Solving the Fermion Flavor Problem using Renormalization Group Flow

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Abstract

A long-standing puzzle of the current Standard Model for particle physics is that both leptons and quarks arise in replicated patterns. Our work suggests that the number of fermion flavors may be directly derived from the dynamics of Renormalization Group (RG) equations. Specifically, we argue that the number of flavors results from demanding stability of the RG flow about its fixed-point solution.

Key words: Renormalization Group; Gauge field theories; QED; QCD; Stability of the laminar flow.

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1. Introduction and motivation

The Standard Model for particle physics (SM) represents a highly successful framework for the description of sub-nuclear particles and their interactions in an energy range bounded by an upper limit of about $200\,GeV$ ([10] and Appendix A). The backbone of SM is relativistic quantum field theory (QFT) whose predictive power rests primarily on the techniques of perturbation theory [1-7, 19]. A key premise of QFT is that the cumulative contribution of arbitrary-order quantum corrections above any energy threshold can be conveniently suppressed. Carrying out this program means that all quantum processes above the threshold can be absorbed into a redefinition of parameters that make up the theory (masses, couplings, fields). It is customary to call this prescription the "renormalization group" approach (RG) and its outcome an "effective field theory". The main outcome of RG is that the parameters of the theory depend on the energy scale at which the physics is probed ([8] and Appendix B). In particular, an important concept in RG is the evolution of coupling with the energy scale, referred to as the coupling flow equation. Since SM is an effective framework for the description of particle physics below $100\,GeV$ [1-7], it is typically assumed that the coupling flow is stable and its approach towards equilibrium develops adiabatically.

Despite its remarkable predictive power, SM cannot explain why both leptons and quarks arise in replicated patterns. This puzzle is referred to as the fermion "flavor problem" [11, 19] and it continues to challenge to the day our understanding of particle physics. Motivated by the relevance of nonlinear dynamics in field theory, this work suggests that the number of fermion flavors may be directly derived from the dynamics of RG flow equations. Specifically, we find that the number of flavors results from demanding stability of the RG flow about its fixed-point solution.

The paper is organized as follows: the next section covers the basics of RG flow theory and section 3 retrieves the number of fermion flavors from a standard stability analysis. Results and concluding remarks are detailed in the last two sections. Three Appendix sections are included for convenience. The deal, respectively, with a brief overview of SM, an introduction to the RG theory of coupling flow equations and a brief presentation of the Routh-Hurwitz criterion.

2. RG flow equations

We start from the set of beta-functions describing the RG flow in the gauge sector of SM [1-7, Appendix B]

$$\frac{dg_i}{dt} = \beta_i(g) = b_i(N,n)g_i^3 + O(g_i^5) \tag{1}$$

in which $i = (1,2,3)$, N is the dimension of the gauge group and n the number of fermion flavors. In particular, the beta-functions for quantum electrodynamics (QED) and non-abelian gauge theories (the weak interaction model and QCD) are respectively supplied by [7]

$$\beta_{QED}(e) = N\frac{ne^3}{12\pi^2} + O(e^5) \tag{2}$$

$$\beta_{NA}(g) = -\frac{11N-2n}{48\pi^2}g^3 + O(g^5) \tag{3}$$

Accounting for the underlying $SU(3)\times SU(2)\times U(1)$ gauge structure of SM, the explicit form of the coefficient vector is

$$\mathbf{b}(N,n) = \begin{vmatrix} b_1(1,n) \\ b_2(2,n) \\ b_3(3,n) \end{vmatrix} \tag{4}$$

with entries

$$\begin{aligned} b_1(1,n) &= \frac{n}{12\pi^2} \quad , \text{ for } N=1 \\ b_2(2,n) &= -\frac{11-n}{24\pi^2} \quad , \text{ for } N=2 \\ b_3(3,n) &= -\frac{33-2n}{48\pi^2} \quad , \text{ for } N=3 \end{aligned} \tag{5}$$

Let us assume in what follows that typical coupling strengths of SM represent fixed-point solutions of (2) and (3). For reference, we also assume that these are computed at the high-energy limit set by the mass of the Z boson [Appendix B, 19]

$$\alpha_{QED}(M_Z) = 1/127.9 \approx 0.00782$$

$$\alpha_2(M_Z) = 0.0338 \tag{6a}$$

$$\alpha_3(M_Z) = 0.123$$

or, in set form

$$\alpha(M_Z) = \{0.00782 \quad 0.0338 \quad 0.123\} \tag{6b}$$

Using (6b) the set of coupling parameters is given by

$$g_0^2(M_Z) = 4\pi\alpha(M_Z) = \{0.098 \quad 0.425 \quad 1.546\} \tag{7}$$

<u>3. Stability analysis</u>

The set of three nonlinear differential equations (2) and (3) based on (5) and (7) depends on the number of flavors n, which plays the role of an independent control parameter. Qualitative changes in the behavior of coupling trajectories are to be expected when n is finely tuned. As pointed out in Section 1, a typical assumption made in QFT is that the coupling flow evolves towards a finite set of attractors consisting of isolated fixed points [17]. On this basis we require that (2) and (3) yield a coupling flow that is *unique* and *stable*. These constraints amount to demanding that all Lyapunov exponents are real and vanishing with the exception of a single one, which is either vanishing or negative. Expanding (2) and (3) about (7) yields the new coefficient vector

$$\mathbf{a}(N,n) = 3\,g_0^2(M_Z)\mathbf{b}(N,n) = 10^{-3}\begin{vmatrix} 2.482\,n \\ -5.382\,(11-n) \\ -9.790(33-2n) \end{vmatrix} \tag{8}$$

Following the Routh-Hurwitz criterion, the set of stability parameters assumes the form (see Appendix C)

$$p(n) = -[a_{11}(n) + a_{22}(n) + a_{33}(n)]$$

$$q(n) = [a_{11}(n)a_{22}(n) + a_{11}(n)a_{33}(n) + a_{22}(n)a_{33}(n)] \tag{9}$$

$$r(n) = -a_{11}(n)a_{22}(n)a_{33}(n)$$

where $a_{kk}(n)$, $k=1,2,3$ are supplied by the components of (8). The characteristic equation is represented by the cubic polynomial

$$\Delta(n) = \lambda^3 + p(n)\lambda^2 + q(n)\lambda + r(n) = 0 \tag{10}$$

The constraint of a unique and stable trajectory implies

$$\lambda_1 = \lambda_2 = 0 \tag{11a}$$

$$\lambda_3 \leq 0 \tag{11b}$$

which yields

$$\begin{gathered} p(n) \geq 0 \\ q(n) = r(n) = 0 \end{gathered} \tag{12}$$

We obtain the least squares solution

$$\boxed{n \cong 7.3} \tag{13}$$

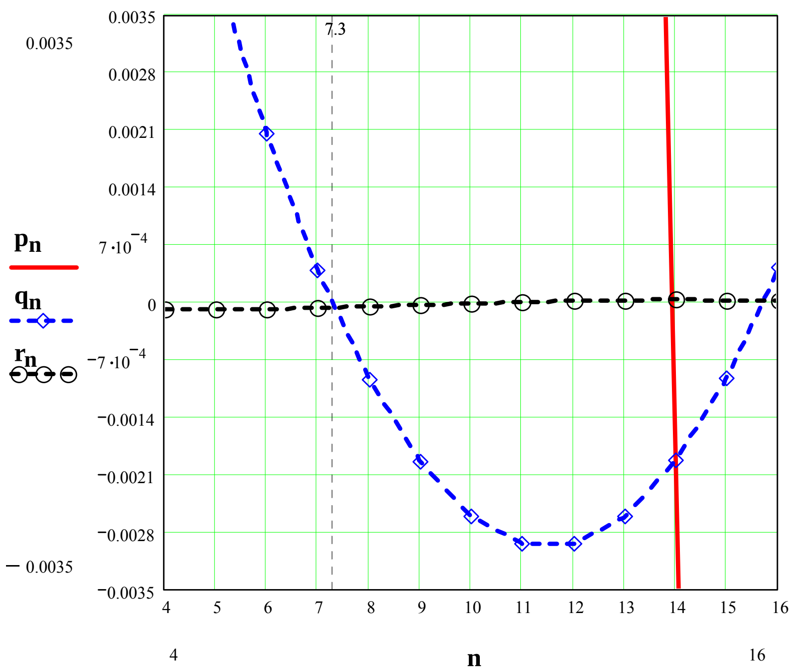

Fig. 1: Variation of the stability parameters with the number of fermion flavors

4. Discussion of results

Fig. 1 graphs the variation of the stability parameters with the flavor number. As expected, the least squares solution lies at the intersection point of $q(n)$ and $r(n)$. There are two distinct interpretations of this result, namely:

1) the actual number of flavors in SM is indeed seven and so we should anticipate an extra fermion flavor to be discovered in future accelerator experiments (such as, but not limited to, the fourth family neutrino [9]).

2) the stability analysis we have developed is only an approximation that needs further revision. One can invoke here, for example, including higher-order corrections to (2) and (3), accounting for the Yukawa sector of the coupling flow [19] or starting from the framework of non-perturbative RG flow equations [20]. The expectation is that, by using one or more of these scenarios, the actual number of SM flavors $n = 6$ may be recovered at the end of calculations.

It is also instructive to note that the condition $p(n) \geq 0$ determines the largest number of flavors that preserves the flow stability. From the graph we see that this number is $n_{MAX} \approx 14$, consistent with the maximum number of quark flavors that maintains asymptotic freedom in QCD.

5. Conclusions

The origin of the six known generations of active fermions continues to be an unresolved issue of SM. We have examined in this work the possibility that the number of fermion generations is rooted in the stability of the RG flow. Constraining the coupling trajectories to settle on a set of isolated stationary points brings the number of flavors to seven. This result *either* makes room for an additional fermion generation in future tests of SM *or* suggests that our stability analysis is valid only up to a first-order approximation. The largest number of flavors for which the coupling trajectory remains stable was found to be fourteen. Future works on the topic may be devoted to the analysis of the gauge coupling flow in the presence of higher-order diagrams and/or random perturbations. A number of excellent studies exist on the subject of stochastic stability for multidimensional nonlinear systems. Although a complete listing is impractical, we believe that the methodology discussed in [22-25] may provide a suitable starting baseline.

Appendix A: on the Standard Model for particle physics [1-8, 10, 19]

SM combines relativity and quantum mechanics in a unified conceptual framework known as relativistic quantum field theory (QFT). Electromagnetic, weak and strong interactions are all included in SM and are described by abelian and non-abelian gauge theories. The structure of SM is a generalization of that of QED – the quantum theory of electromagnetic phenomena – to a larger set of conserved currents and charges. In SM the matter fields have spin $1/2$ and are divided into two groups: quarks (the constituents of protons, neutrons and all hadrons) and leptons. There are six known generations (flavors) of quarks and six generations of leptons. There are eight color charges, which couple quarks in QCD and four electroweak charges, which couple leptons and quarks. All interactions are carried through gauge particles of spin 1. They are, respectively, the photon γ, the three vector bosons of the weak interaction W^+, W^-, Z^0 and the eight gluons of the strong interaction. The set of three interactions can be formulated in terms of unitary groups of different dimensions. It is customary to denote the gauge structure of SM as a product expressed as $SU(3) \times SU(2) \times U(1)$. This notation has the following meaning: a gauge theory described by the group $SU(N)$ is defined in terms of N^2-1 underlying gauge bosons. The group $SU(3)$ is the gauge group of QCD, which carries the $3^2-1=8$ gluons of the strong interaction. The $SU(2)\times U(1)$ group represents the structure of the electro-weak model with $2^2-1=3$ corresponding gauge bosons, namely (γ, W^+, W^-, Z^0).

The interaction amplitude is determined by the magnitude of a coupling constant, generically denoted by g *or* by the magnitude of the coupling strength $\alpha = {g^2}/{4\pi}$. A QFT characterized by a dimensionless coupling constant $g \ll 1$ is said to be weakly coupled and it is well defined by an expansion in powers of g, called perturbation theory. Otherwise, the theory is said to be strongly coupled. Perturbation techniques have limited applicability in strongly coupled theories and various non-perturbative methods have to be implemented in order to derive meaningful results.

Appendix B: the Renormalization Group flow

The underlying idea of renormalization is to avoid divergences that show up in physical predictions of QFT by using systematic rules for performing calculations [1-2, 7-8, 19]. In general, a QFT is called renormalizable if all infinities can be absorbed into a redefinition of a finite number of parameters. There are several technical procedures to renormalize a field theory. One standard way is to cut off the integrals in the calculations at a large but finite value of momentum (Λ). The renormalization is successful if, after taking the limit $\Lambda \to \infty$, the resulting quantities are finite and independent of Λ.

An important consequence of the renormalization program is that all parameters of the theory depend on the energy scale at which the phenomena are recorded (μ). The so-called *beta function* encodes the evolution of a given parameter with the energy scale. For instance, the coupling flow equation is defined by the relation

$$\mu \frac{\partial g}{\partial \mu} = \beta(g) \tag{B1}$$

If the beta-functions of a QFT vanish, then the theory approaches a so-called *fixed point* where it becomes *scale-invariant*. The coupling parameters of a quantum field theory can flow even if the corresponding classical field theory is scale-invariant. In this case, the non-vanishing beta function indicates that the classical scale-invariance is *anomalous*. If a beta-function is *positive*, the corresponding coupling increases with increasing energy. An example is QED, where one finds by using perturbation theory that the beta-function is positive. In particular, at low energies, the fine-structure constant measures $\alpha_{EM} \approx 1/137$, whereas at the scale of the Z boson, about 90 GeV, the same constant becomes $\alpha_{EM} \approx 1/127.9$. In non-abelian gauge theories, the beta function can be *negative*. An example is the beta-function for QCD, and as a result the QCD coupling decreases at high energies. Furthermore, the coupling decreases logarithmically, a phenomenon known as *asymptotic freedom*. This means that the coupling becomes large at low energies, and predictions can no longer rely on perturbation theory.

Appendix C: the Routh-Hurwitz criterion

We review here implementation of the Routh-Hurwitz criterion in the case of a three-dimensional system of nonlinear differential equations. For additional details, the reader is referred to [18]. Consider the three-dimensional system

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + P_1(x_1, x_2, x_3)$$

$$\dot{x}_2 = a_{12}x_1 + a_{22}x_2 + a_{23}x_3 + P_2(x_1, x_2, x_3) \tag{C1}$$

$$\dot{x}_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + P_3(x_1, x_2, x_3)$$

in which the functions P_i contain no linear terms. The characteristic equation of (C1) takes the form of the cubic polynomial

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0 \tag{C2}$$

where the three stability parameters are given by

$$p = -(a_{11} + a_{22} + a_{33})$$

$$q = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} \tag{C3}$$

$$r = -\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

The Routh-Hurwitz stability condition amounts to the following condition

$$p > 0,\ q > 0,\ r > 0 \quad \text{and} \quad R \equiv pq - r > 0 \tag{C4}$$

Boundaries of the stability region are defined by two surfaces $(r = 0, p > 0, q > 0)$ and $(R = 0, p > 0, q > 0)$. Equation (C2) has at least one vanishing root on the surface $r = 0$, and a pair of imaginary roots on the surface $(R = 0, q > 0)$.

9. References

[1] Itzykson C., Zuber J. B., Quantum Field Theory, Mc. Graw-Hill, New York, 1980.

[2] Zinn-Justin J., Quantum Field Theory and Critical Phenomena, Clarendon Press, Oxford, 1996.

[3] Faddeev L. D., Popov V. N., Feynman diagrams for the Yang-Mills field, Phys. Lett. B 1967; 25: 29-30.

[4] Gross D. J., Wilczek F., Ultraviolet behavior of non-abelian gauge theories, Phys. Rev. Lett. 1973; 30:1343-1346.

[5] Ryder L.H. Quantum Field Theory, Cambridge University Press; 1996.

[6] Kaku M. Quantum Field Theory, a modern introduction. Oxford University Press; 1993.

[7] See e. g. van Baal P., Course notes on Quantum Field Theory. Available from http://www.lorentz.leidenuniv.nl/vanbaal/FT/extract.pdf

[8] See e.g. http://plato.stanford.edu/entries-quantum-field-theory

[9] See e.g. Ciftci A. K. *et al.*, Fourth standard model family neutrino at future linear colliders. Phys. Rev. D 2005; 72: 053006.

[10] See e.g. http://www2.slac.stanford.edu/vvc/theory/model.html

[11] See e.g. http://gesalerico.ft.uam.es/forilloph/20040428.ps.

[12] Goldfain E., Complex Dynamics and the High-energy Regime of Quantum Field Theory. Intl. Journal of Nonlinear Sciences and Numerical Simulation 2005; 6:3-223-234.

[13] Goldfain E., Derivation of gauge boson masses from the dynamics of Levy flows. Nonlinear Phenomena in Complex Systems 2005; 8(4): 366-372.

[14] Goldfain E., Derivation of lepton masses from the chaotic regime of the linear σ-model. Chaos, Solitons and Fractals 2002; 14: 1331-1340.

[15] Goldfain E., Feigenbaum scaling, Cantorian space-time and the hierarchical structure of standard model parameters. Chaos, Solitons and Fractals 2006; 30:324-331.

[16] Goldfain E., Fractional dynamics and the TeV regime of field theory. Available from doi:10.1016/j.cnsns.2006.06.001

[17] Morozov A., Niemi A.J., Can Renormalization Group Flow End in a Big Mess? Nucl. Phys. B 2003; 666: 311-336.

[18] Shilnikov L. P. *et al.*, Methods of qualitative theory in nonlinear dynamics (part II), World Scientific, Singapore, 2001.

[19] See e. g. Donoghue J. F. *et al*. Dynamics of the Standard Model. 1994; Cambridge University Press.

[20] Delamotte B., Canet L., What can be learnt from the nonperturbative renormalization group? Condensed Matter Physics 2005; 8(1): 163-179.

[21] See e.g. Skarke H., Renormalization Group flow in a general gauge theory. Available from hep-th/9407086.

[22] Huang D. W. *et al*, On the chaotic motion of some stochastic nonlinear dynamic systems. Chaos, Solitons and Fractals 2007; 31: 242-246.

[23] Zhu W. Q., Lyapunov exponent and stochastic stability of quasi-non-integrable Hamiltonian systems. Intl. Journal of Non-Linear Mechanics 2004; 39: 569-579.

[24] Most T., Bucher C., Stochastic dynamic stability of nonlinear structures. Structural Dynamics, EURODYN 2002, Grundmann & Schueller (Editors).

[25] Ariaratnam S. T., Stochastic Stability of Modes at Rest in Coupled Nonlinear Systems. Nonlinear Stochastic Dynamic Engineering Systems 1987, Ziegler & Schueller (Editors).