{\langle f|_{\{x\}}\rangle x \mid x \in X\}$, thus $f = \bigcup^{\mathsf{FCD}}\{f|_{\{x\}} \mid x \in X\}$. It is trivial that every $f|_{\{x\}}$ is a union of atoms of ComplFCD. □

Theorem 106. A funcoid is complete iff it is a join (on the lattice FCD) of atomic complete funcoids.

Proof. Follows from the theorem 87 and the previous theorem. □

Corollary 107. ComplFCD is join-closed.

Theorem 108. $\mathrm{Compl}(\bigcup^{\mathsf{FCD}} R) = \bigcup^{\mathsf{FCD}}\langle\mathrm{Compl}\rangle R$ for every set R of funcoids.

Proof. $\langle\mathrm{Compl}(\bigcup^{\mathsf{FCD}} R)\rangle X = \bigcup^{\mathfrak{F}}\{\langle\bigcup^{\mathsf{FCD}} R\rangle\{\alpha\} \mid \alpha \in X\} = \bigcup^{\mathfrak{F}}\{\bigcup^{\mathfrak{F}}\{\langle f\rangle\{\alpha\} \mid f \in R\} \mid \alpha \in X\} = \bigcup^{\mathfrak{F}}\{\bigcup^{\mathfrak{F}}\{\langle f\rangle\{\alpha\} \mid \alpha \in X\} \mid f \in R\} = \bigcup^{\mathfrak{F}}\{\langle\mathrm{Compl}\, f\rangle X \mid f \in R\} = \langle\bigcup^{\mathsf{FCD}}\langle\mathrm{Compl}\rangle R\rangle X$ for every set X. □

Lemma 109. Co-completion of a complete funcoid is complete.

Proof. Let f is a complete funcoid.

$\langle\mathrm{CoCompl}\ f\rangle X = \mathrm{Cor}\ \langle f\rangle X = \mathrm{Cor} \bigcup^{\mathfrak{F}}\{\langle f\rangle\{x\} \mid x \in X\} = \bigcup \{\mathrm{Cor}\ \langle f\rangle\{x\} \mid x \in X\} = \bigcup \{\langle\mathrm{CoCompl}\, f\rangle \{x\} \mid x \in X\}$ for every set X. Thus CoCompl f is complete. □

Theorem 110. $\mathrm{Compl}\,\mathrm{CoCompl}\, f = \mathrm{CoCompl}\,\mathrm{Compl}\, f = \mathrm{Cor}\, f$ for every funcoid f.

Proof. Compl CoCompl f is co-complete since (used the lemma) CoCompl f is co-complete. Thus Compl CoCompl f is a discrete funcoid. CoCompl f is the the greatest co-complete funcoid under f and Compl CoCompl f is the greatest complete funcoid under CoCompl f. So Compl CoCompl f is greater than any discrete funcoid under CoCompl f which is greater than any discrete funcoid under f. Thus Compl CoCompl f it is the greatest discrete funcoid under f. Thus $\mathrm{Compl}\,\mathrm{CoCompl}\, f = \mathrm{Cor}\, f$. Similarly $\mathrm{CoCompl}\,\mathrm{Compl}\, f = \mathrm{Cor}\, f$. □

Question 111. Is ComplFCD a co-brouwerian lattice?

3.13 Monovalued funcoids

Following the idea of definition of monovalued morphism let's call *monovalued* such a funcoid f that $f \circ f^{-1} \subseteq I_{\mathrm{im}\, f}$.

Obvious 112. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of funcoids is monovalued iff the funcoid f is monovalued.

Theorem 113. The following statements are equivalent for a funcoid f:

1. f is monovalued.
2. $\forall a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}: \langle f\rangle a \in \mathrm{atoms}^{\mathfrak{F}} \mho \cup \{\emptyset\}$.
3. $\forall \mathcal{I}, \mathcal{J} \in \mathfrak{F}: \langle f^{-1}\rangle(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \langle f^{-1}\rangle\mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{J}$.
4. $\forall I, J \in \mathscr{P}\mho: \langle f^{-1}\rangle(I \cap J) = \langle f^{-1}\rangle I \cap^{\mathfrak{F}} \langle f^{-1}\rangle J$.

Proof.

(2) ⇒ (3). Let $a \in \operatorname{atoms}^{\mathfrak{F}} \mho$, $\langle f \rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathfrak{F}} \mho \cup \{\emptyset\}$

$$(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} b \neq \emptyset \Leftrightarrow \mathcal{I} \cap^{\mathfrak{F}} b \neq \emptyset \wedge \mathcal{J} \cap^{\mathfrak{F}} b \neq \emptyset;$$
$$a[f](\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \Leftrightarrow a[f]\mathcal{I} \wedge a[f]\mathcal{J};$$
$$(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J})\left[f^{-1}\right]a \Leftrightarrow \mathcal{I}\left[f^{-1}\right]a \wedge \mathcal{J}\left[f^{-1}\right]a;$$
$$a \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \neq \emptyset \Leftrightarrow a \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mathcal{I} \neq \emptyset \wedge a \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mathcal{J} \neq \emptyset;$$
$$\left\langle f^{-1}\right\rangle(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \left\langle f^{-1}\right\rangle\mathcal{I} \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mathcal{J}.$$

(4) ⇒ (1). $\left\langle f^{-1}\right\rangle a \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle b = \emptyset$ for every two distinct atomic filter objects a and b. This is equivalent to $\neg(b\left[f^{-1}\right]\left\langle f^{-1}\right\rangle a)$; $\neg(\left\langle f^{-1}\right\rangle a[f]b)$; $b \cap^{\mathfrak{F}} \langle f \rangle \left\langle f^{-1}\right\rangle a = \emptyset$; $b \cap^{\mathfrak{F}} \left\langle f \circ f^{-1}\right\rangle a = \emptyset$; $\neg(a\left[f \circ f^{-1}\right]b)$. So $a\left[f \circ f^{-1}\right]b \Rightarrow a = b$ for every atomic filter objects a and b. This is possible only when $f \circ f^{-1} \subseteq I_{\operatorname{Dst} f}$.

(3) ⇒ (4). Obvious.

¬(2) ⇒ ¬(1). Suppose $\langle f \rangle a \notin \operatorname{atoms}^{\mathfrak{F}} \mathcal{B} \cup \{\emptyset\}$ for some $a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}$. Then there exist two atomic filter objects $p \neq q$ such that $\langle f \rangle a \supseteq p \wedge \langle f \rangle a \supseteq q$. Consequently $p \cap^{\mathfrak{F}} \langle f \rangle a \neq \emptyset$; $a \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle p \neq \emptyset$; $a \subseteq \left\langle f^{-1}\right\rangle p$; $\left\langle f \circ f^{-1}\right\rangle p = \langle f \rangle \left\langle f^{-1}\right\rangle p \supseteq \langle f \rangle a \supseteq q$; $\left\langle f \circ f^{-1}\right\rangle p \nsubseteq p$. So it cannot be $f \circ f^{-1} \subseteq I_{\operatorname{Dst} f}$. □

Corollary 114. A binary relation is a monovalued funcoid iff it is a function.

Remark 115. This corollary can be reformulated as follows: For binary relations the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

Proof. Because $\forall I, J \in \mathscr{P}\mho$: $\left\langle f^{-1}\right\rangle(I \cap J) = \left\langle f^{-1}\right\rangle I \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle J$ is true for a binary relation f if and only if it is a function. □

3.14 T_1- and T_2-separable funcoids

For funcoids can be generalized T_0-, T_1- and T_2- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 116. Let call T_1-*separable* such funcoid f that for every $\alpha, \beta \in \mho$ is true

$$\alpha \neq \beta \Rightarrow \neg(\{\alpha\}[f]\{\beta\})$$

Definition 117. Let call T_0-*separable* such funcoid f that $f \cap^{\mathsf{FCD}} f^{-1}$ is T_1-separable.

Definition 118. Let call T_2-*separable* such funcoid f that the funcoid $f^{-1} \circ f$ is T_1-separable.

For symmetric transitive funcoids T_1- and T_2-separability are the same (see theorem 12).

3.15 Filter objects closed regarding a funcoid

Definition 119. Let's call *closed* regarding a funcoid f such filter object $\mathcal{A}$ that $\langle f \rangle \mathcal{A} \subseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 120. If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding some funcoid), S is a set of closed filter objects, then

1. $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ is a closed filter object;
2. $\bigcap^{\mathfrak{F}} S$ is a closed filter object.

Proof. Let denote the given funcoid as f. $\langle f \rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f \rangle \mathcal{I} \cup^{\mathfrak{F}} \langle f \rangle \mathcal{J} \subseteq \mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$, $\langle f \rangle \bigcap^{\mathfrak{F}} S \subseteq \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle S \subseteq \bigcap^{\mathfrak{F}} S$. Consequently the filter objects $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ and $\bigcap^{\mathfrak{F}} S$ are closed. □

Proposition 121. If S is a set of closed regarding a complete funcoid filter objects, then the filter object $\bigcup^{\mathfrak{F}} S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle \bigcup^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle\langle f\rangle\rangle S \subseteq \bigcup^{\mathfrak{F}} S$ where f is the given funcoid. □

4 Reloids

Definition 122. I will call a *reloid* a filter object on the set of binary relations.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations (the set of binary relations is a subset of the set of reloids, I will call *discrete* these reloids which are binary relations).

Definition 123. The *reverse* reloid of a reloid f is defined by the formula

$$\operatorname{up} f^{-1} = \left\{F^{-1} \mid F \in \operatorname{up} f^{-1}\right\}.$$

Reverse reloid is a generalization of conjugate quasi-uniformity.

I will denote RLD either the set of reloids or the category of reloids (defined below), dependently on context.

4.1 Composition of reloids

Definition 124. Composition of reloids is defined by the formula

$$g \circ f = \bigcap^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.$$

Composition of reloids is a reloid.

Lemma 125. $(h \circ g) \circ f = h \circ (g \circ f)$ for every reloids f, g, h.

Proof. For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow (\forall K \in A \exists L \in B: L \subseteq K) \wedge (\forall K \in B \exists L \in A: L \subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B \circ A = \{L \circ K | K \in A, L \in B\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B: K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ exist $P' \in B \circ C$ such that $P' \subseteq P$; vice verse is analogous. So $A \circ C \sim B \circ C$.

$\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ g) \circ \operatorname{up} f$, $\operatorname{up}(h \circ g) \sim (\operatorname{up} h) \circ (\operatorname{up} g)$. By proven above $\operatorname{up}((h \circ g) \circ f) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

Analogously $\operatorname{up}(h \circ (g \circ f)) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

So $\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ (g \circ f))$ what is possible only if $\operatorname{up}((h \circ g) \circ f) = \operatorname{up}(h \circ (g \circ f))$. □

Theorem 126.

1. $f \circ f = \bigcap^{\mathsf{RLD}} \{F \circ F \mid F \in \operatorname{up} f\}$;
2. $f^{-1} \circ f = \bigcap^{\mathsf{RLD}} \left\{F^{-1} \circ F \mid F \in \operatorname{up} f\right\}$;
3. $f \circ f^{-1} = \bigcap^{\mathsf{RLD}} \left\{F \circ F^{-1} \mid F \in \operatorname{up} f\right\}$.

Proof. I will prove only (1) and (2) because (3) is analogous to (2).

1. Enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f: H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.

2. Enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f \colon H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq G^{-1} \circ F$. □

Conjecture 127. If f, g, h are reloids then

1. $f \circ (g \cup^{\mathsf{RLD}} h) = f \circ g \cup^{\mathsf{RLD}} f \circ h$;
2. $(g \cup^{\mathsf{RLD}} h) \circ f = g \circ f \cup^{\mathsf{RLD}} h \circ f$.

4.2 Direct product of filter objects

In theory of reloids direct product of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \overset{\text{def}}{=} \bigcap^{\mathfrak{F}} \{A \times B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\}.$$

Theorem 128. $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = \bigcup^{\mathfrak{F}} \{a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. Obviously

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \supseteq \bigcup^{\mathfrak{F}} \{a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\}$$

Reversely, let $K \in \operatorname{up} \bigcup^{\mathfrak{F}} \{a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\}$. Then $K \in \operatorname{up}(a \times^{\mathsf{RLD}} b)$ for every $a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}$, $b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}$; $K \supseteq X_a \times^{\mathsf{RLD}} Y_b$ for some $X_a \in \operatorname{up} a$, $Y_b \in \operatorname{up} b$; $K \supseteq \bigcup \{X_a \times Y_b \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{B}\} = \bigcup \{X_a \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}\} \times \bigcup \{Y_b \mid b \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}\} = A \times B$ where $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$; $K \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Theorem 129. $(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap^{\mathsf{RLD}} (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{B}_1)$ for every $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof.

$$\begin{aligned}
(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap^{\mathsf{RLD}} (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) &= \bigcap^{\mathsf{RLD}} \{P \cap Q \mid P \in \operatorname{up}(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0), Q \in \operatorname{up}(\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1)\} \\
&= \bigcap^{\mathsf{RLD}} \{(A_0 \times B_0) \cap (A_1 \times B_1) \mid A_0 \in \operatorname{up} \mathcal{A}_0, B_0 \in \operatorname{up} \mathcal{B}_0, A_1 \in \operatorname{up} \mathcal{A}_1, B_1 \in \operatorname{up} \mathcal{B}_1\} \\
&= \bigcap^{\mathsf{RLD}} \{(A_0 \cap A_1) \times (B_0 \cap B_1) \mid A_0 \in \operatorname{up} \mathcal{A}_0, B_0 \in \operatorname{up} \mathcal{B}_0, A_1 \in \operatorname{up} \mathcal{A}_1, B_1 \in \operatorname{up} \mathcal{B}_1\} \\
&= \bigcap^{\mathsf{RLD}} \{K \times L \mid K \in \operatorname{up}(\mathcal{A}_0 \cap \mathcal{A}_1), L \in \operatorname{up}(\mathcal{B}_0 \cap \mathcal{B}_1)\} \\
&= (\mathcal{A}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{B}_1).
\end{aligned}$$

□

Theorem 130. If $S \in \mathscr{P} \mathfrak{F}^2$ then

$$\bigcap^{\mathsf{RLD}} \{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} = \bigcap^{\mathfrak{F}} \operatorname{dom} S \times^{\mathsf{RLD}} \bigcap^{\mathfrak{F}} \operatorname{im} S.$$

Proof. Let $\mathcal{P} = \bigcap^{\mathfrak{F}} \operatorname{dom} S$, $\mathcal{Q} = \bigcap^{\mathfrak{F}} \operatorname{im} S$; $l = \bigcap^{\mathsf{RLD}} \{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\}$.

$\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q} \subseteq l$ is obvious.

Let $F \in \operatorname{up}(\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q})$. Then exist $P \in \operatorname{up} \mathcal{P}$ and $Q \in \operatorname{up} \mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \cap \ldots \cap P_n$ where $P_i \in \langle \operatorname{up} \rangle \operatorname{dom} S$ and $Q = Q_1 \cap \ldots \cap Q_m$ where $Q_i \in \langle \operatorname{up} \rangle \operatorname{im} S$.

$P \times Q = \bigcap_{i,j} (P_i \times Q_i)$.

$P_i \times Q_i \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some $(\mathcal{A}; \mathcal{B}) \in S$. $P \times Q = \bigcap_{i,j} (P_i \times Q_i) \supseteq l$. $F \in \operatorname{up} l$. □

Conjecture 131. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{RLD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice RLD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Definition 132. I will call a reloid *convex* iff it is a union of direct products.

Example 133. Non-convex reloids exist.

Proof. Let a is a non-trivial atomic f.o. Then $(=)|_a$ is non-convex. This follows from the fact that only direct products which are below $(=)$ are direct products of atomic f.o. and $(=)|_a$ is not their join. □

I will call two filter objects *isomorphic* when the corresponding filters are isomorphic (in the sense defined in [5]).

Theorem 134. The reloid $\{a\} \times^{\mathsf{RLD}} \mathcal{F}$ is isomorphic to the filter object $\mathcal{F}$ for every $a \in \mho$.

Proof. Consider $B = \{a\} \times \mho$ and $f = \{(x; (a; x)) \mid x \in \mho\}$. Then f is a bijection from $\mho$ to B.

If $X \in \operatorname{up} \mathcal{F}$ then $\langle f \rangle X \in B$ and $\langle f \rangle X = \{a\} \times X \in \operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$.

For every $Y \in \operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$ we have $Y = \{a\} \times X$ for some $X \in \operatorname{up} \mathcal{F}$ and thus $Y = \langle f \rangle X$.

So $\langle f \rangle|_{\operatorname{up} \mathcal{F} \cap \mathscr{P} \mho} = \langle f \rangle|_{\operatorname{up} \mathcal{F}}$ is a bijection from $\operatorname{up} \mathcal{F} \cap \mathscr{P} \mho$ to $\operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$.

We have $\operatorname{up} \mathcal{F} \cap \mathscr{P} \mho$ and $\operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$ directly isomorphic and thus $\operatorname{up} \mathcal{F}$ is isomorphic to $\operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$. □

4.3 Restricting reloid to a filter object. Domain and image

Definition 135. I call restricting a reloid f to a filter object $\mathcal{A}$ as $f|_{\mathcal{A}} = f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho)$.

Definition 136. *Domain* and *image* of a reloid f are defined as follows:

$$\operatorname{dom} f = \bigcap^{\mathfrak{F}} \langle \operatorname{dom} \rangle \operatorname{up} f; \quad \operatorname{im} f = \bigcap^{\mathfrak{F}} \langle \operatorname{im} \rangle \operatorname{up} f.$$

Proposition 137. $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$.

Proof.

Direct implication. Follows from $\operatorname{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \operatorname{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.

Reverse implication. $\operatorname{dom} f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} \exists F \in \operatorname{up} f: \operatorname{dom} F \subseteq A$. Analogously

$$\operatorname{im} f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \operatorname{up} \mathcal{B} \exists G \in \operatorname{up} f: \operatorname{im} G \subseteq B.$$

Let $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$, $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$. Then exist $F \in \operatorname{up} f$, $G \in \operatorname{up} f$ such that $\operatorname{dom} F \subseteq A \wedge \operatorname{im} G \subseteq B$. Consequently $F \cap G \in \operatorname{up} f$, $\operatorname{dom}(F \cap G) \subseteq A$, $\operatorname{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. We have exists $H \in \operatorname{up} f$ such that $H \subseteq A \times B$ for every $A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Definition 138. I call *identity reloid* for a filter object $\mathcal{A}$ the reloid $I_{\mathcal{A}} \overset{\text{def}}{=} (=)|_{\mathcal{A}}$.

Theorem 139. $I_{\mathcal{A}} = \bigcap^{\mathfrak{F}} \{I_A \mid A \in \operatorname{up} \mathcal{A}\}$ where I_A is the identity relation on a set A.

Proof. Let $K \in \operatorname{up} \bigcap^{\mathfrak{F}} \{I_A \mid A \in \operatorname{up} \mathcal{A}\}$, then exists $A \in \operatorname{up} \mathcal{A}$ such that $K \supseteq I_A$. Then $(=)|_{\mathcal{A}} = (=) \cap^{\mathsf{RLD}} (\mathcal{A} \times \mho) \subseteq (=) \cap (A \times \mho) = I|_A \subseteq K$; $K \in \operatorname{up} I_{\mathcal{A}}$. Reversely let $K \in \operatorname{up} I_{\mathcal{A}} = \operatorname{up}((=) \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho))$, then exists $A \in \operatorname{up} \mathcal{A}$ such that $K \in \operatorname{up}((=) \cap (A \times \mho)) = \operatorname{up} I_A \subseteq \operatorname{up} I_{\mathcal{A}}$. □

Proposition 140. $I_{\mathcal{A}}{}^{-1} = I_{\mathcal{A}}$.

Proof. Follows from the previous theorem. □

Theorem 141. $f|_{\mathcal{A}} = f \circ I_{\mathcal{A}}$ for every reloid f and filter object $\mathcal{A}$.

Proof. We need to prove that $f \cap^{\mathsf{RLD}} (\mathcal{A} \times \mho) = f \circ \bigcap^{\mathsf{RLD}} \{I_A \mid A \in \operatorname{up} \mathcal{A}\}$. $f \circ \bigcap^{\mathsf{RLD}} \{I_A \mid A \in \operatorname{up} \mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F \circ I_A \mid F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F|_A \mid F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F \cap (A \times \mho) \mid F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F \mid F \in \operatorname{up} f\} \cap \bigcap^{\mathsf{RLD}} \{A \times \mho \mid A \in \operatorname{up} \mathcal{A}\} = f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho)$. □

Theorem 142. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for every reloids f and g and filter object $\mathcal{A}$.

Proof. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ I_{\mathcal{A}} = g \circ (f \circ I_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

Theorem 143. $f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$ for every reloid f and filter objects $\mathcal{A}$ and $\mathcal{B}$.

Proof. $f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = f \cap^{\mathsf{RLD}} (\mathcal{A} \times^{\mathsf{RLD}} \mho) \cap^{\mathsf{RLD}} (\mho \times^{\mathsf{RLD}} \mathcal{B}) = f|_{\mathcal{A}} \cap^{\mathsf{RLD}} (\mho \times \mathcal{B}) = f \circ I_{\mathcal{A}} \cap^{\mathsf{RLD}} (\mho \times \mathcal{B}) = ((f \circ I_{\mathcal{A}})^{-1} \cap^{\mathsf{RLD}} (\mho \times^{\mathsf{RLD}} \mathcal{B})^{-1})^{-1} = ((I_{\mathcal{A}} \circ f^{-1}) \cap^{\mathsf{RLD}} (\mathcal{B} \times^{\mathsf{RLD}} \mho))^{-1} = (I_{\mathcal{A}} \circ f^{-1} \circ I_{\mathcal{B}})^{-1} = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$. □

4.4 Category of reloids

I will define the category RLD of reloids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a reloid such that $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

4.4.1 Monovalued reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f \circ f^{-1} \subseteq I_{\operatorname{im} f}$.

Obvious 144. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloids is monovalued iff the reloid f is monovalued.

Conjecture 145. If a reloid is monovalued then it is a monovalued function restricted to some filter object.

Conjecture 146. A reloid f is monovalued iff $\forall g \in \mathsf{RLD}: (g \subseteq f \Rightarrow \exists \mathcal{A} \in \mathfrak{F}: g = f|_{\mathcal{A}})$.

Conjecture 147. A monovalued reloid restricted to an atomic filter object is atomic or empty.

A weaker conjecture:

Conjecture 148. A (monovalued) function restricted to an atomic filter object is atomic or empty.

4.5 Complete reloids and completion of reloids

Definition 149. A *complete* reloid is a reloid representable as join of direct products $\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Definition 150. A *co-complete* reloid is a reloid representable as join of direct products $a \times^{\mathsf{RLD}} \{\beta\}$ where $\beta \in \mho$ and a is an atomic f.o.

I will denote the sets of complete and co-complete reloids correspondingly as ComplRLD and CoComplRLD.

Obvious 151. Complete and co-complete are dual.

Obvious 152. Complete and co-complete reloids are convex.

Obvious 153. Discrete reloids are complete and co-complete.

Conjecture 154. If a reloid is both complete and co-complete then it is discrete.

Conjecture 155. Composition of complete reloids is complete.

Obvious 156. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 157. $\mathrm{Compl}\mathsf{RLD}$ (with the induced order) is a complete lattice.

Definition 158. *Completion* and *co-completion* of a reloid f are defined by the formulas:

$$\mathrm{Compl}\, f = \mathrm{Cor}^{(\mathsf{RLD};\mathrm{Compl}\mathsf{RLD})} f \quad \text{and} \quad \mathrm{CoCompl}\, f = \mathrm{Cor}^{(\mathsf{RLD};\mathrm{CoCompl}\mathsf{RLD})} f.$$

Theorem 159. Atoms of the lattice $\mathrm{Compl}\mathsf{RLD}$ are exactly direct products of the form $\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\mathrm{Compl}\mathsf{RLD}$. Also $\emptyset$ is an element of $\mathrm{Compl}\mathsf{RLD}$.

$\{\alpha\} \times^{\mathsf{RLD}} b$ are atoms of $\mathrm{Compl}\mathsf{FCD}$ because these are atoms of RLD.

Remain to prove that if f is an atom of $\mathrm{Compl}\mathsf{RLD}$ then $f = \{\alpha\} \times^{\mathsf{RLD}} b$ for some $\alpha \in \mho$ and an atomic f.o. b.

Suppose f is a non-empty complete reloid. Then $\{\alpha\} \times^{\mathsf{RLD}} b \subseteq f$ for some $\alpha \in \mho$ and atomic f.o. b. If f is an atom then $f = \{\alpha\} \times^{\mathsf{FCD}} b$. □

Obvious 160. $\mathrm{Compl}\mathsf{RLD}$ is an atomistic lattice.

Conjecture 161. $\mathrm{Compl}\, f \cap^{\mathsf{RLD}} \mathrm{Compl}\, g = \mathrm{Compl}(f \cap^{\mathsf{RLD}} g)$ for every reloids f and g.

Conjecture 162. $\mathrm{Compl}(\bigcup^{\mathsf{RLD}} R) = \bigcup^{\mathsf{RLD}} \langle \mathrm{Compl} \rangle R$ for every set R of reloids.

Conjecture 163. $\mathrm{Compl}\, \mathrm{CoCompl}\, f = \mathrm{CoCompl}\, \mathrm{Compl}\, f = \mathrm{Cor}\, f$ for every reloid f.

Question 164. Is $\mathrm{Compl}\mathsf{RLD}$ a distributive lattice? Is $\mathrm{Compl}\mathsf{RLD}$ a co-brouwerian lattice?

Conjecture 165. If f is a complete reloid and R is a set of reloids then

$$f \circ \bigcup^{\mathsf{RLD}} R = \bigcup^{\mathsf{RLD}} \langle f \circ \rangle R.$$

This conjecture can be weakened:

Conjecture 166. If f is a discrete reloid and R is a set of reloids then

$$f \circ \bigcup^{\mathsf{RLD}} R = \bigcup^{\mathsf{RLD}} \langle f \circ \rangle R.$$

5 Relationships of funcoids and reloids

5.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD}) f$ by the following formulas:

$$\mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}\, f : \mathcal{X} [F] \mathcal{Y}$$
$$\langle (\mathsf{FCD}) f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \{ \langle F \rangle \mathcal{X} \mid F \in \mathrm{up}\, f \}.$$

We should prove that $(\mathsf{FCD}) f$ is really a funcoid. For this purpose we will additionally define

$$\left\langle (\mathsf{FCD}) f^{-1} \right\rangle \mathcal{Y} = \bigcap^{\mathfrak{F}} \left\{ \left\langle F^{-1} \right\rangle \mathcal{Y} \mid F \in \mathrm{up}\, f \right\}.$$

Proof. We need to prove that

$$\mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle (\mathsf{FCD}) f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle (\mathsf{FCD}) f^{-1} \right\rangle \mathcal{Y} \neq \emptyset.$$

The above formula is equivalent to:

$$\forall F \in \mathrm{up}\, f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}}\{\langle F\rangle\mathcal{X} \mid F \in \mathrm{up}\, f\} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}}\left\{\left\langle F^{-1}\right\rangle\mathcal{Y} \mid F \in \mathrm{up}\, f\right\} \neq \emptyset.$$

We have $\mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}}\{\langle F\rangle\mathcal{X} \mid F \in \mathrm{up}\, f\} = \bigcap^{\mathfrak{F}}\left\{\mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \mid F \in \mathrm{up}\, f\right\}$.

Let's denote $W = \left\{\mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \mid F \in \mathrm{up}\, f\right\}$.

We need to prove that $\bigcap^{\mathfrak{F}} W \neq \emptyset \Leftrightarrow \forall F \in \mathrm{up}\, f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \neq \emptyset$. (The rest follows from symmetry.)

Let's prove that W is a generalized filter base. For this enough to prove that $V = \{\langle F\rangle\mathcal{X} \mid F \in \mathrm{up}\, f\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \langle P\rangle\mathcal{X}$, $\mathcal{B} = \langle Q\rangle\mathcal{X}$ where $P, Q \in \mathrm{up}\, f$. Then for $\mathcal{C} = \langle P \cap Q\rangle\mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base.

From this by the corollary 4 follows that $\bigcap^{\mathfrak{F}} W \neq \emptyset \Leftrightarrow \emptyset \notin W \Leftrightarrow \forall F \in \mathrm{up}\, f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \neq \emptyset$. □

Theorem 167. $\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow (\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) \cap f \neq \emptyset$ for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$ and $f \in \mathsf{RLD}$.

Proof.

$$\begin{aligned}
(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) \cap^{\mathsf{RLD}} f \neq \emptyset &\Leftrightarrow \forall P \in \mathrm{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap^{\mathsf{RLD}} f \neq \emptyset \\
&\Leftrightarrow \forall F \in \mathrm{up}\, f, P \in \mathrm{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap F \neq \emptyset \\
&\Leftrightarrow \forall F \in \mathrm{up}\, f, X \in \mathrm{up}\mathcal{X}, Y \in \mathrm{up}\mathcal{Y}: (X \times^{\mathsf{RLD}} Y) \cap^{\mathsf{RLD}} F \neq \emptyset \\
&\Leftrightarrow \forall F \in \mathrm{up}\, f, X \in \mathrm{up}\mathcal{X}, Y \in \mathrm{up}\mathcal{Y}: X[F]Y \\
&\Leftrightarrow \forall F \in \mathrm{up}\, f: \mathcal{X}[F]\mathcal{Y} \\
&\Leftrightarrow \mathcal{X}[(\mathsf{FCD})f]\mathcal{Y}.
\end{aligned}$$

□

Theorem 168. $(\mathsf{FCD})f = \bigcap^{\mathsf{FCD}}\mathrm{up}\, f$ for every reloid f.

Proof. Let a is an atomic filter object.

$((\mathsf{FCD})f)a = \bigcap^{\mathfrak{F}}\{\langle F\rangle a \mid F \in \mathrm{up}\, f\}$ by the definition of (FCD).

$\left\langle\bigcap^{\mathsf{FCD}}\mathrm{up}\, f\right\rangle a = \bigcap^{\mathfrak{F}}\{\langle F\rangle a \mid F \in \mathrm{up}\, f\}$ by the theorem 55.

So $\langle(\mathsf{FCD})f\rangle a = \left\langle\bigcap^{\mathsf{FCD}}\mathrm{up}\, f\right\rangle a$ for every atomic filter object a. □

Lemma 169. $\langle g\rangle\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle S$ if g is a funcoid and S is a filter base.

Proof. $\mathrm{up}\bigcap^{\mathfrak{F}} S = \bigcup \langle\mathrm{up}\rangle S$ by the theorem 3.

$\langle g\rangle\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle\mathrm{up}\bigcap^{\mathfrak{F}} S$ by the theorem 32.

$\bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle\mathrm{up}\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle\bigcup \langle\mathrm{up}\rangle S$.

Easy to see that $\bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle\bigcup \langle\mathrm{up}\rangle S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle S$ because $S \subseteq \bigcup \langle\mathrm{up}\rangle S$.

Combining these equalities we produce $\langle g\rangle\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle S$. □

Lemma 170. For two sets of binary relations S and T and a set A

$$\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T \Rightarrow \bigcap^{\mathfrak{F}}\{\langle F\rangle A \mid F \in S\} = \bigcap^{\mathfrak{F}}\{\langle G\rangle A \mid G \in T\}$$

Proof. Let $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T$. Suppose $X \in \mathrm{up}\bigcap^{\mathfrak{F}}\{\langle F\rangle A \mid F \in S\}$. Then $X' \in \{\langle F\rangle A \mid F \in S\}$ where $X \supseteq X'$. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$. So $Y' = \langle G\rangle A \subseteq X' \subseteq X$. $Y' \in \{\langle G\rangle A \mid G \in T\}$; $Y' \in \mathrm{up}\bigcap^{\mathfrak{F}}\{\langle G\rangle A \mid G \in T\}$; $X \in \mathrm{up}\bigcap^{\mathfrak{F}}\{\langle G\rangle A \mid G \in T\}$. The reverse is symmetric. □

Theorem 171. $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ for every reloids f and g.

Proof.

$$\begin{aligned}
\langle(\mathsf{FCD})(g \circ f)\rangle X &= \bigcap^{\mathfrak{F}}\{\langle H\rangle X \mid H \in \mathrm{up}(g \circ f)\} \\
&= \bigcap^{\mathfrak{F}}\left\{\langle H\rangle X \mid H \in \mathrm{up}\bigcap^{\mathsf{RLD}}\{G \circ F \mid F \in \mathrm{up}\, f, G \in \mathrm{up}\, g\}\right\}.
\end{aligned}$$

Obviously

$$\bigcap^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{up}f,G\in\operatorname{up}g\}=\bigcap^{\mathsf{RLD}}\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{up}f,G\in\operatorname{up}g\};$$

from this by the lemma 170

$$\bigcap^{\mathfrak{F}}\left\{\langle H\rangle X\mid H\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{up}f,G\in\operatorname{up}g\}\right\}=\bigcap^{\mathfrak{F}}\{\langle G\circ F\rangle X\mid F\in\operatorname{up}f,G\in\operatorname{up}g\}.$$

On the other side

$$\begin{aligned}\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X &= \langle(\mathsf{FCD})g\rangle\langle(\mathsf{FCD})f\rangle X\\ &= \langle(\mathsf{FCD})g\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X\mid F\in\operatorname{up}f\}\\ &= \bigcap^{\mathfrak{F}}\left\{\langle G\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X\mid F\in\operatorname{up}f\}\mid G\in\operatorname{up}g\right\}.\end{aligned}$$

Let's prove that $\{\langle F\rangle X\mid F\in\operatorname{up}f\}$ is a filter base. If $A,B\in\{\langle F\rangle X\mid F\in\operatorname{up}f\}$ then $A=\langle F_1\rangle X$ and $B=\langle F_2\rangle X$ where $F_1,F_2\in\operatorname{up}f$. $A\cap B\supseteq\langle F_1\cap F_2\rangle X\in\{\langle F\rangle X\mid F\in\operatorname{up}f\}$. So $\{\langle F\rangle X\mid F\in\operatorname{up}f\}$ is really a filter base.

By the lemma 169 $\langle G\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X\mid F\in\operatorname{up}f\}=\bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X\mid F\in\operatorname{up}f\}$. So continuing the above equalities,

$$\begin{aligned}\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X &= \bigcap^{\mathfrak{F}}\left\{\bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X\mid F\in\operatorname{up}f\}\mid G\in\operatorname{up}g\right\}\\ &= \bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X\mid F\in\operatorname{up}f,G\in\operatorname{up}g\}\\ &= \bigcap^{\mathfrak{F}}\{\langle G\circ F\rangle X\mid F\in\operatorname{up}f,G\in\operatorname{up}g\}.\end{aligned}$$

Combining these equalities we get $\langle(\mathsf{FCD})(g\circ f)\rangle X=\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X$ for every set X. □

5.2 Reloids induced by funcoid

Every funcoid f induces a reloid in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}}f\overset{\mathrm{def}}{=}\bigcap^{\mathsf{RLD}}\operatorname{up}f$$

$$(\mathsf{RLD})_{\mathrm{in}}f\overset{\mathrm{def}}{=}\bigcup^{\mathsf{RLD}}\left\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid\mathcal{A},\mathcal{B}\in\mathfrak{F},\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\right\}$$

Proposition 172. $\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f=\operatorname{up}f$.

Proof. Because $\operatorname{up}f$ is a filter. □

Theorem 173. $(\mathsf{RLD})_{\mathrm{in}}f=\bigcup^{\mathsf{RLD}}\left\{a\times^{\mathsf{RLD}}b\mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho,a\times^{\mathsf{FCD}}b\subseteq f\right\}$.

Proof. Follows from the theorem 128. □

Lemma 174. $F\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f\Leftrightarrow\forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a[f]b\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b)$ for a funcoid f.

Proof.

$$\begin{aligned}F\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f &\Leftrightarrow F\in\operatorname{up}\bigcup^{\mathfrak{F}}\left\{a\times^{\mathsf{RLD}}b\mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho,a\times^{\mathsf{FCD}}b\subseteq f\right\}\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a\times^{\mathsf{FCD}}b\subseteq f\Rightarrow F\in\operatorname{up}(a\times^{\mathsf{RLD}}b))\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a\times^{\mathsf{FCD}}b\cap^{\mathsf{FCD}}f\neq\emptyset\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b)\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a[f]b\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b).\end{aligned}$$

□

Surprisingly a funcoid is greater inward than outward:

Theorem 175. $(\mathsf{RLD})_{\mathrm{out}}f\subseteq(\mathsf{RLD})_{\mathrm{in}}f$ for a funcoid f.

Proof. We need to prove

$$\bigcap^{\mathsf{RLD}}\{F \mid F\in\mathscr{P}\mho^2, F\supseteq f\}\subseteq\bigcup^{\mathsf{RLD}}\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B} \mid \mathcal{A},\mathcal{B}\in\mathfrak{F}, \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\}.$$

Let

$$K\in\operatorname{up}\bigcup^{\mathfrak{F}}\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B} \mid \mathcal{A},\mathcal{B}\in\mathfrak{F}, \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\}.$$

Then

$$\begin{aligned} K &= \bigcup\{X_{\mathcal{A}}\times Y_{\mathcal{B}} \mid \mathcal{A},\mathcal{B}\in\mathfrak{F}, \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\} \\ &= \bigcup^{\mathsf{RLD}}\{X_{\mathcal{A}}\times Y_{\mathcal{B}} \mid \mathcal{A},\mathcal{B}\in\mathfrak{F}, \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\} \\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}}\in\operatorname{up}\mathcal{A}$, $Y_{\mathcal{B}}\in\operatorname{up}\mathcal{B}$. $K\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{F \mid F\in\mathscr{P}\mho^2, F\supseteq f\}$. □

Theorem 176. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}f=f$ for every funcoid f.

Proof. $\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f=\operatorname{up}f$.

$(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}f=\bigcap^{\mathsf{FCD}}\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f=\bigcap^{\mathsf{FCD}}\operatorname{up}f$.

$\bigcap^{\mathsf{FCD}}\operatorname{up}f=f$ by the theorem 89. So $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}f=f$. □

Conjecture 177. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}f=f$ for every funcoid f.

Conjecture 178. For every funcoid f and reloid g

$$(\mathsf{RLD})_{\mathrm{out}}f\subseteq g\subseteq(\mathsf{RLD})_{\mathrm{in}}f\Leftrightarrow(\mathsf{FCD})g=f.$$

Conjecture 179. For a convex reloid f

1. $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD})f=f$;
2. $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f=f$.

6 Continuous morphisms

This section will use the apparatus from the section "Partially ordered dagger categories".

6.1 Traditional definitions of continuity

6.1.1 Pre-topology

Let μ and ν are funcoids representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall\epsilon\in\operatorname{up}\langle\nu\rangle fa\exists\delta\in\operatorname{up}\langle\mu\rangle\{a\}:\langle f\rangle\delta\subseteq\epsilon.$$

Equivalently transforming this formula we get:

$$\begin{gathered}\forall\epsilon\in\operatorname{up}\langle\nu\rangle fa:\langle f\rangle\langle\mu\rangle\{a\}\subseteq\epsilon;\\ \langle f\rangle\langle\mu\rangle\{a\}\subseteq\langle\nu\rangle fa;\\ \langle f\rangle\langle\mu\rangle\{a\}\subseteq\langle\nu\rangle\langle f\rangle\{a\}.\end{gathered}$$

f is a continuous map from μ to ν in every point of its domain iff $\langle f\rangle\langle\mu\rangle\subseteq\langle\nu\rangle\langle f\rangle$ what is equivalent to $f\circ\mu\subseteq\nu\circ f$.

6.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a nearness-continuous map from μ to ν iff

$$\forall X,Y\in\mathscr{P}\mho:(X[\mu]Y\Rightarrow(\langle f\rangle X)[\nu](\langle f\rangle Y)).$$

Equivalently transforming this formula we get:

$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y \cap \langle \nu\rangle\langle f\rangle X \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y \cap \langle \nu \circ f\rangle X \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X[\nu \circ f]\langle f\rangle Y);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y\left[(\nu \circ f)^{-1}\right]X);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y\left[f^{-1} \circ \nu^{-1}\right]X);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X \cap \left\langle f^{-1} \circ \nu^{-1}\right\rangle\langle f\rangle Y \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X \cap \left\langle f^{-1} \circ \nu^{-1} \circ f\right\rangle Y \neq \emptyset);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow Y\left[f^{-1} \circ \nu^{-1} \circ f\right]X);$$
$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X\left[f^{-1} \circ \nu \circ f\right]Y);$$
$$\mu \subseteq f^{-1} \circ \nu \circ f.$$

So a function f is nearness-continuous iff $\mu \subseteq f^{-1} \circ \nu \circ f$.

6.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu \forall (x; y) \in \delta: (f x; f y) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu \forall (x; y) \in \delta: \{(f x; f y)\} \subseteq \epsilon$$
$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu \forall (x; y) \in \delta: f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon$$
$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu: f \circ \delta \circ f^{-1} \subseteq \epsilon$$
$$\forall \epsilon \in \mathrm{up}\,\nu: f \circ \mu \circ f^{-1} \subseteq \epsilon$$
$$f \circ \mu \circ f^{-1} \subseteq \nu.$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \subseteq \nu$.

6.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \subseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f \in \mathrm{C}'(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge \mu \subseteq f^{\dagger} \circ \nu \circ f;$$
$$f \in \mathrm{C}''(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \circ f^{\dagger} \subseteq \nu.$$

Remark 180. In the examples about funcoids and reloids the "dagger functor" is the inverse of a funcoid or reloid, that is $f^{\dagger} = f^{-1}$.

Proposition 181. Every of these three definitions of continuity forms a sub-precategory (subcategory if the original precategory is a category).

Proof.

C. Let $f \in \mathrm{C}(\mu; \nu)$, $g \in \mathrm{C}(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}(\mu; \mu)$ is obvious.

***C′*.** Let $f \in \mathrm{C}'(\mu; \nu)$, $g \in \mathrm{C}'(\nu; \pi)$. Then $\mu \subseteq f^{\dagger} \circ \nu \circ f$, $\nu \subseteq g^{\dagger} \circ \pi \circ g$;

$$\mu \subseteq f^{\dagger} \circ g^{\dagger} \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^{\dagger} \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}'(\mu; \mu)$ is obvious.

C″. Let $f \in \mathrm{C}''(\mu; \nu)$, $g \in \mathrm{C}''(\nu; \pi)$. Then $f \circ \mu \circ f^{\dagger} \subseteq \nu$, $g \circ \nu \circ g^{\dagger} \subseteq \pi$;

$$g \circ f \circ \mu \circ f^{\dagger} \circ g^{\dagger} \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^{\dagger} \subseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}''(\mu; \mu)$ is obvious. □

Proposition 182. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}'(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof. Let $f \in \mathrm{C}'(\mu; \nu)$. Then $\mu \subseteq f^{\dagger} \circ \nu \circ f$; $f \circ \mu \subseteq f \circ f^{\dagger} \circ \nu \circ f \subseteq 1_{\mathrm{Dst}\,f} \circ \nu \circ f = \nu \circ f$; $f \in \mathrm{C}(\mu; \nu)$.

Let $f \in \mathrm{C}(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f \circ \mu \circ f^{\dagger} \subseteq \nu \circ f \circ f^{\dagger} \subseteq \nu \circ 1_{\mathrm{Dst}\,f} = \nu$; $f \in \mathrm{C}''(\mu; \nu)$. □

Proposition 183. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}''(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}'(\mu; \nu).$$

Proof. Let $f \in \mathrm{C}''(\mu; \nu)$. Then $f \circ \mu \circ f^{\dagger} \subseteq \nu$; $f \circ \mu \circ f^{\dagger} \circ f \subseteq \nu \circ f$; $f \circ \mu \circ 1_{\mathrm{Src}\,f} \subseteq \nu \circ f$; $f \circ \mu \subseteq \nu \circ f$; $f \in \mathrm{C}(\mu; \nu)$.

Let $f \in \mathrm{C}(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f^{\dagger} \circ f \circ \mu \subseteq f^{\dagger} \circ \nu \circ f$; $1_{\mathrm{Src}\,f} \circ \mu \subseteq f^{\dagger} \circ \nu \circ f$; $\mu \subseteq f^{\dagger} \circ \nu \circ f$; $f \in \mathrm{C}'(\mu; \nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 184. If f is a monovalued and entirely defined morphism then

$$f \in \mathrm{C}'(\mu; \nu) \Leftrightarrow f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 185. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu; \nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD}) f \in \mathrm{C}'((\mathsf{FCD})\mu; (\mathsf{FCD})\nu)$.

Exercise 1. I leave a simple exercise for the reader to prove the last theorem.

6.3 Continuousness of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to *identity element* I_A of the semigroup (for example identity funcoid or identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is “dagger” (that is is a dagger precategory) we will require also $(I_A)^{\dagger} = I_A$.

We can define *restricting* an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 186. For every elements f, μ, ν of our semigroup and an object A

1. $f \in \mathrm{C}(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu)$;
2. $f \in \mathrm{C}'(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu)$;
3. $f \in \mathrm{C}''(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof.

1. $f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu; \nu)$.
2. $f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^\dagger \circ \nu \circ f|_A \Leftarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A \Leftarrow \mu \subseteq f^\dagger \circ \nu \circ f \Leftrightarrow f \in \mathrm{C}'(\mu; \nu)$.
3. $f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^\dagger \subseteq \nu \Leftarrow f \circ \mu \circ f^\dagger \subseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu; \nu)$. □

7 Connectedness regarding funcoids and reloids

7.1 Some lemmas

Lemma 187. If $\neg(A[f]B) \wedge A \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$ then f is closed on A for a funcoid f and sets A and B.

Proof. $\neg(A[f]B) \Leftrightarrow B \cap \langle f \rangle A = \emptyset \Leftrightarrow (\operatorname{dom} f \cup \operatorname{im} f) \cap B \cap \langle f \rangle A = \emptyset \Rightarrow (\operatorname{dom} f \cup \operatorname{im} f \setminus A) \cap \langle f \rangle A = \emptyset \Leftrightarrow \langle f \rangle A \subseteq A$. □

Corollary 188. If $\neg(A[f]B) \wedge A \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$ then f is closed on $A \setminus B$ for a funcoid f and sets A and B.

Proof. Let $\neg(A[f]B) \wedge A \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$. Then $\neg((A \setminus B)[f]B) \wedge (A \setminus B) \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$. □

Lemma 189. If $\neg(A[f]B) \wedge A \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$ then $\neg(A[f^n]B)$ for every whole positive n.

Proof. Let $\neg(A[f]B) \wedge A \cup B \supseteq \operatorname{dom} f \cup \operatorname{im} f$. From the above proposition $\langle f \rangle A \subseteq A$. $B \cap \langle f \rangle A = \emptyset$, consequently $\langle f \rangle A \subseteq A \setminus B$. Because (by the above corollary) f is closed on $A \setminus B$, then $\langle f \rangle \langle f \rangle A \subseteq A \setminus B$, $\langle f \rangle \langle f \rangle \langle f \rangle A \subseteq A \setminus B$, etc. So $\langle f^n \rangle A \subseteq A \setminus B$, $B \cap \langle f^n \rangle A = \emptyset$, $\neg(A[f^n]B)$. □

7.2 Endomorphism series

Definition 190. $S_1(\mu) \overset{\text{def}}{=} \mu \cup \mu^2 \cup \mu^3 \cup \ldots$ for an endomorphism μ of a precategory with countable union of morphisms.

Definition 191. $S(\mu) \overset{\text{def}}{=} \mu^0 \cup S_1(\mu)$ where $\mu^0 \overset{\text{def}}{=} I_{\mathrm{Ob}\,\mu}$ (identity morphism for the object Ob μ) where Ob μ is the object of endomorphism μ for an endomorphism μ of a category with countable union of morphisms.

I call S_1 and S *endomorphism series*.

We will consider the collection of all binary relations (on a set $\mho$), as well as the collection of all funcoids and the collection of all reloids, as categories with single object $\mho$ and the identity morphism $(=)$ or $(=)|_{\mho}$.

So if μ is a binary relation or a funcoid or a reloid we have
$S_1(\mu) = \mu \cup \mu^2 \cup \mu^3 \cup \ldots$ and $S(\mu) = (=) \cup \mu \cup \mu^2 \cup \mu^3 \cup \ldots$

Proposition 192. $S(\mu)$ is transitive for the category of binary relations.

Proof.

$$\begin{aligned} S(\mu) \circ S(\mu) &= \mu^0 \circ S(\mu) \cup \mu \circ S(\mu) \cup \mu^2 \circ S(\mu) \cup \ldots \\ &= (\mu^0 \cup \mu^1 \cup \mu^2 \cup \ldots) \cup (\mu^1 \cup \mu^2 \cup \mu^3 \cup \ldots) \cup (\mu^2 \cup \mu^3 \cup \mu^4 \cup \ldots) \\ &= \mu^0 \cup \mu^1 \cup \mu^2 \cup \ldots \\ &= S(\mu). \end{aligned}$$

□

7.3 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations.

Definition 193. A set A is called *(strongly) connected* regarding a binary relation μ when

$$\forall X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}: (X \cup Y = A \Rightarrow X [\mu] Y).$$

Definition 194. *Path* between two elements $a, b \in \mho$ in a set A through binary relation μ is the finite sequence $x_0 \ldots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i (\mu \cap A \times A) x_{i+1}$ for every $i = 0, \ldots, n - 1$. n is called path *length*.

Proposition 195. There exists path between every element $a \in \mho$ and that element itself.

Proof. It is the path consisting of one vertex (of length 0). □

Proposition 196. There is a path from element a to element b in a set A through a binary relation μ iff $a\, (S(\mu \cap A \times A))\, b$ (that is $(a, b) \in S(\mu \cap A \times A)$).

Proof.

- $\Rightarrow$. If exists a path from a to b, then $\{b\} \subseteq \langle (\mu \cap A \times A)^n \rangle \{a\}$ where n is the path length. Consequently $\{b\} \subseteq \langle S(\mu \cap A \times A) \rangle \{a\}$; $a\, (S(\mu \cap A \times A))\, b$.
- $\Leftarrow$. If $a\, (S(\mu \cap A \times A))\, b$ then exists $n \in \mathbb{N}$ such that $a\, (\mu \cap A \times A)^n\, b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \ldots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i\, (\mu \cap A \times A)\, x_{i+1}$ for every $i = 0, \ldots, n - 1$. That is there is path from a to b. □

Theorem 197. The following statements are equivalent for a relation μ and a set A:

1. For every $a, b \in A$ there is a path between a and b in A through μ.
2. $S(\mu \cap A \times A) \supseteq A \times A$.
3. $S(\mu \cap A \times A) = A \times A$.

4. A is connected regarding μ.

Proof.

(1) ⇒ (2). Let for every a, $b \in A$ there is a path between a and b in A through μ. Then $a\,(S(\mu \cap A \times A))\,b$ for every $a, b \in A$. It is possible only when $S(\mu \cap A \times A) \supseteq A \times A$.

(3) ⇒ (1). For every two vertices a and b we have $a\,(S(\mu \cap A \times A))\,b$. So (by the previous theorem) for every two vertices a and b exist path from a to b.

(3) ⇒ (4). Suppose that $\neg(X[\mu \cap A \times A]Y)$ for some $X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg(X[(\mu \cap A \times A)^n]Y)$ for every $n \in \mathbb{N}$. Consequently $\neg(X[S(\mu \cap A \times A)]Y)$. So $S(\mu \cap A \times A) \neq A \times A$.

(4) ⇒ (3). If $\langle S(\mu \cap A \times A)\rangle\{v\} = A$ for every vertex v then $S(\mu \cap A \times A) = A \times A$. Consider the remaining case when $V \overset{\text{def}}{=} \langle S(\mu \cap A \times A)\rangle\{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. Then $V \cup W = A$ and so $V[\mu]W$ what is equivalent to $V[\mu \cap A \times A]W$ that is $\langle \mu \cap A \times A\rangle V \cap W \neq \emptyset$. This is impossible because $\langle \mu \cap A \times A\rangle V = \langle \mu \cap A \times A\rangle\langle S(\mu \cap A \times A)\rangle V = \langle S_1(\mu \cap A \times A)\rangle V \subseteq \langle S(\mu \cap A \times A)\rangle V = V$.

(2) ⇒ (3). Because $S(\mu \cap A \times A) \subseteq A \times A$. □

Corollary 198. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap A \times A$.

Definition 199. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A.

Theorem 200. The set A is partitioned into connected components (regarding every binary relation F).

Proof. Consider the binary relation $a \sim b \Leftrightarrow a\,(S(F))\,b \wedge b\,(S(F))\,a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. □

Proposition 201. A set is connected (regarding a binary relation) iff it has one connected component.

Proof. Direct implication is obvious. Reverse is proved by contradiction. □

7.4 Connectedness regarding funcoids and reloids

Definition 202. $S_1^*(\mu) = \bigcap^{\mathfrak{F}}\{S_1(M) \mid M \in \operatorname{up}\mu\}$ for a reloid μ.

Definition 203. *Connectivity reloid* $S^*(\mu)$ for a reloid μ is defined as follows:

$$S^*(\mu) = \bigcap^{\mathfrak{F}}\{S(M) \mid M \in \operatorname{up}\mu\}.$$

Remark 204. Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

Proposition 205. $S^*(\mu) = (=) \cup^{\mathsf{RLD}} S_1^*(\mu)$ for every reloid μ.

Proof. Follows from the theorem about distributivity of $\cup$ regarding $\bigcap^{\mathfrak{F}}$ (see [5]). □

Proposition 206. $S^*(\mu) = S(\mu)$ if μ is a binary relation.

1. In some math literature these two words are used interchangeably.

Proof. $S^*(\mu) = \bigcap^{\mathfrak{F}}\{S(\mu)\} = S(\mu)$. □

Definition 207. A filter $\mathcal{A}$ is called *connected* regarding a reloid μ when $S^*(\mu \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}) \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Obvious 208. A filter $\mathcal{A}$ is connected regarding a reloid μ when $S^*(\mu \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Definition 209. A filter $\mathcal{A}$ is called *connected* regarding a funcoid μ when

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F} \setminus \{\emptyset\}: (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = \mathcal{A} \Rightarrow \mathcal{X} [\mu] \mathcal{Y}).$$

Proposition 210. A set A is connected regarding a binary relation μ iff it is connected regarding μ considered as a reloid.

Proof. $S^*(\mu \cap^{\mathsf{RLD}} A \times^{\mathsf{RLD}} A) = S^*(\mu \cap A \times A) = S(\mu \cap A \times A)$. So $S^*(\mu \cap^{\mathsf{RLD}} A \times^{\mathsf{RLD}} A) \supseteq A \times^{\mathsf{RLD}} A \Leftrightarrow S(\mu \cap A \times A) \supseteq A \times A$. □

Obvious 211. A filter is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Obvious 212. A filter is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mathcal{A}$.

Theorem 213. A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding every $F \in \operatorname{up} f$ (considered as a reloid).

Proof.

- $\Rightarrow$. Obvious.
- $\Leftarrow$. F is connected iff $S(F) = F^0 \cup F^1 \cup F^2 \cup \ldots \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.
  $S^*(f) = \bigcap^{\mathfrak{F}}\{S(F) \mid F \in \operatorname{up} f\} \supseteq \bigcap^{\mathfrak{F}}\{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A} \mid F \in \operatorname{up} f\} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$. □

Conjecture 214. A filter $\mathcal{A}$ is connected regarding a funcoid μ iff $\mathcal{A}$ is connected for every binary relation $F \in \operatorname{up} \mu$ (considered as a funcoid).

Conjecture 215. A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD})f$.

Conjecture 216. A filter is connected regarding a binary relation considered as a funcoid iff it is connected regarding this binary relation considered as a reloid.

7.5 Algebraic properties of S and S^*

Theorem 217. $S^*(S^*(f)) = S^*(f)$ for every reloid f.

Proof. $S^*(S^*(f)) = \bigcap^{\mathfrak{F}}\{S(R) \mid R \in \operatorname{up} S^*(f)\} \subseteq \bigcap^{\mathfrak{F}}\{S(R) \mid R \in \{S(F) \mid F \in \operatorname{up} f\}\} = \bigcap^{\mathfrak{F}}\{S(S(F)) \mid F \in \operatorname{up} f\} = \bigcap^{\mathfrak{F}}\{S(F) \mid F \in \operatorname{up} f\} = S^*(f)$.

So $S^*(S^*(f)) \subseteq S^*(f)$. $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. □

Corollary 218. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for any reloid f.

Proof. Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.

But $S^*(S(f)) \subseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \subseteq S^*(S^*(f)) = S^*(f)$. □

Conjecture 219. $S(S(f)) = S(f)$ for

1. every reloid f;

2. every funcoid f.

Conjecture 220. For every reloid f

1. $S(f) \circ S(f) = S(f)$;
2. $S^*(f) \circ S^*(f) = S^*(f)$;
3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 221. $S(f) \circ S(f) = S(f)$ for every funcoid f.

8 Postface

8.1 Misc

See this Web page for my research plans: http://www.mathematics21.org/agt-plans.html

I deem that now two most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids.

Also a future research topic are n-ary (where n is an ordinal, or more generally an index set) funcoids and reloids (plain funcoids and reloids are binary by analogy with binary relations).

We should also research relationships between complete funcoids and complete reloids.

8.2 Pointfree funcoids and reloids

I have set wiki site http://funcoids.wikidot.com to write on that site the pointfree variant of the theory of funcoids and reloids (that is generalized funcoids on arbitrary lattices rather than funcoids on a lattice of sets as in this work).

However I consider for me research of pointfree funcoids and pointfree reloids a low priority project. (There are yet enough research topics in the point-set topology and I don't want to meddle into pointfree topology in foreseeable future.)

The work about pointfree funcoids and reloids seems being largely technical and boring. Pointfree theory of funcoids and reloids seems being a trivial generalization of the theory of point-set funcoids and reloids. It is not similar to the traditional pointfree topology which is not an obvious generalization of point-set topology.

But if someone indeed wishes to treat pointfree funcoids, please use the above menioned wiki.

Appendix A Some counter-examples

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap^{\mathfrak{F}} \{(-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}.$$

Example 222. There exist a funcoid f and a set S of funcoids such that $f \cap^{\mathsf{FCD}} \bigcup^{\mathsf{FCD}} S \neq \bigcup^{\mathsf{FCD}} \langle f \cap^{\mathsf{FCD}} \rangle S$.

Proof. Let $f = \Delta \times^{\mathsf{FCD}} \{0\}$ and $S = \{(\varepsilon; +\infty) \times^{\mathsf{FCD}} \{0\} \mid \varepsilon > 0\}$. Then $f \cap^{\mathsf{FCD}} \bigcup^{\mathsf{FCD}} S = (\Delta \times^{\mathsf{FCD}} \{0\}) \cap^{\mathsf{FCD}} ((0; +\infty) \times^{\mathsf{FCD}} \{0\}) = (\Delta \cap^{\mathsf{FCD}} (0; +\infty)) \times^{\mathsf{FCD}} \{0\} \neq \emptyset$ while $\bigcup^{\mathsf{FCD}} \langle f \cap^{\mathsf{FCD}} \rangle S = \bigcup^{\mathsf{FCD}} \{\emptyset\} = \emptyset$. □

Conjecture 223. There exist a set R of funcoids and a funcoid f such that $f \circ \bigcup^{\mathsf{FCD}} R \neq \bigcup^{\mathsf{FCD}} \langle f \circ \rangle R$.

Example 224. There exist a set R of funcoids and f.o. $\mathcal{X}$ and $\mathcal{Y}$ such that

1. $\mathcal{X}\left[\bigcup^{\mathsf{FCD}} R\right]\mathcal{Y} \wedge \nexists f \in R: \mathcal{X}[f]\mathcal{Y}$;
2. $\left\langle\bigcup^{\mathsf{FCD}} R\right\rangle\mathcal{X} \supset \bigcup^{\mathfrak{F}}\{\langle f\rangle\mathcal{X} \mid f \in R\}$.

Proof.

1. Let $\mathcal{X} = \Delta$ and $\mathcal{Y} = \mathbb{R}$. Let $R = \left\{(\varepsilon; +\infty) \times^{\mathsf{FCD}} \mathbb{R} \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\right\}$. Then $\bigcup^{\mathsf{FCD}} R = (0; +\infty) \times^{\mathsf{FCD}} \mathbb{R}$. So $\mathcal{X}\left[\bigcup^{\mathsf{FCD}} R\right]\mathcal{Y}$ and $\forall f \in R: \neg(\mathcal{X}[f]\mathcal{Y})$.
2. With the same $\mathcal{X}$ and R we have $\left\langle\bigcup^{\mathsf{FCD}} R\right\rangle\mathcal{X} = \mathbb{R}$ and $\langle f\rangle\mathcal{X} = \emptyset$ for every $f \in R$, thus $\bigcup^{\mathfrak{F}}\{\langle f\rangle\mathcal{X} \mid f \in R\} = \emptyset$. □

Theorem 225. For a f.o. a we have $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ only in the case if $a = \emptyset$ or a is a trivial atomic f.o. (that is an one-element set).

Proof. If $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ then exists $m \in \operatorname{up}(a \times^{\mathsf{RLD}} a)$ such that $m \subseteq (=)|_{\mho}$. Consequently exist $A, B \in \operatorname{up} a$ such that $A \times B \subseteq (=)|_{\mho}$ what is possible only in the case when $A = B = a$ is an one-element set or empty set. □

Corollary 226. Direct product (in the sense of reloids) of non-trivial atomic filter objects is non-atomic.

Proof. Obviously $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \neq \emptyset$ and $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \subset a \times^{\mathsf{RLD}} a$. □

Example 227. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object and $x \in \mho$. Then

$$(a \times \{x\}) \circ (\{x\} \times a) = \bigcap^{\mathfrak{F}}\{(A \times \{x\}) \circ (\{x\} \times A) \mid A \in \operatorname{up} a\} = \bigcap^{\mathfrak{F}}\{A \times A \mid A \in \operatorname{up} a\} = a \times a$$

is non-atomic despite of $a \times \{x\}$ and $\{x\} \times a$ are atomic. □

Example 228. There exists non-monovalued atomic reloid.

Proof. From the previous example follows that the atomic reloid $\{x\} \times a$ is not monovalued. □

Example 229. $(\mathsf{RLD})_{\mathrm{in}} f \neq (\mathsf{RLD})_{\mathrm{out}} f$ for a funcoid f.

Proof. Let $f = (=)|_{\mho}$. Then $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup^{\mathsf{RLD}}\left\{a \times^{\mathsf{RLD}} a \mid a \in \operatorname{atoms}^{\mathfrak{F}} \mho\right\}$ and $(\mathsf{RLD})_{\mathrm{out}} f = (=)|_{\mho}$. But as we shown above $a \times^{\mathsf{RLD}} a \nsubseteq (=)|_{\mho}$ for non-trivial f.o. a, and so $(\mathsf{RLD})_{\mathrm{in}} f \nsubseteq (\mathsf{RLD})_{\mathrm{out}} f$. □

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