A Note on Unified Statistics Including Fermi-Dirac, Bose-Einstein, and Tsallis Statistics, and Plausible Extension to Anisotropic Effect

Vic Christianto* and Florentin Smarandache†

*Sciprint.org — a Free Scientific Electronic Preprint Server, http://www.sciprint.org
E-mail: admin@sciprint.org

†Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA
E-mail: smarand@unm.edu

In the light of some recent hypotheses suggesting plausible unification of thermostatistics where Fermi-Dirac, Bose-Einstein and Tsallis statistics become its special subsets, we consider further plausible extension to include non-integer Hausdorff dimension, which becomes realization of fractal entropy concept. In the subsequent section, we also discuss plausible extension of this unified statistics to include anisotropic effect by using quaternion oscillator, which may be observed in the context of Cosmic Microwave Background Radiation. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In recent years, there have been some hypotheses suggesting that the spectrum and statistics of Cosmic Microwave Background Radiation has a kind of *scale invariant* character [1], which may be related to non-integer Hausdorff dimension. Interestingly, in this regard there is also proposition sometime ago suggesting that Cantorian spacetime may have deep link with Bose condensate with non-integer Hausdorff dimension [2]. All of these seem to indicate that it is worth to investigate further the non-integer dimension effect of Bose-Einstein statistics, which in turn may be related to Cosmic Microwave Background Radiation spectrum.

In the meantime, some authors also consider a plausible generalization of known statistics, i.e. Fermi-Dirac, Bose-Einstein, and Tsallis statistics, to become more unified statistics [3, 4]. This attempt can be considered as one step forward from what is already known, i.e. to consider anyons as a generalization of bosons and fermions in two-dimensional systems [5, p. 2] Furthermore, it is known that superfluidity phenomena can also be observed in Fermi liquid [6].

First we will review the existing procedure to generalize Fermi-Dirac, Bose-Einstein, and Tsallis statistics, to become more unified statistics [3, 4]. And then we explore its plausible generalization to include fractality of Tsallis' non-extensive entropy parameter.

In the subsequent section, we also discuss plausible extension of this proposed unified statistics to include anisotropic effect, which may be observed in the context of Cosmic Microwave Background Radiation. In particular we consider possibility to introduce quaternionic momentum. To our knowledge this proposition has never been considered before elsewhere.

Further observation is of course recommended in order to verify or refute the propositions outlined herein.

2 Unified statistics including Fermi-Dirac, Bose-Einstein, and Tsallis statistics

In this section we consider a different theoretical framework to generalize Fermi-Dirac and Bose-Einstein statistics, from conventional method using anyons, [5] in particular because this conventional method cannot be generalized further to include Tsallis statistics which has attracted some attention in recent years.

First we write down the standard expression of Bose distribution [9, p. 7]:

$$\bar{n}(\epsilon_i) = \frac{1}{\exp\big(\beta(\epsilon_i - \mu)\big) - 1}\,, \tag{1}$$

where the harmonic energy levels are given by [9, p. 7]:

$$\epsilon_i = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega_0\,. \tag{2}$$

When we assume that bosons and fermions are g-ons obeying fractional exclusion statistics, then we get a very different picture. In accordance with [3], we consider the spectrum of fractal dimension (also called *generalized Renyi dimension* [11]):

$$D_q = \lim_{\delta\to 0} \frac{1}{q-1}\frac{\ln\Omega_q}{\ln\delta}\,, \tag{3}$$

(therefore the spectrum of fractal dimension is equivalent with Hausdorff dimension of the set A [11]).

Then the relation between the entropy and the spectrum of fractal dimension is given by: [3]

$$S_q = -K_B \lim_{\delta\to 0} \ln\delta D_q\,, \tag{4}$$

where K_B is the Boltzmann constant.

The spectrum of fractal dimension may be expressed in terms of p:

$$D_q \approx \frac{1}{q-1}\frac{\sum_{i=1}^{k} p_i^q - 1}{\ln\delta}\,. \tag{5}$$

Then, substituting equation (5) into (4), we get the Tsallis non-extensive entropy [3]:

$$S_q = -K_B\,\frac{\sum_{i=1}^{k} p_i^q - 1}{q-1}\,. \tag{6}$$

After a few more assumptions, and using g-on notation [3], i.e. $g=1$ for generalized Fermi-Dirac statistics and $g=0$ for generalised Bose-Einstein statistics, then one gets the most probable distribution for g-ons [3]:

$$\bar{n}_k(\epsilon_i, g, q) = \frac{1}{\big(1-(q-1)\beta(\epsilon_i-\mu)\big)^{\frac{1}{q-1}} + 2g - 1}\,, \tag{7}$$

Which gives standard Planck distribution for $\mu=0$, $g=0$ and $q=1$ [3, 9]. In other words, we could expect that g-ons gas statistics could yield more generalized statistics than anyons'.

To introduce further generality of this expression (7), one may consider the parameter q as function of another non-integer dimension, therefore:

$$\bar{n}_k(\epsilon_i, g, q, D) = \frac{1}{\big(1-(q^D-1)\beta(\epsilon_i-\mu)\big)^{\frac{1}{q^D-1}} + 2g - 1}\,, \tag{8}$$

where $D=1$ then equation (8) reduces to be (7).

Of course, the picture described above will be different if we introduce non-standard momentum [5, p. 7]:

$$p^2 = -\frac{d^2}{dx^2} + \frac{\lambda}{x^2}\,. \tag{9}$$

In the context of Neutrosophic logic as conceived by one of these writers [8], one may derive a proposition from the arguments presented herein, i.e. apart from common use of anyons as a plausible generalization of fermion and boson, perhaps an alternative method for generalization of fermion and boson can be described as follows:

1. If we denote fermion with (f) and boson with (b), then it follows that there could be a mixture composed of both (f) and (b) $\rightarrow$ (f) $\cap$ (b), which may be called as "anyons";
2. If we denote fermion with (f) and boson with (b), and because $g=1$ for generalized Fermi-Dirac statistics and $g=0$ for generalised Bose-Einstein statistics, then it follows that the wholeness of both (f) and (b) $\rightarrow$ (f) $\cup$ (b), which may be called as "g-on";
3. Taking into consideration of possibility of "neitherness", then if we denote non-fermion with ($\neg$f) and non-boson with ($\neg$b), then it follows that there shall be a mixture composed of both ($\neg$f) and also ($\neg$b) $\rightarrow$ ($\neg$f) $\cap$ ($\neg$b), which may be called as "feynmion" (after physicist the late R. Feynman);
4. Taking into consideration of possibility of "neitherness", then it follows that the wholeness of both ($\neg$f) and ($\neg$b) $\rightarrow$ ($\neg$f) $\cup$ ($\neg$b), which may be called as "anti-g-on".

Therefore, a conjecture which may follow from this proposition is that perhaps in the near future we can observe some new entities corresponding to g-on condensate or feynmion condensate.

3 Further extension to include anisotropic effect

At this section we consider the anisotropic effect which may be useful for analyzing the anisotropy of CMBR spectrum, see Fig. 1 [13].

For anisotropic case, one cannot use again equation (2), but shall instead use [7, p. 2]:

$$\epsilon_i = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z\,, \tag{10}$$

where n_x, n_y, n_z are integers and >0. Or by neglecting the $1/2$ parts and assuming a common frequency, one can re-write (10) as [7a, p.1]:

$$\epsilon_i = (n_x r + n_y s + n_z t)\,\hbar\omega_0\,, \tag{11}$$

where r, s, t is multiplying coefficient for each frequency:

$$r = \frac{\omega_x}{\omega_0}\,, \qquad s = \frac{\omega_y}{\omega_0}\,, \qquad t = \frac{\omega_z}{\omega_0}\,. \tag{12}$$

This proposition will yield a different spectrum compared to isotropic spectrum by assuming isotropic harmonic oscillator (2). See Fig. 2 [7a]. It is interesting to note here that the spectrum produced by anisotropic frequencies yields number of peaks more than 1 (multiple-peaks), albeit this is not near yet to CMBR spectrum depicted in Fig. 1. Nonetheless, it seems clear here that one can expect to predict the anisotropy of CMBR spectrum by using of more anisotropic harmonic oscillators.

In this regard, it is interesting to note that some authors considered half quantum vortices in $p_x + ip_y$ superconductors [14], which indicates that energy of partition function may be generalized to include Cauchy plane, as follows:

$$E = p_x c + i p_y c \approx \hbar\omega_x + i\hbar\omega_y\,, \tag{13}$$

or by generalizing this Cauchy plane to quaternion number [12], one gets instead of (13):

$$E_{qk} = \hbar\omega + i\hbar\omega_x + j\hbar\omega_y + k\hbar\omega_z\,, \tag{14}$$

which is similar to standard definition of quaternion number:

$$Q \equiv a + bi + cj + dk\,. \tag{15}$$

Therefore the partition function with anisotropic harmon-

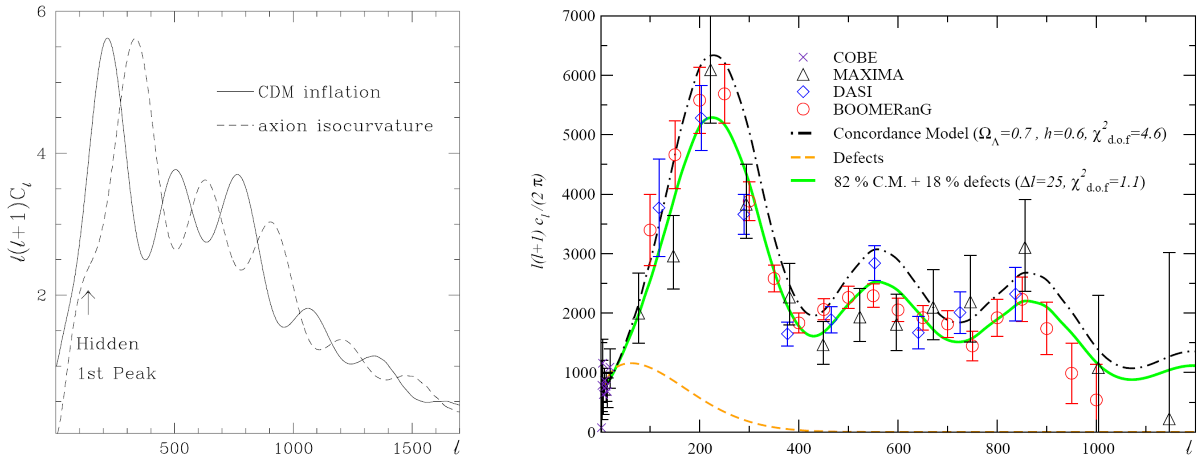

Fig. 1: Anisotropy of CMBR (after Tkachev [13]). Left panel: comparison of CMB power spectra in the models with adiabatic and isocurvature initial perturbations. Right panel: adiabatic power spectra in comparison with spectra appearing in models seeded by topological defects. In this panel some older, pre-WMAP, data are also shown.

ic potential can be written in quaternion form. Therefore instead of (11), we get:

$$\epsilon_i = \left(n_x r + n_y s + n_z t + i n_x r + j n_y s + k n_z t\right) \hbar\omega_0 \,, \quad (16)$$

which can be written as:

$$\epsilon_i = (1 + q_k)(n_k r_k)\,\hbar\omega_0 \,, \quad (17)$$

where $k = 1, 2, 3$ corresponding to index of quaternion number i, j, k. While we don't obtain numerical result here, it can be expected that this generalisation to anisotropic quaternion harmonic potential could yield better prediction, which perhaps may yield to exact CMBR spectrum. Numerical solution of this problem may be presented in another paper.

This proposition, however, may deserve further considerations. Further observation is also recommended in order to verify and also to explore various implications of.

4 Concluding remarks

In the present paper, we review an existing method to generalize Fermi-Dirac, Bose-Einstein, and Tsallis statistics, to become more unified statistics. And then we explore its plausible generalization to include fractality of Tsallis nonextensive entropy parameter .

Therefore, a conjecture which may follow this proposition is that perhaps in the near future we can observe some new entities corresponding to g-on condensate or feynmion condensate.

In the subsequent section, we also discuss plausible extension of this proposed unified statistics to include anisotropic effect, which may be observed in the context of Cosmic Microwave Background Radiation. In particular we consider possibility to introduce quaternionic harmonic oscillator. To our knowledge this proposition has never been considered before elsewhere.

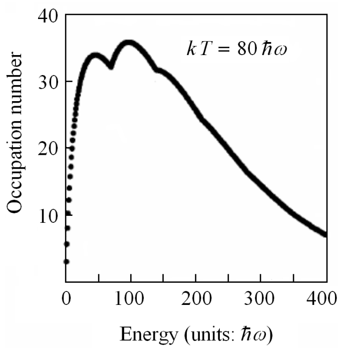

Fig. 2: Spectrum for anisotropic harmonic oscillator potential (after Ligare [7a]).

It is recommended to conduct further observation in order to verify and also to explore various implications of our propositions as described herein.

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