ABSTRACT

Prime Sieve Using Multiplication Operation Table

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Ben Green and Terence Tao showed that for any positive integer $k$, there exist infinitely many arithmetic progressions of length $k$ consisting only of prime numbers. [14] Four parallel proofs of Szemerédi’s theorem have been achieved; one by direct combinatorics, one by ergodic theory, one by hypergraph theory, and one by Fourier analysis and additive combinatorics. Even with so many proofs, Professor T. Tao points out that with this problem, there remains a sense that our understanding of this result is incomplete; for instance, none of the approaches were powerful enough to detect progressions in the primes, mainly due to the sparsity of the prime sequence. [22] Oliver Lonsdale Atkin introduced a prime sieve using irreducible binary quadratic forms and modular arithmetic; the algorithm enumerates representations of integers by certain binary quadratic forms. A way that uses modular arithmetic is widely known: $6n + \delta$, $12n + \delta$, $30n + \delta$, $60n + \delta$. [31] In this paper, we assert that the composite number of the $12n+1,5,7,11$ series as selected by a Modular Arithmetic and Multiplication Table are not random but consist of very structural and regular arithmetic progression groups.
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1. Introduction

Look through a list of prime numbers and you’ll find that it’s impossible to predict when the next prime will appear. The list seems chaotic, random, and offers no clues as to how determine the next number. It is hard to guess at a formula that could generate this kind of pattern. In fact, this procession of primes resembles a random succession of numbers much more than it does a nice orderly pattern. [8] [21]

This paper starts from the question of “Can we express prime numbers and recognize them spatially such as Prime Spiral?” The prime spiral, also known as Ulam's spiral, is a plot in which the positive integers are arranged in a spiral, with primes indicated in some way along the spiral.

![Figure 1](image1.png) Prime numbers are aligned to X-shape of 4 groups with a period of $12n$ (except 2, 3).

![Figure 2](image2.png) A hexagonal prime spiral can also be constructed, as illustrated above (Abbott 2005, wolfram).

1.1. Primes in Modular Arithmetic

If $a$ and $d$ are integers, with $d$ non-zero, then a remainder is an integer $r$ such that $a \equiv qd + r \pmod{d}$ for some integer $q$, and with $0 \leq |r| < |d|$. [46] Putting prime numbers on the regular hexagon, every prime number except 2 and 3 is contained in the $12n + 1, 5, 7, 11$ series, is sorted into 4 kinds of remainder groups—1, 5, 7, and 11—and belongs to at least one of these 4 groups.

1.2 Primes in Arithmetic Progression

Dirichlet's theorem, states that for any two positive coprime integers $a$ and $d$, there are infinitely many primes of the form $a + nd$, where $n \geq 0$. In other words: there are infinitely many primes which are congruent to a modulo $d$. The numbers of the form $a + nd$ form
an arithmetic progression \[37\]
\[a, \ a + d, \ a + 2d, \ a + 3d, \ \cdots a + nd\]

and Dirichlet’s theorem states that this sequence contains infinitely many prime numbers.\[33\] Szemerédi’s theorem generalizes the statement of van der Waerden’s theorem. A theorem of Szemerédi asserts that all subsets of the integers with positive upper density will contain arbitrarily long arithmetic progressions. \[22\] Also any given arithmetic progression of primes has a finite length. Green-Tao settled an Szemerédi’ conjecture by proving the Green-Tao theorem (The primes contain arbitrarily long arithmetic progressions). It follows immediately that there are infinitely many AP-k for any k (integer \(k \geq 3\), an AP-k (also called PAP-k) is k primes in arithmetic progression) \[23\]

2. The Specific Composite Numbers of the \(12n+1,5,7,11\) series

Like prime numbers, composite numbers appear intuitively irregular and seem to be difficult to group into a pattern. But, the specific composite numbers are generated by a certain rule and that rule is that composite numbers consist of sixteen arithmetic progression groups of different lengths; each group and each arithmetic progression included in that group forms that rule.

\[25 = 5 \times 5, \ 35 = 5 \times 7, \ 49 = 7 \times 7, \ 55 = 5 \times 11, \ 65 = 5 \times 13, \ 77 = 7 \times 11,\]
\[85 = 5 \times 17, \ 91 = 7 \times 13, \ 95 = 5 \times 19, \ 115 = 5 \times 23, \ 119 = 7 \times 17\]

2.1 Generating the Composite Number of the \(12n+1,5,7,11\) series

Let us denote the set \(A_n\) is all elements of the \(12n+1,5,7,11\) series\((n=0,1,2,\ldots)\); the set \(P_n\) is all elements of prime numbers as comprised in \(A_n\); the set \(C_n\) is all elements of composite numbers as comprised in \(A_n\).
All elements of the series are not multiples of 2 and 3. Therefore, all prime numbers except 2 and 3 exist in forms of with a period of .

(i) All natural numbers can be represented with a period of 12.

(ii) All even numbers but 2 are not prime numbers

\[ \text{Elements of } 12n + 2, 4, 6, 8, 10, 12 \text{ are all even. } \ (n = 0, 1, 2, \ldots n) \]

Therefore, all \( 12n + 2, 4, 6, 8, 10, 12 \)'s but 2 are not prime numbers.

(iii) All \( 12n + 3 \)'s but 3 are not prime numbers.

\[ 12n + 3 = (3 \times 4)n + 3 \text{ is a multiple of 3.} \]

(iv) \( 12n + 9 \) is not a prime number.

\[ 12n + 9 = (3 \times 4)n + 9 \text{ is a multiple of 3.} \]

As in Theorem 1), every prime number but 2 and 3 is contained in the periodic \( A_n \). So, let us denote this series as follows:
Aₙ =
\[
\begin{cases}
A₁, & \text{if remainder } \equiv 1 \pmod{12} \\
A₅, & \text{if remainder } \equiv 5 \pmod{12} \\
A₇, & \text{if remainder } \equiv 7 \pmod{12} \\
A₁₁, & \text{if remainder } \equiv 11 \pmod{12}
\end{cases}
\]

In Figure 2, Figure 3, the \( Aₙ \) series are multiplied infinitely and we can find 10 basics equations which falls into one of the 4 groups.

In Figure 3, the multiplication table for \( Aₙ \) series is shown. If we assign signs, we can summarize them as follows:

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>Sign</th>
<th>Equation</th>
<th>Remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A₁ \times A₁ )</td>
<td>+ + (( A₁ ))</td>
<td>(12x + 1)(12y + 1)</td>
<td>1 (mod 12)</td>
</tr>
<tr>
<td>( A₅ \times A₅ )</td>
<td>+ + (( A₅ ))</td>
<td>(12x + 5)(12y + 5)</td>
<td>1 (mod 12)</td>
</tr>
<tr>
<td>( A₇ \times A₇ )</td>
<td>+ + (( A₇ ))</td>
<td>(12x + 7)(12y + 7)</td>
<td>1 (mod 12)</td>
</tr>
<tr>
<td>( A₁₁ \times A₁₁ )</td>
<td>+ + (( A₁₁ ))</td>
<td>(12x + 11)(12y + 11)</td>
<td>1 (mod 12)</td>
</tr>
<tr>
<td>( A₁ \times A₅ )</td>
<td>- + (( A₅ ))</td>
<td>(12x + 1)(12y + 5)</td>
<td>5 (mod 12)</td>
</tr>
<tr>
<td>( A₅ \times A₁ )</td>
<td>- + (( A₁ ))</td>
<td>(12x + 5)(12y + 1)</td>
<td>5 (mod 12)</td>
</tr>
<tr>
<td>( A₇ \times A₇ )</td>
<td>- + (( A₇ ))</td>
<td>(12x + 7)(12y + 7)</td>
<td>7 (mod 12)</td>
</tr>
<tr>
<td>( A₁₁ \times A₅ )</td>
<td>- + (( A₅ ))</td>
<td>(12x + 11)(12y + 1)</td>
<td>7 (mod 12)</td>
</tr>
<tr>
<td>( A₁ \times A₇ )</td>
<td>- - (( A₇ ))</td>
<td>(12x + 1)(12y + 7)</td>
<td>7 (mod 12)</td>
</tr>
<tr>
<td>( A₅ \times A₇ )</td>
<td>- - (( A₇ ))</td>
<td>(12x + 5)(12y + 7)</td>
<td>11 (mod 12)</td>
</tr>
<tr>
<td>( A₅ \times A₇ )</td>
<td>- - (( A₇ ))</td>
<td>(12x + 11)(12y + 1)</td>
<td>11 (mod 12)</td>
</tr>
</tbody>
</table>

Table 2: The Result List of Multiplication Table
**Theorem 1.2** All elements of the $A_n \times A_n$ table multiplication are contained in set $A_n$.

**Proof B)**

$$(12x + \alpha)(12y + \beta), (\alpha, \beta \in \{1, 5, 7, 11\}) =$$

$$144xy + 12\beta x + 12\alpha y + \alpha\beta = \begin{cases} 
\alpha\beta \in \{1, 25, 49, 121\}, & \text{if remainder } \equiv 1 \pmod{12} \\
\alpha\beta \in \{5, 77\}, & \text{if remainder } \equiv 5 \pmod{12} \\
\alpha\beta \in \{7, 55\}, & \text{if remainder } \equiv 7 \pmod{12} \\
\alpha\beta \in \{11, 35\}, & \text{if remainder } \equiv 11 \pmod{12} 
\end{cases}$$

Therefore, $\alpha\beta \in \{1, 5, 7, 11, 25, 35, 49, 55, 77, 121\}$

**Theorem 1.3** All elements of the $A_n$ table multiplication are contained in the set of the $A_n \times A_n$ table multiplication.

**Proof C)**

For any element $k$ of the set $A_n$:

- $k \in P_n$ or $k \in C_n$,

if $k$ is:

- $P_n$:
  - $P_n \times 1$, $k \in (12x + \alpha) \times 1$ or $1 \times (12y + \beta)$

- $C_n$:
  - $P_n \times C_n$, $k \in (12x + \alpha) \times (12y + \beta)$
  - $C_n \times C_n$, $k \in (12x + \alpha) \times (12y + \beta)$

Therefore, $C_n \supseteq 25$. Additionally, it is possible to find all values of $C_n$ of the $12n + 1, 5, 7, 11$ series in the results of a matrix-multiplication of $A_n \times A_n$ that are greater than 5.
In the third century B.C., the scholar Eratosthenes came up with a simple algorithm for listing all the prime numbers up to a given \( N \), referred to as the sieve of Eratosthenes. A standard improvement in the sieve of Eratosthenes is to enumerate values of \( x \) not divisible by 2, 3, or 5.

The composite numbers intuitively appear to be irregular and seem to be difficult to group into a pattern. However, the composite numbers are sorted into sixteen arithmetic series.

**2.2 Prime Sieve**

In the following [Attached document 1], the prime numbers under 1000 are being filtered using \( C_n \). More specifically, you can see that arithmetic progressions, \( Ga_n \sim Gp_n \), are intertwined. Of course, since the number of arithmetic progressions of each group increases as \( N \) increases, the complexity of intertwining of further arithmetic progressions increases. The composite numbers intuitively appear to be irregular and seem to be difficult to group into a pattern. However, the composite numbers are sorted into sixteen arithmetic series.
groups \((G_{a_n} - G_{p_n})\) and are generated regularly by arithmetic progressions of each group.

2.3 The Structure of a matrix-multiplication of \(A_n \times A_n\)

Unlike prime numbers, which are unpredictable, the composite numbers are formed by sixteen arithmetic progression groups. This means that composite numbers in principle are predictable because whole composite numbers follow this rule. However, the composite numbers are made up of sixteen arithmetic progressions and it is difficult to see the whole of the arithmetic progressions, whose number increases, without necessary computations and information media that can store the computed results. If you can find the computed results of various arithmetic progressions intuitively, you can predict the rule that governs the composite numbers. This immediately means that you will be able to find the rule that governs the prime numbers. It is not a problem of whether or not the composite numbers are predictable but a problem of human perception.

**Figure - 2.3.a**  Structure of Multiplication Table

- G is a temporary mark used in this paper.
- a ~ p are marked in alphabetical order.
- n is marked to clarify that composite numbers are a set of arithmetic progressions.
Table 3) The composite numbers of the $12n+1, 5, 7, 11$ series are sorted into a total of sixteen groups ($Ga_n \sim Gp_n$).

### 2.4 Symmetric table and Asymmetric table

Analyzing the table of the $12n+1, 5, 7, 11$ series, by the values of horizontal axis ($\alpha$) and vertical axis ($\beta$), the table divides into a symmetric table if $\alpha = \beta$, and into an asymmetric table if $\alpha \neq \beta$. Therefore, we can find the following results.

**i) Symmetric table, $\alpha = \beta$**

$(12x+1) \times (12y+1), \ (x, y \geq 1)$
$(12x+5) \times (12y+5),$
$(12x+7) \times (12y+7),$
$(12x+11) \times (12y+11)$
Since \( \alpha = \beta \), the results are the same, reflecting along the diagonal elements independent of the orders of \( \alpha \) and \( \beta \). So, the symmetric table has four cases.

**ii) Asymmetric table, \( \alpha \neq \beta \)**

\[
\begin{align*}
(12x + 1) \times (12y + 5), & \quad (x \geq 1), \quad (12x + 7) \times (12y + 11) \\
(12x + 1) \times (12y + 7), & \quad (x \geq 1), \quad (12x + 5) \times (12y + 11) \\
(12x + 1) \times (12y + 11), & \quad (x \geq 1), \quad (12x + 5) \times (12y + 7)
\end{align*}
\]

However, the commutative law does not hold if \( \alpha \neq \beta \). So, depending on the orders of \( \alpha \) and \( \beta \), the results are different for the diagonal elements. An asymmetric table has twelve cases.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>Sign</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>++</td>
<td>((12x + 1) \times (12y + 5), (x \geq 1))</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>++</td>
<td>((12x + 5) \times (12y + 1), (x \geq 1))</td>
</tr>
<tr>
<td>7</td>
<td>11</td>
<td>++</td>
<td>((12x + 7) \times (12y + 1), (x \geq 1))</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
<td>++</td>
<td>((12x + 11) \times (12y + 7))</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>--</td>
<td>((12x + 1) \times (12y + 7), (x \geq 1))</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>--</td>
<td>((12x + 7) \times (12y + 1), (x \geq 1))</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>--</td>
<td>((12x + 5) \times (12y + 11))</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>--</td>
<td>((12x + 11) \times (12y + 5))</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>++</td>
<td>((12x + 1) \times (12y + 11), (x \geq 1))</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>++</td>
<td>((12x + 11) \times (12y + 1), (x \geq 1))</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>++</td>
<td>((12x + 5) \times (12y + 7))</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>++</td>
<td>((12x + 7) \times (12y + 5))</td>
</tr>
</tbody>
</table>

*Table 4* The Equation of Asymmetric Table

The composite numbers of the \( 12n + 1, 5, 7, 11 \) series divide into a total of sixteen arithmetic progression groups in the matrix multiplication, \((12x + \alpha) \times (12y + \beta), (\alpha, \beta = 1, 5, 7, 11)\), depending on \( \alpha = \beta \) and \( \alpha \neq \beta \).
3. Composites in Arithmetic Progression

3.1 \((12x+1)(12y+1), (Ga_x)\)

<table>
<thead>
<tr>
<th></th>
<th>13</th>
<th>25</th>
<th>37</th>
<th>49</th>
<th>61</th>
<th>73</th>
<th>85</th>
<th>97</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>169</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>325</td>
<td>625</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>481</td>
<td>925</td>
<td>1369</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>637</td>
<td>1225</td>
<td>1813</td>
<td>2401</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>793</td>
<td>1525</td>
<td>2257</td>
<td>2989</td>
<td>3721</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>73</td>
<td>949</td>
<td>1825</td>
<td>2701</td>
<td>3577</td>
<td>4453</td>
<td>5329</td>
<td></td>
<td></td>
</tr>
<tr>
<td>85</td>
<td>1105</td>
<td>2125</td>
<td>3145</td>
<td>4165</td>
<td>5185</td>
<td>6205</td>
<td>7225</td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>1261</td>
<td>2425</td>
<td>3589</td>
<td>4753</td>
<td>5917</td>
<td>7081</td>
<td>8245</td>
<td>9409</td>
</tr>
</tbody>
</table>

\(12x+1\)

\(12y+1\)

Figure 3.1.a) From the matrix multiplication, each of the diagonal elements becomes the initial terms of the arithmetic progressions.

\(Ga_x: 169(a_1) \quad 325(a_2) \quad 481(a_3) \quad 637(a_4) \quad 793(a_5) \quad 949(a_6) \quad 1105(a_7) \quad 1261(a_8)\)

\(Ga_x: 625(a_1) \quad 925(a_2) \quad 1225(a_3) \quad 1525(a_4) \quad 1825(a_5) \quad 2125(a_6) \quad 2425(a_7)\)

\(Ga_x: 1369(a_1) \quad 1813(a_2) \quad 2257(a_3) \quad 2701(a_4) \quad 3145(a_5) \quad 3589(a_6)\)

\(Ga_x: 2401(a_1) \quad 2989(a_2) \quad 3577(a_3) \quad 4165(a_4) \quad 4753(a_5)\)

\(Ga_x: 3721(a_1) \quad 4453(a_2) \quad 5185(a_3) \quad 5917(a_4)\)

\(Ga_x: 5329(a_1) \quad 6205(a_2) \quad 7018(a_3)\)

\(Ga_x: 7225(a_1) \quad 8545(a_2)\)

\(Ga_x: 9409(a_1)\)

Figure 3.1.b) We have listed the symmetric matrix results for each arithmetic progression. (\(a_1\) = first term, \(d\) = common difference)
Table multiplication results are arithmetic progressions that have initial terms and common differences and have a general formula. ( \( a \) = first term, \( d \) = common difference)

The contour of the arithmetic progression that belongs to \( G_{n} \)

The Figure 3.1.d) shows the contour of the arithmetic progression, cutting by 60 up to 1000. If \( N \) increases, more arithmetic progressions are generated. A point to note here is that they appear as different contours because they have different initial terms and common differences that belong to \( G_{n} \).
**Figure 3.1.e** In $Ga_n$, there exist, under rules, a number of arithmetic progressions that have different initial terms and common differences.

$Ga_1, Ga_2, Ga_3 \sim Ga_n$ make up arithmetic progression groups that have initial terms, $(12x+1)(12y+1), (x, y = n)$ and common differences, $12(12x+1), (x = n)$. Of course, as $n$ increases, the values of the initial term and the common difference increase.

The current and new values for the arithmetic progressions are as follows:

<table>
<thead>
<tr>
<th>$Ga_n : a_i$</th>
<th>$Ga_i$</th>
<th>$Ga_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ga_1 : a_1(169)$</td>
<td>$Ga_1(169)$</td>
<td>$Ga_1$</td>
</tr>
<tr>
<td>$Ga_2 : a_1(625)$</td>
<td>$Ga_1, Ga_2$</td>
<td>$Ga_2$</td>
</tr>
<tr>
<td>$Ga_3 : a_1(1369)$</td>
<td>$Ga_1, Ga_2, Ga_3$</td>
<td>$Ga_3$</td>
</tr>
<tr>
<td>$Ga_4 : a_1(2401)$</td>
<td>$Ga_1, Ga_2, Ga_3, Ga_4$</td>
<td>$Ga_4$</td>
</tr>
<tr>
<td>$Ga_5 : a_1(3721)$</td>
<td>$Ga_1, Ga_2, Ga_3, Ga_4, Ga_5$</td>
<td>$Ga_5$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$Ga_n : a_1$</td>
<td>$Ga_1, Ga_2, Ga_3, Ga_4, Ga_5, \ldots, Ga_n$</td>
<td>$Ga_n$</td>
</tr>
</tbody>
</table>

**Figure 3.1.f** As the natural number $N$, increases, new arithmetic progressions accumulate and keep increasing.

In **Figure 3.1.f**, since new arithmetic progressions accumulate as the natural number $N$ increases, the distribution density of the composite numbers in each region becomes high. For example, the number of composite numbers of the arithmetic progression, $Ga_n$, is higher in the 1,000-2,000 region than in the 11,000-12,000 region. Just like a timer that rings after a certain period of time, for a certain $N$, the corresponding arithmetic progression operates.
**Theorem 2**) As \( N \) increases, the density of composite numbers becomes higher. In other words, it means that, as \( N \) increases, prime number density becomes low. (of the sparsity of the prime sequence.)

When writing the list of prime numbers, you will find that prime numbers become more and more sparse.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \pi(n) )</th>
<th>( \pi(n)/n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>0.4</td>
</tr>
<tr>
<td>( 10^2 )</td>
<td>25</td>
<td>0.25</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>168</td>
<td>0.168</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>1,229</td>
<td>0.1229</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>9,592</td>
<td>0.09592</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>78,498</td>
<td>0.078498</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>664,579</td>
<td>0.066458</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>5,761,455</td>
<td>0.057615</td>
</tr>
<tr>
<td>( 10^9 )</td>
<td>50,847,534</td>
<td>0.050848</td>
</tr>
<tr>
<td>( 10^{10} )</td>
<td>455,052,512</td>
<td>0.045505</td>
</tr>
</tbody>
</table>

*Table 5*) The Sparse of Prime Numbers

The number of prime numbers between 1 and 100 is greater than that between 101 and 200. There are 4 prime numbers (40%) between 0 and 10, 25 prime numbers (25%) between 0 and 100, 168 (16.8%) between 0 and 1000, 1,229 (12.3%) between 0 and 10000, 9592 (9.5%) between 0 and 100000, and 78,498 (7.8%) between 0 and 1000000. The percentage gradually decreases. [9][10]
<table>
<thead>
<tr>
<th>Section</th>
<th>The count of prime</th>
<th>The count of composite</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ~ 10^2</td>
<td>25(0.25)</td>
<td>9(0.09)</td>
<td>34</td>
</tr>
<tr>
<td>10^2 ~ 2×10^2</td>
<td>21(0.21)</td>
<td>13(0.13)</td>
<td>34</td>
</tr>
<tr>
<td>2×10^2 ~ 3×10^2</td>
<td>16(0.16)</td>
<td>17(0.17)</td>
<td>33</td>
</tr>
<tr>
<td>3×10^2 ~ 4×10^2</td>
<td>16(0.16)</td>
<td>17(0.17)</td>
<td>33</td>
</tr>
<tr>
<td>4×10^2 ~ 5×10^2</td>
<td>17(0.17)</td>
<td>17(0.17)</td>
<td>34</td>
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<tr>
<td>5×10^2 ~ 6×10^2</td>
<td>14(0.14)</td>
<td>19(0.19)</td>
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<tr>
<td>6×10^2 ~ 7×10^2</td>
<td>16(0.16)</td>
<td>17(0.17)</td>
<td>33</td>
</tr>
<tr>
<td>7×10^2 ~ 8×10^2</td>
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<td>20(0.20)</td>
<td>34</td>
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<tr>
<td>8×10^2 ~ 9×10^2</td>
<td>15(0.15)</td>
<td>18(0.18)</td>
<td>33</td>
</tr>
<tr>
<td>9×10^2 ~ 10^3</td>
<td>14(0.14)</td>
<td>19(0.19)</td>
<td>33</td>
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</table>

*Table 6* Densities of Prime and Composite Numbers in Each Region

Therefore, if you know reason that the number of composite numbers increases, you will know why prime number density decreases. This is because, at least, we know the rules under which composite numbers are generated. This phenomenon is a natural result because new arithmetic progressions accumulate as a natural number, \( N \), increases. As \( G_{A_n}, G_{B_n}, G_{C_n}, G_{P_n} \) make up groups that all have different initial terms and common differences. Therefore, they make up sixteen groups of arithmetic progressions.
3.2 \((12x+5)(12y+5), (Gb_n)\)

\[
x \times 5 \quad 17 \quad 29 \quad 41 \quad 53 \quad 65 \quad 77 \quad 89 \\
5 \quad 25 \\
17 \quad 85 \quad 289 \\
29 \quad 145 \quad 493 \\
41 \quad 205 \quad 697 \quad 1189 \\
53 \quad 265 \quad 901 \quad 1537 \quad 2173 \\
65 \quad 325 \quad 1105 \quad 1885 \quad 2665 \quad 3445 \\
77 \quad 385 \quad 1309 \quad 2233 \quad 31 \\
89 \quad 445 \quad 1513 \quad 2581 \quad 3649 \quad 4717 \quad 5785 \quad 685 \\
\\
\frac{y}{d} = \frac{12x+5}{12y+5} = 25, \quad \frac{d}{x} = \frac{12y+5}{12x+5} = 60, (x = 0, y = 0)
\]
\[
\text{Ob: } a_1 = (12x+5)(12y+5) = 25, \quad a_2 = (12x+5)12y+5) = 289, \quad a_3 = (12x+5)(12y+5) = 841, \quad a_4 = (12x+5)(12y+5) = 1681, \quad a_5 = (12x+5)(12y+5) = 2809, \quad a_6 = (12x+5)(12y+5) = 4225, \quad a_7 = (12x+5)(12y+5) = 6885, \quad a_8 = (12x+5)(12y+5) = 10688, (x = 7, y = 7)
\]

\[
\text{Gb: } a_1 = (12x+5)(12y+5) = 144x+60y+60, (x = 12x+5), (x = n-1, y = n-1)
\]
3.3 \((12x + 7)(12y + 7), (Gc_x)\)

\[
x \times 7 \quad 19 \quad 31 \quad 43 \quad 55 \quad 67 \quad 79 \quad 91 \\
7 \quad 49 \\
19 \quad 133 \quad 361 \\
31 \quad 217 \quad 589 \\
43 \quad 301 \quad 817 \quad 1333 \\
55 \quad 385 \quad 1045 \quad 1705 \quad 2365 \\
67 \quad 469 \quad 1273 \quad 2077 \quad 2881 \quad 3685 \\
79 \quad 553 \quad 1501 \quad 2449 \quad 3697 \\
91 \quad 637 \quad 1729 \quad 2821 \quad 3913 \quad 5005 \quad 6097 \quad 7189 \quad 8281 \\
\]

\[
Gc_1: 49(a_1) \quad 133(a_2) \quad 217(a_3) \quad 301(a_4) \quad 385(a_5) \quad 469(a_6) \quad 553(a_7) \quad 637(a_8) \\
Gc_2: 36(a_1) \quad 59(a_2) \quad 817(a_3) \quad 1045(a_4) \quad 1273(a_5) \quad 1501(a_6) \\
Gc_3: 90(a_1) \quad 1333(a_2) \quad 2077(a_3) \quad 2449(a_4) \quad 2881(a_5) \\
Gc_4: 366(a_1) \quad 2365(a_2) \quad 2881(a_3) \quad 3397(a_4) \quad 3913(a_5) \\
Gc_5: 4489(a_1) \quad 5293(a_2) \quad 6097(a_3) \\
Gc_6: 6241(a_1) \quad 7189(a_2) \\
Gc_7: 8281(a_1) \\
\]

\[
Gc_1: a_1 = (12x + 7)(12y + 7) = 49, \quad d = 12 \times (12y + 7) = 84, (x = 0, y = 0) \\
Gc_2: a_1 = (12x + 7)(12y + 7) = 361, \quad d = 12 \times (12y + 7) = 228, (x = 1, y = 0) \\
Gc_3: a_1 = (12x + 7)(12y + 7) = 961, \quad d = 12 \times (12y + 7) = 372, (x = 0, y = 1) \\
Gc_4: a_1 = (12x + 7)(12y + 7) = 1849, \quad d = 12 \times (12y + 7) = 516, (x = 1, y = 3) \\
Gc_5: a_1 = (12x + 7)(12y + 7) = 3025, \quad d = 12 \times (12y + 7) = 660, (x = 2, y = 1) \\
Gc_6: a_1 = (12x + 7)(12y + 7) = 4489, \quad d = 12 \times (12y + 7) = 849, (x = 2, y = 3) \\
Gc_7: a_1 = (12x + 7)(12y + 7) = 6241, \quad d = 12 \times (12y + 7) = 948, (x = 2, y = 5) \\
Gc_8: a_1 = (12x + 7)(12y + 7) = 8281, \quad d = 12 \times (12y + 7) = 1092, (x = 2, y = 7) \\
\]

\[
Gc_1: a_1 = (12x + 7)(12y + 7) = 144x + 84x + 84y + 49, \quad d = 12 \times (12x + 7), (x = 0, y = 0) \\
\]
3.4 \((12x+11)(12y+11),(Gd_n)\)

\[
\begin{array}{cccccccc}
1 & 23 & 35 & 47 & 59 & 71 & 83 & 95 \\
11 & 121 \\
23 & 253 & 529 \\
35 & 385 & 805 & 1225 \\
47 & 517 & 1081 & 1645 & 2209 \\
59 & 649 & 1357 & 2065 & 2773 & 3481 \\
71 & 781 & 1633 & 2485 & 3337 & 4189 & 5041 \\
83 & 913 & 1909 & 2905 & 3901 & 4897 & 5893 & 6889 \\
95 & 1045 & 2185 & 3325 & 4465 & 5605 & 6745 & 7885 & 9025 \\
\end{array}
\]

\[
\begin{align*}
G_{n1} &= 12\alpha(n) \quad 253\alpha(n) \quad 517\alpha(n) \quad 649\alpha(n) \quad 781\alpha(n) \quad 913\alpha(n) \quad 1045\alpha(n) \\
G_{n2} &= 529\alpha(n) \quad 805\alpha(n) \quad 1357\alpha(n) \quad 2065\alpha(n) \quad 2773\alpha(n) \quad 3481\alpha(n) \\
G_{n3} &= 1225\alpha(n) \quad 2065\alpha(n) \quad 2485\alpha(n) \quad 2905\alpha(n) \quad 3325\alpha(n) \\
G_{n4} &= 2209\alpha(n) \quad 2773\alpha(n) \quad 3337\alpha(n) \quad 3901\alpha(n) \quad 4465\alpha(n) \\
G_{n5} &= 3481\alpha(n) \quad 4189\alpha(n) \quad 4897\alpha(n) \quad 5605\alpha(n) \\
G_{n6} &= 5041\alpha(n) \quad 5893\alpha(n) \quad 6745\alpha(n) \\
G_{n7} &= 6889\alpha(n) \quad 7885\alpha(n) \\
G_{n8} &= 9025\alpha(n) \\
\end{align*}
\]

\[
\begin{align*}
G_{n1} &= 12(12x+11)(12y+11) = 121, \quad d = 12 \times (12x + 11) = 132, (x = 0, y = 0) \\
G_{n2} &= 529, \quad d = 12 \times (12y + 11) = 276, (x = 1, y = 1) \\
G_{n3} &= 1225, \quad d = 12 \times (12x + 11) = 420, (x = 2, y = 2) \\
G_{n4} &= 2209, \quad d = 12 \times (12y + 11) = 564, (x = 3, y = 3) \\
G_{n5} &= 3481, \quad d = 12 \times (12x + 11) = 708, (x = 4, y = 4) \\
G_{n6} &= 5041, \quad d = 12 \times (12y + 11) = 852, (x = 5, y = 5) \\
G_{n7} &= 6889, \quad d = 12 \times (12x + 11) = 996, (x = 6, y = 6) \\
G_{n8} &= 9025, \quad d = 12 \times (12y + 11) = 1140, (x = 7, y = 7) \\
\end{align*}
\]

\[
G_{n1} = (12x + 11)(12y + 11) = 144xy + 1332x + 132y + 121, \quad d = 12 \times (12y + 11), (x = n, y = n = 1)
\]
3.5 \((12x + 1)(12y + 5), (Ge_n)\)

\[
\times \quad 13 \quad 25 \quad 37 \quad 49 \quad 61 \quad 73 \quad 85 \quad 97
\]

5

29 \quad 377 \quad 725 \quad 1073

41 \quad 533 \quad 1025 \quad 1517 \quad 2009

53 \quad 689 \quad 1325 \quad 1961 \quad 2597 \quad 3233

65 \quad 845 \quad 1625 \quad 2405 \quad 3185 \quad 3965 \quad 4745

77 \quad 1001 \quad 1925 \quad 2849 \quad 3773 \quad 4697 \quad 5621 \quad 6545

89 \quad 1157 \quad 2225 \quad 3293 \quad 4361 \quad 5429 \quad 6497 \quad 7565 \quad 8633

\[
\text{Ge}_1: 65(a_1), 22(a_2), 377(a_3), 533(a_4), 689(a_5), 845(a_6), 1001(a_7), 1157(a_8)
\]

\[
\text{Ge}_2: 425(a_1), 725(a_2), 1025(a_3), 1325(a_4), 1625(a_5), 1925(a_6), 2225(a_7),
\]

\[
\text{Ge}_3: 1073(a_1), 1961(a_2), 2405(a_3), 2849(a_4), 3293(a_5),
\]

\[
\text{Ge}_4: 2009(a_1), 2597(a_2), 3185(a_3), 3773(a_4), 4361(a_5),
\]

\[
\text{Ge}_5: 3233(a_1), 3965(a_2), 4697(a_3), 5429(a_4), 6497(a_5),
\]

\[
\text{Ge}_6: 4745(a_1), 5621(a_2), 6497(a_3),
\]

\[
\text{Ge}_7: 6545(a_1), 7565(a_2),
\]

\[
\text{Ge}_8: 8633(a_1)
\]

\[
\text{Ge}_1: a_1 = (12x + 1)(12y + 5) = 65, \quad d = 12(x + 1) - 156(s - 1, y = 0)
\]

\[
\text{Ge}_2: a_1 = (12x + 1)(12y + 5) = 425, \quad d = 12(x + 1) - 300(s = 2, y = 1)
\]

\[
\text{Ge}_3: a_1 = (12x + 1)(12y + 5) = 1073, \quad d = 12(x + 1) - 444(s = 3, y = 2)
\]

\[
\text{Ge}_4: a_1 = (12x + 1)(12y + 5) = 2009, \quad d = 12(x + 1) - 588(s = 4, y = 3)
\]

\[
\text{Ge}_5: a_1 = (12x + 1)(12y + 5) = 2597, \quad d = 12(x + 1) - 732(s = 5, y = 4)
\]

\[
\text{Ge}_6: a_1 = (12x + 1)(12y + 5) = 3233, \quad d = 12(x + 1) - 876(s = 6, y = 5)
\]

\[
\text{Ge}_7: a_1 = (12x + 1)(12y + 5) = 4745, \quad d = 12(x + 1) - 1020(s = 7, y = 6)
\]

\[
\text{Ge}_8: a_1 = (12x + 1)(12y + 5) = 6545, \quad d = 12(x + 1) - 1164(s = 8, y = 7)
\]

\[
\text{Ge}_9: a_1 = (12x + 1)(12y + 5) = 8633, \quad d = 12(x + 1) - 1308(s = 9, y = n - 1)
\]
### 3.6 \((12x+5)(12y+1),(Gf_n)\)

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<td>125</td>
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<td>37</td>
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</table>

\(Gf; \, 65(a_1), \, 125(a_2), \, 185(a_3), \, 245(a_4), \, 305(a_5), \, 365(a_6), \, 425(a_7), \, 485(a_8)\)

\(Gf; \, 425(a_1), \, 629(a_2), \, 833(a_3), \, 1073(a_4), \, 1421(a_5), \, 185(a_6), \, 2117(a_7), \, 245(a_8)\)

\(Gf; \, 1073(a_1), \, 1421(a_2), \, 185(a_3), \, 2117(a_4), \, 245(a_5), \, 2813(a_6)\)

\(Gf; \, 2009(a_1), \, 245(a_2), \, 2993(a_3), \, 3485(a_4), \, 3977(a_5)\)

\(Gf; \, 3233(a_1), \, 3865(a_2), \, 4505(a_3), \, 5141(a_4)\)

\(Gf; \, 4745(a_1), \, 6305(a_2)\)

\(Gf; \, 6545(a_1), \, 7469(a_2)\)

\(Gf; \, 8633(a_1)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 65, \quad d = 12x+5 \quad (x = 0, \, y = 1)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 425, \quad d = 12x+5 \quad (x = 2, \, y = 1)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 1073, \quad d = 12x+5 \quad (x = 3, \, y = 2)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 2009, \quad d = 12x+5 \quad (x = 4, \, y = 3)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 3233, \quad d = 12x+5 \quad (x = 5, \, y = 4)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 4745, \quad d = 12x+5 \quad (x = 6, \, y = 5)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 6545, \quad d = 12x+5 \quad (x = 7, \, y = 6)\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 8633, \quad d = 12x+5 \quad (x = 8, \, y = 7)\)

\(\vdots\)

\(Gf; \, a_1 = (12x+5)(12y+1) = 144y+12x+60y+5, \quad d = 12x+5 \quad (x = n-1, \, y = n)\)

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3.7 \((12x + 7)(12y + 11), (G_{g_n})\)

\[
\times \quad \begin{array}{cccccccc}
7 & 19 & 31 & 43 & 55 & 67 & 79 & 91 \\
11 & 77 & & & & & & \\
23 & 161 & 437 & & & & & \\
35 & 245 & 665 & 1085 & & & & \\
47 & 329 & 893 & 1457 & & & & \\
59 & 413 & 1121 & 1829 & 2537 & 3245 & & \\
71 & 497 & 1349 & 2201 & 3053 & 3905 & 4757 & \\
83 & 581 & 1577 & 2573 & 3569 & 4565 & 5561 & 6557 \\
95 & 665 & 1805 & 2945 & 4085 & 5225 & 6365 & 7505 & 8645 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
G_1: & 77(a_1) & 161(a_1) & 245(a_1) & 329(a_1) & 413(a_1) & 497(a_1) & 581(a_1) & 665(a_1) \\
G_2: & 437(a_1) & 665(a_1) & 893(a_1) & 1121(a_1) & 1349(a_1) & 1577(a_1) & 1805(a_1) & \\
G_3: & 1085(a_1) & 1457(a_1) & 1829(a_1) & 2201(a_1) & 2537(a_1) & 2945(a_1) & \\
G_4: & 2021(a_1) & 2537(a_1) & 3053(a_1) & 3569(a_1) & 4085(a_1) & \\
G_5: & 3245(a_1) & 3905(a_1) & 4565(a_1) & 5225(a_1) & \\
G_6: & 4757(a_1) & 5561(a_1) & 6365(a_1) & \\
G_7: & 6557(a_1) & 7505(a_1) & \\
G_8: & 8645(a_1) & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
G_1: & a_i = (12x + 7)(12y + 11) = 77, & d = 12 \times (12x + 7) = 84(x = 0, y = 0) \\
G_2: & a_i = (12x + 7)(12y + 11) = 437, & d = 12 \times (12x + 7) = 228(x = 1, y = 1) \\
G_3: & a_i = (12x + 7)(12y + 11) = 1085, & d = 12 \times (12x + 7) = 372(x = 2, y = 2) \\
G_4: & a_i = (12x + 7)(12y + 11) = 2021, & d = 12 \times (12x + 7) = 516(x = 3, y = 3) \\
G_5: & a_i = (12x + 7)(12y + 11) = 2537, & d = 12 \times (12x + 7) = 660(x = 4, y = 4) \\
G_6: & a_i = (12x + 7)(12y + 11) = 4757, & d = 12 \times (12x + 7) = 804(x = 5, y = 5) \\
G_7: & a_i = (12x + 7)(12y + 11) = 6557, & d = 12 \times (12x + 7) = 948(x = 6, y = 6) \\
G_8: & a_i = (12x + 7)(12y + 11) = 8645, & d = 12 \times (12x + 7) = 1092(x = 7, y = 7) \\
\end{array}
\]

\[
\begin{array}{cccccccc}
G_1: & a_i = (12x + 7)(12y + 11) = 144xy + 132x + 84y + 77, & d = 12 \times (12x + 7)(x = n - 1, y = n - 1) \\
\end{array}
\]
3.8 \((12x + 11)(12y + 7), (Gh_n)\)

\[\times \begin{array}{cccccccc}
7 & 11 & 23 & 35 & 47 & 59 & 71 & 83 & 95
\end{array}\]

\[\begin{array}{c}
19 \quad 209 \quad 437 \\
31 \quad 341 \quad 713 \quad 1085 \\
43 \quad 473 \quad 989 \quad 1505 \\
55 \quad 605 \quad 1265 \quad 1925 \quad 2585 \\
67 \quad 737 \quad 1541 \quad 2345 \quad 3149 \quad 3953 \\
79 \quad 869 \quad 1817 \quad 2765 \quad 3613 \quad 4661 \quad 5609 \quad 6557 \\
91 \quad 1001 \quad 2093 \quad 3185 \quad 4277 \quad 5369 \quad 6461 \quad 7553 \quad 8645 \\
\end{array}\]

\[\begin{array}{c}
G_{h_1} (11a_1) = 209(a_1), \quad 341(a_1) \quad 473(a_1) \quad 605(a_1) \quad 737(a_1) \quad 869(a_1) \quad 1001(a_1) \\
G_{h_2} (12a_2) = 1085(a_2) \quad 1265(a_2) \quad 1445(a_2) \quad 1625(a_2) \quad 1805(a_2) \quad 1985(a_2) \\
G_{h_3} (13a_3) = 2585(a_3) \quad 2765(a_3) \quad 2945(a_3) \quad 3125(a_3) \quad 3305(a_3) \quad 3485(a_3) \\
G_{h_4} (14a_4) = 5609(a_4) \quad 6461(a_4) \quad 7313(a_4) \quad 8165(a_4) \quad 9017(a_4) \quad 9869(a_4) \\
G_{h_5} (15a_5) = 1505(a_5) \quad 1925(a_5) \quad 2345(a_5) \quad 2765(a_5) \quad 3185(a_5) \quad 3605(a_5) \\
G_{h_6} (16a_6) = 3953(a_6) \quad 4661(a_6) \quad 5369(a_6) \quad 6077(a_6) \quad 6785(a_6) \quad 7493(a_6) \\
\end{array}\]

\[\begin{array}{c}
G_{h_1}, G_{h_2} = 1001, \quad d = 12 \times \left(12x + 11\right) = 132(x = 0, y = 0) \\
G_{h_1}, G_{h_2} = 209, \quad d = 12 \times \left(12x + 11\right) = 276(x = 1, y = 1) \\
G_{h_1}, G_{h_2} = 341, \quad d = 12 \times \left(12x + 11\right) = 420(x = 2, y = 2) \\
G_{h_1}, G_{h_2} = 473, \quad d = 12 \times \left(12x + 11\right) = 564(x = 3, y = 3) \\
G_{h_1}, G_{h_2} = 605, \quad d = 12 \times \left(12x + 11\right) = 708(x = 4, y = 4) \\
G_{h_1}, G_{h_2} = 737, \quad d = 12 \times \left(12x + 11\right) = 852(x = 5, y = 5) \\
G_{h_1}, G_{h_2} = 869, \quad d = 12 \times \left(12x + 11\right) = 996(x = 6, y = 6) \\
G_{h_1}, G_{h_2} = 1001, \quad d = 12 \times \left(12x + 11\right) = 1140(x = 7, y = 7) \\
\end{array}\]

\[\begin{array}{c}
G_{h_1}, G_{h_2} = 1144, \quad d = 12 \times \left(12x + 11\right) = 1288(x = 8, y = 8) \\
G_{h_1}, G_{h_2} = 132, \quad d = 12 \times \left(12x + 11\right) = 1432(x = 9, y = 9) \\
G_{h_1}, G_{h_2} = 1444, \quad d = 12 \times \left(12x + 11\right) = 1576(x = 10, y = 10) \\
\end{array}\]
3.9 \((12x + 1)(12y + 7), (Gi_n)\)

\[\begin{array}{cccccccccc}
\times & 13 & 25 & 37 & 49 & 61 & 73 & 85 & 97 \\
7 & 91 & 19 & 247 & 475 & 559 & 1075 & 1591 & 2107 \\
19 & 247 & 1147 & 3283 & 4087 & 4891 & 6713 & 8767 & 10201 \\
31 & 403 & 1573 & 3871 & 4819 & 5767 & 8483 & 10529 & 12637 \\
43 & 559 & 2399 & 5927 & 7381 & 8835 & 12307 & 14779 & 17251 \\
55 & 715 & 3455 & 9065 & 11529 & 14093 & 21185 & 24149 & 27113 \\
67 & 871 & 4501 & 12661 & 16825 & 23089 & 38791 & 45859 & 52927 \\
79 & 1027 & 5707 & 17167 & 23631 & 32195 & 56587 & 70161 & 83735 \\
91 & 1183 & 6559 & 20159 & 28323 & 38487 & 64801 & 83263 & 101725 \\
\end{array}\]

\[
\begin{align*}
G_i; & \quad 91(a_{i1}) \quad 247(a_{i2}) \quad 403(a_{i3}) \quad 559(a_{i4}) \quad 715(a_{i5}) \quad 871(a_{i6}) \quad 1027(a_{i7}) \quad 1183(a_{i8}) \\
G_i; & \quad 475(a_{i1}) \quad 775(a_{i2}) \quad 1075(a_{i3}) \quad 1375(a_{i4}) \quad 1675(a_{i5}) \quad 1975(a_{i6}) \quad 2275(a_{i7}) \\
G_i; & \quad 1147(a_{i1}) \quad 1591(a_{i2}) \quad 2035(a_{i3}) \quad 2479(a_{i4}) \quad 2923(a_{i5}) \quad 3367(a_{i6}) \\
G_i; & \quad 2107(a_{i1}) \quad 2695(a_{i2}) \quad 3283(a_{i3}) \quad 3871(a_{i4}) \quad 4459(a_{i5}) \\
G_i; & \quad 3555(a_{i1}) \quad 4087(a_{i2}) \quad 4819(a_{i3}) \quad 5551(a_{i4}) \quad 6383(a_{i5}) \\
G_i; & \quad 4891(a_{i1}) \quad 5767(a_{i2}) \quad 6643(a_{i3}) \\
G_i; & \quad 6715(a_{i1}) \quad 7735(a_{i2}) \\
G_i; & \quad 8827(a_{i1}) \\
\end{align*}
\]

\[
\begin{align*}
G_i; & \quad (12 + 1)(12y + 7) = 91, \quad d = 12 \times (12y + 7) = 156(x = 1, y = 0) \\
G_i; & \quad (12 + 1)(12y + 7) = 475, \quad d = 12 \times (12y + 7) = 558(x = 2, y = 1) \\
G_i; & \quad (12 + 1)(12y + 7) = 1147, \quad d = 12 \times (12y + 7) = 844(x = 3, y = 2) \\
G_i; & \quad (12 + 1)(12y + 7) = 2107, \quad d = 12 \times (12y + 7) = 1588(x = 4, y = 3) \\
G_i; & \quad (12 + 1)(12y + 7) = 3355, \quad d = 12 \times (12y + 7) = 3522(x = 5, y = 4) \\
G_i; & \quad (12 + 1)(12y + 7) = 6715, \quad d = 12 \times (12y + 7) = 6598(x = 6, y = 5) \\
G_i; & \quad (12 + 1)(12y + 7) = 8827, \quad d = 12 \times (12y + 7) = 8764(x = 8, y = 7) \\
\end{align*}
\]
3.10 \((12x + 7)(12y + 1)\), \((G_j)_n\)

\[
x \times 7 \quad 19 \quad 31 \quad 43 \quad 55 \quad 67 \quad 79 \quad 91
\]

13 \quad 91

25 \quad 175 \quad 475

37 \quad 259 \quad 703

49 \quad 343 \quad 931 \quad 1519

61 \quad 427 \quad 1159 \quad 1891 \quad 2623

73 \quad 511 \quad 1387 \quad 2263 \quad 3139 \quad 4015

85 \quad 595 \quad 1615 \quad 2635

97 \quad 679 \quad 1843 \quad 3007 \quad 4171 \quad 5335 \quad 6499 \quad 7663

\[
(12 \times 7)(12 \times 1), 12 (12 \times 7) 228(1, 2), 372(2, 3), 516(3, 4), 660(4, 5), 804(5, 6), 948(6, 7)
\]

\[
G_j : 9(a_0) \quad 175(a_0) \quad 259(a_0) \quad 343(a_0) \quad 427(a_0) \quad 511(a_0) \quad 595(a_0) \quad 679(a_0)
\]

\[
G_j : 475(a_1) \quad 703(a_1) \quad 931(a_1) \quad 1159(a_1) \quad 1387(a_1) \quad 1615(a_1) \quad 1843(a_1)
\]

\[
G_j : 2107(a_2) \quad 2623(a_2) \quad 3139(a_2) \quad 3655(a_2) \quad 4171(a_2)
\]

\[
G_j : 3355(a_3) \quad 4015(a_3) \quad 4675(a_3) \quad 5335(a_3)
\]

\[
G_j : 4892(a_4) \quad 5695(a_4) \quad 6499(a_4)
\]

\[
G_j : 6715(a_5) \quad 7663(a_5)
\]

\[
G_j : 8827(a_6)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 91, \quad d = 12 \times (12x + 7) = 84(x = 0, y = 1)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 475, \quad d = 12 \times (12x + 7) = 228(x = 1, y = 2)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 1147, \quad d = 12 \times (12x + 7) = 372(x = 2, y = 3)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 2107, \quad d = 12 \times (12x + 7) = 516(x = 3, y = 4)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 3355, \quad d = 12 \times (12x + 7) = 660(x = 4, y = 5)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 4892, \quad d = 12 \times (12x + 7) = 847(x = 5, y = 6)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 6715, \quad d = 12 \times (12x + 7) = 948(x = 6, y = 7)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 8827, \quad d = 12 \times (12x + 7) = 1092(x = 7, y = 8)
\]

\[
G_k : a_i = (12x + 7)(12y + 1) = 1444xy + 12x + 84y + 7, \quad d = 12 \times (12x + 7)(x = n - 1, y = n)
\]
3.11 \((12x+5)(12y+11),(Gk_n)\)

\[
\begin{array}{cccccccc}
\times & 5 & 17 & 29 & 41 & 53 & 65 & 77 & 89 \\
11 & 55 & & & & & & & \\
23 & 115 & 391 & & & & & & \\
35 & 175 & 595 & 1015 & & & & & \\
47 & 235 & 799 & 1363 & 2047 & & & & \\
59 & 295 & 1003 & 1711 & 2419 & 3127 & & & \\
71 & 355 & 1207 & 2059 & 2911 & 3763 & 4615 & & \\
83 & 415 & 1411 & 2407 & 3403 & 4399 & 5395 & 6391 & \\
95 & 475 & 1615 & 2755 & 3895 & 5035 & 6175 & 7315 & 8455 \\
\end{array}
\]

\(Gk_5: 55(a_0), 115(a_0), 175(a_0), 235(a_0), 295(a_0), 355(a_0), 415(a_0), 475(a_0)\)

\(Gk_6: 59(a_0), 115(a_0), 175(a_0), 235(a_0), 295(a_0), 355(a_0), 415(a_0), 475(a_0), 539(a_0)\)

\(Gk_7: 1015(a_0), 115(a_0), 175(a_0), 235(a_0), 295(a_0), 355(a_0), 415(a_0), 475(a_0), 539(a_0)\)

\(Gk_8: 391(a_0), 115(a_0), 175(a_0), 235(a_0), 295(a_0), 355(a_0), 415(a_0), 475(a_0), 539(a_0), 6175(a_0)\)

\(Gk_9: 639(a_0), 7315(a_0)\)

\(Gk_{10}: 8455(a_0)\)

\(Gk_{11}: a_0 = (12x+5)(12y+11) = 55, \quad d = 12 \times (12x+5) = 60(x = 0, y = 0)\)

\(Gk_{12}: a_1 = (12x+5)(12y+11) = 391, \quad d = 12 \times (12x+5) = 204(x = 1, y = 1)\)

\(\ldots\)

\(Gk_{25}: a_{25} = (12x+5)(12y+11) = 901, \quad d = 12 \times (12x+5) = 1086(x = 7, y = 7)\)

\(Gk_{26}: a_{26} = (12x+5)(12y+11) = 144xy + 132x + 60y + 55, \quad d = 12 \times (12x+5)(x = n-1, y = n-1)\)
3.12 \((12x+11)(12y+5),(Gl_n)\)

\(\times\) 11 23 35 47 59 71 83 95
5 55
17 187 391
29 319 667 1015
41 451 943 1435
53 583 1219 1855
65 715 1495 2275
77 847 1771 2695
89 979 2047 3115

\(Gl_1: 55(a_{n}) \times 187(a_{n}) 319(a_{n}) 451(a_{n}) 583(a_{n}) 715(a_{n}) 847(a_{n}) 979(a_{n})\)

\(Gl_2: 391(a_{n}) 667(a_{n}) 943(a_{n}) 1219(a_{n}) 1495(a_{n}) 1771(a_{n}) 2047(a_{n})\)

\(Gl_3: 1015(a_{n}) 1435(a_{n}) 1855(a_{n}) 2275(a_{n}) 2695(a_{n}) 3115(a_{n})\)

\(Gl_4: 3127(a_{n}) 3835(a_{n}) 4543(a_{n}) 5251(a_{n})\)

\(Gl_5: 4651(a_{n}) 5467(a_{n}) 6319(a_{n})\)

\(Gl_6: 7307(a_{n})\)

\(Gl_7: 8455(a_{n})\)

\(Gl_1: a_{n} = (12x + 11)(12y + 5) = 55, \quad d = 12 \times (12x + 11) = 132(x = 0, y = 0)\)

\(Gl_2: a_{n} = (12x + 11)(12y + 5) = 391, \quad d = 12 \times (12x + 11) = 276(x = 1, y = 1)\)

\(Gl_3: a_{n} = (12x + 11)(12y + 5) = 1015, \quad d = 12 \times (12x + 11) = 420(x = 2, y = 2)\)

\(Gl_4: a_{n} = (12x + 11)(12y + 5) = 1927, \quad d = 12 \times (12x + 11) = 564(x = 3, y = 3)\)

\(Gl_5: a_{n} = (12x + 11)(12y + 5) = 3217, \quad d = 12 \times (12x + 11) = 708(x = 4, y = 4)\)

\(Gl_6: a_{n} = (12x + 11)(12y + 5) = 4615, \quad d = 12 \times (12x + 11) = 852(x = 5, y = 5)\)

\(Gl_7: a_{n} = (12x + 11)(12y + 5) = 6615, \quad d = 12 \times (12x + 11) = 996(x = 6, y = 6)\)

\(Gl_n: a_{n} = (12x + 11)(12y + 5) = 144x + 60y + 132 + 55, \quad d = 12 \times (12x + 11)(x = n - 1, y = n - 1)\)
3.13 \((12x+1)(12y+11),(Gm_n)\)

\[
\begin{array}{cccccccc}
11 & 143 \\
23 & 299 & 575 \\
35 & 455 & 875 & 1255 \\
47 & 767 & 1475 & 2183 & 2891 & 3599 \\
59 & 923 & 1775 & 2627 & 3479 & 4331 \\
71 & 1079 & 2075 & 3071 & 4067 & 5063 & 6059 & 7055 \\
83 & 1235 & 2375 & 3515 & 4655 & 5795 & 6935 & 8075 & 9215 \\
\end{array}
\]

\[
\begin{align*}
Gm_1: & (12(3x+1)(12(3y+1)) = 143, & d = 12 \times (12x+1) = 156, (x, y = 0) \\
Gm_2: & (12(3x+1)(12(3y+1)) = 575, & d = 12 \times (12x+1) = 300, (x, y = 1) \\
Gm_3: & (12(3x+1)(12(3y+1)) = 1255, & d = 12 \times (12x+1) = 444, (x, y = 2) \\
Gm_4: & (12(3x+1)(12(3y+1)) = 2393, & d = 12 \times (12x+1) = 588, (x, y = 3) \\
Gm_5: & (12(3x+1)(12(3y+1)) = 3991, & d = 12 \times (12x+1) = 732, (x, y = 4) \\
Gm_6: & (12(3x+1)(12(3y+1)) = 5183, & d = 12 \times (12x+1) = 876, (x, y = 5) \\
Gm_7: & (12(3x+1)(12(3y+1)) = 7055, & d = 12 \times (12x+1) = 1020, (x, y = 6) \\
Gm_8: & (12(3x+1)(12(3y+1)) = 9215, & d = 12 \times (12x+1) = 1164, (x, y = 7) \\
\end{align*}
\]
3.14 \((12x+11)(12y+1),(Gn)\)

\[G_{n} = (143(n), 275(n), 407(n), 539(n), 671(n), 803(n), 935(n), 1067(n))\]

\[G_{n} = (575(n), 851(n), 1127(n), 1403(n), 1679(n), 1955(n), 2231(n))\]

\[G_{n} = (1295(n), 1715(n), 2135(n), 2555(n), 2975(n), 3395(n))\]

\[G_{n} = (2303(n), 2867(n), 3431(n), 4075(n), 4559(n))\]

\[G_{n} = (3599(n), 4307(n), 5015(n), 5723(n))\]

\[G_{n} = (5183(n), 6035(n), 6887(n), 7055(n))\]

\[G_{n} = (8051(n), \ldots)\]

\[G_{n} = (9215(n), \ldots)\]

\[G_{n} = (143(n), 275(n), 407(n), 539(n), 671(n), 803(n), 935(n), 1067(n))\]

\[G_{n} = (575(n), 851(n), 1127(n), 1403(n), 1679(n), 1955(n), 2231(n))\]

\[G_{n} = (1295(n), 1715(n), 2135(n), 2555(n), 2975(n), 3395(n))\]

\[G_{n} = (2303(n), 2867(n), 3431(n), 4075(n), 4559(n))\]

\[G_{n} = (3599(n), 4307(n), 5015(n), 5723(n))\]

\[G_{n} = (5183(n), 6035(n), 6887(n), 7055(n))\]

\[G_{n} = (8051(n), \ldots)\]

\[G_{n} = (9215(n), \ldots)\]
3.15 \((12x+5)(12y+7), (Go_n)\)

\[
\begin{array}{ccccccccccc}
\times & 5 & 17 & 29 & 41 & 53 & 65 & 77 & 89 \\
7 & 35 & & & & & & & \\
19 & 95 & 323 & & & & & & \\
31 & 155 & 527 & & & & & & \\
43 & 215 & 731 & 1247 & 1763 & & & & \\
55 & 275 & 935 & 1595 & 2255 & & & & \\
67 & 335 & 1139 & 1943 & 2747 & 3551 & & & \\
79 & 395 & 1343 & 2291 & 32 & 35 & & & \\
\end{array}
\]

\[
G_{o_1} = 35(a_1), \quad 95(a_2), \quad 155(a_3), \quad 215(a_4), \quad 275(a_5), \quad 335(a_6), \quad 395(a_7), \quad 455(a_8)
\]

\[
G_{o_2} = 323(a_1), \quad 731(a_2), \quad 1139(a_3), \quad 1547(a_4), \quad 1943(a_5), \quad 2341(a_6), \quad 2747(a_7), \quad 3155(a_8)
\]

\[
G_{o_3} = 2255(a_1), \quad 2747(a_2), \quad 3239(a_3), \quad 3731(a_4), \quad 4233(a_5), \quad 4735(a_6), \quad 5237(a_7), \quad 5739(a_8)
\]

\[
G_{o_4} = 5135(a_1), \quad 5637(a_2), \quad 6139(a_3), \quad 6641(a_4), \quad 7143(a_5), \quad 7645(a_6), \quad 8147(a_7), \quad 8649(a_8)
\]

\[
G_{o_5} = 8099(a_1), \quad 8603(a_2), \quad 9107(a_3), \quad 9611(a_4)
\]

\[
G_{o_1} = 1 = (12x+5)(12y+7) = 35, \quad d = 12 \times (12x+5) = 60(x=0, y=0)
\]

\[
G_{o_2} = 2 = (12x+5)(12y+7) = 123, \quad d = 12 \times (12x+5) = 204(x=1, y=1)
\]

\[
G_{o_3} = 3 = (12x+5)(12y+7) = 186, \quad d = 12 \times (12x+5) = 306(x=2, y=2)
\]

\[
G_{o_4} = 4 = (12x+5)(12y+7) = 249, \quad d = 12 \times (12x+5) = 429(x=3, y=3)
\]

\[
G_{o_5} = 5 = (12x+5)(12y+7) = 312, \quad d = 12 \times (12x+5) = 504(x=4, y=4)
\]

\[
G_{o_6} = 6 = (12x+5)(12y+7) = 375, \quad d = 12 \times (12x+5) = 666(x=5, y=5)
\]

\[
G_{o_7} = 7 = (12x+5)(12y+7) = 438, \quad d = 12 \times (12x+5) = 848(x=6, y=6)
\]

\[
G_{o_8} = 8 = (12x+5)(12y+7) = 501, \quad d = 12 \times (12x+5) = 1008(x=7, y=7)
\]

\[
G_{o_9} = 9 = (12x+5)(12y+7) = 564, \quad d = 12 \times (12x+5)(x=n-1, y=n-1)
\]
3.16 $({12x + 7})(12y + 5), (G_{p_n})$

\[ x \times 7 \quad 19 \quad 31 \quad 43 \quad 55 \quad 67 \quad 79 \quad 91 \]

\[ 5 \quad 35 \]

\[ 17 \quad 119 \quad 233 \]

\[ 29 \quad 203 \quad 551 \]

\[ 41 \quad 287 \quad 779 \quad 1271 \]

\[ 53 \quad 371 \quad 1007 \quad 1643 \quad 2279 \]

\[ 65 \quad 455 \quad 1235 \quad 2015 \quad 2795 \quad 3575 \]

\[ 77 \quad 539 \quad 1463 \quad 2387 \quad 3579 \quad 4735 \]

\[ 89 \quad 623 \quad 1691 \quad 2759 \quad 3827 \quad 4895 \quad 5963 \quad 7031 \quad 8099 \]

\[ G_{p_1} : 35(a_n) \quad 119(a_n) \quad 287(a_n) \quad 375(a_n) \quad 555(a_n) \quad 623(a_n) \]

\[ G_{p_1} : 352(a_n) \quad 592(a_n) \quad 1080(a_n) \quad 1235(a_n) \quad 1463(a_n) \quad 1691(a_n) \]

\[ G_{p_1} : 896(a_n) \quad 1271(a_n) \quad 1643(a_n) \quad 2015(a_n) \quad 2795(a_n) \]

\[ G_{p_1} : 1763(a_n) \quad 2279(a_n) \quad 2795(a_n) \quad 3311(a_n) \quad 3827(a_n) \]

\[ G_{p_1} : 2915(a_n) \quad 3576(a_n) \quad 4235(a_n) \quad 4895(a_n) \]

\[ G_{p_1} : 4355(a_n) \quad 5159(a_n) \quad 5963(a_n) \]

\[ G_{p_1} : 6083(a_n) \quad 7031(a_n) \]

\[ G_{p_1} : 8099(a_n) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 35, \quad d = 12 \times (12x + 7) = 84(x = 0, y = 0) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 323, \quad d = 12 \times (12x + 7) = 228(x = 1, y = 1) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 899, \quad d = 12 \times (12x + 7) = 372(x = 2, y = 2) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 1763, \quad d = 12 \times (12x + 7) = 516(x = 3, y = 3) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 2915, \quad d = 12 \times (12x + 7) = 660(x = 4, y = 4) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 6085, \quad d = 12 \times (12x + 7) = 948(x = 6, y = 6) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 8099, \quad d = 12 \times (12x + 7) = 1092(x = 7, y = 7) \]

\[ G_{p_1} : a_1 = (12x + 7)(12y + 5) = 144xy + 60x + 84y + 35, \quad d = 12 \times (12x + 7) (x = n - 1, y = n - 1) \]
4. Primality Test

We have seen that all the composite numbers but 2 and 3 can be represented in a form of $(12x + \alpha) \times (12y + \beta)$. Then, how can we determine if a given positive integer, $G$, is prime or composite?

Let $G$ be an arbitrary positive integer.

i) Check if $G$ is a multiple of 2 or 3. If $G$ is a multiple of one of these, it is a composite number.

ii) If $G$ is not a multiple of 2 or 3, $G$’s remainder $R$ when divided by 12 is $R \in \{1, 5, 7, 11\}$ and $R$ is the a number in the table multiplication elements, then $G$ is a composite number. If $R$ is not the same as any of the multiplication elements, then $G$ is a prime number.

4.1 Substitution.

If a given number, $G$, is not a multiple of 2 or 3, we can express it as follows.

$$G = (12x + \alpha) \times (12y + \beta)$$

$$= 144xy + 12\beta x + 12\alpha y + \alpha \beta (\alpha, \beta \in \{1, 5, 7, 11\})$$

If we substitute $xy$, $\beta x + \alpha y$ with $X$ and $Y$, respectively,

\[\beta x + \alpha y = Y, \ xy = X\]

The result is the following.

\[\square(12x + \alpha) \times (12y + \beta)\]

$$= 144X + 12Y + \alpha \beta = G(G \text{ is given number})$$

$$= 12X + Y = C \ (C = \frac{G - \alpha \beta}{12})$$
4.2 Determination of a valid domain.

Let us apply the arithmetic mean and geometric mean to $xy$ and $\beta x + \alpha y$.

From $\beta x + \alpha y = Y$ and $xy = X$, $y = \frac{-\beta x + Y}{\alpha}$.
<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$X(xy)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{-\beta + Y}{\alpha}$</td>
<td>$\frac{-\beta + Y}{\alpha} \times 1$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{-2 \cdot \beta + Y}{\alpha}$</td>
<td>$\frac{-2 \cdot \beta + Y}{\alpha} \times 2$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{-3 \cdot \beta + Y}{\alpha}$</td>
<td>$\frac{-3 \cdot \beta + Y}{\alpha} \times 3$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{-4 \cdot \beta + Y}{\alpha}$</td>
<td>$\frac{-4 \cdot \beta + Y}{\alpha} \times 4$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{-5 \cdot \beta + Y}{\alpha}$</td>
<td>$\frac{-5 \cdot \beta + Y}{\alpha} \times 5$</td>
</tr>
<tr>
<td>$k$</td>
<td>$\frac{-k \cdot \beta + Y}{\alpha}$</td>
<td>$\frac{-k \cdot \beta + Y}{\alpha} \times k$</td>
</tr>
</tbody>
</table>

When $x = 1$, $X(xy)$ is the minimum. Therefore, the minimum of $X(xy)$ is

$X(xy) = \frac{-\beta + Y}{\alpha}$.

$12X + Y = C$
$\rightarrow 12X + \alpha \cdot X + \beta = C$ (substitution: $Y = \alpha \cdot X + \beta$)
$\rightarrow (12 + \alpha) \cdot X = C - \beta$
$\rightarrow X = \frac{C - \beta}{12 + \alpha}$

The maximum of $X(xy)$ is

(i) If $x + y$ is even, then the maximum is achieved when $x = y$. 

34
\[ 12X + Y = C \]
\[ \rightarrow 12x^2 + (\alpha + \beta) \cdot x - C = 0 \]
\[ \rightarrow x = \frac{-(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 + 4 \cdot 12 \cdot C}}{24} \]
\[ \rightarrow X(xy, x = y) = \left( \frac{-(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 + 4 \cdot 12 \cdot C}}{24} \right)^2 \]
\[ \rightarrow X(positive) = \left( \frac{-(\alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4 \cdot 12 \cdot C}}{24} \right)^2 \]

(ii) If \( x + y \) is odd, then the maximum is achieved when \( x + 1 = y \).

\[ 12X + Y = C \]
\[ \rightarrow 12x(x+1) + \beta \cdot x + \alpha(x+1) - C = 0 \]
\[ \rightarrow 12x^2 + (\alpha + \beta + 12)x + \alpha - C = 0 \]
\[ \rightarrow x = \frac{-(\alpha + \beta + 12) \pm \sqrt{(\alpha + \beta + 12)^2 + 4 \cdot 12 \cdot (C - \alpha)}}{24} \]
\[ \rightarrow X(positive) = \left( \frac{-(\alpha + \beta + 12) + \sqrt{(\alpha + \beta + 12)^2 + 4 \cdot 12 \cdot (C - \alpha)}}{24} \right)^2 \]

- If the maximum and minimum of \( X(xy) \) is not an integer, then we can make it an integer by rounding it up.
4.3 Finding integer values of \( x \) and \( y \).

If we can determine the valid domain, we can make the following table list of \((X, Y)\).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>( a_5 )</th>
<th>( a_6 )</th>
<th>( \cdots )</th>
<th>( \cdots )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>( b_4 )</td>
<td>( b_5 )</td>
<td>( b_6 )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( b_n )</td>
</tr>
</tbody>
</table>

In \( 12X + Y = C \), \( X \) increases by 1 and \( Y \) decreases by 12. So, \( X \) and \( Y \) have properties of an arithmetic progression. Let two arithmetic progressions, \( X \) and \( Y \), be \( X = n + a \) and \( Y = -12n + b \), respectively.

\((X, Y)\) Tables pairs

Is there an efficient method to find a valid set of \((X, Y)\), which has an integer root, from \((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5), (a_6, b_6), \cdots (a_n, b_n)\).
4.3.1 First Method- Pell’s equation

$x, n, r$ is positive integer $(x, r$ contains zero $)$

Since $X = \frac{-\beta \cdot x^2 + Y \cdot x}{\alpha}$

$n + a = \frac{-\beta \cdot x^2 + (-12n + b) \cdot x}{\alpha}$, (substitution : $X = n + a$, $Y = -12n + b$)

$\rightarrow \alpha \cdot (n + a) = -\beta \cdot x^2 + (-12n + b) \cdot x$

$\rightarrow \beta \cdot x^2 - (b - 12n) \cdot x + \alpha \cdot (n + a) = 0$

$\rightarrow x = \frac{(b - 12n) \pm \sqrt{(b - 12n)^2 - 4 \cdot \alpha \beta \cdot (n + a)}}{2 \cdot \beta}$

Since $x$ is zero or a positive integer

We can find integer roots $(n, r)$ from $r' = (b - 12n)^2 - 4 \cdot \alpha \beta \cdot (n + a)$

$\rightarrow 36r^2 = 36 \cdot \{(12n - b)^2 - 4 \cdot \alpha \beta \cdot (n + a)\}$

$\rightarrow 36r^2 = 36 \cdot (144n^2 - 24b \cdot n + b^2 - 4 \cdot \alpha \beta \cdot n - 4 \cdot \alpha \beta \cdot a)$

$\rightarrow 36r^2 = 36 \cdot (144n^2 - 24b \cdot n - 4 \cdot \alpha \beta \cdot n + b^2 - 4 \cdot \alpha \beta \cdot a)$

$\rightarrow 36r^2 = (72n - (6b + \alpha \beta))^2 - (6b + \alpha \beta)^2 + 36b^2 - 4 \cdot 36 \cdot \alpha \beta \cdot a$

$\rightarrow (72n - (6b + \alpha \beta))^2 - 36r^2 = (6b + \alpha \beta)^2 - 36b^2 + 4 \cdot 36 \cdot \alpha \beta \cdot a$

$\rightarrow (72n - (6b + \alpha \beta) + 6r) \times (72n - (6b + \alpha \beta) - 6r) = 12(12a + b) \cdot \alpha \beta + (\alpha \beta)^2$

If an integer root, $(n, r)$, exists, then we can find integer $(x, y)$.

Example)

Let the given number $G = 32,185$.

From $G = 2,682 \times 12 + 1$, $R = 1$. Since it is included in $A_1$, we can perform the following table multiplications. $A_1 \times A_1$, $A_2 \times A_2$, $A_3 \times A_3$, $A_4 \times A_3$, $A_1 \times A_1$

When $\alpha = \beta = 1$, $G = (12x + 1) \times (12y + 1)$ (This paper will deal with only one example.)
$X = 207 + n - 1 = n + 206. \ Y = 198 - 12(n - 1) = -12n + 210$

$\rightarrow a = 206, \ b = 210, \ \alpha \beta = 1$

$r^2 = (12n - b)^2 - 4 \cdot \alpha \beta \cdot (n + a)$

$r^2 = (12n - 210)^2 - 4 \cdot (n + 206)$

$\rightarrow r^2 = 144n^2 - 5,044n + 43,276 = 38,376 - \sum_{k=1}^{n-1} (4,612 - 288(k - 1))$

$\rightarrow 36(144n^2 - 5,044n + 43,276) = 36r^2$

$\rightarrow (72n - 1261)^2 - 32,185 = 36r^2$

$\rightarrow (72n - 1261)^2 - 36r^2 = 32,185$

$\rightarrow (72n - 1,261 - 6r) \times (72n - 1,261 + 6r) = (\pm 1, \pm 32,185), \ (\pm 5, \pm 6, 437), (\pm 41, \pm 785), (\pm 157, \pm 205)$

The pair that has an integer value is $(-157, -205)$

$\rightarrow (72n - 1,261 - 6r) \times (72n - 1,261 + 6r) = (-157, -205)$
The integer roots are \((n, r) = (15, 4)\).

4.3.2 Second Method

In order to find \((x, n)\) pairs that have integer roots, let us do the substitutions

\[ \beta x + \alpha y = Y, \quad xy = X \]

\[ x, \quad y = \frac{Y - \beta \cdot x}{\alpha} \]

\[ \rightarrow X = \frac{-\beta \cdot x^2 + Y \cdot x}{\alpha}, \]

\[ \rightarrow n + a = \frac{-\beta \cdot x^2 + (-12n + b) \cdot x}{\alpha}, \text{(substitution: } X = n + a, \ Y = -12n + b) \]

\[ \rightarrow (12x + \alpha)n = -\beta \cdot x^2 + b \cdot x - \alpha \cdot a, \]

\[ \rightarrow n = \frac{-\beta \cdot x^2 + b \cdot x - \alpha \cdot a}{12x + \alpha} \]

\[ n - x \text{ Curve} \] has the shape of a long sickle.
In order to find an integer root pair, \((x, n)\), 270 thousand iteration is required. But, this method is inefficient because all \((X, Y)\) tables need to be iterated. However, from the graph of the function, we can discover the following properties. When the value of \(x\) is small, \(n\) increases faster. However, for a certain domain, the rate at which \(n\) increases is greatly reduced as \(x\) increases, and when \(n\) approaches its limit, \(n\) does not either increase or decrease as \(x\) increases. Therefore, we can confirm that the values of \(n\) congregate at certain domains.

So, we have come up with the following idea to find integer root pair \((x, n)\).

If we do the iteration in the domain \(0 \sim x\), on the x-axis (up to the point where \(n_{k+1} - n_k\) is greater than 1) and in the range \(n_1 \sim n_2\), on the n-axis at the points where \(n_{k+1} - n_k\) becomes less than 1, then we have the same effect as that of inspecting all the whole numbers.

Therefore, the number of iterations becomes \(H = (0 \sim x_i) + (n_1 \sim n_2)\).

But, we have to multiply by 4 because it is hard to find the value of \(G\) from the 16 subgroups (the greater the value of \(G\), the more the computational complexity increases)
(Each of A1, A5, A7, and A11 has 4 subgroups.)
Total number of iterations \(H_{total} = 4 \times H\)

Example)
Let the given number be \(G = 32,185\). (the same as in the previous example)

\[
\begin{array}{cccccccccc}
X & 207 & 208 & 209 & 210 & 211 & 212 & \cdots & \cdots & 221 \\
Y & 198 & 186 & 174 & 162 & 150 & 138 & \cdots & \cdots & 30 \\
\end{array}
\]

\(X = 207 + n - 1 = n + 206, \quad Y = 198 - 12(n - 1) = -12n + 210\)

When \(\rightarrow a = 206, \quad b = 210, \quad \alpha \beta = 1\),

\[
\begin{align*}
  n &= \frac{-\beta \cdot x^2 + b \cdot x - \alpha \cdot a}{12x + \alpha} \\
  n &= \frac{-x^2 + 210 \cdot x - 206}{12x + 1},
\end{align*}
\]
In the A domain, the values of x are such that \( x_n - x_{n-1} \geq 1 \). So, we count by x.

In the B domain, the values of x are such that \( x_n - x_{n-1} \leq 1 \). So, we count by n.

In the A domain, the number of x count is, 5 (1, 2, 3, 4, 5) and, in the B domain, that of n count is 2 (14, 15). So, \( H = 5 + 2 = 7 \)

When \( G = 505,488,001 \), \( H = 800 \) approximately. When \( G = 5,391,508,801 \), \( H = 2,700 \) approximately.

So, \( H = O(\frac{\sqrt{x}}{20}) \), approximately.

5. Uncertainty Principle

Given a number \( G \), how can we be sure that it is prime or composite? If it is a composite number, it must, of course, exist on the multiplication table. If it is a prime number, it must not exist on the multiplication table.

From \( G = (12x + \alpha) \times (12y + \beta) \), it is thought to be very easy to find positive integers, \( x \) and \( y \), given a composite number, \( G \). However, the coordinates of the matrix can go to infinity. When \( G \) is a very large number, it is not easy to find positive integers for \( x \) and \( y \). In the worst case, we have to inspect the whole matrix. We can recognize that it is impossible to assign arbitrary integers to get positive integers for \( x \) and \( y \). In other words, if we can determine the value of the positive integer \( x \), we cannot find the value of \( y \) (which turns out to be a real or complex number). On the other hand, if we can determine the value of positive integer \( y \), we cannot find the value of \( x \) (which turns out to be real or complex number).

If \( G \) is a composite number, the positive integer values \( x \) and \( y \), are determined independent of ourselves and we, as observers, are unable to find the values until we do the necessary calculation.
6. Impossibility Principle

From $G = (12x + \alpha) \times (12y + \beta)$, can we find the values of $x$ and $y$, simultaneously? It was once thought that there was a method to find them intuitively, but this was a fantasy or phantom. It is impossible. The principle is very simple.

If a given number $G$, exists on the matrix, then it is a composite number. If not, then it is a prime number. If $G$ exists on the matrix (a composite number), then it is represented as $G = (12x + \alpha) \times (12y + \beta)$. Also, the matrix is structured in arithmetic progression groups. So, the composite number $G$, an element of matrix, is an element of an arithmetic progression. Therefore, we can express it as follows.

$$G = a_1 + (n-1)d$$

Since $IG$ is an arithmetic progression, element $G$ can be expressed as $G = I + (n-1)d$.

$I = (12x' + \alpha) \times (12y' + \beta)$, \textit{(first term)}

$d = 12 \times (12x' + \alpha)$, \textit{(common difference)}

$n = y'' - y' + 1, (n = 1, 2, 3, 4, \ldots)$
So, after some manipulation,
\[ G = I + (n - 1)d \]
\[ \rightarrow (12x' + \alpha) \cdot (12y' + \beta) + (y'' - y') \cdot (12 \times (12x' + \alpha)) \]
\[ \rightarrow (144x'y' + 12 \cdot \beta \cdot x' + 12 \cdot \alpha \cdot y' + \alpha \beta) + (144x'y' + 12 \cdot \alpha \cdot y' - 144x'y' - 12 \cdot \alpha \cdot y') \]
\[ \rightarrow 144x'y' + 12 \cdot \beta \cdot x' + 12 \cdot \alpha \cdot y' + \alpha \beta \]
\[ \rightarrow (12x' + \alpha)(12y'' + \beta) \]

When \( G \) is given, we can see that it is equivalent to the problem of finding positive integers \( x \) and \( y \). Given \( I \), \( n \), and \( d \), it is easy to find the value of \( G \). However, is it easy to find the value of \( I \), \( n \), and \( d \) once we are given the value of \( G \)? It is not easy. Determining if a given number is a prime or a composite is the same as finding the initial term, difference, and general term of \( G \), an element of arithmetic progression. Is it possible? No, it isn’t.

7. Conclusion

This paper asserts that the composite numbers of the \( 12n + 1 \), \( 5 \), \( 7 \), \( 11 \) series make up sixteen arithmetic progression groups. We were able indirectly to deduce the uncertainty of prime numbers through the composite numbers. It is true that, unlike the prime numbers, the composite numbers are governed by a rule that is structural and regular. Using the matrix, we have shown it possible to significantly reduce perceptive complexity of Pseudoprimes (composite numbers of the \( 12n + 1 \), \( 5 \), \( 7 \), \( 11 \) and, at the same time, to express them in quadratic algebra. Applying arithmetic and geometric means, we reduced some of the computational complexity by converting the matrix into a quadratic equation and computing the valid domains. But, the uncertainty of the matrix directly reflects the irregularity of prime numbers (whether a given number is a prime number or a composite number and what the next prime number is). We can see that, since the complexity of a matrix as a given number, \( G \), becomes larger and larger, it becomes harder to predict the next prime number. Then, is there a method to reduce the complexity of the matrix? It is impossible. To find the given number, we need to solve \( G = (12x + \alpha) \times (12y + \beta) \). Since \( G \) is an element of the arithmetic progression, \( G = a_1 + (n - 1)d \).

\[ (a_1 = \text{first term}, \ d = \text{common difference}), \] it is the same problem as finding the initial term, common difference from a given number, \( G \). Is this possible? It is not possible.
Acknowledgements

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External links

[41] http://en.wikipedia.org/wiki/Modular_Arithmetic
[45] In this modular art article, one can learn more about applications of modular arithmetic in art.
[46] Modular Arithmetic and patterns in addition and multiplication tables
<Appendix 1> Certainty of Mathematics

(a) Platonism and Mathematical Reality

According to Platonism, mathematical objects are real. Their existence is an objective fact, quite independent of our knowledge of them. These objects are, of course, not physical or material. They exist outside the space and time of physical existence. They are immutable; they were not created, and they will not change or disappear. Any meaningful question about a mathematical object has a definite answer, whether we are able to determine it or not. According to Platonism, a mathematician is an empirical scientist like a geologist; he cannot invent anything, because it is all there already. All he can do is discover. (The Mathematical Experience, 318)

Since mathematical objects are what they are, in defiance of our ignorance or preferences, they must be real in a sense independent of human minds. (The Mathematical Experience, 408)

The Later Pythagoreans and the Platonists distinguished sharply between the world of things and the world of ideas. Objects and relationships in the material world were subject to imperfections, change, and decay and hence did not represent the ultimate truth, but there was an ideal world in which there were absolute and unchanging truths. (The Loss of Certainty, 16)

The roots of the philosophy of mathematics, as of mathematics itself, are in classical Greece. For the Greeks, mathematics meant geometry, and the philosophy of mathematics in Plato and Aristotle is the philosophy of geometry. For Plato, the mission of philosophy was to discover true knowledge behind the veil of opinion and appearance, the change and illusion of the temporal world. (The Mathematical Experience, 325)
(b) Debacle of Certainty

A classic statement to this effect was made by Pascal(1623–1662) in his Pensees long before the modern controversies arose. “Truth is so subtle a point that our instruments are too blunt to touch it exactly. When they do reach it, they crush the point and bear down around it, more on the false than on the true.” (The Loss of Certainty, 324)

Certainly experience did not vouch for the behavior of infinite straight lines, whereas axioms were supposed to be self-evident truths about the physical world. The parallel axiom in the form stated by Euclid(BC 330?–BC 275?) was thought to be somewhat too complicated. It lacked the simplicity of the other axioms. Apparently even Euclid did not like his version of the parallel axiom because he did not call upon it until he had proved all the theorems he could without it.

A related problem, which did not bother many people but ultimately came to the fore as a vital concern, is whether one can be sure of the existence of infinite straight lines in physical space. Euclid was careful to postulate only that one can extend a finite line segment as far as necessary, so that even the extended segment was still finite. Nevertheless Euclid did imply the existence of infinite straight lines for, were they finite, they could not be extended as far as necessary in any given context. (The Loss of Certainty P 78-79)

The historical development of non-Euclidean geometry was a result of attempts to deal with this axiom (Euclid’s Fifth). (The Mathematical Experience, 218)

However, the material in Gauss’s notes became available after his death in 1855 when his reputation was unexcelled and the publication in 1868 of Riemann’s 1854 paper convinced many mathematicians that a non-Euclidean geometry
could be the geometry of physical space and that we could no longer be sure which geometry was true. The mere fact that there can be alternative geometries was in itself a shock. But the greater shock was that one could no longer be sure which geometry was true or whether any one of them was true. (The Loss of Certainty, 88)

The loss of certainty in geometry was philosophically intolerable, because it implied the loss of all certainty in human knowledge. Geometry had served, from the time of Plato, as the supreme exemplar of the possibility of certainty in human knowledge. (The Mathematical Experience, 331)

But by 1900 Euclidean geometry was recognized to be just a logical structure erected on a set of twenty or so man-made axioms, and it was indeed possible that contradictory theorems could turn up. (The Loss of Certainty, 198)

This confidence that truths would be discovered in all fields was shattered by the recognition that there is no truth in mathematics. The hope and perhaps even the belief that truths can be obtained in politics, ethics, religion, economics, and many other fields may still persist in human minds, but the best support for the hope has been lost. Mathematics offered to the world proof that man can acquire truths and then destroyed the proof. It was non-Euclidean geometry and quaternions, both triumphs of reason, that proved the way for this intellectual disaster. With the loss of truth, man lost his intellectual center, his frame of reference, the established authority for all thought. The “pride of human reason” suffered a fall which brought down with it the house of truth. (The Loss of Certainty, 99)

At first Gauss(1777–1855) who has discovered non-Euclid geometry seems to have concluded that there is no truth in all of mathematics. In a letter to Bassel of November 21, 1811, he said,
“One should never forget that the functions [of a complex variable], like all mathematical constructions, are only our own creations, and that when the definition with which one begins ceases to make sense, one should not ask, What is, but what is it convenient to assume in order that it remain significant.”

“According to my most sincere conviction the theory of space has an entirely different place in knowledge from that occupied by pure mathematics [the mathematics built on number]. There is lacking throughout our knowledge of it the complete persuasion of necessity(also of absolute truth) which is common to the latter; we must add in humility, that if number is exclusively the product of our mind, space has a reality outside our mind and we cannot completely prescribe its laws.” (The Loss of Certainty, 87)

Gauss’ thoughts about the number and space become a decisive motive to give a birth of non-Euclid geometry and Georg Friedrich Bernhard Riemann(1826-1866), who are very familiar with his study, gives birth to Riemann geometry. One of Riemann’s objectives was to show that Euclid’s axioms were indeed empirical rather than self-evident truths. (The Loss of Certainty, 86)

Albert Einstein(1879~1955) adopted Riemann’s mathematical discoveries by giving them a precise physical interpretation. (The Elegant Universe, 233)

It is not obvious whether an infinite straight line, which exists in the human imagination, is applicable to the real physical world. Einstein rather adopts Riemannian geometry in his General Theory of Relativity and, on this basis, he successfully explained the relationship between light and gravity. However, there is no heuristic evidence that reality in the physical world agrees with reality in the mathematical world.
(c) Withering of Mathematical Truth

Due to the loss of certainty in geometry, mathematicians tend to discover the truth from the arithmetic.

The immediate forerunner of modern intuitionism is Leopold Kronecker (1823~1891). His epigram (delivered in an after-dinner speech), “God made the integers; all the rest is the work of man,” is well known. The complicated logical derivation of the ordinary whole numbers such as Cantor and Dedekind presented through a general theory of sets seemed less reliable than direct acceptance of the integers. (The Loss of Certainty, 232)

As for the consistency of arithmetic, no one doubted it. (The Loss of Certainty, 196)

David Hilbert (1862~1943) showed through the medium of analytic geometry that Euclidean geometry is consistent if the science of arithmetic is consistent. Hence, in his second problem he asked for a proof that the science of arithmetic is consistent. (The Loss of Certainty, 196)

However, Kurt Gödel (1906-1978) tells that it is impossible to prove the non-contradiction of arithmetic in his first incompleteness theorem (Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but no provable in the theory (Kleene 1967, p.250)) and second incompleteness theorem (For any formal effectively generated theory T including basic arithmetical truth and also certain truths about formal provability, T includes a statement of its own consistency if and only if T is inconsistent.) Both of Gödel's results were shattering. The inability to prove consistency dealt
a death blow most directly to Hibert’s formalist philosophy because he had planned such a proof in his meta-mathematics and was confident it would succeed. Hence mathematicians were working under a threat of doom. (The Loss of Certainty, 263)

There are mathematicians who are more seriously skeptical about the self-evident truth.

Arend Heyting, leading intuitions, affirmed that no one today can speak of the true mathematics, that is, true in the sense of a correct, unique body of knowledge. Hermann Hankel (1839~1873), Richard Dedekind (1831~1916), and Karl Weierstrass (1815~1897) all believed that mathematics is a human creation. And Ludwig Wittgenstein (1889-1951), a student of Russell and an authority in his own right, believed that the mathematician is an inventor not a discoverer. Hermann Weyl (1885~1955), too, was rather ironic about eternal truths. The Nobel prize-winning physicist Percy W. Bridgman (1882~1961), in The Logic of Modern Physics (1946), rejected flatly any objective world of mathematics. “It is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention.” Theoretical science is a game of mathematical make-believe. All these men contend that mathematics is not only man-made but very much influenced by the cultures in which it was developed. (The Loss of Certainty, 324-325)

Plato did believe that mathematics exists in some ideal world independent of human beings, his doctrines included much that does not apply to the current views, and the use of the appellation Platonist is more unsuitable than helpful. These assertions about the existence of an objective, unique body of mathematics do not explain where mathematics resides. They say merely that mathematics exists in some extra-human world, a castle in the air, and is merely detected by man. (The Loss of Certainty, 323)
The typical mathematician is both a Platonist and a formalist-a secret Platonist with a formalist mask that he puts on when the occasion calls for it. (The Mathematical Experience, 322)

Today many, perhaps most, mathematicians have no such conviction of the objective existence of the objects they study. (The Mathematical Experience, 252)

<Appendix 2> Discovery of Number

Aside from philosophers', physicists', and mathematicians' efforts to investigate truth, human being have made mathematical meanings and used them as means to distinguish and conceive objects and all the academic systems nowadays are being helped from its departmentalized and instrumental properties. Rapid scientific achievements further diminished scientists' motives to search for the truth. Seeing the nature as an object to take advantage of, human have created mathematics as necessary, altered, and applied. Despite of brilliant achievements, they have lost the spirits to correctly understand the nature. (The Loss of Certainty, 279–354)

We have seen that in number theory, there may be heuristic evidence so strong that it carries conviction even without rigorous proof. (The Mathematical Experience, 369)

(a) Natural Numbers

The idea that the whole numbers derive from the intuition of time had been maintained by Immanuel Kant(1724–1804), William R. Hamilton(1805–1865) in his article “Algebra as a Science of Time,” and the philosopher Arthur
Schopenhauer (1788–1860).
(The Loss of Certainty, 234 Morris Kline)

From the concept of time, Hamilton derived properties of the positive whole numbers and then extended this development to rational numbers (positive and negative whole numbers and fractions) and irrational numbers. (The Loss of Certainty, 178 Morris Kline)

However, there are opinions that oppose the statement.

The Platonist does not attempt to describe it, let alone analyze its nature. How does one acquire mathematical intuition? Evidently it varies from one person to another, even from one mathematical genius to another; and it has to be developed mathematical genius to another; and it has to be developed and refined, since it seems to be inadequate at present. But then by whom, according to what criteria, does one train or develop it? (The Mathematical Experience, 394)

The natural number system seems an innate intuition only to mathematicians so sophisticated they cannot remember or conceive of the tie before they acquired it; and so isolated that they never have to communicate seriously with people (still no doubt the majority of the human race) who have not internalized this set of ideas and made it intuitive. (The Mathematical Experience, 395)

The concept of natural numbers which we can naturally derive from intuitions about time immediately leads us to intuitively accept that natural numbers exist sequentially on a straight line. However, this problem leads us to Euclid’s Axiom 5 of the parallel postulate and we have gained only certainty without real evidence that natural numbers exist on an infinite straight line.
(b) Prime Numbers

The ancient Greeks also liked to attribute sexual qualities to numbers, but it was they who first discovered, in the 4th century BC, the primes' true potency as the building blocks for all numbers. They saw that every number could be constructed by multiplying prime numbers together.

(The Music of the Primes, 26)

In mathematics, a prime number (or a prime) is a natural number which has exactly two distinct natural number divisors: 1 and itself. The first twenty-five prime numbers are:


The property of being prime is called primality. Verifying the primality of a given number \( n \) can be done by trial divisions, that is to say dividing \( n \) by all smaller numbers \( m \), thereby checking whether \( n \) is a multiple of \( m \), and therefore not prime, or composite. For big primes, increasingly sophisticated algorithms which are faster than that technique have been devised.

There is no known formula yielding all primes and no composites. However, the distribution of primes, that is to say, the statistical behavior of primes in the large can be modeled. The first result in that direction is the prime number theorem which says that the probability that a given, randomly chosen number \( N \) is prime is inversely proportional to its number of digits, or the logarithm of \( N \). This statement has been proved at the end of the 19th century. The unproven Riemann hypothesis dating from 1859 implies a refined statement concerning the distribution of primes.
Despite being intensely studied, many fundamental questions around prime numbers remain open. For example, Goldbach's conjecture which asserts that any even natural number bigger than two is the sum of two primes, or the twin prime conjecture which says that there are infinitely many twin primes (pairs of primes whose difference is two), have been unresolved for more than a century, notwithstanding the simplicity of their statements. Prime numbers give rise to various generalizations in other mathematical domains, mainly algebra, notably the notion of prime ideals.

[http://en.wikipedia.org/wiki/Prime_number#cite_note-1]

The logical argument (actually, the dilemma, which forces one to the same conclusion whichever path one is compelled to take) tells us that the list of primes never ends. The second feature of the list of primes that strikes one is the absence of any noticeable pattern or regularity. Of course all the prime numbers except 2 are odd, so the gap between any two successive primes has to be an even number. (The Mathematical Experience, 212)

Yet despite their apparent simplicity and fundamental character, prime numbers remain the most mysterious objects studied by mathematicians. Look through a list of prime numbers and you’ll find that it’s impossible to predict when the next prime will appear. The list seems chaotic, random, and offers no clues as to how determine the next number. It is hard to guess at a formula that could generate this kind of pattern. In fact, this procession of primes resembles a random succession of numbers much more than it does a nice orderly pattern. (The Music of The Primes, 5-6)

Like the value of $\pi$ (The Mathematical Experience, 369), the properties of prime numbers are already determined regardless of human will, knowledge, reason, experience and, even though they follow no pattern and seem to be random,
their properties do not change or perish.

The following are reasons why we conceive prime numbers as uncertain:

- It is not possible to know if a number is a prime number without calculation. Ex) It is hard to tell which of the numbers 7866041, 7866047, and 7866051 is a prime number.
- Once we find a prime number, we are not able to predict the values (cycle, clock) of the prime number which comes after. Ex) It is hard to guess what the next prime number is after 19,999,909.
- Since we are not able to find the factor which determines the value of a prime number, We can’t find a rule that applies to all prime numbers.

The study of prime numbers goes back to the ancient Greeks but recent discoveries have been very limited. There has been no clear significant progress on the most famous unsolved problems in mathematics such as Goldbach’s conjecture, the Riemann hypothesis, twin prime conjectures, etc. Problems related to prime numbers remain a great challenge for the human spirit to conquer. The great physicist and mathematician, Johann Carl Friedrich Gauss (1777-1855), recognized the value of this problem. This is why he turned the focus of his research from light and space to prime numbers, strongly confident that mathematical objects and physical objects coincide.

Appendix 3> Duality(Prime Number Theorem)

The German mathematician Gauss speculated that \( \pi(N) \), which represents a smaller number of prime numbers than natural number \( N \), gradually approaches \( N / \log_e N \) as the natural number \( N \) becomes bigger. This was termed the Prime Number Theorem. (The Music of Primes P19–58)(Prime
Table 3) Prime Number Theorem

Table 3) shows that $Li(x)$ is central to our whole inquiry. In fact, the PNT is most often stated as $\pi(N) \sim Li(N)$, rather than as $\pi(N) \sim N / \log N$. Because the
twiddle sign is transitive, the two things are equivalent, as can be seen in Fig

Figure 6. Out of Riemann’s 1859 paper came a precise, though unproven, expression for \( \pi(N) \), and \( \text{Li}(x) \) leads off that expression. Note just one more thing about Table Table-3. For all the values \( N \) shown in the table, \( N/\log N \) gives a low estimate for \( \pi(N) \), while \( \text{Li}(N) \) gives a high one. I am just going to leave that lying there as a comment, for future reference. This is merely true; it is, in a manner of speaking, truer, I mean, \( \text{Li}(N) \) is actually a better estimate of \( \pi(N) \) than \( N/\log N \) is. A much better estimate. (Prime Obsession, 116-117)

Fig

Figure 6) shows that \( \pi(N) \) is smaller than \( \text{Li}(N) \) and greater than \( N/\log N \). However, similar to Fig

Figure 6, mathematicians have only been able to speculate regarding \( \pi(N) \)’s transitional path and have not been able to define its meaning or significance.

This thesis considers that \( N \) and \( \pi(N) \) of the prime number theorem are already determined regardless of human’s reason, knowledge, experience and apperception. Even if we cannot reveal the factor which determines the relationship between \( \pi(N) \) and \( N \), we should pay attention to the fact that the base of the log function, \( \pi(N) \sim N/\log N \), changes dynamically with \( N \). Table-4) shows the dynamic value of \( \nabla \pi \).
(1) $\pi(N) = N / \log_{\pi} N$

(2) $\log_{\pi} N = N / \pi(N)$

(3) $\nabla \pi^{N/\pi(N)} = N$

(4) $\nabla \pi = N^{\pi(N)/N}$

<table>
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<tr>
<th>$N$</th>
<th>$\pi(N)$</th>
<th>$\pi(N)/N$</th>
<th>$\nabla \pi$</th>
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<td>$\pi''(N)$</td>
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</table>

*Table 4) $\pi(N)$ and $\pi'(N)$ Distribution (Prime Number Theorem, [wikipedia](https://en.wikipedia.org/wiki/Prime_number_theorem))

*Figure 7) Dynamic model of prime number distribution*

*Figure 8) Dynamic curve of prime number distribution*
From the data shown in Table 4, we have discovered that the overall prime number distribution is surprisingly very similar to the spectrum curve of light as shown in Figure 8. What does this curve mean? Based on the curve in Figure 8, I have applied a model of radius 1 and found, in spite of small differences, that it approaches the model of the contraction of the space after an expansion.

(1) \( \pi(N) = N/\log \pi N \),
(2) \( \nabla \pi = N^{\pi(N)/N} \),
(3) \( \lim_{N \to \infty} \nabla \pi = \sqrt{3}\pi/2 \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \pi(N) )</th>
<th>( \pi(N) - N/\log \pi N )</th>
<th>( \pi(N) - N/\log_{\sqrt{3}\pi/2} N )</th>
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<td>( 10^3 )</td>
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</table>
Table – 5) shows that the limit of $\nabla \pi$ is not the value $e (2.71828)$ defined in the Prime Number Theorem but it approaches $\sqrt{\pi}/2 (2.720699)$. Based on this fact, the limiting value of the expansion and contraction of $\nabla \pi$ is determined with certainty.

How certain on the limiting value? We have no alternative but to confirm.

But for conjectures such as those about the distribution of primes, no one believes that the behavior we observe in our sample will suddenly change to something radically different in another sample, taken farther out toward infinity.

(The Mathematical Experience, 367)
Figure 9) Local (Partial) Properties of Prime Number Distribution

In Figure 8), the distribution of prime numbers appears to take the shape of a smooth curve, but Figure 9) shows that they fluctuate with their distribution being locally unpredictable.

In Figure 9), the spacing between natural numbers is set to 10. By measuring the number of prime numbers in intervals of 10, we find the value of $\nu \pi$.

In 1010 ~ 1500, 10010 ~ 10500, 10000010 ~ 10000500, we applied 3 different numerical models as a sample.
In this thesis, I have found the dual properties that the distribution of prime numbers is locally fluctuating but wholly approaches a single point asymptotically in a smooth curve. This duality exists because a prime number is determined uncertainly by the properties of the matrix.

<Appendix 4> Addition of Primes(Goldbach’s conjecture)

The problem of prime number addition has a deep relationship with Goldbach’s famous conjecture.

“Every even integer greater than 2 can be written as the sum of two primes”

In this section, we will examine how even numbers are combined using the addition of odd numbers, the addition of the $12n+1, 5, 7, 11$ sequence, and the addition of prime numbers.

In Figure 10) and Figure 11), the additions of odd numbers from the sequence yield a non-changing pattern in the equilibrium area independent of the magnitude of the vector in the multiplication table.
In Figure 12) and Figure 13), the additions of the $12n+1,5,7,11$ sequence yield a repetition with an oscillating pattern in the equilibrium area independent of the magnitude of the vector in the multiplication table.

In Figure 14), we have modeled 4 groups of the $12n+1,5,7,11$ sequence. Based on Figure 14), we have added the tables in Figure 16). Figure 15) is a modeling of the
results from

\textit{Figure – 16}.

<table>
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<td>1 + 7</td>
<td>5 + 5</td>
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<td>5 + 3</td>
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<td>(12x+1)+(12y+5)</td>
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<td>(12x+5)+(12y+7)</td>
<td>(12x+1)+(12y+11)</td>
</tr>
</tbody>
</table>

\textit{Equilibrium_Area}\n
\textit{Equilibrium_Area}\n
Addition of the 12n+1,5,7,11 sequence has the property that when it is combined into any of the even numbers, the ratios of the combined even numbers are uniform.

Since the distribution of the 12n+1,5,7,11 sequence in \textit{Table – 6) is uniform, the ratios of the combined even numbers are also uniform.
Figure – 17) Addition of Prime Numbers  Figure – 18) Model of Prime number

Figure – 17) and Figure – 18) ; With the additions of prime numbers, as the magnitude of vectors in the multiplication table increases, the intertwined phenomena become more complicated in the equilibrium area. In recent times, Goldbach’s conjecture has been proven to be true up to $12 \times 10^{17}$ using computer calculations. In Figure – 17), as the vector in the multiplication table increases, the number of even numbers is increasing. This finding agrees with existing statistical data. Oliveira e Silva (Jul. 14, 2008).

In 3(c), due to the uncertainty of the prime number distribution, it is impossible to express a generalized equation regarding the addition of prime numbers. However, we can apply it infinitely if we are certain of the heuristic evidence that ① the prime numbers can be sorted into 4 groups in 3(a); ② there is a factor which determines even numbers in 3(e); ③ the prime numbers follow the rules of the multiplication table; and ④ the distribution of prime numbers approaches a certain point.

<Appendix 5> Seeking Truth

The concept of symmetry is exciting and mysterious in many ways, and it is a very natural phenomenon for us to understand. It has become an object of physics and sometimes of mathematics but the concept itself is not easy to understand. The reason it is both easy and difficult is that it is metaphysically and physically too broad and that each person has a different conception of what it means. There is also a big reason why today’s physicists have a deep interest in symmetry. This concept is being applied to many natural phenomena examined today. But the most important point is that it is recognized deeply by physicists as a physical law which never varies.
(a) Meaning of Symmetry

Professor Hermann Weyl has given this definition of symmetry: a thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation. For instance, if we look at a vase that is left-and-right symmetrical, then turn it 180° around the vertical axis, it looks the same. We shall adopt the definition of symmetry in Weyl’s more general form, and in that form we shall discuss symmetry of physical laws. (Six Not So Easy Pieces, 1)

In fact, we will see that the history of the universe is, to a large extent, the history of symmetry.
This symmetry is known as translational symmetry or translational invariance. It applies not only to Newton’s laws but also to Maxwell’s laws of electromagnetism, to Einstein’s special and general relativities, to quantum mechanics, and to just about any proposal in modern physics that anyone has taken seriously. Considerations of symmetry have clearly been indispensable in the development of modern cosmological theory. (THE FABRIC OF THE COSMOS, 219–250)

A fact that most physicists still find somewhat staggering, a most profound and beautiful thing, is that, in quantum mechanics, for each of the rules of symmetry there is a corresponding conservation law; there is a definite connection between the laws of conservation and the symmetries of physical laws. (Six Not So Easy Pieces, 29)

The marvelous thing about it all is that for such a wide range of important, strong phenomena-nuclear forces, electrical phenomena, and even weak ones like gravitation-over a tremendous range of physics, all the laws for these seem to be symmetrical. (Six Not So Easy Pieces, 46)
Now our problem is to explain why they are nearly symmetrical by looking at tidal forces and so on. Richard Phillips Feynman (1918–1988) (Six Not So Easy Pieces, 47)

Heigenberg (1971) states in his book, 'Physics And Beyond', that it is natural to say that the symmetry is the fundamental element which the nature have plan to create. ‘In the beginning was symmetry’ is certainly a better expression than Democritus ‘In the beginning was the particle.’
Elementary particles embody symmetries; they are their simplest representations, and yet they are merely their consequence. (Physics And Beyond, 240)

What are the differences between mathematical symmetry and physical symmetry? What are the differences between reality in mathematics and that in physics? What are the differences between the determined properties of $\pi$ (The Mathematical Experience, 369-374), those of prime number distribution, and those of spaces? What we are certain of is that reality in both mathematics and physics are already determined and invariant regardless of human experience and reason.

**(b) Search for the Truth**

Today mathematical description and not physical explanation is the goal in science.
At best mathematics describes some processes of nature, but its symbols do not contain all of it. Even in the physical realm, mathematics deals with simplifications which merely touch reality as a tangent touches a curve at one point. (The Loss of Certainty, 25)

In his Philosophy of Mathematics and Natural Science (1949), Weyl conceded:
How much more convincing and closer to facts are the heuristic arguments and the subsequent systematic constructions in Einstein’s general relativity theory, or the Heisenberg-Schrödinger quantum mechanics. A truly realistic mathematics should be conceived, in line with physics, as a branch of the theoretical construction of the one real world, and should adopt the same sober and cautious attitude toward hypothetic extensions of its foundation as is exhibited by physics. (The Loss of Certainty, 330)

Godfrey H. Hardy (1877-1947) expressed the same view in his book *A Mathematician’s Apology*:
I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our “creations.” are simply our notes of our observations. (The Loss of Certainty, 322)

Hermite offered this explanation of the accord between mathematics and science: There exists, if I am not deceived, a world which is the collection of mathematical truths, to which we have access only through our intellects, just as there is the world of physical reality; the one and the other independent of us, both of divine creation, which appear distinct because of the weakness of our minds, but for a more powerful mode of thinking are one and the same thing. The synthesis of the two is revealed partially in the marvelous correspondence between abstract mathematics on the one hand and all the branches of physics on the other. (The Loss of Certainty, 345)

Later in life Eddington (1882-1944) too became convinced that nature is mathematically designed and he affirmed categorically in *Fundamental Theory* (1946) that our minds can build up a pure science of nature from a priori knowledge. This science is the only one possible; any other one would contain logical inconsistencies. (The Loss of Certainty, 346)
In another passage Einstein reaffirmed his belief through the now famous phrase about God: “I, at any rate, am convinced that He does not throw dice.” And if He does, then, as Ralph Waldo Emerson once suggested, “The dice of God are always loaded.” Einstein is not affirming here that the mathematical laws we now have are the correct ones but that there are such, and we can hope to come closer and closer to them. As he put it, “God is subtle; He is not malicious.” (The Loss of Certainty, 347)

More recently (1945) Erwin Schrödinger(1887–1961) in What Is Life said that the miracle of man’s discovering laws of nature may well be beyond human understanding. Another physicist, the highly distinguished Freeman Dyson agrees: “We are probably not close yet to understanding the relation between the physical and the mathematical worlds.” (The Loss of Certainty, 349)

What is surprising is that even some of the leaders in the work on foundations-Hilbert, Alnozo Church, and the members of the Bourbaki school-affirm that the mathematical concepts and properties exist in some objective sense and that they can be apprehended by human minds. Thus mathematical truth is discovered not invented. What evolves is not mathematics but man’s knowledge of mathematics. (The Loss of Certainty, 323)

As Erwin Schrödinger has pointed out, it may be beyond the scope of the human understanding to discover the laws of nature. But, we need to remind of what Rene Descartes(1596–1650) says on the other hand. “I shall preserver until I find something that is certain or, at least, until I find for certain that nothing is certain”. (The Loss of Certainty, 327)

If we accept the indications of leading scholars, our objective becomes obvious.
Self-evident truth is not an invention but a discovery. This thesis has found that symmetry, a physical attribute, and prime numbers, a mathematical object, coexist. Also, from the fact that the distribution of prime numbers exists in spatial, symmetric, and periodic dimensions, we have found that the set of all natural numbers, which also includes the set of all prime numbers, is not separated from space and in fact coexists with space, as opposed to the claims of Immanuel Kant and Hamilton that we accept the concept of natural numbers from intuitions about time. It is hard to say here how to interpret this finding and what implications it has. However, as a result of this discovery, there absolutely exists an object of self-evident truth and it is very different from the mathematical and physical conception of it we have known until now. Human beings are incomplete existences who can easily believe in the existence of a thing which does not exist. It is impossible to know which objects are out there and which objects are not, and which is possible and which is impossible. Perhaps here we should recall the words of Francis Bacon (1561~1626), who said that our conception receives information with endless distortion and that as a result we lose certainty. The reason human beings search for truth is closer to instinct than any other and the goal is to obtain a higher dimensional certainty through discoveries of truth.

<Appendix 6> Attached document

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If you knew the composite numbers less than 1000, you can filter out the prime numbers.