

# APPLICATIONS OF WALLIS THEOREM

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**Abstract:** In this paper we present theorems and applications of Wallis theorem related to trigonometric integrals.

Let's recall Wallis Theorem:

**Theorem 1.** (Wallis, 1616-1703)

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx = \frac{2 \cdot 4 \cdot \dots \cdot (2n)}{1 \cdot 3 \cdot \dots \cdot (2n+1)}.$$

*Proof:* Using the integration by parts, we obtain

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x \sin x dx = -\cos x \cdot \sin 2nx \Big|_0^{\frac{\pi}{2}} + \\ &+ 2n \int_0^{\frac{\pi}{2}} \sin^{2n-1} x (1 - \sin^2 x) dx = 2n I_{n-1} - 2n I_n \end{aligned}$$

from where:

$$I_n = \frac{2n}{2n+1} I_{n-1}.$$

By multiplication, we obtain the statement.

We prove in the same manner for  $\cos x$ .

**Theorem 2.**

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x dx = \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot (2n)} \cdot \frac{\pi}{2}.$$

*Proof:* Same as the first theorem.

**Theorem 3.** If  $f(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$ , then

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx = \frac{\pi}{2} a_0 + \frac{\pi}{2} \sum_{k=1}^{\infty} a_{2k} \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot (2k)}.$$

*Proof:* In the function  $f(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$  we substitute  $x$  by  $\sin x$  and then integrate from 0 to  $\frac{\pi}{2}$ , and we use the second theorem.

**Theorem 4.** If  $g(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$ , then

$$\int_0^{\frac{\pi}{2}} g(\sin x) dx = \int_0^{\frac{\pi}{2}} g(\cos x) dx = a_1 + \sum_{k=1}^{\infty} a_{2k+1} \frac{2 \cdot 4 \cdot \dots \cdot (2k)}{1 \cdot 3 \cdot \dots \cdot (2k+1)}.$$

**Theorem 5.** If  $h(x) = \sum_{k=0}^{\infty} a_k x^k$ , then

$$\int_0^{\frac{\pi}{2}} h(\sin x) dx = \int_0^{\frac{\pi}{2}} h(\cos x) dx = \frac{\pi}{2} a_0 + a_1 + \sum_{k=1}^{\infty} \left( \frac{\pi}{2} a_{2k} \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot \dots \cdot (2k)} + a_{2k+1} \frac{2 \cdot 4 \cdot \dots \cdot (2k)}{1 \cdot 3 \cdot \dots \cdot (2k+1)} \right).$$

**Application 1.**

$$\int_0^{\frac{\pi}{2}} \sin(\sin x) dx = \int_0^{\frac{\pi}{2}} \sin(\cos x) dx = \sum_{k=0}^{\infty} (-1)^k \frac{1}{1^2 \cdot 3^2 \cdot \dots \cdot (2k+1)^2}$$

*Proof:* We use that  $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ .

**Application 2.**

$$\int_0^{\frac{\pi}{2}} \cos(\sin x) dx = \int_0^{\frac{\pi}{2}} \cos(\cos x) dx = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2}.$$

*Proof:* We use that  $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ .

**Application 3.**

$$\int_0^{\frac{\pi}{2}} sh(\sin x) dx = \int_0^{\frac{\pi}{2}} sh(\cos x) dx = \sum_{k=0}^{\infty} \frac{1}{1^2 \cdot 3^2 \cdot \dots \cdot (2k+1)^2}.$$

*Proof:* We use that  $shx = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

**Application 4.**

$$\int_0^{\frac{\pi}{2}} ch(\sin x) dx = \int_0^{\frac{\pi}{2}} ch(\cos x) dx = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1}{4^k (k!)^2}.$$

*Proof:* We use that  $chx = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ .

**Application 5.**

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{\pi^2}{6}$$

*Proof:* In the expression of  $\arcsin x = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)x^{2k+1}}{2 \cdot 4 \cdot \dots \cdot (2k)(2k+1)}$  we substitute  $x$  by  $\sin x$ , and use theorem 4. It results that  $\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$ .

Because:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

we obtain:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

**Application 6.**

$$\int_0^{\frac{\pi}{2}} \sin x \operatorname{ctg}(\sin x) dx = \int_0^{\frac{\pi}{2}} \cos x \operatorname{ctg}(\cos x) dx = \frac{\pi}{2} - \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{B_k}{(k!)^2}$$

where  $B_k$  is the  $k$ -th Bernoulli type number (see [1]).

*Proof:* We use that  $x \operatorname{ctg} x = 1 - \sum_{k=1}^{\infty} \frac{4^k B_k x^{2k}}{(2k)!}$ .

**Application 7.**

$$\int_0^{\frac{\pi}{2}} \operatorname{arctg}(\sin x) dx = \int_0^{\frac{\pi}{2}} \operatorname{arctg}(\cos x) dx = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2 \cdot 4 \cdot \dots \cdot (2k)}{1 \cdot 3 \cdot \dots \cdot (2k-1)(2k+1)^2}.$$

*Proof:* We use that  $\operatorname{arctg} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ .

**Application 8.**

$$\int_0^{\frac{\pi}{2}} \operatorname{arg th}(\sin x) dx = \int_0^{\frac{\pi}{2}} \operatorname{arg th}(\cos x) dx = 1 + \sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot \dots \cdot (2k)}{1 \cdot 3 \cdot \dots \cdot (2k-1)(2k+1)^2}.$$

*Proof:* We use that  $\operatorname{arg th} x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$ .

**Application 9.**

$$\int_0^{\frac{\pi}{2}} \arg sh(\sin x) dx = \int_0^{\frac{\pi}{2}} \arg sh(\cos x) dx = \sum_{k=1}^{\infty} (-1)^k \frac{1}{(2k+1)^2}.$$

*Proof:* We use that  $\arg sh x = \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)x^{2k+1}}{2 \cdot 4 \cdot \dots \cdot (2k)(2k+1)}$ .

**Application 10.**

$$\int_0^{\frac{\pi}{2}} \operatorname{tg}(\sin x) dx = \int_0^{\frac{\pi}{2}} \operatorname{tg}(\cos x) dx = \sum_{k=1}^{\infty} \frac{2^{2k-1}(4^k-1)B_k}{1^2 \cdot 3^2 \cdot \dots \cdot (2k-1)^2 k}.$$

*Proof:* We use that  $\operatorname{tg} x = \sum_{k=1}^{\infty} \frac{2^{2k}(4^k-1)B_k}{(2k)!} x^{2k-1}$ .

**Application 11.**

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(\sin x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin(\cos x)} dx = \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} \frac{(2^{2k-1}-1)B_k}{2^{2k}(k!)^2}$$

*Proof:* We use that  $\frac{x}{\sin x} = 1 + 2 \sum_{k=1}^{\infty} \frac{(2^{2k-1}-1)B_k}{(2k)!} x^{2k}$ .

**Application 12.**

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{sh(\sin x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{sh(\cos x)} dx = \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} \frac{(2^{2k-1}-1)B_k}{2^{2k}(k!)^2}.$$

*Proof:* We use that  $\frac{x}{sh x} = 1 + 2 \sum_{k=1}^{\infty} (-1)^k \frac{(2^{2k-1}-1)B_k}{(2k)!} x^{2k}$ .

**Application 13.**

$$\int_0^{\frac{\pi}{2}} \sec(\sin x) dx = \int_0^{\frac{\pi}{2}} \sec(\cos x) dx = \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} \frac{E_k}{2^{2k+1}(k!)^2},$$

where  $E_k$  is the  $k$ -th Euler type number (see [1]).

*Proof:* We use that  $\sec x = 1 + \sum_{k=1}^{\infty} \frac{E_k}{(2k)!} x^{2k}$

**Application 14.**

$$\int_0^{\frac{\pi}{2}} \operatorname{sech}(\sin x) dx = \int_0^{\frac{\pi}{2}} \operatorname{sech}(\cos x) dx = \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} (-1)^k \frac{E_k}{2^{2k+1}(k!)^2}.$$

*Proof:* We use that  $\operatorname{sech} x = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{E_k}{(2k)!} x^{2k}$ .

## REFERENCES

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