Thanks for many comments.
Most people give me the comment that "theorem (2.1) is wrong or impossible".
However, they didn't understand me using exact wrong points.
Look at the below equations.

$$\text{Equation (0.1)} : \prod_{k=1}^{\infty}(1-\frac{1}{p_k^s})$$
$$=(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})(1-\frac{1}{7^s})(1-\frac{1}{11^s})(1-\frac{1}{13^s})(1-\frac{1}{17^s})(1-\frac{1}{19^s})\dots$$
$$=(1-\frac{1}{2^s}-\frac{1}{3^s}-\frac{1}{5^s}-\frac{1}{7^s}-\cdots)+(\frac{1}{6^s}+\frac{1}{10^s}+\frac{1}{14^s}\cdots)+(-\frac{1}{30^s}-\frac{1}{42^s}-\frac{1}{66^s}-\cdots)+\cdots$$
$$=1-\frac{1}{2^s}-\frac{1}{3^s}-\frac{1}{5^s}+\frac{1}{6^s}-\frac{1}{7^s}+\frac{1}{10^s}-\frac{1}{11^s}-\frac{1}{13^s}+\frac{1}{14^s}+\frac{1}{15^s}-\frac{1}{17^s}-\frac{1}{19^s}+\frac{1}{21^s}+\cdots$$
$$=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$

(0.2)

$$\text{Equation (0.2)} : \prod_{k=1}^{\infty}(1-\frac{p_k^s}{p_k^s})$$
$$=(1-\frac{2^s}{2^s})(1-\frac{3^s}{3^s})(1-\frac{5^s}{5^s})(1-\frac{7^s}{7^s})(1-\frac{11^s}{11^s})(1-\frac{13^s}{13^s})(1-\frac{17^s}{17^s})(1-\frac{19^s}{19^s})\cdots$$
$$=(1-\frac{2^s}{2^s}-\frac{3^s}{3^s}-\frac{5^s}{5^s}-\frac{7^s}{7^s}-\cdots)+(\frac{6^s}{6^s}+\frac{10^s}{10^s}+\frac{14^s}{14^s}\cdots)+(-\frac{30^s}{30^s}-\frac{42^s}{42^s}-\frac{66^s}{66^s}-\cdots)+\cdots$$
$$=1-\frac{2^s}{2^s}-\frac{3^s}{3^s}-\frac{5^s}{5^s}+\frac{6^s}{6^s}-\frac{7^s}{7^s}+\frac{10^s}{10^s}-\frac{11^s}{11^s}-\frac{13^s}{13^s}+\frac{14^s}{14^s}+\frac{15^s}{15^s}-\frac{17^s}{17^s}-\frac{19^s}{19^s}+\frac{21^s}{21^s}+\cdots$$
$$=\sum_{n=1}^{\infty}\frac{n^s\mu(n)}{n^s}=\sum_{n=1}^{\infty}\mu(n)$$

Equation (0.1) is exactly right. Is it right?
Because, it is a very famous equation in Riemann's hypothesis.
Please, if you think that equation (0.2) is wrong,
explain me the exact wrong and different positions using the comparison of equation (0.1) and (0.2).

A PROOF OF RIEMANN HYPOTHESIS USING THE GROWTH OF MERTENS FUNCTION

YOUNG-MOOK KANG

Abstract. A study of growth of $M(x)$ as $x \to \infty$ is one of the most useful approach to the Riemann hypophotesis(RH). It is very known that the RH is equivalent to which $M(x) = O(x^{1/2+\varepsilon})$ for $\varepsilon > 0$. Also Littlewood proved that "the RH is equivalent to the statement that $\lim_{x\to\infty} M(x)x^{-1/2-\varepsilon} = 0$, for every $\varepsilon > 0$".[1] To use growth of $M(x)$ approaches zero as $x \to \infty$, I simply prove that the Riemann hypothesis is valid. Now Riemann hypothesis is not hypothesis any longer.

1. Introduction

The Riemann zeta-function $\zeta(s)$ is the function of complex numbers s ($s \neq 1$). There are infinitely many zeros at the negative even integers such that at ($s = -2$, $s = -4$, $s = -6$, $\cdots$) These are called the trivial zeros. The Riemann hypothesis(RH) is related the non-trivial zeros, and states that:

"All non-trivial zeros of Riemann zeta-function $\zeta(s)$ have real part $\frac{1}{2}$."

The RH has been implied strong bounds on the growth of many arithmetic functions. Among them, our most interesting function is Mertens function.

1.1. **Mertens function :** $M(n)$ is defined as follows :

$$M(n) = \sum_{k=1}^{n} \mu(k)$$

where $\mu(k)$ is the Möbius function. [1, 2]

The inverse of the Riemann zeta function is expressed that the Dirichlet series generates the Möbius function by Euler product.

$$\frac{1}{\zeta(s)} = \prod_{k=1}^{\infty}(1 - \frac{1}{p_k^s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{1.1}$$

where $\Re(s) > 1$, p_k is the k-th prime number

Mertens function, $M(x)$ is closely linked with the positions of zeroes of the Riemann zeta-function, $\zeta(s)$. When we define $M(0) = 0$, their relation is expressed as follows : [3]

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{M(n)}{n^s} - \sum_{n=1}^{\infty} \frac{M(n-1)}{(n)^s} = \sum_{n=1}^{\infty} \frac{M(n)}{n^s} - \sum_{n=1}^{\infty} \frac{M(n)}{(n+1)^s}$$

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$$= \sum_{n=1}^{\infty} M(n)(\frac{1}{n^s} - \frac{1}{(n+1)^s}) = \sum_{n=1}^{\infty} M(n) \int_n^{n+1} \frac{s}{x^{s+1}} dx$$

$$= s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{M(x)}{x^{s+1}} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx$$

since $M(x)$ is constant on each interval $[n, n+1)$

$$\frac{1}{\zeta(s)} = s \int_1^{\infty} M(x) x^{-s-1} dx \tag{1.2}$$

The equation (1.2) shows that a relation of the Mertens function and zeros of the Riemann zeta-function very well.

If $|M(x)| < C|x^{1/2}|$ for $C > 0$, then

$$|\frac{M(x)}{x^{s+1}}| < |\frac{C\sqrt{x}}{x^{s+1}}| = \frac{C}{\sqrt{x}} |\frac{1}{x^s}| = \frac{C}{\sqrt{x}} \frac{1}{x^{\Re(s)}} = \frac{C}{x^{\Re(s)+1/2}}$$

This means that $\Re(s) > 1/2$ because, the right integral in equation (1.2) would converge provided which $\Re(s) + 1/2 > 1$. According to this result, it can define a function analytic in $\Re(s) > 1/2$ and extend an analytic continuation of $1/\zeta(s)$ from $\Re(s) > 1$ to $\Re(s) > 1/2$. It means that $\zeta(s)$ have no zeros for $\Re(s) > 1/2$ and also for $\Re(s) < 1/2$ by symmetry.[3] Thus, all non-trivial zeros must have real part one-half.$|M(x)| < C|x^{1/2}|$ called Mertens conjecture is a condition stronger than RH. Actually, the RH is equivalent to a condition that $M(x) = O(x^{1/2+\varepsilon})$ for all $\varepsilon > 0$.[2, 4] Also according to a chapter 12 in the reference[1], a necessary and sufficient condition for the RH is

$$\lim_{x\to\infty} \frac{M(x)}{x^{1/2+\varepsilon}} = 0, \text{ for every } \varepsilon > 0 \text{ , proven by Littlewood.} \tag{1.3}$$

I just will prove that equation (1.3) is valid using the growth of $M(x)$, for a proof of the RH.

2. The Growth of Mertens Function

While I was studying about the growth of $M(x)$ as $x \to \infty$, I found a fact that the equation (1.1) is very similar to $\sum_{n=1}^{\infty} \mu(n)$. If we can remove $\frac{1}{n^s}$ in the equation (1.1), can we know about $\sum_{n=1}^{\infty} \mu(n)$? The solution was found very easily. Look at the equation (2.1).

$$\prod_{k=1}^{\infty} (1 - \frac{p_k}{p_k}) = 0 \text{ , where } p_k \text{ is the } k\text{-th prime number.} \tag{2.1}$$

Actually, it is seem that means nothing at all. However, I want to call that it is one of the Golden Keys for opening locked RH. Because, it shows that the growth of $M(x)$ approaches zero as $x \to \infty$.

$$\prod_{k=1}^{\infty} (1 - \frac{1}{p_k^s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \text{ vs } \prod_{k=1}^{\infty} (1 - \frac{p_k^s}{p_k^s}) = \sum_{n=1}^{\infty} \frac{n^s \mu(n)}{n^s}$$

Theorem 2.1. *A Golden Key of the Riemann Hypothesis*

$$\prod_{k=1}^{\infty}(1-\frac{p_k}{p_k})=\sum_{n=1}^{\infty}\mu(n)=\lim_{n\to\infty}M(n)=0$$

Proof.

$$\prod_{k=1}^{\infty}(1-\frac{p_k}{p_k})=0$$

$$=(1-\frac{2}{2})(1-\frac{3}{3})(1-\frac{5}{5})(1-\frac{7}{7})(1-\frac{11}{11})(1-\frac{13}{13})(1-\frac{17}{17})(1-\frac{19}{19})\dots$$

$$=1-\frac{2}{2}-\frac{3}{3}-\frac{5}{5}+\frac{6}{6}-\frac{7}{7}+\frac{10}{10}-\frac{11}{11}-\frac{13}{13}+\frac{14}{14}+\frac{15}{15}-\frac{17}{17}-\frac{19}{19}+\frac{21}{21}+\dots$$

$$=1+\frac{-2}{2}+\frac{-3}{3}+\frac{0}{4}+\frac{-5}{5}+\frac{6}{6}+\frac{-7}{7}+\frac{0}{8}+\frac{0}{9}+\frac{10}{10}+\frac{-11}{11}+\frac{0}{12}+\frac{-13}{13}+\frac{14}{14}+\frac{15}{15}+\frac{0}{16}+\dots$$

$$=\frac{1\times 1}{1}+\frac{2\times -1}{2}+\frac{3\times -1}{3}+\frac{4\times 0}{4}+\frac{5\times -1}{5}+\frac{6\times 1}{6}+\frac{7\times -1}{7}+\frac{8\times 0}{8}+\frac{9\times 0}{9}+\frac{10\times 1}{10}+\dots$$

$$=\frac{1\mu(1)}{1}+\frac{2\mu(2)}{2}+\frac{3\mu(3)}{3}+\frac{4\mu(4)}{4}+\frac{5\mu(5)}{5}+\frac{6\mu(6)}{6}+\frac{7\mu(7)}{7}+\frac{8\mu(8)}{8}+\frac{9\mu(9)}{9}+\frac{10\mu(10)}{10}+\dots$$

$$=\sum_{n=1}^{\infty}\frac{n\mu(n)}{n}=\sum_{n=1}^{\infty}\mu(n)=\lim_{n\to\infty}M(n)=0$$

□

How do you think about the convergence of the growth of $M(x)$? Maybe most people have believed that the growth of $M(x)$ must be diverged as $x\to\infty$. However, the theorem(2.1) shows that the growth of $M(x)$ approaches zero as $x\to\infty$.

3. The Probability of Möbius Function

The theorem(2.1) shows some results about probability of Möbius function as following :

Corollary 3.1.

$$Pr(\mu(n)=+1)=Pr(\mu(n)=-1)$$

where n is the natural number.

Proof.

$$\text{Because, }\lim_{n\to\infty}M(n)=0$$

Therefore, the numbers of -1 and $+1$ of $\mu(n)$ are equal.

□

Corollary 3.2.

$$Pr(\mu(n)=+1)=\frac{3}{\pi^2}\ ,\ Pr(\mu(n)=-1)=\frac{3}{\pi^2}\ \textit{and}\ Pr(\mu(n)=0)=1-\frac{6}{\pi^2}$$

where n is the natural number.

Proof.

Using the inclusion-exclusion principle,

a probability of the total square-free numbers is defined as follows :

$$Pr(\mu(n) \neq 0) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{5^2})(1 - \frac{1}{7^2})(1 - \frac{1}{11^2}) \cdots$$

$$= \prod_{k=1}^{\infty}(1 - \frac{1}{p_k^2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

Because, $Pr(\mu(n) = -1) = Pr(\mu(n) = +1)$

and

$$Pr(\mu(n) = -1) + Pr(\mu(n) = +1) + Pr(\mu(n) = 0) = 1$$

Therefore, $Pr(\mu(n) = +1) = \frac{3}{\pi^2}$, $Pr(\mu(n) = -1) = \frac{3}{\pi^2}$, $Pr(\mu(n) = 0) = 1 - \frac{6}{\pi^2}$

□

Denjoy's proposal an another probabilistic condition that is equilvalent to RH with probability one.[1] It has some suppositions which square-free numbers are random sequences and independent events with symmetrical distribution. In other words if a square-free number is taken at random and has an equal probability of containing an odd or an even number of distinct prime divisors, $M(x) = O(x^{1/2+\varepsilon})$ and the RH is true with probability one. From corollary (3.1) and (3.2), we can verify a fact that $Pr(\mu(n) = +1)$ and $Pr(\mu(n) = -1)$ are equal. These are providing the plausible evidences for the Riemann Hypothesis.

4. A Proof of Riemann's Hypothesis

Theorem 4.1.

All non-trivial zeros of $\zeta(s)$ have real part one-half.

Proof.

Using theorem (2.1), $\lim_{x \to \infty} M(x) = 0$

$$\Downarrow$$

$$\lim_{x \to \infty} \frac{M(x)}{x^{1/2+\varepsilon}} = 0, \text{ for every } \varepsilon > 0$$

This condition is equivalent to the Riemann hypothesis.[1]

Therefore, the Riemann hypothesis is true.

□

5. Conclusion

I very simply prove the RH using the growth of $M(x)$ approaches zero as $x \to \infty$. From now on, Riemann hypothesis is not his hypothesis any longer. It is reborn an obvious theorem.

The $M(x)$ closely linked with the positions of zeroes of $\zeta(s)$ have some questions still. Their relation has been very known that the RH is equivalent to $M(x) = O(x^{1/2+\varepsilon})$.[2, 4] I think that this relation is very similar to $|\pi(x) - Li(x)| = O(\sqrt{x}\ logx)$ called Koch's result. RH is proven using the growth of $M(x)$ approaches zero as $x \to \infty$. This condition is fairly stronger than $O(x^{1/2+\varepsilon})$. If Koch's result and $M(x)$ are closely related, I conjecture that $\lim_{x\to\infty} |\pi(x) - Li(x)| = 0$ alike the growth of $M(x)$.

Conjecture 1.

$$|\pi(x) - Li(x)| \leq C\sqrt{x}\ logx\ ,\ where\ C \geq 0$$

$$\lim_{x\to\infty} C = 0$$

Today, the precise version of Koch's result is that $|\pi(x) - Li(x)| < \pi/8\sqrt{x}\ logx$ where $x > 2657$ proven by Schoenfeld.

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