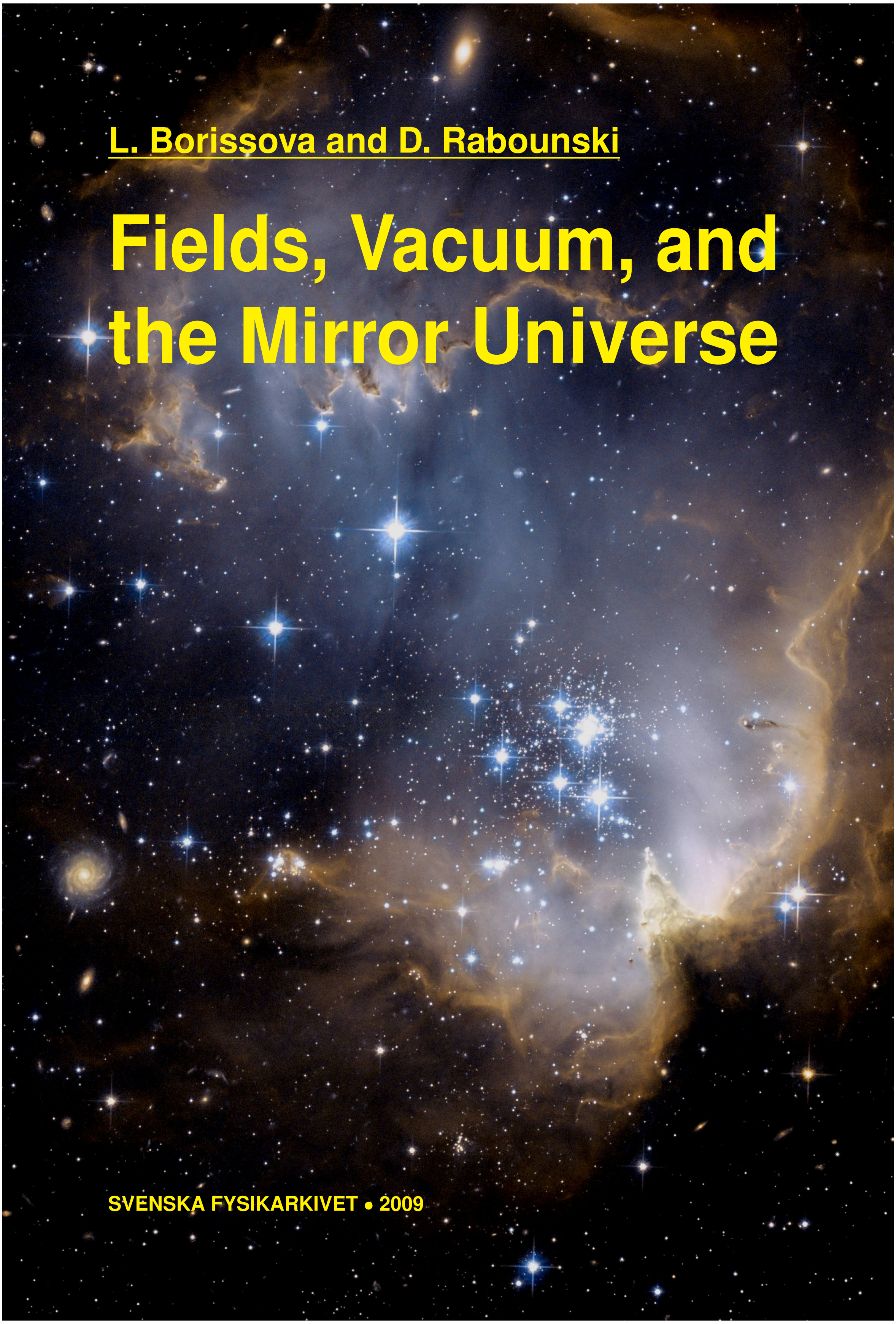
L. Borissova and D. Rabounski

Fields, Vacuum, and the Mirror Universe

SVENSKA FYSIKARKIVET • 2009



Larissa Borissova and Dmitri Rabounski

Fields, Vacuum, and the Mirror Universe

Fält, vakuum och spegeluniversum

2009

Swedish physics archive
Svenska fysikarkivet

Svenska fysikarkivet (that means the Swedish physics archive) is a publisher registered with the Royal National Library of Sweden (Kungliga biblioteket), Stockholm.

Postal address for correspondence:
Svenska fysikarkivet, Näsbydalsvägen 4/11, 183 31 Täby, Sweden

Edited by Chifu Ebenezer Ndikilar

Cover image: This image taken with NASA's Hubble Space Telescope depicts bright, blue, newly formed stars that are blowing a cavity in the center of a star-forming region in the Small Magellanic Cloud. This image is a courtesy of the Hubble Space Telescope Science Institute (STScI) and NASA. This image is a public domain product; see http://hubblesite.org/copyright for details. We are thankful to STScI and NASA for the image.

This book was typeset using teTEX typesetting system and Kile, a TEX/LATEX editor for the KDE desktop. Powered by Ubuntu Linux.

Signed to print on May 28, 2009.

ISBN: 978-91-85917-09-9

Printed in the United States of America

Contents

Foreword to the 2nd Edition

This is the English translation of our *Theory of Non-Geodesic Motion of Particles*, originally published in Russian in 1999, with some recent amendments.

The cornerstone of this book is that when tackling the problems of the General Theory of Relativity we had to amend the existing theory with some new mathematical techniques. In their famous *The Classical Theory of Fields*, which has already become a de facto standard for a university reference book on the General Theory of Relativity, Lev Landau and Evgeny Lifshitz give an excellent account of the theory of motion of particles in gravitational and electromagnetic fields. They however, employed only the usual generally covariant method of analysis. The mathematical method of chronometric invariants (physically observable quantities in the General Theory of Relativity) had not yet been developed at that time (the middle of the 1930's). We now feel that this method should also be taken into account. Therefore, in the process of writing this book, we have had to face the necessity to introduce the mathematical method of chronometric invariants into the existing theory of motion of particles in gravitational and electromagnetic fields. Moreover, the motion of spin-particles was not covered by Landau and Lifshitz. Therefore, in our book, a separate consideration has been given to the motion of particles with inner rotational momentum (spin). We have also added a chapter with an account of tensor algebra and analysis in terms of chronometric invariants. All these make our book a contemporary supplement to *The Classical Theory of Fields.*

In conclusion of this foreword, we would like to express our sincere gratitude to our teachers, Dr. Abraham Zelmanov (1913–1987) and Prof. Kyril Stanyukovich (1916–1989). Many years of acquaintance and countless hours of friendly conversations with them have planted seeds of fundamental ideas which by now have grown up in our minds to be reflected on these pages. We are also grateful to Kyril Dombrovski whose talks and friendly discussions greatly influenced our outlooks. Special thanks go to Chifu Ebenezer Ndikilar, for careful editing of the book that has made the complicated subjects of the theory of relativity much more accessible to the reader.

May 15, 2009 *Larissa Borissova and Dmitri Rabounski*

Chapter 1 Introduction

§1.1 Geodesic motion of particles

Numerous experiments aimed at proving theoretical conclusions of the General Theory of Relativity have also proved that its basic space-time (the four-dimensional pseudo-Riemannian space) is the basis of our real world geometry. So, despite the progress in experimental physics and astronomy, with the discovery of new effects, the four-dimensional pseudo-Riemannian space will remain the corner-stone for further widening of the basic geometry of the General Theory of Relativity and will become one of its particular cases. Therefore, when building the mathematical theory of motion of particles, we are considering their motion in the four-dimensional pseudo-Riemannian space.

At this point, it is necessary to take note of the following terminology. Generally, the basic space-time in the General Theory of Relativity is a *Riemannian space** with four dimensions with Minkowski's sign-alternating label $(+---)$ or $(-+++)$. The later implies a (3+1)-split of coordinate axes in the Riemannian space into three spatial coordinate axes and the time axis. For convenience of calculations, we consider a Riemannian space of the signature $(+---)$, where time is real while spatial coordinates are imaginary. Also, some theories, largely the General Theory of Relativity, employ the label $(-+++)$, in which time is imaginary and spatial coordinates are real. In general, Riemannian spaces may have non-alternating signatures, e. g. $(++++)$. Therefore, a Riemannian space with alternating signature label is commonly referred to as a *pseudo-Riemannian space*, to emphasize the split of coordinate axes into two different types, referred to as time and spatial coordinates. Nonetheless, in this case, all its geometric properties are still properties of Riemannian geometry and the prefix "pseudo" is not absolutely proper from the mathematical point of view. Nevertheless, we are going to use this notation as a long-established and traditionally understood one.

*A metric space whose geometry is defined by the metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ is known as Riemann's metric. Bernhard Riemann (1826–1866), a German mathematician, the founder of Riemannian geometry (1854).

We consider motion of a particle in the four-dimensional pseudo-Riemannian space. A particle affected by gravitation only falls freely and it moves along the shortest (*geodesic*) line. Such motion is referred to as *free* or *geodesic motion*. If the particle is also affected by additional non-gravitational forces, they deviate the particle from its geodesic trajectory and the motion becomes *non-geodesic*.

From the geometric viewpoint, motion of a particle in the four-dimensional pseudo-Riemannian space is parallel transfer of its own four-dimensional vector Q^α, which is therefore tangential to the trajectory in any of its points. Consequently, equations of motion of this particle actually define parallel transfer of the vector Q^α along its four-dimensional trajectory and they are equations of the absolute derivative of this vector with respect to a parameter ρ, which is non-zero all along

$$\frac{\mathrm{D}Q^\alpha}{d\rho} = \frac{dQ^\alpha}{d\rho} + \Gamma^\alpha_{\mu\nu} Q^\mu \frac{dx^\nu}{d\rho}\,, \qquad \alpha, \mu, \nu = 0, 1, 2, 3. \tag{1.1}$$

Here, $\mathrm{D}Q^\alpha = dQ^\alpha + \Gamma^\alpha_{\mu\nu} Q^\mu dx^\nu$ is the absolute differential (the absolute increment in the pseudo-Riemannian space) of the vector Q^α. The absolute differential is different from a regular differential dQ^α by the presence of Christoffel's symbols of the 2nd kind $\Gamma^\alpha_{\mu\nu}$ (the coherence coefficients of the given Riemannian space), which are calculated through Christoffel's symbols (the coherence coefficients) of the 1st kind $\Gamma_{\mu\nu,\rho}$ and they are functions of the first derivatives of the fundamental metric tensor $g_{\alpha\beta}$*

$$\Gamma^\alpha_{\mu\nu} = g^{\alpha\rho}\,\Gamma_{\mu\nu,\rho}\,, \quad \Gamma_{\mu\nu,\rho} = \frac{1}{2}\left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right). \tag{1.2}$$

When moving along a geodesic trajectory (free motion) the parallel transfer occurs in the meaning of Levi-Civita†. Here the absolute derivative of any transferred vector equals zero, in particular it is true for the four-dimensional vector of the particle

$$\frac{dQ^\alpha}{d\rho} + \Gamma^\alpha_{\mu\nu} Q^\mu \frac{dx^\nu}{d\rho} = 0\,, \tag{1.3}$$

*Coherence coefficients of a Riemannian space (the Christoffel symbols) are named after German mathematician Elwin Bruno Christoffel (1829–1900), who obtained them in 1869. In the space-time of the Special Theory of Relativity (Minkowski's space) one can always set an inertial reference frame, where the matrix of the fundamental metric tensor becomes a unit diagonal, so all the Christoffel symbols become zeroes.

†Tullio Levi-Civita (1873–1941), an Italian mathematician, who was the first to study such a parallel transfer [1].

so the square of the transferred vector remains unchanged $Q_\alpha Q^\alpha = const$ along the trajectory. Such equations are referred to as equations of free motion.

Kinematic motion of a free particle is characterized by the four-dimensional vector of its acceleration, referred to as the *kinematic vector*

$$Q^\alpha = \frac{dx^\alpha}{d\rho}\,, \tag{1.4}$$

so the Levi-Civita parallel transfer of this vector gives equations of the four-dimensional trajectory of the particle (*equations of geodesic lines*)

$$\frac{d^2x^\alpha}{d\rho^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\rho}\frac{dx^\nu}{d\rho} = 0\,. \tag{1.5}$$

The necessary condition $\rho \neq 0$ along the trajectory implies that the derivation parameters ρ are not the same along trajectories of different kinds. In the pseudo-Riemannian space, three kinds of trajectories are principally possible, each kind corresponds to a specific kind of particles, namely:

1) *Non-isotropic real trajectories*, which lay "within" the light hyper-cone. Along such trajectories the square of the space-time interval is $ds^2 > 0$, thus, the interval ds is real. These are trajectories of regular sub-light particles with non-zero rest-masses and real relativistic masses;
2) *Non-isotropic imaginary trajectories*, which lay "outside" the light hyper-cone. Along such trajectories the square of the space-time interval is $ds^2 < 0$, hence, ds is imaginary. These are trajectories of super-light particles with imaginary relativistic masses, known as *tachyons**;
3) *Isotropic trajectories*, which lay on the surface of the light hyper-cone and are trajectories of particles with zero rest-mass (massless light-like particles), which travel at the light velocity. Along the isotropic trajectories the space-time interval is zero, $ds^2 = 0$, but the three-dimensional interval is not zero.

*Tachyons — faster-than-light particles. The possibility of tachyons and faster-than-light signals was first considered in the framework of the Special Theory of Relativity in 1958, by Tangherlini, in his dissertation [2]. As was pointed out by Malykins [3], most studies on the history of tachyons missed this fact. Meanwhile, the most important surveys of this theme such as [4,5] referred to Tangherlini. Tachyons were first illuminated in the journal publications on the theory of relativity in the principal paper of 1960 [6], authored by Terletskii, and in the more detailed paper of 1962 [7], authored by Bilaniuk, Deshpande, and Sudarshan. The term "tachyons" was first used later, in 1967 by Feinberg [8]. See Malykins' survey [3] for detail.

The space-time interval, ds, is commonly used as a derivation parameter along non-isotropic trajectories. Nevertheless, it can not be used as a derivation parameter for trajectories of massless particles, because, in this case, $ds=0$. For this reason, Zelmanov [9] proposed another variable, which does not turn into zero along isotropic trajectories, to be used as the derivation parameter. It is a three-dimensional (spatial) physical observable interval

$$d\sigma^2 = \left(-\,g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}\right) dx^i dx^k, \tag{1.6}$$

which differs from a three-dimensional regular coordinate interval. Landau and Lifshitz also arrived at the same conclusion in §84 of their *The Classical Theory of Fields* [10].

Substituting respective differentiation parameters into the generalized equations of geodesic lines (1.5), we arrive at equations of non-isotropic geodesic lines (trajectories of mass-bearing particles)

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,, \tag{1.7}$$

and equations of isotropic geodesic lines (light-like particles)

$$\frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0\,. \tag{1.8}$$

But, in order to make the whole picture of motion of a particle clear, we have to build dynamic equations of motion, which contain physical properties of this particle (namely — its mass, energy, etc.).

Motion of a free mass-bearing particle (a non-isotropic geodesic trajectory) is characterized by its own four-dimensional momentum vector

$$P^\alpha = m_0\frac{dx^\alpha}{ds}\,, \tag{1.9}$$

where m_0 is the rest-mass of this particle. From geometric viewpoint, parallel transfer in the meaning of Levi-Civita of the vector P^α gives dynamic equations of motion of the mass-bearing particle

$$\frac{dP^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}P^\mu\frac{dx^\nu}{ds} = 0\,, \qquad P_\alpha P^\alpha = m_0^2 = const. \tag{1.10}$$

Motion of a massless light-like particle (an isotropic geodesic line) is characterized by its own four-dimensional wave vector

$$K^\alpha = \frac{\omega}{c}\frac{dx^\alpha}{d\sigma}\,, \tag{1.11}$$

where ω is a cyclic frequency, specific for this massless particle. Respectively, the Levi-Civita parallel transfer of the vector K^α gives dynamic equations of motion of the massless particle

$$\frac{dK^\alpha}{d\sigma} + \Gamma^\alpha_{\mu\nu} K^\mu \frac{dx^\nu}{d\sigma} = 0\,, \qquad K_\alpha K^\alpha = 0\,. \tag{1.12}$$

So, we have got dynamic equations of motion for free particles. Here, we present the equations in four-dimensional general covariant form. This form has its own advantage as well as a substantial drawback. The advantage is their invariance in all transitions from one reference frame to another. The drawback is that, in general covariant form, terms of the equations do not contain actual three-dimensional quantities, which can be measured in experiments or observations (namely — *physical observable quantities*). This implies that, in general covariant form, equations of motion are merely an intermediate theoretical result, not applicable in practice. Therefore, in order to make results of any physical mathematical theory applicable in practice, we need to formulate its equations with physical observable quantities. Namely, to calculate trajectories of a particle we have to formulate general covariant equations of its motion through physical observable properties of an actual physical reference frame of the observer.

In the same time, to define physical observable quantities is not a trivial problem. For instance, for a four-dimensional vector Q^α (with few components-four) we may *heuristically assume* that its three spatial components form a three-dimensional observable vector, while the temporal component is observable potential of the vector field (which generally does not prove they can be actually observed, though). However, a contravariant tensor of the 2nd rank $Q^{\alpha\beta}$ (with as many as 16 components) makes the problem much more indefinite. For tensors of higher rank the problem of heuristic definition of observable components is more complicated. Besides, there is an obstacle related to the definition of observable components of covariant tensors (with lower indices) and mixed kind tensors (with both lower and upper indices).

Therefore, the most reasonable way out of the labyrinth of heuristic guesses is creating a strict mathematical theory to enable calculation of observable components for any tensor quantity. Such a theory had been built by Zelmanov in 1944 [9]. It should be noted that, many researchers were working on the theory of observable quantities in the 1940's. For example, Landau and Lifshitz in their famous *The Classical Theory of Fields* [10] introduced observable time and observable three-dimensional interval similar to those introduced by Zelmanov. But, they limited themselves only to this particular case and they did not arrive at general mathematical methods to define physical observable quantities in pseudo-Riemannian spaces.

Over the next decades, Zelmanov improved his mathematical apparatus of physical observable quantities (the theory of chronometric invariants), setting forth the results in the publications [11–13]. Similar results had also been obtained by Cattaneo [14–17], an Italian mathematician, independently from Zelmanov. However, Cattaneo published his first study on the theme (the study was far from a complete theory) only in 1958 [14].

In the next section, §1.2, we will give just a brief overview of the Zelmanov theory of physical observable quantities, which is necessary for understanding it and using the mathematical methods in practice.

In §1.3, we will present the results of studying geodesic motion of particles using the mathematical methods. In §1.4, we will focus on setting the problem of building equations of particles along non-geodesic trajectories, i. e. under the action of non-gravitational external forces.

§1.2 Physical observable quantities

This section introduces the basics of Zelmanov's mathematical apparatus of chronometric invariants.

To determine which components of any four-dimensional quantity are physical observable quantities, we consider a real reference frame of a real observer, which includes *coordinate nets*, spanned over his *reference body* (which is a real physical body), at each point of which a *real clock* is installed. The reference body, being a real physical body possesses a gravitational field, may be rotating and deforming, making the reference space inhomogeneous and anisotropic. Actually, the reference body and its attributed reference space may be considered as a set of real physical references, to which the observer compares all results of his measurements. Therefore, physical observable quantities shall be obtained as a result of projecting four-dimensional quantities on time lines and the three-dimensional space of the observer's reference body.

From geometric viewpoint, the observer's three-dimensional space is the *spatial section* $x^0 = ct = const$. At any point of the space-time, a local spatial section (a local space) can be placed orthogonal to the *time line*. If there exists a space-time enveloping curve to such local spaces, then it is a spatial section everywhere orthogonal to the time lines. Such a space is known as *holonomic space*. If no enveloping curve exists to such local spaces, only spatial sections locally orthogonal to the time lines exist, such a space is known as *non-holonomic space*.

We assume that the observer is at rest with respect to his physical references (his reference body). The reference frame of such an observer accompanies the reference body in any displacements, so such a system is called the *accompanying reference frame*. Any coordinate net which is at rest with respect to the same reference body is related to another one through the transformation

$$\left.\begin{aligned} &\tilde{x}^0 = \tilde{x}^0\left(x^0,\, x^1,\, x^2,\, x^3\right) \\ &\tilde{x}^i = \tilde{x}^i\left(x^1,\, x^2,\, x^3\right), \qquad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\}, \tag{1.13}$$

where the later equation implies that spatial coordinates in the tilde-marked net are independent of time in the non-tilded net, which is equivalent to setting a coordinate net of fixed lines of time $x^i = const$ in any point of the net. Transformation of spatial coordinates is nothing but only transition from one coordinate net to another within the same spatial section. Transformation of time implies changing the whole set of clocks, so this is transition to another spatial section (to another three-dimensional reference space). In practice, this means replacement of one reference body with all of its physical references with another reference body that has its own physical references. But when using different references, the observer will obtain different results (other observable quantities). Therefore, physical observable quantities must be invariant with respect to transformations of time, so they become *chronometrically invariant quantities*.

Because transformations (1.13) define a set of fixed lines of time, chronometric invariants (physical observable quantities) are all those quantities, which are invariant with respect to the transformations.

In practice, to obtain physical observable quantities in the accompanying reference frame of a real observer, we have to calculate chronometrically invariant projections of four-dimensional quantities on time lines and the spatial section of his physical reference body and formulate them with chronometrically invariant (physical observable) properties of his reference space.

We project four-dimensional quantities using operators, which characterize properties of the observer's reference space. The operator of projection on the time line, b^α, is a unit vector of the four-dimensional velocity of the observer with respect to his reference body, namely — the vector

$$b^\alpha = \frac{dx^\alpha}{ds}\,, \tag{1.14}$$

which is tangential to the observer's world-trajectory at every point. Because any reference frame is described by its own tangential unit vector b^α, Zelmanov referred to the b^α as the *monad vector*. The operator of projection on the spatial section is defined as the four-dimensional symmetric tensor

$$\left.\begin{aligned} h_{\alpha\beta} &= -\,g_{\alpha\beta} + b_\alpha b_\beta \\ h^{\alpha\beta} &= -\,g^{\alpha\beta} + b^\alpha b^\beta \end{aligned}\right\}, \tag{1.15}$$

whose mixed components are

$$h_\alpha^\beta = -\,g_\alpha^\beta + b_\alpha b^\beta. \tag{1.16}$$

As it was shown [9], the vector b^α and the tensor $h_{\alpha\beta}$ possess all necessary properties of the projection operators, namely — the properties $b_\alpha b^\alpha = 1$ and $h_\alpha^\beta b^\alpha = 0$. Projection of a tensor quantity on the time line is a result of its contraction with the monad vector b^α. Projection on the spatial section is contraction with the tensor $h_{\alpha\beta}$.

The observer's three-dimensional velocity with respect to his reference body, in the accompanying reference frame, is zero $b^i = 0$. The remaining components of this monad vector are

$$b^0 = \frac{1}{\sqrt{g_{00}}}\,, \quad b_0 = g_{0\alpha} b^\alpha = \sqrt{g_{00}}\,, \quad b_i = g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}}\,. \tag{1.17}$$

Respectively, in the accompanying reference frame ($b^i = 0$), components of the tensor of projection on the spatial section are

$$\left.\begin{aligned} & h_{00} = 0\,, && h^{00} = -g^{00} + \frac{1}{g_{00}}\,, && h_0^0 = 0 \\ & h_{0i} = 0\,, && h^{0i} = -g^{0i}, && h_0^i = \delta_0^i = 0 \\ & h_{i0} = 0\,, && h^{i0} = -g^{i0}, && h_i^0 = \frac{g_{i0}}{g_{00}} \\ & h_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}\,, && h^{ik} = -g^{ik}, && h_k^i = -\,g_k^i = \delta_k^i \end{aligned}\right\}. \tag{1.18}$$

The tensor $h_{\alpha\beta}$ in the three-dimensional space of the accompanying reference frame of the observer possesses all properties of the fundamental metric tensor

$$h_\alpha^i h_k^\alpha = \delta_k^i - b_k b^i = \delta_k^i\,, \qquad \delta_k^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.19}$$

where δ_k^i is the unit three-dimensional tensor*. For this reason, in the accompanying reference frame the three-dimensional chr.inv.-tensor h_{ik} can lift or lower indices in chr.inv.-quantities.

Projections on time lines and the spatial section of an arbitrary vector Q^α in the accompanying reference frame ($b^i = 0$) are

$$T = b^\alpha Q_\alpha = b^0 Q_0 = \frac{Q_0}{\sqrt{g_{00}}}\,, \tag{1.20}$$

$$L^0 = h^0_\beta Q^\beta = -\frac{g_{0k}}{g_{00}}\,Q^k, \quad L^i = h^i_\beta Q^\beta = \delta^i_k Q^k = Q^k. \tag{1.21}$$

Projections of an arbitrary tensor of the 2nd rank $Q^{\alpha\beta}$ are

$$T = b^\alpha b^\beta Q_{\alpha\beta} = b^0 b^0 Q_{00} = \frac{Q_{00}}{g_{00}}\,, \tag{1.22}$$

$$L^{00} = h^0_\alpha h^0_\beta Q^{\alpha\beta} = -\frac{g_{0i}\,g_{0k}}{g_{00}^2}\,Q^{ik}, \quad L^{ik} = h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}. \tag{1.23}$$

After testing the obtained quantities by the transformations (1.13), we see that chronometrically invariant (physical observable) quantities are the projection on time lines and spatial components of the projection on the spatial section. We will refer to the observable quantities as chr.inv.-projections.

Hence, projecting four-dimensional coordinates x^α in the accompanying reference frame, we obtain the chr.inv.-invariant of *physical observable time*

$$\tau = \sqrt{g_{00}}\,t + \frac{g_{0i}}{c\sqrt{g_{00}}}\,x^i, \tag{1.24}$$

and the chr.inv.-vector of *physical observable coordinates*, which coincide the spatial coordinates x^i. In the same way, projection of an elementary interval of four-dimensional coordinates dx^α gives an elementary interval of physical observable time, which is the chr.inv.-invariant

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i, \tag{1.25}$$

and also the chr.inv.-vector of an elementary interval of physical observable coordinates dx^i. Thus, the *physical observable velocity* of a particle is the three-dimensional chr.inv.-vector

$$\mathrm{v}^i = \frac{dx^i}{d\tau}\,, \tag{1.26}$$

*This tensor δ_k^i is the three-dimensional part of the four-dimensional unit tensor δ_β^α, which can be used to replace indices in four-dimensional quantities.

which is different from its coordinate velocity $u^i = \frac{dx^i}{dt}$.

Projecting the fundamental metric tensor, we deduce that h_{ik} is the *metric chr.inv.-tensor*, or, in other words, the *observable metric tensor* in the accompanying reference frame

$$h^i_\alpha h^k_\beta\, g^{\alpha\beta} = g^{ik} = -\,h^{ik}, \qquad h^\alpha_i h^\beta_k g_{\alpha\beta} = g_{ik} - b_i b_k = -\,h_{ik}\,, \quad (1.27a)$$

whose components are

$$h_{ik} = -\,g_{ik} + b_i b_k\,, \quad h^{ik} = -\,g^{ik}, \quad h^i_k = -\,g^i_k = \delta^i_k\,. \qquad (1.27b)$$

So, the square of an observable spatial interval $d\sigma$ is

$$d\sigma^2 = h_{ik}\, dx^i dx^k. \qquad (1.28)$$

Space-time interval formulated with physical observable quantities can be obtained by substituting $g_{\alpha\beta}$ from (1.15), namely

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \qquad (1.29)$$

Apart from their projections on time lines and the spatial section, four-dimensional quantities of the 2nd rank and above also have mixed components which have both upper and lower indices at the same time. How do we find physical observable quantities among them, if any? The best approach is to develop a generalized method to calculate physical observable quantities, based solely on their property of chronometric invariance. Such a method had been developed by Zelmanov, who set forth the method in a theorem:

Zelmanov's theorem

We assume that $Q^{ik\dots p}_{00\dots 0}$ are components of a four-dimensional tensor $Q^{\mu\nu\dots\rho}_{00\dots 0}$ of r-th rank, in which all upper indices are not zero, while all m lower indices are zeroes. Then tensor quantities

$$T^{ik\dots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\dots p}_{00\dots 0} \qquad (1.30)$$

make up three-dimensional contravariant chr.inv.-tensor of $(r-m)$-th rank. Hence, the tensor $T^{ik\dots p}$ is a result of m-fold projection on time lines by indices $\alpha, \beta \dots \sigma$ and also, projection on the spatial section by $r-m$ indices $\mu, \nu \dots \rho$ of the initial tensor $Q^{\mu\nu\dots\rho}_{\alpha\beta\dots\sigma}$.

An immediate result of this theorem is that, for any vector Q^α two quantities are physical observable, which were obtained earlier

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = Q^i. \qquad (1.31)$$

For any symmetric tensor of the 2nd rank $Q^{\alpha\beta}$, three quantities are physical observable, namely

$$b^{\alpha}b^{\beta}Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \quad h^{i\alpha}b^{\beta}Q_{\alpha\beta} = \frac{Q_0^i}{\sqrt{g_{00}}}\,, \quad h^i_{\alpha}h^k_{\beta}Q^{\alpha\beta} = Q^{ik}, \tag{1.32}$$

and in an antisymmetric tensor of the 2nd rank, the first quantity is zero, because $Q_{00} = Q^{00} = 0$.

The physical observable quantities (chr.inv.-projections) must be compared to the observer's references — observable properties of his reference space, which are specific for any particular body of reference. Therefore, we will now consider the basic properties of his accompanying reference space, with which the final equations of theory must be formulated.

Physical observable properties of the accompanying reference space can be obtained with the help of chr.inv.-operators of derivation with respect to time and the spatial coordinates. The mentioned operators had been introduced by Zelmanov as follows [9]

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}\,, \tag{1.33}$$

they are non-commutative, so the difference between the 2nd derivatives is not zero

$$\frac{{}^*\partial^2}{\partial x^i\,\partial t} - \frac{{}^*\partial^2}{\partial t\,\partial x^i} = \frac{1}{c^2}\,F_i\,\frac{{}^*\partial}{\partial t}\,, \tag{1.34}$$

$$\frac{{}^*\partial^2}{\partial x^i\partial x^k} - \frac{{}^*\partial^2}{\partial x^k\partial x^i} = \frac{2}{c^2}\,A_{ik}\,\frac{{}^*\partial}{\partial t}\,. \tag{1.35}$$

Here, A_{ik} is the *three-dimensional antisymmetric chr.inv.-invariant tensor of angular velocities of the space rotation*

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{1.36}$$

where v_i is the linear velocity of this rotation

$$\left.\begin{aligned} v_i &= -\,c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \qquad & v^i &= -\,c\,g^{0i}\sqrt{g_{00}} \\ v_i &= h_{ik}v^k, \qquad & v^2 &= v_k v^k = h_{ik}\,v^i v^k \end{aligned}\right\}. \tag{1.37}$$

The tensor A_{ik}, equated to zero, is the necessary and sufficient condition of holonomity of this space [9]. In this case, $g_{0i} = 0$ and $v_i = 0$. In a non-holonomic space $A_{ik} \neq 0$. For this reason, the tensor A_{ik} is also the tensor of the space non-holonomity*.

Hence forth, F_i is the *three-dimensional chr.inv.-vector of gravitational inertial force*

$$F_i = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \tag{1.38}$$

where w is a gravitational potential

$$\mathrm{w} = c^2\,(1-\sqrt{g_{00}}\,)\,, \tag{1.38a}$$

an origin of which is the gravitational field of the observer's reference body†. In quasi-Newtonian approximation, i. e. in a weak gravitational field at velocities much lower than the light velocity and in the absence of rotations of the space, the quantity F_i becomes a non-relativistic gravitational force

$$F_i = \frac{\partial \mathrm{w}}{\partial x^i}\,. \tag{1.39}$$

The observer's reference body is a real physical body and so coordinate nets spanned over it may be deformed. So, his real reference space may be deformed as well. Therefore, real physical references must take the space deformations into account. Namely, as a result of the deformations, the observable metric h_{ik} of the reference space must be non-stationary. This can be accounted for by introducing the *three-dimensional symmetric chr.inv.-tensor of the rate of the space deformations*

$$\left.\begin{aligned} D_{ik} &= \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t} \\ D &= h^{ik}D_{ik} = D^n_n = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \qquad h = \det\|h_{ik}\| \end{aligned}\right\}. \tag{1.40}$$

With the given definitions, we can generally formulate any property of geometric objects located in a space with observable parameters of the space. For instance, the Christoffel symbols, which appear in equations of motion, are not tensors [18]. Nevertheless, they can be formulated as well with physical observable quantities. The formulae obtained by Zel-

*The space-time of the Special Theory of Relativity (the Minkowski space) in a Galilean reference frame and also numerous cases in the space-time of the General Theory of Relativity are examples of holonomic spaces $A_{ik} = 0$.

†The quantities w and v_i do not possess the property of chronometric invariance, while the gravitational inertial force vector and the tensor of angular velocities of the space rotation, built using them, are chr.inv.-quantities.

manov [9] are

$$\Gamma_{00}^{0} = -\frac{1}{c^3}\left[\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right) v_k F^k\right], \tag{1.41}$$

$$\Gamma_{00}^{k} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F^k, \tag{1.42}$$

$$\Gamma_{0i}^{0} = \frac{1}{c^2}\left[-\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial x^i} + v_k\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\, v_i F^k\right)\right], \tag{1.43}$$

$$\Gamma_{0i}^{k} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\, v_i F^k\right), \tag{1.44}$$

$$\begin{aligned}\Gamma_{ij}^{0} = &-\frac{1}{c\left(1-\frac{\mathrm{w}}{c^2}\right)}\left\{-D_{ij} + \frac{1}{c^2}\, v_n \times\right. \\ &\times \left[v_j\left(D_i^n + A_{i\cdot}^{\cdot n}\right) + v_i\left(D_j^n + A_{j\cdot}^{\cdot n}\right) + \frac{1}{c^2}\, v_i v_j F^n\right] + \\ &\left. + \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right) - \Delta_{ij}^n v_n\right\},\end{aligned} \tag{1.45}$$

$$\Gamma_{ij}^{k} = \Delta_{ij}^{k} - \frac{1}{c^2}\left[v_i\left(D_j^k + A_{j\cdot}^{\cdot k}\right) + v_j\left(D_i^k + A_{i\cdot}^{\cdot k}\right) + \frac{1}{c^2}\, v_i v_j F^k\right], \tag{1.46}$$

where Δ_{ij}^k are the *chr.inv.-Christoffel symbols*, which are defined as well as the regular Christoffel symbols (1.2) but through the metric chr.inv.-tensor h_{ik} and chr.inv.-operators of derivation

$$\Delta_{jk}^{i} = h^{im}\Delta_{jk,m} = \frac{1}{2}\, h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right). \tag{1.47}$$

So, we have discussed the basics of the mathematical apparatus of chronometric invariants. Now, having any equations obtained using general covariant methods we can calculate their chr.inv.-projections onto the time line and spatial section of any particular body of reference and formulate them with its real physical observable properties. From here, we arrive at equations containing only measurable quantities in practice.

Naturally, the first possible application of this mathematical apparatus that comes to our mind is the deduction of chr.inv.-equations of motion of free particles and studying the results. Particular solution of this problem had been obtained by Zelmanov [9]. The next section, §1.3, will focus on the general solution of the problem.

§1.3 Dynamic equations of motion of free particles

The absolute derivative of the four-dimensional vector of a particle with respect to a non-zero scalar parameter along its trajectory is actually a four-dimensional vector

$$N^{\alpha} = \frac{dQ^{\alpha}}{d\rho} + \Gamma^{\alpha}_{\mu\nu} Q^{\mu} \frac{dx^{\nu}}{d\rho}\,, \tag{1.48}$$

whose chr.inv.-projections are defined in the same way as the projections of any four-dimensional vector (1.31)

$$\frac{N_0}{\sqrt{g_{00}}} = \frac{g_{0\alpha} N^{\alpha}}{\sqrt{g_{00}}} = \frac{1}{\sqrt{g_{00}}} \left(g_{00} N^0 + g_{0i} N^i\right), \tag{1.49}$$

$$N^i = h^i_{\beta} N^{\beta} = h^i_0 N^0 + h^i_k N^k. \tag{1.50}$$

From the geometric viewpoint, these are the projection of the vector N^{α} on the time line and spatial components of its projection on the spatial section in the accompanying reference frame.

So, projecting general covariant equations of motion of a free mass-bearing particle (1.10) and of a free massless particle (1.12), we obtain chr.inv.-equations of their motion. For the mass-bearing particle the equations are

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0\,, \tag{1.51}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \tag{1.52}$$

while for the massless particle we have

$$\frac{dk}{d\tau} - \frac{k}{c^2} F_i c^i + \frac{k}{c^2} D_{ik} c^i c^k = 0\,, \tag{1.53}$$

$$\frac{d\left(kc^i\right)}{d\tau} + 2k\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k - kF^i + k\Delta^i_{nk}c^n c^k = 0\,, \tag{1.54}$$

where m is the relativistic mass of the mass-bearing particle, $k = \frac{\omega}{c}$ is the wave number of the massless particle, and c^i is the three-dimensional chr.inv.-vector of the light velocity. As it is easy to see, in contrast to general covariant dynamic equations of motion (1.10, 1.12), the chr.inv.-equations have a single derivation parameter for both mass-bearing and massless particles. This universal parameter is physical observable time τ.

These chr.inv.-equations were first obtained by Zelmanov [9]. As we have shown in our study [19], the equations that include the time function $\frac{dt}{d\tau}$ are strictly positive, so physical time has strictly direct flow $d\tau > 0$ here. The flow of coordinate time dt shows change of time coordinates of the particle $x^0 = ct$ with respect to the observer's clock. Hence, the sign of the time function shows where the particle travels to in time with respect to the observer.

The time function $\frac{dt}{d\tau}$ is derived from the condition that the square of the four-dimensional velocity of the particle remains unchanged along its world-trajectory $u_\alpha u^\alpha = g_{\alpha\beta}\, u^\alpha u^\beta = const$. Equations of $\frac{dt}{d\tau}$ are the same for sub-light mass-bearing particles, massless particles and super-light mass-bearing particles. The equations have two solutions which are given here by the common formula

$$\left(\frac{dt}{d\tau}\right)_{1,2} = \frac{v_i \mathrm{v}^i \pm c^2}{c^2\left(1 - \frac{\mathrm{w}}{c^2}\right)}\,. \tag{1.55}$$

As it was shown in our study [19], time has direct flow if $v_i \mathrm{v}^i \pm c^2 > 0$, time has reverse flow if $v_i \mathrm{v}^i \pm c^2 < 0$, and the flow of time stops if $v_i \mathrm{v}^i \pm c^2 = 0$. Therefore, there exists a whole range of solutions for various kinds of particles and directions they travel in time with respect to the observer. For instance, the relativistic mass of a mass-bearing particle* $\frac{P_0}{\sqrt{g_{00}}} = \pm m$ is positive if this particle travels into the future, and it is negative if the particle travels into the past. The wave number of a massless particle $\frac{K_0}{\sqrt{g_{00}}} = \pm k$ is also positive for motion into the future, and negative for motion into the past.

As a result, for a free mass-bearing particle, which moves into the past, we obtain chr.inv.-equations of motion

$$-\frac{dm}{d\tau} - \frac{m}{c^2}\, F_i \mathrm{v}^i + \frac{m}{c^2}\, D_{ik} \mathrm{v}^i \mathrm{v}^k = 0\,, \tag{1.56}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + mF^i + m\Delta^i_{nk} \mathrm{v}^n \mathrm{v}^k = 0\,, \tag{1.57}$$

while for a free massless particle we have

$$-\frac{dk}{d\tau} - \frac{k}{c^2}\, F_i c^i + \frac{k}{c^2}\, D_{ik} c^i c^k = 0\,, \tag{1.58}$$

$$\frac{d\left(kc^i\right)}{d\tau} + kF^i + k\Delta^i_{nk} c^n c^k = 0\,. \tag{1.59}$$

*The relativistic mass is the projection of the particle's four-dimensional vector on the observer's time line.

For a super-light mass-bearing particle chr.inv.-equations of motion are similar to those for sub-light velocities, except that the relativistic mass
m is multiplied by imaginary unit i.

As it easy to see, chr.inv.-equations of motion into future and into past are not symmetric due to different physical conditions in the cases of the direct and reverse time flows, so some terms in equations will be missing.

Besides, we have considered the motion of mass-bearing and massless particles within the wave-particle concept, assuming their motion propagates as waves in geometric optics approximation [19]. As it is well-known, in the frames of the wave-particle concept, the dynamic vector of a massless particle is [10]

$$K_\alpha = \frac{\partial\psi}{\partial x^\alpha}\,, \tag{1.60}$$

where ψ is the wave phase (eikonal). In the same way, we introduced the dynamic vector for a mass-bearing particle

$$P_\alpha = \frac{\hbar}{c}\frac{\partial\psi}{\partial x^\alpha}\,, \tag{1.61}$$

where $\hbar$ is Planck's constant. The wave phase equation (the eikonal equation) in the geometric optics approximation is the condition $K_\alpha K^\alpha = 0$. Hence the eikonal chr.inv.-equation for the massless particle is

$$\frac{1}{c^2}\left(\frac{^*\partial\psi}{\partial t}\right)^2 - h^{ik}\frac{^*\partial\psi}{\partial x^i}\frac{^*\partial\psi}{\partial x^k} = 0\,, \tag{1.62}$$

and for the mass-bearing particle we have

$$\frac{1}{c^2}\left(\frac{^*\partial\psi}{\partial t}\right)^2 - h^{ik}\frac{^*\partial\psi}{\partial x^i}\frac{^*\partial\psi}{\partial x^k} = \frac{m_0^2c^2}{\hbar^2}\,. \tag{1.63}$$

Substituting the wave form of the dynamic vector into general covariant equations of motion (1.10, 1.12), after their projection in the accompanying reference frame we obtain chr.inv.-equations of motion in their "wave form". For the mass-bearing particle, the resulting equations are

$$\pm\frac{d}{d\tau}\left(\frac{^*\partial\psi}{\partial t}\right) + F^i\frac{^*\partial\psi}{\partial x^i} - D^i_k \mathrm{v}^k\frac{^*\partial\psi}{\partial x^i} = 0\,, \tag{1.64}$$

$$\begin{gathered}\frac{d}{d\tau}\left(h^{ik}\frac{^*\partial\psi}{\partial x^k}\right) - \left(D^i_k + A^{\cdot i}_{k\cdot}\right)\left(\pm\frac{1}{c^2}\frac{^*\partial\psi}{\partial t}\mathrm{v}^k - h^{km}\frac{^*\partial\psi}{\partial x^m}\right) \pm \\ \pm\frac{1}{c^2}\frac{^*\partial\psi}{\partial t}F^i + h^{mn}\Delta^i_{mk}\mathrm{v}^k\frac{^*\partial\psi}{\partial x^n} = 0\,,\end{gathered} \tag{1.65}$$

where "plus" in alternating terms stands for motion of the particle from the past into the future (the direct flow of time), while "minus" stands for its motion into the past (the reverse flow of time). Noteworthy, in contrast to the "corpuscular form" of chr.inv.-equations of motion (1.51, 1.52) and (1.56, 1.57), the equations in "wave form" (1.64, 1.65) are symmetric with respect to the direction of motion in time. For the massless particle chr.inv.-equations of motion in "wave form" show the only difference: instead of the particle's chr.inv.-velocity v^i the equations include the chr.inv.-vector of the light velocity c^i.

The fact that corpuscular equations of motion into the past and into the future are asymmetric leads to the evident conclusion that in the four-dimensional inhomogeneous space-time of the General Theory of Relativity there exists a fundamental asymmetry of directions in time. To understand the physical sense of this fundamental asymmetry, we had introduced the *mirror principle* or, in otherwords — the *observable effect of the mirror Universe* [19].

Let us imagine a *mirror* in the four-dimensional space-time which coincides the spatial section, so this mirror separates the past from the future. Then, particles and waves travelling from the past into the future (positive relativistic masses and frequencies) hit the mirror and bounce back in time into the past. Hence, their properties take negative numerical values. Conversely, particles and waves travelling into the past (negative relativistic masses and frequencies) bounce from the mirror to give positive numerical values to their properties and begin travelling into the future. When bouncing from the mirror, the quantity $\frac{^*\partial\psi}{dt}$ changes sign, and so equations of propagation of a wave into the future become equations of propagation of this wave into the past (and vice versa). Noteworthy, when reflecting from the mirror, chr.inv.-equations of wave propagation transform into each other *completely* without contracting or adding new terms. In other word, the wave form of matter undergoes *full reflection* from the mirror. On the contrary, corpuscular chr.inv.-equations of motion *do not transform completely* in reflection from the mirror. Spatial components of the equations for mass-bearing and massless particles, travelling from the past into the future, have an additional term

$$2m\left(D_k^i + A_{k\cdot}^{\cdot i}\right)\mathrm{v}^k, \qquad 2k\left(D_k^i + A_{k\cdot}^{\cdot i}\right)c^k, \tag{1.66}$$

not found in the equations of motion from the future into the past. The equations of motion into the past gain the additional term on reflection. Conversely, the equations of motion into the future lose the term when the particle hits the mirror. This implies that, either in the case of motion of a particle-ball (the corpuscular equations) as well as in the case of propagation of a wave (the wave equations), we come across a situation which is not a simple "bouncing" from the mirror, but rather *passing* through the mirror itself into another world — into a *world beyond the mirror.*

In this *mirror world* all particles have negative masses or frequencies, so they travel (from our viewpoint) from the future into the past. The wave form of matter in our world does not affect events in the mirror world, while the mirror world's matter in wave form does not affect events in our world. To the contrary, the corpuscular form of matter (particles) in our world may produce significant effect on events in the mirror world, while the mirror world's particles may affect events in our world. Our world is fully isolated from the mirror world (no mutual effect between particles from the two worlds) under the evident condition $D^i_k \mathrm{v}^k = -A^{\cdot i}_{k\cdot} \mathrm{v}^k$, at which the additional term in corpuscular chr.inv.-equations of motion becomes zero. This becomes true, in particular, when $D^i_k = 0$ and $A^{\cdot i}_{k\cdot} = 0$, i. e. when there are no deformations and rotation in the space.

So far, we have considered motion of particles along non-isotropic trajectories, where $ds^2 = c^2 d\tau^2 - d\sigma^2 > 0$, and motion along isotropic (light-like) trajectories, where $ds^2 = 0$ and $c^2 d\tau^2 = d\sigma^2 \neq 0$. Besides, we considered trajectories of the third kind [19], which, apart from $ds^2 = 0$, meet even more strict conditions $c^2 d\tau^2 = d\sigma^2 = 0$

$$d\tau = \left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right] dt = 0\,, \tag{1.67}$$

$$d\sigma^2 = h_{ik}\, dx^i dx^k = 0\,. \tag{1.68}$$

We called such fully degenerate trajectories *zero-trajectories*, because from the viewpoint of a regular sub-light observer, any interval of observable time and any observable spatial interval are zeroes along them. We can as well show that along zero-trajectories the determinant of the fundamental metric tensor is zero $g = 0$. In Riemannian spaces, by their definition we have $g < 0$, so the Riemannian metric is strictly non-degenerate. We called a space, a metric of which is fully degenerate, *zero-space.* For the same reason, we called particles, which move along trajectories in such a space *zero-particles.*

Actually, formulae (1.67, 1.68) show physical conditions, under which total degeneration of the four-dimensional space-time occurs. We can re-write the *physical conditions of the degeneration* as follows

$$\mathrm{w} + v_i u^i = c^2, \tag{1.69}$$

$$g_{ik}\,u^i u^k = c^2\left(1-\frac{\mathrm{w}}{c^2}\right)^2. \tag{1.70}$$

Respectively, formula for the mass of a zero-particle M, including the degeneration conditions, is different from the relativistic mass m of a regular particle in a non-degenerate area

$$M = \frac{m}{1-\frac{1}{c^2}\left(\mathrm{w}+v_i u^i\right)}\,, \tag{1.71}$$

so that it is a ratio between two quantities, each one equals zero in the case where the metric is degenerate, but the ratio is not zero*.

The dynamic vector of a zero-particle, represented in its corpuscular and wave forms, is

$$P^{\alpha} = \frac{M}{c}\frac{dx^{\alpha}}{dt}\,, \qquad P_{\alpha} = \frac{\hbar}{c}\frac{\partial\psi}{\partial x^{\alpha}}\,. \tag{1.72}$$

Then, dynamic chr.inv.-equations of motion in the zero-space, taken in their corpuscular form, are

$$MD_{ik}u^i u^k = 0\,, \tag{1.73}$$

$$\frac{d}{dt}\left(Mu^i\right) + M\Delta^i_{nk}u^n u^k = 0\,, \tag{1.74}$$

while the wave form of the equations is

$$D^m_k u^k \frac{{}^*\partial\psi}{\partial x^m} = 0\,, \tag{1.75}$$

$$\frac{d}{dt}\left(h^{ik}\frac{{}^*\partial\psi}{\partial x^k}\right) + h^{mn}\Delta^i_{mk}u^k\frac{{}^*\partial\psi}{\partial x^n} = 0\,. \tag{1.76}$$

The eikonal chr.inv.-equation for the zero-particle is

$$h^{ik}\frac{{}^*\partial\psi}{\partial x^i}\frac{{}^*\partial\psi}{\partial x^k} = 0\,, \tag{1.77}$$

so it is a standing wave equation, which describes the zero-particle to be in the form of an information ring. Therefore, from the viewpoint of a regular sub-light observer, the whole zero-space is filled with a system of standing light-like waves (zero-particles) — a *standing-light hologram*. Besides, in the zero-space, observable time has the same numerical value for any two events (1.67). This implies that from the viewpoint of a regular observer, the velocity of any zero-particle is infinite; so zero-particles can instantly transfer information from one point of our regular world to another, performing the *long-range action* [19].

*This is similar to the case of massless particles, because given $\mathrm{v}^2=c^2$ we have that $m_0=0$ and $\sqrt{1-\mathrm{v}^2/c^2}=0$ are zeroes, but their ratio is $m=\frac{m_0}{\sqrt{1-\mathrm{v}^2/c^2}}\neq 0$.

§1.4 Non-geodesic motion of particles. Problem statement

It is well-known that, free motion of a particle (along its own geodesic line) leaves the absolute derivative of the dynamic world-vector of this particle (its four-dimensional momentum) zero, so the square of the vector remains unchanged along the trajectory of the motion. In other words, the vector is parallel transferred in the meaning of Levi-Civita.

In non-free (non-geodesic) motion of a particle, the absolute derivative of its four-dimensional momentum is not zero. But, the absolute derivative of the sum of its four-dimensional momentum P^{α} is equal to zero. Also, the absolute derivative of an additional momentum vector L^{α} gained by this particle from interaction with external fields, which deviate its motion from geodesic line is zero. Superposition of any number of vectors can be subjected to parallel transfer [18]. Hence, building equations of non-geodesic motion first of all requires the definition of non-gravitational perturbation fields.

Naturally, an external field will only interact with the particle and deviate it from geodesic line if the particle has a physical property of the same kind as the external field does. As of today, we know of three fundamental physical properties of particles, not related to each other. These are *mass*, *electric charge* and *spin*. If fundamental character of the former two was under no doubt, the spin of an electron over a few years after experiments by Stern and Gerlach (1921) and their interpretation by Gaudsmith and Ulenbek (1925), was considered as a specific momentum of the electron caused by its rotation around its own axis. But experiments done over the next decades, in particular, discovery of spin in other elementary particles, proved that views of spin-particles as rotating gyroscopes were wrong. Spin proved to be a fundamental property of particles just like mass and charge, though it has dimension of angular momentum and in interactions manifest as the specific rotation momentum inside the particle.

Gravitational fields by now have received geometric interpretation due to Einstein's equations. In the theory of chronometric invariants, gravitational force and the potential (1.38) are obtained as functions of only geometric properties of the space itself. Therefore, considering motion of a particle in a pseudo-Riemannian space, we actually consider its motion in a gravitational field.

But we still do not know whether Lorentz' electromagnetic force and the electromagnetic field potential can be expressed through geometric properties of the space. Therefore, electromagnetic fields at the moment have no geometric interpretation. An electromagnetic field is introduced into a pseudo-Riemannian space as an external tensor field (the field of Maxwell's tensor). By now the main equations of the theory of electromagnetic fields have been obtained in general covariant form*. In this theory, a charged particle gains a four-dimensional momentum $\frac{e}{c^2}A^{\alpha}$ from the acting electromagnetic fields, where A^{α} is the four-dimensional potential of the field, and e is the electric charge of the particle [10, 20]. Adding this extra-momentum to the specific momentum vector of the particle and applying the Levi-Civita parallel transfer, we can obtain general covariant equations of motion of the particle in the space, filled with gravitational and electromagnetic fields.

The case of spin-particles is far more complicated. To deduce a momentum a particle gains due to its spin, we need to define the external field that interacts with the spin as a fundamental property of the particle. Initially, this problem was approached using methods of Quantum Mechanics only (Dirac's equations, 1928). Geometric methods of the General Theory of Relativity were first used by Papapetrou and Corinaldesi [21, 22] for studying the problem. Their approach relied on general view of particles as mechanical monopoles and the dipoles. From this viewpoint, a regular mass-bearing particle is a *mechanical monopole*. If a particle can be represented as two masses co-rotating around a common centre of gravity, then the particle is a *mechanical dipole*. Therefore, proceeding from representation of a spin-particle as a rotating gyroscope we can consider it as a mechanical dipole, whose centre of gravity lays over the particle's surface. Papapetrou and Corinaldesi considered motion of such a mechanic dipole in a pseudo-Riemannian space with Schwarzschild's metric — a very particular case, where rotation of the space is zero and the metric is stationary (the tensor of the space deformations rate is zero).

There is no doubt that Papapetrou's method is noteworthy, but it has a significant drawback. Being developed in the 1940's, it fully relied on the view of spin-particles as swiftly rotating gyroscopes, which does not match experimental data of the recent decades†.

There is another way to solve the problem of motion of spin-particles.

---

*Despite this positive fact, due to complicated calculations of the energy-momentum tensor for an electromagnetic field in the space-time of the General Theory of Relativity, specific problems are commonly solved either for particular cases of the General Theory of Relativity, or in a Galilean reference frame in the Minkowski space (the space-time of the Special Theory of Relativity).

†As a matter of fact, considering an electron as a ball with radius of $r_{\mathrm{e}} = 2.8\times10^{-13}$ cm implies that the linear velocity of its rotation on the surface is $u = \frac{\hbar}{2m_0 r_{\mathrm{e}}} = 2\times10^{11}$ cm/sec, which is $\sim$70 times as high as the light velocity. Experiments show there are no such velocities in electrons.

In Riemannian spaces, the fundamental metric tensor is symmetric, $g_{\alpha\beta} = g_{\beta\alpha}$. Nevertheless, we can build a space in which the metric tensor will have arbitrary form $g_{\alpha\beta} \neq g_{\beta\alpha}$ (such a space will have non-Riemannian geometry). Then, a non-zero antisymmetric part can be found in the metric tensor*. Appropriate additions will appear in Christoffel's symbols $\Gamma^{\alpha}_{\mu\nu}$ and in Riemann-Christoffel's curvature tensor $R_{\alpha\beta\mu\nu}$. These additions will be as a result of the fact that, a vector transferred along a closed contour does not to return to its initial point, so the trajectory becomes twisted like a spiral. Such a space is known as twisted space. In such a space, the spin-rotation of a particle can be considered as transfer of the rotation vector along its surface contour, that generates a local field of the space twist.

Nonetheless, this method has got significant drawbacks as well. Firstly, if we have $g_{\alpha\beta} \neq g_{\beta\alpha}$, then functions of the components $g_{\alpha\beta}$ with different order of indices may be varied. The functions have been fixed somehow in to order to set a specific field of this twist, which dramatically narrows the range of possible solutions, enabling only the building of equations for a range of specific cases. Secondly, this method fully relies on assumption of the spin-rotation of a particle as a local field of a twist, produced by transfer of the vector of the particle's rotation along a contour. This again implies the view of spin-particles as rotating gyroscopes with limited radii (like Papapetrou's method), which does not match experimental data.

Nevertheless, there is no doubt that, an additional momentum gained by a spin-particle can be represented with methods of the General Theory of Relativity. Adding it to the specific dynamic vector of this particle (the effect of gravitation) and undergoing parallel transfer, we can obtain general covariant equations of motion of the particle†.

Once we have obtained general covariant equations of motion of a spin-particle and an electric charged particle, we shall project them on time lines and the spatial section in the accompanying reference

*Generally, in any tensor of the 2nd rank and of high ranks symmetric and antisymmetric parts can be distinguished. For instance, in the fundamental metric tensor $g_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\beta} + g_{\beta\alpha}\right) + \frac{1}{2}\left(g_{\alpha\beta} - g_{\beta\alpha}\right) = S_{\alpha\beta} + N_{\alpha\beta}$ we have the symmetric part $S_{\alpha\beta}$ and the antisymmetric part $N_{\alpha\beta}$. Because the metric tensor of any Riemannian space is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$, its antisymmetric part is zero.

†We wrote this in the mid-1990's, in the 1st edition of this book. In 2007, a new and highly original approach to spin-particle was developed by Suhendro [23, 24] on the basis of the views onto spin as an elementary curl of the space itself. We should agree that his approach, having a purely geometrical nature, is more close to Einstein's ideology (geometrization of matter and interactions) than our approach realized in Chapter 4 of this book on the basis of the Lagrangian method.

frame, then we shall express their chr.inv.-projections through physical observable properties of the reference space. As a result, we shall arrive at chr.inv.-equations of non geodesic motion.

Therefore, the problem we are going to solve in this book falls into few stages. At first stage, we shall build the chr.inv.-theory of an electromagnetic field in the four-dimensional pseudo-Riemannian space. Also, we shall arrive at chr.inv.-equations of motion of a charged particle in the field. This problem will be solved in Chapter 3.

Then, we shall create the theory of motion of a spin-particle. We will approach this problem in its most general form, assuming spin is a fundamental property of matter (like mass or electric charge). In Chapter 4, detailed study will show that the field of non-holonomity of the space (the space rotations field) interacts with the spin of a particle, giving the particle additional momentum.

In Chapter 5, we are going to discuss chr.inv.-projections of Einstein's equations. Proceeding from them we will study properties of physical vacuum and how they are applied in cosmology.

In Chapter 6, we shall consider the theory of the mirror world and also the physical conditions to move to it, through the membrane.

Before turning to these studies, in Chapter 2 we would like to have a look at tensor algebra and analysis in terms of physical observable quantities (chronometric invariants). Mainly, we recommend Chapter 2 to readers who are going to use the mathematical apparatus in their own theoretical studies. For general understanding of our book, reading the next Chapter may not be necessary.

Chapter 2 Tensor Algebra and Analysis

§2.1 Tensors and tensor algebra

We assume a space (not necessarily a metric one) with an arbitrary reference frame x^α located in it. In an area of this space, there exists an object G defined by n functions f_n of the coordinates x^α. We know the transformation rule to calculate these n functions in any other reference frame $\tilde{x}^\alpha$ in this space. If the n functions f_n and also the transformation rule have been given, then G is a *geometric object*, which in the system x^α has axial components $f_n(x^\alpha)$, while in any other system $\tilde{x}^\alpha$ it has components $\tilde{f}_n(\tilde{x}^\alpha)$.

We assume that a tensor object (*tensor*) of zero rank is any geometric object φ, transformable according to the rule

$$\tilde{\varphi} = \varphi\, \frac{\partial x^\alpha}{\partial \tilde{x}^\alpha}\,, \tag{2.1}$$

where the index one-by-one takes numbers of all coordinate axes (this notation is also known as *by-component notation* or *tensor notation*). Any tensor of zero rank has a single component and is also known as *scalar*. From the geometric viewpoint, any scalar is a point to which a certain number is attributed.

Consequently, a scalar field* is a set of points of the space, which have a common property. For instance, a point mass is a scalar, while a distributed mass (a gas, for instance) makes up a scalar field.

Contravariant tensors of the 1st rank A^α are geometric objects with components, transformable according to the rule

$$\tilde{A}^\alpha = A^\mu\, \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\,. \tag{2.2}$$

From the geometric viewpoint, such an object is an n-dimensional vector. For instance, the vector of an elementary displacement dx^α is a contravariant tensor of the 1st rank.

*Algebraic notations for a tensor and a tensor field are the same. The field of a tensor is represented as the tensor in a given point of the space, but its presence in other points in this area of the space is assumed.

Contravariant tensors of the 2nd rank $A^{\alpha\beta}$ are geometric objects with components, transformable according to the rule

$$\tilde{A}^{\alpha\beta} = A^{\mu\nu} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} \,. \tag{2.3}$$

From the geometric viewpoint, such an object is an area (parallelogram) constructed by two vectors. For this reason, contravariant tensors of the 2nd rank are also known as *bivectors*.

Thus, contravariant tensors of higher ranks are

$$\tilde{A}^{\alpha\ldots\sigma} = A^{\mu\ldots\tau} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \cdots \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\tau}} \,. \tag{2.4}$$

A vector field or a higher rank tensor field are space distributions of the tensor quantities. For instance, because a mechanical strength characterizes both its own magnitude and the direction, its distribution in a physical body can be presented by a vector field.

Covariant tensors of the 1st rank A_{α} are geometric objects, transformable according to the rule

$$\tilde{A}_{\alpha} = A_{\mu} \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \,. \tag{2.5}$$

So, the gradient of a scalar field φ, i. e. the quantity $A_{\alpha} = \frac{\partial \varphi}{\partial x^{\alpha}}$, is a covariant tensor of the 1st rank. That is, because for a regular invariant we have $\tilde{\varphi} = \varphi$, then

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{x}^{\alpha}} = \frac{\partial \tilde{\varphi}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} = \frac{\partial \varphi}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \,. \tag{2.6}$$

Covariant tensors of the 2nd rank $A_{\alpha\beta}$ are geometric objects with transformation rule

$$\tilde{A}_{\alpha\beta} = A_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}} \,. \tag{2.7}$$

Hence, covariant tensors of higher ranks are

$$\tilde{A}_{\alpha\ldots\sigma} = A_{\mu\ldots\tau} \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \cdots \frac{\partial x^{\tau}}{\partial \tilde{x}^{\sigma}} \,. \tag{2.8}$$

Mixed tensors are tensors of the 2nd rank or of higher ranks with both upper and lower indices. For instance, any mixed symmetric tensor A^{α}_{β} is a geometric object, transformable according to the rule

$$\tilde{A}^{\alpha}_{\beta} = A^{\mu}_{\nu} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}} \,. \tag{2.9}$$

Tensor objects exist both in metric and non-metric spaces*. Any tensor has a^n components, where a is its dimension and n is the rank. For instance, a four-dimensional tensor of zero rank has 1 component, a tensor of the 1st rank has 4 components, a tensor of the 2nd rank has 16 components and so on.

Indices in a geometric object, marking its axial components, are found not in tensors only, but in other geometric objects as well. For this reason, if we come across a quantity in by-component notation, this is not necessarily a tensor quantity.

In practice, to know whether a given object is a tensor or not, we need to know a formula for this object in a reference frame and to transform it to any other reference frame. For instance, we consider this classic question: are Christoffel's symbols (i. e. the space coherence coefficients) tensors?

To answer this question, we need to calculate the quantities in a tilde-marked reference frame

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \tilde{g}^{\alpha\sigma}\,\widetilde{\Gamma}_{\mu\nu,\sigma}\,, \quad \widetilde{\Gamma}_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^{\nu}} + \frac{\partial\tilde{g}_{\nu\sigma}}{\partial\tilde{x}^{\mu}} - \frac{\partial\tilde{g}_{\mu\nu}}{\partial\tilde{x}^{\sigma}}\right) \tag{2.10}$$

proceeding from the quantities in a non-marked reference frame.

We calculate the terms in the brackets (2.10). The fundamental metric tensor like any other covariant tensor of the 2nd rank, is transformable to the tilde-marked reference frame according to the rule

$$\tilde{g}_{\mu\sigma} = g_{\varepsilon\tau}\,\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\,. \tag{2.11}$$

Because the $g_{\varepsilon\tau}$ depends on non-tilde-marked coordinates, its derivative with respect to tilde-marked coordinates (which are functions of non-tilded ones) is calculated according to the rule

$$\frac{\partial g_{\varepsilon\tau}}{\partial\tilde{x}^{\nu}} = \frac{\partial g_{\varepsilon\tau}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\,. \tag{2.12}$$

Then the first term in the brackets (2.10), taking the rule of transformation of the fundamental metric tensor into account, is

$$\frac{\partial\tilde{g}_{\mu\sigma}}{\partial\tilde{x}^{\nu}} = \frac{\partial g_{\varepsilon\tau}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial\tilde{x}^{\nu}}\frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}} + g_{\varepsilon\tau}\left(\frac{\partial x^{\tau}}{\partial\tilde{x}^{\sigma}}\frac{\partial^2 x^{\varepsilon}}{\partial\tilde{x}^{\nu}\partial\tilde{x}^{\mu}} + \frac{\partial x^{\varepsilon}}{\partial\tilde{x}^{\mu}}\frac{\partial^2 x^{\tau}}{\partial\tilde{x}^{\nu}\partial\tilde{x}^{\sigma}}\right). \tag{2.13}$$

*In non-metric spaces, as it is known, the distance between any two points can not be measured. This is in contrast to metric spaces. In theories of space-time-matter, such as the General Theory of Relativity and its extensions, metric spaces are taken under consideration. This is because the core of the theories is measurement for time durations and spatial lengths, that is nonsense in a non-metric space.

Hence, calculating the remaining terms of the tilde-marked Christoffel symbols (2.10), after transposition of free indices we obtain

$$\widetilde{\Gamma}_{\mu\nu,\sigma} = \Gamma_{\varepsilon\rho,\tau}\,\frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial \tilde{x}^{\nu}}\frac{\partial x^{\tau}}{\partial \tilde{x}^{\sigma}} + g_{\varepsilon\tau}\,\frac{\partial x^{\tau}}{\partial \tilde{x}^{\sigma}}\frac{\partial^2 x^{\varepsilon}}{\partial \tilde{x}^{\mu}\partial \tilde{x}^{\nu}}\,, \tag{2.14}$$

$$\widetilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\gamma}_{\varepsilon\rho}\,\frac{\partial \tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\mu}}\frac{\partial x^{\rho}}{\partial \tilde{x}^{\nu}} + \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\gamma}}\frac{\partial^2 x^{\gamma}}{\partial \tilde{x}^{\mu}\partial \tilde{x}^{\nu}}\,. \tag{2.15}$$

So, we see that the Christoffel symbols are not transformed in the same way as tensors, hence they are not tensors.

Tensors can be represented as matrices. But in practice, this form may be possible for only tensors of the 1st or 2nd rank (single-row and flat matrices, respectively). For instance, the tensor of an elementary four-dimensional displacement is

$$dx^{\alpha} = \left(dx^0,\, dx^1,\, dx^2,\, dx^3\right), \tag{2.16}$$

while the four-dimensional fundamental metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{2.17}$$

Tensors of the 3rd rank are three-dimensional matrices. Representing tensors of higher ranks as matrices is more problematic.

Now we turn to tensor algebra — a section of tensor calculus, which focuses on algebraic operations over tensors.

Only same-type tensors of the same rank with indices in the same position can be added or subtracted. *Adding up* two same-type tensors of the n-rank gives a new tensor of the same type and rank with components being sums of respective components of the tensors added up. For instance

$$A^{\alpha} + B^{\alpha} = D^{\alpha}, \qquad A^{\alpha}_{\beta} + B^{\alpha}_{\beta} = D^{\alpha}_{\beta}\,. \tag{2.18}$$

Multiplication is permitted not only for same-type, but for any tensors of any ranks. *External multiplication* of tensors of n-rank and m-rank gives a tensor of $(n+m)$-rank

$$A_{\alpha\beta}\,B_{\gamma} = D_{\alpha\beta\gamma}\,, \qquad A_{\alpha}\,B^{\beta\gamma} = D^{\beta\gamma}_{\alpha}. \tag{2.19}$$

Contraction is multiplication of the same-rank tensors, when indices are the same. Contraction of tensors by all indices gives scalar

$$A_{\alpha}\,B^{\alpha} = C\,, \qquad A^{\gamma}_{\alpha\beta}\,B^{\alpha\beta}_{\gamma} = D\,. \tag{2.20}$$

Often multiplication of tensors implies contraction of some indices. Such multiplication is known as *internal multiplication*, which implies contraction of some indices inside the multiplication. This is an example of internal multiplication

$$A_{\alpha\sigma}B^{\sigma} = D_{\alpha}\,, \qquad A^{\gamma}_{\alpha\sigma}B^{\beta\sigma}_{\gamma} = D^{\beta}_{\alpha}\,. \tag{2.21}$$

Using internal multiplication of geometric objects we can determine whether they are tensors or not. This is the so-called theorem of fractions, which is given here according to [9]:

THEOREM OF FRACTIONS
If $B^{\sigma\beta}$ is a tensor and its internal multiplication with a geometric object $A\,(\alpha,\sigma)$ is a tensor $D\,(\alpha,\beta)$

$$A\,(\alpha,\sigma)\,B^{\sigma\beta} = D\,(\alpha,\beta)\,, \tag{2.22}$$

then this object $A\,(\alpha,\sigma)$ is also a tensor.

According to the theorem, if internal multiplication of an object $A_{\alpha\sigma}$ with a tensor $B^{\sigma\beta}$ gives a tensor D^{β}_{α}

$$A_{\alpha\sigma}B^{\sigma\beta} = D^{\beta}_{\alpha}\,, \tag{2.23}$$

then this object $A_{\alpha\sigma}$ is a tensor. Or, if internal multiplication of an object A^{α}_{σ} and a tensor $B^{\sigma\beta}$ gives a tensor $D^{\alpha\beta}$

$$A^{\alpha\cdot}_{\cdot\sigma}B^{\sigma\beta} = D^{\alpha\beta}, \tag{2.24}$$

then the object $A^{\alpha\cdot}_{\cdot\sigma}$ is a tensor.

Geometric properties of any metric space are defined by its fundamental metric tensor $g_{\alpha\beta}$, which can lower or lift indices in objects of this metric space*. For instance,

$$g_{\alpha\beta}A^{\beta} = A_{\alpha}\,, \qquad g^{\mu\nu}g^{\sigma\rho}A_{\mu\nu\sigma} = A^{\rho}. \tag{2.25}$$

In Riemannian spaces, the mixed fundamental metric tensor g^{β}_{α} equals the unit tensor $g^{\beta}_{\alpha} = g_{\alpha\sigma}g^{\sigma\beta} = \delta^{\beta}_{\alpha}$. Diagonal components of the unit tensor are units, while the rest are zeroes. Using the unit tensor we can replace indices in four-dimensional quantities, so that

$$\delta^{\beta}_{\alpha}A_{\beta} = A_{\alpha}\,, \qquad \delta^{\nu}_{\mu}\delta^{\sigma}_{\rho}A^{\mu\rho} = A^{\nu\sigma}. \tag{2.26}$$

*In Riemannian spaces the metric has square form $ds^2 = g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}$, known also as the *Riemannian metric form*, so the fundamental metric tensor of a Riemannian space is the tensor of the 2nd rank, $g_{\alpha\beta}$.

Contraction of any tensor of the 2nd rank with the fundamental metric tensor gives a scalar quantity, known as the *tensor spur* or its *trace*

$$g^{\alpha\beta} A_{\alpha\beta} = A_{\sigma}^{\sigma}\,, \tag{2.27}$$

For instance, the spur of the fundamental metric tensor in a four-dimensional pseudo-Riemannian space of the signature $(+---)$ is

$$g_{\alpha\beta}\, g^{\alpha\beta} = g_{\sigma}^{\sigma} = g_0^0 + g_1^1 + g_2^2 + g_3^3 = -2. \tag{2.28}$$

The metric chr.inv.-tensor h_{ik} (1.27) possesses all properties of the fundamental metric tensor $g_{\alpha\beta}$ in the observer's three-dimensional space. Therefore, h_{ik} can lower, lift or replace indices in chr.inv.-quantities. Respectively, the spur of a three-dimensional chr.inv.-tensor is obtained by means of its contraction with the metric chr.inv.-tensor h_{ik}.

For instance, the spur of the tensor of the rate of the space deformations D_{ik} (1.40) is

$$h^{ik} D_{ik} = D_m^m\,, \tag{2.29}$$

stands for the rate of relative expansion of an elementary volume of the space.

Of course, this brief account can not fully cover such a vast field like tensor algebra. Moreover, there is even no need in doing that here. Detailed accounts of tensor algebra can be found in numerous mathematical books not related to the General Theory of Relativity. Besides, many specific techniques of this science, which occupy substantial part of mathematical textbooks, are not used in theoretical physics. Therefore our goal was to give only a basic introduction into tensors and tensor algebra, necessary for understanding this book. For the same reasons we have not covered issues like weight of tensors or many others not used in calculations in this book.

§ 2.2 Scalar product of vectors

Scalar product of two vectors A^{α} and B^{α} in a four-dimensional pseudo-Riemannian space is

$$g_{\alpha\beta}\, A^{\alpha} B^{\beta} = A_{\alpha} B^{\alpha} = A_0 B^0 + A_i B^i. \tag{2.30}$$

Scalar product is a contraction, because multiplication of vectors contracts all indices at the same time. Therefore, scalar product of two vectors (tensors of the 1st rank) is always scalar (tensor of zero rank). If both vectors are the same, their scalar product

$$g_{\alpha\beta}\, A^{\alpha} A^{\beta} = A_{\alpha} A^{\alpha} = A_0 A^0 + A_i A^i \tag{2.31}$$

is the square of the given vector A^α. Consequently, the length of this vector A^α is

$$A = |A^\alpha| = \sqrt{g_{\alpha\beta}A^\alpha A^\beta}. \tag{2.32}$$

Because the four-dimensional pseudo-Riemannian space by its definition has the sign-alternating metric (the sign-alternating signature $(+---)$ or $(-+++)$), then lengths of four-dimensional vectors may be real, imaginary or zero. Vectors with non-zero (real or imaginary) lengths are known as *non-isotropic*. Vectors with zero length are known as *isotropic*. Isotropic vectors are tangential to trajectories of light-like particles (isotropic trajectories).

In three-dimensional Euclidean space, scalar product of two vectors is a scalar quantity with magnitude equal to the product of their lengths, multiplied by cosine of the angle between them

$$A_iB^i = |A^i|\,|B^i|\cos\left(A^i;B^i\right). \tag{2.33}$$

Theoretically, at every point of any Riemannian space a tangential flat space can be set, whose basic vectors will be tangential to basic vectors of the Riemannian space at this point. Then, the metric of the tangential flat space will be the metric of the Riemannian space at this point. Therefore, this statement is also true in the Riemannian space, if we consider the angle between coordinate lines and replace Roman (three-dimensional) indices with Greek (four-dimensional) ones.

From here, we can see that the scalar product of two vectors is zero, if the vectors are orthogonal. In other words, scalar product from geometric viewpoint is the projection of one vector on the other. If the vectors are the same, then the vector is projected on itself, so the result of this projection is the square of its length.

Denote chr.inv.-projections of arbitrary vectors A^α and B^α as follows

$$a = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad a^i = A^i, \tag{2.34}$$

$$b = \frac{B_0}{\sqrt{g_{00}}}\,, \qquad b^i = B^i, \tag{2.35}$$

then their remaining components are

$$A^0 = \frac{a + \frac{1}{c}\,v_i\,a^i}{1-\frac{\mathrm{w}}{c^2}}\,, \qquad A_i = -\,a_i - \frac{a}{c}\,v_i\,, \tag{2.36}$$

$$B^0 = \frac{b + \frac{1}{c}\,v_i\,b^i}{1-\frac{\mathrm{w}}{c^2}}\,, \qquad B_i = -\,b_i - \frac{b}{c}\,v_i\,. \tag{2.37}$$

Substituting the chr.inv.-projections into the formulae for $A_\alpha B^\alpha$ and $A_\alpha A^\alpha$, we obtain

$$A_\alpha B^\alpha = ab - a_i b^i = ab - h_{ik}\, a^i b^k, \tag{2.38}$$

$$A_\alpha A^\alpha = a^2 - a_i a^i = a^2 - h_{ik}\, a^i a^k. \tag{2.39}$$

From here, we see that the square of any vector's length is the difference between the squares of the lengths of its time and spatial chr.inv.-projections. If both projections are equal, then the vector's length is zero, so the vector is isotropic. Hence, any isotropic vector equally belongs to the time line and the spatial section. Equality of the time and spatial chr.inv.-projections also implies that the vector is orthogonal to itself. If its time projection is "longer", then the vector is real. If the spatial projection is "longer", then the vector is imaginary.

Scalar product of any four-dimensional vector with itself can be illustrated by the square of the length of the space-time interval

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = dx_\alpha dx^\alpha = dx_0 dx^0 + dx_i dx^i. \tag{2.40}$$

In terms of physical observable quantities, it can be represented as follows

$$ds^2 = c^2 d\tau^2 - dx_i dx^i = c^2 d\tau^2 - h_{ik}\, dx^i dx^k = c^2 d\tau^2 - d\sigma^2. \tag{2.41}$$

Its length $ds = \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta}$ may be real, imaginary or zero, depending on whether ds is time-like $c^2 d\tau^2 > d\sigma^2$ (sub-light real trajectories), space-like $c^2 d\tau^2 < d\sigma^2$ (imaginary super-light trajectories), or isotropic $c^2 d\tau^2 = d\sigma^2$ (light-like trajectories).

§2.3 Vector product of vectors. Antisymmetric tensors. Pseudotensors

Vector product of two vectors A^α and B^α is a tensor of the 2nd rank $V^{\alpha\beta}$, obtained from their external multiplication according to the specific rule

$$V^{\alpha\beta} = \left[A^\alpha ; B^\beta\right] = \frac{1}{2}\left(A^\alpha B^\beta - A^\beta B^\alpha\right) = \frac{1}{2}\begin{vmatrix} A^\alpha & A^\beta \\ B^\alpha & B^\beta \end{vmatrix}. \tag{2.42}$$

As it is easy to see, the order in which vectors are multiplied matters, i.e. the order in which we write down tensor indices is important. Therefore, tensors obtained as vector products are *antisymmetric*. In an antisymmetric tensor $V^{\alpha\beta} = -V^{\beta\alpha}$; indices being moved "reserve" their places as dots, $g_{\alpha\sigma} V^{\sigma\beta} = V_{\alpha\cdot}^{\cdot\beta}$, thus showing from where the index was moved. In symmetric tensors there is no need to "reserve" places for moved indices, because the order in which they appear does not matter. In particular, the fundamental metric tensor is symmetric $g_{\alpha\beta}=g_{\beta\alpha}$, while the tensor of the space curvature $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is symmetric in respect to transposition by pair of its indices and is antisymmetric inside each pair of the indices. It is evident that, only tensors of the 2nd rank or of higher ranks may be symmetric or antisymmetric.

All diagonal components of any antisymmetric tensor by its definition are zeroes. For instance, in an antisymmetric tensor of the 2nd rank we have

$$V^{\alpha\alpha}=[A^{\alpha};B^{\alpha}]=\frac{1}{2}\left(A^{\alpha}B^{\alpha}-A^{\alpha}B^{\alpha}\right)=0\,. \tag{2.43}$$

In a three-dimensional Euclidean space, the numerical value of the vector product of two vectors is defined as the area of the parallelogram they make and equals the product of their moduli, multiplied by sine of the angle between them

$$V^{ik}=|\,A^{i}\,|\,|\,B^{k}\,|\sin\left(A^{i};B^{k}\right). \tag{2.44}$$

This implies that the vector product of two vectors (i. e. an antisymmetric tensor of the 2nd rank) is a pad, oriented in the space according to the directions of its forming vectors.

Contraction of an antisymmetric tensor $V_{\alpha\beta}$ with any symmetric tensor $A^{\alpha\beta}=A^{\alpha}A^{\beta}$ is zero, because $V_{\alpha\alpha}=0$ and $V_{\alpha\beta}=-V_{\beta\alpha}$ so that we have

$$V_{\alpha\beta}A^{\alpha}A^{\beta}=V_{00}A^{0}A^{0}+V_{0i}A^{0}A^{i}+V_{i0}A^{i}A^{0}+V_{ik}A^{i}A^{k}=0\,. \tag{2.45}$$

According to the theory of chronometric invariants, chr.inv.-projections of an antisymmetric tensor of the 2nd rank $V^{\alpha\beta}$ are

$$\frac{V_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}=-\frac{V_{\cdot 0}^{i\cdot}}{\sqrt{g_{00}}}=\frac{1}{2}\left(a\,b^{i}-b\,a^{i}\right), \tag{2.46}$$

$$V^{ik}=\frac{1}{2}\left(a^{i}b^{k}-a^{k}b^{i}\right), \tag{2.47}$$

where the third chr.inv.-projection $\frac{V_{00}}{g_{00}}$ (1.32) is zero, because in any antisymmetric tensor all diagonal components are zeroes.

Physical observable components V^{ik} (the projections of $V^{\alpha\beta}$ on the observer's spatial section) are analogous to a vector product in a three-dimensional space, while the quantity $\frac{V_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}$, which is the space-time (mixed) projection of the tensor $V^{\alpha\beta}$, has no equivalent among components of a regular three-dimensional vector product.

The square of an antisymmetric tensor of the 2nd rank, formulated with chr.inv.-projections of its forming vectors, is

$$V_{\alpha\beta}V^{\alpha\beta} = \frac{1}{2}\left(a_i a^i b_k b^k - a_i b^i a_k b^k\right) + ab a_i b^i - \frac{1}{2}\left(a^2 b_i b^i - b^2 a_i a^i\right). \quad (2.48)$$

The last two terms in this formula contain quantities a (2.34) and b (2.35), which are chr.inv.-projections of the multiplied vectors A^α and B^α on the observer's time line, so the terms have no equivalent in a vector product in a three-dimensional Euclidean space.

Asymmetry of tensor fields is defined by reference antisymmetric tensors. In a Galilean reference frame* such references are Levi-Civita's tensors. For four-dimensional quantities, this is the *four-dimensional completely antisymmetric unit tensor* $e^{\alpha\beta\mu\nu}$, while for three-dimensional quantities, this is the *three-dimensional completely antisymmetric unit tensor* e^{ikm}. Components of the Levi-Civita tensors, which have all indices different, are either $+1$ or -1 depending on the number of transpositions of their indices. All the remaining components, i. e. those having at least two coinciding indices, are zeroes. Moreover, for the signature $(+---)$ we are using all non-zero components having a sign opposite to their respective covariant components†. For instance, in the Minkowski space we have

$$\left.\begin{aligned} g_{\alpha\sigma}g_{\beta\rho}g_{\mu\tau}g_{\nu\gamma}e^{\sigma\rho\tau\gamma} &= g_{00}g_{11}g_{22}g_{33}e^{0123} = -e^{0123} \\ g_{i\alpha}g_{k\beta}g_{m\gamma}e^{\alpha\beta\gamma} &= g_{11}g_{22}g_{33}e^{123} = -e^{123} \end{aligned}\right\}, \quad (2.49)$$

because of the signature conditions $g_{00}=1$ and $g_{11}=g_{22}=g_{33}=-1$ we have accepted. Therefore, components of the tensor $e^{\alpha\beta\mu\nu}$ are

$$\left.\begin{aligned} &e^{0123}=+1, \quad e^{1023}=-1, \quad e^{1203}=+1, \quad e^{1230}=-1 \\ &e_{0123}=-1, \quad e_{1023}=+1, \quad e_{1203}=-1, \quad e_{1230}=+1 \end{aligned}\right\} \quad (2.50)$$

and components of the tensor e^{ikm} are

$$\left.\begin{aligned} &e^{123}=+1, \quad e^{213}=-1, \quad e^{231}=+1 \\ &e_{123}=-1, \quad e_{213}=+1, \quad e_{231}=-1 \end{aligned}\right\}. \quad (2.51)$$

---

*A Galilean frame of reference is the one that does not rotate, is not subject to deformation and falls freely in the flat space-time of the Special Theory of Relativity (the Minkowski space). The lines of time are linear and so are three-dimensional coordinate axes.

†If the space-time signature is $(-+++)$, this is true for only the four-dimensional tensor $e^{\alpha\beta\mu\nu}$. Components of the three-dimensional tensor e^{ikm} will have the same sign as well as the respective components of e_{ikm}.

Because we have an arbitrary choice for the sign of the first component, we assume $e^{0123}=-1$ and $e^{123}=-1$. Consequently, the remaining components will change. In general, the tensor $e^{\alpha\beta\mu\nu}$ is related to the tensor e^{ikm} as follows $e^{0ikm}=e^{ikm}$.

Multiplying the four-dimensional antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$ by itself we obtain a regular tensor of the 8th rank with non-zero components, which are given in the matrix

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\rho\gamma} = -\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} & \delta^{\alpha}_{\rho} & \delta^{\alpha}_{\gamma} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} & \delta^{\beta}_{\rho} & \delta^{\beta}_{\gamma} \\ \delta^{\mu}_{\sigma} & \delta^{\mu}_{\tau} & \delta^{\mu}_{\rho} & \delta^{\mu}_{\gamma} \\ \delta^{\nu}_{\sigma} & \delta^{\nu}_{\tau} & \delta^{\nu}_{\rho} & \delta^{\nu}_{\gamma} \end{pmatrix}. \tag{2.52}$$

The remaining properties of the tensor $e^{\alpha\beta\mu\nu}$ are derived from the previous by means of contraction of indices

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\rho\nu} = -\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} & \delta^{\alpha}_{\rho} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} & \delta^{\beta}_{\rho} \\ \delta^{\mu}_{\sigma} & \delta^{\mu}_{\tau} & \delta^{\mu}_{\rho} \end{pmatrix}, \tag{2.53}$$

$$e^{\alpha\beta\mu\nu}e_{\sigma\tau\mu\nu} = -2\begin{pmatrix} \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} \\ \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} \end{pmatrix} = -2\left(\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} - \delta^{\beta}_{\sigma}\delta^{\alpha}_{\tau}\right), \tag{2.54}$$

$$e^{\alpha\beta\mu\nu}e_{\sigma\beta\mu\nu} = -6\delta^{\alpha}_{\sigma}\,, \quad e^{\alpha\beta\mu\nu}e_{\alpha\beta\mu\nu} = -6\delta^{\alpha}_{\alpha} = -24. \tag{2.55}$$

Multiplying the three-dimensional antisymmetric unit tensor e^{ikm} by itself we obtain a regular tensor of the 6th rank

$$e^{ikm}e_{rst} = \begin{pmatrix} \delta^{i}_{r} & \delta^{i}_{s} & \delta^{i}_{t} \\ \delta^{k}_{r} & \delta^{k}_{s} & \delta^{k}_{t} \\ \delta^{m}_{r} & \delta^{m}_{s} & \delta^{m}_{t} \end{pmatrix}. \tag{2.56}$$

The remaining properties of the tensor e^{ikm} are

$$e^{ikm}e_{rsm} = -\begin{pmatrix} \delta^{i}_{r} & \delta^{i}_{s} \\ \delta^{k}_{r} & \delta^{k}_{s} \end{pmatrix} = \delta^{i}_{s}\delta^{k}_{r} - \delta^{i}_{r}\delta^{k}_{s}\,, \tag{2.57}$$

$$e^{ikm}e_{rkm} = 2\delta^{i}_{r}\,, \quad e^{ikm}e_{ikm} = 2\delta^{i}_{i} = 6. \tag{2.58}$$

The completely antisymmetric unit tensor defines for a tensor object its respective *pseudotensor*, marked with asterisk. For instance, any four-dimensional scalar, vector and tensors of the 2nd, 3rd, and 4th ranks have respective four-dimensional pseudotensors of the following ranks

$$\left.\begin{aligned} V^{*\alpha\beta\mu\nu}=e^{\alpha\beta\mu\nu}V, \quad V^{*\alpha\beta\mu}=e^{\alpha\beta\mu\nu}V_{\nu}\,, \quad V^{*\alpha\beta}=\frac{1}{2}\,e^{\alpha\beta\mu\nu}V_{\mu\nu} \\ V^{*\alpha}=\frac{1}{6}\,e^{\alpha\beta\mu\nu}V_{\beta\mu\nu}\,, \quad V^{*}=\frac{1}{24}\,e^{\alpha\beta\mu\nu}V_{\alpha\beta\mu\nu} \end{aligned}\right\}, \tag{2.59}$$

where pseudotensors of the 1st rank $V^{*\alpha}$ are called *pseudovectors*, while pseudotensors of zero rank V^{*} are called *pseudoscalars*. Any tensor and its respective pseudotensor are known as *dual* to each other to emphasize their common genesis. So, three-dimensional tensors have respective three-dimensional pseudotensors

$$\left.\begin{aligned} V^{*ikm} &= e^{ikm}V, \qquad V^{*ik} = e^{ikm}V_{m} \\ V^{*i} &= \frac{1}{2}\,e^{ikm}V_{km}\,, \quad V^{*} = \frac{1}{6}\,e^{ikm}V_{ikm} \end{aligned}\right\}. \tag{2.60}$$

Pseudotensors are called such because, in contrast to regular tensors, they do not change when reflected with respect to one of the axes. For instance, when reflected with respect to the abscissa axis $x^1=-\tilde{x}^1$, $x^2=\tilde{x}^2$, $x^3=\tilde{x}^3$. The reflected component of an antisymmetric tensor V_{ik}, orthogonal to x^1, is $\widetilde{V}_{23}=-V_{23}$, while its dual component of the pseudovector V^{*i} is

$$\left.\begin{aligned} V^{*1}=\frac{1}{2}\,e^{1km}V_{km}=\frac{1}{2}\left(e^{123}V_{23}+e^{132}V_{32}\right)=V_{23} \\ \widetilde{V}^{*1}=\frac{1}{2}\,\tilde{e}^{1km}\widetilde{V}_{km}=\frac{1}{2}\,e^{k1m}\widetilde{V}_{km}=\frac{1}{2}\left(e^{213}\widetilde{V}_{23}+e^{312}\widetilde{V}_{32}\right)=V_{23} \end{aligned}\right\}. \tag{2.61}$$

Because a four-dimensional antisymmetric tensor of the 2nd rank and its dual pseudotensor are of the same rank, their contraction yields a pseudoscalar, so that

$$V_{\alpha\beta}V^{*\alpha\beta} = V_{\alpha\beta}\,e^{\alpha\beta\mu\nu}V_{\mu\nu} = e^{\alpha\beta\mu\nu}B_{\alpha\beta\mu\nu} = B^{*}. \tag{2.62}$$

The square of a pseudotensor $V^{*\alpha\beta}$ and the square of a pseudovector V^{*i}, expressed through their dual tensors, are

$$V_{*\alpha\beta}V^{*\alpha\beta} = e_{\alpha\beta\mu\nu}V^{\mu\nu}e^{\alpha\beta\rho\sigma}V_{\rho\sigma} = -24V_{\mu\nu}V^{\mu\nu}, \tag{2.63}$$

$$V_{*i}V^{*i} = e_{ikm}V^{km}e^{ipq}V_{pq} = 6V_{km}V^{km}. \tag{2.64}$$

In inhomogeneous anisotropic pseudo-Riemannian spaces, we can not set a Galilean reference frame, so references of asymmetry of tensor fields will depend on inhomogeneity and anisotropy of the space itself, which are defined by the fundamental metric tensor. In this general case, a reference antisymmetric tensor is the *four-dimensional completely antisymmetric discriminant tensor*

$$E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}\,, \qquad E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}\,. \tag{2.65}$$

Here is the proof. Transformation of the unit completely antisymmetric tensor from a Galilean (non-tilde-marked) reference frame into an arbitrary (tilde-marked) reference frame is

$$\tilde{e}_{\alpha\beta\mu\nu} = \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\alpha}}\frac{\partial x^{\gamma}}{\partial \tilde{x}^{\beta}}\frac{\partial x^{\varepsilon}}{\partial \tilde{x}^{\mu}}\frac{\partial x^{\tau}}{\partial \tilde{x}^{\nu}}\, e_{\sigma\gamma\varepsilon\tau} = J e_{\alpha\beta\mu\nu}\,, \tag{2.66}$$

where $J = \det\left\|\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\sigma}}\right\|$ is called the *Jacobian of the transformation* (the determinant of Jacobi's matrix)

$$J = \det\begin{Vmatrix} \frac{\partial x^0}{\partial \tilde{x}^0} & \frac{\partial x^0}{\partial \tilde{x}^1} & \frac{\partial x^0}{\partial \tilde{x}^2} & \frac{\partial x^0}{\partial \tilde{x}^3} \\ \frac{\partial x^1}{\partial \tilde{x}^0} & \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \frac{\partial x^1}{\partial \tilde{x}^3} \\ \frac{\partial x^2}{\partial \tilde{x}^0} & \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \frac{\partial x^2}{\partial \tilde{x}^3} \\ \frac{\partial x^3}{\partial \tilde{x}^0} & \frac{\partial x^3}{\partial \tilde{x}^1} & \frac{\partial x^3}{\partial \tilde{x}^2} & \frac{\partial x^3}{\partial \tilde{x}^3} \end{Vmatrix}. \tag{2.67}$$

Because the fundamental metric tensor $g_{\alpha\beta}$ is transformable according to the rule

$$\tilde{g}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}}\, g_{\mu\nu}\,, \tag{2.68}$$

its determinant in the tilde-marked reference frame is

$$\tilde{g} = \det\left\|\frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}}\, g_{\mu\nu}\right\| = J^2 g\,. \tag{2.69}$$

Because in the Galilean (non-tilde-marked) reference frame

$$g = \det\|g_{\alpha\beta}\| = \det\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} = -1, \tag{2.70}$$

then $J^2 = -\tilde{g}^2$. Expressing $\tilde{e}_{\alpha\beta\mu\nu}$ in an arbitrary reference frame as $E_{\alpha\beta\mu\nu}$ and writing down the metric tensor in a regular non-tilde-marked form, we obtain $E_{\alpha\beta\mu\nu}=e_{\alpha\beta\mu\nu}\sqrt{-g}$ (2.65). In the same way, we obtain transformation rules for the components $E^{\alpha\beta\mu\nu}$, because for them $g=\tilde{g}\,\tilde{J}^2$, where $\tilde{J}=\det\left\|\frac{\partial\tilde{x}^\alpha}{\partial x^\sigma}\right\|$.

The discriminant tensor $E^{\alpha\beta\mu\nu}$ is not a physical observable quantity. A physical observable reference of asymmetry of tensor fields is the *three-dimensional discriminant chr.inv.-tensor*

$$\varepsilon^{\alpha\beta\gamma}=h^\alpha_\mu h^\beta_\nu h^\gamma_\rho b_\sigma E^{\sigma\mu\nu\rho}=b_\sigma E^{\sigma\alpha\beta\gamma}, \tag{2.71}$$

$$\varepsilon_{\alpha\beta\gamma}=h^\mu_\alpha h^\nu_\beta h^\rho_\gamma b^\sigma E_{\sigma\mu\nu\rho}=b^\sigma E_{\sigma\alpha\beta\gamma}\,, \tag{2.72}$$

which in the accompanying reference frame ($b^i=0$), taking into account that $\sqrt{-g}=\sqrt{h}\,\sqrt{g_{00}}$, takes the form

$$\varepsilon^{ikm}=b_0 E^{0ikm}=\sqrt{g_{00}}\,E^{0ikm}=\frac{e^{ikm}}{\sqrt{h}}\,, \tag{2.73}$$

$$\varepsilon_{ikm}=b^0 E_{0ikm}=\frac{E_{0ikm}}{\sqrt{g_{00}}}=e_{ikm}\sqrt{h}\,. \tag{2.74}$$

With its help, we can transform chr.inv.-tensors into chr.inv.-pseudotensors. For instance, taking the antisymmetric chr.inv.-tensor of angular velocities of the space rotation A_{ik} (1.36), we obtain the chr.inv.-pseudovector of this rotation $\Omega^{*i}=\frac{1}{2}\,\varepsilon^{ikm}A_{km}$.

§2.4 Differential and derivative to a direction

In geometry, a *differential* of a function is its variation between two infinitely close points with coordinates x^α and $x^\alpha+dx^\alpha$. Respectively, the *absolute differential* in an n-dimensional space is the variation of an n-dimensional quantity between two infinitely close points of n-dimensional coordinates in this space. For continuous functions, which we commonly deal with in practice, their variations between infinitely close points are infinitesimal. But in order to define infinitesimal variation of a tensor quantity, we can not use simple "difference" between its numerical values in the points x^α and $x^\alpha+dx^\alpha$, because tensor algebra does not define the ratio between the numerical values of a tensor in different points in space. This ratio can be defined only using rules of transformation of tensors from one reference frame into another. As a consequence, differential operators and the results of their application to tensors must be tensors.

For instance, the absolute differential of a tensor quantity is a tensor of the same rank as the original tensor itself. For a scalar φ it is the scalar

$$\mathrm{D}\varphi = \frac{\partial\varphi}{\partial x^\alpha}\,dx^\alpha, \tag{2.75}$$

which in the accompanying reference frame ($b^i = 0$) is

$$\mathrm{D}\varphi = \frac{^*\partial\varphi}{\partial t}\,d\tau + \frac{^*\partial\varphi}{\partial x^i}\,dx^i. \tag{2.76}$$

As it is easy to see, apart from three-dimensional observable differential there is an additional term which takes into account the dependence of the absolute displacement $\mathrm{D}\varphi$ on the flow of physical observable time $d\tau$.

The absolute differential of a contravariant vector A^α, formulated with the operator of absolute derivation ∇ (nabla), is

$$\mathrm{D}A^\alpha = \nabla_\sigma\,A^\alpha dx^\sigma = \frac{\partial A^\alpha}{\partial x^\sigma}\,dx^\sigma + \Gamma^\alpha_{\mu\sigma}A^\mu dx^\sigma = dA^\alpha + \Gamma^\alpha_{\mu\sigma}A^\mu dx^\sigma, \tag{2.77}$$

where $\nabla_\sigma\,A^\alpha$ is the absolute derivative of A^α with respect to x^σ, and d stands for regular differentials

$$\nabla_\sigma\,A^\alpha = \frac{\partial A^\alpha}{\partial x^\sigma} + \Gamma^\alpha_{\mu\sigma}A^\mu, \tag{2.78}$$

$$d = \frac{\partial}{\partial x^\alpha}\,dx^\alpha. \tag{2.79}$$

Formulating the absolute differential with physical observable quantities is equivalent to projecting its general covariant form on time lines and the spatial section in the accompanying reference frame

$$T = b_\alpha\,\mathrm{D}A^\alpha = \frac{g_{0\alpha}\mathrm{D}A^\alpha}{\sqrt{g_{00}}}\,, \qquad B^i = h^i_\alpha \mathrm{D}A^\alpha. \tag{2.80}$$

Denoting chr.inv.-projections of the vector A^α as

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i, \tag{2.81}$$

we have its remaining components

$$A_0 = \varphi\left(1 - \frac{\mathrm{w}}{c^2}\right), \quad A^0 = \frac{\varphi + \frac{1}{c}\,v_i q^i}{1 - \frac{\mathrm{w}}{c^2}}\,, \quad A_i = -q_i - \frac{\varphi}{c}\,v_i\,. \tag{2.82}$$

Because a regular differential in chr.inv.-form is

$$d = \frac{^*\partial}{\partial t}\,d\tau + \frac{^*\partial}{\partial x^i}\,dx^i\,, \tag{2.83}$$

after substituting it and the Christoffel symbols, taken in the accompanying reference frame (1.41–1.46), into T and B^i (2.80), we obtain the chr.inv.-projections of the absolute differential of the vector A^α

$$T = b_\alpha \mathrm{D}A^\alpha = d\varphi + \frac{1}{c}\left(-F_i q^i d\tau + D_{ik}\, q^i dx^k\right), \tag{2.84}$$

$$\begin{aligned} B^i = h^i_\sigma\, \mathrm{D}A^\sigma = dq^i + \Big(\frac{\varphi}{c}\, dx^k + q^k d\tau\Big)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \\ - \frac{\varphi}{c}\, F^i d\tau + \Delta^i_{mk}\, q^m dx^k. \end{aligned} \tag{2.85}$$

To build chr.inv.-equations of motion, we will also need chr.inv.-projections of the absolute derivative of a vector to the direction, tangential to the trajectory. From geometric viewpoint a *derivative to a given direction* of a function is its change with respect to elementary displacement along the given direction. The *absolute derivative to the given direction* in an n-dimensional space is a change of an n-dimensional quantity with respect to an elementary n-dimensional interval along the given direction. For instance, the absolute derivative of a scalar function φ to a direction, defined by a curve $x^\alpha = x^\alpha(\rho)$, where ρ is a non-zero monotone parameter along this curve, shows the "rate" of change of this function

$$\frac{\mathrm{D}\varphi}{d\rho} = \frac{d\varphi}{d\rho}\,. \tag{2.86}$$

In the accompanying reference frame it is

$$\frac{\mathrm{D}\varphi}{d\rho} = \frac{^*\partial\varphi}{\partial t}\frac{d\tau}{d\rho} + \frac{^*\partial\varphi}{\partial x^i}\frac{dx^i}{d\rho}\,. \tag{2.87}$$

The absolute derivative of a vector A^α to the given direction of a curve $x^\alpha = x^\alpha(\rho)$ is

$$\frac{\mathrm{D}A^\alpha}{d\rho} = \nabla_\sigma A^\alpha \frac{dx^\sigma}{d\rho} = \frac{dA^\alpha}{d\rho} + \Gamma^\alpha_{\mu\sigma} A^\mu \frac{dx^\sigma}{d\rho}\,, \tag{2.88}$$

its chr.inv.-projections are

$$b_\alpha \frac{\mathrm{D}A^\alpha}{d\rho} = \frac{d\varphi}{d\rho} + \frac{1}{c}\left(-F_i q^i \frac{d\tau}{d\rho} + D_{ik}\, q^i \frac{dx^k}{d\rho}\right), \tag{2.89}$$

$$\begin{aligned} h^i_\sigma \frac{\mathrm{D}A^\sigma}{d\rho} = \frac{dq^i}{d\rho} + \left(\frac{\varphi}{c}\frac{dx^k}{d\rho} + q^k \frac{d\tau}{d\rho}\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \\ - \frac{\varphi}{c}\, F^i \frac{d\tau}{d\rho} + \Delta^i_{mk}\, q^m \frac{dx^k}{d\rho}\,. \end{aligned} \tag{2.90}$$

Actually, the projections are "generic" chr.inv.-equations of motion. But once we define a particular vector for the motion of a particle, we calculate its chr.inv.-projections and substitute them into the given equations, we immediately obtain chr.inv.-equations of the motion.

§2.5 Divergence and curl

The *divergence* of a tensor field is its "change" along a coordinate axis. Respectively, the *absolute divergence* of an n-dimensional tensor field is its divergence in an n-dimensional space. The divergence of a tensor field is a result of contraction of the field tensor with the operator of absolute derivation ∇. The divergence of a vector field is the scalar

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} A^\mu, \tag{2.91}$$

while the divergence of a field of the 2nd rank tensor is the vector

$$\nabla_\sigma F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu} F^{\sigma\mu}, \tag{2.92}$$

where, as it can be proved, $\Gamma^\sigma_{\sigma\mu}$ is

$$\Gamma^\sigma_{\sigma\mu} = \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}\,. \tag{2.93}$$

To prove this, we will use the definition of the Christoffel symbols. Then we write down $\Gamma^\sigma_{\sigma\mu}$ in details

$$\Gamma^\sigma_{\sigma\mu} = g^{\sigma\rho}\Gamma_{\mu\sigma,\rho} = \frac{1}{2}\, g^{\sigma\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^\sigma} + \frac{\partial g_{\sigma\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\rho}\right). \tag{2.94}$$

Because σ and ρ are free indices here, they can change their sites. As a result, after contraction with the tensor $g^{\rho\sigma}$ the first and the last terms cancel each other, so $\Gamma^\sigma_{\sigma\mu}$ takes the form

$$\Gamma^\sigma_{\sigma\mu} = \frac{1}{2}\, g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}\,. \tag{2.95}$$

The quantities $g^{\rho\sigma}$ are components of a tensor reciprocal to the tensor $g_{\rho\sigma}$. Therefore, each component of the matrix $g^{\rho\sigma}$ is

$$g^{\rho\sigma} = \frac{a^{\rho\sigma}}{g}\,, \qquad g = \det\|g_{\rho\sigma}\|\,, \tag{2.96}$$

where $a^{\rho\sigma}$ is the algebraic co-factor of the matrix element with indices $\rho\sigma$, equal to $(-1)^{\rho+\sigma}$, multiplied by the determinant of the matrix obtained by crossing the row and the column with numbers σ and ρ out of the matrix $g_{\rho\sigma}$. As a result, we obtain $a^{\rho\sigma}=gg^{\rho\sigma}$. Because the determinant of the fundamental metric tensor $g=\det\|g_{\rho\sigma}\|$ by definition is

$$g=\sum_{\alpha_0\dots\alpha_3}(-1)^{N(\alpha_0\dots\alpha_3)}\,g_{0(\alpha_0)}g_{1(\alpha_1)}g_{2(\alpha_2)}g_{3(\alpha_3)}\,, \tag{2.97}$$

then the quantity dg will be $dg=a^{\rho\sigma}dg_{\rho\sigma}=gg^{\rho\sigma}dg_{\rho\sigma}$, or

$$\frac{dg}{g}=g^{\rho\sigma}dg_{\rho\sigma}\,. \tag{2.98}$$

Integration of the left hand side gives $\ln(-g)$, because the g is negative while logarithm is defined for only positive functions. Then, we have $d\ln(-g)=\frac{dg}{g}$. Taking into account that $(-g)^{\frac{1}{2}}=\frac{1}{2}\ln(-g)$, we obtain

$$d\ln\sqrt{-g}=\frac{1}{2}\,g^{\rho\sigma}dg_{\rho\sigma}\,, \tag{2.99}$$

so $\Gamma^{\sigma}_{\sigma\mu}$ (2.95) takes the form

$$\Gamma^{\sigma}_{\sigma\mu}=\frac{1}{2}\,g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^{\mu}}=\frac{\partial\ln\sqrt{-g}}{\partial x^{\mu}}\,, \tag{2.100}$$

which has been proved (2.93).

Now, we are going to deduce chr.inv.-projections of the divergence of a vector field (2.91) and of a tensor field of the 2nd rank (2.92). The divergence of a vector field A^{α} is scalar, hence $\nabla_{\sigma}A^{\sigma}$ can not be projected on time lines and the spatial section, but, this is enough to express through chr.inv.-projections of A^{α} and through observable properties of the reference space. Besides, regular operators of derivation shall be replaced with the chr.inv.-operators.

Assuming notations φ and q^i for chr.inv.-projections of the vector A^{α} (2.81), we express the remaining components of the vector through them (2.82). Then, substituting regular operators of derivations, expressed through the chr.inv.-operators

$$\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}=\frac{^*\partial}{\partial t}\,,\qquad \sqrt{g_{00}}=1-\frac{\mathrm{w}}{c^2}\,, \tag{2.101}$$

$$\frac{^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{1}{c^2}\,v_i\frac{^*\partial}{\partial t}\,, \tag{2.102}$$

into (2.91), and taking into account that $\sqrt{-g}=\sqrt{h}\,\sqrt{g_{00}}$ after some algebra we obtain

$$\nabla_{\sigma}A^{\sigma}=\frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t}+\varphi D\right)+\frac{^*\partial q^i}{\partial x^i}+q^i\,\frac{^*\partial\ln\sqrt{h}}{\partial x^i}-\frac{1}{c^2}\,F_i\,q^i. \tag{2.103}$$

In the third term the quantity

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} = \Delta^j_{ji} \tag{2.104}$$

stands for the Christoffel chr.inv.-symbols Δ^k_{ji} (1.47), contracted by two symbols. Hence, similarly, to the definition of the absolute divergence of a vector field (2.91), the quantity

$$\frac{{}^*\partial q^i}{\partial x^i} + q^i \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} = \frac{{}^*\partial q^i}{\partial x^i} + q^i \Delta^j_{ji} = {}^*\nabla_i \, q^i \tag{2.105}$$

is the *chr.inv.-divergence* of a three-dimensional vector field q^i. Consequently, we call the *physical chr.inv.-divergence* of the vector field q^i the chr.inv.-quantity

$${}^*\widetilde{\nabla}_i \, q^i = {}^*\nabla_i \, q^i - \frac{1}{c^2} F_i \, q^i, \tag{2.106}$$

in which the 2nd term takes into account the fact that the pace of time is different on the opposite walls of an elementary volume [9]. As a matter of fact, that in calculation of divergence we consider an elementary volume of the space, so we calculate the difference between the amounts of a "substance" which flows in and out of the volume over an elementary time interval. But the presence of gravitational inertial force F^i (1.38) results in different pace of time at different points in the space. Therefore, if we measure durations of time intervals at the opposite walls of the volume, the beginnings and the ends of the interval will not coincide making them invalid for comparison. Synchronization of clocks at the opposite walls of the volume will give the true picture — the measured durations of the intervals will be different.

The final equation for $\nabla_\sigma \, A^\sigma$ will be

$$\nabla_\sigma \, A^\sigma = \frac{1}{c}\left(\frac{{}^*\partial \varphi}{\partial t} + \varphi D\right) + {}^*\widetilde{\nabla}_i \, q^i. \tag{2.107}$$

The second term in this formula is a physical observable analogous to a regular divergence in the observer's three-dimensional space. The first term has no equivalent, it is made up of two parts: $\frac{{}^*\partial \varphi}{\partial t}$ is the variation in time of the time projection φ of the vector A^α, while $D\,\varphi$ is the variation in time of a volume of the three-dimensional vector field q^i, because the spur of the chr.inv.-tensor of the rate of the space deformations $D = h^{ik} D_{ik} = D^n_n$ is the rate of relative expansion of an elementary volume of the space.

Applying $\nabla_\sigma \, A^\sigma = 0$, to the four-dimensional vector potential A^α of an electromagnetic field gives Lorentz' condition for the field. The Lorentz condition in chr.inv.-form is

$$^*\widetilde{\nabla}_i\, q^i = -\,\frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t} + \varphi D\right). \tag{2.108}$$

Now we are going to deduce chr.inv.-projections of the divergence of an arbitrary antisymmetric tensor $F^{\alpha\beta} = -F^{\beta\alpha}$ (later we will need them to obtain Maxwell's equations in chr.inv.-form)

$$\nabla_\sigma\, F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} F^{\alpha\mu} + \Gamma^\alpha_{\sigma\mu} F^{\sigma\mu} = \frac{\partial F^{\sigma\alpha}}{\partial x^\sigma} + \frac{\partial \ln\sqrt{-g}}{\partial x^\mu}\, F^{\alpha\mu}, \tag{2.109}$$

where the third term $\Gamma^\alpha_{\sigma\mu}F^{\sigma\mu}$ is zero, because of contraction of the Christoffel symbols $\Gamma^\alpha_{\sigma\mu}$ (which are symmetric by their lower indices) and an antisymmetric tensor $F^{\sigma\mu}$ is zero as in the case of any symmetric and antisymmetric tensor.

The term $\nabla_\sigma\, F^{\sigma\alpha}$ is a four-dimensional vector, so its chr.inv.-projections are

$$T = b_\alpha \nabla_\sigma\, F^{\sigma\alpha}, \qquad B^i = h^i_\alpha \nabla_\sigma\, F^{\sigma\alpha} = \nabla_\sigma\, F^{i\alpha}. \tag{2.110}$$

We denote chr.inv.-projections of the tensor $F^{\alpha\beta}$ as follows

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}\,, \qquad H^{ik} = F^{ik}, \tag{2.111}$$

then the remaining non-zero components of the tensor are

$$F_{0\cdot}^{\cdot 0} = \frac{1}{c}\, v_k E^k, \tag{2.112}$$

$$F_{k\cdot}^{\cdot 0} = \frac{1}{\sqrt{g_{00}}}\left(E_i - \frac{1}{c}\, v_n H_{k\cdot}^{\cdot n} - \frac{1}{c^2}\, v_k v_n E^n\right), \tag{2.113}$$

$$F^{0i} = \frac{E^i - \frac{1}{c}\, v_k H^{ik}}{\sqrt{g_{00}}}\,, \qquad F_{0i} = -\sqrt{g_{00}}\, E_i\,, \tag{2.114}$$

$$F_{i\cdot}^{\cdot k} = -H_{i\cdot}^{\cdot k} - \frac{1}{c}\, v_i E^k, \quad F_{ik} = H_{ik} + \frac{1}{c}\left(v_i E_k - v_k E_i\right), \tag{2.115}$$

and the square of this tensor $F^{\alpha\beta}$ is

$$F_{\alpha\beta}F^{\alpha\beta} = H_{ik}H^{ik} - 2E_i E^i. \tag{2.116}$$

Substituting the components into (2.110) and replacing regular operators of derivation with the chr.inv.-operators, after some algebra we obtain

$$T = \frac{\nabla_\sigma F_{0\cdot}^{\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial E^i}{\partial x^i} + E^i \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c}\, H^{ik} A_{ik}\,, \tag{2.117}$$

$$\begin{aligned} B^i = \nabla_\sigma F^{\sigma i} = \frac{{}^*\partial H^{ik}}{\partial x^k} + H^{ik} \frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\, F_k H^{ik} - \\ - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right), \end{aligned} \tag{2.118}$$

where A_{ik} is the antisymmetric chr.inv.-tensor of non-holonomity of the space. Taking into account that

$$\frac{{}^*\partial E^i}{\partial x^i} + E^i \frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} = {}^*\nabla_i\, E^i \tag{2.119}$$

is the chr.inv.-divergence of the vector E^i, and also that

$${}^*\nabla_k\, H^{ik} - \frac{1}{c^2}\, F_k H^{ik} = {}^*\widetilde{\nabla}_k\, H^{ik} \tag{2.120}$$

is the physical chr.inv.-divergence of the tensor H^{ik} we arrive at the final equations for chr.inv.-projections of the divergence of an arbitrary antisymmetric tensor $F^{\alpha\beta}$

$$T = {}^*\nabla_i\, E^i - \frac{1}{c}\, H^{ik} A_{ik}\,, \tag{2.121}$$

$$B^i = {}^*\widetilde{\nabla}_k\, H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right). \tag{2.122}$$

Hence, we calculate chr.inv.-projections of the divergence of the pseudotensor $F^{*\alpha\beta}$, which is dual to the given antisymmetric tensor $F^{\alpha\beta}$, namely

$$F^{*\alpha\beta} = \frac{1}{2}\, E^{\alpha\beta\mu\nu} F_{\mu\nu}\,, \qquad F_{*\alpha\beta} = \frac{1}{2}\, E_{\alpha\beta\mu\nu} F^{\mu\nu}. \tag{2.123}$$

We denote its chr.inv.-projections as follows

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}\,, \qquad E^{*ik} = F^{*ik}, \tag{2.124}$$

so there are evident relations $H^{*i} \sim H^{ik}$ and $E^{*ik} \sim E^i$ between the chr.inv.-quantities and chr.inv.-projections of the antisymmetric tensor $F^{\alpha\beta}$ (2.111), because of duality of the given quantities $F^{\alpha\beta}$ and $F^{*\alpha\beta}$.

Therefore, given that

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{1}{2}\,\varepsilon^{ipq}H_{pq}\,, \qquad F^{*ik} = -\,\varepsilon^{ikp}E_p\,, \tag{2.125}$$

the remaining components of the pseudotensor $F^{*\alpha\beta}$, formulated with the chr.inv.-projections of its dual tensor $F^{\alpha\beta}$ (2.111) are

$$F_{0\cdot}^{*\cdot 0} = \frac{1}{2c}\,v_k\,\varepsilon^{kpq}\left[H_{pq} + \frac{1}{c}\left(v_pE_q - v_qE_p\right)\right], \tag{2.126}$$

$$\begin{gathered} F_{i\cdot}^{*\cdot 0} = \frac{1}{2\sqrt{g_{00}}}\left[\varepsilon_{i\cdot}^{\cdot pq}H_{pq} + \frac{1}{c}\,\varepsilon_{i\cdot}^{\cdot pq}\left(v_pE_q - v_qE_p\right) - \right. \\ \left. - \frac{1}{c^2}\,\varepsilon^{kpq}v_iv_kH_{pq} - \frac{1}{c^3}\,\varepsilon^{kpq}v_iv_k\left(v_pE_q - v_qE_p\right)\right], \end{gathered} \tag{2.127}$$

$$F^{*0i} = \frac{1}{2\sqrt{g_{00}}}\,\varepsilon^{ipq}\left[H_{pq} + \frac{1}{c}\left(v_pE_q - v_qE_p\right)\right], \tag{2.128}$$

$$F_{*0i} = \frac{1}{2}\sqrt{g_{00}}\,\varepsilon_{ipq}H^{pq}, \tag{2.129}$$

$$F_{i\cdot}^{*\cdot k} = \varepsilon_{i\cdot}^{\cdot kp}E_p - \frac{1}{2c}\,v_i\,\varepsilon^{kpq}H_{pq} - \frac{1}{c^2}\,v_iv_m\,\varepsilon^{mkp}E_p\,, \tag{2.130}$$

$$F_{*ik} = \varepsilon_{ikp}\left(E^p - \frac{1}{c}\,v_qH^{pq}\right), \tag{2.131}$$

while its square is

$$F_{*\alpha\beta}F^{*\alpha\beta} = \varepsilon^{ipq}\left(E_pH_{iq} - E_iH_{pq}\right), \tag{2.132}$$

where ε^{ipq} is the three-dimensional discriminant chr.inv.-tensor (2.73, 2.74). Then the chr.inv.-projections of the divergence of the pseudotensor $F^{*\alpha\beta}$ are

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = \frac{{}^*\partial H^{*i}}{\partial x^i} + H^{*i}\,\frac{{}^*\partial \ln\sqrt{h}}{\partial x^i} - \frac{1}{c}\,E^{*ik}A_{ik}\,, \tag{2.133}$$

$$\begin{gathered} \nabla_\sigma F^{*\sigma i} = \frac{{}^*\partial E^{*ik}}{\partial x^i} + E^{*ik}\,\frac{{}^*\partial \ln\sqrt{h}}{\partial x^k} - \frac{1}{c^2}\,F_kE^{*ik} - \\ - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right), \end{gathered} \tag{2.134}$$

or, using respective formulae which determine the chr.inv.-divergence ${}^*\nabla_i\,H^{*i}$ and also the physical chr.inv.-divergence ${}^*\widetilde{\nabla}_k\,E^{*ik}$, as well as

(2.119, 2.120), we obtain

$$\frac{\nabla_\sigma F_{0\cdot}^{*\cdot\sigma}}{\sqrt{g_{00}}} = {}^*\nabla_i H^{*i} - \frac{1}{c}\, E^{*ik} A_{ik}\,, \tag{2.135}$$

$$\nabla_\sigma F^{*\sigma i} = {}^*\widetilde{\nabla}_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right). \tag{2.136}$$

Apart from the divergence of vectors, antisymmetric tensors and pseudotensors of the 2nd rank, we need to deduce chr.inv.-projections of the divergence of a symmetric tensor of the 2nd rank (we will need them to obtain the conservation laws in chr.inv.-form). We will uplift them fully from Zelmanov [9]. Like Zelmanov did in his theory, we denote chr.inv.-projections of a symmetric tensor $T^{\alpha\beta}$ as follows

$$\frac{T_{00}}{g_{00}} = \rho\,, \qquad \frac{T_0^i}{\sqrt{g_{00}}} = K^i, \qquad T^{ik} = N^{ik}, \tag{2.137}$$

according to [9] we have

$$\frac{\nabla_\sigma T_0^\sigma}{\sqrt{g_{00}}} = \frac{{}^*\partial\rho}{\partial t} + \rho D + D_{ik}N^{ik} + c\,{}^*\nabla_i K^i - \frac{2}{c}\,F_i K^i, \tag{2.138}$$

$$\begin{gathered}\nabla_\sigma T^{\sigma i} = c\,\frac{{}^*\partial K^i}{\partial t} + c\,DK^i + 2c\left(D_k^i + A_{k\cdot}^{\cdot i}\right) K^k + \\ + c^2\,{}^*\nabla_k N^{ik} - F_k N^{ik} - \rho F^i.\end{gathered} \tag{2.139}$$

Among the internal (scalar) product of a tensor with the operator of absolute derivation ∇, which is the divergence of this tensor field, we can consider a difference between the covariant derivatives of the field. This quantity is known as a *curl* of the field, because from geometric viewpoint, it is the vortex (rotation) of the field. The *absolute curl* is the curl of a n-dimensional tensor field in a n-dimensional space. The curl of an arbitrary four-dimensional vector field A^α is a covariant antisymmetric 2nd rank tensor, defined as follows*

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\,, \tag{2.140}$$

where $\nabla_\mu A_\nu$ is the absolute derivative of the A_α with respect to the coordinate x^μ

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\sigma A_\sigma\,. \tag{2.141}$$

*See §98 in the well-known book authored by Peter Raschewski [18]. Actually, curl is not the tensor (2.140), but its dual pseudotensor (2.142), because the invariance with respect to reflection is necessary for any rotations.

The curl, contracted with the four-dimensional absolutely antisymmetric discriminant tensor $E^{\alpha\beta\mu\nu}$ (2.65), is the pseudotensor

$$F^{*\alpha\beta} = E^{\alpha\beta\mu\nu} \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right) = E^{\alpha\beta\mu\nu} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right). \qquad (2.142)$$

In electrodynamics $F_{\mu\nu}$ (2.140) is the tensor of an electromagnetic field (Maxwell's tensor), which is the curl of the four-dimensional potential A^α of this electromagnetic field. Therefore, later, we will need formulae for chr.inv.-projections of the four-dimensional curl $F_{\mu\nu}$ and its dual pseudotensor $F^{*\alpha\beta}$, expressed through chr.inv.-projections of the four-dimensional vector potential A^α (2.81), which forms them.

Let us calculate components of the curl $F_{\mu\nu}$, taking into account that $F_{00} = F^{00} = 0$ just like for any other antisymmetric tensor. As a result, after some algebra we obtain

$$F_{0i} = \left(1 - \frac{\mathrm{w}}{c^2}\right)\left(\frac{\varphi}{c^2}\,F_i - \frac{{}^*\partial\varphi}{\partial x^i} - \frac{1}{c}\frac{{}^*\partial q_i}{\partial t}\right), \qquad (2.143)$$

$$\begin{gathered} F_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} + \frac{\varphi}{c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right) + \\ + \frac{1}{c}\left(v_i\,\frac{{}^*\partial\varphi}{\partial x^k} - v_k\,\frac{{}^*\partial\varphi}{\partial x^i}\right) + \frac{1}{c^2}\left(v_i\,\frac{{}^*\partial q_k}{\partial t} - v_k\,\frac{{}^*\partial q_i}{\partial t}\right), \end{gathered} \qquad (2.144)$$

$$F_{0\cdot}^{\cdot 0} = -\frac{\varphi}{c^3}\,v_k F^k + \frac{1}{c}\,v^k\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right), \qquad (2.145)$$

$$\begin{gathered} F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}}\left[\frac{\varphi}{c^2}\,F_k - \frac{{}^*\partial\varphi}{\partial x^k} - \frac{1}{c}\frac{{}^*\partial q_k}{\partial t} + \right. \\ + \frac{2\varphi}{c^2}\,v^m A_{mk} + \frac{1}{c^2}\,v_k v^m\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \\ \left. - \frac{1}{c}\,v^m\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{\varphi}{c^4}\,v_k v_m F^m\right], \end{gathered} \qquad (2.146)$$

$$\begin{gathered} F_{k\cdot}^{\cdot i} = h^{im}\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{1}{c}\,h^{im} v_k\,\frac{{}^*\partial\varphi}{\partial x^m} - \\ - \frac{1}{c^2}\,h^{im} v_k\,\frac{{}^*\partial q_m}{\partial t} + \frac{\varphi}{c^3}\,v_k F^i + \frac{2\varphi}{c}\,A_{k\cdot}^{\cdot i}\,, \end{gathered} \qquad (2.147)$$

$$\begin{gathered} F^{0k} = \frac{1}{\sqrt{g_{00}}}\left[h^{km}\left(\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) - \frac{\varphi}{c^2}\,F^k + \right. \\ \left. + \frac{1}{c}\,v^n h^{mk}\left(\frac{{}^*\partial q_n}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^n}\right) - \frac{2\varphi}{c^2}\,v_m A^{mk}\right], \end{gathered} \qquad (2.148)$$

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha}F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik}\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right) - \frac{\varphi}{c^2}\,F^i, \tag{2.149}$$

$$F^{ik} = g^{i\alpha}g^{k\beta}F_{\alpha\beta} = h^{im}h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}\,A^{ik}, \tag{2.150}$$

where (2.149, 2.150) are chr.inv.-projections of the curl $F_{\mu\nu}$. Respectively, chr.inv.-projections of its dual pseudotensor $F^{*\alpha\beta}$ are

$$\frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = \frac{g_{0\alpha}F^{*\alpha i}}{\sqrt{g_{00}}} = \varepsilon^{ikm}\left[\frac{1}{2}\left(\frac{{}^*\partial q_k}{\partial x^m} - \frac{{}^*\partial q_m}{\partial x^k}\right) - \frac{\varphi}{c}\,A_{km}\right], \tag{2.151}$$

$$F^{*ik} = \varepsilon^{ikm}\left(\frac{\varphi}{c^2}\,F_m - \frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right), \tag{2.152}$$

where $F_{0\cdot}^{*\cdot i} = g_{0\alpha}F^{*\alpha i} = g_{0\alpha}E^{\alpha i\mu\nu}F_{\mu\nu}$ can be calculated using already mentioned components of the curl $F_{\mu\nu}$ (2.143–2.148).

§2.6 Laplace's operator and d'Alembert's operator

Laplace's operator is the three-dimensional operator of derivation

$$\Delta = \nabla\,\nabla = \nabla^2 = -g^{ik}\,\nabla_i\,\nabla_k\,. \tag{2.153}$$

Its four-dimensional generalization in a pseudo-Riemannian space is *d'Alembert's general covariant operator*

$$\Box = g^{\alpha\beta}\,\nabla_\alpha\,\nabla_\beta\ . \tag{2.154}$$

In the Minkowski space, the operators take the form

$$\Delta = \frac{\partial^2}{\partial x^1\partial x^1} + \frac{\partial^2}{\partial x^2\partial x^2} + \frac{\partial^2}{\partial x^3\partial x^3}\,, \tag{2.155}$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^1\partial x^1} - \frac{\partial^2}{\partial x^2\partial x^2} - \frac{\partial^2}{\partial x^3\partial x^3} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\ . \tag{2.156}$$

Our goal is to apply d'Alembert operator to scalar and vector fields, located in a pseudo-Riemannian space, and also to present the results in chr.inv.-form. At first, we apply d'Alembert operator to a four-dimensional scalar field φ, because in this case the calculations will be much simpler (the absolute derivative of a scalar field $\nabla_\alpha\,\varphi$ does not contain the Christoffel symbols, so it becomes regular derivative)

$$\Box\varphi = g^{\alpha\beta}\,\nabla_\alpha\,\nabla_\beta\ \varphi = g^{\alpha\beta}\frac{\partial\varphi}{\partial x^\alpha}\left(\frac{\partial\varphi}{\partial x^\beta}\right) = g^{\alpha\beta}\,\frac{\partial^2\varphi}{\partial x^\alpha\partial x^\beta}\,. \tag{2.157}$$

Henceforth, we formulate components of the fundamental metric tensor in terms of chronometric invariants. For g^{ik} from (1.18) we obtain $g^{ik} = -\,h^{ik}$. Components g^{0i} are obtained from the linear velocity of the space rotation $v^i = -\,c\,g^{0i}\sqrt{g_{00}}$

$$g^{0i} = -\frac{1}{c\sqrt{g_{00}}}\,v^i. \tag{2.158}$$

Component g^{00} can be obtained from the main property of the fundamental metric tensor $g_{\alpha\sigma}g^{\beta\sigma} = g^{\beta}_{\alpha}$. Setting $\alpha = \beta = 0$, gives

$$g_{0\sigma}\,g^{0\sigma} = g_{00}\,g^{00} + g_{0i}\,g^{0i} = \delta^0_0 = 1, \tag{2.159}$$

then, taking into account that

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \qquad g_{0i} = -\frac{1}{c}\,v_i\left(1 - \frac{\mathrm{w}}{c^2}\right), \tag{2.160}$$

we obtain the formula

$$g^{00} = \frac{1}{\left(1 - \frac{\mathrm{w}}{c^2}\right)^2}\left(1 - \frac{1}{c^2}\,v_i v^i\right), \qquad v_i v^i = h_{ik}\,v^i v^k = v^2. \tag{2.161}$$

Substituting the obtained formulae into $\Box\,\varphi$ (2.157) and replacing regular operators of derivation with the chr.inv.-operators, we obtain d'Alembertian of the scalar field in chr.inv.-form

$$\Box\,\varphi = \frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{{}^*\partial^2\varphi}{\partial x^i\partial x^k} = {}^*\Box\,\varphi\,, \tag{2.162}$$

where, in contrast to the regular operators, ${}^*\Box$ is the d'Alembert chr. inv.-operator, and ${}^*\Delta$ is the Laplace chr.inv.-operator

$${}^*\Box = \frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2} - h^{ik}\frac{{}^*\partial^2}{\partial x^i\partial x^k}\,, \tag{2.163}$$

$${}^*\Delta = -g^{ik}\,{}^*\nabla_i\,{}^*\nabla_k = h^{ik}\frac{{}^*\partial^2}{\partial x^i\partial x^k}\,. \tag{2.164}$$

Now, we apply d'Alembert operator to an arbitrary four-dimensional vector field A^α

$$\Box\,A^\alpha = g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,A^\alpha. \tag{2.165}$$

Since $\Box\,A^\alpha$ is a four-dimensional vector, chr.inv.-projections of this quantity are

$$T = b_\sigma\,\Box\,A^\sigma = b_\sigma\,g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,A^\sigma, \tag{2.166}$$

$$B^i = h^i_\sigma\,\Box\,A^\sigma = h^i_\sigma\,g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,A^\sigma. \tag{2.167}$$

In general, to obtain d'Alembertian in chr.inv.-form for a vector field in a pseudo-Riemannian space is not a trivial task, because the Christoffel symbols are not zeroes, so formulae for projections of the second derivatives take dozens of pages*.

After some difficult algebra, we obtain required formulae for the chr.inv.-projections of the d'Alembertian of the vector field A^α in a pseudo-Riemannian space

$$\begin{aligned} T = {}^*\Box\varphi &- \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(F_k q^k\right) - \frac{1}{c^3}F_i\frac{{}^*\partial q^i}{\partial t} + \frac{1}{c^2}F^i\frac{{}^*\partial\varphi}{\partial x^i} + \\ &+ h^{ik}\Delta^m_{ik}\frac{{}^*\partial\varphi}{\partial x^m} - h^{ik}\frac{1}{c}\frac{{}^*\partial}{\partial x^i}\left[\left(D_{kn}+A_{kn}\right)q^n\right] + \frac{D}{c^2}\frac{{}^*\partial\varphi}{\partial t} - \\ &- \frac{1}{c}D^k_m\frac{{}^*\partial q^m}{\partial x^k} + \frac{2}{c^3}A_{ik}F^i q^k + \frac{\varphi}{c^4}F_iF^i - \frac{\varphi}{c^2}D_{mk}D^{mk} - \\ &- \frac{D}{c^3}F_m q^m - \frac{1}{c}\Delta^m_{kn}D^k_m q^n + \frac{1}{c}h^{ik}\Delta^m_{ik}\left(D_{mn}+A_{mn}\right)q^n, \end{aligned} \tag{2.168}$$

$$\begin{aligned} B^i = {}^*\Box A^i &+ \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left[\left(D^i_k + A_{k\cdot}^{\cdot i}\right)q^k\right] + \frac{D}{c^2}\frac{{}^*\partial q^i}{\partial t} + \\ &+ \frac{1}{c^2}\left(D^i_k + A_{k\cdot}^{\cdot i}\right)\frac{{}^*\partial q^k}{\partial t} - \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(\varphi F^i\right) - \frac{1}{c^3}F^i\frac{{}^*\partial\varphi}{\partial t} + \\ &+ \frac{1}{c^2}F^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{1}{c}\left(D^{mi} + A^{mi}\right)\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c^4}q^kF_kF^i + \\ &+ \frac{1}{c^2}\Delta^i_{km}q^mF^k - \frac{\varphi}{c^3}DF^i + \frac{D}{c^2}\left(D^i_n + A_{n\cdot}^{\cdot i}\right)q^n - \\ &- h^{km}\left\{\frac{{}^*\partial}{\partial x^k}\left(\Delta^i_{mn}q^n\right) + \frac{1}{c}\frac{{}^*\partial}{\partial x^k}\left[\varphi\left(D^i_m + A_{m\cdot}^{\cdot i}\right)\right] + \right. \\ &+ \left(\Delta^i_{kn}\Delta^n_{mp} - \Delta^n_{km}\Delta^i_{np}\right)q^p + \frac{\varphi}{c}\left[\Delta^i_{kn}\left(D^n_m + A_{m\cdot}^{\cdot n}\right) - \right. \\ &\left.\left. - \Delta^n_{km}\left(D^i_n + A_{n\cdot}^{\cdot i}\right)\right] + \Delta^i_{kn}\frac{{}^*\partial q^n}{\partial x^m} - \Delta^n_{km}\frac{{}^*\partial q^i}{\partial x^n}\right\}, \end{aligned} \tag{2.169}$$

where ${}^*\Box\varphi$ and ${}^*\Box q^i$ are results from application of d'Alembert chr.inv.-operator (2.163) to the quantities $\varphi = \frac{A_0}{\sqrt{g_{00}}}$ and $q^i = A^i$, which are chr.

*This is one of the reasons why practical applications of the theory of electromagnetic field are mainly calculated in a Galilean reference frame in the Minkowski space (the space-time of the Special Theory of Relativity), where the Christoffel symbols are zeroes. As a matter of fact, general covariant notation hardly permits unambiguous interpretation of calculation results, unless they are formulated with physical observable quantities (chronometric invariants) or demoted to a simple specific case, like that in the Minkowski space, for instance.

inv.-projections (physical observable components) of the vector A^α,

$$^*\Box\varphi = \frac{1}{c^2}\frac{^*\partial^2\varphi}{\partial t^2} - h^{ik}\frac{^*\partial^2\varphi}{\partial x^i \partial x^k}\,, \tag{2.170}$$

$$^*\Box q^i = \frac{1}{c^2}\frac{^*\partial^2 q^i}{\partial t^2} - h^{km}\frac{^*\partial^2 q^i}{\partial x^k \partial x^m}\,. \tag{2.171}$$

The main criterion for correct calculations in such a complicate case as here (the chr.inv.-projections of the d'Alembertian of a vector field, which resulted formulae 2.168 and 2.169) is Zelmanov's rule of chronometric invariance: "Correct calculations make all terms in the final equations chronometrically invariant quantities. That is, they consist of chr.inv.-quantities themselves, their chr.inv.-derivatives and also chr.inv.-properties of the reference space. If any single mistake is made during calculations, the terms of the final equations will not be chronometric invariants".

D'Alembert operator from a tensor field, equated to zero or not zero, gives *d'Alembert equations* for this field. From the physical viewpoint, these are equations of propagation of waves of the field. If d'Alembertian is not zero, these are equations of propagation of waves enforced by the field-inducing sources (d'Alembert equations with sources). For instance, the sources in electromagnetic fields are electric charges and currents. If d'Alembert operator for a field is zero, then these are equations of propagation of waves of the field not related to any sources. If the space-time area under consideration, aside from the tensor field in this question, is filled with another medium, then d'Alembert equations will gain an additional term to characterize the media, which can be obtained from the equations which define it.

§ 2.7 Conclusions

We are now ready to outline the results of this Chapter. Apart from general knowledge of tensors and tensor algebra, we have obtained some tools to facilitate our calculations in the next Chapters. Equality to zero of the absolute derivative of the dynamic vector of a particle to its direction of motion sets the equations of motion of this particle. Equality to zero of the divergence of a vector field sets the Lorentz condition and the continuity equation for this field. Equality to zero of the divergence of a symmetric tensor of the 2nd rank sets the conservation law, while equality to zero of an antisymmetric tensor of the 2nd rank (and of its dual pseudotensor) set the Maxwell equations. The curl of a vector field, applied to an electromagnetic field, is the field tensor (the Maxwell tensor). The d'Alembert equations for a given field are equations of propagation of the field waves.

So, we have a brief list of possible applications of the mathematical apparatus in our possession. Hence, if we now come across an antisymmetric tensor or a differential operator, we may simply use templates already obtained in this Chapter.

Chapter 3 Motion of Charged Particles

§3.1 Problem statement

In this Chapter, we will set forth the theory of electromagnetic field and moving charged particles in a four-dimensional pseudo-Riemannian space. The peculiarity, which makes this theory different from regular relativistic electrodynamics, is that all equations here will be given in chr.inv.-form (in other words, expressed through physical observable quantities).

An electromagnetic field is commonly studied as a vector field of the electromagnetic four-dimensional potential A^{α}, located in the four-dimensional pseudo-Riemannian space. Its time component is known as the *scalar potential* φ of the field, while its spatial components make up the so-called *vector-potential* A^{i}. The four-dimensional electromagnetic potential A^{α} in CGSE and Gaussian systems of units has the dimensions

$$A^{\alpha}\ [\,\text{gram}^{1/2}\,\text{cm}^{1/2}\,\text{sec}^{-1}\,]. \tag{3.1}$$

As it is evident, its components φ and A^{i} have the same dimensions. Therefore, studying electromagnetic fields is substantially different from studying gravitational fields: according to the theory of chronometric invariants, gravitational inertial force F^{i} and gravitational potential w (1.38) are functions of geometric properties of the space only, while electromagnetic fields (the fields of the electromagnetic potential A^{α}) has not been "geometrically interpreted" yet, so we have to study electromagnetic fields just as external vector fields introduced into the space.

Equations of Classical Electrodynamics — Maxwell's equations, which define the relationship between the electric and magnetic components of the given field, — had been obtained long before theoretical physics accepted the terms of Riemannian geometry and even Minkowski's space of the Special Theory of Relativity. Later, when electrodynamics was set forth in the Minkowski space under the name of *relativistic electrodynamics*, the Maxwell equations had been obtained in four-dimensional form. Then, the Maxwell equations in general covariant form, acceptable for any pseudo-Riemannian space had been obtained. But having accepted general covariant form, the Maxwell equa-

equations became less illustrative, which used to be an advantage of Classical Electrodynamics. On the other hand, four-dimensional equations in the Minkowski space can be simply presented as their scalar (time) and vector (spatial) components, because in a Galilean reference frame they are observable quantities by definition. But when we turn to an inhomogeneous, anisotropic, curved, and deforming pseudo-Riemannian space, the problem of comparing the vector and scalar components in general covariant equations with equations of Classical Electrodynamics becomes non-trivial. In other words, a question arises on which quantities in relativistic electrodynamics can be assumed as physical observables.

Thus, the equations of relativistic electrodynamics in a pseudo-Riemannian space shall be formulated with physical observable components (chr.inv.-projections) of the electromagnetic field potential and also observable properties of the space. We are going to tackle the problem using the mathematical apparatus of chronometric invariants, namely — projecting general covariant quantities on time lines and the spatial section of a real observer. The results we are going to obtain using this method will help us to arrive at *observable generalization* of the basic quantities and the laws of relativistic electrodynamics. Also, Classical Electrodynamics, which will take into account the effects of physical and geometric properties of the observer's reference space will be obtained.

§3.2 Observable components of the electromagnetic field tensor. The field invariants

By definition, the tensor of an electromagnetic field is the curl of its four-dimensional potential A^{α}. This field tensor is also referred to as *Maxwell's tensor*

$$F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} = \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}\,. \tag{3.2}$$

As it is easy to see, this formula is a general covariant generalization of three-dimensional quantities in Classical Electrodynamics

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\,, \qquad \vec{H} = \mathrm{curl}\,\vec{A}\,, \tag{3.3}$$

where $\vec{E}$ and $\vec{H}$ are the strength vectors of the electric and magnetic components of the field, respectively. Here φ is the scalar potential and $\vec{A}$ is the spatial vector-potential of the field, and

$$\vec{\nabla} = \vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \tag{3.4}$$

is the gradient operator in three-dimensional Euclidean space.

At first, in this section, we are going to determine the components of the electromagnetic field tensor $F_{\alpha\beta}$ that are physical observable quantities in a given pseudo-Riemannian space. Then, we are going to find a relationship between the observable quantities and the electric strength $\vec{E}$ and the magnetic strength $\vec{H}$ of the field in Classical Electrodynamics. The strength vectors will also be obtained in the pseudo-Riemannian space, which in general is inhomogeneous, anisotropic, curved, and deformed.

It is important to take note of this. Since in the Minkowski space (the space-time of the Special Theory of Relativity) in an inertial reference frame (the one, which moves linearly at a constant velocity) the metric is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2, \tag{3.5}$$

and components of the fundamental metric tensor are

$$g_{00} = 1\,, \qquad g_{0i} = 0\,, \qquad g_{11} = g_{22} = g_{33} = -1\,, \tag{3.6}$$

no difference exists between covariant and contravariant components of A^α (in particular, this is why all calculations in the Minkowski space are much simpler)

$$\varphi = A_0 = A^0, \qquad A_i = -A^i. \tag{3.7}$$

In the pseudo-Riemannian space (and in Riemannian spaces in general) there is a difference, because the metric has more general form. Therefore, the scalar potential and the vector-potential of the electromagnetic field we are considering shall be defined as chr.inv.-projections (physical observable components) of the four-dimensional potential A^α

$$\varphi = b^\alpha A_\alpha = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = h^i_\sigma A^\sigma = A^i. \tag{3.8}$$

The remaining components of A^α, are not chr.inv.-quantities. They are formulated with φ and q^i as follows

$$A^0 = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\varphi + \frac{1}{c}\,v_i q^i\right), \qquad A_i = -q_i - \frac{\varphi}{c}\,v_i\,. \tag{3.9}$$

Note, in accordance with the theory of chronometric invariants, the covariant chr.inv.-vector q_i is obtained from the contravariant chr.inv.-vector q^i as a result of lowering the index using the metric chr.inv.-tensor h_{ik} as follows; $q_i = h_{ik}q^k$. On the contrary, the regular covariant vector A_i, which is not a chr.inv.-quantity, is obtained as a result of lowering the index using the fundamental metric tensor, so that $A_i = g_{i\alpha} A^\alpha$.

According to the general formula for the square of a vector (2.39), the square of the potential A^α in the accompanying reference frame is

$$A_\alpha A^\alpha = g_{\alpha\beta} A^\alpha A^\beta = \varphi^2 - h_{ik} q^i q^k = \varphi^2 - q^2, \tag{3.10}$$

and the quantity is real, if $\varphi^2 > q^2$; imaginary, if $\varphi^2 < q^2$; zero (isotropic), if $\varphi^2 = q^2$.

Now, using components of the potential A^α (3.8, 3.9) in the definition of the electromagnetic field tensor $F_{\alpha\beta}$ (3.2), formulating regular derivatives with chr.inv.-derivatives (1.33), and using formulae for components of the curl of an arbitrary vector field (2.143–2.150), we obtain chr.inv.-projections of the tensor $F_{\alpha\beta}$

$$\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = \frac{g^{i\alpha} F_{0\alpha}}{\sqrt{g_{00}}} = h^{ik}\left(\frac{{}^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{{}^*\partial q_k}{\partial t}\right) - \frac{\varphi}{c^2} F^i, \tag{3.11}$$

$$F^{ik} = g^{i\alpha} g^{k\beta} F_{\alpha\beta} = h^{im} h^{kn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c} A^{ik}. \tag{3.12}$$

We denote the chr.inv.-projections of the electromagnetic field tensor in a classic way as follows

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}}, \qquad H^{ik} = F^{ik}, \tag{3.13}$$

so the covariant (lower-index) chr.inv.-quantities are

$$E_i = h_{ik} E^k = \frac{{}^*\partial\varphi}{\partial x^i} + \frac{1}{c}\frac{{}^*\partial q_i}{\partial t} - \frac{\varphi}{c^2} F_i\,, \tag{3.14}$$

$$H_{ik} = h_{im} h_{kn} H^{mn} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} - \frac{2\varphi}{c} A_{ik}\,, \tag{3.15}$$

while the mixed components $H_{k\cdot}^{\cdot m} = -H_{\cdot k}^{m\cdot}$ are obtained from H^{ik} using the metric chr.inv.-tensor h_{ik}, so that $H_{k\cdot}^{\cdot m} = h_{ki} H^{im}$. In this case, the space deformation tensor $D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}$ (1.40) is also present in the formulae, but in an implicit way and appears when we substitute the components $q_k = h_{km} q^m$ into the time derivatives.

Besides, we may as well formulate other components of the electromagnetic field tensor $F_{\alpha\beta}$ with its chr.inv.-projections E^i and H^{ik} (3.11) using formulae for components of an arbitrary antisymmetric tensor (2.112–2.115). This is possible because the generalized formulae (2.112–2.115) contain E^i and H^{ik} in "implicit" form, irrespective of whether they are components of a curl or of an antisymmetric tensor of any other kind.

In the Minkowski space, with no acceleration F^i, rotation A_{ik} and deformations D_{ik}, the formula for E_i becomes

$$E_i = \frac{\partial\varphi}{\partial x^i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\,, \tag{3.16}$$

or in three-dimensional vector form

$$\vec{E} = \vec{\nabla}\varphi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\,, \tag{3.17}$$

which, apart from the sign, matches the formula for $\vec{E}$ in Classical Electrodynamics.

Now, we formulate the electric and magnetic strengths through components of the field pseudotensor $F^{*\alpha\beta}$, which is dual to the Maxwell tensor of this field $F^{*\alpha\beta} = \frac{1}{2}\,E^{\alpha\beta\mu\nu}F_{\mu\nu}$ (2.123). So, in accordance with (2.124), chr.inv.-projections of this pseudotensor are

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}}\,, \qquad\qquad E^{*ik} = F^{*ik}. \tag{3.18}$$

Using formulae for components of an arbitrary pseudotensor $F^{*\alpha\beta}$, obtained in Chapter 2 (2.125–2.131), and also formulae for E_i and H_{ik} (3.14, 3.15), we obtain expanded formulae for H^{*i} and E^{*ik}, namely

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn}\right) = \frac{1}{2}\,\varepsilon^{imn}H_{mn}\,, \tag{3.19}$$

$$E^{*ik} = \varepsilon^{ikn}\left(\frac{\varphi}{c^2}\,F_n - \frac{{}^*\partial\varphi}{\partial x^n} - \frac{1}{c}\frac{{}^*\partial q_n}{\partial t}\right) = -\,\varepsilon^{ikn}E_k\,. \tag{3.20}$$

It is easy to see that, the following pairs of tensors are dual conjugates: H^{*i} and H_{mn}, E^{*ik} and E_m. The chr.inv.-pseudovector H^{*i} (3.19) includes the term

$$\frac{1}{2}\,\varepsilon^{imn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) = \frac{1}{2}\,\varepsilon^{imn}\left({}^*\nabla_n\, q_m - {}^*\nabla_m\, q_n\right), \tag{3.21}$$

which is the chr.inv.-curl of the three-dimensional vector field q_m. Here is also the term

$$\frac{1}{2}\,\varepsilon^{imn}\frac{2\varphi}{c}\,A_{mn} = \frac{2\varphi}{c}\,\Omega^{*i}, \tag{3.22}$$

where $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}$ is the chr.inv.-pseudovector of angular velocities of the space rotation. In a Galilean reference frame in the Minkowski space (because there is no acceleration, rotations, and deformations in that space), the obtained formula for the magnetic strength chr.inv.-pseudovector H^{*i} (3.19) takes the form

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m}\right), \tag{3.23}$$

or in three-dimensional vector form, is

$$\vec{H} = \mathrm{curl}\,\vec{A}. \tag{3.24}$$

Therefore, the structure of a pseudo-Riemannian space affects electromagnetic fields, located in it, due to the fact that chr.inv.-vectors of the electric strength E_i (3.14) and the magnetic strength H^{*i} (3.19) depend on gravitational potential and rotation of this space.

The same will be true as well in the Minkowski space, if a non-inertial reference frame, which rotates and moves with acceleration, is assumed as the observer's reference frame. But in the Minkowski space, we can always find a Galilean reference frame (that is not true in a pseudo-Riemannian space), because the Minkowski space itself does not accelerate the reference frame and neither rotates nor deforms it. Therefore, such effects in the Minkowski space are strictly relative.

In relativistic electrodynamics we introduce invariants, which characterize the electromagnetic field we are considering — in other words, the *field invariants*

$$J_1 = F_{\mu\nu}F^{\mu\nu} = 2F_{0i}F^{0i} + F_{ik}F^{ik}, \tag{3.25}$$

$$J_2 = F_{\mu\nu}F^{*\mu\nu} = 2F_{0i}F^{*0i} + F_{ik}F^{*ik}. \tag{3.26}$$

The first invariant is scalar, while the second is pseudoscalar. Formulating them with components of the field tensor, we obtain

$$J_1 = H_{ik}H^{ik} - 2E_iE^i, \qquad J_2 = \varepsilon^{imn}\left(E_mH_{in} - E_iH_{nm}\right), \tag{3.27}$$

and using formulae for components of the field pseudotensor $F^{*\mu\nu}$ obtained in Chapter 2 we write down the field invariants as follows

$$J_1 = -2\left(E_iE^i - H_{*i}H^{*i}\right), \qquad J_2 = -4E_iH^{*i}. \tag{3.28}$$

Because the quantities J_1 and J_2 are invariants, we conclude:

a) If in a reference frame, the squares of the electric and magnetic strengths are equal $E^2 = H^{*2}$, then this equality remains unchanged in any other reference frame;

b) If in a reference frame, the electric and magnetic strengths are orthogonal $E_i H^{*i} = 0$, then this orthogonality remains unchanged in any other reference frame.

An electromagnetic field, where the conditions $E^2 = H^{*2}$ and $E_i H^{*i} = 0$ are true, that is one or both of the field invariants (3.28) are zeroes, is known as *isotropic*. Here the term "isotropic" does not stand for location of this field in light-like area of the space (as it is assumed in geometry), but rather for the field's property of equal emissions at any direction in the three-dimensional space (the spatial section).

The electromagnetic field invariants can be also formulated with chr.inv.-derivatives of the scalar chr.inv.-potential φ and the vector chr.inv.-potential q^i (3.8) as well as with chr.inv.-properties of the observer's reference space. So, we have

$$\begin{aligned} J_1 = 2\Big[& h^{im}h^{kn}\left(\frac{^*\partial q_i}{\partial x^k} - \frac{^*\partial q_k}{\partial x^i}\right)\frac{^*\partial q_m}{\partial x^n} - h^{ik}\frac{^*\partial \varphi}{\partial x^i}\frac{^*\partial \varphi}{\partial x^k} - \\ & - \frac{2}{c}\,h^{ik}\frac{^*\partial \varphi}{\partial x^i}\frac{^*\partial q_k}{\partial t} - \frac{1}{c^2}\,h^{ik}\frac{^*\partial q_i}{\partial t}\frac{^*\partial q_k}{\partial t} + \frac{8\varphi}{c^2}\,\Omega_i\Omega^i - \\ & - \frac{2\varphi}{c}\,\varepsilon^{imn}\Omega_m\frac{^*\partial q_i}{\partial x^n} + \frac{2\varphi}{c^2}\frac{^*\partial \varphi}{\partial x^i}F^i + \frac{2\varphi}{c^3}\frac{^*\partial q_i}{\partial t}F^i - \frac{\varphi}{c^4}\,F_iF^i\Big], \end{aligned} \tag{3.29}$$

$$J_2 = \frac{1}{2}\left[\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m}\right) - \frac{4\varphi}{c}\,\Omega^{*i}\right]\left(\frac{^*\partial \varphi}{\partial x^i} + \frac{1}{c}\frac{^*\partial q_i}{\partial t} - \frac{\varphi}{c^2}\,F_i\right). \tag{3.30}$$

We can know physical conditions in isotropic electromagnetic fields, by setting the formulae (3.29, 3.30) to zero. Doing this, we can see that the conditions of equality of the lengths of the electric and magnetic strengths $E^2 = H^{*2}$ and their orthogonality $E_i H^{*i} = 0$ in a pseudo-Riemannian space depend on not only properties of the field itself (the scalar potential φ and the vector-potential q^i) but also on acceleration F^i, rotation A_{ik} and deformations D_{ik} of the space itself. In particular, the vectors E_i and H^{*i} are orthogonal if the space is holonomic $\Omega^{*i} = 0$, while the spatial field of the vector-potential q^i is rotation-free $\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^m} - \frac{^*\partial q_n}{\partial x^n}\right) = 0$.

§3.3 Maxwell's equations, their observable components. Conservation of electric charge. Lorentz' condition

In Classical Electrodynamics, correlations of the electric strength of an electromagnetic field $\vec{E}$ [gram$^{1/2}$ cm$^{-1/2}$ sec^{-1}] to its magnetic strength $\vec{H}$ [gram$^{1/2}$ cm$^{-1/2}$ sec^{-1}] are set forth in *Maxwell's equations*, which had originally been derived from generalization of experimental data. In the middle of the 19th century, Maxwell showed that if an electromagnetic field is induced in vacuum by given charges and currents, then the resulting field is defined by two groups of equations [20]

$$\left.\begin{aligned} &\operatorname{curl}\vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \\ &\operatorname{div}\vec{E} = 4\pi\rho \end{aligned}\right\} \mathrm{I}\,, \tag{3.31a}$$

$$\left.\begin{aligned} &\operatorname{curl}\vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = 0 \\ &\operatorname{div}\vec{H} = 0 \end{aligned}\right\} \mathrm{II}\,, \tag{3.31b}$$

where ρ $[\mathrm{gram}^{1/2}\,\mathrm{cm}^{-3/2}\,\mathrm{sec}^{-1}]$ is the electric charge density (namely — the amount e $[\mathrm{gram}^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{sec}^{-1}]$ of the charge within 1 cm^3) and $\vec{j}$ $[\mathrm{gram}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{sec}^{-2}]$ is the current density vector. Equations containing the field-inducing sources ρ and $\vec{j}$ are known as the *1st group of the Maxwell equations*, while equations, which do not contain the sources are known as the *2nd group of the Maxwell equations*.

The first equation in the 1st group is Biot-Savart's law, the second is Gauss' theorem, both in differential notation. The first and the second equations in the 2nd group are differential notation of Faraday's law of electromagnetic induction and the condition that no magnetic charges exist, respectively. In total, there are 8 equations (four vector and four scalar) in 10 unknowns: three components of $\vec{E}$, three components of $\vec{H}$, three components of $\vec{j}$, and one component of ρ.

A correlation between the field sources ρ and $\vec{j}$ is set by the *law of conservation of electric charge*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0\,, \tag{3.32}$$

which is a mathematical notation of the experimental fact that an electric charge can not be destroyed, but is merely re-distributed between charged bodies in contact.

Now we have a system of 9 equations in 10 unknowns, so the system defining the field and its sources is still indefinite. The 10th equation, which makes the system definite (the number of equations should be the same as that of the unknowns), is *Lorentz' condition*, which constructs the scalar and vector potentials of the field as follows

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \operatorname{div}\vec{A} = 0\,. \tag{3.33}$$

The Lorentz condition is derived from the fact that the scalar potential φ and the vector potential $\vec{A}$ of any given electromagnetic field, related to the strength vectors $\vec{E}$ and $\vec{H}$ with (3.3), are defined ambiguously from them: $\vec{E}$ and $\vec{H}$ in (3.3) remain unchanged if we replace

$$\vec{A} = \vec{A}' + \vec{\nabla}\Psi\,, \qquad \varphi = \varphi' - \frac{1}{c}\frac{\partial\Psi}{\partial t}\,, \tag{3.34}$$

where Ψ is an arbitrary scalar. Evidently, ambiguous definitions of φ and $\vec{A}$ permit other correlations between the quantities except for the Lorentz condition. Nevertheless, it is the Lorentz condition, which enables transformation of the Maxwell equations into wave equations.

This is how the Lorentz condition does the transformation.

The equation $\mathrm{div}\vec{H} = 0$ (3.31) is satisfied completely, if we assume $\vec{H} = \mathrm{curl}\,\vec{A}$. In this case, the first equation in the 1st group (3.31) takes the form

$$\mathrm{curl}\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0\,, \tag{3.35}$$

which has the solution

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\,. \tag{3.36}$$

Substituting $\vec{H} = \mathrm{curl}\,\vec{A}$ and $\vec{E}$ (3.36) into the 1st group of the Maxwell equations, we obtain

$$\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\mathrm{div}\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) = -\frac{4\pi}{c}\,\vec{j}\,, \tag{3.37}$$

$$\Delta\varphi + \frac{1}{c}\frac{\partial}{\partial t}\left(\mathrm{div}\vec{A}\right) = -\,4\pi\rho\,, \tag{3.38}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is Laplace's regular operator.

Constructing the potentials φ and $\vec{A}$ with the Lorentz condition (3.33), we bring equations in the 1st group to the form

$$\Box\varphi = -\,4\pi\rho\,, \tag{3.39}$$

$$\Box\vec{A} = -\frac{4\pi}{c}\,\vec{j}\,, \tag{3.40}$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ is d'Alembert regular operator.

Applying d'Alembert operator to a field yields equations of propagation of waves of this field (see §2.6). For this reason, the obtained result implies that if the Lorentz condition is true, then the 1st group of the Maxwell equations (3.31) is a system of equations of propagation of waves of the scalar and vector electromagnetic potentials (in the presence of the field-inducing sources — charges and currents). The equations will be obtained in the next section, §3.4.

Henceforth, we are going to consider the Maxwell equations in a pseudo-Riemannian space to obtain them in chr.inv.-form, i. e. formulated with physical observable quantities.

In a four-dimensional pseudo-Riemannian space, the Lorentz condition has general covariant form

$$\nabla_\sigma A^\sigma = \frac{\partial A^\sigma}{\partial x^\sigma} + \Gamma^\sigma_{\sigma\mu} A^\mu = 0\,, \tag{3.41}$$

so it is a condition of conservation of the four-dimensional potential of a given electromagnetic field under consideration. The law of conservation of electric charge (the *continuity equation*) is

$$\nabla_\sigma j^\sigma = 0\,, \tag{3.42}$$

where j^α is the four-dimensional *current vector*, also known as the *shift current*. Chr.inv.-projections of the current vector j^α are the electric charge density

$$\rho = \frac{1}{c}\frac{j_0}{\sqrt{g_{00}}}\,, \tag{3.43}$$

and the spatial current density j^i. Using the chr.inv.-formula for the divergence of an arbitrary vector field (2.107), we obtain the Lorentz condition (3.41) in chr.inv.-form

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t} + \frac{\varphi}{c}\,D + {}^*\nabla_i\, q^i - \frac{1}{c^2}\,F_i\, q^i = 0\,, \tag{3.44}$$

and also the continuity equation in chr.inv.-form

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\nabla_i\, j^i - \frac{1}{c^2}\,F_i\, j^i = 0\,. \tag{3.45}$$

Here, $D = h^{ik}D_{ik} = D^n_n = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}$ is the spur of the tensor of the space deformations rate (1.40). Actually, the spur is the rate of relative expansion of an elementary volume, while ${}^*\nabla_i$ is the operator of chr.inv.-divergence (2.105).

Because F_i (1.38) contains the first derivative of gravitational potential $\mathrm{w} = c^2(1-\sqrt{g_{00}})$, the term $\frac{1}{c^2}\,F_i q^i$ takes into account that the pace of time is different at the opposite walls of the elementary volume. The mentioned formula for gravitational inertial force F_i (1.38) also takes into account the non-stationary nature of the space rotation. Besides, because the operators of chr.inv.-derivation (1.33) are

$$\frac{^*\partial}{\partial t} = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial}{\partial t}\,, \qquad \frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{1}{c^2}\,v_i\frac{^*\partial}{\partial t}\,, \tag{3.46}$$

the condition of conservation of the vector field A^α, namely — the equations (3.44, 3.45), directly depend on gravitational potential and the velocity of the space rotation.

Chr.inv.-quantities $\frac{^*\partial\varphi}{\partial t}$ and $\frac{^*\partial\rho}{\partial t}$ are observable changes in time of the chr.inv.-quantities φ and ρ. Chr.inv.-quantities φD and ρD are observable changes in time of spatial volumes, filled with the quantities φ and ρ.

If there are no gravitational inertial forces, rotation and deformations in the space, then the obtained chr.inv.-formulae for the Lorentz condition (3.44) and the charge conservation law (3.45) become

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \frac{\partial q^i}{\partial x^i} - \frac{\partial\ln\sqrt{h}}{\partial x^i}\,q^i = 0\,, \tag{3.47}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial j^i}{\partial x^i} - \frac{\partial\ln\sqrt{h}}{\partial x^i}\,j^i = 0\,, \tag{3.48}$$

which in a Galilean reference frame in the Minkowski space are

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \frac{\partial q^i}{\partial x^i} = 0\,, \qquad \frac{\partial\rho}{\partial t} + \frac{\partial j^i}{\partial x^i} = 0\,, \tag{3.49}$$

or, in a regular vector notation

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \mathrm{div}\vec{A} = 0\,, \qquad \frac{\partial\rho}{\partial t} + \mathrm{div}\,\vec{j} = 0\,, \tag{3.50}$$

which fully matches notations of the Lorentz condition (3.33) and the charge conservation law (3.32) in Classical Electrodynamics.

Let us turn to the Maxwell equations. In a pseudo-Riemannian space each pair of the equations merge into a single general covariant equation

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,, \tag{3.51}$$

where $F^{\mu\sigma}$ is contravariant (upper-index) form of the electromagnetic field tensor, $F^{*\mu\sigma}$ is its dual pseudotensor. Using chr.inv.-formulae for the divergence of an arbitrary antisymmetric tensor of the 2nd rank (2.121, 2.122) and for its dual pseudotensor (2.135, 2.136), we arrive at the Maxwell equations in chr.inv.-form

$$\left.\begin{aligned} &{}^*\nabla_i\, E^i - \frac{1}{c}\, H^{ik} A_{ik} = 4\pi\rho \\ &{}^*\nabla_k\, H^{ik} - \frac{1}{c^2}\, F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + DE^i\right) = \frac{4\pi}{c}\, j^i \end{aligned}\right\}\ \mathrm{I}\,, \tag{3.52}$$

$$\left.\begin{aligned} &{}^*\nabla_i\, H^{*i} - \frac{1}{c}\, E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k\, E^{*ik} - \frac{1}{c^2}\, F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right) = 0 \end{aligned}\right\}\ \mathrm{II}\,. \tag{3.53}$$

The Maxwell equations in this chr.inv.-notation were first obtained by Jose del Prado and Nikolai Pavlov [25] independently (Zelmanov asked these students to do this, and explained how to do).

Now, let us transform the Maxwell chr.inv.-equations in a way that they include E^i and H^{*i} as unknowns. Obtaining the quantities from their definitions (2.125, 2.124, 2.111)

$$H_{*i} = \frac{1}{2}\, \varepsilon_{imn} H^{mn}, \tag{3.54}$$

$$E^{*ik} = \varepsilon^{ikm}\left(\frac{\varphi}{c^2}\, F_m - \frac{{}^*\partial\varphi}{\partial x^m} - \frac{1}{c}\frac{{}^*\partial q_m}{\partial t}\right) = -\,\varepsilon^{ikm} E_m\,, \tag{3.55}$$

and multiplying the first equation by ε^{ipq}, we obtain

$$\varepsilon^{ipq} H_{*i} = \frac{1}{2}\, \varepsilon^{ipq}\varepsilon_{imn} H^{mn} = \frac{1}{2}\left(\delta^p_m \delta^q_n - \delta^q_m \delta^p_n\right) H^{mn} = H^{pq}. \tag{3.56}$$

Substituting the result as $H^{ik} = \varepsilon^{mik} H_{*m}$ into the first equation in the 1st group (3.52) we bring it to the form

$${}^*\nabla_i\, E^i - \frac{2}{c}\, \Omega_{*m} H^{*m} = 4\pi\rho\,, \tag{3.57}$$

where $\Omega^{*i} = \frac{1}{2}\, \varepsilon^{imn} A_{mn}$ is the chr.inv.-pseudovector of angular velocities of the space rotation. Substituting $E^{*ik} = -\varepsilon^{ikm} E_m$ (3.55) into the first equation of the 2nd group (3.53), we obtain

$${}^*\nabla_i\, H^{*i} + \frac{2}{c}\, \Omega_{*m} E^m = 0\,. \tag{3.58}$$

Then, substituting $H^{ik} = \varepsilon^{mik} H_{*m}$ into the second equation in the 2nd group (3.52) we obtain

$${}^*\nabla_k \left(\varepsilon^{mik} H_{*m}\right) - \frac{1}{c^2}\, F_k \varepsilon^{mik} H_{*m} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\, E^i\right) = \frac{4\pi}{c}\, j^i \tag{3.59}$$

and, after multiplying both sides by $\sqrt{h}$ and taking ${}^*\nabla_k\, \varepsilon^{mik} = 0$ into account, we bring this formula (3.59) to the form

$$\varepsilon^{ikm}\, {}^*\nabla_k \big(H_{*m}\sqrt{h}\big) - \frac{1}{c^2}\, \varepsilon^{ikm} F_k H_{*m}\sqrt{h} - \frac{1}{c}\frac{{}^*\partial}{\partial t}\big(E^i\sqrt{h}\big) = \frac{4\pi}{c}\, j^i\sqrt{h} \tag{3.60}$$

or, in the other notation

$$\varepsilon^{ikm}\, {}^*\widetilde{\nabla}_k \big(H_{*m}\sqrt{h}\big) - \frac{1}{c}\frac{{}^*\partial}{\partial t}\big(E^i\sqrt{h}\big) = \frac{4\pi}{c}\, j^i\sqrt{h}\,, \tag{3.61}$$

where $j^i\sqrt{h}$ is the current's volume density and ${}^*\widetilde{\nabla}_k = {}^*\nabla_k - \frac{1}{c^2} F_k$ is physical chr.inv.-divergence (2.106), which takes into account the fact that the pace of time accounts is different at the opposite walls of the elementary volume.

The obtained equation (3.60) is chr.inv.-notation for the Biot-Savart law in the pseudo-Riemannian space.

Substituting $E^{*ik} = -\varepsilon^{ikm} E_m$ (3.55) into the second equation in the 2nd group (3.53), after similar transformations we obtain

$$\varepsilon^{ikm}\, {}^*\widetilde{\nabla}_k \left(E_m\sqrt{h}\right) + \frac{1}{c}\frac{{}^*\partial}{\partial t}\big(H^{*i}\sqrt{h}\big) = 0\,, \tag{3.62}$$

which is chr.inv.-notation for the Faraday law of electromagnetic induction in the pseudo-Riemannian space.

So, the final system of 10 chr.inv.-equations in 10 unknowns (two groups of the Maxwell equations, the Lorentz condition, and the continuity equation), which define an electromagnetic field and its sources in the pseudo-Riemannian space, is

$$\left.\begin{aligned} &{}^*\nabla_i\, E^i - \frac{2}{c}\,\Omega_{*m} H^{*m} = 4\pi\rho \\ &\varepsilon^{ikm}\, {}^*\widetilde{\nabla}_k \left(H_{*m}\sqrt{h}\right) - \frac{1}{c}\frac{{}^*\partial}{\partial t}\big(E^i\sqrt{h}\big) = \frac{4\pi}{c}\, j^i\sqrt{h} \end{aligned}\right\}\ \mathrm{I}\,, \tag{3.63}$$

$$\left.\begin{aligned} &{}^*\nabla_i\, H^{*i} + \frac{2}{c}\,\Omega_{*m} E^m = 0 \\ &\varepsilon^{ikm}\, {}^*\widetilde{\nabla}_k \left(E_m\sqrt{h}\right) + \frac{1}{c}\frac{{}^*\partial}{\partial t}\big(H^{*i}\sqrt{h}\big) = 0 \end{aligned}\right\}\ \mathrm{II}\,, \tag{3.64}$$

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t} + \frac{\varphi}{c}\, D + {}^*\widetilde{\nabla}_i\, q^i = 0 \qquad \text{the Lorentz condition,} \tag{3.65}$$

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\widetilde{\nabla}_i\, j^i = 0 \qquad \text{the continuity equation.} \tag{3.66}$$

In a Galilean reference frame in the Minkowski space, the determinant of the metric chr.inv.-tensor $\sqrt{h}=1$, so it is not subject to deformations $D_{ik}=0$, rotation $\Omega_{*m}=0$ or acceleration $F_i=0$ in the space. Then the Maxwell chr.inv.-equations (3.63, 3.64), we have obtained in the pseudo-Riemannian space of the General Theory of Relativity, bring us directly to the Maxwell equations of Classical Electrodynamics written in tensor form

$$\left.\begin{aligned} &\frac{\partial E^i}{\partial x^i}=4\pi\rho \\ &e^{ikm}\left(\frac{\partial H_{*m}}{\partial x^k}-\frac{\partial H_{*k}}{\partial x^m}\right)-\frac{1}{c}\frac{\partial E^i}{\partial t}=\frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \mathrm{I}\,, \tag{3.67}$$

$$\left.\begin{aligned} &\frac{\partial H^{*i}}{\partial x^i}=0 \\ &e^{ikm}\left(\frac{\partial E_m}{\partial x^k}-\frac{\partial E_k}{\partial x^m}\right)-\frac{1}{c}\frac{\partial H^{*i}}{\partial t}=0 \end{aligned}\right\}\ \mathrm{II}\,. \tag{3.68}$$

The same equations, put in vector notation, will be similar to Maxwell's classic equations in three-dimensional Euclidean space (3.31). Besides, the obtained Maxwell chr.inv.-equations in the four-dimensional pseudo-Riemannian space (3.64) show that in the absence of the space rotation the chr.inv.-divergence of the magnetic strength is zero ${}^*\nabla_i\, H^{*i}=0$. In other word, the field magnetic component remains unchanged, if the space is holonomic. In the same time, the divergence of the electric strength in this case is not zero ${}^*\nabla_i\, E^i=4\pi\rho$ (3.63), so the electric component is linked directly to the charge density ρ. Hence a conclusion on "magnetic charge", if it actually exists, should be linked directly to the field of rotation of the space itself.

§3.4 D'Alembert's equations for the electromagnetic potential, and their observable components

As we have already mentioned, d'Alembert's operator, applied to a field, gives equations of propagation of waves of this field. For this reason, d'Alembert's equations for the scalar electromagnetic potential φ are equations of propagation of waves of this scalar field, while for the spatial vector-potential $\vec{A}$ these are equations of propagation of waves of this vector field $\vec{A}$.

General covariant form of d'Alembert equations for the electromagnetic field potential A^α were obtained by Stanyukovich [26], using the 1st group of the Maxwell general covariant equations $\nabla_\sigma\, F^{\mu\sigma}=\frac{4\pi}{c}\, j^\mu$

(3.51) and the Lorentz condition $\nabla_\sigma A^\sigma = 0$ (3.41). His equations are

$$\Box A^\alpha - R^\alpha_\beta A^\beta = -\frac{4\pi}{c}\, j^\alpha, \tag{3.69}$$

where $R^\alpha_\beta = g^{\alpha\mu} R^{\sigma}_{\cdot\mu\beta\sigma}$ is Ricci's tensor, while $R^{\alpha}_{\cdot\mu\beta\sigma}$ is Riemann-Christoffel's tensor of the space curvature. The term $R^\alpha_\beta A^\beta$ is absent in the left part, if the Ricci tensor is zero, so the space metric satisfies Einstein's equations away from gravitating masses. This term can be neglected in that case, where the space curvature is not significant. But, even in the Minkowski space, this problem can be considered in the presence of acceleration and rotation. Even this approximation may reveal, for instance, effects of acceleration and rotation of the observer's reference body on the observable velocity of propagation of electromagnetic waves.

The reason for the above discussion is that obtaining chr.inv.-projections of d'Alembert equations in full is a very difficult task. The resulting equations will be so bulky to make any unambiguous conclusions. Therefore, we will limit the scope of our work to transforming d'Alembert equations into chr.inv.-tensor form for an electromagnetic field in a non-inertial reference frame in the Minkowski space. But this does not affect other sections in this Chapter, where we will go back to the pseudo-Riemannian space of the General Theory of Relativity.

So forth, calculating chr.inv.-projections of d'Alembert equations

$$\Box A^\alpha = -\frac{4\pi}{c}\, j^\alpha \tag{3.70}$$

using general formulae (2.168, 2.169), we obtain

$$\begin{gathered} {}^*\Box\varphi - \frac{1}{c^3}\frac{{}^*\partial}{\partial t}\left(F_k q^k\right) - \frac{1}{c^3}\,F_i\,\frac{{}^*\partial q^i}{\partial t} + \frac{1}{c^2}\,F^i\,\frac{{}^*\partial\varphi}{\partial x^i} + h^{ik}\,\Delta^m_{ik}\,\frac{{}^*\partial\varphi}{\partial x^m} - \\ - h^{ik}\,\frac{1}{c}\frac{{}^*\partial}{\partial x^i}\left(A_{kn}q^n\right) + \frac{1}{c}\,h^{ik}\Delta^m_{ik}A_{mn}q^n = 4\pi\rho\,, \end{gathered} \tag{3.71}$$

$$\begin{gathered} {}^*\Box A^i + \frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left(A_{k\cdot}^{\cdot i}q^k\right) + \frac{1}{c^2}\,A_{k\cdot}^{\cdot i}\,\frac{{}^*\partial q^k}{\partial t} - \frac{1}{c^3}\,\frac{{}^*\partial\left(\varphi F^i\right)}{\partial t} - \\ - \frac{1}{c^3}F^i\,\frac{{}^*\partial\varphi}{\partial t} + \frac{1}{c^2}\,F^k\,\frac{{}^*\partial q^i}{\partial x^k} - \frac{1}{c}\,A^{mi}\,\frac{{}^*\partial\varphi}{\partial x^m} + \frac{1}{c^2}\,\Delta^i_{km}q^m F^k - \\ - h^{km}\left\{\frac{{}^*\partial}{\partial x^k}\left(\Delta^i_{mn}q^n\right) + \frac{1}{c}\frac{{}^*\partial}{\partial x^k}\left(\varphi A_{m\cdot}^{\cdot i}\right) + \right. \\ + \left(\Delta^i_{kn}\Delta^n_{mp} - \Delta^n_{km}\Delta^i_{np}\right)q^p + \frac{\varphi}{c}\left(\Delta^i_{kn}A_{m\cdot}^{\cdot n} - \Delta^n_{km}A_{n\cdot}^{\cdot i}\right) + \\ \left. + \Delta^i_{kn}\,\frac{{}^*\partial q^n}{\partial x^m} - \Delta^n_{km}\,\frac{{}^*\partial q^i}{\partial x^n}\right\} = \frac{4\pi}{c}\,j^i. \end{gathered} \tag{3.72}$$

where we take into account the observable charge density $\rho = \frac{1}{c\sqrt{g_{00}}}\, g_{0\alpha}\, j^{\alpha}$ in the space out of dynamic deformations, and in the linear approximation (with higher order terms ignored — we assume that fields of gravitation and the space rotation are weak).

We see that physical observable properties of the reference space, namely — the quantities F^i, A_{ik}, D_{ik}, and Δ^i_{km} constitute some additional "sources" that together with the sources φ and j^i induce waves travelling through the given electromagnetic field.

Let us now analyze the results. At first, we consider the obtained equations (3.71, 3.72) in a Galilean reference frame in the Minkowski space. Here the metric takes the form as in formula (3.5) and therefore d'Alembert chr.inv.-operator $^*\Box$ (2.163) transforms into d'Alembert regular operator $^*\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta = \Box$. Then the obtained equations (3.71, 3.72) will be

$$\Box\varphi = 4\pi\rho\,, \qquad \Box q^i = -\frac{4\pi}{c}\, j^i, \tag{3.73}$$

which fully matches the respective equations of Classical Electrodynamics (3.39, 3.40).

Now we return to the obtained d'Alembert chr.inv.-equations (3.39, 3.40). To make their analysis easier we denote all terms in the left hand sides of the scalar equation (3.39) as T and of the vector equation (3.40) as B^i. Transpositioning the variables into their rightful positions and expanding the formulae for $^*\Box$ (2.173) we obtain

$$\frac{1}{c^2}\frac{^*\partial^2\varphi}{\partial t^2} - h^{ik}\, {}^*\nabla_i\, {}^*\nabla_k\, \varphi = T + 4\pi\rho\,, \tag{3.74}$$

$$\frac{1}{c^2}\frac{^*\partial^2 q^i}{\partial t^2} - h^{mk}\, {}^*\nabla_m {}^*\nabla_k\, q^i = B^i + \frac{4\pi}{c} j^i, \tag{3.75}$$

where $h^{ik}\, {}^*\nabla_i\, {}^*\nabla_k = {}^*\Delta$ is Laplace chr.inv.-operator. As it is easy to see, if the potentials φ and q^i are stationary (they don't depend on time), the d'Alembert wave equations become the Laplace equations

$$^*\Delta\varphi = T + 4\pi\rho\,, \tag{3.76}$$

$$^*\Delta q^i = B^i + \frac{4\pi}{c}\, j^i, \tag{3.77}$$

which characterize static states of this field.

A field is homogeneous along a direction, if its regular derivative with respect to this direction is zero. In Riemannian spaces, a field is homogeneous if its general covariant derivative is zero. If a tensor field located in a Riemannian space is considered in the accompanying reference frame, then observable inhomogeneity of this field is characterized by the difference of chr.inv.-operator ${}^*\nabla_i$ taken from the field potential from zero [9, 11–13]. In other words, if for a scalar quantity A the condition ${}^*\nabla_i A = 0$ is true, then the field A *is observed* as homogeneous.

Therefore, the d'Alembert chr.inv.-operator ${}^*\Box$ is the difference between the 2nd derivatives of the operator $\frac{1}{c}\frac{{}^*\partial}{\partial t}$, which characterizes observable non-stationarity of the field, and the operator ${}^*\nabla_i$, which characterizes its observable spatial inhomogeneity. If the field is stationary and homogeneous, then the left hand sides of the d'Alembert equations (3.74, 3.75) are zeroes, so this field does not generate waves — it is not a wave field.

In an inhomogeneous stationary field (${}^*\nabla_i \neq 0$, $\frac{1}{c}\frac{{}^*\partial}{\partial t} = 0$) the d'Alembert equations (3.74, 3.75) characterize a standing wave

$$-\,h^{ik}\,{}^*\nabla_i\,{}^*\nabla_k\,\varphi = T + 4\pi\rho\,, \tag{3.78}$$

$$-\,h^{mk}\,{}^*\nabla_m{}^*\nabla_k\,q^i = B^i + \frac{4\pi}{c}\,j^i. \tag{3.79}$$

In a homogeneous non-stationary field (${}^*\nabla_i = 0$, $\frac{1}{c}\frac{{}^*\partial}{\partial t} \neq 0$) the d'Alembert equations describe changes of the field with time depending on the field-inducing sources (charges and currents)

$$\frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} = T + 4\pi\rho\,, \tag{3.80}$$

$$\frac{1}{c^2}\frac{{}^*\partial^2 q^i}{\partial t^2} = B^i + \frac{4\pi}{c}\,j^i. \tag{3.81}$$

In an inertial reference frame (the Christoffel symbols are zero) general covariant derivative equals to the regular one ${}^*\nabla_i\,\varphi = \frac{{}^*\partial\varphi}{\partial x^i}$, so the d'Alembert scalar chr.inv.-equation (3.74) is

$$\frac{1}{c^2}\frac{{}^*\partial^2\varphi}{\partial t^2} - h^{ik}\,\frac{{}^*\partial^2\varphi}{\partial x^i\,\partial x^k} = T + 4\pi\rho\,. \tag{3.82}$$

Here, the left hand side takes the most simple form, which facilitates more detailed study of it. As it is known from the theory of oscillations in mathematical physics, in the d'Alembert equations in their regular form

$$\Box\varphi = \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} + g^{ik}\,\frac{\partial^2\varphi}{\partial x^i\partial x^k} \tag{3.83}$$

the term a is the absolute value of the three-dimensional velocity of elastic oscillations which spread across the field φ.

Expanding chr.inv.-derivatives by spatial coordinates (3.46) we bring the d'Alembert scalar equation (3.82) to the form

$$\begin{aligned}\frac{1}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{{}^*\partial^2\varphi}{\partial t^2}-h^{ik}\frac{\partial^2\varphi}{\partial x^i\partial x^k}+\frac{2v^k}{c^2-\mathrm{w}}\frac{\partial^2\varphi}{\partial x^k\partial t}+\\+\frac{1}{c^2-\mathrm{w}}\,h^{ik}\frac{\partial v_k}{\partial x^i}\frac{\partial\varphi}{\partial t}+\frac{1}{c^2}\,v^kF_k\frac{\partial\varphi}{\partial t}=T+4\pi\rho\,,\end{aligned}\tag{3.84}$$

where $v^2 = h_{ik}v^iv^k$ and the second chr.inv.-derivative with respect to time formulates with regular derivatives as follows

$$\frac{{}^*\partial^2\varphi}{\partial t^2}=\frac{1}{\left(1-\frac{\mathrm{w}}{c^2}\right)^2}\frac{\partial^2\varphi}{\partial t^2}+\frac{1}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)^3}\frac{\partial\mathrm{w}}{\partial t}\frac{\partial\varphi}{\partial t}\,.\tag{3.85}$$

We can now see that, the square of the linear velocity of the space rotation v^2 has a greater effect, while the observable non-stationarity of the field (the term $\frac{{}^*\partial\varphi}{\partial t}$) has a lesser effect on propagation of the waves. In the ultimate case, where $v\rightarrow c$, the d'Alembert operator becomes the Laplace operator, so the d'Alembert wave equations becomes the Laplace stationary equations. At low velocities of the space rotation, $v \ll c$, one assumes that observable waves of electromagnetic waves propagate at the light velocity.

Generally, the absolute value of the observable velocity of waves of the scalar electromagnetic potential $\mathrm{v}_{(\varphi)}$ becomes

$$\mathrm{v}_{(\varphi)}=\frac{c}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,.\tag{3.86}$$

It is evident that the chr.inv.-quantity (3.85), which is the observable acceleration of the scalar potential φ, is quite different from the analogous "coordinate" quantity; the higher the gravitational potential, the higher the rate of change of the gravitational potential with time

$$\frac{\partial^2\varphi}{\partial t^2}=\left(1-\frac{\mathrm{w}}{c^2}\right)^2\frac{{}^*\partial^2\varphi}{\partial t^2}+\frac{1}{c^2-\mathrm{w}}\frac{\partial\mathrm{w}}{\partial t}\frac{\partial\varphi}{\partial t}\,.\tag{3.87}$$

In the ultimate case, where $\mathrm{w}\rightarrow c^2$ (approaching gravitational collapse as on the surface of a gravitational collapsar), observable accelerations of the scalar potential become infinitesimal, while the coordinate rate of growth of the potential $\frac{\partial\varphi}{\partial t}$, to the contrary, becomes infinitely large. But under regular conditions, gravitational potential w needs only smaller corrections into the acceleration and the velocity of growth of the potential φ.

All what has been said above about the chr.inv.-scalar quantity $\frac{{}^*\partial^2\varphi}{\partial t^2}$ is also true for the chr.inv.-vector $\frac{{}^*\partial^2 q^i}{\partial t^2}$, because the d'Alembert chr.inv.-operator ${}^*\Box = \frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2} - h^{ik}\frac{{}^*\partial^2}{\partial x^i \partial x^k}$ is different from the scalar and vector functions in only the second term — the Laplace operator, in which chr.inv.-derivatives of the scalar and vector quantities are different from each other, i. e.

$$ {}^*\nabla_i\,\varphi = \frac{{}^*\partial\varphi}{\partial x^i}\,, \qquad {}^*\nabla_i\, q^k = \frac{{}^*\partial q^k}{\partial x^i} + \Delta^k_{im}\, q^m. \tag{3.88} $$

If the space rotation and gravitational potential are infinitesimal, the d'Alembert chr.inv.-operator for the scalar potential becomes the d'Alembert regular operator

$$ {}^*\Box\,\varphi = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - h^{ik}\frac{\partial^2\varphi}{\partial x^i\partial x^k}\,, \tag{3.89} $$

so in this case electromagnetic waves, produced by the scalar potential φ, propagate at the light velocity.

§3.5 Lorentz' force. The energy-momentum tensor of an electromagnetic field

In this section, we are going to deduce chr.inv.-projections (physical observable components) of the four-dimensional force, which results from the fact that electromagnetic fields affect an electric charge in a pseudo-Riemannian space. This problem will be solved for two following cases: a) a point charge; b) a charge distributed in the space. In addition, we are going to deduce chr.inv.-projections of the energy-momentum tensor for an electromagnetic field.

In a three-dimensional Euclidean space of Classical Electrodynamics, motion of a charged particle is characterized by the vector equation

$$ \frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\left[\,\vec{u};\vec{H}\,\right], \tag{3.90} $$

where $\vec{p} = m\vec{u}$ is the particle's three-dimensional momentum vector and m is its relativistic mass. The right hand side of this equation is referred to as *Lorentz' force.*

The equation, characterizing the change of the kinetic (relativistic) energy of the particle

$$ E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{3.91} $$

due to work accomplished by the field's electric strength to displace it within unit time, takes the vector form

$$\frac{dE}{dt} = e\vec{E}\,\vec{u}, \tag{3.92}$$

and is also known as the *live forces theorem.*

In four-dimensional form, thanks to unification of energy and momentum, in a Galilean reference frame in the Minkowski space, both equations (3.90, 3.92) take the form

$$m_0 c\,\frac{dU^\alpha}{ds} = \frac{e}{c}\,F^{\alpha\cdot}_{\cdot\sigma}U^\sigma, \qquad U^\alpha = \frac{dx^\alpha}{ds}\,, \tag{3.93}$$

and are known as the *Minkowski equations* ($F^{\alpha\cdot}_{\cdot\sigma}$ is the electromagnetic field tensor). Because the metric here is diagonal (3.5), hence

$$ds = cdt\sqrt{1-\frac{u^2}{c^2}}\,, \qquad u^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \tag{3.94}$$

and components of the particle's four-dimensional velocity U^α are

$$U^0 = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\,, \qquad U^i = \frac{u^i}{c\sqrt{1-\frac{u^2}{c^2}}}\,, \tag{3.95}$$

where $u^i = \frac{dx^i}{dt}$ is its three-dimensional coordinate velocity. Once components of $\frac{e}{c}F^{\alpha\cdot}_{\cdot\sigma}U^\sigma$ in the Galilean reference frame are

$$\frac{e}{c}\,F^{0\cdot}_{\cdot\sigma}U^\sigma = -\frac{e}{c^2}\frac{E_i u^i}{\sqrt{1-\frac{u^2}{c^2}}}\,, \tag{3.96}$$

$$\frac{e}{c}\,F^{i\cdot}_{\cdot\sigma}U^\sigma = -\frac{1}{c\sqrt{1-\frac{u^2}{c^2}}}\left(eE^i + \frac{e}{c}\,e^{ikm}u_k H_{*m}\right), \tag{3.97}$$

then, in the Galilean reference frame as well, the time and spatial components of the Minkowski equations (3.93) are

$$\frac{dE}{dt} = -\,eE_i u^i, \tag{3.98}$$

$$\frac{dp^i}{dt} = -\left(eE^i + \frac{e}{c}\,e^{ikm}u_k H_{*m}\right), \qquad p^i = mu^i. \tag{3.99}$$

The above relativistic equations, except for the sign at the right positions, match the live forces theorem and the equations of motion of a charged particle in Classical Electrodynamics (3.90, 3.91). Note that difference in signs in the right positions is determined only by choice of the space signature. We use the signature $(+---)$, but if we accept the signature $(-+++)$, then the sign in the right positions of the equations will be the opposite.

We now turn to this problem not in the Minkowski space, but in the pseudo-Riemannian space of the General Theory of Relativity. So forth, chr.inv.-projections of the four-dimensional momentum vector $\Phi^{\alpha} = \frac{e}{c} F^{\alpha\cdot}_{\cdot\sigma} U^{\sigma}$, which the charged particle gains in the pseudo-Riemannian space from interaction of its charge e with the electromagnetic field, are

$$T = \frac{e}{c}\,\frac{F_{0\sigma}U^{\sigma}}{\sqrt{g_{00}}}\,, \tag{3.100}$$

$$B^i = \frac{e}{c}\,F^{i\cdot}_{\cdot\sigma}U^{\sigma} = \frac{e}{c}\left(F^{i\cdot}_{\cdot 0}U^0 + F^{i\cdot}_{\cdot k}U^k\right). \tag{3.101}$$

Given that components of U^{α} are

$$U^0 = \frac{\frac{1}{c^2}\,v_i \mathrm{v}^i \pm 1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\left(1-\frac{\mathrm{w}}{c^2}\right)}\,, \qquad U^i = \frac{\mathrm{v}^i}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{3.102}$$

then, taking into account formulae for components of an arbitrary curl (2.143–2.159), we arrive at

$$T = -\frac{e}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(\frac{^*\partial\varphi}{\partial x^i} + \frac{1}{c}\frac{^*\partial q_i}{\partial t} - \frac{\varphi}{c^2}\,F_i\right)\mathrm{v}^i, \tag{3.103}$$

$$\begin{gathered} B^i = -\frac{e}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left\{\pm\left(\frac{^*\partial\varphi}{\partial x^k} + \frac{1}{c}\frac{^*\partial q_k}{\partial t} - \frac{\varphi}{c^2}\,F_k\right)h^{ik} + \right. \\ \left. + \left[h^{im}h^{kn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}\,A^{ik}\right]\mathrm{v}_k\right\}. \end{gathered} \tag{3.104}$$

Chr.inv.-scalar quantity T, to within the multiplier $-\frac{1}{c^2}$, is the work done by the field to displace this charge e. Chr.inv.-vector quantity B^i, to within the multiplier $\frac{1}{c}$, in a non-relativistic case is a force which acts on the particle due to the electromagnetic field

$$\Phi^i = cB^i = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}H_{*m}\mathrm{v}_k\right), \tag{3.105}$$

and it is the Lorentz observable force. Note that alternating sign is derived here from the fact that in pseudo-Riemannian spaces the square equation with respect to $\frac{dt}{d\tau}$ has two roots (1.55). Respectively, "plus" in the Lorentz force stands for the particle's motion into future (in respect of the observer), while "minus" denotes the motion into past. In a Galilean reference frame in the Minkowski space there is no difference between physical observable time τ and coordinate time t. So, the Lorentz force (3.99) obtained from the Minkowski equations will have no alternating signs.

If the charge is not a point charge but is spatially distributed matter, then the Lorentz force $\Phi^{\alpha} = \frac{e}{c} F^{\alpha\cdot}_{\cdot\sigma} U^{\sigma}$ in the Minkowski equations (3.93) will be replaced by the four-dimensional vector of the *Lorentz force density*

$$f^{\alpha} = \frac{1}{c}\, F^{\alpha\cdot}_{\cdot\sigma}\, j^{\sigma}, \tag{3.106}$$

where the four-dimensional current density $j^{\sigma} = \left\{c\rho;\, j^{i}\right\}$ is defined by the 1st group of the Maxwell equations (3.51)

$$j^{\sigma} = \frac{c}{4\pi}\, \nabla_{\mu}\, F^{\sigma\mu}. \tag{3.107}$$

Chr.inv.-projections of the Lorentz force density f^{α}

$$\frac{f_0}{\sqrt{g_{00}}} = -\frac{1}{c}\, E_i\, j^{i}, \tag{3.108}$$

$$f^{i} = -\left(\rho E^{i} + \frac{1}{c}\, H^{i\cdot}_{\cdot k}\, j^{k}\right) = -\left(\rho E^{i} + \frac{1}{c}\, \varepsilon^{ikm} H_{*m}\, j_k\right). \tag{3.109}$$

in three-dimensional Euclidean space the projections are

$$\frac{f_0}{\sqrt{g_{00}}} = \frac{q}{c} = \frac{1}{c}\, \vec{E}\, \vec{j}\,, \tag{3.110}$$

$$\vec{f} = \rho \vec{E} + \frac{1}{c}\, \left[\, \vec{j}; \vec{H}\,\right], \tag{3.111}$$

where q is the density of a heat power released into a current conductor.

Now, we transform the Lorentz force density (3.106), using the Maxwell equations. Substituting j^{σ} (3.107) we arrive at

$$f_{\nu} = \frac{1}{c} F_{\nu\sigma} j^{\sigma} = \frac{1}{4\pi} F_{\nu\sigma} \nabla_{\mu} F^{\sigma\mu} = \frac{1}{4\pi} \Big[\nabla_{\mu} \left(F_{\nu\sigma} F^{\sigma\mu}\right) - F^{\sigma\mu} \nabla_{\mu} F_{\nu\sigma}\Big]. \tag{3.112}$$

Transpositioning the mute indices μ and σ, by which we add-up, and taking into account that the Maxwell tensor $F_{\alpha\beta}$ is antisymmetric, we transform the second term to the form

$$F^{\sigma\mu}\nabla_\mu F_{\nu\sigma} = \frac{1}{2}F^{\sigma\mu}\left(\nabla_\mu F_{\nu\sigma} + \nabla_\sigma F_{\mu\nu}\right) = \\ = -\frac{1}{2}F^{\sigma\mu}\nabla_\nu F_{\mu\sigma} = \frac{1}{2}F^{\sigma\mu}\nabla_\nu F_{\sigma\mu}\,. \tag{3.113}$$

As a result, for f_ν (3.112) and its contravariant form we obtain

$$f_\nu = \frac{1}{4\pi}\nabla_\mu\left(-F^{\mu\sigma}F_{\nu\sigma} + \frac{1}{4}\,\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}\right), \tag{3.114}$$

$$f^\nu = \nabla_\mu\left[\frac{1}{4\pi}\left(-F^{\mu\sigma}F_{\cdot\sigma}^{\nu\cdot} + \frac{1}{4}\,g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)\right]. \tag{3.115}$$

Denoting the term

$$\frac{1}{4\pi}\left(-\,F^{\mu\sigma}F_{\cdot\sigma}^{\nu\cdot} + \frac{1}{4}\,g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) = T^{\mu\nu}, \tag{3.116}$$

we obtain the expression

$$f^\nu = \nabla_\mu T^{\mu\nu}, \tag{3.117}$$

so the four-dimensional vector of the Lorentz force density f^ν equals the absolute divergence of a quantity $T^{\mu\nu}$, referred to as the *energy-momentum tensor* of the electromagnetic field. Its structure shows that it is symmetric $T^{\mu\nu} = T^{\nu\mu}$, while its spur (given that the spur of the fundamental metric tensor is $g_{\mu\nu}g^{\mu\nu} = \delta^\nu_\nu = 4$) is zero

$$T^\nu_\nu = g_{\mu\nu}T^{\mu\nu} = \frac{1}{4\pi}\left(-\,F^{\mu\sigma}F_{\mu\sigma} + \frac{1}{4}\,g_{\mu\nu}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) = \\ = \frac{1}{4\pi}\left(-\,F^{\mu\sigma}F_{\mu\sigma} + F^{\alpha\beta}F_{\alpha\beta}\right) = 0\,. \tag{3.118}$$

Chr.inv.-projections of the energy-momentum tensor are

$$q = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2T^{ik}, \tag{3.119}$$

where the chr.inv.-scalar q is of the *observable density of the field*, the chr.inv.-vector J^i is the *observable density of the field's momentum*, and the chr.inv.-tensor U^{ik} is the *observable density of the field's momentum flux*. For the energy-momentum tensor of the electromagnetic field (3.116) we obtain the expressions

$$q = \frac{E^2 + H^{*2}}{8\pi}\,, \tag{3.120}$$

$$J^i = \frac{c}{4\pi}\,\varepsilon^{ikm} E_k H_{*m}\,, \tag{3.121}$$

$$U^{ik} = qc^2 h^{ik} - \frac{c^2}{4\pi}\left(E^i E^k + H^{*i} H^{*k}\right), \tag{3.122}$$

where $E^2 = h_{ik}\,E^i E^k$ and $H^{*2} = h_{ik}\,H^{*i} H^{*k}$. Comparing the obtained formula for q (3.120) with that for the energy density of the electromagnetic field from Classical Electrodynamics we have

$$W = \frac{E^2 + H^2}{8\pi}\,, \tag{3.123}$$

where $E^2 = (\vec{E};\vec{E})$ and $H^2 = (\vec{H};\vec{H})$; we can see that q, the chr.inv.-quantity, is the *observable energy density* of the electromagnetic field in the pseudo-Riemannian space. Comparing the obtained formula for the chr.inv.-vector J^i (3.121) with that for Poynting's vector in Classical Electrodynamics we have

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E};\vec{H}\,\right), \tag{3.124}$$

we can see that the J^i is the *Poynting observable vector* in the pseudo-Riemannian space. Correspondence of the third observable component U^{ik} (3.122) to quantities in Classical Electrodynamics can be established using similarities with mechanics of continuous medias, where the three-dimensional tensor of similar structure is the stress tensor for an elementary volume of a media. Therefore, U^{ik} is the *observable stress tensor* of the electromagnetic field in the pseudo-Riemannian space.

Now, we can obtain identities for the chr.inv.-projections of the Lorentz force density (3.108, 3.109), formulating them with chr.inv.-components of the energy-momentum tensor of this field (3.120–3.122). Taking the equation $f^\nu = \nabla_\mu T^{\mu\nu}$ and using ready formulae for chr.inv.-components of the absolute divergence of an arbitrary symmetric tensor of the 2nd rank (2.138, 2.139), we obtain

$$\frac{{}^*\partial q}{\partial t} + qD + \frac{1}{c^2}\,D_{ij}U^{ij} + {}^*\widetilde{\nabla}_i\,J^i - \frac{1}{c^2}\,F_i\,J^i = -\frac{1}{c}\,E_i\,j^i, \tag{3.125}$$

$$\begin{gathered}\frac{{}^*\partial J^k}{\partial t} + DJ^k + 2\left(D^k_i + A^{k\cdot}_{\cdot i}\right)J^i + {}^*\widetilde{\nabla}_i\,U^{ik} - qF^k = \\ = -\left(\rho E^k + \frac{1}{c}\,\varepsilon^{kim} H_{*i}\,j_m\right).\end{gathered} \tag{3.126}$$

The first chr.inv.-identity (3.125) shows that if the observable vector of the current density j^i is orthogonal to the observable electric strength of the field E^i, the right hand side turns to zero. Generally, i. e. in the case of an arbitrary orientation of the vectors j^i and E^i, observable change of the electromagnetic field density with time (the quantity $\frac{{}^*\partial q}{\partial t}$) depends on the following factors:

a) The rate of changes of the observable volume of the space, filled with the electromagnetic field (the term qD);

b) Effect of forces of the space deformations (the term $D_{ij}U^{ij}$);

c) Effect of gravitational inertial force on the electromagnetic field momentum density (the term $F_i J^i$);

d) The observable "spatial variation" (physical divergence) of the electromagnetic field momentum density (the term ${}^*\widetilde{\nabla}_i J^i$);

e) Magnitudes and mutual orientation of the current density vector j^i and the electric strength vector E^i (the right hand side).

The second chr.inv.-identity (3.126) shows that observable change of the electromagnetic field momentum density with time (i. e. the quantity $\frac{{}^*\partial J^k}{\partial t}$) depends on the following factors:

a) The rate of changes of the observable volume of the space, filled with the electromagnetic field (the term DJ^k);

b) Forces of the space deformation and Coriolis' forces, which are designated by the term $2\left(D_i^k + A_{\cdot i}^{k\cdot}\right)J^i$;

c) Effect of gravitational inertial force on the observable density of the electromagnetic field (the term qF^k);

d) The observable "spatial variation" of the field stress ${}^*\widetilde{\nabla}_i U^{ik}$;

e) Effect of the Lorentz force observable density — the right hand side, defined by the quantity $f^k = -\left(\rho E^k + \frac{1}{c}\,\varepsilon^{kim} H_{*i}\, j_m\right)$.

In conclusion, we consider a particular case, where the electromagnetic field is isotropic. A formal definition of isotropic fields made with the Maxwell tensor [20] is a set of two conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.127}$$

which implies that both field invariants $J_1 = F_{\mu\nu}F^{\mu\nu}$ and $J_2 = F_{\mu\nu}F^{*\mu\nu}$ (3.25, 3.26) are zeroes. In chr.inv.-notation, taking (3.28) into account, the conditions take the form

$$E^2 = H^{*2}, \qquad E_i H^{*i} = 0\,. \tag{3.128}$$

We see that an electromagnetic field in a pseudo-Riemannian space is observed as isotropic, if the observable lengths of its electric and magnetic strength vectors are equal, while the Poynting vector J^i expressed with (3.121) is

$$J^i = \frac{c}{4\pi}\,\varepsilon^{ikm}E_k H_{*m}\,. \tag{3.129}$$

In the terms of chr.inv.-components of the energy-momentum tensor (3.120, 3.121) the obtained conditions (3.128) also imply that

$$J = c\,q\,, \tag{3.130}$$

where $J = \sqrt{J^2}$ and $J^2 = h_{ik}J^iJ^k$. In other words, the length J of the momentum density chr.inv.-vector of any isotropic electromagnetic field depends only on the field density q.

§3.6 Equations of motion of a charged particle, obtained using the parallel transfer method

In this section, we will obtain chr.inv.-equations of motion of a charged mass-bearing test-particle in an electromagnetic field, located in a four-dimensional pseudo-Riemannian space*.

The equations are chr.inv.-projections of parallel transfer equations of the four-dimensional summary vector

$$Q^\alpha = P^\alpha + \frac{e}{c^2}\,A^\alpha, \tag{3.131}$$

where $P^\alpha = m_0\frac{dx^\alpha}{ds}$ is the four-dimensional momentum vector of the particle, and $\frac{e}{c^2}A^\alpha$ is a part of the previous — an additional four-dimensional momentum which the particle gains from interaction of its charge e with the electromagnetic field potential A^α deviating its trajectory from a geodesic line. Given this problem statement, parallel transfer of superposition on the non-geodesic momentum of the particle and the deviating vector is also geodesic, so that

$$\frac{d}{ds}\left(P^\alpha + \frac{e}{c^2}\,A^\alpha\right) + \Gamma^\alpha_{\mu\nu}\left(P^\mu + \frac{e}{c^2}\,A^\mu\right)\frac{dx^\nu}{ds} = 0\,. \tag{3.132}$$

By definition, a geodesic line is a *line of constant direction*, so the one for which any vector tangential to it in a given point will remain tangential along the line being subjected to parallel transfer [9].

Equations of motion may be obtained in another way, namely — by considering motion along a line of the least (extremum) length using the least action principle. Extremum length lines are also lines of constant

*Generally, using the method described herein we can also obtain equations of motion for a particle, which is not a test one. A test particle is one with charge and mass so small that they do not affect electromagnetic and gravitational fields in which it moves.

direction. But, for instance, in spaces with non-metric geometry, length is not defined as category. Therefore, lines of extremum lengths are neither defined and we can not use the least action method to obtain the equations. Nevertheless, even in non-metric spaces we can define lines of constant direction and non-zero derivation parameter along them. Hence, one can assume that in metric spaces, to which Riemannian spaces belong, lines of extremum length are merely a particular case of constant direction lines.

In accordance with general formulae we have obtained in Chapter 2, chr.inv.-projections of the parallel transfer equations (3.132) are defined as follows

$$\frac{d\tilde{\varphi}}{ds} + \frac{1}{c}\left(-F_i\,\tilde{q}^i\,\frac{d\tau}{ds} + D_{ik}\,\tilde{q}^i\,\frac{dx^k}{ds}\right) = 0\,, \tag{3.133}$$

$$\frac{d\tilde{q}^i}{ds}+\left(\frac{\tilde{\varphi}}{c}\frac{dx^k}{ds}+\tilde{q}^k\frac{d\tau}{ds}\right)\left(D^i_k+A^{\cdot i}_{k\cdot}\right)-\frac{\tilde{\varphi}}{c}\,F^i\,\frac{d\tau}{ds}+\Delta^i_{mk}\,\tilde{q}^m\,\frac{dx^k}{ds}=0\,, \tag{3.134}$$

where the space-time interval s is assumed as the derivation parameter along the trajectory, $\tilde{\varphi}$ and $\tilde{q}^i$ are chr.inv.-projections of the dynamic vector Q^α (3.131) of this particle

$$\tilde{\varphi} = b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}} = \frac{1}{\sqrt{g_{00}}}\left(P_0 + \frac{e}{c^2}\,A_0\right), \tag{3.135}$$

$$\tilde{q}^i = h^i_\alpha Q^\alpha = Q^i = P^i + \frac{e}{c^2}\,A^i. \tag{3.136}$$

Chr.inv.-projections of the momentum vector are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm\, m\,, \qquad P^i = \frac{1}{c}\,m\mathrm{v}^i = \frac{1}{c}\,p^i, \tag{3.137}$$

where "plus" stands for motions into the future (with respect to the observer), while "minus" appears if the particle moves into the past, and $p^i = m\frac{dx^i}{d\tau}$ is the three-dimensional momentum chr.inv.-vector of the particle. Chr.inv.-projections of the additional momentum vector $\frac{e}{c^2}A^\alpha$ are as follows

$$\frac{e}{c^2}\frac{A_0}{\sqrt{g_{00}}} = \frac{e}{c^2}\,\varphi\,, \qquad \frac{e}{c^2}\,A^i = \frac{e}{c^2}\,q^i, \tag{3.138}$$

where φ is the scalar potential and q^i is the vector-potential of the acting electromagnetic field — these are chr.inv.-components of the four-dimensional field potential A^α (3.8). Then the quantities $\tilde{\varphi}$ (3.135) and $\tilde{q}^i$ (3.136), which actually are chr.inv.-projections of the summary vector

Q^α, take the form

$$\tilde{\varphi} = \pm\, m + \frac{e}{c^2}\,\varphi\,, \tag{3.139}$$

$$\tilde{q}^i = \frac{1}{c}\left(p^i + \frac{e}{c^2}\,q^i\right). \tag{3.140}$$

We now substitute the quantities $\tilde{\varphi}$ and $\tilde{q}^i$ into general formulae for chr.inv.-equations of motion (3.133, 3.134). Moving the terms, which characterize electromagnetic interaction, into the right positions we arrive at the chr.inv.-equations of motion for the our-world charged particle (the particle moves into the future with respect to a regular observer)

$$\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\,\mathrm{v}^i\mathrm{v}^k = -\frac{e}{c^2}\frac{d\varphi}{d\tau} + \frac{e}{c^3}\left(F_i q^i - D_{ik}\,q^i\mathrm{v}^k\right), \tag{3.141}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{e}{c}\frac{dq^i}{d\tau} - \frac{e}{c}\left(\frac{\varphi}{c}\,\mathrm{v}^k + q^k\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + \frac{e\varphi}{c^2}F^i - \frac{e}{c}\,\Delta^i_{nk}\,q^n\mathrm{v}^k,\end{gathered} \tag{3.142}$$

while for the analogous particle located in the mirror-world (it moves into the past with respect to the observer) the equations are

$$-\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k = -\frac{e}{c^2}\frac{d\varphi}{d\tau} + \frac{e}{c^3}\left(F_i\,q^i - D_{ik}\,q^i\mathrm{v}^k\right), \tag{3.143}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{e}{c}\frac{dq^i}{d\tau} - \frac{e}{c}\left(\frac{\varphi}{c}\,\mathrm{v}^k + q^k\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + \frac{e\varphi}{c^2}\,F^i - \frac{e}{c}\,\Delta^i_{nk}\,q^n\mathrm{v}^k.\end{gathered} \tag{3.144}$$

As it is easy to see, the left hand side of the equations fully match those of the chr.inv.-equations of motion of this particle, provided the particle is free. The only difference is that the equations include terms, which characterize its non-geodesic motion. Therefore, the right hand sides here are not zeroes. The right hand sides account for the effect that the electromagnetic field produces on the particle, as well as the effect from physical and geometric properties of the space (F^i, A_{ik}, D_{ik}, Δ^i_{nk}). It is evident that, if the particle becomes charge-free, $e = 0$, the right hand sides turn to zero and the resulting equations fully match the chr.inv.-equations of motion of a free mass-bearing particle (see formulae 1.51, 1.52 and also 1.56, 1.57).

Let us consider the right hand sides in details. The obtained equations are absolutely symmetric for motions either into the future or the past and they change their sign once the sign of the charge changes. We denote the right hand sides of the scalar chr.inv.-equations of motion (3.141, 3.143) as T. Given that

$$\frac{d\varphi}{d\tau} = \frac{^*\partial\varphi}{\partial t} + \mathrm{v}^i \frac{^*\partial\varphi}{\partial x^i}\,, \tag{3.145}$$

then using the formula for the electric strength in covariant form E_i (3.14), we can represent T as follows

$$\begin{aligned} T = -\frac{e}{c^2}\,E_i\mathrm{v}^i - \frac{e}{c^2}\frac{^*\partial\varphi}{\partial t} + \\ + \frac{e}{c^3}\left(\frac{^*\partial q_i}{\partial t} - D_{ik}q^k\right)\mathrm{v}^i + \frac{e}{c^3}\left(q^i - \frac{\varphi}{c}\,\mathrm{v}^i\right)F_i\,. \end{aligned} \tag{3.146}$$

Substituting this formula into (3.141, 3.143) and multiplying the results by c^2, we obtain the equation for the relativistic energy $E = \pm mc^2$ of the charged particle, which moves into the future and into the past, respectively

$$\begin{aligned} \frac{dE}{d\tau} - mF_i\mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -eE_i\mathrm{v}^i - e\,\frac{^*\partial\varphi}{\partial t} + \\ + \frac{e}{c}\left(\frac{^*\partial q_i}{\partial t} - D_{ik}q^k\right)\mathrm{v}^i + \frac{e}{c}\left(q^i - \frac{\varphi}{c}\,\mathrm{v}^i\right)F_i\,, \end{aligned} \tag{3.147}$$

$$\begin{aligned} -\frac{dE}{d\tau} - mF_i\mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -eE_i\mathrm{v}^i - e\,\frac{^*\partial\varphi}{\partial t} + \\ + \frac{e}{c}\left(\frac{^*\partial q_i}{\partial t} - D_{ik}q^k\right)\mathrm{v}^i + \frac{e}{c}\left(q^i - \frac{\varphi}{c}\,\mathrm{v}^i\right)F_i\,, \end{aligned} \tag{3.148}$$

where $eE_i\mathrm{v}^i$ is the work done by the electric component of the field to displace the particle in unit time.

The scalar chr.inv.-equations of motion of a charged particle (3.147, 3.148) make the *theorem of live forces* in the pseudo-Riemannian space, represented in chr.inv.-form. As it is easy to see, in a Galilean reference frame the scalar equation for the particle which moves into the future (3.147) matches the time component of the Minkowski equations (3.98). In three-dimensional Euclidean space, the equation (3.147) transforms into the theorem of live forces from Classical Electrodynamics which is $\frac{dE}{dt} = e\vec{E}\vec{u}$ (3.92).

Let us turn to the right hand sides of the vector chr.inv.-equations of motion (3.142, 3.144). We denote them as M^i. Because of

$$\frac{dq^i}{d\tau} = \frac{^*\partial q^i}{\partial t} + \mathrm{v}^k \frac{^*\partial q^i}{\partial x^k}\,, \tag{3.149}$$

and in it, taking into account, that $\frac{{}^*\partial h^{ik}}{\partial t} = -2D^{ik}$ (1.40)

$$\frac{{}^*\partial q^i}{\partial t} = \frac{{}^*\partial}{\partial t}\left(h^{ik}q_k\right) = -2D^i_k q^k + h^{ik}\frac{{}^*\partial q_k}{\partial t}\,, \tag{3.150}$$

then M^i takes the form

$$\begin{gathered} M^i = -\frac{e}{c}\,h^{ik}\frac{{}^*\partial q_k}{\partial t} + \frac{e\varphi}{c^2}\left(F^i + A^{ik}\mathrm{v}_k\right) + \frac{e}{c}\,A^{ik}q_k + \\ + \frac{e}{c}\left(q^k - \frac{\varphi}{c}\,\mathrm{v}^k\right)D^i_k - \frac{e}{c}\,\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{c}\,\Delta^i_{nk}q^n\mathrm{v}^k. \end{gathered} \tag{3.151}$$

Using formulae for chr.inv.-components E^i (3.11) and H^{ik} (3.12) of the Maxwell tensor $F_{\alpha\beta}$, we write down the first two terms from M^i (3.151) and the third term as follows

$$-\frac{e}{c}\,h^{ik}\frac{{}^*\partial q_k}{\partial t} + \frac{e\varphi}{c^2}\,F^i = -\,eE^i + eh^{ik}\frac{{}^*\partial\varphi}{\partial x^k}\,, \tag{3.152}$$

$$\frac{e\varphi}{c^2}\,A^{ik}\mathrm{v}_k = \frac{e}{2c}\,h^{im}\mathrm{v}^n\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{e}{2c}\,H^{ik}\mathrm{v}_k\,. \tag{3.153}$$

We write down the quantity H^{ik} as $H^{ik} = \varepsilon^{mik}H_{*m}$ (3.56). Then we have the following

$$\frac{e\varphi}{c^2}\,A^{ik}\mathrm{v}_k = \frac{e}{2c}\,h^{im}\mathrm{v}^n\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{e}{2c}\,\varepsilon^{ikm}H_{*m}\mathrm{v}_k\,, \tag{3.154}$$

$$\begin{gathered} M^i = -e\left(E^i + \frac{1}{2c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right) + \frac{e}{c}\left(q^k - \frac{\varphi}{c}\,\mathrm{v}^k\right)D^i_k + \\ + eh^{ik}\frac{{}^*\partial\varphi}{\partial x^k} + \frac{e}{c}\,A^{ik}q_k + \frac{e}{2c}\,h^{im}\mathrm{v}^k\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \\ - \frac{e}{c}\,\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{c}\,\Delta^i_{nk}q^n\mathrm{v}^k, \end{gathered} \tag{3.155}$$

and the sum of the latter three terms in M^i equals

$$\begin{gathered} \frac{e}{2c}\,h^{im}\mathrm{v}^k\left(\frac{{}^*\partial q_m}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^m}\right) - \frac{e}{c}\,\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{c}\,\Delta^i_{nk}q^n\mathrm{v}^k = \\ = -\frac{e}{2c}\,h^{im}\mathrm{v}_k\frac{{}^*\partial q^k}{\partial x^m} - \frac{e}{2c}\,\mathrm{v}^k\frac{{}^*\partial q^i}{\partial x^k} - \frac{e}{2c}\,h^{im}q^n\mathrm{v}^k\frac{{}^*\partial h_{km}}{\partial x^n}\,. \end{gathered} \tag{3.156}$$

At last, the vector chr.inv.-equations of motion of the charged particle (3.142, 3.144) which moves into the future and into the past take the form, respectively

$$\begin{aligned}
&\frac{d\left(m\mathrm{v}^i\right)}{d\tau} - mF^i + 2m\left(D_k^i + A_{k\cdot}^{\cdot i}\right)\mathrm{v}^k + m\Delta_{nk}^i \mathrm{v}^n \mathrm{v}^k = \\
&= -\,e\left(E^i + \frac{1}{2c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right) + \\
&+ \frac{e}{c}\left(q^k - \frac{\varphi}{c}\,\mathrm{v}^k\right)D_k^i + eh^{ik}\frac{^*\partial\varphi}{\partial x^k} + \frac{e}{c}\,A^{ik}q_k - \\
&- \frac{e}{2c}\,h^{im}\mathrm{v}_k\frac{^*\partial q^k}{\partial x^m} - \frac{e}{2c}\,\mathrm{v}^k\frac{^*\partial q^i}{\partial x^k} - \frac{e}{2c}\,h^{im}q^n\mathrm{v}^k\frac{^*\partial h_{km}}{\partial x^n}\,,
\end{aligned} \tag{3.157a}$$

$$\begin{aligned}
&\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + mF^i + m\Delta_{nk}^i \mathrm{v}^n \mathrm{v}^k = \\
&= -\,e\left(E^i + \frac{1}{2c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right) + \\
&+ \frac{e}{c}\left(q^k - \frac{\varphi}{c}\,\mathrm{v}^k\right)D_k^i + eh^{ik}\frac{^*\partial\varphi}{\partial x^k} + \frac{e}{c}\,A^{ik}q_k - \\
&- \frac{e}{2c}\,h^{im}\mathrm{v}_k\frac{^*\partial q^k}{\partial x^m} - \frac{e}{2c}\,\mathrm{v}^k\frac{^*\partial q^i}{\partial x^k} - \frac{e}{2c}\,h^{im}q^n\mathrm{v}^k\frac{^*\partial h_{km}}{\partial x^n}\,.
\end{aligned} \tag{3.157b}$$

From here we see that the first term $-e\big(E^i + \frac{1}{2c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\big)$ in their right hand sides is different from the Lorentz chr.inv.-force, which is $\Phi^i = -e\big(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\big)$, by the coefficient $\frac{1}{2}$ on the term that stands for the magnetic component of the force. This fact is very surprising, because regular equations of motion of a charged particle, being three-dimensional components of the general covariant equations, contain the Lorentz force in full form. In §3.9 we are going to show the structure of the electromagnetic field potential A^α at which the other terms in the M^i fully compensate this coefficient $\frac{1}{2}$ so that only the Lorentz force is left.

§3.7 Equations of motion, obtained using the least action principle as a particular case of the previous equations

In this section, we are going to deduce chr.inv.-equations of motion of a mass-bearing charged particle, using the least action principle. The principle says that an action S to displace a particle along the shortest trajectory is the least, so the variation of the action is zero

$$\delta\int_a^b dS = 0\,. \tag{3.158}$$

Therefore, equations of motion, obtained from the least action principle are equations of the shortest lines.

The elementary action of gravitational and electromagnetic fields to displace a charged particle at an elementary space-time interval ds is [10]

$$dS = -m_0 c\, ds - \frac{e}{c}\, A_\alpha dx^\alpha. \tag{3.159}$$

We see that this quantity is only applicable to characterize particles which move along non-isotropic trajectories ($ds \neq 0$). On the other hand, obtaining equations of motion through the parallel transfer method (constant direction lines) is equally applicable to both non-isotropic ($ds \neq 0$) and isotropic trajectories ($ds = 0$). Moreover, parallel transfer is as well applicable to non-metric geometries, in particular, to obtain equations of motion of particles in a fully degenerate space-time (zero-space). Therefore, equations of the shortest length lines, because they are obtained through the least action method, are merely a narrow particular case of constant direction lines, which result from parallel transfer.

But we are returning to the least action principle (3.158). For the charged particle we are considering the condition takes the form

$$\delta\int_a^b dS = -\,\delta\int_a^b m_0 c\, ds - \delta\int_a^b \frac{e}{c}\, A_\alpha dx^\alpha = 0\,, \tag{3.160}$$

where the first term can be denoted as follows

$$\begin{aligned} -\,\delta\int_a^b m_0 c\, ds = -\int_a^b m_0 c\, \mathrm{D}U_\alpha \delta x^\alpha = \\ = \int_a^b m_0 c\, (dU_\alpha\, ds - \Gamma_{\alpha,\mu\nu} U^\mu dx^\nu)\, \delta x^\alpha. \end{aligned} \tag{3.161}$$

We represent the variation of the second integral from the initial formula (3.160) as the sum

$$-\frac{e}{c}\,\delta\int_a^b A_\alpha dx^\alpha = -\,\frac{e}{c}\left(\int_a^b \delta A_\alpha dx^\alpha + \int_a^b A_\alpha\, d\delta x^\alpha\right). \tag{3.162}$$

Integrating the second term, we obtain

$$\int_a^b A_\alpha\, d\delta x^\alpha = A_\alpha\, \delta x^\alpha\Big|_a^b - \int_a^b dA_\alpha \delta x^\alpha. \tag{3.163}$$

Here, the first term is zero, as the integral is varied with the given numerical values of coordinates of the integration limits. Taking into account that the variation of any covariant vector is

$$\delta A_\alpha = \frac{\partial A_\alpha}{\partial x^\beta}\,\delta x^\beta, \qquad dA_\alpha = \frac{\partial A_\alpha}{\partial x^\beta}\,dx^\beta, \tag{3.164}$$

we obtain the variation of the electromagnetic part of the action

$$-\frac{e}{c}\,\delta\int_a^b A_\alpha\,dx^\alpha = -\frac{e}{c}\int_a^b \left(\frac{\partial A_\alpha}{\partial x^\beta}\,dx^\alpha\delta x^\beta - \frac{\partial A_\alpha}{\partial x^\beta}\,\delta x^\alpha dx^\beta\right). \tag{3.165}$$

Transpositioning free indices α and β in the first term of this formula and accounting for the variation of the gravitational part of the action (3.161) we arrive at the variation of the total action (3.160) as follows

$$\delta\int_a^b dS = \int_a^b \left[m_0 c\left(dU_\alpha - \Gamma_{\alpha,\mu\nu}U^\mu dx^\nu\right) - \frac{e}{c}\,F_{\alpha\beta}\,dx^\beta\right]\delta x^\alpha, \tag{3.166}$$

where $F_{\alpha\beta} = \frac{A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$ is the Maxwell tensor, and $U^\mu = \frac{dx^\mu}{ds}$ is the four-dimensional velocity of the particle. Because the quantity δx^α is arbitrary, the formula under the integral is always zero. Hence, we arrive at general covariant equations of motion of the charged particle in their covariant (lower-index) form

$$m_0 c\left(\frac{dU_\alpha}{ds} - \Gamma_{\alpha,\mu\nu}\,U^\mu U^\nu\right) = \frac{e}{c}\,F_{\alpha\beta}\,U^\beta, \tag{3.167}$$

or, lifting the index α, we arrive at the contravariant form of the equations

$$m_0 c\left(\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\,U^\mu U^\nu\right) = \frac{e}{c}\,F^{\alpha\cdot}_{\cdot\beta}\,U^\beta. \tag{3.168}$$

The equations (3.168) actually are the Minkowski equations in the pseudo-Riemannian space. In a Galilean reference frame in the Minkowski space (the Special Theory of Relativity), the obtained equations transform into regular relativistic equations (3.93).

Therefore, chr.inv.-projections of the obtained equations (3.168) may be called the Minkowski chr.inv.-equations in the pseudo-Riemannian space. For an our-world charged particle (it moves into the future w ith respect to a regular observer) the Minkowski chr.inv.-equations are

$$\frac{dE}{d\tau} - mF_i\mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -\,eE_i\mathrm{v}^i, \tag{3.169}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\Delta^i_{nk}\,\mathrm{v}^n\mathrm{v}^k = \\ = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right),\end{gathered} \tag{3.170}$$

and for the analogous particle in the mirror world (it moves into the past) the equations are

$$-\frac{dE}{d\tau} - mF_i \mathrm{v}^i + mD_{ik}\mathrm{v}^i \mathrm{v}^k = -\,eE_i \mathrm{v}^i, \tag{3.171}$$

$$\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + mF^i + m\Delta^i_{nk}\mathrm{v}^n \mathrm{v}^k = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right). \tag{3.172}$$

The scalar chr.inv.-equations of motion, both in our world and the mirror world, represent the live forces theorem. The right hand sides of the vector chr.inv.-equations represent the Lorentz chr.inv.-force in the pseudo-Riemannian space. As it is easy to see, in a Galilean reference frame in the Minkowski space the obtained equations become the regular theorem of live forces (3.92) and the regular three-dimensional equations of motion (3.90) accepted in Classical Electrodynamics.

It is evident that, the right hand sides of the equations of motion (3.169–3.172), obtained through the least action method, are different from the right hand sides of the equations (3.146, 3.157), obtained by the parallel transfer method. The difference here, is in the absence in (3.169–3.172) of numerous terms, which characterize the structure of the acting electromagnetic field and the space itself. But as we have already mentioned, shortest length lines are only a particular case of constant direction lines, defined by parallel transfer. Therefore, there is little surprise in that the equations of parallel transfer, as more general ones, have additional terms, which account for the structure of the acting electromagnetic field and of the space.

§ 3.8 The geometric structure of the four-dimensional electromagnetic potential

In this section, we are going to find the structure of the acting electromagnetic field potential A^α, under which the length of any charged particle's summary vector $Q^\alpha = P^\alpha + \frac{e}{c^2}A^\alpha$ remains unchanged in its parallel transfer in the Levi-Civita meaning (so, a pseudo-Riemannian space is assumed).

As it is known, the Levi-Civita parallel transfer conserves the length of any transferred vector Q^α, so the condition $Q_\alpha Q^\alpha = const$ is true. Given that the square of the length of any n-dimensional vector is invariant in the n-dimensional pseudo-Riemannian space where the vector is located, this condition must be true in any reference frame, including the case of any observer who accompanies his reference body. Hence, we can analyze the condition $Q_\alpha Q^\alpha = const$, formulating it with physical observable quantities in the accompanying reference frame, in chr.inv.-

form in other words.

Components of the vector Q^α in the accompanying reference frame are

$$Q_0 = \left(1 - \frac{\mathrm{w}}{c^2}\right)\left(\pm m + \frac{e\varphi}{c^2}\right), \tag{3.173}$$

$$Q^0 = \frac{1}{1 - \frac{\mathrm{w}}{c^2}}\left[\left(\pm m + \frac{e\varphi}{c^2}\right) + \frac{1}{c^2}\, v_i \left(m\mathrm{v}^i + \frac{e}{c}\, q^i\right)\right], \tag{3.174}$$

$$Q_i = -\frac{1}{c}\left(m\mathrm{v}_i + \frac{e}{c}\, q_i\right) - \frac{1}{c}\left(\pm m + \frac{e\varphi}{c^2}\right) v_i\,, \tag{3.175}$$

$$Q^i = \frac{1}{c}\left(m\mathrm{v}^i + \frac{e}{c}\, q^i\right), \tag{3.176}$$

and its square is

$$Q_\alpha Q^\alpha = m_0^2 + \frac{e^2}{c^4}\left(\varphi^2 - q_i q^i\right) + \frac{2me}{c^2}\left(\pm\varphi - \frac{1}{c}\,\mathrm{v}_i\, q^i\right). \tag{3.177}$$

From here, we can see that the square of the charged particle's summary vector can be split into the following quantities:

a) The square of the four-dimensional momentum of the particle $P_\alpha P^\alpha = m_0^2$;
b) The square of the four-dimensional additional momentum $\frac{e}{c^2} A^\alpha$ which the particle gains from the acting electromagnetic field (the second term);
c) The term $\frac{2me}{c^2}(\pm\varphi - \frac{1}{c}\,\mathrm{v}_i\, q^i)$, which describes interaction between the mass of this particle m and its electric charge e.

In the formula for $Q_\alpha Q^\alpha$ (3.177), the first term m_0^2 remains unchanged. In other words, it is an invariant and does not depend on the reference frame. Our goal is to deduce the conditions, under which the whole formula (3.177) remains unchanged.

Hence, let us propose that the field vector-potential has the structure

$$q^i = \frac{\varphi}{c}\,\mathrm{v}^i. \tag{3.178}$$

In this case* the second term of (3.177) is

$$\frac{e^2}{c^4}\, A_\alpha A^\alpha = \frac{e^2\varphi^2}{c^4}\left(1 - \frac{\mathrm{v}^2}{c^2}\right). \tag{3.179}$$

*A similar problem could be solved, assuming that $q^i = \pm\frac{\varphi}{c}\,\mathrm{v}^i$. But in comparative analysis of two groups of the equations only positive numerical values of $q^i = \frac{\varphi}{c}\,\mathrm{v}^i$ will be important, because the observer's physical time τ, by definition, flows from the past into the future only, so the interval of physical observable time $d\tau$ is always positive.

Transforming the third term in the same way, we obtain the square of the vector Q^{α} (3.177) in the form

$$Q_{\alpha}Q^{\alpha} = m_0^2 + \frac{e^2\varphi^2}{c^4}\left(1-\frac{\mathrm{v}^2}{c^2}\right) + \frac{2m_0\,e}{c^2}\,\varphi\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,. \tag{3.180}$$

Then introducing notation for the field scalar potential

$$\varphi = \frac{\varphi_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{3.181}$$

we can represent the obtained formula (3.180) as follows

$$Q_{\alpha}Q^{\alpha} = m_0^2 + \frac{e^2\varphi_0^2}{c^4} + \frac{2m_0\,e\varphi_0}{c^2} = const. \tag{3.182}$$

So, the length of the summary vector Q^{α} remains unchanged in its parallel transfer, if the observable potentials φ and q^i of the field are related to its four-dimensional potential A^{α} as follows

$$\frac{A_0}{\sqrt{g_{00}}} = \varphi = \frac{\varphi_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad A^i = q^i = \frac{\varphi}{c}\,\mathrm{v}^i. \tag{3.183}$$

Then for the vector $\frac{e}{c^2}A^{\alpha}$, which characterizes interaction of the particle's charge with the electromagnetic field we have

$$\frac{e}{c^2}\frac{A_0}{\sqrt{g_{00}}} = \frac{e\varphi_0}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad \frac{e}{c^2}A^i = \frac{e\varphi_0}{c^3}\frac{\mathrm{v}^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,. \tag{3.184}$$

Dimensions of the vectors $\frac{e}{c^2}A^{\alpha}$ and $P^{\alpha} = m_0\frac{dx^{\alpha}}{ds}$ in CGSE and Gaussian systems of units are the same and equal to mass m [gram].

Comparing chr.inv.-projections of both vectors, we can see that a similar quantity for the relativistic mass m in interactions between the particle's charge and the acting electromagnetic field is the quantity

$$\frac{e\varphi}{c^2} = \frac{e\varphi_0}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{3.185}$$

where $e\varphi$ is the potential energy of the particle moving at the observable velocity $\mathrm{v}^i = \frac{dx^i}{d\tau}$ with respect to the acting electromagnetic field (this particle is at rest with respect to the observer and his reference body). In general, the scalar potential φ is the potential energy of the field itself, divided by unit charge. Then, $e\varphi$ is the potential *relativistic-energy* of the particle with charge e in this electromagnetic field, while

$e\varphi_0$ is the particle's rest-energy in the field. When the particle is at rest with respect to the field, its potential rest-energy equals the potential relativistic-energy.

Comparing $E = mc^2$ and $W = e\varphi$, we arrive at the same conclusion. Respectively, $\frac{W_0}{c^2} = \frac{e\varphi_0}{c^2}$ is an electromagnetic quantity analogous to the rest-mass m_0. Then, the chr.inv.-quantity $\frac{e}{c^2}A^i = \frac{e\varphi}{c^2}\mathrm{v}^i$ is similar to the observable momentum chr.inv.-vector $p^i = m\mathrm{v}^i$. Therefore, when the particle is at rest with respect to the field, its "electromagnetic projection" on the observer's spatial section (the chr.inv.-vector) is zero, while only the time projection (the potential rest-energy $e\varphi_0 = const$) is observable. But if the particle moves in the field at the velocity v^i, its observable "electromagnetic projections" will be the potential relativistic-energy $e\varphi$ and the three-dimensional momentum $\frac{e\varphi}{c^2}\mathrm{v}^i$.

Having obtained chr.inv.-projections of the vector $\frac{e}{c^2}A^\alpha$ calculated for the given structure (3.183), we can restore the vector A^α in general covariant form. Taking into account that its spatial component A^i is

$$A^i = q^i = \frac{\varphi}{c}\,\mathrm{v}^i = \frac{\varphi}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dx^i}{d\tau} = \varphi_0\,\frac{dx^i}{ds}\,, \tag{3.186}$$

we obtain the desired general covariant notation for A^α

$$A^\alpha = \varphi_0\,\frac{dx^\alpha}{ds}\,, \qquad \frac{e}{c^2}A^\alpha = \frac{e\varphi_0}{c^2}\frac{dx^\alpha}{ds}\,. \tag{3.187}$$

In the same time, taking chr.inv.-projections the final formula for the A^α (3.187)

$$\frac{A_0}{\sqrt{g_{00}}} = \pm\,\varphi = \pm\,\frac{\varphi_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad A^i = q^i = \frac{\varphi}{c}\,\mathrm{v}^i, \tag{3.188}$$

we obtain alternating signs in the time chr.inv.-projection, which was not the case in the initial formula (3.183). Naturally, a question arises: how did the scalar observable component of the vector A^α, initially defined as φ, at the given structure of the A^α (3.187) accept the alternating sign? The answer is that in the first case φ and q^i were *defined* proceeding from the general rule of building chr.inv.-quantities. But without knowing the structure of the projected vector A^α itself, we can not calculate them. Therefore, in the formulae for the time and spatial projections (3.183) the symbols φ and q^i merely *denote* the quantities without revealing their structure. On the contrary, in the formulae (3.188) the quantities φ and q^i were *calculated* using formulae $\varphi = \sqrt{g_{00}}\,A^0 + \frac{g_{0i}}{\sqrt{g_{00}}}A^i$ and $q^i = A^i$, where detailed formulae for the components A^0 and A^i were given. Hence, in the second case, the quantity $\pm\varphi$ results from calculation and sets forth the specific formula

$$\varphi = \pm\frac{\varphi_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,. \tag{3.189}$$

Therefore, the *calculated* chr.inv.-projections of the vector $\frac{e}{c^2}A^\alpha$ have the form

$$\frac{e}{c^2}\frac{A_0}{\sqrt{g_{00}}} = \pm\frac{e\varphi}{c^2} = \pm\frac{e\varphi_0}{c^2\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad \frac{e}{c^2}A^i = \frac{e\varphi}{c^3}\,\mathrm{v}^i, \tag{3.190}$$

where "plus" stands if the particle is located in our world, so it travels from the past into the future, while "minus" stands if the particle is located in the mirror world, travelling into the past with respect to us. The square of the vector's length is

$$\frac{e^2}{c^4}A_\alpha A^\alpha = \frac{e^2\varphi^2}{c^4}\left(1-\frac{\mathrm{v}^2}{c^2}\right) = \frac{e^2\varphi_0^2}{c^4} = const\,. \tag{3.191}$$

This vector, $\frac{e}{c^2}A^\alpha$, has real length at $\mathrm{v}^2 < c^2$, zero length at $\mathrm{v}^2 = c^2$ and imaginary length at $\mathrm{v}^2 > c^2$. However, we limit our study to real form of the vector (sub-light velocities), because light-like or super-light charged particles are unknown.

Comparing formulae for $P^\alpha = m_0\frac{dx^\alpha}{ds}$ and $\frac{e}{c^2}A^\alpha = \frac{e\varphi_0}{c^2}\frac{dx^\alpha}{ds}$ we can see that both vectors are collinear, so they are tangential to the same non-isotropic trajectory, to which the derivation parameter s is assumed. Hence, in this case, the momentum vector of the particle P^α is co-directed with the acting electromagnetic field, so the particle moves "along" the field.

We are going to consider the general case, where the vectors are not co-directed. From the square of the summary vector $Q_\alpha Q^\alpha$ (3.177) we see that the third term there is the doubled scalar product of the vectors P^α and $\frac{e}{c^2}A^\alpha$. Parallel transfer of the vectors leaves their scalar product unchanged

$$\mathrm{D}\left(P_\alpha A^\alpha\right) = A^\alpha\,\mathrm{D}P_\alpha + P_\alpha\mathrm{D}A^\alpha = 0\,, \tag{3.192}$$

because the absolute increment of each vector is zero. Hence, we obtain

$$\frac{2e}{c^2}\,P_\alpha A^\alpha = \frac{2me}{c^2}\left(\pm\varphi - \frac{1}{c}\,\mathrm{v}_i\,q^i\right) = const, \tag{3.193}$$

that is, the scalar product of P^α and $\frac{e}{c^2}A^\alpha$ remains unchanged. Consequently, the lengths of both vectors remain unchanged as well. In particular, we have

$$A_\alpha A^\alpha = \varphi^2 - q_i q^i = const. \tag{3.194}$$

As it is known, the scalar product of two vectors is the product of their lengths multiplied by the cosine of the angle between them. Therefore, parallel transfer also leaves the angle between the transferred vectors unchanged

$$\cos\left(P^\alpha; A^\alpha\right) = \frac{P_\alpha A^\alpha}{m_0\sqrt{\varphi^2 - q_i q^i}} = const. \tag{3.195}$$

Taking into account the formula for relativistic mass m, we can rewrite the condition (3.193) as follows

$$\frac{2e}{c^2}\,P_\alpha A^\alpha = \pm\frac{2m_0 e}{c^2}\frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - \frac{2m_0 e}{c^2}\frac{\mathrm{v}_i\, q^i}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = const, \tag{3.196}$$

or as the relationship between the scalar and vector potentials

$$\pm\frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - \frac{\mathrm{v}_i\, q^i}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = const. \tag{3.197}$$

For instance, we can find the relationship between the potentials φ and q^i for that case, where the momentum vector of the particle P^α is orthogonal to the additional momentum $\frac{e}{c^2}A^\alpha$, away from the electromagnetic field.

Because parallel transfer leaves the angle between transferred vectors unchanged (3.195), then cosine of the angle between transferred orthogonal vectors is zero. So, we have

$$P_\alpha A^\alpha = \pm\,\varphi - \frac{1}{c}\,\mathrm{v}_i\, q^i = 0\,. \tag{3.198}$$

Consequently, if the particle travels in the electromagnetic field so that the vectors P^α and A^α are orthogonal, then the scalar potential of the field is

$$\varphi = \pm\,\frac{1}{c}\,\mathrm{v}_i\, q^i, \tag{3.199}$$

so it is the scalar product of the particle's observable velocity v^i and the spatial observable vector-potential of the field q^i.

Now, we are going to obtain the formula for the square of the summary vector Q^α, assuming that the structure of the electromagnetic field potential is $A^\alpha = \varphi_0 \frac{dx^\alpha}{ds}$ (3.187), so the field vector A^α is collinear to the particle's momentum vector P^{α}. Then

$$Q_{\alpha}Q^{\alpha} = m^2 - \frac{m^2}{c^2}\,\mathrm{v}_i\mathrm{v}^i + \frac{e^2}{c^4}\left(\varphi^2 - q_i q^i\right) = m_0^2 + \frac{e^2}{c^4}\,\varphi_0^2\,. \qquad (3.200)$$

Multiplying both sides of the equation by c^4 and denoting the relativistic energy of the particle as $E = mc^2$, we obtain

$$E^2 - c^2p^2 + e^2\varphi^2 - e^2 q_i q^i = E_0^2 + e^2\varphi_0^2\,. \qquad (3.201)$$

§ 3.9 Minkowski's equations as a particular case of the obtained equations of motion

In §3.6 we considered a charged particle of non-zero mass in a pseudo-Riemannian space. There, general covariant equations of its motion were obtained by applying the *parallel transfer method*. So, we have obtained chr.inv.-projections of the general covariant equations.

We showed that their time chr.inv.-projection (3.147) in a Galilean reference frame takes the form of the time component of the Minkowski equations (3.98), becoming the live forces theorem of Classical Electrodynamics (3.92) in three-dimensional Euclidean space. However, the right hand sides of the spatial chr.inv.-projections have the term $-e\left(E^i + \frac{1}{2c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right)$, instead of the Lorentz chr.inv.-force, which is $\Phi^i = -e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right)$, and numerous other additional terms which depend on observable characteristics of the acting electromagnetic field and the space itself. Therefore, for the spatial chr.inv.-projections, the principle of correspondence with three-dimensional components of the Minkowski equations is set non-trivially.

On the other hand, equations of constant direction lines, obtained through parallel transfer in a pseudo-Riemannian space, are a more general case of the shortest length lines, obtained with the least action method. Equations of motion, obtained from the least action principle in §3.7, have the structure matching that of the Minkowski equations. Hence, we can suppose that chr.inv.-projections of the equations of motion in §3.6, are more general ones; in a particular case, they can be transformed into chr.inv.-projections of the equations of motion, obtained from the least action principle in §3.7.

To find out exactly under what conditions this can be true, we are going to consider the spatial chr.inv.-projections of the equations of motion (3.157), which contain the mismatch with the Lorentz force. For the convenience of analysis, we considered the right hand side of (3.157) as a separate formula denoted as M^i. Substituting the magnetic strength H^{ik} (3.12) into the term $\frac{e\varphi}{c^2}A^{ik}\mathrm{v}_k$ from M^i, we write down the term as follows

$$\frac{e\varphi}{c^2}A^{ik}\mathrm{v}_k = \frac{e}{2c}\,h^{im}\mathrm{v}^n\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m}\right) - \frac{e}{2c}\,\varepsilon^{ikm}H_{*m}\mathrm{v}_k\,, \tag{3.202}$$

where $\varepsilon^{ikm}H_{*m} = H^{ik}$. Now, we substitute chr.inv.-components of the electromagnetic field potential A^α as in (3.188) into (3.157). With this potential, the momentum vector $\frac{e}{c^2}A^\alpha$ which the electrically charged particle gains from this electromagnetic field is tangential to the particle's trajectory.

Using the first formula, $q_m = \frac{\varphi}{c}\mathrm{v}_m$, we arrive at the dependence of the right hand side under consideration on only the scalar potential of the field

$$\begin{gathered} M^i = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right) + \\ +\,e h^{ik}\left(1-\frac{\mathrm{v}^2}{c^2}\right)\frac{^*\partial\varphi}{\partial x^k} + \frac{e\varphi}{2}\,h^{ik}\frac{^*\partial}{\partial x^k}\left(1-\frac{\mathrm{v}^2}{c^2}\right). \end{gathered} \tag{3.203}$$

Substituting the relativistic formula of the scalar potential φ (3.181) into this formula we see that the sum of the last two terms becomes zero

$$-\frac{e\varphi}{2}\,h^{ik}\frac{^*\partial}{\partial x^k}\left(1-\frac{\mathrm{v}^2}{c^2}\right) + \frac{e\varphi}{2}\,h^{ik}\frac{^*\partial}{\partial x^k}\left(1-\frac{\mathrm{v}^2}{c^2}\right) = 0\,. \tag{3.204}$$

Then M^i takes the form of the Lorentz chr.inv.-force

$$M^i = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right), \tag{3.205}$$

which is exactly what we had to prove.

Now, we are going to consider the right hand side c^2T of the scalar chr.inv.-equation of motion (3.147) under the condition that the vector A^α has the structure as mentioned in the above and the vector is tangential to the particle's trajectory. Substituting chr.inv.-projections φ and q^i of the vector A^α of the given structure into (3.146), we transform the quantity T to the form

$$\begin{gathered} c^2T = -\,eE_i\mathrm{v}^i - e\,\frac{^*\partial\varphi}{\partial t} + \frac{e}{c^2}\left[\frac{^*\partial}{\partial t}\left(\varphi h_{ik}\mathrm{v}^k\right) - \varphi D_{ik}q^k\right]\mathrm{v}^i = \\ = -\,eE_i\mathrm{v}^i - e\,\frac{^*\partial\varphi}{\partial t}\left(1-\frac{\mathrm{v}^2}{c^2}\right) + \frac{e\varphi}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k + \frac{e\varphi}{c^2}\,\mathrm{v}_k\frac{^*\partial \mathrm{v}^k}{\partial t}. \end{gathered} \tag{3.206}$$

Substituting the relativistic definition of φ (3.181) into the first derivative and after derivation returning to φ again, we obtain

$$\begin{gathered}c^2T = -\,eE_i\mathrm{v}^i - \frac{e\varphi}{2c^2}\frac{{}^*\partial}{\partial t}\left(h_{ik}\mathrm{v}^i\mathrm{v}^k\right) + \frac{e\varphi}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k + \\ + \frac{e\varphi}{c^2}\,\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t} = -\,eE_i\mathrm{v}^i - \frac{e\varphi}{2c^2}\left(\frac{{}^*\partial h_{ik}}{\partial t}\,\mathrm{v}^i\mathrm{v}^k + 2\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t}\right) + \\ + \frac{e\varphi}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k + \frac{e\varphi}{c^2}\,\mathrm{v}_k\frac{{}^*\partial\mathrm{v}^k}{\partial t} = -\,eE_i\mathrm{v}^i,\end{gathered} \tag{3.207}$$

because we took into account that $\frac{{}^*\partial h_{ik}}{\partial t} = 2D_{ik}$ by definition of the tensor of the space deformations rate D_{ik} (1.40).

So, chr.inv.-equations of motion of a charged particle, obtained using the parallel transfer method in a pseudo-Riemannian space, match the equations, obtained using the least action principle in a particular case, where:

a) The electromagnetic field potential A^α has the following structure $A^\alpha = \varphi_0\frac{dx^\alpha}{ds}$ (3.187);

b) The field potential A^α is tangential to the four-dimensional trajectory of the moved particle.

Consequently, given such an electromagnetic potential in a Galilean reference frame in the Minkowski space, the obtained chr.inv.-equations of motion fully match the live force theorem (which is the scalar chr.inv.-equation of motion) and the Minkowski equations (the vector chr.inv.-equations) in three-dimensional Euclidean space, taking the well-known form in Classical Electrodynamics.

Noteworthy, this is another illustration of the geometric fact that the shortest length lines, obtained from the least action principle, are merely a particular case of constant direction lines, which result from the parallel transfer method.

§3.10 Structure of a space filled with a stationary electromagnetic field

It is evident that, setting a particular structure of electromagnetic fields imposes certain limits on motion of charges, which, in their turn, imposes limitations on the structure of a pseudo-Riemannian space where the motions take place. We are going to find out what kind of the structure the pseudo-Riemannian space should have so that a charged particle can move in a stationary electromagnetic field.

Chr.inv.-equations of motion of a charged particle of non-zero mass in our world have the form

$$\frac{dE}{d\tau} - mF_i \mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = -e\,\frac{d\varphi}{d\tau} + \frac{e}{c}\left(F_i\, q^i - D_{ik}q^i\mathrm{v}^k\right), \tag{3.208}$$

$$\begin{gathered}\frac{d\left(m\mathrm{v}^i\right)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{e}{c}\frac{dq^i}{d\tau} - \frac{e}{c}\left(\frac{\varphi}{c}\,\mathrm{v}^k + q^k\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) + \frac{e\varphi}{c^2}\,F^i - \frac{e}{c}\,\Delta^i_{nk}q^n\mathrm{v}^k.\end{gathered} \tag{3.209}$$

Because we assume the electromagnetic field to be stationary, the field potentials φ and q^i depend on spatial coordinates, but not time. In this case chr.inv.-components of the electromagnetic field tensor are

$$E_i = \frac{{}^*\partial\varphi}{\partial x^i} - \frac{\varphi}{c^2}\,F_i = \frac{\partial\varphi}{\partial x^i} - \varphi\,\frac{\partial}{\partial x^i}\ln\!\left(1 - \frac{\mathrm{w}}{c^2}\right), \tag{3.210}$$

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn} = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn}\right). \tag{3.211}$$

From here, we can arrive at limitations on the space metric, imposed by the stationary state of the acting electromagnetic field.

The formulae for E_i and H^{*i}, together with chr.inv.-derivatives of the scalar and vector electromagnetic potentials, also include properties of the space, namely — the chr.inv.-vector of gravitational inertial force F_i and the chr.inv.-tensor of the space non-holonomity A_{ik}. It is evident that, in stationary electromagnetic fields the mentioned properties of the space should be stationary as well

$$\frac{{}^*\partial F_i}{\partial t} = 0\,, \qquad \frac{{}^*\partial F^i}{\partial t} = 0\,, \qquad \frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \qquad \frac{{}^*\partial A^{ik}}{\partial t} = 0\,. \tag{3.212}$$

From these definitions, we see that the quantities F_i and A_{ik} are stationary (do not depend on time), if the linear velocity of the space rotation is as well stationary, $\frac{\partial v_i}{\partial t} = 0$. So, the condition $\frac{\partial v_i}{\partial t} = 0$, namely — stationary rotation of the space, turns chr.inv.-derivative with respect to spatial coordinates into the regular derivative

$$\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{1}{c^2}\frac{{}^*\partial}{\partial t} = \frac{\partial}{\partial x^i}\,. \tag{3.213}$$

Because chr.inv.-derivative with respect to time is differs from the regular derivative only by the multiplier $\frac{\partial}{\partial t} = \left(1 - \frac{\mathrm{w}}{c^2}\right)\frac{{}^*\partial}{\partial t}$, the regular derivative of stationary quantity is zero as well.

For the tensor of the space deformations rate D_{ik} under a stationary rotation of the space we have

$$\frac{{}^*\partial D_{ik}}{\partial t} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = \frac{1}{2}\frac{{}^*\partial}{\partial t}\left(-\,g_{ik} + \frac{1}{c^2}\,v_i v_k\right) = -\frac{1}{2}\frac{{}^*\partial g_{ik}}{\partial t}\,. \qquad (3.214)$$

Because in the case under consideration the right hand sides of equations of motion are stationary, the left hand sides should be the same as well. This implies, that the space does not deform. Then according to (3.124), the three-dimensional coordinate metric g_{ik} does not depend on time, so the Christoffel chr.inv.-symbols Δ^i_{jk} (1.47) are stationary as well.

Using chr.inv.-components of the Maxwell tensor (3.210, 3.211), we transform the Maxwell equations (3.63, 3.64) for the stationary electromagnetic field. As a result we have

$$\left.\begin{aligned} &\frac{\partial E^i}{\partial x^i} + \frac{\partial\ln\sqrt{h}}{\partial x^i}\,E^i - \frac{2}{c}\,\Omega_{*m}H^{*m} = 4\pi\rho \\ &\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\big(H_{*m}\sqrt{h}\big) = \frac{4\pi}{c}\,j^i\sqrt{h} \end{aligned}\right\}\ \mathrm{I}\,, \qquad (3.215)$$

$$\left.\begin{aligned} &\frac{\partial H^{*i}}{\partial x^i} + \frac{\partial\ln\sqrt{h}}{\partial x^i}\,H^{*i} + \frac{2}{c}\,\Omega_{*m}E^m = 0 \\ &\varepsilon^{ikm}\,{}^*\widetilde{\nabla}_k\big(E_m\sqrt{h}\big) = 0 \end{aligned}\right\}\ \mathrm{II}\,. \qquad (3.216)$$

Then the Lorentz condition (3.65) and the continuity equation (3.66) respectively take the form

$${}^*\widetilde{\nabla}_i\,q^i = 0\,, \qquad {}^*\widetilde{\nabla}_i\,j^i = 0\,. \qquad (3.217)$$

So, we have found the way in which any stationary state of an electromagnetic field, located in a pseudo-Riemannian space, affects physical observable properties of the space itself and hence the main equations of electrodynamics.

In the next sections, §3.11–§3.13, we will use the results for solving equations of motion of a charged particle (3.208, 3.209) in stationary electromagnetic fields of three kinds:

1) A stationary electric field (the magnetic strength is zero);
2) A stationary magnetic field (the electric strength is zero);
3) A stationary electromagnetic field (both components are nonzeroes).

§3.11 Motion in a stationary electric field

We are going to consider motion of a charged mass-bearing particle in a pseudo-Riemannian space, filled with a stationary electromagnetic field of strictly electric kind. The magnetic component of the field does not reveal itself for the observer, so the component is absent, in other words.

What conditions should the space satisfy to allow existence of a stationary electromagnetic field of strictly electric kind? From the formula for a stationary state of the magnetic strength

$$H_{ik} = \frac{\partial q_i}{\partial x^k} - \frac{\partial q_k}{\partial x^i} - \frac{2\varphi}{c} A_{ik} \tag{3.218}$$

we see that $H_{ik} = 0$ in this case provided the following two conditions are satisfied:

a) The vector-potential q^i is irrotational $\frac{\partial q_i}{\partial x^k} = \frac{\partial q_k}{\partial x^i}$;

b) The space is holonomic $A_{ik} = 0$.

The stationary electric strength E_i (3.210) is the sum of the spatial derivative of the scalar potential φ and the term $\frac{\varphi}{c^2} F_i$. But on the Earth surface, the ratio of the gravitational potential and the square of the light velocity is nothing but only

$$\frac{\mathrm{w}}{c^2} = \frac{GM_{\oplus}}{c^2 R_{\oplus}} \approx 10^{-10}. \tag{3.219}$$

Therefore, in a real Earth laboratory, the second term in (3.210) may be neglected so that the E_i will only depend on spatial distribution of the scalar potential

$$E_i = \frac{\partial \varphi}{\partial x^i}\,. \tag{3.220}$$

Because the right hand sides of the equations of motion that stand for the Lorentz force are stationary, the left hand sides should be stationary too. Under the conditions we are considering, this is true if the tensor of the space deformation rate is zero (the space does not deform). So, if a stationary electromagnetic field has non-zero electric component and zero magnetic component, then the pseudo-Riemannian space where the field is located should satisfy the following conditions:

a) Potential w of the acting gravitational field is negligible $\mathrm{w} \approx 0$;

b) The space does not rotate $A_{ik} = 0$;

c) The space does not deform $D_{ik} = 0$.

To make further calculations easier, we assume that our three-dimensional space is close to Euclidean one, so we assume $\Delta^i_{nk} \approx 0$.

Then chr.inv.-equations of motion of a particle of electric charge e (3.208, 3.209) take the form

$$\frac{dm}{d\tau} = -\frac{e}{c^2}\frac{d\varphi}{d\tau}\,, \tag{3.221}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) = -\frac{e}{c}\frac{dq^i}{d\tau}\,. \tag{3.222}$$

From the scalar chr.inv.-equation of motion (the live forces theorem), we can see that change of the particle's relativistic energy $E=mc^2$ is due to work done by the field electric component E_i.

From the vector chr.inv.-equations of motion, we can see that the particle's observable momentum has changed under change of the field vector-potential q^i. Assuming that the field four-dimensional potential is tangential to the four-dimensional trajectory of the particle, we (as it was shown in §3.9) get the Lorentz three-dimensional force

$$\Phi^i = -\,eE^i \tag{3.223}$$

on the right hand side. That is, in this case, the particle's observable momentum has changed under the action of the electric strength of the field.

Both groups of the Maxwell chr.inv.-equations for a stationary field (3.215, 3.216) in this case become very simple

$$\left.\begin{aligned} &\frac{\partial E^i}{\partial x^i} = 4\pi\rho \\ &j^i = 0 \end{aligned}\right\}\ \mathrm{I}\,, \tag{3.224}$$

$$\left.\varepsilon^{ikm}\frac{\partial E_m}{\partial x^k} = 0\ \right\}\ \mathrm{II}\,. \tag{3.224}$$

Integrating the scalar chr.inv.-equation of motion (the live forces theorem) we arrive at the so-called *live forces integral*

$$m + \frac{e\varphi}{c^2} = B = const, \tag{3.225}$$

where B is the integration constant.

Another consequence from the Maxwell chr.inv.-equations is that in this case, the scalar potential of the field satisfies either 1) or 2) below:

1) Poisson's equation $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 4\pi\rho$, if $\rho\neq 0$;

2) Laplace's equation $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$, if $\rho = 0$.

So, we have found out the properties of the pseudo-Riemannian space that allows motion of charged particles in a stationary electric field. It would be natural now to obtain exact solutions of chr.inv.-equations of motion for such a particle, namely — the equations (3.221, 3.222). But, unless a particular structure of the field itself is set by the Maxwell equations this can not be done. For this reason, to simplify the calculations, we assume that the electric field is homogeneous.

We assume that the covariant chr.inv.-vector of the electric strength E_i is directed along the x axis. Following Landau and Lifshitz (see §20 of *The Classical Theory of Fields* [10]) we are going to consider the case of a charged particle *repelled* by the field — the case of a negative numerical value of the electric strength and increasing coordinate x of the particle*. Then components of the vector E_i are

$$E_1 = E_x = -E = const, \qquad E_2 = E_3 = 0\,. \tag{3.226}$$

Because the field homogeneity implies $E_i = \frac{\partial\varphi}{\partial x^i} = const$, the scalar potential φ is a function of x that satisfies the Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial E}{\partial x} = 0\,. \tag{3.227}$$

This implies that, the homogeneous stationary electric field satisfies the condition of the absence of charges $\rho = 0$.

We assume that the particle moves along the electric strength E_i, so it is directed along x. Then chr.inv.-equations of its motion are

$$\frac{dm}{d\tau} = -\frac{e}{c^2}\frac{d\varphi}{d\tau} = -\frac{e}{c^2}\frac{d\varphi}{dx^i}\,\mathrm{v}^i = \frac{e}{c^2}\,E\,\frac{dx}{d\tau}\,, \tag{3.228}$$

$$\frac{d}{d\tau}\left(m\frac{dx}{d\tau}\right) = eE\,, \quad \frac{d}{d\tau}\left(m\frac{dy}{d\tau}\right) = 0\,, \quad \frac{d}{d\tau}\left(m\frac{dz}{d\tau}\right) = 0\,. \tag{3.229}$$

Integrating the scalar chr.inv.-equation of motion (the live forces theorem), we arrive at the live forces integral

$$m = \frac{eE}{c^2}\,x + B\,, \qquad B = const. \tag{3.230}$$

This constant B can be obtained from the initial conditions of integration $m|_{\tau=0} = m_{(0)}$ and $x|_{\tau=0} = x_{(0)}$

$$B = m_{(0)} - \frac{eE}{c^2}\,x_{(0)}\,, \tag{3.231}$$

*Naturally, in the case of the particle *attracted* by the field the electric strength is positive while the coordinate of the particle decreases.

so the solution (3.230) takes the form

$$m = \frac{eE}{c^2}\left(x - x_{(0)}\right) + m_{(0)}\,. \tag{3.232}$$

Substituting the obtained integral of live forces into the vector chr. inv.-equations of motion (3.229), we bring them to the form*

$$\left.\begin{aligned} &\frac{eE}{c^2}\,\dot{x}^2 + \left(B + \frac{eE}{c^2}\,x\right)\ddot{x} = eE \\ &\frac{eE}{c^2}\,\dot{x}\dot{y} + \left(B + \frac{eE}{c^2}\,x\right)\ddot{y} = 0 \\ &\frac{eE}{c^2}\,\dot{x}\dot{z} + \left(B + \frac{eE}{c^2}\,x\right)\ddot{z} = 0 \end{aligned}\right\}. \tag{3.233}$$

From here, we realize that the last two equations in (3.233) are equations with separable variables

$$\frac{\ddot{y}}{y} = \frac{-\frac{eE}{c^2}\,\dot{x}}{B + \frac{eE}{c^2}\,x}\,, \qquad \frac{\ddot{z}}{z} = \frac{-\frac{eE}{c^2}\,\dot{x}}{B + \frac{eE}{c^2}\,x}\,, \tag{3.234}$$

which can be integrated. Their solutions are

$$\dot{y} = \frac{C_1}{B + \frac{eE}{c^2}\,x}\,, \qquad \dot{z} = \frac{C_2}{B + \frac{eE}{c^2}\,x}\,, \tag{3.235}$$

where C_1 and C_2 are integration constants which can be found by setting the initial conditions $\dot{y}|_{\tau=0} = \dot{y}_{(0)}$ and $\dot{x}|_{\tau=0} = \dot{x}_{(0)}$ and using the formula for B (3.121). As a result, we obtain

$$C_1 = m_{(0)}\,\dot{y}_{(0)}\,, \qquad C_2 = m_{(0)}\,\dot{z}_{(0)}\,. \tag{3.236}$$

Let us solve the equation of motion along x — the first equation from (3.233). So, we set $\dot{x} = \frac{dx}{d\tau} = p$. Then

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt} = pp', \tag{3.237}$$

and the above equation of motion along x transforms into an equation with separable variables

$$\frac{p\,dp}{1 - \frac{p^2}{c^2}} = \frac{eE\,dx}{B + \frac{eE}{c^2}\,x}\,, \tag{3.238}$$

*Dot stands for derivation with respect to physical observable time τ.

which is a standard integral. After integration, we arrive at the solution

$$\sqrt{1-\frac{p^2}{c^2}}=\frac{C_3}{B+\frac{eE}{c^2}\,x}\,,\qquad C_3=const. \tag{3.239}$$

Assuming $p=\dot{x}|_{\tau=0}=\dot{x}_{(0)}$ and substituting B from (3.231) we find the integration constant

$$C_3=m_{(0)}\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}\,. \tag{3.240}$$

In the case under consideration, we can replace the interval of physical observable time $d\tau$ with the interval of coordinate time dt. We explain why in the next section.

In *The Classical Theory of Fields* Landau and Lifshitz solved equations of motion of a charged particle in a Galilean reference frame in the Minkowski space of the Special Theory of Relativity [10]. Naturally, to be able to compare our solutions with theirs we consider the same particular case — motion in a homogeneous stationary electric field (see §20 in *The Classical Theory of Fields*). But in this case, as we showed earlier in section, using the methods of chronometric invariants, we have $F_i=0$ and $A_{ik}=0$, hence we obtain that in this case

$$d\tau=\left(1-\frac{\mathrm{w}}{c^2}\right)dt-\frac{1}{c^2}\,v_i\,dx^i=dt\,. \tag{3.241}$$

In other words, in the four-dimensional area in this study where the particle travels, the metric is Galilean.

Substituting the variable $p=\frac{dx}{dt}$ into the formula (3.239) we arrive at the last equation with separable variables

$$\frac{dx}{dt}=c\,\frac{\sqrt{\left(B+\frac{eE}{c^2}\,x\right)^2-C_3^2}}{B+\frac{eE}{c^2}\,x}\,, \tag{3.242}$$

whose solution is the function

$$ct=\frac{c^2}{eE}\sqrt{\left(B+\frac{eE}{c^2}\,x\right)^2-C_3^2}\;+C_4\,,\quad C_4=const, \tag{3.243}$$

where the integration constant C_4, taking into account the initial conditions at the moment $t=0$, is

$$C_4=-\frac{m_{(0)}c}{eE}\,\dot{x}_{(0)}\,. \tag{3.244}$$

Now formulating coordinate x explicitly from (3.243) with t we obtain the final solution of the spatial chr.inv.-equations of motion of the charged particle along x

$$x = \frac{c^2}{eE}\left[\sqrt{\frac{e^2E^2}{c^4}\left(ct - C_4\right)^2 + C_3^2} - B\right], \qquad (3.245)$$

or, after substituting integration constants

$$x = \sqrt{\left(ct + \frac{m_{(0)}c\dot{x}_{(0)}}{eE}\right)^2 + \left(\frac{m_{(0)}c^2}{eE}\right)^2\left(1 - \frac{\dot{x}_{(0)}^2}{c^2}\right)} - \frac{m_{(0)}c^2}{eE} + x_{(0)}\,. \quad (3.246)$$

If the field *attracts* the particle (the electric strength is positive $E_1 = E_x = E = const$), we will obtain the same solution for x but having the opposite sign

$$x = \frac{c^2}{eE}\left[B - \sqrt{\frac{e^2E^2}{c^4}\left(ct - C_4\right)^2 + C_3^2}\right]. \qquad (3.247)$$

In *The Classical Theory of Fields* [10] a similar problem is considered, but Landau and Lifshitz solved it through integration of three-dimensional components of general covariant equations of motion (the Minkowski three-dimensional equations) without accounting for the live forces theorem. Their formula for x is

$$x = \frac{1}{eE}\sqrt{\left(m_0c^2\right)^2 + \left(ceEt\right)^2}\,. \qquad (3.248)$$

This formula matches our solution (3.245) if $x_{(0)} - \frac{m_{(0)}c^2}{eE} = 0$ and the initial velocity of the particle is zero $\dot{x}_{(0)} = 0$. The latter stands for significant simplifications accepted in *The Classical Theory of Fields*, according to which some integration constants are assumed zeroes.

As it is easy to see, even when solving equations of motion in a Galilean reference frame in the Minkowski space, the mathematical methods of chronometric invariants give certain advantages revealing hidden factors which are left unnoticed when solving regular three-dimensional components of general covariant equations of motion. This means that, even when physical observable quantities coincide coordinate quantities, it is geometrically correct to solve *a system* of chr.inv.-equations of motion, because the live forces theorem, being their scalar part, inevitably affects the solution of the vector equations.

Of course, in the case of an inhomogeneous non-stationary electric field some additional terms will appear in our solution to reflect the more complicated and time varying field structure.

Now, let us calculate three-dimensional trajectory of the particle in the homogeneous stationary electric field we are considering. To obtain it, we integrate the equations of motion along the axes y and z (3.235), formulate time from there and substitute it into the solution for x we have obtained.

First, substituting the obtained solution for x (3.245) into the equation for $\dot{y}$, we obtain the equation with separable variables

$$\frac{dy}{dt} = \frac{C_1}{\sqrt{\frac{e^2E^2}{c^4}\left(ct - C_4\right)^2 + C_3^2}}\,, \tag{3.249}$$

integrating we have

$$y = \frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE}\,\operatorname{arc\,sinh}\frac{eEt + m_{(0)}\,\dot{x}_{(0)}}{m_{(0)}\,c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}} + C_5\,, \tag{3.250}$$

where C_5 is integration constant. From $y = y_{(0)}$ at $t = 0$ we find

$$C_5 = y_{(0)} - \frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE}\,\operatorname{arc\,sinh}\frac{\dot{x}_{(0)}}{c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}}\,. \tag{3.251}$$

Substituting the constant into y (3.250) we finally have

$$y = y_{(0)} + \frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE}\times$$
$$\times\left\{\operatorname{arc\,sinh}\frac{eEt + m_{(0)}\dot{x}_{(0)}}{m_{(0)}c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}} - \operatorname{arc\,sinh}\frac{\dot{x}_{(0)}}{c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}}\right\}. \tag{3.252}$$

Formulating from here t with y and $y_{(0)}$ and taking into account that $a = \operatorname{arc\,sinh} b$ if $b = \sinh a$, after substituting formula $\operatorname{arc\,sinh} b = \ln\left(b + \sqrt{b^2+1}\right)$ into the second term we have

$$t = \frac{1}{eE}\left\{m_{(0)}\,c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}\;\times\right.$$
$$\left.\times\,\sinh\left[\frac{y - y_{(0)}}{m_{(0)}\,\dot{y}_{(0)}\,c}\,eE + \ln\frac{\dot{x}_{(0)} + c}{c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}}\right] - m_{(0)}\,\dot{x}_{(0)}\right\}. \tag{3.253}$$

Now, we substitute it into our solution for x (3.246). As a result we obtain the desired equation for the three-dimensional trajectory of the particle

$$\begin{gathered} x = x_{(0)} + \frac{m_{(0)}c^2}{eE}\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}} \times \\ \times \cosh\left\{\frac{y-y_{(0)}}{m_{(0)}\dot{y}_{(0)}c}\,eE + \ln\frac{\dot{x}_{(0)}+c}{c\sqrt{1-\frac{\dot{x}^2_{(0)}}{c^2}}}\right\} - \frac{m_{(0)}c^2}{eE}\,. \end{gathered} \tag{3.254}$$

The obtained formula implies that a charged particle in a homogeneous stationary electric field, located in our world, travels along a curve based on *chain line*, while factors which deviate it from "pure" chain line are functions of the initial conditions.

Our formula (3.254) fully matches the result from *The Classical Theory of Fields*

$$x = \frac{m_{(0)}c^2}{eE}\cosh\frac{eEy}{m_{(0)}\dot{y}_{(0)}c} \tag{3.255}$$

(which is formula 20.5 in [10]) once we assume that $x_{(0)} - \frac{m_{(0)}c^2}{eE} = 0$, and the initial velocity of the particle $\dot{x}_{(0)} = 0$ as well. The latter condition suggests that the integration constant in the scalar chr.inv.-equation of motion (the live forces theorem) is zero, which is not always true but may be assumed only in a particular case.

At low velocities after equalling relativistic terms to zero and expanding hyperbolic cosine into series $\cosh b = 1 + \frac{b^2}{2!} + \frac{b^4}{4!} + \frac{b^6}{6!} + \ldots$ our formula for the three-dimensional trajectory of the particle (3.254), having higher order terms ignored, takes the form

$$x = x_{(0)} + \frac{eE\left(y-y_{(0)}\right)^2}{2m_{(0)}\dot{y}^2_{(0)}}\,, \tag{3.256}$$

so the particle travels along *parabola*. Thus, once the initial coordinates of the particle are assumed zeroes, (3.256) matches the result from *The Classical Theory of Fields*

$$x = \frac{eEy^2}{2m_{(0)}\dot{y}^2_{(0)}}\,. \tag{3.257}$$

Integration of the equation of motion along the axis z gives the same results. This is because the only difference between the equations with respect to $\dot{y}$ and $\dot{z}$ (3.235) is a fixed coefficient — the integration constant (3.236), which equals to the initial momentum of the particle along y (in the equation for $\dot{y}$) and along z (in the equation for $\dot{z}$).

Let us find properties of the particle (its energy and momentum) affected by the acting homogeneous stationary electric field. Calculating the relativistic square root (accounting for the assumptions we made)

$$\sqrt{1-\frac{\mathrm{v}^2}{c^2}} = \sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}} = \frac{m_{(0)}\sqrt{1-\frac{\dot{x}^2_{(0)}+\dot{y}^2_{(0)}+\dot{z}^2_{(0)}}{c^2}}}{m_{(0)}+\frac{eE}{c^2}\left(x-x_{(0)}\right)}\,, \tag{3.258}$$

we obtain the energy of the particle

$$E = \frac{m_{(0)}c^2}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = \frac{m_{(0)}c^2 + eE\left(x-x_{(0)}\right)}{\sqrt{1-\frac{\dot{x}^2_{(0)}+\dot{y}^2_{(0)}+\dot{z}^2_{(0)}}{c^2}}}\,, \tag{3.259}$$

which at the velocity much lower than the light velocity is

$$E = m_{(0)}c^2 + eE\left(x-x_{(0)}\right). \tag{3.260}$$

The relativistic momentum of the particle is obtained in the same way, but since the formula is bulky we would not include it here.

So, we have studied motion of a charged particle in a homogeneous stationary electric field, located in our world. Now we consider motion of an analogous particle of the mirror world under the same conditions.

Chr.inv.-equations of motion of the mirror-world particle, taking into account the constraints imposed here on the geometric structure of the space, are

$$\frac{dm}{d\tau} = \frac{e}{c^2}\frac{d\varphi}{d\tau}\,, \tag{3.261}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) = -\frac{e}{c}\frac{dq^i}{d\tau}\,. \tag{3.262}$$

In other words, the only difference from the equations in our world (3.221, 3.222) is the sign in the live forces theorem.

We assume that the electric strength is negative (i. e. the field *repulses* the particle) and that the particle moves along the field strength, so it is co-directed with the axis x.

Then integrating the live forces theorem for the mirror-world particle (3.261) we obtain the live forces integral

$$m = -\frac{eE}{c^2}\,x + B\,, \tag{3.263}$$

where the integration constant, calculated from the initial conditions, is

$$B = m_{(0)} + \frac{eE}{c^2}\, x_{(0)} \,. \tag{3.264}$$

Substituting the results into the vector chr.inv.-equations of motion (3.262), we have (compare them with 3.233)

$$\left.\begin{aligned} -\frac{eE}{c^2}\,\dot{x}^2 + \left(B - \frac{eE}{c^2}\,x\right)\ddot{x} = eE \\ -\frac{eE}{c^2}\,\dot{x}\dot{y} + \left(B - \frac{eE}{c^2}\,x\right)\ddot{y} = 0 \\ -\frac{eE}{c^2}\,\dot{x}\dot{z} + \left(B - \frac{eE}{c^2}\,x\right)\ddot{z} = 0 \end{aligned}\right\}. \tag{3.265}$$

After some algebra similar to that done to obtain the trajectory of the our-world charged particle, we arrive at

$$x = \frac{c^2}{eE}\left[B - \sqrt{C_3^2 - \frac{e^2E^2}{c^4}\left(ct - C_4\right)^2}\,\right], \tag{3.266}$$

where $C_3 = m_{(0)}\sqrt{1 + \frac{\dot{x}^2_{(0)}}{c^2}}$ and $C_4 = -\frac{c\,m_{(0)}\,\dot{x}_{(0)}}{eE}$. Or,

$$x = -\sqrt{\left(\frac{m_{(0)}c^2}{eE}\right)^2\left(1 + \frac{\dot{x}^2_{(0)}}{c^2}\right) - \left(ct + \frac{m_{(0)}c\dot{x}_{(0)}}{eE}\right)^2} + \frac{m_{(0)}c^2}{eE} + x_{(0)} \,. \tag{3.267}$$

The obtained coordinate x of the mirror-world charged particle, *repelled* by the field, is similar to that for the our-world particle *attracted* by the field (3.247) when the electric strength is positive $E_1 = E_x = E = const$. Hence an interesting conclusion: transition of a charged particle from our world into the mirror world (where there is the reverse flow of time) is the same as changing the sign of its charge.

Noteworthy, the similar conclusion can be drawn in respect of particles' masses: purported transition of a particle from our world into the mirror world is the same as changing the sign of its mass. Hence, our-world particles and mirror-world particles are mass and charge complementary.

Let us find the three-dimensional trajectory of the charged particle in the homogeneous stationary electric field, located in the mirror world.

Calculating y in the same manner as for the our-world particle, we have

$$\begin{gathered} y = y_{(0)} + \frac{m_{(0)}\,\dot{y}_{(0)}\,c}{eE} \times \\ \times \left\{ \arcsin \frac{eEt + m_{(0)}\,\dot{x}_{(0)}}{m_{(0)}\,c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}} - \arcsin \frac{\dot{x}_{(0)}}{c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}} \right\}. \end{gathered} \tag{3.268}$$

In contrast to the formula for the our-world particle (3.252), this formula has a regular arcsine and "plus" sign under the square root.

Formulating time t from here with the coordinates y and $y_{(0)}$

$$\begin{gathered} t = \frac{1}{eE} \left\{ m_{(0)}\,c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}} \times \right. \\ \left. \times \sin\left[\frac{y - y_{(0)}}{m_{(0)}\,\dot{y}_{(0)}\,c}\, eE + \ln \frac{\dot{x}_{(0)} + c}{c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}} \right] - m_{(0)}\,\dot{x}_{(0)} \right\}, \end{gathered} \tag{3.269}$$

and substituting it into our formula for x (3.267), we obtain the final formula for the trajectory

$$\begin{gathered} x = x_{(0)} - \frac{m_{(0)}\,c^2}{eE}\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}} \times \\ \times \cos\left\{ \frac{y - y_{(0)}}{m_{(0)}\,\dot{y}_{(0)}\,c}\, eE + \arcsin \frac{\dot{x}_{(0)}}{c\sqrt{1+\frac{\dot{x}^2_{(0)}}{c^2}}} \right\} - \frac{m_{(0)}\,c^2}{eE}\,. \end{gathered} \tag{3.270}$$

In other words, motion of the particle is *harmonic oscillation*. Once we assume the initial coordinates of the particle equal to zero, as well as its initial velocity $\dot{x}_{(0)} = 0$ and the integration constant $B = 0$, the obtained equation of the trajectory takes a simpler form

$$x = -\frac{m_{(0)}\,c^2}{eE}\cos\frac{eEy}{m_{(0)}\,\dot{y}_{(0)}\,c}\,. \tag{3.271}$$

At low velocities, after equating relativistic terms to zero and expanding into the cosine series $\cos b = 1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \ldots \approx 1 - \frac{b^2}{2!}$ (this is always possible within a smaller part of the trajectory), our formula (3.270) becomes

$$x = x_{(0)} + \frac{eE\left(y - y_{(0)}\right)^2}{2m_{(0)}\,\dot{y}^2_{(0)}}\,, \tag{3.272}$$

which is the equation of a *parabola*. So, the charged particle in the mirror world at low velocity travels along a parabola, as does the our-world particle in the same conditions in the field.

Therefore, a charged particle of our world travels in homogeneous stationary electric fields along a chain line, which at low velocities becomes a parabola. An analogous mirror-world particle travels along a harmonic trajectory, smaller parts of which at low velocities becomes a parabola (as is the case for the our-world particle).

§3.12 Motion in a stationary magnetic field

Let us consider motion of a charged particle when the electric component of the electromagnetic field is absent, while the magnetic component is present and it is stationary. In this case chr.inv.-vectors of the electric and magnetic strengths are

$$E_i = \frac{^*\partial\varphi}{\partial x^i} - \frac{\varphi}{c^2}\,F_i = \frac{\partial\varphi}{\partial x^i} - \frac{\varphi}{c^2}\,\frac{1}{1-\frac{\mathrm{w}}{c^2}}\,\frac{\partial\mathrm{w}}{\partial x^i} = 0\,, \tag{3.273}$$

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn} = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{\partial q_m}{\partial x^n} - \frac{\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn}\right) \neq 0 \tag{3.274}$$

because if the field is strictly magnetic $\varphi = const$ ($E_i = 0$), then gravitational effect can be neglected. From (3.274) we can see that the magnetic strength H^{*i} is not zero, if at least one of the following conditions is true:

a) The potential q^i is rotational;

b) The space is non-holonomic $A_{ik} \neq 0$.

We are going to consider motion of the particle in a general case, when both conditions are true (we will use the non-holonomic space later as the basic space for spin-particles). As we did in the previous section,§3.11, we assume deformations of the space to be zero and the three-dimensional metric to be Euclidean $g_{ik} = \delta_{ik}$. The observable metric $h_{ik} = -g_{ik} + \frac{1}{c^2}\,v_i v_k$ in this case is not Galilean, because in non-holonomic spaces we have $h_{ik} \neq -g_{ik}$.

We assume that the space rotates about the z axis at the constant angular velocity $\Omega_{12} = -\Omega_{21} = \Omega$. Then the linear velocity of this rotation

$v_i = \Omega_{ik} x^k$ has two non-zero components $v_1 = \Omega y$ and $v_2 = -\Omega x$, while the non-holonomity tensor has the only non-zero component $A_{12} = -A_{21} = -\Omega$. In this case, the metric takes the form

$$ds^2 = c^2 dt^2 - 2\Omega y\, dt\, dx + 2\Omega x\, dt\, dy - dx^2 - dy^2 - dz^2. \tag{3.275}$$

In this space we have $F_i = 0$ and $D_{ik} = 0$. In the previous section §3.11, which focused on a charged particle in a stationary electric field, we assumed that the Christoffel symbols are zeroes. In other words, we considered its motion in a Galilean reference frame in the Minkowski space. But in this section, the three-dimensional observable metric h_{ik} is not Euclidean, because the space rotation and the Christoffel symbols Δ^i_{jk} (1.47) are not zeroes.

If the linear velocity of the space rotation is not infinitesimal compared to the light velocity, components of the metric chr.inv.-tensor h_{ik} are

$$h_{11} = 1 + \frac{\Omega^2 y^2}{c^2}\,, \quad h_{22} = 1 + \frac{\Omega^2 x^2}{c^2}\,, \quad h_{12} = -\frac{\Omega^2 xy}{c^2}\,, \quad h_{33} = 1\,, \tag{3.276}$$

so its determinant and components of h^{ik} are

$$h = \det\|h_{ik}\| = h_{11}h_{22} - h_{12}^2 = 1 + \frac{\Omega^2\left(x^2 + y^2\right)}{c^2}\,, \tag{3.277}$$

$$\left.\begin{array}{ll} h^{11} = \dfrac{1}{h}\left(1 + \dfrac{\Omega^2 x^2}{c^2}\right), & h^{22} = \dfrac{1}{h}\left(1 + \dfrac{\Omega^2 y^2}{c^2}\right) \\[2ex] h^{12} = \dfrac{\Omega^2 xy}{hc^2}\,, & h^{33} = 1 \end{array}\right\}. \tag{3.278}$$

Respectively, from here we obtain non-zero components of the Christoffel chr.inv.-symbols Δ^i_{jk} (1.47), namely

$$\Delta^1_{11} = -\frac{2\Omega^4 xy^2}{c^4\left[1 + \frac{\Omega^2(x^2+y^2)}{c^2}\right]}\,, \tag{3.279}$$

$$\Delta^1_{12} = \frac{\Omega^2 y\left(1 + \frac{2\Omega^2 x^2}{c^2}\right)}{c^2\left[1 + \frac{\Omega^2(x^2+y^2)}{c^2}\right]}\,, \tag{3.280}$$

$$\Delta^1_{22} = -\frac{2\Omega^2 x}{c^2}\,\frac{1 + \frac{\Omega^2 x^2}{c^2}}{1 + \frac{\Omega^2(x^2+y^2)}{c^2}}\,, \tag{3.281}$$

$$\Delta_{11}^{2} = -\frac{2\Omega^2 y}{c^2}\,\frac{1+\frac{\Omega^2 y^2}{c^2}}{1+\frac{\Omega^2(x^2+y^2)}{c^2}}\,, \tag{3.282}$$

$$\Delta_{12}^{2} = \frac{\Omega^2 x\left(1+\frac{\Omega^2 y^2}{c^2}\right)}{c^2\left[1+\frac{\Omega^2(x^2+y^2)}{c^2}\right]}\,, \tag{3.283}$$

$$\Delta_{22}^{2} = -\frac{2\Omega^4 x^2 y}{c^4\left[1+\frac{\Omega^2(x^2+y^2)}{c^2}\right]}\,. \tag{3.284}$$

We are going to solve chr.inv.-equations of motion of a charged particle in the stationary magnetic field, located in the pseudo-Riemannian space. To make the calculations easier, we assume that the field four-dimensional potential A^{α} is tangential to the four-dimensional trajectory of the particle. Because the field electric component is zero $E_i = 0$, the component does not perform any work, so the right hand sides of the scalar chr.inv.-equation of motion turn into zeroes. Applying chr.inv.-equations of motion of a charged particle (3.208, 3.209) to the particle in the stationary magnetic field located in our world, we obtain

$$\frac{dm}{d\tau} = 0\,, \tag{3.285}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2mA_{k\cdot}^{\cdot i}\mathrm{v}^k + m\Delta_{nk}^{i}\mathrm{v}^n\mathrm{v}^k = -\frac{e}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,, \tag{3.286}$$

while for the analogous charged particle which moves in the same stationary magnetic field located in the mirror world, we have

$$-\frac{dm}{d\tau} = 0\,, \tag{3.287}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + m\Delta_{nk}^{i}\mathrm{v}^n\mathrm{v}^k = -\frac{e}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{3.288}$$

Integrating the live forces theorem for the our-world particle and the mirror-world particle we obtain, respectively

$$m = \frac{m_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = const = B\,, \qquad -m = \frac{m_0}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = const = \widetilde{B}\,, \tag{3.289}$$

where B and $\widetilde{B}$ are integration constants. This implies that, $\mathrm{v}^2 = const$, so the modulus of the particle's observable velocity remains unchanged in the absence of the electric component of the electromagnetic field. Then the vector chr.inv.-equations of motion for the our-world particle

(3.286) are

$$\frac{d\mathrm{v}^i}{d\tau}+2A_{k\cdot}^{\cdot i}\mathrm{v}^k+\Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k=-\frac{e}{mc}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,, \tag{3.290}$$

while for the mirror-world particle (3.288) we have the same equation but without the term $2A_{k\cdot}^{\cdot i}\mathrm{v}^k$, namely

$$\frac{d\mathrm{v}^i}{d\tau}+\Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k=-\frac{e}{mc}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{3.291}$$

The magnetic strength here is defined by the Maxwell equations for stationary fields (3.215, 3.216), which in the absence of the electric strength and under the constraints we assumed in this section are

$$\left.\begin{aligned} &\Omega_{*m}H^{*m}=-2\pi c\rho\\ &\varepsilon^{ikm}\,{}^*\nabla_k\big(H_{*m}\sqrt{h}\big)=\frac{4\pi}{c}\,j^i\sqrt{h}\end{aligned}\right\}\ \mathrm{I}\,, \tag{3.292}$$

$$\left.{}^*\nabla_i\,H^{*i}=\frac{\partial H^{*i}}{\partial x^i}+\frac{\partial\ln\sqrt{h}}{\partial x^i}\,H^{*i}=0\ \right\}\ \mathrm{II}\,. \tag{3.293}$$

From the first equation of the 1st group, we see that the scalar product of the space non-holonomity pseudovector and the magnetic strength pseudovector is a function of the charge density. Hence, if the charge density is $\rho=0$, then the pseudovectors Ω_{*i} and H^{*i} are orthogonal.

Henceforth, we consider two possible orientations of the magnetic strength with respect to the space non-holonomity pseudovector.

a) Magnetic field is co-directed with non-holonomity field

We assume that the magnetic strength pseudovector H^{*i} is directed along the z axis, i.e. in the same direction with the pseudovector of angular velocities of the space rotation $\Omega^{*i}=\frac{1}{2}\,\varepsilon^{ikm}A_{km}$. Then the space rotation pseudovector has one non-zero component $\Omega^{*3}=\Omega$, while the magnetic strength pseudovector has

$$\begin{aligned} H^{*3}=\frac{1}{2}\,\varepsilon^{3mn}H_{mn}=\frac{1}{2}\left(\varepsilon^{312}H_{12}+\varepsilon^{321}H_{21}\right)=H_{12}=\\ =\frac{\varphi}{c}\left(\frac{\partial\mathrm{v}_1}{\partial x}-\frac{\partial\mathrm{v}_2}{\partial y}\right)+\frac{2\varphi}{c}\,\Omega\,.\end{aligned} \tag{3.294}$$

The condition $\varphi=const$ is derived from the absence of the field electric component. Hence the 1st group of the Maxwell equations (2.392) in this case are

$$\left.\begin{aligned} &\Omega_{*3}H^{*3}=\frac{\Omega\varphi}{c}\left(\frac{\partial \mathrm{v}_1}{\partial x}-\frac{\partial \mathrm{v}_2}{\partial y}\right)+\frac{2\varphi\,\Omega^2}{c}=-2\pi c\rho\\ &\frac{\partial}{\partial y}\left(H_{*3}\sqrt{h}\right)=\frac{4\pi}{c}\,j^1\sqrt{h}\\ &-\frac{\partial}{\partial x}\left(H_{*3}\sqrt{h}\right)=\frac{4\pi}{c}\,j^2\sqrt{h}\\ &j^3=0 \end{aligned}\right\}. \tag{3.295}$$

The 2nd group of the equations (3.293) will be trivial turning into simple relationship $\frac{\partial H^{*3}}{\partial z}=0$, so that $H^{*3}=const$. Actually this implies that the stationary magnetic field we are considering is homogeneous along z. Next, we assume that the stationary magnetic field is strictly homogeneous $H^{*i}=const$. Then from the first equation of the 1st group (3.295) we see that the field is homogeneous provided that two conditions

$$\left(\frac{\partial \mathrm{v}_1}{\partial x}-\frac{\partial \mathrm{v}_2}{\partial y}\right)=const, \tag{3.296a}$$

$$\rho=-\frac{\varphi\,\Omega^2}{\pi c^2}=const. \tag{3.296b}$$

Hence, the charge density $\rho>0$, if the field scalar potential $\varphi<0$. Then the other equations from the 1st group (3.295) are

$$\left.\begin{aligned} &j^1=\frac{c}{4\pi}\frac{\partial\ln\sqrt{h}}{\partial y}\\ &j^2=\frac{c}{4\pi}\frac{\partial\ln\sqrt{h}}{\partial x}\\ &j^3=0 \end{aligned}\right\}. \tag{3.297}$$

Because of $h=1+\frac{\Omega^2(x^2+y^2)}{c^2}$ (3.277), this implies: the current vector in the homogeneous stationary magnetic field is non-zero in only the *strong* field of the space non-holonomity, i. e. where the space rotation velocity is comparable to the light velocity. In a weak field of the space non-holonomity we have $h=1$, hence $j^1=j^2=0$.

Now, expressing the magnetic strength from the Maxwell equations (3.295) we write down the vector chr.inv.-equations of motion for the our-world particle (3.290, 3.291) in the form

$$\left.\begin{aligned} &\ddot{x}+\frac{2\Omega}{h}\left[\frac{\Omega^2 xy\dot{x}}{c^2}+\left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\right]+\Delta^1_{11}\dot{x}^2+2\Delta^1_{12}\dot{x}\dot{y}+\\ &\qquad\qquad+\Delta^1_{22}\,\dot{y}^2=-\frac{eH}{mc}\left[-\frac{\Omega^2xy\dot{x}}{c^2}+\left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\right]\\ &\ddot{y}-\frac{2\Omega}{h}\left[\frac{\Omega^2 xy\dot{y}}{c^2}+\left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\right]+\Delta^2_{11}\dot{x}^2+2\Delta^2_{12}\dot{x}\dot{y}+\\ &\qquad\qquad+\Delta^2_{22}\,\dot{y}^2=\frac{eH}{mc}\left[-\frac{\Omega^2xy\dot{y}}{c^2}+\left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\right]\\ &\ddot{z}=0 \end{aligned}\right\}, \qquad (3.298)$$

while those for the mirror-world particle they are

$$\left.\begin{aligned} &\ddot{x}+\Delta^1_{11}\dot{x}^2+2\Delta^1_{12}\dot{x}\dot{y}+\Delta^1_{22}\,\dot{y}^2=\\ &\qquad\qquad=-\frac{eH}{mc}\left[-\frac{\Omega^2xy\dot{x}}{c^2}+\left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\right]\\ &\ddot{y}+\Delta^2_{11}\dot{x}^2+2\Delta^2_{12}\dot{x}\dot{y}+\Delta^2_{22}\,\dot{y}^2=\\ &\qquad\qquad=\frac{eH}{mc}\left[-\frac{\Omega^2xy\dot{y}}{c^2}+\left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\right]\\ &\ddot{z}=0. \end{aligned}\right\}. \qquad (3.299)$$

The terms in the right hand sides which contain $\frac{\Omega^2}{c^2}$ appear, because in the space rotation the observable chr.inv.-metric h_{ik} is not Euclidean. Hence, in the case under consideration there is a difference between the contravariant form of the observable velocity and its covariant form. The right hand sides include the covariant components

$$\mathrm{v}_2=h_{21}\mathrm{v}^1+h_{22}\mathrm{v}^2=-\frac{\Omega^2xy}{c^2}\,\dot{x}+\left(1+\frac{\Omega^2x^2}{c^2}\right)\dot{y}\,, \qquad (3.300)$$

$$\mathrm{v}_1=h_{11}\mathrm{v}^1+h_{12}\mathrm{v}^2=-\frac{\Omega^2xy}{c^2}\,\dot{y}+\left(1+\frac{\Omega^2y^2}{c^2}\right)\dot{x}\,. \qquad (3.301)$$

If there is no space rotation, $\Omega=0$, then the chr.inv.-equations of motion of the our-world particle (3.298) to within their sign match the equations of motion in a homogeneous stationary magnetic field given by Landau and Lifshitz (see form. 21.2 in *The Classical Theory of Fields*)

$$\ddot{x} = \frac{eH}{mc}\,\dot{y}\,, \qquad \ddot{y} = -\frac{eH}{mc}\,\dot{x}\,, \qquad \ddot{z} = 0\,, \tag{3.302}$$

while our equations (3.298) imply that

$$\ddot{x} = -\frac{eH}{mc}\,\dot{y}\,, \qquad \ddot{y} = \frac{eH}{mc}\,\dot{x}\,, \qquad \ddot{z} = 0\,. \tag{3.303}$$

The difference is derived from the fact that Landau and Lifshitz assumed the magnetic strength in the Lorentz force to have a "plus" sign, while in our equations it has "minus" sign, which is not that important though, because it only depends on the choice of the space signature.

If the space rotates (it is non-holonomic), the equations of motion will include the terms that contain Ω, $\frac{\Omega^2}{c^2}$, and $\frac{\Omega^4}{c^4}$.

In a strong field of the space non-holonomity, solving the equations we have obtained is a non-trivial task, which is likely to be tackled in future with computer-aided numerical methods. Hopefully, the results will be quite interesting.

Let us find their exact solutions in a weak field of the space non-holonomity, namely — neglecting terms of the second order. In this case, the equations of motion we have obtained (3.298, 3.299) for the our-world particle are

$$\ddot{x} + 2\Omega\,\dot{y} = -\frac{eH}{mc}\,\dot{y}\,, \qquad \ddot{y} - 2\Omega\dot{x} = \frac{eH}{mc}\,\dot{x}\,, \qquad \ddot{z} = 0\,, \tag{3.304}$$

and for the mirror-world particle they are

$$\ddot{x} = -\frac{eH}{mc}\,\dot{y}\,, \qquad \ddot{y} = \frac{eH}{mc}\,\dot{x}\,, \qquad \ddot{z} = 0\,. \tag{3.305}$$

First we approach the equations for the our-world particle. The equation along z can be integrated straightaway. The solution is

$$z = \dot{z}_{(0)}\,\tau + z_{(0)}\,. \tag{3.306}$$

From here we see that if at the initial moment of time the particle's velocity along z is zero, so the particle moves within $x\,y$ plane only. We re-write the remaining two equations of (3.304) as follows

$$\frac{d\dot{x}}{d\tau} = -\left(2\Omega + \omega\right)\dot{y}\,, \qquad \frac{d\dot{y}}{d\tau} = \left(2\Omega + \omega\right)\dot{x}\,, \tag{3.307}$$

where we denote $\omega = \frac{eH}{mc}$ for convenience. The same notation was used in §21 of *The Classical Theory of Fields*. Then, formulating $\dot{x}$ from the second equation, we derive it to the observable time $\dot{x}$ and substitute the result into the first equation. So, we obtain

$$\frac{d^2\dot{y}}{d\tau^2} + (2\Omega+\omega)^2\,\dot{y} = 0\,, \tag{3.308}$$

which is the equation of oscillations; with solution

$$\dot{y} = C_1 \cos{(2\Omega+\omega)\,\tau} + C_2 \sin{(2\Omega+\omega)\,\tau}\,, \tag{3.309}$$

where $C_1 = \dot{y}_{(0)}$ and $C_2 = \frac{\ddot{y}_{(0)}}{2\Omega+\omega}$ are integration constants. Substituting $\dot{y}$ (3.309) into the first equation (3.307) we obtain

$$\frac{d\dot{x}}{d\tau} = -\,(2\Omega+\omega)\,\dot{y}_{(0)}\cos{(2\Omega+\omega)\,\tau} - \ddot{y}_{(0)}\sin{(2\Omega+\omega)\,\tau}\,, \tag{3.310}$$

or, after integration,

$$\dot{x} = \dot{y}_{(0)}\sin{(2\Omega+\omega)\,\tau} - \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos{(2\Omega+\omega)\,\tau} + C_3\,, \tag{3.311}$$

where the integration constant is $C_3 = \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}$.

Having all the constants substituted, the obtained formulae for $\dot{x}$ (3.311) and $\dot{y}$ (3.309) finally transform into

$$\dot{x} = \dot{y}_{(0)}\sin{(2\Omega+\omega)\,\tau} - \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos{(2\Omega+\omega)\,\tau} + \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\,, \tag{3.312}$$

$$\dot{y} = \dot{y}_{(0)}\cos{(2\Omega+\omega)\,\tau} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\sin{(2\Omega+\omega)\,\tau}\,. \tag{3.313}$$

Hence, the formulae for components of the particle's velocity $\dot{x}$ and $\dot{y}$ in the homogeneous stationary magnetic field are the equations of harmonic oscillations. The frequency in a weak field of the space non-holonomity is $2\Omega+\omega = 2\Omega+\frac{eH}{mc}$.

From the live forces integral in the stationary magnetic field (3.289) we see that the square of the particle's velocity is a constant quantity. Calculating $\mathrm{v}^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$ for the our-world particle we obtain that this quantity

$$\begin{aligned}\mathrm{v}^2 &= \dot{x}^2_{(0)} + \dot{y}^2_{(0)} + \dot{z}^2_{(0)} + 2\left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\right)\times \\ &\times\left[\frac{\ddot{y}_{(0)}}{2\Omega+\omega} + \dot{y}_{(0)}\sin{(2\Omega+\omega)\,\tau} - \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos{(2\Omega+\omega)\,\tau}\right]\end{aligned} \tag{3.314}$$

is constant $\mathrm{v}^2 = const$, provided that $C_3 = \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega} = 0$.

Integrating $\dot{x}$ and $\dot{y}$ to τ (namely — integrating the equations 3.312, 3.313), we obtain coordinates of the our-world particle which moves in the homogeneous stationary magnetic field

$$x = \left[\frac{\ddot{y}_{(0)}}{2\Omega+\omega}\sin(2\Omega+\omega)\,\tau - \dot{y}_{(0)}\cos(2\Omega+\omega)\,\tau\right]\frac{1}{2\Omega+\omega} + \left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\right)\tau + C_4\,, \tag{3.315}$$

$$y = \left[\dot{y}_{(0)}\sin(2\Omega+\omega)\,\tau + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos(2\Omega+\omega)\,\tau\right]\frac{1}{2\Omega+\omega} + C_5\,, \tag{3.316}$$

where the integration constants are

$$C_4 = x_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega+\omega}\,, \qquad C_5 = y_{(0)} + \frac{\ddot{y}_{(0)}}{(2\Omega+\omega)^2}\,. \tag{3.317}$$

From (3.315) we see that the particle performs *harmonic oscillations* along x provided that the equation $\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{2\Omega+\omega} = 0$ is true. This is also the condition for the constant square of the particle's velocity (3.314), i. e. it satisfies the live forces integral. Taking this result into account we arrive at the equation of the particle's trajectory within $x\,y$ plane

$$\begin{aligned} x^2 + y^2 = {} & \frac{1}{(2\Omega+\omega)^2}\left[\dot{y}^2_{(0)} + \frac{\ddot{y}^2_{(0)}}{(2\Omega+\omega)^2}\right] - \frac{2C_4}{2\Omega+\omega}\times \\ & \times\left[\dot{y}_{(0)}\cos(2\Omega+\omega)\,\tau + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\sin(2\Omega+\omega)\,\tau\right] + \\ & + \left[\dot{y}_{(0)}\sin(2\Omega+\omega)\,\tau + \frac{\ddot{y}_{(0)}}{2\Omega+\omega}\cos(2\Omega+\omega)\,\tau\right]\times \\ & \times\frac{2C_5}{2\Omega+\omega} + C_4^2 + C_5^2\,. \end{aligned} \tag{3.318}$$

Assuming that for the initial moment of time, $\ddot{y}_{(0)} = 0$ and the integration constants C_4 and C_5 to be zeroes, we can simplify the obtained formulae (3.315, 3.316), namely

$$x = -\frac{1}{2\Omega+\omega}\,\dot{y}_{(0)}\cos(2\Omega+\omega)\,\tau\,, \tag{3.319}$$

$$y = \frac{1}{2\Omega+\omega}\,\dot{y}_{(0)}\sin(2\Omega+\omega)\,\tau\,. \tag{3.320}$$

Given the formulae, our equation of the trajectory (3.318) transforms into a simple equation of the circle

$$x^2 + y^2 = \frac{\dot{y}^2_{(0)}}{(2\Omega + \omega)^2} \,. \tag{3.321}$$

Hence, if the initial velocity of the our-world charged particle with respect to the direction of the homogeneous magnetic field (the axis z) is zero, then the particle moves within $x\,y$ plane along a *circle* of radius

$$r = \frac{\dot{y}_{(0)}}{2\Omega + \omega} = \frac{\dot{y}_{(0)}}{2\Omega + \frac{eH}{mc}} \,, \tag{3.322}$$

which depends on the field strength and the angular velocity of the space rotation.

If the initial velocity of the particle along the magnetic field direction is not zero, then it moves along *a spiral line* of the radius r along the field. In a general case, the particle moves along an *ellipse* within $x\,y$ plane (3.318), whose shape deviates from that of a circle depending on the initial conditions of this motion.

As it is easy to see, our results match those in §21 of *The Classical Theory of Fields*

$$x = -\frac{1}{\omega}\,\dot{y}_{(0)} \cos\omega\tau \,, \qquad y = \frac{1}{\omega}\,\dot{y}_{(0)} \sin\omega\tau \,, \tag{3.323}$$

once we assume $\Omega = 0$, i. e. in the absence of the space rotation. In this particular case, the radius $r = \frac{\dot{y}_{(0)}}{\omega} = \frac{mc}{eH}\,\dot{y}_{(0)}$ of the particle's trajectory does not depend on the velocity of the space rotation. If $\Omega \neq 0$, then the non-holonomity field disturbs the particle from moving in the magnetic field adding up with the magnetic strength, due to which the correction quantity 2Ω to the term $\omega = \frac{eH}{mc}$ appears in the equations. In a strong field of the space non-holonomity, where Ω can not be neglected compared to the light velocity, the disturbance is even stronger.

On the other hand, in a non-holonomic space the argument of trigonometric functions in our equations contains a *sum of two terms*, one of which is derived from interaction of the particle's charge with the magnetic strength, while the other is a result of the space rotation, which depends neither on the electric charge of this particle, nor on the presence of the magnetic field. This allows us to consider two special cases of motion of a charged particle in a homogeneous stationary magnetic field, located in a non-holonomic space.

In the first case, where the particle is electrically neutral or the magnetic field is absent, its motion will be the same as that under action of the magnetic component of the Lorentz force, except for the fact that this motion will be caused by the space rotation 2Ω, comparable to $\omega = \frac{eH}{mc}$.

How real is this case? To answer this question, we need at least an approximate assessment of the ratio between the angular velocity of the space rotation Ω and the magnetic strength H in a special case. The best example may be an atom, because on the scales of electronic orbits electromagnetic interactions are a few orders of magnitude stronger than the others and besides, orbital velocities of electrons are relatively high.

Such assessment can be made proceeding from the second case of the special motions, where

$$\frac{eH}{mc} = -2\Omega\,, \tag{3.324}$$

is true and hence the argument of trigonometric functions in the equations of motion becomes zero.

We consider the observer's reference frame, whose reference space is attributed to the nucleus in an atom. Then the ratio in the question (in CGSE and Gaussian systems of units) for an electron in this atom is

$$\begin{aligned}\frac{\Omega}{H} = -\frac{e}{2m_{\mathrm{e}}c} = -\frac{4.8\times 10^{-10}}{18.2\times 10^{-28}\; 3.0\times 10^{10}} = \\ = -\,8.8\times 10^{6}\ \mathrm{cm}^{1/2}\,\mathrm{gram}^{-1/2},\end{aligned} \tag{3.325}$$

where their "minus" sign is derived from the fact that Ω and H in (3.324) are oppositely directed.

Now, let us solve the equations of motion of the mirror-world particle in the homogeneous stationary magnetic field (3.305), which match the equations in the absence of the space non-holonomity

$$\ddot{x} = -\,\omega\dot{y}\,, \qquad \ddot{y} = \omega\dot{x}\,, \qquad \ddot{z} = 0\,. \tag{3.326}$$

The solution of the third equation of motion (along z) is a simpler integral $z = \dot{z}_{(0)}\tau + z_{(0)}$.

The equations of motion along x and y are similar to those for the our-world particle, save for the fact that the argument of trigonometric functions has ω instead of $\omega + 2\Omega$:

$$\dot{x} = \dot{y}_{(0)}\sin\omega\tau - \frac{\ddot{y}_{(0)}}{\omega}\cos\omega\tau + \dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}\,, \tag{3.327}$$

$$\dot{y} = \dot{y}_{(0)}\cos\omega\tau + \frac{\ddot{y}_{(0)}}{\omega}\sin\omega\tau\,. \tag{3.328}$$

Hence the formulae for components of the velocity of the mirror-world particle $\dot{x}$ and $\dot{y}$ are the equations of harmonic oscillations at the frequency $\omega = \frac{eH}{mc}$.

Consequently, their solutions, namely — formulae for coordinates of the mirror-world particle moving in the homogeneous stationary magnetic field are

$$x = \frac{1}{\omega}\left(\frac{\ddot{y}_{(0)}}{\omega}\sin\omega\tau - \dot{y}_{(0)}\cos\omega\tau\right) + \left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}\right)\tau + C_4\,, \tag{3.329}$$

$$y = \frac{1}{\omega}\left(\dot{y}_{(0)}\sin\omega\tau + \frac{\ddot{y}_{(0)}}{\omega}\cos\omega\tau\right) + C_5\,, \tag{3.330}$$

where the integration constants are

$$C_4 = x_{(0)} + \frac{\dot{y}_{(0)}}{\omega}\,, \qquad C_5 = y_{(0)} + \frac{\ddot{y}_{(0)}}{\omega^2}\,. \tag{3.331}$$

As we have already mentioned, the live forces integral in stationary magnetic fields (3.289) implies the constant relativistic mass of a particle moved in the fields and hence the constant square of its observable velocity. Then putting the solutions for the velocities of the mirror-world particle, namely — squaring the quantities $\dot{x}$, $\dot{y}$, $\dot{z}$, and adding them up we obtain that

$$\begin{gathered}\mathrm{v}^2 = \dot{x}^2_{(0)} + \dot{y}^2_{(0)} + \dot{z}^2_{(0)} + \\ +\, 2\left(\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega}\right)\left(\frac{\ddot{y}_{(0)}}{\omega} + \dot{y}_{(0)}\sin\omega\tau - \frac{\ddot{y}_{(0)}}{\omega}\cos\omega\tau\right)\end{gathered} \tag{3.332}$$

is constant $\mathrm{v}^2 = const$ provided that

$$\dot{x}_{(0)} + \frac{\ddot{y}_{(0)}}{\omega} = 0\,. \tag{3.333}$$

From the formula for x (3.329), we see that the particle performs strictly harmonic oscillations along x provided that the same condition (3.333) is true. Taking this fact into account, squaring and adding up x (3.329) and y (3.330) for the mirror-world particle in the homogeneous stationary magnetic field, we obtain its trajectory within $x\,y$ plane

$$\begin{gathered}x^2 + y^2 = \frac{1}{\omega^2}\left(\dot{y}^2_{(0)} + \frac{\ddot{y}^2_{(0)}}{\omega^2}\right) - \frac{2C_4}{\omega}\left(\dot{y}_{(0)}\cos\omega\tau + \frac{\ddot{y}_{(0)}}{\omega}\sin\omega\tau\right) + \\ +\left(\dot{y}_{(0)}\sin\omega\tau + \frac{\ddot{y}_{(0)}}{\omega}\cos\omega\tau\right)\frac{2C_5}{\omega} + C_4^2 + C_5^2\,,\end{gathered} \tag{3.334}$$

which only differs from the our-world particle trajectory (3.318) by $\omega+2\Omega$ replaced with ω and by numerical values of integration constants (3.331). Therefore a mirror-world charged particle of zero initial velocity along z (the direction of the magnetic strength), moves along an *ellipse* within $x\,y$ plane.

Once we assume $\ddot{y}_{(0)}$, as well as the constants C_4 and C_5 to be zeroes, the solutions become simpler

$$x=-\frac{1}{\omega}\,\dot{y}_{(0)}\cos\omega\tau\,,\qquad y=\frac{1}{\omega}\,\dot{y}_{(0)}\sin\omega\tau\,. \tag{3.335}$$

In such a simplified case, the mirror-world particle which is at rest with respect to the field direction makes a *circle* within $x\,y$ plane

$$x^2+y^2=\frac{\dot{y}^2_{(0)}}{\omega^2} \tag{3.336}$$

with radius $r=\frac{\dot{y}_{(0)}}{\omega}=\frac{mc}{eH}\,\dot{y}_{(0)}$. Consequently, if the initial velocity of the particle along the magnetic field direction (the axis z) is not zero, then the particle moves along a *spiral line* around the magnetic field direction. Hence, motion of mirror-world charged particles in homogeneous stationary magnetic fields is the same as that of our-world charged particles in the absence of the space non-holonomity.

b) Magnetic field is orthogonal to non-holonomity field

We are going to consider the case, where the magnetic strength pseudovector H^{*i} is orthogonal to the pseudovector $\Omega^{*i}=\frac{1}{2}\,\varepsilon^{ikm}A_{km}$ of the space non-holonomity field. Then the first equation from the 1st group of the Maxwell equations we have obtained for stationary magnetic fields (3.292) implies that the charge density is zero $\rho=0$.

We assume that the magnetic strength is directed along y (only the component $H^{*2}=H$ is not zero), while the non-holonomity field is directed along z (only the component $\Omega^{*3}=\Omega$ is not zero). We also assume that the magnetic field is stationary and homogeneous. Hence, the non-zero component of the magnetic strength is

$$H^{*2}=H_{31}=\frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_3}{\partial x}-\frac{\partial \mathrm{v}_1}{\partial z}\right)=const\,. \tag{3.337}$$

Then, if the non-holonomity field is weak, the equations of motion of the our-world particle are

$$\ddot{x}+2\Omega\dot{y}=\frac{eH}{mc}\,\dot{z}\,,\quad \ddot{y}-2\Omega\dot{x}=0\,,\quad \ddot{z}=-\frac{eH}{mc}\,\dot{x}\,, \tag{3.338}$$

or, denoting $\omega = \frac{eH}{mc}$,

$$\ddot{x} + 2\Omega\dot{y} = \omega\dot{z}\,, \qquad \ddot{y} - 2\Omega\dot{x} = 0\,, \qquad \ddot{z} = -\omega\dot{x}\,. \tag{3.339}$$

Differentiating the first equation with respect to τ and substituting $\ddot{y}$ and $\ddot{z}$ into it from the second and the third equations we have

$$\dddot{x} + \left(4\Omega^2 + \omega^2\right)\dot{x} = 0\,. \tag{3.340}$$

Setting $\dot{x} = p$, we arrive at the equation of oscillations

$$\ddot{p} + \widetilde{\omega}^2 p = 0\,, \qquad \widetilde{\omega} = \sqrt{4\Omega^2 + \omega^2} = \sqrt{4\Omega^2 + \left(\frac{eH}{mc}\right)^2}\,, \tag{3.341}$$

which solves as follows

$$p = C_1 \cos\widetilde{\omega}\tau + C_2 \sin\widetilde{\omega}\tau\,, \tag{3.342}$$

where $C_1 = \dot{x}_{(0)}$ and $C_2 = \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}$ are integration constants. Integrating $\dot{x} = p$ with respect to τ we obtain the expression for x as follows

$$x = \frac{\dot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\cos\widetilde{\omega}\tau + x_{(0)} + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\,, \tag{3.343}$$

where $x_{(0)} + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2} = C_3$ is integration constant.

Substituting $\dot{x} = p$ (3.342) into the equations of motion in terms of y and z (3.339) and integrating we obtain

$$\dot{y} = \frac{2\Omega}{\widetilde{\omega}}\,\dot{x}_{(0)}\sin\widetilde{\omega}\tau - \frac{2\Omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\cos\widetilde{\omega}\tau + \dot{y}_{(0)} + \frac{2\Omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\,, \tag{3.344}$$

$$\dot{z} = \frac{\omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\cos\widetilde{\omega}\tau - \frac{\omega}{\widetilde{\omega}}\,\dot{x}_{(0)}\sin\widetilde{\omega}\tau + \dot{z}_{(0)} - \frac{\omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\,, \tag{3.345}$$

where $\dot{y}_{(0)} + \frac{2\Omega\ddot{x}_{(0)}}{\widetilde{\omega}^2} = C_4$ and $\dot{z}_{(0)} - \frac{\omega\ddot{x}_{(0)}}{\widetilde{\omega}^2} = C_5$ are new integration constants. Then integrating these equations (3.344, 3.345) with respect to τ we obtain final formulae for y and z

$$\begin{aligned} y = -\frac{2\Omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right) + \dot{y}_{(0)}\tau + {} \\ + \frac{2\Omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\tau + y_{(0)} + \frac{2\Omega}{\widetilde{\omega}^2}\,\dot{x}_{(0)}\,, \end{aligned} \tag{3.346}$$

$$\begin{aligned} z = \frac{\omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right) + \dot{z}_{(0)}\tau - {} \\ - \frac{\omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\tau + z_{(0)} - \frac{\omega}{\widetilde{\omega}^2}\,\dot{x}_{(0)}\,, \end{aligned} \tag{3.347}$$

where $y_{(0)} + \frac{2\Omega\dot{x}_{(0)}}{\widetilde{\omega}^2} = C_6$ and $z_{(0)} - \frac{\omega\dot{x}_{(0)}}{\widetilde{\omega}^2} = C_7$.

Provided that $\Omega = 0$ (the space rotation is absent), and that some integration constants are zeroes, the above equations fully match well-known formulae of relativistic electrodynamics for the case, where a stationary magnetic field is directed along the axis z

$$x = \frac{\dot{x}_{(0)}}{\omega}\sin\widetilde{\omega}\tau\,, \quad y = y_{(0)} + \dot{y}_{(0)}\tau\,, \quad z = \frac{\dot{x}_{(0)}}{\omega}\cos\widetilde{\omega}\tau\,. \tag{3.348}$$

Because the live forces integral implies that the square of the observable velocity of a charged particle in stationary magnetic fields is constant, we have a possibility to calculate $\mathrm{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. We obtain

$$\begin{gathered}\mathrm{v}^2 = \dot{x}^2_{(0)} + \dot{y}^2_{(0)} + \dot{z}^2_{(0)} + \frac{2}{\widetilde{\omega}}\left(\ddot{x}_{(0)} + 2\Omega\,\dot{y}_{(0)} - \omega\dot{z}_{(0)}\right)\times \\ \times\left(\frac{\ddot{x}_{(0)}}{\widetilde{\omega}} + \dot{x}_{(0)}\sin\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\cos\widetilde{\omega}\tau\right),\end{gathered} \tag{3.349}$$

so $\mathrm{v}^2 = const$, provided that

$$\ddot{x}_{(0)} + 2\Omega\,\dot{y}_{(0)} - \omega\dot{z}_{(0)} = 0\,. \tag{3.350}$$

The spatial trajectory of the particle can be found, calculating $x^2 + y^2 + z^2$, so that we obtain the equation

$$\begin{gathered}x^2 + y^2 + z^2 = \frac{1}{\widetilde{\omega}^2}\left(\dot{x}^2_{(0)} + \frac{\ddot{x}^2_{(0)}}{\widetilde{\omega}^2}\right) + C_3^2 + C_6^2 + C_7^2 + \\ + \left(C_4^2 + C_5^2\right)\tau^2 + 2\left(C_4C_6 + C_5C_7\right)\tau + \Big[\left(\omega C_7 - 2\Omega C_6\right) + \\ + 2\left(\omega C_5 - 2\Omega C_6\right)\tau\Big]\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right)\frac{1}{\widetilde{\omega}^2} + \\ + \frac{2C_3}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right),\end{gathered} \tag{3.351}$$

which includes a linear (with respect to time) term and a square term, as well as a parametric term and two harmonic terms. In a particular case, if we assume integration constants to be zeroes, the obtained formula (3.351) takes the form of a regular equation of a *sphere*

$$x^2 + y^2 + z^2 = \frac{1}{\widetilde{\omega}^2}\left(\dot{x}^2_{(0)} + \frac{\ddot{x}^2_{(0)}}{\widetilde{\omega}^2}\right), \tag{3.352}$$

whose radius is

$$r = \frac{1}{\widetilde{\omega}}\sqrt{\dot{x}^2_{(0)} + \frac{\ddot{x}^2_{(0)}}{\widetilde{\omega}^2}}\,, \tag{3.353}$$

where $\widetilde{\omega} = \sqrt{4\Omega^2 + \omega^2} = \sqrt{4\Omega^2 + \left(\frac{eH}{mc}\right)^2}$.

So, an our-world charged particle in a homogeneous stationary magnetic field, orthogonal to the space non-holonomity field, moves on a surface of a *sphere* whose radius depends on the magnetic strength and the angular velocity of the space rotation.

In a particular case, where the non-holonomity field is absent and the initial acceleration is zero, our equation of the trajectory simplifies significantly to an equation of the sphere

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2}\,\dot{x}^2_{(0)}\,, \qquad r = \frac{1}{\omega}\,\dot{x}_{(0)} = \frac{mc}{eH}\,\dot{x}_{(0)} \tag{3.354}$$

with radius depending only on interaction of the particle's charge with the magnetic field — the result, well-known in electrodynamics (see §21 in *The Classical Theory of Fields*).

For a mirror-world charged particle which moves in a homogeneous stationary magnetic field, orthogonal to the non-holonomity field, the equations of motion are

$$\ddot{x} = \frac{eH}{mc}\,\dot{z}, \qquad \ddot{y} = 0, \qquad \ddot{z} = -\frac{eH}{mc}\,\dot{x}\,. \tag{3.355}$$

These are only different from the equations for the our-world particle (3.338) by the absence of the terms which include the angular velocity Ω of the space rotation.

§3.13 Motion in a stationary electromagnetic field

In this section, we are going to focus on motion of a charged particle under action of both magnetic and electric components of a stationary electromagnetic field.

As a "background" we will consider a non-holonomic space which rotates about the z axis at a constant angular velocity $\Omega_{12} = -\,\Omega_{21} = \Omega$, so the space is of the metric (3.275). In such a space, $F_i = 0$ and $D_{ik} = 0$.

We will solve the problem assuming that the non-holonomity field is *weak* and hence the three-dimensional space has the Euclidean metric. Here the Maxwell equations for stationary fields (3.215, 3.216) are

$$\left.\begin{aligned} &\Omega_{*m}H^{*m} = -\,2\pi c\rho \\ &\varepsilon^{ikm}\,\nabla_k\big(H_{*m}\sqrt{h}\big) = \frac{4\pi}{c}\,j^i\sqrt{h} = 0 \end{aligned}\right\}\ \mathrm{I}\,, \tag{3.356}$$

$$\left.\begin{aligned} &\Omega_{*m}E^m = 0 \\ &\varepsilon^{ikm}\,\nabla_k\big(E_m\sqrt{h}\big) = 0 \end{aligned}\right\}\ \mathrm{II}\,, \tag{3.357}$$

because the condition of observable homogeneity of the field is the equality to zero of its chr.inv.-derivative [9,11–13], while in the particular case under consideration the Christoffel chr.inv.-symbols equal zero (the metric is Galilean) so the chr.inv.-derivative is the same as that in a regular case. Hence, the Maxwell equations imply that the following conditions will be true here:

a) The space non-holonomity pseudovector and the electric strength are orthogonal to each other $\Omega_{*m}E^m = 0$;

b) The space non-holonomity pseudovector and the magnetic strength are orthogonal to each other. Here, the charge density is zero, i.e $\rho = 0$;

c) The electromagnetic field current is absent, $j^i = 0$.

The last condition implies that the presence of the electromagnetic field currents $j^i \neq 0$ is derived from inhomogeneity of the acting magnetic strength.

Given that the non-holonomity pseudovector is orthogonal to the electric strength, we can consider motion of the particle in two cases of mutual orientation of the fields:

1) $\vec{H} \perp \vec{E}$ and $\vec{H} \parallel \vec{\Omega}$;

2) $\vec{H} \parallel \vec{E}$ and $\vec{H} \perp \vec{\Omega}$.

In either case, we assume that the electric strength is co-directed with the x axis . In the background metric (3.275) the space rotation pseudovector is co-directed with z. Hence in the first case, the magnetic strength is co-directed with z, while in the second case it is co-directed with x.

Chr.inv.-equations of motion of a charged particle in the stationary electromagnetic field, where the electric strength is co-directed with x are as follows. For the our-world particle

$$\frac{dm}{d\tau} = -\frac{eE_1}{c^2}\frac{dx}{d\tau}\,, \tag{3.358}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2mA_{k\cdot}^{\cdot i}\mathrm{v}^k = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right), \tag{3.359}$$

and for the mirror-world particle

$$\frac{dm}{d\tau} = \frac{eE_1}{c^2}\frac{dx}{d\tau}\,, \tag{3.360}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) = -\,e\left(E^i + \frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right). \tag{3.361}$$

As was done before, we consider the case of a particle *repelled* by the field. Then components of the electric strength E_i, co-directed with x, are (in a Galilean reference frame covariant and contravariant indices of tensor quantities are the same)

$$E_1 = E_x = \frac{\partial \varphi}{\partial x} = const = -E\,, \qquad E_2 = E_3 = 0\,. \tag{3.362}$$

Integration of the live forces theorem gives the live forces integral for our world and the mirror world, respectively as

$$m = \frac{eE}{c^2}\,x + B\,, \qquad m = -\frac{eE}{c^2}\,x + \widetilde{B}\,. \tag{3.363}$$

Here B is our-world integration constant and $\widetilde{B}$ is the mirror-world integration constant. Calculating these constants from the initial conditions at the moment $\tau = 0$, yields

$$B = m_{(0)} - \frac{eE}{c^2}\,x_{(0)}\,, \qquad \widetilde{B} = m_{(0)} + \frac{eE}{c^2}\,x_{(0)}\,, \tag{3.364}$$

where $m_{(0)}$ is the relativistic mass of the particle and $x_{(0)}$ is its displacement at the initial moment of time.

From the obtained integrals of live forces (3.363), we see that the differences between the three cases under this study, due to different orientations of the magnetic strength $\vec{H}$ to the electric strength $\vec{E}$ and to the angular velocity $\vec{\Omega}$ of the space rotation (the space non-holonomity field), will only reveal themselves in the vector chr.inv.-equations of motion, while the scalar chr.inv.-equations (3.358, 3.360) and their solutions (3.363) will be the same.

Note that the vector $\vec{E}$ can also be directed along y, but can not be directed along z. This is true, because in the space with such a metric co-directed with z is the non-holonomity pseudovector $\vec{\Omega}$, while the 2nd group of the Maxwell equations require $\vec{E}$ to be orthogonal to $\vec{\Omega}$.

Now, taking into account the integration results from the live forces theorem (3.363), we will write down the vector chr.inv.-equations for all cases under this study.

Case 1. We assume that $\vec{H} \perp \vec{E}$ and $\vec{H} \parallel \vec{\Omega}$, so the magnetic strength $\vec{H}$ is directed along z (parallel to the non-holonomity field).

Then out of all components of the magnetic strength the only non-zero component is

$$H^{*3} = H_{12} = \frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_1}{\partial y} - \frac{\partial \mathrm{v}_2}{\partial x}\right) + \frac{2\varphi}{c}\,A_{12} = const = H\,. \tag{3.365}$$

Consequently, the vector chr.inv.-equations of motion for the our-world particle are

$$\left.\begin{aligned} &\frac{eE}{c^2}\,\dot{x}^2 + \left(B + \frac{eE}{c^2}\,x\right)(\ddot{x} + 2\Omega\dot{y}) = eE - \frac{eH}{c}\,\dot{y} \\ &\frac{eE}{c^2}\,\dot{x}\dot{y} + \left(B + \frac{eE}{c^2}\,x\right)(\ddot{y} - 2\Omega\dot{x}) = \frac{eH}{c}\,\dot{x} \\ &\frac{eE}{c^2}\,\dot{x}\dot{z} + \left(B + \frac{eE}{c^2}\,x\right)\ddot{z} = 0 \end{aligned}\right\}, \tag{3.366}$$

while for the mirror-world particle we have

$$\left.\begin{aligned} &\frac{eE}{c^2}\,\dot{x}^2 + \left(\widetilde{B} - \frac{eE}{c^2}\,x\right)\ddot{x} = eE - \frac{eH}{c}\,\dot{y} \\ &\frac{eE}{c^2}\,\dot{x}\dot{y} + \left(\widetilde{B} - \frac{eE}{c^2}\,x\right)\ddot{y} = \frac{eH}{c}\,\dot{x} \\ &\frac{eE}{c^2}\,\dot{x}\dot{z} + \left(\widetilde{B} - \frac{eE}{c^2}\,x\right)\ddot{z} = 0 \end{aligned}\right\}. \tag{3.367}$$

Besides, the 1st group of the Maxwell equations require that in the case under study, the next condition must be true

$$\Omega_{*3}H^{*3} = -2\pi c\rho\,, \tag{3.368}$$

where $\Omega_{*3} = \Omega = const$ and $H^{*3} = H = const$. From this formula we arrive at the obvious conclusion: this mutual orientation of the space non-holonomity pseudovector and the magnetic strength is only possible in the case, where electric charges are present in the space so the charge density is $\rho \neq 0$.

Case 2. $\vec{H} \parallel \vec{E}$, $\vec{H} \perp \vec{\Omega}$, and $\vec{E} \perp \vec{\Omega}$, so the magnetic and electric strengths are co-directed with x, while the non-holonomity field is still directed along z.

Here, out of all components of the magnetic strength only the first component is non-zero

$$H^{*1} = H_{23} = \frac{\varphi}{c}\left(\frac{\partial \mathrm{v}_2}{\partial z} - \frac{\partial \mathrm{v}_3}{\partial y}\right) = const = H\,, \tag{3.369}$$

With this formula, we obtain the vector chr.inv.-equations of motion for the our-world particle and those for the mirror-world prticle. For the our-world particle the equaions are

$$\left.\begin{aligned} &\frac{eE}{c^2}\,\dot{x}^2 + \left(B + \frac{eE}{c^2}\,x\right)(\ddot{x} + 2\Omega\dot{y}) = eE \\ &\frac{eE}{c^2}\,\dot{x}\dot{y} + \left(B + \frac{eE}{c^2}\,x\right)(\ddot{y} - 2\Omega\dot{x}) = -\frac{eH}{c}\,\dot{z} \\ &\frac{eE}{c^2}\,\dot{x}\dot{z} + \left(B + \frac{eE}{c^2}\,x\right)\ddot{z} = \frac{eH}{c}\,\dot{y} \end{aligned}\right\}, \tag{3.370}$$

while the equations for the mirror-world particle are

$$\left.\begin{aligned} &\frac{eE}{c^2}\,\dot{x}^2 + \left(\widetilde{B} - \frac{eE}{c^2}\,x\right)\ddot{x} = eE \\ &\frac{eE}{c^2}\,\dot{x}\dot{y} + \left(\widetilde{B} - \frac{eE}{c^2}\,x\right)\ddot{y} = -\frac{eH}{c}\,\dot{z} \\ &\frac{eE}{c^2}\,\dot{x}\dot{z} + \left(\widetilde{B} - \frac{eE}{c^2}\,x\right)\ddot{z} = \frac{eH}{c}\,\dot{y} \end{aligned}\right\}. \tag{3.371}$$

Now that we have equations of motion of the charged particle for all three cases of mutual orientation of the acting stationary fields (the electromagnetic field and the space non-holonomity field) we can turn to solving them.

a) Magnetic field is orthogonal to electric field and is parallel to non-holonomity field

Let us solve the vector chr.inv.-equations of motion of the charged particle (3.366, 3.367) in non-relativistic approximation, i. e. assuming the absolute value of its observable velocity is negligible compared to the light velocity. Hence, we can also assume that the particle's mass at the initial moment of time is equal to its rest-mass

$$m_{(0)} = \frac{m_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \cong m_0\,. \tag{3.372}$$

We further assume that the electric strength E is negligible as well, thus the term $\frac{eEx}{c^2}$ can be truncated. Under these conditions, the vector chr.inv.-equations of motion will be transformed as follows: For the our-world particle they become

$$m_0\,(\ddot{x}+2\Omega\dot{y}) = eE-\frac{eH}{c}\,\dot{y}\,, \quad m_0\,(\ddot{y}-2\Omega\dot{x}) = \frac{eH}{c}\,\dot{x}\,, \quad m_0\ddot{z} = 0\,, \tag{3.373}$$

while for the mirror-world particle we have

$$m_0\ddot{x} = eE - \frac{eH}{c}\,\dot{y}\,, \qquad m_0\ddot{y} = \frac{eH}{c}\,\dot{x}\,, \qquad m_0\ddot{z} = 0\,. \tag{3.374}$$

These equations match those obtained in §22 in *The Classical Theory of Fields* [10] in the case, where the space non-holonomity is absent $\Omega = 0$ and the electric strength is co-directed with x.

The obtained equations for the mirror-world particle are a particular case of the our-world equations at $\Omega = 0$. Therefore, we can only integrate the our-world equations, while the mirror-world solutions are obtained automatically by assuming $\Omega = 0$. Integrating the equation of motion along z we arrive at

$$z = \dot{z}_{(0)}\,\tau + z_{(0)}\,. \tag{3.375}$$

Integrating the equation along y) we arrive at

$$\dot{y} = \left(2\Omega + \frac{eH}{m_0 c}\right) x + C_1\,, \tag{3.376}$$

where the integration constant is $C_1 = \dot{y}_{(0)} - \left(2\Omega + \frac{eH}{m_0 c}\right) x_{(0)}$.

Substituting $\dot{y}$ into the first equation (3.373) we obtain second-order differential equation with respect to x

$$\ddot{x} + \omega^2 x = \frac{eE}{m_0} + \omega^2 x_{(0)} - \omega \dot{y}_{(0)}\,, \tag{3.377}$$

where $\omega = 2\Omega + \frac{eH}{m_0 c}$. Introducing a new variable

$$u = x - \frac{A}{\omega^2}\,, \qquad A = \frac{eE}{m_0} + \omega^2 x_{(0)} - \omega \dot{y}_{(0)}\,, \tag{3.378}$$

we obtain the equation of harmonic oscillations

$$\ddot{u} + \omega^2 u = 0\,, \tag{3.379}$$

with solution

$$u = C_2 \cos\omega\tau + C_3 \sin\omega\tau\,, \tag{3.380}$$

where the integration constants are $C_2 = u_{(0)}$, $C_3 = \frac{\dot{u}_{(0)}}{\omega}$. Returning to the variable x by reverse substitution of variables we finally obtain a formula for x

$$x = \frac{1}{\omega}\left(\dot{y}_{(0)} - \frac{eE}{m_0\omega}\right)\cos\omega\tau + \frac{\dot{x}_{(0)}}{\omega}\sin\omega\tau + \frac{eE}{m_0\omega^2} + x_{(0)} - \frac{\dot{y}_{(0)}}{\omega}\,. \tag{3.381}$$

Substituting the formula into the obtained equation for $\dot{y}$ (3.376), and integrating gives a formula for y as

$$y=\frac{1}{\omega}\left(\dot{y}_{(0)}-\frac{eE}{m_0\omega}\right)\sin\omega\tau-\frac{\dot{x}_{(0)}}{\omega}\cos\omega\tau+\frac{eE}{m_0\omega^2}+y_{(0)}+\frac{\dot{x}_{(0)}}{\omega}\,. \quad (3.382)$$

The vector chr.inv.-equations in the mirror world have the same solutions, but because for them $\Omega=0$, the frequency is $\omega=\frac{eH}{m_0c}$.

Energies of our-world and mirror-world particles are $E=mc^2$ and $E=-mc^2$, respectively.

Finally, we obtain the three-dimensional momentum of the our-world particle

$$\left.\begin{aligned}
&p^1=m_0\dot{x}=\left(\frac{eE}{\omega}-m_0\dot{y}_{(0)}\right)\sin\omega\tau+m_0\dot{x}_{(0)}\cos\omega\tau\\
&p^2=m_0\dot{y}=\left(\frac{2\Omega m_0}{\omega}+\frac{eH}{\omega c}\right)\left(\frac{eE}{m_0\omega}-\dot{y}_{(0)}\right)+m_0\,\dot{y}_{(0)}+\\
&\quad+\left(\frac{2\Omega m_0}{\omega}+\frac{eH}{\omega c}\right)\left[\left(\dot{y}_{(0)}-\frac{eE}{m_0\omega}\right)\cos\omega\tau+\dot{x}_{(0)}\sin\omega\tau\right]\\
&p^3=m_0\dot{z}=m_0\dot{z}_{(0)}
\end{aligned}\right\}, \quad (3.383)$$

while in the mirror world we have

$$\left.\begin{aligned}
&p^1=\left(\frac{eE}{\omega}-m_0\dot{y}_{(0)}\right)\sin\omega\tau+m_0\dot{x}_{(0)}\cos\omega\tau\\
&p^2=\frac{eE}{\omega}+m_0\left[\left(\dot{y}_{(0)}-\frac{eE}{m_0\omega}\right)\cos\omega\tau+\dot{x}_{(0)}\sin\omega\tau\right]\\
&p^3=m_0\dot{z}_{(0)}
\end{aligned}\right\}, \quad (3.384)$$

so in contrast to our world, the frequency is $\omega=\frac{eH}{m_0c}$.

From here we see that the momentum of an our-world charged particle in the given configuration of the acting fields performs harmonic oscillations along x and y, while along z it is a linear function of the observable time τ (if the initial velocity $\dot{z}\neq 0$). Within xy plane the oscillation frequency is $\omega=2\Omega+\frac{eH}{m_0c}$.

It should be noted that obtaining exact solutions of the equations of motion in the presence of both electric and magnetic components is problematic, because we need to solve elliptic integrals in the process. It may be possible to solve them in future, when the solutions will be obtained on computers, but this problem evidently stays beyond the goal of this book. Presumably, Landau and Lifshitz faced a similar problem, because in §22 of *The Classical Theory of Fields* where considering a similar problem* they obtained equations of motion and solved them assuming the velocity to be non-relativistic and the electric strength to be weak $\frac{eEx}{c^2} \approx 0$.

b) Magnetic field is parallel to electric field and is orthogonal to non-holonomity field

Let us solve the vector chr.inv.-equations of motion of the charged particle (3.370, 3.371) in the same approximation as we did in the first case. Then for our world and for the mirror world the equation are, respectively

$$\ddot{x} + 2\Omega\dot{y} = \frac{eE}{m_0}\,, \quad \ddot{y} - 2\Omega\dot{x} = -\frac{eH}{m_0 c}\,\dot{z}\,, \quad \ddot{z} = \frac{eH}{m_0 c}\,\dot{y}\,, \tag{3.385}$$

$$\ddot{x} = \frac{eE}{m_0}\,, \qquad \ddot{y} = -\frac{eH}{m_0 c}\,\dot{z}\,, \qquad \ddot{z} = \frac{eH}{m_0 c}\,\dot{y}\,. \tag{3.386}$$

Integrating the first equation of motion in our world (3.385), which is along x, we obtain

$$\dot{x} = \frac{eE}{m_0}\,\tau - 2\Omega y + C_1\,, \qquad C_1 = const = \dot{x}_{(0)} + 2\Omega y_{(0)}\,. \tag{3.387}$$

Integrating the third equation (along z) we have

$$\dot{z} = \frac{eH}{m_0 c}\,y + C_2\,, \qquad C_2 = const = \dot{z}_{(0)} - \frac{eH}{m_0 c}\,y_{(0)}\,. \tag{3.388}$$

Substituting the obtained formulae for $\dot{x}$ and $\dot{z}$ into the second equation of motion (3.385) we obtain the linear differential equation of the 2nd order with respect to y

$$\ddot{y} + \left(4\Omega^2 + \frac{e^2H^2}{m_0^2c^2}\right) y = \frac{2\Omega eE}{m_0}\,\tau + 2\Omega C_1 - \frac{eH}{m_0 c}\,C_2\,. \tag{3.389}$$

We are going to solve it, using the method of change of variables. Introducing a new variable u

$$u = y + \frac{1}{\omega^2}\left(\frac{eH}{m_0 c}\,C_2 - 2\Omega C_1\right), \quad \omega^2 = 4\Omega^2 + \frac{e^2H^2}{m_0^2c^2}\,, \tag{3.390}$$

*But in contrast to this book, they used general covariant methods and did not account for the space non-holonomity.

we obtain an equation of forced oscillations

$$\ddot{u} + \omega^2 u = \frac{2\Omega eE}{m_0}\,\tau\,, \tag{3.391}$$

whose solution is the sum of the general solution of the equation of free oscillations

$$\ddot{u} + \omega^2 u = 0\,, \tag{3.392}$$

and of a particular solution of the inhomogeneous equation

$$\tilde{u} = M\tau + N\,, \tag{3.393}$$

where $M = const$ and $N = const$. Differentiating $\tilde{u}$ twice with respect to τ and substituting the results into the inhomogeneous equation (3.391) and then equating the obtained coefficients for τ we obtain the linear coefficients

$$M = \frac{2\Omega eE}{m_0\omega^2}\,, \qquad N = 0\,. \tag{3.394}$$

Then the general solution of the initial inhomogeneous equation (3.391) becomes

$$u = C_3\cos\omega\tau + C_4\sin\omega\tau + \frac{2\Omega eE}{m_0\omega^2}\,\tau\,, \tag{3.395}$$

where the integration constants can be obtained by substituting the initial conditions at $\tau = 0$ into the obtained formula. As a result, we have $C_3 = u_{(0)}$ and $C_4 = \frac{\dot{u}_{(0)}}{\omega}$.

Returning to the old variable y (3.390) we find the final solution for this coordinate

$$\begin{aligned} y = {} & \left[y_{(0)} + \frac{1}{\omega^2}\left(\frac{eH}{m_0c}\,C_2 + 2\Omega C_1\right)\right]\cos\omega\tau + \\ & + \frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau - \frac{1}{\omega^2}\left(\frac{eH}{m_0c}\,C_2 + 2\Omega C_1\right) + \frac{2\Omega eE}{m_0\omega^2}\,\tau\,. \end{aligned} \tag{3.396}$$

Then substituting this formula into equations for $\dot{x}$ and $\dot{z}$ after integration we arrive at the solutions for x and z

$$\begin{aligned} x = {} & \frac{eE}{2m_0}\left(1 - \frac{4\Omega^2}{\omega^2}\right)\tau^2 - \frac{2\Omega}{\omega}\left(y_{(0)} + A\right)\sin\omega\tau + \\ & + \frac{2\Omega\,\dot{y}_{(0)}}{\omega}\cos\omega\tau + (C_1 + 2\Omega A)\,\tau + C_5\,, \end{aligned} \tag{3.397}$$

$$z = \frac{eH}{m_0 c\,\omega}\left[\big(y_{(0)} + A\big)\sin\omega\tau - \frac{\dot{y}_{(0)}}{\omega}\cos\omega\tau\right] - \\ - \left(\frac{eH}{m_0 c}\,A - C_2\right)\tau + C_6\,, \tag{3.398}$$

where (for convenient notation),

$$A = \frac{1}{\omega^2}\left(\frac{eH}{m_0 c}\,C_2 - 2\Omega C_1\right), \tag{3.399}$$

while the new integration constants are

$$C_5 = x_0 - \frac{2\Omega\,\dot{y}_{(0)}}{\omega}\,, \qquad C_6 = z_{(0)} + \frac{eH\,\dot{y}_{(0)}}{m_0 c\,\omega^2}\,. \tag{3.400}$$

If we assume $\Omega = 0$, then from coordinates of the our-world charged particle (3.396–3.398) we immediately obtain the solutions for the analogous charged particle in the mirror world

$$x = \frac{eE}{2m_0}\,\tau^2 + \dot{x}_{(0)}\tau + x_{(0)}\,, \tag{3.401}$$

$$y = \frac{\dot{z}_{(0)}}{\omega}\cos\omega\tau + \frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau - \frac{\dot{z}_{(0)}}{\omega} + y_{(0)}\,, \tag{3.402}$$

$$z = \frac{\dot{z}_{(0)}}{\omega}\sin\omega\tau - \frac{\dot{y}_{(0)}}{\omega}\cos\omega\tau + \frac{\dot{y}_{(0)}}{\omega} + z_{(0)}\,. \tag{3.403}$$

Consequently, components of the three-dimensional momentum of the our-world particle under the considered configuration of the acting fields take the form

$$\left.\begin{aligned} & p^1 = m_0\dot{x}_{(0)} + eE\left(1 - \frac{4\Omega^2}{\omega^2}\right)\tau - \\ & - 2m_0\Omega\left[\frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau + \big(y_{(0)} + A\big)\cos\omega\tau - \frac{\dot{y}_{(0)}}{\omega} - A\right] \\ & p^2 = m_0\Big[\,\dot{y}_{(0)}\cos\omega\tau - \omega\,\big(y_{(0)} + A\big)\sin\omega\tau\,\Big] + \frac{2\Omega\,eE}{\omega^2} \\ & p^3 = m_0\,\dot{z}_{(0)} + \\ & + \frac{eH}{c}\left[\big(y_{(0)}{+}A\big)\cos\omega\tau{+}\frac{\dot{y}_{(0)}}{\omega}\sin\omega\tau{-}A{+}\frac{2\Omega\,eE}{m_0\omega^2}\,\tau{-}y_{(0)}\right] \end{aligned}\right\}, \tag{3.404}$$

where the frequency is $\omega = \sqrt{4\Omega^2 + \frac{e^2H^2}{m_0^2 c}}$. In the mirror world, given this configuration of the acting fields, components of the three-dimensional momentum of the analogous charged particle are

$$\left.\begin{aligned} p^1 &= m_0 \dot{x}_{(0)} + 2eE\tau \\ p^2 &= m_0 \left(\dot{y}_{(0)} \cos\omega\tau - \dot{z}_{(0)} \sin\omega\tau\right) \\ p^3 &= m_0 \left(\dot{z}_{(0)} \cos\omega\tau - \dot{y}_{(0)} \sin\omega\tau\right) \end{aligned}\right\}, \qquad (3.405)$$

where in contrast to our world the frequency is $\omega = \frac{eH}{m_{(0)}c}$.

§3.14 Conclusions

In fact the theory we have built in this Chapter can be more precisely referred to as the *chronometrically invariant representation of electrodynamics in a pseudo-Riemannian space.* In other words, because the mathematical apparatus of physical observable quantities initially assumes the four-dimensional space-time of the General Theory of Relativity, we can simply refer to it as the *chronometrically invariant electrodynamics* (CED). Here we have obtained only the basics of this theory:

- The chr.inv.-components of the electromagnetic field tensor (the Maxwell tensor);
- Maxwell's equations in chr.inv.-form;
- The law of conservation of electric charge in chr.inv.-form;
- Lorentz' condition in chr.inv.-form;
- D'Alembert's equations in chr.inv.-form (the wave propagation equations) for the scalar potential and the vector-potential of the electromagnetic field;
- Lorentz' force in chr.inv.-form;
- The energy-momentum tensor of an electromagnetic field, and its chr.inv.-components;
- The chr.inv.-equations of motion of a charged test-particle;
- The geometric structure of the four-dimensional potential of an electromagnetic field.

It is evident that, the whole scope of the chr.inv.-electrodynamics is much wider. In addition to what has been said we could obtain chr.inv.-equations of motion of a spatially distributed charge or study motion of a particle which bears its own electromagnetic emission, interacting the field or, at last, deduce equations of motion for a particle which travels at an arbitrary angle to the field strength (either for an individual particle or a distributed charge), or tackle scores of other interesting problems.

Chapter 4 Motion of Spin-Particles

§4.1 Problem statement

In this Chapter we are going to obtain equation of motion of a particle with an inner rotational momentum (*spin*). As we mentioned in Chapter 1, these are equations of parallel transfer of the four-dimensional dynamic vector of the particle Q^α, which is the sum of vectors

$$Q^\alpha = P^\alpha + S^\alpha, \tag{4.1}$$

where $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ is the four-dimensional momentum vector of this particle. The four-dimensional vector S^α is an additional momentum which this particle gains from its inner momentum (spin), so this momentum makes motion of the particle non-geodesic. Therefore, we will refer to S^α as the *spin-momentum.* Because we know components of the momentum vector P^α, to define summary dynamic vector Q^α we only need to obtain components of the spin-momentum vector S^α.

Hence our first step will be defining a particle's spin as geometric quantity in the four-dimensional pseudo-Riemannian space of the General Theory of Relativity. Then in §4.2, we are going to deduce the spin-momentum vector S^α itself. In §4.3 our goal will be to obtain equations of motion of a spin-particle in the pseudo-Riemannian space and their chr.inv.-projections. Other sections of this Chapter will focus on motion of elementary particles.

The numerical value of spin is $\pm n\hbar$, measured in fractions of Planck's constant, where n is the so-called *spin quantum number.* As of today, it is known that for various kinds of elementary particles this number may be $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. Alternating sign $\pm$ stands for possible right-wise or left-wise inner rotation of the particle under consideration. Besides, the Planck constant $\hbar$ has dimension of angular momentum [gram cm^2/sec]. This alone hints that spin's tensor by its geometric structure should be similar to the tensor of angular momentum, i. e. should be an antisymmetric tensor of the 2nd rank. We are going to check if another source can prove this.

Bohr's second postulate says that the length of the orbit of an electron should comprise the *integer* number of de Broglie wavelengths

$\lambda = \frac{h}{p}$, which stands for the electron according to the wave-particle concept. In other words, the length of the electron orbit $2\pi r$ comprises k de Broglie wavelengths

$$2\pi r = k\lambda = k\,\frac{h}{p}\,, \tag{4.2}$$

where p is the orbital momentum of the electron. Taking into account that Planck's constant is $\hbar = \frac{h}{2\pi}$, this equation (4.2) should be

$$rp = k\hbar\,. \tag{4.3}$$

Because the radius-vector of the electron orbit r^i is orthogonal to the vector of its orbital momentum p^k, this formula in tensor notation is a vector product, namely

$$\left[\,r^i; p^k\right] = k\hbar^{ik}\,. \tag{4.4}$$

From here we see that the Planck constant deduced from Bohr's second postulate in tensor notation is present with an antisymmetric 2nd rank tensor.

But this representation of the Planck constant is linked to orbital model of an atom — of the system more complicated than electron or any other elementary particle. Nevertheless, spin also defined by this constant, is an inner property of elementary particles themselves. Therefore, according to Bohr's second postulate we have to consider the geometric structure of the Planck constant proceeding from another experimental relationship which is related to inner structure of electron only.

We have such an opportunity thanks to classical experiments performed by Stern and Gerlach in 1921. One of their results is that any electron bears inner magnetic momentum L_{m}, which is proportional to its inner rotational momentum (spin)

$$\frac{m_{\mathrm{e}}}{e}\,L_{\mathrm{m}} = n\hbar\,, \tag{4.5}$$

where e is the charge of the electron, m_{e} is its mass and n is the spin quantum number (for electron $n = \frac{1}{2}$). The magnetic momentum of a contour with an area $S = \pi r^2$, which conducts a current I, is $L_{\mathrm{m}} = IS$. The current equals to the charge e divided by its period of circulation $T = \frac{2\pi r}{u}$ along this contour

$$I = \frac{eu}{2\pi r}\,, \tag{4.6}$$

where u is the linear velocity of the charge circulation. Hence, the inner magnetic momentum of the electron is

$$L_{\mathrm{m}} = \frac{1}{2}\, eur\,, \tag{4.7}$$

or in tensor notation*

$$L^{ik}_{\mathrm{m}} = \frac{1}{2}\, e\left[\, r^i; u^k \,\right] = \frac{1}{2}\left[\, r^i; p^k_{\mathrm{m}} \,\right]\,, \tag{4.8}$$

where r^i is the radius-vector of the inner current circulation provided by the electron, and u^k is the vector of the circulation velocity.

From here we see that the Planck constant, being calculated from the inner magnetic momentum of an electron (4.5), is also the vector product of two vectors. So it is an antisymmetric tensor of the 2nd rank, namely

$$\frac{m_{\mathrm{e}}}{2e}\left[\, r^i; p^k_{\mathrm{m}} \,\right] = n\hbar^{ik}, \tag{4.9}$$

which proves similar conclusion based on the Bohr second postulate.

Subsequently, considering inter-electronic quantum relationships in the four-dimensional pseudo-Riemannian space, we arrive at the *Planck four-dimensional antisymmetric tensor* $\hbar^{\alpha\beta}$, whose spatial components are three-dimensional quantities $\hbar^{ik}$

$$\hbar^{\alpha\beta} = \begin{pmatrix} \hbar^{00} & \hbar^{01} & \hbar^{02} & \hbar^{03} \\ \hbar^{10} & \hbar^{11} & \hbar^{12} & \hbar^{13} \\ \hbar^{20} & \hbar^{21} & \hbar^{22} & \hbar^{23} \\ \hbar^{30} & \hbar^{31} & \hbar^{32} & \hbar^{33} \end{pmatrix}. \tag{4.10}$$

This antisymmetric tensor $\hbar^{\alpha\beta}$ corresponds to dual the *Planck pseudotensor* $\hbar^{*\alpha\beta} = \frac{1}{2}\, E^{\alpha\beta\mu\nu}\hbar_{\mu\nu}$. Subsequently, spin of a particle in the four-dimensional pseudo-Riemannian space is characterized by the antisymmetric tensor $n\,\hbar^{\alpha\beta}$, or by its dual pseudotensor $n\,\hbar^{*\alpha\beta}$. Note that physical nature of spin does not matter here, it is sufficient that this fundamental property of particles is characterized by a tensor (or a pseudotensor) of a certain kind. Thanks to this approach, we can solve the problem of motion of spin-particles without any preliminary assumption on their inner structure, i. e. using strictly formal mathematical method.

*Equations (4.8) and (4.9) are given for the Minkowski space, which is quite acceptable for the above experiments. In Riemannian spaces the result of integration depends on the integration path. Therefore the radius-vector of a finite length is not defined in Riemannian spaces, because its length depends on constantly varying direction.

Hence from the geometric point of view, the Planck constant is an antisymmetric tensor of the 2nd rank, whose dimension is angular momentum irrespective of the quantities from which it was obtained: mechanical or electromagnetic. The latter also implies that the Planck tensor does not characterize rotation of masses inside atoms or any masses inside elementary particles, but it is derived from some fundamental quantum rotation of the space itself and sets all "elementary" rotations in the space irrespective of their nature.

The rotation of the space is characterized by the chr.inv.-tensor A_{ik} (1.36), which results from lowering indices $A_{ik} = h_{im}h_{kn}A^{mn}$ in the components A^{mn} of the contravariant four-dimensional tensor

$$A^{\alpha\beta} = c\,h^{\alpha\mu}h^{\beta\nu}a_{\mu\nu}\,, \qquad a_{\mu\nu} = \frac{1}{2}\left(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu}\right). \tag{4.11}$$

In the accompanying reference frame ($b^i = 0$) the auxiliary quantity $a_{\mu\nu}$ has the components

$$a_{00} = 0\,, \quad a_{0i} = \frac{1}{2c^2}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad a_{ik} = \frac{1}{2c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right), \tag{4.12}$$

so we have

$$\left.\begin{aligned} &A_{00} = 0\,, \qquad A_{0i} = -A_{i0} = 0\,, \\ &A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right). \end{aligned}\right\} \tag{4.13}$$

In the absence of gravitational fields, the tensor of angular velocities of the space rotation depends only on the linear velocity of this rotation v_i, hence we denote it as $A_{\alpha\beta} = \Omega_{\alpha\beta}$

$$\Omega_{00} = 0\,, \quad \Omega_{0i} = -\,\Omega_{i0} = 0\,, \quad \Omega_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right). \tag{4.14}$$

On the other hand, according to the wave-particle concept, any particle corresponds to a wave with the energy $E = mc^2 = \hbar\omega$, where m is the relativistic mass of the particle and ω is its specific frequency. In other words, from geometric viewpoint any particle can be considered as a wave defined within infinite proximity of geometric location of the particle, whose specific frequency depends on certain distribution of the angular velocities $\omega_{\alpha\beta}$, also defined within this proximity. As a result, the above quantum relationship in tensor notation becomes $mc^2 = \hbar^{\alpha\beta}\omega_{\alpha\beta}$.

Because the Planck tensor is antisymmetric, all of its diagonal elements are zeroes. Its space-time (mixed) components in the accompanying reference frame also should be zero similar to respective components of the four-dimensional tensor of the space rotation (4.14). Numerical values of spatial (three-dimensional) components of the Planck tensor, observable in experiments, are $\pm\hbar$ depending on the rotational direction and make the *Planck three-dimensional chr.inv.-tensor* $\hbar^{ik}$. In the case of left-wise rotations the components $\hbar^{12}$, $\hbar^{23}$, $\hbar^{31}$ are positive, while the components $\hbar^{13}$, $\hbar^{32}$, $\hbar^{21}$ are negative.

Then the geometric structure of the Planck four-dimensional tensor, represented as matrix, becomes

$$\hbar^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hbar & -\hbar \\ 0 & -\hbar & 0 & \hbar \\ 0 & \hbar & -\hbar & 0 \end{pmatrix}. \tag{4.15}$$

In the case of right-wise rotations the components $\hbar^{12}$, $\hbar^{23}$, $\hbar^{31}$ change their sign to become negative, while the components $\hbar^{13}$, $\hbar^{32}$, $\hbar^{21}$ become positive

$$\hbar^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\hbar & \hbar \\ 0 & \hbar & 0 & -\hbar \\ 0 & -\hbar & \hbar & 0 \end{pmatrix}. \tag{4.16}$$

The square of the Planck four-dimensional tensor can be deduced as follows

$$\begin{aligned} \hbar_{\alpha\beta}\,\hbar^{\alpha\beta} &= 2\hbar^2 \big[\big(g_{11}g_{22}-g_{12}^2\big)+\big(g_{11}g_{33}-g_{13}^2\big)+\big(g_{22}g_{33}-g_{23}^2\big)+ \\ &\quad + 2\left(g_{12}g_{23}-g_{22}g_{13}-g_{12}g_{33}+g_{13}g_{23}-g_{11}g_{23}+g_{12}g_{13}\right)\big]\,, \end{aligned} \tag{4.17}$$

and in the Minkowski space, where the reference frame is Galilean and the metric is diagonal (2.70), it equals $\hbar_{\alpha\beta}\hbar^{\alpha\beta}=6\hbar^2$. In the pseudo-Riemannian space the quantity $\hbar_{\alpha\beta}\hbar^{\alpha\beta}$ can be deduced by substitution of dependency of spatial components of the fundamental metric tensor from the metric chr.inv.-tensor $h_{ik}=-\,g_{ik}+\frac{1}{c^2}v_i v_k$ and the space rotation velocity into (4.17). Hence, though the physical observable components $\hbar^{ik}$ of the Planck tensor are constant (have opposite signs for left and right-wise rotations), its square in a general case depends on the angular velocity of the space rotation.

Now having components of the Planck tensor defined, we can approach deduction of a momentum that a particle gains from its spin as well as equations of motion of the spin-particle travelling in the pseudo-Riemannian space. This will be the focus of the next section, §4.2.

§4.2 A particle's spin-momentum in equations of motion

The additional momentum S^{α} that a particle gains from its spin can be obtained from considering *action* for this spin-particle.

Action S for a particle that possesses an inner scalar field k, with which an external scalar field A interacts and thus displaces the particle at an elementary interval ds, is

$$S = \alpha_{(kA)} \int_a^b kA\, ds\,, \tag{4.18}$$

where $\alpha_{(kA)}$ is a scalar constant, which characterizes properties of the particle in a given interaction and equates dimensions [10, 20]. If the inner scalar field of the particle k corresponds to an external field of the tensor of the 1st rank A_{α}, then the action required to displace the particle by the field is

$$S = \alpha_{(kA_{\alpha})} \int_a^b kA_{\alpha}\, dx^{\alpha}. \tag{4.19}$$

In interaction of the particle's inner scalar field k with an external field of the tensor of the 2nd rank $A_{\alpha\beta}$, action to displace the particle by that field is

$$S = \alpha_{(kA_{\alpha\beta})} \int_a^b kA_{\alpha\beta}\, dx^{\alpha} dx^{\beta}. \tag{4.20}$$

And so forth. For instance, if the specific vector potential of the particle k^{α} corresponds to an external vector field A_{α}, then action of this interaction to displace the particle is

$$S = \alpha_{(k^{\alpha}A_{\alpha})} \int_a^b k^{\alpha}A_{\alpha}\, ds\,. \tag{4.21}$$

Besides, the action can be represented as follows irrespective of the nature of inner properties of particles and external fields

$$S = \int_{t_1}^{t_2} L dt\,, \tag{4.22}$$

where L is the so-called *Lagrange's function.* Because the dimension of action S is [erg sec = gram cm^2/ sec], then the Lagrange function has dimension of energy [erg = gram cm^2/sec^2]. And the derivative of the Lagrange function with respect to the three-dimensional coordinate velocity $u^i = \frac{dx^i}{dt}$ of the particle

$$\frac{\partial L}{\partial u^i} = p_i \tag{4.23}$$

is the covariant notation of its three-dimensional momentum $p^i = cP^i$ which can be used to restore full notation for the four-dimensional momentum vector of the particle P^{α}.

Hence having action for the particle, having the Lagrange function outlined and differentiated with respect to the coordinate velocity of particle, we can calculate the additional momentum which the particle gains from its spin.

As it is known, action to displace a free particle in the pseudo-Riemannian space is*

$$S = \int_a^b m_0 c\, ds\,. \tag{4.24}$$

In a Galilean reference frame in the Minkowski space, because non-diagonal terms of the fundamental metric tensor are zeroes, the space-time interval is

$$ds = \sqrt{g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}} = cdt\sqrt{1 - \frac{u^2}{c^2}}\,, \tag{4.25}$$

and hence the action (4.24) becomes

$$S = \int_a^b m_0 c\, ds = \int_{t_1}^{t_2} m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}\; dt\,. \tag{4.26}$$

Therefore the Lagrange function of the free particle in a Galilean reference frame in the Minkowski space is

$$L = m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}\,. \tag{4.27}$$

---

*In *The Classical Theory of Fields* [10] Landau and Lifshitz put "minus" before the action, while we always have "plus" before the integral of the action and also before the Lagrange function. This is because the sign of action depends on the signature of the pseudo-Riemannian space. Landau and Lifshitz use the signature $(-+++)$, where time is imaginary, spatial coordinates are real and three-dimensional coordinate momentum is positive (see in the below). To the contrary, we stick to the signature $(+---)$ as used by Zelmanov [9, 11–13], because in this case time is real and spatial coordinates are imaginary, so three-dimensional *observable* momentum is positive in this case.

Differentiating it with respect to the particle's coordinate velocity we arrive at the covariant form of its three-dimensional momentum

$$p_i = \frac{\partial L}{\partial u^i} = m_0 c^2 \, \frac{\partial \sqrt{1-\frac{u^2}{c^2}}}{\partial u^i} = -\frac{m_0\, u_i}{\sqrt{1-\frac{u^2}{c^2}}}\,, \tag{4.28}$$

from which, after lifting indices, we arrive at the four-dimensional momentum vector of the free particle as follows

$$P^{\alpha} = \frac{m_0}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dx^{\alpha}}{dt} = m_0 \frac{dx^{\alpha}}{ds}\,. \tag{4.29}$$

In the final formula both multipliers, m_0 and $\frac{dx^{\alpha}}{ds}$, are general covariant quantities, so they do not depend on choice of a particular reference frame. For this reason, this formula obtained in a Galilean reference frame in the Minkowski space is also true in any other arbitrary reference frame in any pseudo-Riemannian space.

Now let us consider motion of a particle that possesses inner structure, which in experiments reveals itself like its *spin*. Inner rotation (spin) of the particle $n\hbar^{\alpha\beta}$ in the four-dimensional pseudo-Riemannian space corresponds to the external field $A_{\alpha\beta}$ of the space rotation. Therefore summary action to displace this spin-particle is

$$S = \int_a^b \bigl(m_0 c\, ds + \alpha_{(\mathrm{s})} \hbar^{\alpha\beta} A_{\alpha\beta}\, ds\bigr)\,, \tag{4.30}$$

where $\alpha_{(\mathrm{s})}$ [sec/cm] is a scalar constant, which characterizes the particle in spin-interaction. Because action constants may include only this particle's properties or fundamental physical constants, $\alpha_{(\mathrm{s})}$ is evidently the spin quantum number n, which is the function of inner properties of the particle, divided by the light velocity $\alpha_{(\mathrm{s})} = \frac{n}{c}$. Then the action to displace the particle, produced by interaction of its spin with the space non-holonomity field $A_{\alpha\beta}$ is

$$S = \alpha_{(\mathrm{s})} \int_a^b \hbar^{\alpha\beta} A_{\alpha\beta}\, ds = \frac{n}{c}\int_a^b \hbar^{\alpha\beta} A_{\alpha\beta}\, ds\,. \tag{4.31}$$

A note should be taken that building the four-dimensional momentum vector for a spin-particle using the same method as for a free particle is impossible. As it is known, we first obtained the momentum of a free particle in a Galilean reference frame in the Minkowski space, where a formula for ds expressed through dt and substituted into the action had simple form (4.25). It was shown that the obtained formula (4.29) due to its property of general covariance was true in any reference frame in the pseudo-Riemannian space. But as we can see from the formula of the action for a spin-particle, spin affects motion of the particle in the non-holonomic space $A_{\alpha\beta} \neq 0$ only, i. e. where non-diagonal terms g_{0i} of the fundamental metric tensor are not zeroes. In a Galilean reference frame, by definition, all non-diagonal terms in the metric tensor are zeroes, hence zeroes are components of the linear velocity of the space rotation $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ and also components of the non-holonomity tensor $A_{\alpha\beta}$. Therefore this is worthless and cannot be used to deduce the desired formula for the spin-particle momentum in a Galilean reference frame in the Minkowski space (where it obviously is zero), instead we should deduce it directly in the pseudo-Riemannian space.

The space-time interval ds, in an arbitrary accompanying reference frame in the pseudo-Riemannian space, is

$$ds = cd\tau\sqrt{1-\frac{\mathrm{v}^2}{c^2}} = cdt\left(1-\frac{\mathrm{w}+v_i u^i}{c^2}\right)\sqrt{1-\frac{u^2}{c^2\left(1-\frac{\mathrm{w}+v_i u^i}{c^2}\right)^2}}\,, \qquad (4.32)$$

where the coordinate velocity of the particle $u^i = \frac{dx^i}{dt}$ can be expressed with its observable velocity $\mathrm{v}^i = \frac{dx^i}{d\tau}$ as follows

$$\mathrm{v}^i = \frac{u^i}{1-\frac{\mathrm{w}+v_i u^i}{c^2}}\,, \qquad \mathrm{v}^2 = \frac{h_{ik}\,u^i u^k}{\left(1-\frac{\mathrm{w}+v_i u^i}{c^2}\right)^2}\,. \qquad (4.33)$$

Then the additional action (4.31), produced by interaction of spin with the space non-holonomity field, becomes

$$S = n\int_{t_1}^{t_2}\hbar^{\alpha\beta}A_{\alpha\beta}\sqrt{\left(1-\frac{\mathrm{w}+v_i u^i}{c^2}\right)^2-\frac{u^2}{c^2}}\,dt\,. \qquad (4.34)$$

Therefore, the Lagrange function for this action is

$$L = n\hbar^{\alpha\beta}A_{\alpha\beta}\sqrt{\left(1-\frac{\mathrm{w}+v_i u^i}{c^2}\right)^2-\frac{u^2}{c^2}}\,. \qquad (4.35)$$

Now to deduce the spin-momentum we only have to differentiate the Lagrange function (4.35) with respect to the coordinate velocity of the particle. Taking into account that $\hbar^{\alpha\beta}$, the tensor of inner rotations of the particle, and $A_{\alpha\beta}$ (4.13), the tensor of the space rotation, are not functions of the particle's velocity, after differentiating we obtain

$$p_i = \frac{\partial L}{\partial u^i} = n\hbar^{mn} A_{mn} \frac{\partial}{\partial u^i} \sqrt{\left(1 - \frac{\mathrm{w} + v_i u^i}{c^2}\right)^2 - \frac{u^2}{c^2}} = \\ = - \frac{n\hbar^{mn} A_{mn}}{c^2 \sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \left(v_i + \mathrm{v}_i\right), \tag{4.36}$$

where $\mathrm{v}_i = h_{ik} \mathrm{v}^k$. We compare (4.36) with the spatial covariant component $p_i = cP_i$ of the four-dimensional momentum vector $P^\alpha = m_0 \frac{dx^\alpha}{ds}$ of the particle in pseudo-Riemannian space*. If the particle is located in our world, so it travels from the past into the future with respect to us, its three-dimensional covariant momentum is

$$p_i = cP_i = cg_{i\alpha} P^\alpha = - m \left(v_i + \mathrm{v}_i\right) = - \frac{m_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} \left(v_i + \mathrm{v}_i\right). \tag{4.37}$$

From here we see that the four-dimensional momentum S^α, which the particle gains from its spin (the spin-momentum) is

$$S^\alpha = \frac{1}{c^2}\, n\hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}\,, \tag{4.38}$$

or, introducing notation $\eta_0 = n\hbar^{\mu\nu} A_{\mu\nu} = n\hbar^{mn} A_{mn}$ to make the formula simpler, we obtain

$$S^\alpha = \frac{1}{c^2}\, \eta_0 \frac{dx^\alpha}{ds}\,. \tag{4.39}$$

Then the summary vector Q^α (4.1), which characterizes motion of the spin-particle is

$$Q^\alpha = P^\alpha + S^\alpha = m_0 \frac{dx^\alpha}{ds} + \frac{1}{c^2}\, n\hbar^{\mu\nu} A_{\mu\nu} \frac{dx^\alpha}{ds}\,. \tag{4.40}$$

So, any spin-particle in a non-holonomic space ($A_{\mu\nu} \neq 0$) actually gains an additional momentum, which deviates its motion from geodesic line and makes it non-geodesic. In the absence of the space rotation, i. e. where the space is holonomic, we have $A_{\mu\nu} = 0$, so the particle's spin does not affect its motion. But there is hardly an area in the space where rotation is fully absent. Therefore, spin most often, affects motion of particles in the subject domain of atomic physics, where rotations are especially strong.

*In this comparism we mean a non-zero mass particle.

§4.3 Equations of motion of a spin-particle

Equations of motion of a spin-particle are equations of parallel transfer of the summary vector $Q^\alpha = P^\alpha + S^\alpha$ (4.40) along the trajectory of the particle (its parallel transfer in the four-dimensional pseudo-Riemannian space), namely

$$\frac{d}{ds}\left(P^\alpha + S^\alpha\right) + \Gamma^\alpha_{\mu\nu}\left(P^\mu + S^\mu\right)\frac{dx^\nu}{ds} = 0\,, \tag{4.41}$$

where the square of the vector remains unchanged $Q_\alpha Q^\alpha = const$ in its parallel transfer along the trajectory.

Our goal is to deduce chr.inv.-projections of the equations. The projections in general notation, as obtained in Chapter 2, should be

$$\frac{d\varphi}{ds} - \frac{1}{c}\,F_i q^i\,\frac{d\tau}{ds} + \frac{1}{c}\,D_{ik}\,q^i\,\frac{dx^k}{ds} = 0\,, \tag{4.42}$$

$$\frac{dq^i}{ds} + \left(\frac{\varphi}{c}\frac{dx^k}{ds} + q^k\frac{d\tau}{ds}\right)\left(D^i_k + A^{\cdot i}_{k\cdot}\right) - \frac{\varphi}{c}\,F^i\frac{d\tau}{ds} + \Delta^i_{mk}\,q^m\frac{dx^k}{ds} = 0\,, \tag{4.43}$$

where φ is the projection of the summary vector Q_α on the observer's time line and q^i is its projection on the spatial section

$$\varphi = b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}} = \frac{P_0}{\sqrt{g_{00}}} + \frac{S_0}{\sqrt{g_{00}}}\,, \tag{4.44}$$

$$q^i = h^i_\alpha Q^\alpha = Q^i = P^i + S^i. \tag{4.45}$$

Therefore attaining the goal requires deducing φ and q^i, substituting them into (4.42, 4.43) and cancelling similar terms. Chr.inv.-projections of the momentum vector $P^\alpha = m_0\frac{dx^\alpha}{ds}$ are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \qquad P^i = \frac{1}{c}\,m\mathrm{v}^i, \tag{4.46}$$

and now we have to deduce chr.inv.-projections of the spin-momentum vector S^α. Taking into account in the formula for S^α (4.39) that the space-time interval, formulated with physical observable quantities, is $ds = cd\tau\sqrt{1-\mathrm{v}^2/c^2}$, we obtain components of the S^α, which are

$$S^0 = \frac{1}{c^2}\,\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,\frac{\left(v_i\mathrm{v}^i \pm c^2\right)}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)}\,, \tag{4.47}$$

$$S^i = \frac{1}{c^3}\,\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,\mathrm{v}^i, \tag{4.48}$$

$$S_0 = \pm \frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{4.49}$$

$$S_i = -\frac{1}{c^3}\frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\left(v_i \pm \mathrm{v}_i\right), \tag{4.50}$$

also formulated with physical observable quantities. Respectively, chr. inv.-projections of the particle's spin-momentum vector are

$$\frac{S_0}{\sqrt{g_{00}}} = \pm\frac{1}{c^2}\,\eta\,, \qquad S^i = \frac{1}{c^3}\,\eta\,\mathrm{v}^i, \tag{4.51}$$

where the quantity η is

$$\eta = \frac{n\hbar^{mn}A_{mn}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \tag{4.52}$$

while alternating signs, which results from substituting the time function $\frac{dt}{d\tau}$ (1.55) indicate motion of the particle into the future (the upper sign) or into the past (the lower sign). Then the square of the spin-momentum vector is

$$S_\alpha S^\alpha = g_{\alpha\beta}\,S^\alpha S^\beta = \frac{1}{c^4}\,\eta_0^2\,g_{\alpha\beta}\,\frac{dx^\alpha dx^\beta}{ds^2} = \frac{1}{c^4}\,\eta_0^2\,, \tag{4.53}$$

and the square of the summary vector Q^α is

$$Q_\alpha Q^\alpha = g_{\alpha\beta}\,Q^\alpha Q^\beta = m_0^2 + \frac{2}{c^2}\,m_0\eta_0 + \frac{1}{c^4}\,\eta_0^2\,. \tag{4.54}$$

Therefore the square of the summary vector of any spin-particle falls apart into three parts, namely:

a) The square of the momentum vector of the particle $P_\alpha P^\alpha = m_0^2$;

b) The square of its spin-momentum vector $S_\alpha S^\alpha = \frac{1}{c^4}\,\eta_0^2$;

c) The term $\frac{2}{c^2}\,m_0\eta_0$, describing spin-gravitational interactions.

To effect parallel transfer (4.41), it is necessary that the square of the transferred summary vector remains unchanged along the entire path. But the obtained formula (4.54) implies that (because $m_0 = const$) the square of the spin-particle's summary vector Q^α remains unchanged if only $\eta_0 = const$, i.e.

$$d\eta_0 = \frac{\partial\eta_0}{\partial x^\alpha}\,dx^\alpha = 0\,. \tag{4.55}$$

Dividing both sides of the equation by $d\tau$, which is always possible because an elementary interval of the observer's physical time is greater than zero*, we obtain the chr.inv.-condition of conservation of the square of the spin-particle's summary vector

$$\frac{d\eta_0}{d\tau} = \frac{^*\partial\eta_0}{\partial t} + \mathrm{v}^k \frac{^*\partial\eta_0}{\partial x^k} = 0\,. \tag{4.56}$$

Substituting $\eta_0 = n\hbar^{mn}A_{mn}$ we have

$$n\hbar^{mn}\left(\frac{^*\partial A_{mn}}{\partial t} + \mathrm{v}^k \frac{^*\partial A_{mn}}{\partial x^k}\right) = 0\,. \tag{4.57}$$

To illustrate the results, we formulate the space non-holonomity tensor A_{ik}, which is actually the tensor of angular velocities of the space rotation, with the pseudovector of angular velocities of this rotation

$$\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}\,, \tag{4.58}$$

which is also chr.inv.-quantity. Multiplying Ω^{*i} by ε_{ipq}

$$\Omega^{*i}\,\varepsilon_{ipq} = \frac{1}{2}\,\varepsilon^{imn}\varepsilon_{ipq}A_{mn} = \frac{1}{2}\left(\delta^m_p\delta^n_q - \delta^n_p\delta^m_q\right)A_{mn} = A_{pq}\,, \tag{4.59}$$

we obtain (4.57) as follows

$$\begin{gathered} n\hbar^{mn}\left[\frac{^*\partial}{\partial t}\big(\varepsilon_{imn}\Omega^{*i}\big) + \mathrm{v}^k\frac{^*\partial}{\partial x^k}\big(\varepsilon_{imn}\Omega^{*i}\big)\right] = \\ = n\hbar^{mn}\varepsilon_{imn}\left[\frac{1}{\sqrt{h}}\frac{^*\partial}{\partial t}\big(\sqrt{h}\,\Omega^{*i}\big) + \mathrm{v}^k\frac{1}{\sqrt{h}}\frac{^*\partial}{\partial x^k}\big(\sqrt{h}\,\Omega^{*i}\big)\right] = 0\,. \end{gathered} \tag{4.60}$$

The vector of gravitational inertial force and the space non-holonomity tensor are related through Zelmanov's identities, one of which (see formula 13.20 in [9]) is

$$\frac{2}{\sqrt{h}}\frac{^*\partial}{\partial t}\big(\sqrt{h}\,\Omega^{*i}\big) + \varepsilon^{ijk}\,{}^*\nabla_j\,F_k = 0\,, \tag{4.61}$$

or, in the other notation

$$\frac{^*\partial A_{ik}}{\partial t} + \frac{1}{2}\big({}^*\nabla_k\,F_i - {}^*\nabla_i\,F_k\big) = \frac{^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{^*\partial F_k}{\partial x^i} - \frac{^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{4.62}$$

*The condition $d\tau = 0$ only has sense in a generalized space-time, where degeneration of the fundamental metric tensor $g_{\alpha\beta}$ is possible. In this case the above condition defines fully degenerate domain (zero-space) that hosts zero-particles, which are capable of instant displacement, so they are carriers of long-range action.

where $\varepsilon^{ijk}\,{}^*\nabla_j F_k$ is the chr.inv.-curl of the gravitational inertial force field F_k. From here we see that non-stationarity of the space rotation A_{ik} is due to *curl character* of the acting field of gravitational inertial force F_{ik}. Hence taking into account equation (4.61), our formula (4.60) becomes

$$-\,n\hbar^{mn}\,{}^*\nabla_m F_n + n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\,\frac{1}{\sqrt{h}}\,\frac{{}^*\partial}{\partial x^k}\big(\sqrt{h}\,\Omega^{*i}\big) = 0\,, \tag{4.63}$$

or in the other notation

$$n\hbar^{mn}\,{}^*\nabla_m F_n = n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\left(\Omega^{*i}\,\frac{{}^*\partial\ln\sqrt{h}}{\partial x^k} + \frac{{}^*\partial\Omega^{*i}}{\partial x^k}\right). \tag{4.64}$$

Now lets recall that this formula is nothing but the expanded chr. inv.-notation of the conservation condition of the summary vector (4.57). The left hand side of (4.64) equals

$$\pm\,2n\hbar\left({}^*\nabla_1 F_2 - {}^*\nabla_2 F_1 + {}^*\nabla_1 F_3 - {}^*\nabla_3 F_1 + {}^*\nabla_2 F_3 - {}^*\nabla_3 F_2\right), \tag{4.65}$$

where "plus" and "minus" stand for right-rotating and left-rotating reference frames, respectively. Therefore, the left hand side of equation (4.64) is the chr.inv.-curl of gravitational inertial force. The right hand side of (4.64) depends on the spatial orientation of the space rotation pseudovector Ω^{*i}.

Hence to conserve the square of the spin-particle's transferred vector, it is necessary that the right and the left hand sides of (4.64) are equal to each other along the trajectory. In a general case, with no additional assumptions on the geometric structure of the space, this requires that there should be a balance between the vortical field of the acting gravitational inertial force and the spatial distribution of the space rotation pseudovector.

If the field of gravitational inertial force is vortexless, then the left hand side of the conservation condition (4.64) is zero and this condition becomes

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\,\frac{1}{\sqrt{h}}\,\frac{{}^*\partial}{\partial x^k}\big(\sqrt{h}\,\Omega^{*i}\big) = 0\,. \tag{4.66}$$

Introducing chr.inv.-derivative $\frac{{}^*\partial}{\partial x^k} = \frac{\partial}{\partial x^k} + \frac{1}{c^2}v_k\frac{{}^*\partial}{\partial t}$, we have

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\,\frac{1}{\sqrt{h}}\left[\frac{\partial}{\partial x^k}\big(\sqrt{h}\,\Omega^{*i}\big) - \frac{1}{c^2}\,v_k\,\frac{{}^*\partial}{\partial t}\big(\sqrt{h}\,\Omega^{*i}\big)\right] = 0\,. \tag{4.67}$$

Since the force field F_i is vortexless, then because of (4.66) the second term in this formula is zero. Therefore the square of the summary vector of the spin-particle remains unchanged in the vortexless force field F_i, provided that chr.inv.-formula (4.66) and the formula with regular derivatives are zeroes

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k\,\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\,\Omega^{*i}\right)=0\,. \tag{4.68}$$

For non-zero mass particles this is the case, for instance, where $\mathrm{v}^k=0$, so this is when they are at rest with respect to the observer and his reference body. In this case equality to zero of derivatives in (4.68) is not essential. But massless particles travel at the light velocity, hence for them in the vortexless field of force F_i the derivatives $\frac{\partial}{\partial x^k}(\sqrt{h}\,\Omega^{*i})$ must be zeroes.

Let us obtain chr.inv.-equations of motion of a spin-particle in the pseudo-Riemannian space. Substituting (4.46) and (4.51) into (4.44) and (4.45) we arrive at chr.inv.-projections of the summary vector of the spin-particle

$$\varphi=\pm\Big(m+\frac{1}{c^2}\,\eta\Big),\qquad q^i=\frac{1}{c}\,m\,\mathrm{v}^i+\frac{1}{c^3}\,\eta\,\mathrm{v}^i. \tag{4.69}$$

Having the quantities substituted for $\varphi>0$ into (4.42, 4.43) we obtain chr.inv.-equations of motion for a non-zero mass spin-particle located in our world (the particle travels from the past into the future)

$$\frac{dm}{d\tau}-\frac{m}{c^2}\,F_i\,\mathrm{v}^i+\frac{m}{c^2}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k=-\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}\,F_i\,\mathrm{v}^i-\frac{\eta}{c^4}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k, \tag{4.70}$$

$$\begin{gathered}\frac{d}{d\tau}\left(m\mathrm{v}^i\right)+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k-mF^i+m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\=-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\,\mathrm{v}^i\right)-\frac{2\eta}{c^2}\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k+\frac{\eta}{c^2}\,F^i-\frac{\eta}{c^2}\,\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k,\end{gathered} \tag{4.71}$$

while for the mirror-world particle (which moves into the past), having the quantities (4.69) substituted for $\varphi<0$, we have

$$-\frac{dm}{d\tau}-\frac{m}{c^2}\,F_i\mathrm{v}^i+\frac{m}{c^2}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k=\frac{1}{c^2}\frac{d\eta}{d\tau}+\frac{\eta}{c^4}\,F_i\,\mathrm{v}^i-\frac{\eta}{c^4}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k, \tag{4.72}$$

$$\begin{gathered}\frac{d}{d\tau}\left(m\mathrm{v}^i\right)+mF^i+m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\=-\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\,\mathrm{v}^i\right)-\frac{\eta}{c^2}\,F^i-\frac{\eta}{c^2}\,\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k,\end{gathered} \tag{4.73}$$

We write down the obtained equations in a way that their left hand sides have the *geodesic part*, which describes free (geodesic) motion of this particle, while the right hand sides have the terms produced by the particle's spin, which makes the motion non-geodesic (the *non-geodesic part*). Hence for a spin-free particle the right hand sides become zeroes and we obtain chr.inv.-equations of free motion. This form of the equations will facilitate their analysis.

Within the wave-particle concept, a massless particle is described by the four-dimensional wave vector $K^\alpha = \frac{\omega}{c}\frac{dx^\alpha}{d\sigma}$, where the quantity $d\sigma^2 = h_{ik}\,dx^i dx^k$ is the square of the spatial physical observable interval, not equal to zero along isotropic trajectories. Because massless particles travel along isotropic trajectories (the light propagation trajectories), the vector K^α is also isotropic: its square is zero. But because the vector's dimension K^α is $[\,\mathrm{cm}^{-1}]$, the equations have dimension different from that of equations of motion of non-zero mass particles. Besides, this fact does not permit us to build a uniform formula of action for both massless and non-zero mass particles [9].

On the other hand, spin is a physical property, possessed by both non-zero mass and massless particles. Therefore, deduction of equations of motion for spin-particles require using a uniform vector for both kinds of particles. Such a vector can be obtained by applying physical conditions which are true along isotropic trajectories,

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0\,, \qquad c\,d\tau = d\sigma \neq 0\,, \tag{4.74}$$

to the four-dimensional momentum vector of a mass-bearing particle

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{m}{c}\frac{dx^\alpha}{d\tau} = m\,\frac{dx^\alpha}{d\sigma}\,. \tag{4.75}$$

As a result the observable spatial interval, not equal to zero along isotropic trajectories, becomes the derivation parameter, while the dimension of the formula, in contrast to the four-dimensional wave vector $K^\alpha\,[\,\mathrm{cm}^{-1}]$, coincides that of the four-dimensional momentum vector $P^\alpha\,[\,\mathrm{gram}]$. Relativistic mass m, not equal to zero for massless particles, can be obtained from the energy equivalent using $E = mc^2$ formula. For instance, a photon energy of $E = 1\,\mathrm{MeV} = 1.6\times10^{-6}$ erg corresponds to its relativistic mass of $m = 1.8\times10^{-28}$ gram.

Therefore the four-dimensional momentum vector (4.75), depending on its form, may describe motion of either non-zero mass particles (non-isotropic trajectories) or massless particles (isotropic trajectories). As a matter of fact, for massless particles $m_0 = 0$ and $ds = 0$, therefore their ratio in (4.75) is a $\frac{0}{0}$ indeterminacy. However the transition (4.75) $\frac{m_0}{ds}$ to $\frac{m}{d\sigma}$ solves the indeterminacy, because the relativistic mass of any massless particle is $m \neq 0$, so along their trajectory we have $d\sigma \neq 0$.

It is evident that, in the form applicable to massless particles (i. e. along isotropic trajectories) the square of P^α (4.75) is zero

$$P_\alpha P^\alpha = g_{\alpha\beta}\, P^\alpha P^\beta = m^2 g_{\alpha\beta}\,\frac{dx^\alpha}{d\sigma}\,\frac{dx^\beta}{d\sigma} = m^2\,\frac{ds^2}{d\sigma^2} = 0\,. \tag{4.76}$$

Chr.inv.-projections of the four-dimensional momentum vector of a massless particle $P^\alpha = m\,\frac{dx^\alpha}{d\sigma}$ are

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \qquad P^i = \frac{1}{c}\,mc^i, \tag{4.77}$$

where c^i is the chr.inv.-vector of the light velocity. In this case, the spin-momentum vector of this particle (4.39) is as well isotropic

$$S^\alpha = \frac{1}{c^2}\,\eta_0\,\frac{dx^\alpha}{ds} = \frac{1}{c^2}\,\eta\,\frac{dx^\alpha}{cd\tau} = \frac{1}{c^2}\,\eta\,\frac{dx^\alpha}{d\sigma}\,, \tag{4.78}$$

because its square is zero

$$S_\alpha S^\alpha = g_{\alpha\beta}\, S^\alpha S^\beta = \frac{1}{c^4}\,\eta^2 g_{\alpha\beta}\,\frac{dx^\alpha dx^\beta}{d\sigma^2} = \frac{1}{c^4}\,\eta^2\,\frac{ds^2}{d\sigma^2} = 0\,, \tag{4.79}$$

so the square of the particle's summary vector $Q^\alpha = P^\alpha + S^\alpha$ is also zero. Chr.inv.-projections of the isotropic spin-momentum are

$$\frac{S_0}{\sqrt{g_{00}}} = \pm\,\frac{1}{c^2}\,\eta\,, \qquad S^i = \frac{1}{c^3}\,\eta c^i, \tag{4.80}$$

so its spatial observable projection coincides that for a mass-bearing particle (4.51), which instead of the particle's observable velocity v^i (4.51) has the light velocity chr.inv.-vector c^i. Subsequently, cr.inv.-projections of the summary vector of the massless spin-particle are

$$\varphi = \pm\Big(m + \frac{1}{c^2}\,\eta\Big), \qquad q^i = \frac{1}{c}\,mc^i + \frac{1}{c^3}\,\eta c^i. \tag{4.81}$$

Having these quantities substituted for the positive φ into the initial formulae (4.42, 4.43), we arrive at chr.inv.-equations of motion of the massless spin-particle located in our world (the particle travels from the past into the future), namely

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i c^i + \frac{m}{c^2}\,D_{ik} c^i c^k = -\frac{1}{c^2}\frac{d\eta}{d\tau} + \frac{\eta}{c^4}\,F_i c^i - \frac{\eta}{c^4}\,D_{ik} c^i c^k, \tag{4.82}$$

$$\begin{gathered}\frac{d}{d\tau}\left(mc^i\right) + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k - mF^i + m\Delta^i_{nk} c^n c^k = \\ = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta c^i\right) - \frac{2\eta}{c^2}\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k + \frac{\eta}{c^2}\,F^i - \frac{\eta}{c^2}\,\Delta^i_{nk} c^n c^k,\end{gathered} \tag{4.83}$$

while for the analogous particle in the mirror world (the particle travels from the future into the past), having the quantities (4.81) substituted for $\varphi < 0$, the chr.inv.-equations of motion are

$$-\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i c^i + \frac{m}{c^2}\,D_{ik}c^i c^k = \frac{1}{c^2}\frac{d\eta}{d\tau} + \frac{\eta}{c^4}\,F_i c^i - \frac{\eta}{c^4}\,D_{ik}c^i c^k, \quad (4.84)$$

$$\begin{aligned}\frac{d}{d\tau}\left(mc^i\right) + mF^i + m\Delta^i_{nk}c^n c^k =\\ = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta c^i\right) - \frac{\eta}{c^2}\,F^i - \frac{\eta}{c^2}\,\Delta^i_{nk}c^n c^k.\end{aligned} \quad (4.85)$$

§4.4 The physical conditions of spin-interaction

As we have shown, spin of a particle (its inner rotational momentum) interacts with an external field of the space rotation, described by the space non-holonomity tensor $A^{\alpha\beta} = \frac{1}{2}\,ch^{\alpha\mu}h^{\beta\nu}\left(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu}\right)$, depending on the curl of the four-dimensional velocity vector b^α of the observer with respect to his reference body. In electromagnetic phenomena a particle's charge interacts with an external electromagnetic field — the field of Maxwell's tensor $F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$. Therefore, it seems natural to compare chr.inv.-projections of the Maxwell tensor $F_{\alpha\beta}$ to chr.inv.-projections of the space non-holonomity tensor $A_{\alpha\beta}$.

In Chapter 3, we showed that the electromagnetic field tensor $F_{\alpha\beta}$ (Maxwell's tensor), yields two groups of chr.inv.-projections, produced by the tensor itself and by its dual pseudotensor* $F^{*\alpha\beta} = \frac{1}{2}\,E^{\alpha\beta\mu\nu}F_{\mu\nu}$:

$$\left.\begin{aligned}\frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = E^i, \qquad & F^{ik} = H^{ik}\\ \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = H^{*i}, \qquad & F^{*ik} = E^{*ik}\end{aligned}\right\}. \quad (4.86)$$

On the other hand, chr.inv.-projections of the non-holonomity tensor $A_{\alpha\beta}$ (4.11) and of its dual pseudotensor $A^{*\alpha\beta} = \frac{1}{2}\,E^{\alpha\beta\mu\nu}A_{\mu\nu}$ are

$$\left.\begin{aligned}\frac{A_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = 0\,, \qquad & A^{ik} = h^{im}h^{kn}A_{mn}\\ \frac{A_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = 0\,, \qquad & A^{*ik} = 0\end{aligned}\right\}. \quad (4.87)$$

*Here $E^{\alpha\beta\mu\nu}$ is the four-dimensional completely antisymmetric discriminant tensor, which produce pseudotensors in the four-dimensional pseudo-Riemannian space, see §2.3 in Chapter 2 for details.

Comparing the formulae we see that spin-interaction gives an analogy for only the "magnetic" component $\mathcal{H}^{ik} = A^{ik} = h^{im}h^{kn}A_{mn}$ of the space non-holonomity field. The "electric" component of the field in spin-interaction turns to be zero $\mathcal{E}^i = \frac{A_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = 0$. This is no surprise, because the inner rotational field of a particle (its spin, in other words) interacts with the space non-holonomity field in the same way like an external field, and both fields are produced by motion.

Besides, for the "magnetic" component of the non-holonomity field $\mathcal{H}^{ik} = A^{ik} \neq 0$ can not be dual to zero value $\mathcal{H}^{*i} = \frac{A_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = 0$. So, similarity with electromagnetic fields turns out to be incomplete. But full matching could not even be expected, because the space non-holonomity tensor and the electromagnetic field tensor have somewhat different structures: the Maxwell tensor is a "pure" curl $F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$, while the non-holonomity tensor $A^{\alpha\beta} = \frac{1}{2}\, c\, h^{\alpha\mu}h^{\beta\nu}\big(\frac{\partial b_\nu}{\partial x^\mu} - \frac{\partial b_\mu}{\partial x^\nu}\big)$ is an "add-on" curl. On the other hand, we have no doubts that in the future comparative analysis of these fields will produce a theory of spin-interactions, similar to electrodynamics.

Incomplete similarity of the space non-holonomity field and electromagnetic fields also leads to another result. If we define the force in spin-interaction in the same way that we define the Lorentz force $\Phi^\alpha = \frac{e}{c} F_{\cdot\sigma}^{\alpha\cdot} U^\sigma$, the obtained formula $\Phi^\alpha = \frac{\eta_0}{c^2} A_{\cdot\sigma}^{\alpha\cdot} U^\sigma$ will *not* include *all* the terms from the right hand sides of equations of motion of a spin-particle. But an external force acting on the particle, by definition, must include all those factors which deviate the particle from geodesic line, i. e. all terms in the right hand sides of the equations of motion. In other words, the four-dimensional force of spin-interaction Φ^α [gram/sec] is defined by the formula

$$\Phi^\alpha = \frac{DS^\alpha}{ds} = \frac{dS^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} S^\mu \frac{dx^\nu}{ds}\,, \tag{4.88}$$

whose chr.inv.-projection on the spatial section, after being divided by c, gives the three-dimensional observable force of the interaction Φ^i [gram cm/sec^2]. For instance, for a mass-bearing particle located in our world, having (4.71) as a base, we have

$$\Phi^i = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta \mathrm{v}^i\right) - \frac{2\eta}{c^2}\left(D^i_k + A_{k\cdot}^{\cdot i}\right)\mathrm{v}^k + \frac{\eta}{c^2}F^i - \frac{\eta}{c^2}\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k. \tag{4.89}$$

From further comparison between electromagnetic interaction and spin-interaction, using similarity with the electromagnetic field invariants (3.25, 3.26) we deduce the invariants of the space non-holonomity field as

$$J_1 = A_{\alpha\beta}A^{\alpha\beta} = A_{ik}A^{ik} = \varepsilon_{ikm}\,\varepsilon^{ikn}\Omega^{*m}\Omega_{*n} = 2\Omega_{*i}\Omega^{*i}, \tag{4.90}$$

$$J_2 = A_{\alpha\beta}A^{*\alpha\beta} = 0\,. \tag{4.91}$$

Hence the scalar invariant $J_1 = 2\Omega_{*i}\Omega^{*i}$ is always non-zero, otherwise the space would be holonomic (not rotating) and spin-interaction would be absent.

Now we are approaching physical conditions of motion of elementary spin-particles. Using the definition of the chr.inv.-vector of gravitational inertial force

$$F_i = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = -\,c^2\,\frac{\partial \ln\!\left(1-\frac{\mathrm{w}}{c^2}\right)}{\partial x^i} - \frac{{}^*\partial v_i}{\partial t} \tag{4.92}$$

we formulate the non-holonomity tensor A_{ik} as

$$A_{ik} = \frac{1}{2}\left(\frac{{}^*\partial v_k}{\partial x^i} - \frac{{}^*\partial v_i}{\partial x^k}\right) + v_i\,\frac{\partial \ln\sqrt{1-\frac{\mathrm{w}}{c^2}}}{\partial x^k} - v_k\,\frac{\partial \ln\sqrt{1-\frac{\mathrm{w}}{c^2}}}{\partial x^i}\,. \tag{4.93}$$

From here we see that the non-holonomity tensor A_{ik} is the three-dimensional observable curl of the linear velocity of the space rotation with two additional terms, produced by both the gravitational potential w and the space rotation.

On the other hand, because of the small numerical value of the Planck constant, spin-interaction only affects elementary particles. And as it is known, in the scales of such small masses and distances gravitational interaction is a few orders of magnitude weaker than electromagnetic interactions, weak (spin) interactions, or strong interactions. We therefore can assume that $\mathrm{w} \to 0$ for spin-interaction in the formula for A_{ik} (4.93). Then in the microscopic scales of elementary particles the tensor A_{ik} is the physical observable curl in "strict" notation

$$A_{ik} = \frac{1}{2}\left(\frac{{}^*\partial v_k}{\partial x^i} - \frac{{}^*\partial v_i}{\partial x^k}\right), \tag{4.94}$$

while the acting gravitational inertial force (4.92) has only its inertial part

$$F_i = -\frac{{}^*\partial v_i}{\partial t} = -\frac{1}{1-\frac{\mathrm{w}}{c^2}}\,\frac{\partial v_i}{\partial t} = -\frac{\partial v_i}{\partial t}\,. \tag{4.95}$$

Zelmanov's identities

$$\frac{2}{\sqrt{h}}\,\frac{{}^*\partial}{\partial t}\left(\sqrt{h}\,\Omega^{*i}\right) + \varepsilon^{ijk}\,{}^*\nabla_j\,F_k = 0\,, \quad {}^*\nabla_k\,\Omega^{*k} + \frac{1}{c^2}\,F_k\Omega^{*k} = 0\,, \tag{4.96}$$

which link the space rotation to the gravitational inertial force, acting on it, for elementary particles ($\mathrm{w} \to 0$) become

$$\left.\begin{aligned} & \frac{1}{\sqrt{h}}\frac{\partial}{\partial t}\left(\sqrt{h}\,\Omega^{*i}\right) + \frac{1}{2}\,\varepsilon^{ijk}\left(\frac{^*\partial^2 v_k}{\partial x^j \partial t} - \frac{^*\partial^2 v_j}{\partial x^k \partial t}\right) = 0 \\ & {}^*\nabla_k\,\Omega^{*k} - \frac{1}{c^2}\frac{^*\partial v_k}{\partial t}\,\Omega^{*k} = 0 \end{aligned}\right\}. \tag{4.97}$$

If we substitute $\frac{^*\partial v_k}{\partial t} = 0$ here, so we are assuming that the observable rotation of the space is stationary, we obtain ${}^*\nabla_k\,\Omega^{*k} = 0$, so the space rotation pseudovector remains unchanged. Then the Zelmanov 1st identity becomes

$$\Omega^{*i} D + \frac{^*\partial\Omega^{*i}}{\partial t} = 0\,, \tag{4.98}$$

from which we see that $D = \det\|D^n_n\| = \frac{^*\partial \ln\sqrt{h}}{\partial t}$, so the rate of relative expansion of the space elementary volume is zero $D = 0$.

Therefore, the equations suggest that for elementary particles (i. e. with $\mathrm{w} \to 0$) at stationary rotations of the space $\frac{^*\partial v_k}{\partial t} = 0$ the tensor of angular velocities of this rotation remains unchanged ${}^*\nabla_k\,\Omega^{*k} = 0$ and the space relative expansions (deformations) are absent $D = 0$.

It is possible, that stationarity of the space non-holonomity field (the external field in spin-interaction) is the necessary condition of stability of the elementary particle under this action. Out of this we may conclude that long-living spin-particles should possess stable inner rotations, while short-living particles must be unstable spatial vortexes.

To study motion of short-living particles is pretty problematic as we do not have experimental data on the structure of unstable vortexes, which may produce them. In the same time, the study for long-living particles, i. e. in the stationary field of the space rotation, can give exact solutions of their equations of motion. We will focus on these issues in the next section, §4.5.

§4.5 Motion of elementary spin-particles

As we have mentioned, the Planck constant, being a small absolute value, only "works" for elementary particles, where gravitational interactions is a few orders of magnitude weaker than electromagnetic, weak and strong ones. Hence assuming $\mathrm{w} \to 0$ in chr.inv.-equations of motion of spin-particles (4.70–4.73) and (4.82–4.85), we will arrive at chr.inv.-equations of motion of elementary particles.

Besides, as we have obtained in the previous section, §4.4, under stationary rotations of the space $\frac{{}^*\partial v_k}{\partial t} = 0$, in the scale of elementary particles the spur of the space deformations tensor is zero $D = 0$. Of course, zero spur of a tensor does not necessarily imply the tensor itself is zero. On the other hand, the space deformation is a rare phenomenon, so for our study of motion of elementary particles we will assume $D_{ik} = 0$.

In §4.3 we have showed that under stationary rotations of the space, the conservation condition for the spin-momentum vector of an arbitrary spin-particle S^α becomes (4.68) so that

$$n\hbar^{mn}\varepsilon_{imn}\mathrm{v}^k \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\,\Omega^{*i}\right) = 0\,. \tag{4.99}$$

On the other hand, under $\frac{{}^*\partial v_k}{\partial t} = 0$ the Zelmanov identities we applied for elementary particles (4.97) imply that

$${}^*\nabla_k\,\Omega^{*k} = \frac{\partial\Omega^{*k}}{\partial x^k} + \frac{\partial\sqrt{h}}{\partial x^k}\,\Omega^{*k} = \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^k}\left(\sqrt{h}\,\Omega^{*k}\right) = 0\,. \tag{4.100}$$

The first condition is true provided that $\frac{\partial}{\partial x^k}(\sqrt{h}\,\Omega^{*k}) = 0$. This is true if the space rotation pseudovector is

$$\Omega^{*i} = \frac{\Omega^{*i}_{(0)}}{\sqrt{h}}\,, \qquad \Omega^{*i}_{(0)} = const\,, \tag{4.101}$$

in this case the second condition (4.100) is also true.

Taking what has been said above into account, from (4.70, 4.71) after some algebra we obtain chr.inv.-equations of motion of a non-zero mass elementary particle. For the our-world particle (it travels into the future with respect to a regular observer), the equations are

$$\frac{dm}{d\tau} = -\frac{1}{c^2}\frac{d\eta}{d\tau}\,, \tag{4.102}$$

$$\begin{gathered}\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + 2mA_{k\cdot}^{\cdot i}\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right) - \frac{2\eta}{c^2}\,A_{k\cdot}^{\cdot i}\mathrm{v}^k - \frac{\eta}{c^2}\,\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k,\end{gathered} \tag{4.103}$$

while for the particle, which is located in the mirror world so it travels into past, from (4.72, 4.73) we obtain

$$-\frac{dm}{d\tau} = \frac{1}{c^2}\frac{d\eta}{d\tau}\,, \tag{4.104}$$

$$\frac{d}{d\tau}\left(m\mathrm{v}^i\right) + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\mathrm{v}^i\right) - \frac{\eta}{c^2}\,\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k. \tag{4.105}$$

As it is easy to see, the scalar chr.inv.-equations of motion are the same for both our-world particles and mirror-world particles. Integrating the scalar equation for an our-world particle (the direct flow of time), namely — taking the integral

$$\int_{\tau_1=0}^{\tau_2} \frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)d\tau = 0\,, \tag{4.106}$$

we obtain

$$m+\frac{\eta}{c^2} = const = B\,, \tag{4.107}$$

where B is the constant of integration which can be defined from the initial conditions.

To illustrate the physical sense of the obtained live forces integral, we use the analogy between chr.inv.-projections

$$\left.\begin{aligned} \frac{P_0}{\sqrt{g_{00}}} &= \pm\, m\,, \qquad & P^i &= \frac{1}{c}\, m\mathrm{v}^i = \frac{1}{c}\,p^i \\ \frac{S_0}{\sqrt{g_{00}}} &= \pm\,\frac{1}{c^2}\,\eta\,, \qquad & S^i &= \frac{1}{c^3}\,\eta\mathrm{v}^i \end{aligned}\right\} \tag{4.108}$$

of the particle's four-dimensional momentum vector and those of its spin-momentum vector, i. e. of $P^\alpha = m_0\frac{dx^\alpha}{ds}$ and $S^\alpha = \frac{\eta_0}{c^2}\frac{dx^\alpha}{ds}$. Using analogy with relativistic mass $\pm m$ we will refer to the quantity $\pm\frac{1}{c^2}\,\eta$ as relativistic *spin-mass*, so the quantity $\frac{1}{c^2}\,\eta_0$ is rest spin-mass. Further, live forces theorem for the elementary spin-particle (4.107) implies that with the assumptions we made, the sum of the particle's relativistic mass and of its spin-mass remains unchanged along the trajectory.

Now using the live forces integral*, we approach the vector chr.inv.-equations of motion of the mass-bearing elementary particle, located in our world, namely — the equations (4.103). Substituting the live force integral (4.107) into (4.103), having the constant cancelled, we obtain pure kinematic equations of motion

$$\frac{d\mathrm{v}^i}{d\tau} + 2A_{k\cdot}^{\cdot i}\mathrm{v}^k + \Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k = 0\,, \tag{4.109}$$

which in this case, are non-geodesic. The term $\Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k$, which is contraction of the Christoffel chr.inv.-symbols with the particle's observable velocity, is relativistic in the sense that it is a square function of the velocity. Therefore it can be neglected, provided that the observable

*The solution of the scalar chr.inv.-equation of motion.

metric $h_{ik} = -\,g_{ik} + \frac{1}{c^2} v_i v_k$ along the trajectory is close to that in the Euclidean space. Such a case is possible, if the linear velocity of the space rotation is much lower than the light velocity, while the three-dimensional coordinate metric g_{ik} is Euclidean as well. Then diagonal components of the metric chr.inv.-tensor are

$$h_{11} = h_{22} = h_{33} = +1\,, \tag{4.110}$$

while the others are $h_{ik} = 0$ if $i \neq k$. Noteworthy, the four-dimensional metric can not be Galilean here, because the spatial section rotates with respect to time. In other words, though the observable three-dimensional space (the spatial section) in this case is a flat Euclidean space, the four-dimensional space-time is not the Minkowski space but it is a pseudo-Riemannian space whose metric is

$$\begin{aligned} ds^2 &= g_{00}\, dx^0 dx^0 + 2 g_{0i}\, dx^0 dx^i + g_{ik}\, dx^i dx^k = \\ &= c^2 dt^2 + 2 g_{0i}\, c\, dt\, dx^i - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 . \end{aligned} \tag{4.111}$$

We assume that the space rotates at a constant angular velocity $\Omega = const$ around the x^3 axis, for instance. Then the linear velocity of the space rotation $v_i = \Omega_{ik} x^k$ becomes

$$v_1 = \Omega_{12} x^2 = \Omega\, y\,, \qquad v_2 = \Omega_{21} x^1 = -\Omega\, x\,, \tag{4.112}$$

where $A_{ik} = \Omega_{ik}$. Then the space non-holonomity tensor A_{ik} has only two non-zero components

$$A_{12} = -A_{21} = -\,\Omega\,. \tag{4.113}$$

Thus the vector chr.inv.-equations of motion (4.109) become

$$\frac{d\mathrm{v}^1}{d\tau} + 2\Omega \mathrm{v}^2 = 0\,, \qquad \frac{d\mathrm{v}^2}{d\tau} - 2\Omega \mathrm{v}^1 = 0\,, \qquad \frac{d\mathrm{v}^3}{d\tau} = 0\,, \tag{4.114}$$

where the third equation can be solved immediately and solution is

$$\mathrm{v}^3 = \mathrm{v}^3_{(0)} = const. \tag{4.115}$$

Taking into account that $\mathrm{v}^3 = \frac{dx^3}{d\tau}$, we represent x^3 as follows

$$x^3 = \mathrm{v}^3_{(0)} \tau + x^3_{(0)}\,, \tag{4.116}$$

where $x^3_{(0)}$ is the numerical value of the x^3 coordinate at the initial moment $\tau = 0$. We formulate v^2 from the first equation (4.114)

$$\mathrm{v}^2 = -\frac{1}{2\Omega}\frac{d\mathrm{v}^1}{d\tau}\,, \tag{4.117}$$

differentiating (4.117) with respect to $d\tau$ yields

$$\frac{d\mathrm{v}^2}{d\tau} = -\frac{1}{2\Omega}\frac{d^2\mathrm{v}^1}{d\tau^2}\,, \tag{4.118}$$

and substituting (4.118) into the second equation (4.114) we obtain

$$\frac{d^2\mathrm{v}^1}{d\tau^2} + 4\Omega^2\mathrm{v}^1 = 0\,, \tag{4.119}$$

i. e. the equation of free oscillations. Its solution is

$$\mathrm{v}^1 = C_1\cos\left(2\Omega\tau\right) + C_2\sin\left(2\Omega\tau\right), \tag{4.120}$$

where C_1 and C_2 are integration constants (4.119), which can be defined from the conditions at the moment $\tau = 0$

$$\left.\begin{aligned} \mathrm{v}^1_{(0)} &= C_1 \\ \left.\frac{d\mathrm{v}^1}{d\tau}\right|_{\tau=0} &= -\,2\Omega C_1\sin\left(2\Omega\tau\right)|_{\tau=0} + 2\Omega C_2\cos\left(2\Omega\tau\right)|_{\tau=0} \end{aligned}\right\}. \tag{4.121}$$

Thus $C_1 = \mathrm{v}^1_{(0)}$, $C_2 = \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}$, where $\dot{\mathrm{v}}^1_{(0)} = \frac{d\mathrm{v}^1}{d\tau}\big|_{\tau=0}$. Then, we finally obtain the equation for v^1 as

$$\mathrm{v}^1 = \mathrm{v}^1_{(0)}\cos\left(2\Omega\tau\right) + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\sin\left(2\Omega\tau\right), \tag{4.122}$$

so the velocity of the mass-bearing elementary spin-particle along x^1 performs sinusoidal oscillations at the frequency equal to the double angular velocity of the space rotation.

Taking into account that $\mathrm{v}^1 = \frac{dx^1}{d\tau}$, we integrate the obtained formula (4.122) with respect to $d\tau$. We obtain

$$x^1 = \frac{\mathrm{v}^1_{(0)}}{2\Omega}\sin\left(2\Omega\tau\right) - \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2}\cos\left(2\Omega\tau\right) + C_3\,. \tag{4.123}$$

Assuming that at the initial moment $\tau = 0$ we have $x^1 = x^1_{(0)}$, we obtain the integration constant $C_3 = x^1_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2}$. Then we have

$$x^1 = \frac{\mathrm{v}^1_{(0)}}{2\Omega}\sin\left(2\Omega\tau\right) - \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2}\cos\left(2\Omega\tau\right) + x^1_0 + \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2}\,, \tag{4.124}$$

so the x^1 coordinate of the elementary particle also performs free oscillations at the frequency 2Ω.

Now having the obtained v^1 (4.122) substituted into the second equation (4.114), we arrive at

$$\frac{d\mathrm{v}^2}{d\tau} = 2\Omega \mathrm{v}^1_{(0)} \cos(2\Omega\tau) + \dot{\mathrm{v}}^1_{(0)} \sin(2\Omega\tau)\,, \tag{4.125}$$

which after integration gives v^2

$$\mathrm{v}^2 = \mathrm{v}^1_{(0)} \sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + C_4\,. \tag{4.126}$$

Assuming for the moment $\tau = 0$ the value $\mathrm{v}^2 = \mathrm{v}^2_{(0)}$, we obtain the constant $C_3 = \mathrm{v}^2_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}$. Then

$$\mathrm{v}^2 = \mathrm{v}^1_{(0)} \sin(2\Omega\tau) - \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + \mathrm{v}^2_{(0)} + \frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\,. \tag{4.127}$$

Taking into account that $\mathrm{v}^2 = \frac{dx^2}{d\tau}$, we integrate the formula with respect to $d\tau$. Then we obtain the formula for the coordinate x^2 of this particle, namely

$$x^2 = -\frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2} \sin(2\Omega\tau) - \frac{\mathrm{v}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + \mathrm{v}^2_{(0)}\tau + \frac{\dot{\mathrm{v}}^1_{(0)}\tau}{2\Omega} + C_5\,. \tag{4.128}$$

Integration constant can be found from the conditions $x^2 = x^2_{(0)}$ at $\tau = 0$ as $C_5 = x^2_{(0)} + \frac{\mathrm{v}^1_{(0)}}{2\Omega}$. Then, finally, the x^2 coordinate is

$$\begin{aligned} x^2 = \mathrm{v}^2_{(0)}\tau + \frac{\dot{\mathrm{v}}^1_{(0)}\tau}{2\Omega} - \frac{\dot{\mathrm{v}}^1_{(0)}}{4\Omega^2} \sin(2\Omega\tau) \; - \\ - \frac{\mathrm{v}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + x^2_{(0)} + \frac{\mathrm{v}^1_{(0)}}{2\Omega}\,. \end{aligned} \tag{4.129}$$

From this formula we see: if at the initial moment of observable time $\tau = 0$ the mass-bearing elementary spin-particle had the velocity $\mathrm{v}^2_{(0)}$ along x^2 and the acceleration $\dot{\mathrm{v}}^1_{(0)}$ along x^1, then this particle, together with free oscillations of the x^2 coordinate at the frequency, equal to the double angular velocity of the space rotation Ω, is subjected to a linear displacement at $\Delta x^2 = \mathrm{v}^2_{(0)}\tau + \frac{\dot{\mathrm{v}}^1_{(0)}\tau}{2\Omega}$.

Looking back at the live forces integral (the solution of the scalar chr.inv.-equation of motion) for this particle $m + \frac{\eta}{c^2} = B = const$ (4.107), we find the integration constant B. Writing (4.107) in the form

$$m_0 + \frac{\eta_0}{c^2} = B\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}\,, \tag{4.130}$$

so the square of the particle's observable velocity is $\mathrm{v}^2 = const$. Because components of the velocity have already been defined, we can present the formula for its square as follows

$$\begin{aligned}\left[\mathrm{v}^1\right]^2+\left[\mathrm{v}^2\right]^2+\left[\mathrm{v}^3\right]^2 = \left[\mathrm{v}^1_{(0)}\right]^2+\left[\mathrm{v}^2_{(0)}\right]^2+\left[\mathrm{v}^3_{(0)}\right]^2+\frac{\left[\dot{\mathrm{v}}^1_{(0)}\right]^2}{2\Omega^2}+\\ +\frac{\dot{\mathrm{v}}^1_{(0)}\dot{\mathrm{v}}^2_{(0)}}{\Omega}+2\left[\mathrm{v}^2_{(0)}+\frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\right]\left[\mathrm{v}^1_{(0)}\sin\left(2\Omega\tau\right)-\frac{\dot{\mathrm{v}}^1_{(0)}}{2\Omega}\cos\left(2\Omega\tau\right)\right]\end{aligned} \tag{4.131}$$

(this is with takinf into account that the three-dimensional metric in question is Euclidean).

We see that the square of the velocity remains unchanged, if $\dot{\mathrm{v}}^2_{(0)} = 0$ and $\dot{\mathrm{v}}^1_{(0)} = 0$. The constant B from the live forces integral is

$$B = \frac{m_0 + \frac{\eta_0}{c^2}}{\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,, \qquad \left[\mathrm{v}^2_{(0)}\right]^2 = \left[\mathrm{v}^1_{(0)}\right]^2 + \left[\mathrm{v}^3_{(0)}\right]^2 = const\,, \tag{4.132}$$

while the live forces integral itself (4.170) becomes

$$m + \frac{\eta}{c^2} = \frac{m_0 + \frac{\eta_0}{c^2}}{\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,, \tag{4.133}$$

so it is the conservation condition for the sum of the relativistic mass of the particle m and of its spin-mass $\frac{\eta}{c^2}$.

A note should be taken here concerning all that has been said in the above on elementary particles. Taking into account in the definition $\eta_0 = n\hbar^{mn}A_{mn}$ that $A_{mn} = \varepsilon_{mnk}\Omega^{*k}$, we obtain

$$\eta_0 = n\hbar^{mn}A_{mn} = 2n\hbar_{*k}\Omega^{*k}. \tag{4.134}$$

where $\hbar_{*k} = \frac{1}{2}\,\varepsilon_{nmk}\hbar^{mn}$. Here, $\hbar_{*k}$ is the three-dimensional pseudovector of the inner momentum of the elementary particle. Hence η_0 is the scalar product of three-dimensional pseudovectors: that of the particle's inner momentum $\hbar_{*k}$ and that of the angular velocity of the space rotation Ω^{*k}. Hence spin-interaction is absent if pseudovectors of the particle's inner rotation and the external rotation of the space are collinear.

Now we refer back to equations of motion of spin-particles. Taking into account the integration constants we have obtained, the vector chr.inv.-equations of motion of a mass-bearing spin-particle, located in our world, have solutions as

$$\left.\begin{array}{ll} \mathrm{v}^1 = \mathrm{v}^1_{(0)} \cos(2\Omega\tau)\,, & x^1 = \dfrac{\mathrm{v}^1_{(0)}}{2\Omega} \sin(2\Omega\tau) + x^1_{(0)} \\ \mathrm{v}^2 = \mathrm{v}^2_{(0)} \sin(2\Omega\tau)\,, & x^2 = -\dfrac{\mathrm{v}^1_{(0)}}{2\Omega} \cos(2\Omega\tau) + \dfrac{\mathrm{v}^1_{(0)}}{2\Omega} + x^2_{(0)} \\ \mathrm{v}^3 = \mathrm{v}^3_{(0)}\,, & x^3 = \mathrm{v}^3_{(0)}\tau + x^3_{(0)} \end{array}\right\}. \tag{4.135}$$

Let us look at the form of a spatial curve along which the particle moves. We set the observer's reference frame so that the initial displacement of the particle is zero $x^1_{(0)} = x^2_{(0)} = x^3_{(0)} = 0$. Now all its spatial coordinates at an arbitrary moment of time are

$$x^1 = x = a\sin(2\Omega\tau)\,, \quad x^2 = y = a\big[1 - \cos(2\Omega\tau)\big], \quad x^3 = z = b\tau\,, \tag{4.136}$$

where $a = \frac{\mathrm{v}^1_{(0)}}{2\Omega}$, $b = \mathrm{v}^3_{(0)}$. The obtained solutions for coordinates are parametric equations of a surface, along which the particle travels. To illustrate what kind of surface it is, we switch from parametric to coordinate notation, removing the parameter τ from the equations. Squaring the equations for x and y we obtain

$$x^2 + y^2 = 2a^2\big[1 - \cos(2\Omega\tau)\big] = 4a^2\sin^2(\Omega\tau) = 4a^2\sin^2\frac{z\Omega}{b}\,. \tag{4.137}$$

The obtained result looks like a spiral line equation $x^2 + y^2 = a^2$, $z = b\tau$. However, the similarity is not complete — the particle travels along the surface of a cylinder at a constant velocity $b = \mathrm{v}^3_{(0)}$ along its z axis , while its radius oscillates at a frequency Ω within the range* from zero up to the maximum $2a = \frac{\mathrm{v}^1_{(0)}}{\Omega}$ at $z = \frac{\pi k b}{2\Omega}$.

So the trajectory of the mass-bearing elementary spin-particle, located in our world, looks like a spiral line "wound" over an oscillating cylinder. The particle's life span is the length of the cylinder divided by its velocity along z (the cylinder's axis). Oscillations of the cylinder are energy "breath ins" and "breath outs" of the particle.

This means that the cylinder we have obtained is the *cylinder of events* of the particle from its birth in our world (the act of materialization) through its death (the dematerialization). But even after the death, its event cylinder does not disappear completely, but the cylinder *splits* into few event cylinders of other particles, produced by this decay

*Where $k = 0, 1, 2, 3, \ldots$ If $\mathrm{v}^3_{(0)} = 0$, the particle simply oscillates within the $x\,y$ plane (the plane of the cylinder's section).

either in our world or in the mirror world.

Therefore analysis of births and decays of elementary particles in the General Theory of Relativity implies analysis of branch points of event cylinders of the particles, taking into account possible branches that lead into the mirror world.

If we consider motion of two linked spin-particles, which rotate around a common centre of masses, for instance, that of positronium (dumb-bell shaped system of an electron and a positron), we obtain a double DNA-like spiral — a twisted "rope ladder" with a number of steps (links of the particles), wound over an oscillating cylinder of their events.

Let us solve chr.inv.-equations of motion of a mass-bearing spin-particle, which moves in the mirror world, a world with the reverse flow of time. With the physical conditions under consideration* the mentioned equations (4.104, 4.105) become

$$-\frac{dm}{d\tau} = \frac{1}{c^2}\frac{d\eta}{d\tau}\,, \tag{4.138}$$

$$\frac{d}{d\tau}\left(m\text{v}^i\right) = -\frac{1}{c^2}\frac{d}{d\tau}\left(\eta\text{v}^i\right). \tag{4.139}$$

Solution of the scalar chr.inv.-equation is the live forces integral $m+\frac{\eta}{c^2}=B=const$, as was the case for the analogous our-world particle (4.107). Substituting it into the vector chr.inv.-equations (4.139) we obtain the solution

$$\frac{d\text{v}^i}{d\tau} = 0\,, \tag{4.140}$$

hence $\text{v}^i=\text{v}^i_{(0)}=const$. This result implies that from the viewpoint of a regular observer the mass-bearing spin-particle travels in the mirror world linearly at a constant velocity in contrast to observable motion of the analogous our-world particle, which travels along an oscillating "spiral" line.

On the other hand, from the viewpoint of a hypothetical observer, who is located in the mirror world, motion of mass-bearing spin-particles in our world will be linear and even, while mirror-world particles will travel along oscillating "spiral" lines.

We could also get an analysis of motion of massless (light-like) spin-particles in a similar way, but we don't know how adequate our assumption that the linear velocity of the space rotation is much smaller com-

*Namely — stationary rotation of the space at a low velocity, the absence of the space deformation, and the Euclidean three-dimensional metric.

pared to the light velocity would be. And it was thanks to this assumption that we were able to obtain exact solutions of chr.inv.-equations of motion for mass-bearing elementary spin-particles. Though in general, the methods to solve chr.inv.-equations of motion are the same for mass-bearing and massless particles.

§4.6 A spin-particle in an electromagnetic field

In this section, we are going to deduce chr.inv.-equations of motion of a particle which has both spin and electric charge, and travels in an external electromagnetic field located in the four-dimensional pseudo-Riemannian space. The particle's summary vector is

$$Q^{\alpha} = P^{\alpha} + \frac{e}{c^2}\,A^{\alpha} + S^{\alpha}, \tag{4.141}$$

were P^{α} is the four-dimensional momentum vector of the particle. The remaining two four-dimensional vectors are an additional momentum, which the particle gains from interaction of its charge with the electromagnetic field, and an additional momentum gained from interaction of the particle's spin with the space non-holonomity field. Note, because the vectors P^{α} and S^{α} are tangential to the four-dimensional trajectory of the particle, we assume that the vector A^{α} (the electromagnetic field potential) is also tangential to the trajectory. In this case the vector is $A^{\alpha} = \varphi_0 \frac{dx^{\alpha}}{ds}$, while the formula $q^i = \frac{\varphi}{c}\,\mathrm{v}^i$ (see §3.8) sets the relationship between the scalar potential φ and the vector potential q^i of the electromagnetic field.

Then chr.inv.-projections $\tilde{\varphi}$ and $\tilde{q}^i$ of the particle's summary vector Q^{α} (4.141) under consideration are

$$\tilde{\varphi} = \pm\left(m + \frac{e\varphi}{c^2} + \frac{\eta}{c^2}\right), \quad \tilde{q}^i = \frac{1}{c^2}\,m\mathrm{v}^i + \frac{1}{c^3}\left(\eta + e\varphi\right)\mathrm{v}^i, \tag{4.142}$$

where m is the relativistic mass of the particle, φ is the scalar potential of the acting electromagnetic field, while η describes interaction of the particle's spin with the space non-holonomity field

$$m = \frac{m_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad \varphi = \frac{\varphi_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad \eta = \frac{\eta_0}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,. \tag{4.143}$$

Generally speaking, the equations can be deduced in the same way as those for a charged particle and a charge-free spin-particle, except for the fact that we have to project the absolute derivative of the sum of the three vectors. Using formulae for $\tilde{\varphi}$ and $\tilde{q}^i$ (4.142), we obtain chr.inv.-

equations of motion of the charged mass-bearing spin-particle located in our world (it travels from the past into the future)

$$\begin{aligned}&\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=\\&=-\frac{1}{c^2}\frac{d}{d\tau}(\eta+e\varphi)+\frac{\eta+e\varphi}{c^4}F_i\mathrm{v}^i-\frac{\eta+e\varphi}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k,\end{aligned}\tag{4.144}$$

$$\begin{aligned}&\frac{d}{d\tau}\left(m\mathrm{v}^i\right)+2m\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k-mF^i+m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\&=-\frac{1}{c^2}\frac{d}{d\tau}\left[(\eta+e\varphi)\,\mathrm{v}^i\right]-\frac{2\,(\eta+e\varphi)}{c^2}\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k+\\&\qquad+\frac{\eta+e\varphi}{c^2}F^i-\frac{\eta+e\varphi}{c^2}\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k,\end{aligned}\tag{4.145}$$

while for the analogous particle located in the mirror world (it travels from the future into the past) the equations are

$$\begin{aligned}&-\frac{dm}{d\tau}-\frac{m}{c^2}F_i\mathrm{v}^i+\frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k=\\&=\frac{1}{c^2}\frac{d}{d\tau}(\eta+e\varphi)+\frac{\eta+e\varphi}{c^4}F_i\mathrm{v}^i-\frac{\eta+e\varphi}{c^4}D_{ik}\mathrm{v}^i\mathrm{v}^k,\end{aligned}\tag{4.146}$$

$$\begin{aligned}&\frac{d}{d\tau}\left(m\mathrm{v}^i\right)+mF^i+m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\&=-\frac{1}{c^2}\frac{d}{d\tau}\left[(\eta+e\varphi)\,\mathrm{v}^i\right]-\frac{\eta+e\varphi}{c^2}F^i-\frac{\eta+e\varphi}{c^2}\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k.\end{aligned}\tag{4.147}$$

Parallel transfer in Riemannian spaces leaves the length of any transferred vector unchanged. Hence its square is invariant in any reference frame. In particular, in the accompanying reference frame it is constant as well

$$\begin{aligned}&Q_\alpha Q^\alpha=g_{\alpha\beta}\left(P^\alpha+\frac{e}{c^2}A^\alpha+S^\alpha\right)\left(P^\beta+\frac{e}{c^2}A^\beta+S^\beta\right)=\\&=g_{\alpha\beta}\left(m_0+\frac{e\varphi_0}{c^2}+\frac{\eta_0}{c^2}\right)^2\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=\left(m_0+\frac{e\varphi_0}{c^2}+\frac{\eta_0}{c^2}\right)^2.\end{aligned}\tag{4.148}$$

In §3.9 we showed that introducing a specific direction of the four-dimensional electromagnetic potential A^α with respect to the trajectory of a charged particle makes the field to move and we substantially simplified the right hand sides of chr.inv.-equations of its motion. The right hand side of the vector chr.inv.-equations of motion becomes the Lorentz chr.inv.-force $\Phi^i=-e\left(E^i+\frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\right)$, while the right hand side of the scalar chr.inv.-equation is obtained as scalar product of the electric strength vector E_i and the observable velocity of the particle. Keeping this in mind, we present the obtained chr.inv.-equations (4.144–4.147) in a more specific form. For the particle located in our world we obtain

$$\begin{aligned}\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)-\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)F_i\mathrm{v}^i+\\ +\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)D_{ik}\mathrm{v}^i\mathrm{v}^k=-\frac{e}{c^2}\,E_i\mathrm{v}^i,\end{aligned}\tag{4.149}$$

$$\begin{aligned}\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right]+2\left(m+\frac{\eta}{c^2}\right)\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k-\left(m+\frac{\eta}{c^2}\right)F^i+\\ +\left(m+\frac{\eta}{c^2}\right)\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=-\,e\left(E^i+\frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right),\end{aligned}\tag{4.150}$$

while for the analogous particle in the mirror world we have

$$\begin{aligned}-\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)-\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)F_i\mathrm{v}^i+\\ +\frac{1}{c^2}\left(m+\frac{\eta}{c^2}\right)D_{ik}\mathrm{v}^i\mathrm{v}^k=-\frac{e}{c^2}\,E_i\mathrm{v}^i,\end{aligned}\tag{4.151}$$

$$\begin{aligned}\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right]+\left(m+\frac{\eta}{c^2}\right)F^i+\left(m+\frac{\eta}{c^2}\right)\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k=\\ =-\,e\left(E^i+\frac{1}{c}\,\varepsilon^{ikm}\mathrm{v}_kH_{*m}\right).\end{aligned}\tag{4.152}$$

To make precise conclusions on motion of charged mass-bearing spin-particles in the pseudo-Riemannian space we have to set a concrete geometric structure of the space. As we did in the previous section, §4.5, where we analyzed motion of charge-free spin-particles, we now assume that:

a) Because gravitational interactions in the scales of elementary particles are infinitesimal, we can assume that $\mathrm{w}\to 0$;
b) The space rotation is stationary, so $\frac{^*\partial v_k}{\partial t}=0$;
c) The are no space deformations, so $D_{ik}=0$;
d) The three-dimensional coordinate metric $g_{ik}dx^idx^k$ is Euclidean, so $g_{ik}=\left|\begin{matrix}-1, & i=k\\ 0, & i\neq k\end{matrix}\right.$;
e) The space rotates at a constant angular velocity Ω around the axis $x^3=z$, so components of the linear velocity of the rotation are $v_1=\Omega_{12}\,x^2=\Omega y$, $v_2=\Omega_{21}x^1=-\Omega x$.

Keeping these constraints in mind, we obtain a formula for ds^2 for elementary particles as

$$ds^2 = c^2 dt^2 - 2\Omega y\, dt\, dx + 2\Omega x\, dt\, dy - dx^2 - dy^2 - dz^2, \qquad (4.153)$$

while physical observable characteristics of the reference space under this metric are

$$F_i = 0\,, \quad D_{ik} = 0\,, \quad A_{12} = -A_{21} = -\,\Omega\,, \quad A_{23} = A_{31} = 0\,. \quad (4.154)$$

As we did in the previous section, §4.5, looking at motion of elementary spin-particles, we assume that the linear velocity of the space rotation is much less than the light velocity (a weak field of the space non-holonomity). In such a case, the metric chr.inv.-tensor h_{ik} is Euclidean and all the Christoffel chr.inv.-symbols Δ^i_{jk} become zeroes, which dramatically simplifies the algebra involved. Then chr.inv.-equations of motion of the particle located in our world become

$$\frac{d}{d\tau}\left(m + \frac{\eta}{c^2}\right) = -\frac{e}{c^2}\, E_i\, \frac{dx^i}{d\tau}\,, \qquad (4.155)$$

$$\left.\begin{aligned} &\frac{d\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^1}{d\tau} + 2\left(m+\frac{\eta}{c^2}\right)\Omega\,\mathrm{v}^2 = -\,e\left(E^1 + \frac{1}{c}\,\varepsilon^{1km}\mathrm{v}_k H_{*m}\right)\\ &\frac{d\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^2}{d\tau} - 2\left(m+\frac{\eta}{c^2}\right)\Omega\,\mathrm{v}^1 = -\,e\left(E^2 + \frac{1}{c}\,\varepsilon^{2km}\mathrm{v}_k H_{*m}\right)\\ &\frac{d\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^3}{d\tau} = -\,e\left(E^3 + \frac{1}{c}\,\varepsilon^{3km}\mathrm{v}_k H_{*m}\right) \end{aligned}\right\}, \quad (4.156)$$

while for the mirror-world particle these are

$$\frac{d}{d\tau}\left(m + \frac{\eta}{c^2}\right) = \frac{e}{c^2}\, E_i\, \frac{dx^i}{d\tau}\,, \qquad (4.157)$$

$$\left.\begin{aligned} &\frac{d\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^1}{d\tau} = -\,e\left(E^1 + \frac{1}{c}\,\varepsilon^{1km}\mathrm{v}_k H_{*m}\right)\\ &\frac{d\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^2}{d\tau} = -\,e\left(E^2 + \frac{1}{c}\,\varepsilon^{2km}\mathrm{v}_k H_{*m}\right)\\ &\frac{d\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^3}{d\tau} = -\,e\left(E^3 + \frac{1}{c}\,\varepsilon^{3km}\mathrm{v}_k H_{*m}\right) \end{aligned}\right\}. \qquad (4.158)$$

Let us look at the scalar chr.inv.-equation of motion in our world (4.155) and in the mirror world (4.157). From here, we see that the sum of the particle's relativistic mass and its spin-mass equals the work done by the electric component of the acting electromagnetic field to displace this charged particle by an elementary interval dx^i. From the vector chr.inv.-equations of motion we see that in our world (4.156) as well as in the mirror world (4.158) the sum of the spatial momentum vector of the particle and its spin-momentum vector along $x^3 = z$ is defined only by the Lorentz force's component along the same axis.

Now our goal is to obtain the trajectory of an elementary charged spin-particle in an electromagnetic field with the given particular properties. As we did in Chapter 3, we assume the field is constant, so its electric and magnetic strengths E_i and H^{*i} are

$$E_i = \frac{\partial\varphi}{dx^i}\,, \tag{4.159}$$

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn} H_{mn} = \frac{1}{2c}\,\varepsilon^{imn}\left[\frac{\partial\,(\varphi \mathrm{v}_m)}{dx^n} - \frac{\partial\,(\varphi \mathrm{v}_n)}{dx^m} - 2\varphi A_{mn}\right]. \tag{4.160}$$

In Chapter 3 we tackled a similar problem — solving chr.inv.-equations of motion for a charged mass-bearing particle, but without taking its spin into account. It is evident that, in a particular case of a spin-free charged particle (spin is zero), solutions of chr.inv.-equations of motion of a charged spin-particle, as a general case, should coincide those obtained in Chapter 3 within "pure" electrodynamics.

To compare our results with those obtained in electrodynamics, it would be reasonable to analyze motion of the mass-bearing spin-particle in three typical kinds of electromagnetic fields, which were under study in Chapter 3 as well as in *The Classical Theory of Fields* by Landau and Lifshitz [10]:

a) A homogeneous stationary electric field (i. e. the field magnetic strength is zero);

b) A homogeneous stationary magnetic field (i. e. the field electric strength is zero);

c) A homogeneous stationary electromagnetic field (both components are non-zeroes).

On the other hand, electrodynamics studies motion of regular macro-particles and it is not evident that all three cases mentioned above are applicable, given the metric constraints, typical for micro-world. This is why.

Firstly, spin of an elementary particle affects its motion only if an external field of the space non-holonomity exists, hence the non-holonomity tensor is $A_{ik} \neq 0$. But from the formulae for the electric and magnetic strengths E_i and H^{*i} (4.159, 4.160) we see that the space non-holonomity only affects the magnetic strength. Hence we will largely focus on motion of the elementary spin-particle in an electromagnetic field of strictly magnetic kind.

Secondly, the scalar chr.inv.-equation of motion of a mass-bearing charged spin-particle (4.155)

$$\left(m_0 + \frac{\eta_0}{c^2}\right) \frac{d}{d\tau} \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} = -\frac{e}{c^2} E_i \mathrm{v}^i \tag{4.161}$$

in a non-relativistic case, where the particle's velocity is much less than the light velocity, becomes

$$E_i \mathrm{v}^i = 0\,, \tag{4.162}$$

so the electric component of the field *does not perform work* to displace the particle under constraints on the metric, typical for the world of elementary particles. Because we are looking at stationary fields, the obtained condition (4.162) can be presented as follows

$$E_i \mathrm{v}^i = \frac{\partial \varphi}{\partial x^i} \mathrm{v}^i = \frac{\partial \varphi}{\partial x^i} \frac{dx^i}{d\tau} = \frac{d\varphi}{d\tau} = 0\,, \tag{4.163}$$

which implies that the field scalar potential $\varphi = const$, so that

$$H^{*i} = \frac{\varphi}{2c}\, \varepsilon^{imn} \left[\frac{\partial \mathrm{v}_m}{\partial x^n} - \frac{\partial \mathrm{v}_n}{\partial x^m} - 2\left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right)\right]. \tag{4.164}$$

For a relativistic case, the electric component reveals itself (it performs work to displace the particle), provided that the absolute value of the particle's velocity is not stationary

$$\frac{1}{2\,c^2\left(1 - \frac{\mathrm{v}^2}{c^2}\right)^{3/2}} \left(m_0 + \frac{\eta_0}{c^2}\right) \frac{d\mathrm{v}^2}{d\tau} = -\frac{e}{c^2} E_i \mathrm{v}^i \neq 0\,. \tag{4.165}$$

Hence the electric component of the acting electromagnetic field, given the constraints on the metric, typical for elementary particles, reveals itself only for relativistic particles, whose velocity is not constant along the trajectory. Hence all "slow-moving" particles fall out of our consideration in the field of strictly electric kind.

Therefore, the general case* should be studied only for a stationary electromagnetic field of *strictly magnetic kind*, where the electric component is absent. This will be done in §4.7.

*Motion of an elementary charged spin-particle at an arbitrary velocity, either low or relativistic.

§4.7 Motion in a stationary magnetic field

In this section, we are going to look at motion of a charged spin-particle in a homogeneous stationary electromagnetic field of strictly magnetic kind.

As we did in the previous section, §4.6, we assume that the space-time has the metric (4.153), so $F_i = 0$ and $D_{ik} = 0$. The non-holonomity field is stationary. In the space rotation around z, out of all components of the non-holonomity tensor only the components $A_{12} = -A_{21} = -\Omega = const$ are not zeroes, so the space rotates within the $x\,y$ plane at a constant velocity Ω.

Under the considered conditions the quantity $\eta_0 = n\hbar^{mn}A_{mn}$, which describes interaction between the particle's spin (its inner rotation) and an external field of the space non-holonomity, is

$$\eta_0 = n\hbar^{mn}A_{mn} = n\left(\hbar^{12}A_{12} + \hbar^{21}A_{21}\right) = -2n\hbar\Omega\,, \tag{4.166}$$

where the sign before the product $\hbar\,\Omega$ depends only on mutual orientation of the $\hbar$ and Ω. "Plus" stands for co-directed $\hbar$ and Ω. "Minus" implies that they are oppositely directed.

In this case* chr.inv.-equations of motion of the particle located in our world become

$$\frac{d}{d\tau}\left(m + \frac{\eta}{c^2}\right) = 0\,, \tag{4.167}$$

$$\begin{gathered}\frac{d}{d\tau}\left[\left(m+\frac{\eta}{c^2}\right)\mathrm{v}^i\right] + 2\left(m+\frac{\eta}{c^2}\right)A_{k\cdot}^{\cdot i}\mathrm{v}^k + \left(m+\frac{\eta}{c^2}\right)\Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k = \\ = -\frac{e}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,,\end{gathered} \tag{4.168}$$

while for the analogous particle located in the mirror world we obtain

$$-\frac{d}{d\tau}\left(m + \frac{\eta}{c^2}\right) = 0\,, \tag{4.169}$$

$$\frac{d}{d\tau}\left[\left(m + \frac{\eta}{c^2}\right)\mathrm{v}^i\right] + \left(m + \frac{\eta}{c^2}\right)\Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k = -\frac{e}{c}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{4.170}$$

Having the live forces theorem (the scalar chr.inv.-equation of motion) integrated, we obtain the live forces integral. In our world and in the mirror world it is, respectively

$$m + \frac{\eta}{c^2} = B = const\,, \qquad m + \frac{\eta}{c^2} = -\widetilde{B} = const\,, \tag{4.171}$$

*Provided that the electromagnetic field potential A^α is directed along the four-dimensional trajectory of the particle.

where B and $\widetilde{B}$ are integration constants in our world and in the mirror world, respectively. We can obtain these constants having the initial conditions at $\tau = 0$ substituted into (4.171). As a result, we obtain

$$B = m_0 + \frac{\eta_0}{c^2} = m_0 + \frac{n\hbar^{mn}A_{mn}}{c^2}\,, \tag{4.172}$$

$$\widetilde{B} = -m_0 - \frac{\eta_0}{c^2} = -m_0 - \frac{n\hbar^{mn}A_{mn}}{c^2}\,. \tag{4.173}$$

Formulae for the live forces integrals (4.171) imply that, in the absence of the electric component of the acting electromagnetic field, the square of the velocity of the charged spin-particle remains unchanged $\mathrm{v}^2 = h_{ik}\mathrm{v}^i\mathrm{v}^k = const.$

Having the formulae for the live forces integrals substituted into (4.168, 4.170), we arrive at the vector chr.inv.-equations of motion in our world and in the mirror world, respectively

$$\frac{d\mathrm{v}^i}{d\tau} + 2A_{k\cdot}^{\cdot i}\mathrm{v}^k + \Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k = -\frac{e}{cB}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,, \tag{4.174}$$

$$\frac{d\mathrm{v}^i}{d\tau} + \Delta_{nk}^i\mathrm{v}^n\mathrm{v}^k = -\frac{e}{c\widetilde{B}}\,\varepsilon^{ikm}\mathrm{v}_k H_{*m}\,. \tag{4.175}$$

These are similar to chr.inv.-equations of motion of a charged macro-particle (that is a spin-free charged particle) in a homogeneous stationary magnetic field (3.290, 3.291), except for the fact that here the integration constant from the live forces integral, found in the right hand side, is not equal to the relativistic mass m of the particle, as it was in electrodynamics (3.290, 3.291), but to the formula (4.171), which accounts for interaction of the particle's spin with the space non-holonomity field. The same is true for the vector chr.inv.-equations (3.298, 3.299).

For our readers with special interest in the method of chronometric invariants we will make a remark related to this notation of chr.inv.-equations of motion.

When obtaining components of the term $A_{k\cdot}^{\cdot i}\mathrm{v}^k$, found only in the our-world equations, we have, for instance, for $i = 1$

$$A_{k\cdot}^{\cdot 1}\mathrm{v}^k = A_{1\cdot}^{\cdot 1}\mathrm{v}^1 + A_{2\cdot}^{\cdot 1}\mathrm{v}^2 = h^{12}A_{12}\mathrm{v}^1 + h^{11}A_{21}\mathrm{v}^2, \tag{4.176}$$

where $A_{12} = -A_{21} = -\Omega$. Then obtaining $A_{1\cdot}^{\cdot 1}$ and $A_{2\cdot}^{\cdot 1}$ we have

$$A_{1\cdot}^{\cdot 1} = h^{1m}A_{1m} = h^{11}A_{11} + h^{12}A_{12} = h^{12}A_{12}\,, \tag{4.177}$$

$$A_{2\cdot}^{\cdot 1} = h^{1m} A_{2m} = h^{11} A_{21} + h^{12} A_{22} = h^{11} A_{21}\,, \tag{4.178}$$

where h^{ik} are elements of a matrix reciprocal to the matrix h_{ik}, so the required components of h^{ik} are calculated as

$$h^{11} = \frac{h_{22}}{h}\,, \qquad h^{12} = -\frac{h_{12}}{h}\,. \tag{4.179}$$

Then, because the determinant of the metric chr.inv.-tensor (see §3.12 for details) is

$$h = \det\|h_{ik}\| = 1 + \frac{\Omega^2\left(x^2+y^2\right)}{c^2}\,, \tag{4.180}$$

the unknown quantity $A_{k\cdot}^{\cdot 1}\mathrm{v}^k$ (4.176) is

$$A_{k\cdot}^{\cdot 1}\mathrm{v}^k = \frac{\Omega}{h}\left[\frac{\Omega^2}{c^2}\,xy\dot{x} + \left(1+\frac{\Omega^2 x^2}{c^2}\right)\dot{y}\right]. \tag{4.181}$$

The component $A_{k\cdot}^{\cdot 2}\mathrm{v}^k$, found in the equation of motion along y, can be found in a similar way.

Let us get back to the vector chr.inv.-equations of motion of the charged spin-particle in the homogeneous stationary magnetic field. We approach them in two possible cases of mutual orientation of the magnetic strength and the space non-holonomity pseudovector, when they are co-directed and are orthogonal to each other.

a) Magnetic field is co-directed with non-holonomity field

We assume that the space non-holonomity field, the space non-holonomity pseudovector, is directed along z and the field is weak. Then the vector chr.inv.-equations of motion of the mass-bearing charged spin-particle located in our world are

$$\ddot{x} + 2\Omega\dot{y} = -\frac{eH}{cB}\,\dot{y}\,, \quad \ddot{y} - 2\Omega\dot{x} = -\frac{eH}{cB}\,\dot{x}\,, \quad \ddot{z} = 0\,, \tag{4.182}$$

while for the analogous particle located in the mirror world we have

$$\ddot{x} = -\frac{eH}{c\widetilde{B}}\,\dot{y}\,, \quad \ddot{y} = -\frac{eH}{c\widetilde{B}}\,\dot{x}\,, \quad \ddot{z} = 0\,. \tag{4.183}$$

The equations differ from those for a spin-free charged particle under the same conditions (3.104, 3.305) only by having on the right hand side the integration constant from the live forces integral, which describes interaction of the particle's spin with the space non-holonomity field, instead of the relativistic mass of the particle.

Using ready solutions from §3.12 we can immediately obtain the formulae for coordinates of the our-world charged spin-particle

$$\begin{aligned} x = -\left[\,\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau + \dot{x}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,\right]\frac{1}{2\Omega+\omega} + \\ + x_{(0)} + \frac{\dot{y}_{(0)}}{2\Omega+\omega}\,, \end{aligned} \tag{4.184}$$

$$\begin{aligned} y = \left[\,\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau - \dot{x}_{(0)}\cos\left(2\Omega+\omega\right)\tau\,\right]\frac{1}{2\Omega+\omega} + \\ + y_{(0)} - \frac{\dot{x}_{(0)}}{2\Omega+\omega}\,, \end{aligned} \tag{4.185}$$

and those for the mirror-world particle

$$x = -\frac{1}{\omega}\left[\,\dot{y}_{(0)}\cos\omega\tau + \dot{x}_{(0)}\sin\omega\tau\,\right] + x_{(0)} + \frac{\dot{y}_{(0)}}{\omega}\,, \tag{4.186}$$

$$y = \frac{1}{\omega}\left[\,\dot{y}_{(0)}\sin\omega\tau - \dot{x}_{(0)}\cos\omega\tau\,\right] + y_{(0)} - \frac{\dot{x}_{(0)}}{\omega}\,, \tag{4.187}$$

which are different from solutions for a charged particle in electrodynamics only by the fact that the frequency ω accounts for interaction of the particle's spin with the space non-holonomity field.

In our world masses of particles are positive, so ω is

$$\omega = \frac{eH}{mc + \frac{\eta}{c}} = \frac{eH\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}{m_0 c + \frac{\eta_0}{c}} = \frac{eH\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}{m_0 c \mp \frac{2n\hbar\,\Omega}{c}}\,, \tag{4.188}$$

where the sign in the denominator depends on mutual orientation of the $\hbar$ and Ω — "minus" stands for the co-directed $\hbar$ and Ω (their scalar product is positive), while "plus" implies that they are oppositely directed, irrespective of our choice of right or left-hand reference frames.

Masses of particles, which inhabit the mirror world, are always negative

$$m = -\frac{m_0}{\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}} < 0\,, \tag{4.189}$$

so in the mirror world ω is

$$\omega = \frac{eH}{mc + \frac{\eta}{c}} = \frac{eH\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}{-m_0 c + \frac{\eta_0}{c}} = \frac{eH\sqrt{1-\frac{\mathrm{v}_{(0)}^2}{c^2}}}{-m_0 c \mp \frac{2n\hbar\,\Omega}{c}}\,. \tag{4.190}$$

Note that the obtained formulae for coordinates (4.184–4.187) already took account of the fact that the square of the particle's velocity remains unchanged both in our world and in the mirror world, that is presented with the conditions (respectively)

$$\dot{x}_{(0)} + \frac{\ddot{y}_0}{2\Omega + \omega} = 0\,, \qquad \dot{x}_{(0)} + \frac{\ddot{y}_0}{\omega} = 0\,, \tag{4.191}$$

which results from the live forces integral (§3.12).

The third equation of motion (along z) has solution as

$$z = \dot{z}_{(0)}\tau + z_{(0)}\,. \tag{4.192}$$

The obtained formulae for coordinates (4.184–4.187) reveal that a mass-bearing charged spin-particle in a homogeneous stationary magnetic field, parallel to a weak field of the space non-holonomity, performs *harmonic oscillations* along x and y. In our world the frequency of the oscillations is

$$\widetilde{\omega} = 2\Omega + \omega = 2\Omega + \frac{eH}{m_0 c \mp \frac{2n\hbar\Omega}{c}}\sqrt{1 - \frac{\mathrm{v}_{(0)}^2}{c^2}}\,, \tag{4.193}$$

while in the mirror world the analogous particle performs similar oscillations at a frequency ω obtained in (4.190).

In a weak field of the space non-holonomity the quantity $n\hbar\Omega$ is much less than the energy m_0c^2, because for any small quantity α it is true that $\frac{1}{1\mp\alpha} \cong 1 \pm \alpha$, for low velocities we have

$$\widetilde{\omega} \cong 2\Omega + \frac{eH}{m_0 c}\left(1 \pm \frac{2n\hbar\Omega}{m_0 c^2}\right). \tag{4.194}$$

If at the initial moment of time the displacement and the velocity of the our-world particle satisfy the conditions

$$x_{(0)} + \frac{\dot{y}_0}{2\Omega + \omega} = 0\,, \qquad y_{(0)} - \frac{\dot{x}_0}{2\Omega + \omega} = 0\,, \tag{4.195}$$

it will travel, like a charged spin-free particle, within $x\,y$ plane along a *circle**

$$x^2 + y^2 = \frac{\dot{y}_0^2}{(2\Omega + \omega)^2}\,. \tag{4.196}$$

*We set the y axis along the initial momentum of the particle, which is always possible. Then all formulae for coordinates will have zero initial velocity of the particle along x.

But in this case its radius, which is equal to

$$r = \frac{\dot{y}_0}{2\Omega + \omega} = \frac{\dot{y}_0}{2\Omega + \frac{eH}{m_0 c \mp \frac{2n\hbar\Omega}{c}}\sqrt{1 - \frac{\mathrm{v}_{(0)}^2}{c^2}}}\,, \tag{4.197}$$

will depend on the absolute value and the orientation of the spin. If the initial velocity of a charged particle with spin, directed along the magnetic strength (along z), is not zero, the particle travels along the magnetic strength along a *spiral line* with the same radius r.

An analogous mirror-world particle, provided its displacement and the velocity at the initial moment of time satisfy the conditions

$$x_{(0)} + \frac{\dot{y}_0}{\omega} = 0\,, \qquad y_{(0)} - \frac{\dot{x}_0}{\omega} = 0\,, \tag{4.198}$$

will also travel along a circle

$$x^2 + y^2 = \frac{\dot{y}_0^2}{\omega^2}\,, \tag{4.199}$$

with the radius

$$r = \frac{\dot{y}_0}{\omega} = \frac{\dot{y}_0}{\frac{eH}{-m_0 c \mp \frac{2n\hbar\Omega}{c}}\sqrt{1 - \frac{\mathrm{v}_{(0)}^2}{c^2}}}\,. \tag{4.200}$$

In general, where no additional conditions (4.195, 4.198) are imposed, the trajectory within the (xy)-plane will not be circular.

Let us obtain the energy and the momentum of the particle. Using formulae for the live forces integrals, we find the quantity η_0, which is $\eta_0 = n\hbar^{mn}A_{mn} = n\,(\hbar^{12}A_{12} + \hbar^{21}A_{21}) = -2n\hbar\,\Omega$. Then for the particle located in our world we have

$$E_{\mathrm{tot}} = Bc^2 = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{\sqrt{1 - \frac{\mathrm{v}_{(0)}^2}{c^2}}} = const\,, \tag{4.201}$$

while in the mirror-world we have

$$E_{\mathrm{tot}} = \widetilde{B}c^2 = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{\sqrt{1 - \frac{\mathrm{v}_{(0)}^2}{c^2}}} = const\,. \tag{4.202}$$

Since in this section, §4.7, we have assumed that the electric component of the acting electromagnetic field is absent, the field does not contribute to the total energy of the particle (as it is known, the magnetic component of the field does not perform work to displace electric charges).

From the obtained formulae (4.201, 4.202) we see that the total energy of the particle remains unchanged along the trajectory, while its numerical value depends on mutual orientation of the particle's inner momentum $\hbar$ and the angular velocity of the space rotation Ω.

The latter statement requires some comments to be made. By definition the scalar quantity n (the absolute value of spin in the $\hbar$ units) is always positive, while $\hbar$ and Ω are numerical values of components of the antisymmetric tensors h^{ik} and Ω_{ik}, which take opposite signs in right or left-handed reference frames. But because we are dealing with the product of the quantities, only their mutual orientation matters, which does not depend on our choice of a right or left-handed reference frames.

If $\hbar$ and Ω are co-directed, then the total energy of the our-world particle E_{tot} (4.201) is the sum of its relativistic energy $E = mc^2$ and its "spin-energy"

$$E_{\text{s}} = \frac{2n\hbar\Omega}{\sqrt{1-\frac{\text{v}^2_{(0)}}{c^2}}}\,, \tag{4.203}$$

so the total energy becomes greater than $E = mc^2$.

If $\hbar$ and Ω are oppositely directed, then E_{tot} is the difference between the relativistic energy and the spin-energy. This orientation permits a specific case, where $m_0 c^2 = 2n\hbar\Omega$ and therefore the total energy becomes zero (this case will be discussed in the next section, §4.8, concerning proper fields of elementary particles).

For charged spin-particles having negative masses, which inhabit the mirror world, the situation is different. The total energy E_{tot} (4.202) is negative and by its absolute value is greater than the relativistic energy $E = -mc^2$, provided that $\hbar$ and Ω are oppositely directed.

So forth, for the total spatial observable momentum of the our-world particle we have

$$p^i_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\Omega}{c^2\sqrt{1-\frac{\text{v}^2_{(0)}}{c^2}}}\,\text{v}^i = m\text{v}^i \mp \frac{2n\hbar\Omega}{c^2\sqrt{1-\frac{\text{v}^2_{(0)}}{c^2}}}\,\text{v}^i, \tag{4.204}$$

so it is an algebraic sum of the particle's relativistic observable momentum $p^i = m\text{v}^i$ and the spin-momentum that the particle gains from the space non-holonomity field. The particle's total momentum is greater than its relativistic momentum, if $\hbar$ and Ω are co-directed, and it is less otherwise.

In the case of opposite mutual orientation of $\hbar$ and Ω the total momentum becomes zero (so does the total energy), provided that the condition $m_0 c^2 = 2n\hbar\,\Omega$ is true.

For the mirror-world particle the quantity p^i_{tot} is

$$p^i_{\text{tot}} = \frac{-\,m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\mathrm{v}^i = -\,m\mathrm{v}^i \mp \frac{2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\mathrm{v}^i, \tag{4.205}$$

so the particle moves more slowly if the $\hbar$ and Ω are co-directed, and it is faster otherwise.

Components of the velocity of the charged spin-particle in the magnetic field co-directed with the space non-holonomity field, taking into account the conditions (4.191), in our world are

$$\dot{x} = \dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau - \dot{x}_{(0)}\cos\left(2\Omega+\omega\right)\tau\,, \tag{4.206}$$

$$\dot{y} = \dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau + \dot{x}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,, \tag{4.207}$$

while for the analogous particle located in the mirror world we have

$$\dot{x} = \dot{y}_{(0)}\sin\omega\tau - \dot{x}_{(0)}\cos\omega\tau\,, \tag{4.208}$$

$$\dot{y} = \dot{y}_{(0)}\cos\omega\tau + \dot{x}_{(0)}\sin\omega\tau\,. \tag{4.209}$$

Then components of the total momentum of the particle* in our world are

$$p^1_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\sin\left(2\Omega+\omega\right)\tau\,, \tag{4.210}$$

$$p^2_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\cos\left(2\Omega+\omega\right)\tau\,, \tag{4.211}$$

$$p^3_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{z}_{(0)}\,, \tag{4.212}$$

*The initial momentum of the particle within $x\,y$ plane is directed along y.

where ω is as in (4.188). In the mirror world we have

$$p^1_{\text{tot}} = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\sin\omega\tau\,, \tag{4.213}$$

$$p^2_{\text{tot}} = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)}\cos\omega\tau\,, \tag{4.214}$$

$$p^3_{\text{tot}} = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{z}_{(0)}\,, \tag{4.215}$$

where ω is as of (4.190). Noteworthy, though the magnetic strength does not appear in the total energy E_{tot}, it appears in the total momentum, as a term of the formula for ω (4.190).

b) Magnetic field is orthogonal to non-holonomity field

Now we are going to approach motion of a mass-bearing charged spin-particle in a magnetic field, which is orthogonal to the space non-holonomity field. So, the magnetic field is homogeneous and stationary. The non-holonomity field is directed along z and it is weak, so the magnetic field is directed along y. Then vector chr.inv.-equations of motion will be similar to those for a charged spin-free particle under the above conditions in our world (3.338), namely

$$\ddot{x} + 2\Omega\dot{y} = \frac{eH}{cB}\,\dot{z}\,, \qquad \ddot{y} - 2\Omega\dot{x} = 0\,, \qquad \ddot{z} = -\frac{eH}{cB}\,\dot{x}\,. \tag{4.216}$$

The difference from (3.338) is that here the denominator of the right hand side contains the integration constant from the live forces integral instead of the relativistic mass, which accounts for interaction between the particle's spin and the non-holonomity field. After integration the equations yields

$$x = \frac{\dot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau - \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\cos\widetilde{\omega}\tau + x_{(0)} + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}^2}\,, \tag{4.217}$$

$$\begin{gathered} y = -\frac{2\Omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right) + \dot{y}_{(0)}\tau + \\ + \frac{2\Omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\tau + y_{(0)} + \frac{2\Omega}{\widetilde{\omega}^2}\,\dot{x}_{(0)}\,, \end{gathered} \tag{4.218}$$

$$z = \frac{\omega}{\widetilde{\omega}^2}\left(\dot{x}_{(0)}\cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}}\sin\widetilde{\omega}\tau\right) + \dot{z}_{(0)}\tau - \\ - \frac{\omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\tau + z_{(0)} - \frac{\omega}{\widetilde{\omega}^2}\,\dot{x}_{(0)}\,, \tag{4.219}$$

which are different from the respective solutions for a charged spin-free particle by the fact that the frequency $\widetilde{\omega}$ here depends on the spin and its mutual orientation with the non-holonomity field. Namely, he frequency $\widetilde{\omega}$ is expressed as follows

$$\widetilde{\omega} = \sqrt{4\Omega^2 + \omega^2} = \sqrt{4\Omega^2 + \frac{e^2H^2\left(1 - \frac{\mathrm{v}^2_{(0)}}{c^2}\right)^2}{\left(m_0c^2 \mp \frac{2n\hbar\,\Omega}{c}\right)^2}}\;. \tag{4.220}$$

Subsequently, an equation of the trajectory of the charged spin-particle is similar to that of the spin-free particle. In a particular case, namely — under certain initial conditions, the trajectory equation is that of a *sphere*

$$x^2 + y^2 + z^2 = \frac{1}{\widetilde{\omega}^2}\,\dot{x}^2_{(0)}\,, \tag{4.221}$$

whose radius, in contrast to the radius of the trajectory of the spin-free particle, depends on the particle's orientation with respect to the non-holonomity field

$$r = \frac{1}{\sqrt{4\Omega^2 + \frac{e^2H^2\left(1 - \frac{\mathrm{v}^2_{(0)}}{c^2}\right)^2}{\left(m_0c^2 \mp \frac{2n\hbar\,\Omega}{c}\right)^2}}}\,\dot{x}_{(0)}\,. \tag{4.222}$$

Let us look at an analogous particle, located in the mirror world, moves in a weak field of the space non-holonomity, directed along y and orthogonal to the magnetic field. For the particle, the vector chr.inv.-equations of motion are

$$\ddot{x} = \frac{eH}{c\widetilde{B}}\,\dot{z}\,, \qquad \ddot{y} = 0\,, \qquad \ddot{z} = -\frac{eH}{c\widetilde{B}}\,\dot{x}\,, \tag{4.223}$$

so they are different from the equations for the our-world particle (4.216) by the absence of the terms which contain the angular velocity of the space rotation Ω. As a result their solutions can be obtained from the solutions for our world (4.217–4.219), if we assume $\widetilde{\omega} = \omega$. Subsequently, an equation of the trajectory of the charged spin-particle located in the mirror world is

$$x^2 + y^2 + z^2 = \frac{1}{\omega^2}\,\dot{x}^2_{(0)}\,, \qquad r = \frac{-m_0 c^2 \mp \dfrac{2n\hbar\,\Omega}{c}}{eH\sqrt{1 - \frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{x}_{(0)}\,. \tag{4.224}$$

The total energy of the particle E_{tot} in this case, where the magnetic field is orthogonal to the space non-holonomity field, is the same as it was for the case of parallel orientation of the fields. But the formulae for components of the total momentum (4.201, 4.205) are different, because they include the particle's velocity which depends on mutual orientation of the magnetic field and the non-holonomity field. In the particular case, where the fields are orthogonal to each other, components of the particle's velocity (obtained by derivation of the formulae for 4.217–4.219) in our world are

$$\dot{x} = \dot{x}_{(0)} \cos\widetilde{\omega}\tau + \frac{\ddot{x}_{(0)}}{\widetilde{\omega}} \sin\widetilde{\omega}\tau\,, \tag{4.225}$$

$$\dot{y} = \frac{2\Omega}{\widetilde{\omega}}\,\dot{x}_{(0)} \sin\widetilde{\omega}\tau - \frac{2\Omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)} \cos\widetilde{\omega}\tau + \dot{y}_{(0)} + \frac{2\Omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\,, \tag{4.226}$$

$$\dot{z} = \frac{\omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)} \cos\widetilde{\omega}\tau - \frac{\omega}{\widetilde{\omega}}\,\dot{x}_{(0)} \sin\widetilde{\omega}\tau + \dot{z}_{(0)} - \frac{\omega}{\widetilde{\omega}^2}\,\ddot{x}_{(0)}\,, \tag{4.227}$$

while in the mirror world we obtain

$$\dot{x} = \dot{x}_{(0)} \cos\omega\tau + \frac{\ddot{x}_{(0)}}{\omega} \sin\omega\tau\,, \tag{4.228}$$

$$\dot{y} = \dot{y}_{(0)}\,, \tag{4.229}$$

$$\dot{z} = \frac{1}{\omega}\,\ddot{x}_{(0)} \cos\widetilde{\omega}\tau - \dot{x}_{(0)} \sin\omega\tau + \dot{z}_{(0)} - \frac{1}{\omega}\,\ddot{x}_{(0)}\,. \tag{4.230}$$

Now we assume that the initial acceleration of the particle and the integration constants are zeroes. We also set the axis x along the initial momentum of the particle. In the frames of this consideration we obtain components of the total momentum for the particle located in our world

$$p^1_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1 - \frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{x}_{(0)} \cos\widetilde{\omega}\tau\,, \tag{4.231}$$

$$p^2_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1 - \frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\frac{2\Omega}{\widetilde{\omega}}\,\dot{x}_{(0)} \sin\widetilde{\omega}\tau\,, \tag{4.232}$$

$$p^3_{\text{tot}} = \frac{m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\frac{\omega}{\widetilde{\omega}}\,\dot{x}_{(0)}\sin\widetilde{\omega}\tau\,, \tag{4.233}$$

and for the analogous particle located in the mirror world

$$p^1_{\text{tot}} = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{x}_{(0)}\cos\widetilde{\omega}\tau\,, \tag{4.234}$$

$$p^2_{\text{tot}} = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{y}_{(0)} = 0\,, \tag{4.235}$$

$$p^3_{\text{tot}} = \frac{-m_0 c^2 \mp 2n\hbar\,\Omega}{c^2\sqrt{1-\frac{\mathrm{v}^2_{(0)}}{c^2}}}\,\dot{x}_{(0)}\sin\widetilde{\omega}\tau\,. \tag{4.236}$$

As it easy to see, the obtained solutions can be transformed into respective ones from electrodynamics (§3.12) by assuming $\hbar \to 0$.

§4.8 The quantization law for masses of elementary particles

As obtained before, scalar chr.inv.-equations of motion of a charged spin-particle in an electromagnetic field, located in our world and in the mirror world, are

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right) = -\frac{e}{c^2}\,E_i\mathrm{v}^i, \quad -\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right) = -\frac{e}{c^2}\,E_i\mathrm{v}^i. \tag{4.237}$$

The equations can be easily integrated to produce the live forces integrals

$$m+\frac{\eta}{c^2} = B\,, \qquad -\left(m+\frac{\eta}{c^2}\right) = \widetilde{B}\,, \tag{4.238}$$

where B is integration constant in our world and $\widetilde{B}$ is that in the mirror world. The constants depend only on the initial conditions. Hence it is possible to choose them as to make the integration constants zeroes.

We find out under what initial conditions the integration constants become zeroes. For charged spin-particles, located in our world and in the mirror world (4.238), we obtain, respectively

$$m+\frac{\eta}{c^2} = 0\,, \qquad -\left(m+\frac{\eta}{c^2}\right) = 0\,, \tag{4.239}$$

while the right hand sides of the vector chr.inv.-equations of motion (4.150, 4.152), which contain the Lorentz chr.inv.-force, also become zeroes. In other words, with the integration constants in the scalar chr.inv.-equations equal to zero the acting electromagnetic field does no work to displace the particles.

Having relativistic square root cancelled in (4.239), which is always possible for any particle having non-zero rest-masses, we can present these formulae in a notation that does not depend on the particle's velocity. Then for mass-bearing particles located in our world we have

$$m_0 c^2 = -n\hbar^{mn} A_{mn}\,, \tag{4.240}$$

while for mirror-world particles of non-zero masses we have

$$m_0 c^2 = n\hbar^{mn} A_{mn}\,. \tag{4.241}$$

We will refer to the formulae (4.240, 4.241) as the *law of quantization of masses of elementary particles*:

> Rest-mass of any mass-bearing spin-particle is proportional to energy of interaction between its spin and the field of the space non-holonomity, taken with the opposite sign.

Or, in other words:

> Rest-energy of any mass-bearing spin-particle equals energy of interaction between its spin and the field of the space non-holonomity, taken with the opposite sign.

Because in the mirror world the energy of any particle is negative, "plus" in the right hand side of (4.241) stands for the energy of interaction in the mirror world taken with the opposite sign. The same is true for "minus" in (4.240) for our world.

Evidently, these quantum formulae are not applicable to non-spin particles.

Let us make some quantitative estimates, which are derived from the obtained law. Considering an elementary particle, we will obtain numerical values of the quantity* $\eta_0 = n\hbar^{mn} A_{mn}$ as follows. We formulate the tensor of angular velocities of the space rotation A_{mn} with the pseudovector $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn} A_{mn}$

$$\Omega^{*i}\varepsilon_{imn} = \frac{1}{2}\,\varepsilon^{ipq}\varepsilon_{imn} A_{pq} = \frac{1}{2}\left(\delta^p_m \delta^q_n - \delta^p_n \delta^q_m\right) A_{pq} = A_{mn}\,, \tag{4.242}$$

*This quantity characterize the energy of interaction between the particle's spin and the space non-holonomity field — the "spin-energy", in other words.

so we have $A_{mn} = \varepsilon_{imn}\Omega^{*i}$. Then because

$$\frac{1}{2}\,\varepsilon_{imn}\hbar^{mn} = \hbar_{*i} \tag{4.243}$$

is the Planck pseudovector, the quantity $\eta_0 = n\hbar^{mn}\varepsilon_{imn}\Omega^{*i}$ is

$$\eta_0 = 2n\hbar_{*i}\Omega^{*i}, \tag{4.244}$$

so it is the double scalar product of the Planck three-dimensional pseudovector and the three-dimensional pseudovector of angular velocities of the space rotation, multiplied by the particle's spin quantum number. If $\hbar_{*i}$ and Ω^{*i} are co-directed, then the cosine is positive, hence

$$\eta_0 = 2n\hbar_{*i}\Omega^{*i} = 2n\hbar\,\Omega\cos\big(\vec{\hbar}\,;\vec{\Omega}\big) > 0\,, \tag{4.245}$$

while if they are oppositely directed, then

$$\eta_0 = 2n\hbar_{*i}\Omega^{*i} = 2n\hbar\,\Omega\cos\big(\vec{\hbar}\,;\vec{\Omega}\big) < 0\,. \tag{4.246}$$

Therefore for any mass-bearing elementary particle, located in our world, the integration constant from the live forces integral becomes zero, provided that the pseudovectors $\hbar_{*i}$ and Ω^{*i} are oppositely directed. In the case of any mass-bearing elementary particle, located in the mirror world, the constant becomes zero if the pseudovectors $\hbar_{*i}$ and Ω^{*i} are co-directed.

This implies that if the energy of interaction of a mass-bearing elementary particle with the space non-holonomity field becomes equal to its rest-energy $E = m_0c^2$, then the momentum of the particle neither reveals itself in our world nor in the mirror world.

We assume that the z axis is co-directed with the pseudovector of angular velocities of the space rotation Ω^{*i}. Then out of all three components of the Ω^{*i} the only non-zero component is

$$\Omega^{*3} = \frac{1}{2}\,\varepsilon^{3mn}A_{mn} = \frac{1}{2}\big(\varepsilon^{312}A_{12} + \varepsilon^{321}A_{21}\big) = \varepsilon^{312}A_{12} = \frac{e^{312}}{\sqrt{h}}A_{12}\,. \tag{4.247}$$

To simplify the algebra we assume that the three-dimensional coordinate metric g_{ik} is Euclidean and the space rotates at a constant angular velocity Ω. Then components of the linear velocity of the space rotation are $v_1 = \Omega x$, $v_2 = -\,\Omega y$, and $A_{12} = -\,\Omega$. Hence

$$\Omega^{*3} = \frac{e^{312}}{\sqrt{h}}A_{12} = \frac{A_{12}}{\sqrt{h}} = -\frac{\Omega}{\sqrt{h}}\,. \tag{4.248}$$

The square root of the determinant of the metric chr.inv.-tensor, as defined in (4.180) is

$$\sqrt{h} = \sqrt{\det \|h_{ik}\|} = \sqrt{1 + \frac{\Omega^2 \left(x^2 + y^2\right)}{c^2}}\,. \tag{4.249}$$

Because we are dealing with very small coordinate values in the scales of elementary particles, we can assume $\sqrt{h} \approx 1$ and, according to (4.248) also $\Omega^{*3} = -\Omega = const$. Then the law of quantization of masses of elementary particles (4.240), taken in our world and in the mirror world, respectively, becomes

$$m_0 = \frac{2n\hbar\Omega}{c^2}\,, \qquad m_0 = -\frac{2n\hbar\Omega}{c^2}\,. \tag{4.250}$$

Hence for any elementary particle of non-zero mass, located in our world, the following relationship between its rest-mass m_0 and the angular velocity of the space rotation Ω is eminent

$$\Omega = \frac{m_0 c^2}{2n\hbar}\,. \tag{4.251}$$

This implies that the rest-mass (the true mass) of an observable object, under regular conditions does not depend on properties of the observer's reference space; but for elementary particles it becomes strictly dependent on these properties, in particular it depends on the angular velocity of the space rotation.

Hence, proceeding from the quantization law, we can calculate frequencies of rotation of the observer's space, corresponding to rest-masses of elementary particles, located in our world.

The results, proceeding from the calculations for elementary particles of known kinds, are given in Table 4.1.

These results show that for elementary particles, the observer's space is always non-holonomic. So forth for instance, in observation of an electron $r_{\mathrm{e}} = 2.8 \times 10^{-13}$ cm the linear velocity of rotation of the observer's space is $v = \Omega r = 2200$ km/sec*. Because other elementary particles are even smaller, this linear velocity seems to be the upper limit†.

---

*This value of v equals the velocity of an electron in the Bohr 1st orbit, though when calculating the velocity of the space rotation (see Table 1) we considered a *free electron*, i. e. the one not related to an atomic nucleus and quantization of orbits in an atom of hydrogen. The reason is that the "genetic" quantum non-holonomity of the space seems not only to define rest-masses of elementary particles, but to be the reason of rotation of electrons in atoms.

†It is interesting, the angular velocities of the space rotation in barions (see Table 1) up within the order of the magnitude match the frequency $\sim 10^{23}$ sec^{-1} which characterizes elementary particles as oscillators [27, 28].

Elementary particles	Rest-mass	Spin	Ω, sec^{-1}
LEPTONS			
electron e^-, positron e^+	1	1/2	7.782×10^{20}
electron neutrino ν_e and			
electron anti-neutrino $\tilde{\nu}_e$	$<4\times10^{-4}$	1/2	$<3\times10^{17}$
μ-meson neutrino ν_μ and			
μ-meson anti-neutrino $\tilde{\nu}_\mu$	<8	1/2	$<6\times10^{21}$
μ^--meson, μ^+-meson	206.766	1/2	1.609×10^{23}
BARIONS			
nuclons			
proton p, anti-proton $\tilde{p}$	1836.09	1/2	1.429×10^{24}
neutron n, anti-neutron $\tilde{n}$	1838.63	1/2	1.431×10^{24}
hyperons			
Λ^0-hyperon, anti-Λ^0-hyperon	2182.75	1/2	1.699×10^{24}
Σ^+-hyperon, anti-Σ^+-hyperon	2327.6	1/2	1.811×10^{24}
Σ^--hyperon, anti-Σ^--hyperon	2342.6	1/2	1.823×10^{24}
Σ^0-hyperon, anti-Σ^0-hyperon	2333.4	1/2	1.816×10^{24}
Ξ^--hyperon, anti-Ξ^--hyperon	2584.7	1/2	2.011×10^{24}
Ξ^0-hyperon, anti-Ξ^0-hyperon	2572	1/2	2.00×10^{24}
Ω^--hyperon, anti-Ω^--hyperon	3278	3/2	8.50×10^{23}

Table 4.1: Frequencies of rotation of the observer's reference space, which correspond to elementary particles of non-zero mass.

So, what did we get? Generally, the observer compares results of his measurements with special standards located in his reference body. But the body and himself are not related to the observed object and do not affect it during observations. Hence in macro-world there is no dependence of the true properties of observed bodies (rest-mass, rest-energy, etc.) on properties of the reference body and the reference space — these are properties of objects *non-related* to each other.

In other words, though observed images are distorted by influence from physical properties of the observer's reference frame, the observer himself and his reference body in macro-world do not affect measured objects anyhow.

But the world of elementary particles presents a big difference. In this section, we have seen that once we reach the scale of elementary particles, where spin, a quantum property of the particles, significantly affects their motion, physical properties of the reference body (the reference space) and those of the particles become tightly linked to each other, so the reference body *affects* the observed particles. In other words, the observer does not just compare properties of the observed particles to those of his references any longer, but instead directly *affects* the observed particles. The observer shapes their properties in a tight quantum relationship with properties of the references he possesses.

We can explain the above in other words as follows. When looking at effects in the world of elementary particles, *there is no border* between the observer (his reference body and the reference space) and the observed particle. Hence we have an opportunity to define a *relationship* between the space non-holonomity field, linked to the observer, and rest-masses of the observed particles — objects of his observations, which in macro-world are not related to the reference body. So, the obtained law of quantization of masses is only true for elementary particles.

Please note that we have obtained the result using only geometric methods of the General Theory of Relativity, and not methods of Quantum Mechanics. In future, this result may possibly become a "bridge" between these two theories.

§4.9 The Compton wavelength

So, we have obtained that in observation of an elementary particle with rest-mass m_0 the rotation frequency of the observer's space is $\Omega = \frac{m_0 c^2}{2n\hbar}$ (4.251). We are going to find the wavelength which corresponds to that frequency. Assuming that this wave, i. e. the wave of the space non-holonomity, propagates at the light velocity $\lambda\Omega = c$, we have

$$\lambda = \frac{c}{\Omega} = 2n\frac{\hbar}{m_0 c}\,. \tag{4.252}$$

In other words, if we observe a mass-bearing particle with spin $n = \frac{1}{2}$ the length of the space non-holonomity wave equals Compton's wavelength of this particle $\lambdabar_{\mathrm{c}} = \frac{\hbar}{m_0 c}$.

What does this mean? Compton effect, named after A. Compton who discovered it in 1922, is "diffraction" of a photon on a free electron, which results in decrease of its own frequency

$$\Delta\lambda = \lambda_2 - \lambda_1 = \frac{h}{m_{\mathrm{e}} c}\,(1 - \cos\vartheta) = \lambda_{\mathrm{c}}^{\mathrm{e}}\,(1 - \cos\vartheta)\,, \tag{4.253}$$

where λ_1 and λ_2 are the photon's wavelengths before and after the encounter, ϑ is the angle of "diffraction". The multiplier $\lambda_{\mathrm{c}}^{\mathrm{e}}$, specific to the electron, at first was called the *Compton wavelength* of the electron. Later it was found out that other elementary particles during "diffraction" of photons reveal as well the specific wavelengths $\lambda_{\mathrm{c}} = \frac{h}{m_0 c}$, or,

$\lambda\!\!\!^{-}_{\rm c} = \frac{\hbar}{m_0 c}$. That is, elementary particles of every kind (electrons, protons, neutrons etc.) have their own Compton wavelengths. The physical sense behind the quantity will be explained later. It was obtained, within an area smaller than $\lambda\!\!\!^{-}_{\rm c}$, any elementary particle is no longer a point object and its interaction with other particles (and with the observer) is described by Quantum Mechanics. Hence the $\lambda\!\!\!^{-}_{\rm c}$-sized area is sometimes interpreted as the "size" of the elementary particle.

As for the results we have obtained in the previous section, §4.8, these can be interpreted as follows. In observation of a mass-bearing elementary particle the observer's space rotates so fast that the angular velocity of its rotation makes a specific wavelength equal to the Compton wavelength of the observed particle, so to the "size" inside which the particle is no longer a point object. In other words, it is the angular velocity of the space rotation (the wavelength in the space non-holonomity field), which defines the Compton observable wavelength (the specific "size") of the particle.

§ 4.10 Massless spin-particles

Because massless particles do not have electric charge, their scalar chr. inv.-equations of motion in our world and in the mirror world are as follows, respectively,

$$\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=0\,, \qquad -\frac{d}{d\tau}\left(m+\frac{\eta}{c^2}\right)=0\,. \tag{4.254}$$

Their integration always gives a constant equal to zero, hence we always obtain the formulae (4.239). Hence for massless particles in our world and in the mirror world, respectively

$$mc^2=-\,\eta\,, \qquad mc^2=\eta\,. \tag{4.255}$$

On the other hand, it is obvious that the term "rest-mass" is not applicable to massless particles — they are always on the move. Their relativistic masses are defined from energy equivalent $E=mc^2$, measured in electron-volts. Consequently, massless particles have no rest spin-energy $\eta_0 = n\hbar^{mn}A_{mn}$.

Nevertheless, the Planck tensor found in spin-energy η enables quantization of relativistic masses of massless particles and angular velocities of the space rotation. Hence to obtain angular velocities of the space rotation for massless particles we need an expanded formula of their relativistic spin-energy η, which would not contain the relativistic square root.

Quantum Mechanics speaks of "spirality" of massless particles — the projection of spin on the direction of momentum. The reason for introducing this term is the fact that massless particles can not be at rest in respect of any regular observer, as they always travel at the light velocity with respect to him. Hence we can assume that spin of any massless particle is tangential to its light-like trajectory (either co-directed or oppositely directed to it).

Keeping in mind that the spin quantum number n of any massless particle is 1, we assume that for the massless particles

$$\eta = \hbar^{mn} \tilde{A}_{mn} \,, \tag{4.256}$$

where $\tilde{A}_{mn}$ is the angular velocities chr.inv.-tensor of their space rotation (the light-like space).

Hence to obtain the relativistic spin-energy of a massless particle (4.256) we need to find components of the angular velocities chr.inv.-tensor of the light-like space rotation. We are going to build the tensor similar to the four-dimensional tensor of the space rotation $A^{\alpha\beta}$ (4.11), which describes rotation of the space of a frame of reference, which travels with respect to the observer at an arbitrary velocity (a non-accompanying reference frame). As a result we obtain

$$\tilde{A}^{\alpha\beta} = \frac{1}{2}\, c\tilde{h}^{\alpha\mu}\tilde{h}^{\beta\mu}\tilde{a}_{\mu\nu} \,, \qquad \tilde{a}_{\mu\nu} = \frac{\partial \tilde{b}_{\nu}}{\partial x^{\mu}} - \frac{\partial \tilde{b}_{\mu}}{\partial x^{\nu}} \,, \tag{4.257}$$

where $\tilde{b}^{\alpha}$ is the four-dimensional velocity of a light-like reference frame with respect to the observer and

$$\tilde{h}^{\alpha\mu} = -\, g^{\alpha\mu} + \tilde{b}^{\alpha}\tilde{b}^{\mu} \tag{4.258}$$

is the four-dimensional generalization of the metric chr.inv.-tensor for the light-like space and a reference frame located in it.

The space inhabited by massless particles is a space-time area, which corresponds to the four-dimensional light-like (isotropic) cone set by the equation $g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = 0$. This cone exists at any point of the four-dimensional pseudo-Riemannian space with the signature $(+---)$.

The four-dimensional velocity vector of the light-like reference frame of massless particles is

$$\tilde{b}^{\alpha} = \frac{dx^{\alpha}}{d\sigma} = \frac{dx^{\alpha}}{cd\tau} \,, \qquad \tilde{b}_{\alpha}\tilde{b}^{\alpha} = 0 \,, \tag{4.259}$$

so its chr.inv.-projections in the reference frame of a regular "sub-light" observer are

$$\frac{\tilde{b}_0}{\sqrt{g_{00}}} = \pm 1, \qquad \tilde{b}^{i} = \frac{1}{c}\frac{dx^{i}}{d\tau} = \frac{1}{c}\, c^{i}, \tag{4.260}$$

while the other components of this isotropic vector (4.259) are

$$\tilde{b}^0 = \frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\,v_i c^i \pm 1\right), \qquad \tilde{b}_i = -\frac{1}{c}\left(c_i \pm v_i\right), \tag{4.261}$$

where c^i is the chr.inv.-vector of the light velocity.

Let us consider properties of the light-like space of massless particles in details. The isotropy condition of the particles' four-dimensional velocity $b_\alpha b^\alpha = 0$ in chr.inv.-form becomes

$$h_{ik}\,c^i c^k = c^2 = const, \tag{4.262}$$

where h_{ik} is the metric chr.inv.-tensor of a regular "sub-light" observer's reference space. Components of the four-dimensional light-like metric tensor $\tilde{h}^{\alpha\beta}$ (4.258), whose three-dimensional components make up the light-like space's metric chr.inv.-tensor $\tilde{h}^{ik}$, are

$$\left.\begin{aligned} \tilde{h}^{00} &= \frac{v_k v^k \pm 2v_k c^k + \frac{1}{c^2}\,v_k v_n c^k c^n}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)^2} \\ \tilde{h}^{0i} &= \frac{v^i \pm c^i + \frac{1}{c^2}\,v_k c^k c^i}{c\left(1-\frac{\mathrm{w}}{c^2}\right)}\,, \qquad \tilde{h}^{ik} = h^{ik} + \frac{1}{c^2}\,c^i c^k \end{aligned}\right\}, \tag{4.263}$$

where "plus" stands for the light-like space with the direct flow of time (our world) and "minus" stands for the reverse-time (mirror) world.

Now we have to deduce components of the curl of the four-dimensional velocity vector of massless particles, found in the formula (4.257). After some algebra we obtain

$$\left.\begin{aligned} \tilde{a}_{00} &= 0\,, \quad \tilde{a}_{0i} = \frac{1}{2c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(\pm F_i - \frac{^*\partial c_i}{\partial t}\right) \\ \tilde{a}_{ik} &= \frac{1}{2c}\left(\frac{\partial c_i}{\partial x^k} - \frac{\partial c_k}{\partial x^i}\right) \pm \frac{1}{2c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right) \end{aligned}\right\}. \tag{4.264}$$

Generally, to define the spin-energy of a massless particle (4.256) we need covariant spatial components of the tensor of its space rotation, namely — lower-indices components $\tilde{A}_{ik}$. To deduce them we take the formula for contravariant components $\tilde{A}^{ik}$ and lower their indices, as for any chr.inv.-quantity using the metric chr.inv.-tensor of the observer's reference space.

Substituting into

$$\tilde{A}^{ik} = c\left(\tilde{h}^{i0}\tilde{h}^{k0}\,\tilde{a}_{00} + \tilde{h}^{i0}\tilde{h}^{km}\,\tilde{a}_{0m} + \tilde{h}^{im}\tilde{h}^{k0}\,\tilde{a}_{m0} + \tilde{h}^{im}\tilde{h}^{kn}\,\tilde{a}_{mn}\right) \tag{4.265}$$

the obtained components $\tilde{h}^{\alpha\beta}$ and $\tilde{a}_{\alpha\beta}$, we arrive at

$$\begin{aligned}\tilde{A}^{ik} = h^{im}h^{kn}\left[\frac{1}{2}\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n c_m - F_m c_n\right)\right] \pm \\ \pm\, h^{im}h^{kn}\left[\frac{1}{2}\left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n v_m - F_m v_n\right)\right] + \\ + \left(\frac{1}{c^2}\, v_n c^n \pm 1\right)\left(c^k h^{im} - c^i h^{km}\right)\frac{{}^*\partial c_m}{\partial t} - \\ - \left(v^k h^{im} - v^i h^{km}\right)\frac{{}^*\partial c_m}{\partial t} + \frac{1}{2c^2}\, c^m\left(c^i h^{kn} - c^k h^{in}\right) \times \\ \times \left[\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right) \pm \left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right)\right].\end{aligned} \tag{4.266}$$

In this formula, the quantity $\frac{1}{2}\left(\frac{\partial v_m}{\partial x^n} - \frac{\partial v_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n v_m - F_m v_n\right)$, by definition, is the chr.inv.-tensor of angular velocities of the observer's space rotation A_{mn}, which is the non-holonomity tensor of the *non-isotropic space* * in the same time.

The quantity $\frac{1}{2}\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right) + \frac{1}{2c^2}\left(F_n c_m - F_m c_n\right)$ by its structure is similar to the tensor A_{mn}, but instead of the linear velocity of the non-isotropic space rotation v_i it has components of the covariant chr.inv.-vector of the light velocity $c_m = h_{mn}c^n$. The vector c_m is a physical observable quantity, because it was obtained by lowering indices in the chr.inv.-vector c^n using the metric chr.inv.-tensor h_{mn}. We denote that tensor as $\breve{A}_{mn}$, where the inward curved cap means the quantity belongs to the *isotropic space*† with the direct flow of time — the "upper" part of the light cone, which in a twisted space-time gets "round" shape. Then we obtain

$$\breve{A}_{mn} = \frac{1}{2}\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right) + \frac{1}{2\,c^2}\left(F_n c_m - F_m c_n\right). \tag{4.267}$$

*We will refer to an area in the four-dimensional space-time, where particles with non-zero rest-masses exist as a *non-isotropic space*. This is the area of world-trajectories along which $ds \neq 0$. Subsequently, if the interval ds is real, then the particles travel at sub-light velocities (regular particles); if it is imaginary, then the particles travel at super-light velocities (tachyons). So, the space of both sub-light particles and super-light tachyons is non-isotropic by definition.

†We will refer as the *isotropic space* to an area of the four-dimensional space-time, inhabited by massless (light-like) particles. This area can be also-called the *light membrane*. From geometric viewpoint the light membrane is the surface of the isotropic cone, i. e. the set of its four-dimensional elements (world-lines of the light propagation).

In a particular case, where gravitational potential is negligible (i. e. where $\mathrm{w} \approx 0$) the tensor becomes

$$\breve{A}_{mn} = \frac{1}{2}\left(\frac{\partial c_m}{\partial x^n} - \frac{\partial c_n}{\partial x^m}\right), \tag{4.268}$$

so it is the chr.inv.-curl of the light velocity. Therefore we will refer to $\breve{A}_{mn}$ as the *isotropic space curl*.

The following example gives geometric illustration of the isotropic space curl. As it is known, the necessary and sufficient condition of the equality $A_{mn}=0$ (the space holonomity condition) is equality to zero of all components $v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}$, i. e. the absence of space rotation. The tensor $\breve{A}_{mn}$ is defined only in the isotropic space, inhabited by massless particles. Outside the isotropic space it is senseless, because the "interior" of the light cone is inhabited by sub-light particles, while tachyons inhabit its "exterior".

Our subject here is massless particles (photons). From (4.268) it is seen that non-holonomity of the isotropic space is linked to curl nature of the linear velocity of massless particles c_m. Hence any photon is a spatial curl of the isotropic space, while the photon's spin results from interaction between its inner curl field and the external tensor field $\breve{A}_{mn}$.

To make the explanations even more illustrative, we depict areas of existence of different kinds of particles. The light cone exists in every point of the space. The light cone equation $g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}=0$ in chr.inv.-notation is

$$c^2\tau^2 - h_{ik}\,x^i x^k = 0\,, \qquad h_{ik}\,x^i x^k = \sigma^2. \tag{4.269}$$

On Minkowski's diagram the light cone "interior" is filled with the non-isotropic space, where sub-light particles exist. Outside there is also an area of the non-isotropic space, inhabited by super-light particles (tachyons). The specific space of massless particles is a *space-time membrane* between these two non-isotropic areas. The picture is mirror-symmetric: in the upper part of the cone there is the sub-light space with the direct flow of time (our world), separated with the observer's spatial section from the lower part — the sub-light space with the reverse flow of time (the mirror world). In other words, the upper part is inhabited by real particles with positive masses and energies, while the lower part is inhabited by their mirror "counterparts", whose masses and energies are negative (from our viewpoint).

Therefore, rotation of the sub-light non-isotropic space "inside" the cone involves the surrounding light membrane (the isotropic space). As a result, the light cone begins rotation described by the tensor $\breve{A}_{mn}$

— the isotropic space curl. Of course we can assume a reverse order of events, where rotation of the light cone involves "the content" of its inner part. But because particles "inside" the cone have non-zero rest-masses, they are "heavier" than massless particles on the light membrane. Hence the inner "content" of the light cone is also an inertial media.

Now we return to the formula for the relativistic spin-energy of a massless particle $\eta = \hbar^{mn}\tilde{A}_{mn}$ (4.256). By lowering indices in the non-holonomity tensor of the isotropic space $\tilde{A}^{ik}$ (4.266), we obtain

$$\begin{aligned}\tilde{A}_{ik} = \pm A_{ik} + \breve{A}_{ik} + \frac{1}{2c^2}c^m\Bigg\{c_i\left[\frac{\partial\,(c_m \pm v_m)}{\partial x^k} - \frac{\partial\,(c_k \pm v_k)}{\partial x^m}\right] - \\ - c_k\left[\frac{\partial\,(c_m \pm v_m)}{\partial x^i} - \frac{\partial\,(c_i \pm v_i)}{\partial x^m}\right]\Bigg\} + \left(v_i\frac{{}^*\partial c_k}{\partial t} - v_k\frac{{}^*\partial c_i}{\partial t}\right) + \\ + \left(\frac{1}{c^2}v_n v^n \pm 1\right)\left(c_k\frac{{}^*\partial c_i}{\partial t} - c_i\frac{{}^*\partial c_k}{\partial t}\right).\end{aligned} \tag{4.270}$$

Having $\tilde{A}_{ik}$ contracted with the Planck tensor $\hbar^{ik}$, we have

$$\begin{aligned}\eta = \eta_0 + n\hbar^{ik}\breve{A}_{ik} + \left[\left(\frac{1}{c^2}v_n v^n \pm 1\right)\left(c_k\frac{{}^*\partial c_i}{\partial t} - c_i\frac{{}^*\partial c_k}{\partial t}\right) + \right. \\ \left. + \left(v_i\frac{{}^*\partial c_k}{\partial t} - v_k\frac{{}^*\partial c_i}{\partial t}\right)\right] n\hbar^{ik} + \frac{1}{2c^2}n\hbar^{ik}c^m\Bigg\{c_i\left[\frac{\partial\,(c_m \pm v_m)}{\partial x^k} - \right. \\ \left. - \frac{\partial\,(c_k \pm v_k)}{\partial x^m}\right] - c_k\left[\frac{\partial\,(c_m \pm v_m)}{\partial x^i} - \frac{\partial\,(c_i \pm v_i)}{\partial x^m}\right]\Bigg\},\end{aligned} \tag{4.271}$$

where "plus" stands for our world and "minus" — for the mirror world.

The quantity $\eta_0 = \eta\sqrt{1 - \mathrm{v}^2/c^2}$ for massless particles is zero, because they travel at the light velocity. Hence keeping in mind that $\eta_0 = n\hbar^{mn}A_{mn}$, we obtain an additional condition imposed on the non-holonomity tensor of the isotropic space $\tilde{A}_{ik}$: at any point of the trajectory of any massless particle the condition

$$\hbar^{mn}A_{mn} = 2\hbar\,(A_{12} + A_{23} + A_{31}) = 0\,, \tag{4.272}$$

must be true. Or, in the other notation, $\Omega^1 + \Omega^2 + \Omega^3 = 0$.

Therefore, in an area, where the observer "sees" the massless particle, the angular velocity of rotation of the observer's non-isotropic space equals zero. Other terms consisting the particle's relativistic spin-energy (4.271) are due to possible non-stationary nature of the light velocity $\frac{{}^*\partial c_i}{\partial t}$ and other dependencies which include squares of the light velocity.

We analyze the obtained formula (4.271) to make two simplification assumptions:

a) Gravitational potential is negligible ($\mathrm{w} \approx 0$);

b) The three-dimensional chr.inv.-velocity of light is stationary.

In this case the quantities A_{ik} and $\breve{A}_{ik}$ (the observer's space non-holonomity tensor and the isotropic space curl) become

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right), \qquad \breve{A}_{ik} = \frac{1}{2}\left(\frac{\partial c_k}{\partial x^i} - \frac{\partial c_i}{\partial x^k}\right), \tag{4.273}$$

and the massless particle's relativistic spin-energy (4.271) becomes

$$\eta = n\left(\hbar^{ik}\breve{A}_{ik} + \frac{1}{c^2}\, c_i c^m \hbar^{ik}\breve{A}_{km}\right). \tag{4.274}$$

Therefore this quantity η, describing action of the massless particle's spin, is defined (aside from the spin) only by the isotropic space curl and in no way depends on the observer's space non-holonomity (the rotation).

To make further deductions simpler, we transform η (4.274) as follows. Similar to the space rotation pseudovector $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm} A_{km}$ we introduce a pseudovector

$$\breve{\Omega}^{*i} = \frac{1}{2}\,\varepsilon^{ikm}\breve{A}_{km}\,, \tag{4.275}$$

which can be formally interpreted as the pseudovector of rotation angular velocity of the isotropic space.

Subsequently, $\breve{A}_{km} = \varepsilon_{kmn}\breve{\Omega}^{*n}$. Then the formula for η (4.274) can be presented as follows

$$\eta = n\left(\hbar_{*i}\breve{\Omega}^{*i} + \frac{1}{c^2}\, c_i c^m \hbar^{ik}\,\varepsilon_{kmn}\breve{\Omega}^{*n}\right). \tag{4.276}$$

This means that the inner mechanical curl (spin) of a massless particle only reveals itself in interaction with the isotropic space curl. The result of the interaction is the scalar product $\hbar_{*i}\breve{\Omega}^{*i}$, to which the massless particle's spin is attributed. Hence massless particles are elementary light-like curls of the isotropic space itself.

Let us estimate rotations of the isotropic space for massless particles having different energies. At present we know for sure that among massless particles are photons — the quanta of an electromagnetic field.

Any photon's spin quantum number is 1. Besides, its energy $E = \hbar\,\omega$ is positive in our world. Hence taking into account the live forces integral (4.255), for observable our-world photons we have

$$\hbar\omega = \hbar_{*i}\breve{\Omega}^{*i} + \frac{1}{c^2}\, c_i c^m \hbar^{ik}\,\varepsilon_{kmn}\breve{\Omega}^{*n}. \tag{4.277}$$

Kind of photons	Frequency $\breve{\Omega}$, s^{-1}
Radiowaves	$10^3 \div 10^{11}$
Infra-red rays	$10^{11} \div 1.2\times10^{15}$
Visible light	$1.2\times10^{15} \div 2.4\times10^{15}$
Ultraviolet rays	$2.4\times10^{15} \div 10^{17}$
X-rays	$10^{17} \div 10^{19}$
Gamma rays	$10^{19} \div 10^{23}$ and above

Table 4.2: Rotation frequencies of the isotropic space, which correspond to photons.

We assume that the rotation pseudovector of the isotropic space Ω^{*i} is directed along the z axis, while the light velocity is directed along y. Then the relationship (4.277) obtained for photons becomes $\hbar\omega = 2\hbar\breve{\Omega}$, or, after having the Planck constant cancelled,

$$\breve{\Omega} = \frac{\omega}{2} = \frac{2\pi\nu}{2} = \pi\nu\,, \tag{4.278}$$

so the frequency $\breve{\Omega}$ of the isotropic space rotation for massless particle is constant and coincides the particle's own frequency ν. Thanks to this formula, which results from the quantization law for relativistic masses of massless particles, we can estimate the isotropic space's angular velocities, which correspond to photons of different energy levels. Table 4.2 gives the results.

From Table 4.2 we see that angular velocities of the isotropic space rotation in photons, taken in the gamma range, are of the order of frequencies of the regular space rotation in electrons and other elementary particles (see Table 4.1).

§4.11 Conclusions

Here is what we have obtained in this Chapter.

Spin of any particle is characterized by the four-dimensional antisymmetric tensor of the 2nd rank called the Planck tensor. Its diagonal and space-time components are zeroes, while non-diagonal spatial components are $\pm\hbar$ depending on the spatial direction of the spin and our choice of a right or left-handed frame of reference.

The spin (the inner vortical field of the particle) interacts with an external field of the space non-holonomity. As a result, the particle gains an additional momentum, which deviates the moving particle from geodesic line. This interaction energy is found from the scalar chr.inv.-equation of motion of the particle (the live forces theorem), so the equation must be taken into account when solving the vector chr.inv.-equations of motion.

Particular solution of the scalar chr.inv.-equation is the law of quantization of masses of elementary spin-particles, which unambiguously links rest-masses of mass-bearing elementary particles with angular velocities of the observer's space rotation, as well as between relativistic masses of photons and angular velocities of rotation of their inner light-like space. Because an area, where light-like particles exist, is the area of four-dimensional isotropic trajectories, the terms "isotropic space" and "light-like space" can be used as synonyms.

Please note that we have obtained the result using only geometric methods of the General Theory of Relativity, not Quantum Mechanics' methods. In future, this result may possibly become a "bridge" between these two theories.

Chapter 5 Physical Vacuum

§5.1 Introduction

According to recent data, the average density of matter in our Universe is $\sim 5\div 10\times 10^{-30}$ gram/cm^3. The average density of substances concentrated in galaxies is higher, $\sim 10^{-24}$ gram/cm^3 in our Galaxy. Astronomical observations show that most part of the cosmic mass is accumulated in compact objects, e. g. in stars, whose total volume is incomparable to that of the whole Universe (so-called "island" distribution of substance). We can therefore assume that our Universe is predominantly empty.

For a long time the words "emptiness" and "vacuum" have been considered synonyms. But since the 1920's the geometric methods of the General Theory of Relativity have showed that they are different states of matter.

Distribution of matter in the Universe is characterized by the energy-momentum tensor, which is linked to the geometric structure of the space-time (the fundamental metric tensor) by the *equations of gravitational field*. In Einstein's theory of gravitation, which is an application of mathematical methods of Riemannian geometry, the equations referred to as *Einstein's equations* are*

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{5.1}$$

These equations, except for the energy-momentum tensor and the fundamental metric tensor, include other quantities, namely:

1) $R_{\alpha\sigma} = R_{\alpha\beta\sigma\cdot}^{\cdot\cdot\cdot\beta}$ is Ricci's tensor†, which is the result of contraction of Riemann-Christoffel's curvature tensor $R_{\alpha\beta\gamma\delta}$ by two indices;
2) $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature;

---

*The left hand side of the field equations (5.1) is often referred to as the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R$, in notation $G_{\alpha\beta} = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$.

†Gregorio Ricci-Curbastro (1853–1925), an Italian mathematician who was the teacher of Tullio Levi-Civita in Padua in the 1890's.

3) $\varkappa = \frac{8\pi G}{c^2} = 1.862 \times 10^{-27}$ [cm/gram] is Einstein's gravitational constant, while $G = 6.672 \times 10^{-8}$ [$\text{cm}^3/\text{gram sec}^2$] is Gauss' gravitational constant. Note that some researchers such as Landau and Lifshitz [10] prefer to use $\varkappa = \frac{8\pi G}{c^4}$ instead of $\varkappa = \frac{8\pi G}{c^2}$ as used by Zelmanov and other people. To understand the reason, why not $\varkappa = \frac{8\pi G}{c^4}$ is in our study, we have to look at chr.inv.-projections of the energy-momentum tensor $T_{\alpha\beta}$, namely: $\frac{T_{00}}{g_{00}} = \rho$ is the chr.inv.-scalar of the observable mass density, $\frac{c\,T_0^i}{\sqrt{g_{00}}} = J^i$ is the chr.inv.-vector of the observable momentum density, and $c^2 T^{ik} = U^{ik}$ is the chr.inv.-tensor of the observable momentum flux density [9]. Accordingly, the scalar chr.inv.-projection of the Einstein equations is $\frac{G_{00}}{g_{00}} = -\frac{\varkappa T_{00}}{g_{00}} + \lambda$. As it is known, the Ricci tensor has dimension [cm^{-2}], hence the Einstein tensor $G_{\alpha\beta}$ and the quantity $\frac{\varkappa T_{00}}{g_{00}} = \frac{8\pi G\rho}{c^2}$ have the same dimension. Consequently, it is evident that the dimension of the energy-momentum tensor $T_{\alpha\beta}$ is that of mass density [gram/cm^3]. This implies that when we use $\frac{8\pi G}{c^4}$ on the right hand side of the Einstein equations, we actually use not the energy-momentum tensor itself, but the quantity $c^2 T_{\alpha\beta}$, whose scalar and vector chr.inv.-projections are the observable energy density $\frac{c^2 T_{00}}{g_{00}} = \rho c^2$ and the observable energy flux $\frac{c^3 T_0^i}{\sqrt{g_{00}}} = c^2 J^i$;

4) λ [cm^{-2}] is the so-called *cosmological term*, which describes non-Newtonian forces of attraction or repulsion, depending on the sign before λ ($\lambda > 0$ stands for repulsion, $\lambda < 0$ stands for attraction). The term is referred to as "cosmological", because it is assumed that forces described by λ grow up proportionally to distance and therefore reveal themselves in a full scale at "cosmological" distances comparable to the size of the Universe. Because the non-Newtonian gravitational fields (λ-fields) have never been observed, for our Universe in general the cosmological term is $|\lambda| < 10^{-56}\,\text{cm}^{-2}$ (as of today's measurement accuracy).

From the Einstein equations (5.1) we see that the energy-momentum tensor describing distribution of matter is genetically linked to both the metric tensor and the Ricci tensor, and hence to the Riemann-Christoffel curvature tensor. Equality of the Riemann-Christoffel tensor to zero is the necessary and sufficient condition for the given space-time to be flat. The Riemann-Christoffel tensor is not zero for curved spaces only. It reveals itself as an increment of an arbitrary vector V^α in its parallel transfer along a closed contour

$$\Delta V^{\mu} = -\frac{1}{2}\, R_{\alpha\beta\gamma\cdot}^{\cdot\cdot\cdot\mu}\, V^{\alpha} \Delta\sigma^{\beta\gamma}, \tag{5.2}$$

where $\Delta\sigma^{\beta\gamma}$ is the area within this contour. As a result, the initial vector V^{α} and the vector $V^{\alpha}+\Delta V^{\alpha}$ have different directions. From quantitative viewpoint the difference is described by a quantity K, referred to as the *four-dimensional curvature* of the pseudo-Riemannian space along the given parallel transfer (see [9] for details)

$$K = \lim_{\Delta\sigma\to 0} \frac{\tan\varphi}{\Delta\sigma}\,, \tag{5.3}$$

where $\tan\varphi$ is the tangent of the angle between the vector V^{α} and the projection of the vector $V^{\alpha}+\Delta V^{\alpha}$ on the area constructed by the transfer contour. For instance, we consider a surface and a "geodesic" triangle on it, produced by crossing three geodesic lines. We transfer a vector, defined in any arbitrary point of that triangle, parallel to itself along the sides of the triangle. The summary rotation angle φ after the vector returns to the initial point is $\varphi=\Sigma-\pi$ (where Σ is the sum of the inner angles of the triangle). We assume the surface curvature K is equal at all its points, then

$$K = \lim_{\Delta\sigma\to 0} \frac{\tan\varphi}{\Delta\sigma} = \frac{\varphi}{\sigma} = const, \tag{5.4}$$

where σ is the triangle's area and $\varphi=K\sigma$ is called *spherical excess*. If $\varphi=0$, then the curvature is $K=0$, so the surface is flat. In this case the sum of all inner angles of the geodesic triangle is π (a flat space). If $\Sigma>\pi$ (the transferred vector is rotated towards the circuit), then there is positive spherical excess, so the curvature $K>0$. An example of such a space is the surface of a sphere: a triangle on the surface is convex. If $\Sigma<\pi$ (the transferred vector is rotated counter the circuit), the spherical excess is negative and the curvature is $K<0$.

Einstein postulated that gravitation is the space-time curvature. He understood the curvature as not equality to zero of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}\neq 0$ (the same is assumed in Riemannian geometry). This concept fully includes Newtonian gravitational concept, so Einstein's four-dimensional gravitation-curvature for a regular physical observer can reveal itself as follows:

a) Newtonian gravitation;

b) Rotation of the three-dimensional space (the spatial section);

c) Deformation of the three-dimensional space;

d) The three-dimensional curvature, so that there are non-zero first derivatives of Christoffel's symbols.

According to Mach's Principle, on which the Einstein theory of gravitation rests, "...the property of inertia is fully determined by interaction of matter" [29], so the space-time curvature is produced by matter which fills it. Proceeding from that and from the Einstein equations (5.1) we can give *mathematical definitions of emptiness and vacuum*:

EMPTINESS is the state of a given space-time, for which the Ricci tensor is $R_{\alpha\beta}=0$, this implies the absence of any substance $T_{\alpha\beta}=0$ and of non-Newtonian gravitational fields $\lambda=0$. The field equations(5.1) in emptiness* are as simple as $R_{\alpha\beta}=0$;

VACUUM is the state in which any substance is absent $T_{\alpha\beta}=0$, but $\lambda\neq 0$ and hence $R_{\alpha\beta}\neq 0$. Emptiness is a particular case of vacuum in the absence of λ-fields. The field equations in vacuum are

$$R_{\alpha\beta}-\frac{1}{2}\,g_{\alpha\beta}R=\lambda g_{\alpha\beta}\,. \tag{5.5}$$

The Einstein equations are applicable to the most varied cases of distribution of matter, except for the cases where the density is close to that of substance in atomic nuclei. It is hard to give accurate mathematical description to all cases of distribution of matter because such a problem is so general and it can not be approached *per se*. On the other hand, the average density of substance in our Universe is so small $5\div 10\times 10^{-30}$ gram/cm^3, that we can assume it near vacuum. The Einstein equations say that the energy-momentum tensor is functionally dependent on the metric tensor and the Ricci tensor (i. e. from the curvature tensor, contracted by two indices). At such small numerical values of density we can assume the energy-momentum tensor proportional to the metric tensor $T_{\alpha\beta}\sim g_{\alpha\beta}$ and hence proportional to the Ricci tensor. Therefore, besides the field equations in vacuum (5.5) we can consider the equations

$$R_{\alpha\beta}=k\,g_{\alpha\beta}\,,\qquad k=const, \tag{5.6}$$

i. e. where the energy-momentum tensor is different from the metric tensor only by a constant. This case, including the absence of masses

*If we put down the Einstein equations for an empty space $R_{\alpha\beta}-\frac{1}{2}\,g_{\alpha\beta}R=0$ in the mixed form $R^{\beta}_{\alpha}-\frac{1}{2}\,g^{\beta}_{\alpha}R=0$, then after contraction ($R^{\alpha}_{\alpha}-\frac{1}{2}g^{\alpha}_{\alpha}R=0$) we obtain $R-\frac{1}{2}\,4R=0$. So the scalar curvature in emptiness is $R=0$. Hence the field equations (the Einstein equations) in the empty space are $R_{\alpha\beta}=0$.

(i. e. in vacuum) and some conditions close to it and related to our Universe, were studied in details by Petrov [30,31]. He called spaces for which the energy-momentum tensor is proportional to the metric tensor (and, hence, to the Ricci tensor) *Einstein spaces.*

Substance in spaces with $R_{\alpha\beta}=kg_{\alpha\beta}$ (namely — Einstein spaces) is homogeneous at every point, have no mass fluxes, while the density of matter which fills them (including any substances) is everywhere constant. In this case

$$R = g^{\alpha\beta}R_{\alpha\beta} = k\,g_{\alpha\beta}g^{\alpha\beta} = 4k\,, \tag{5.7}$$

while the Einstein tensor takes the form

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}R = -\,k\,g_{\alpha\beta}\,, \tag{5.8}$$

where $kg_{\alpha\beta}$ is the analog of the energy-momentum tensor for that matter which fills Einstein spaces.

To find out what kinds of matter fill Einstein spaces, Petrov studied the algebraic structure of the energy-momentum tensor. This is what he did: the tensor $T_{\alpha\beta}$ was compared to the metric tensor in an arbitrary point; for this point the difference $T_{\alpha\beta}-\xi g_{\alpha\beta}$ is calculated, where ξ are the so-called eigenvalues of the matrix $T_{\alpha\beta}$; the difference is equated to zero to find the values of ξ, which make the equality true. This problem is also referred to as the matrix eigenvalues problem*. The set of the matrix eigenvalues allows us to define the algebraic kind of this matrix. For a sign-constant metric this problem had been solved already, but Petrov proposed a method to bring any matrix to canonical form for the sign-alternating metric, which allowed using it in the pseudo-Riemannian space, in particular, to study the algebraic structure of the energy-momentum tensor. This can be illustrated as follows. Eigenvalues of elements of the matrix $T_{\alpha\beta}$ are similar to basic vectors of the metric tensor matrix, so the eigenvalues define a kind of "skeleton" of the tensor $T_{\alpha\beta}$ (the skeleton of matter); but even if we know what the skeleton is, we may not know exactly what the muscles are. Nevertheless, the structure of such a skeleton (the length and mutual direction of the basic vectors)can be depicted based on the properties of matter, such as homogeneity or isotropy, and their relation to the space curvature.

As a result, Petrov had shown that Einstein spaces have three basic algebraic kinds of the energy-momentum tensor and a few subtypes.

*Generally, the problem should be solved at a given point, but the obtained result is applicable to any point of the space.

According to his algebraic classification of the energy-momentum tensor and the curvature tensor, all Einstein spaces are sub-divided into three basic kinds so-called *Petrov's classification**.

Einstein spaces of the kind I are best intuitively comprehensible, because the field of gravitation there is produced by a massive island (the "island" distribution of substance), while the space itself may be empty or filled with vacuum. The curvature of such a space is created by the island mass and by vacuum. At the infinite distance from the island mass, in the absence of vacuum, this space remains flat. Devoid of any island masses but filled with vacuum, the space of the kind I also has curvature (e. g. de Sitter's space). An empty space of the kind I, i. e. the one devoid of any island masses or vacuum, is flat.

Einstein spaces of the kind II and of the kind III are more exotic, because they are curved by themselves. Their curvature is not related to the island distribution of masses or the presence of vacuum. The kind II and the kind III are generally attributed to radiation fields, for instance, to gravitational waves.

A few years later Gliner [33–35] in his study of the algebraic structure of the energy-momentum tensor of vacuum-like states of matter ($T_{\alpha\beta} \sim g_{\alpha\beta}$, $R_{\alpha\beta} = k g_{\alpha\beta}$) outlined its special kind for which all four eigenvalues are the same, so three space vectors and the time vector of the "ortho-reference" of the tensor $T_{\alpha\beta}$ are equal to each other†. The matter which corresponds to the energy-momentum tensor of such a structure has a constant density $\mu = const$, equal to coinciding eigenvalues of the energy-momentum tensor $\mu = \xi$ (the dimension of μ is the same as that of $T_{\alpha\beta}$ [gram/cm^3]). The energy-momentum tensor in this case is‡

$$T_{\alpha\beta} = \mu\, g_{\alpha\beta}\,. \tag{5.9}$$

The field equations under $\lambda = 0$ are

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa \mu\, g_{\alpha\beta}\,, \tag{5.10}$$

*Chr.inv.-interpretation of this algebraic classification of Einstein spaces (or, in other words, of Petrov's gravitational fields) had been obtained in 1970 by a co-author of this book (Borissova, née Grigoreva [32]).

†If we introduce a local flat space, tangential to the given Riemannian space at a given point, then the eigenvalues ξ of the tensor $T_{\alpha\beta}$ are the quantities in an ortho-reference, corresponded to this tensor, in contrast to the eigenvalues of the metric tensor $g_{\alpha\beta}$ in an ortho-reference, defined in this tangential space.

‡Gliner used the signature $(-+++)$, hence he had $T_{\alpha\beta} = -\mu g_{\alpha\beta}$. So because the observable density is positive $\rho = \frac{T_{00}}{g_{00}} = -\mu > 0$, he had negative numerical values of the μ. In our book we use the signature $(+---)$, because in this case three-dimensional observable interval is positive. Hence we have $\mu > 0$ and $T_{\alpha\beta} = \mu g_{\alpha\beta}$.

and, under the cosmological term $\lambda \neq 0$, are

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\mu\, g_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{5.11}$$

Gliner called this state of matter *μ-vacuum* [33–35]: this is because the state is related to vacuum-like states of substance ($T_{\alpha\beta} \sim g_{\alpha\beta}$, $R_{\alpha\beta} = k g_{\alpha\beta}$), but is not exactly vacuum (in vacuum $T_{\alpha\beta} = 0$). At the same time Gliner showed that spaces filled with μ-vacuum are Einstein spaces, so three basic kinds of μ-vacuum exist, which correspond to three basic algebraic kinds of the energy-momentum tensor (and the curvature tensor). In other words, an Einstein space of each kind (I, II, and III), provided matter is present in them, is filled with μ-vacuum of the corresponding kind (I, II, or III).

Actually, because being taken in the "ortho-reference" of the energy-momentum tensor of μ-vacuum all three space vectors and the time vector are the same (all the four directions have the same significance), μ-vacuum is the highest degree of isotropic matter. Besides, because Einstein spaces are homogeneous, so that the matter density at every point is everywhere equal [30, 31], then μ-vacuum that fills them does not only have a constant density, but is homogeneous as well.

As we have seen, Einstein spaces can be filled with μ-vacuum, with regular vacuum $T_{\alpha\beta} = 0$ or with emptiness. Besides, there may exist isolated "islands" of mass, which also produce the space curvature. Therefore Einstein spaces of the kind I are the best illustration of our knowledge of our Universe as a whole. And thus to study geometry of our Universe and physical states of matter, which fills it, is the same as studying Einstein spaces of the kind I.

Petrov has proposed and proved a theorem (see §13 in [30]), we will refer to it as *Petrov's theorem*:

Petrov's theorem

Any space of a constant curvature is an Einstein space. < So that > . . . Einstein spaces of the kind II and of the kind III can not be constant curvature spaces.

Hence constant curvature spaces are Einstein spaces of the kind I, according to the Petrov classification. If $K = 0$ an Einstein space of the kind I is flat. This makes our study of vacuum and vacuum-like states of matter in the Universe even simpler, because by today we have well studied constant curvature spaces. These are *de Sitter spaces*, or, in other words, spaces with de Sitter's metric.

In any de Sitter space we have $T_{\alpha\beta} = 0$ and $\lambda \neq 0$, so it is filled with regular vacuum and does not contain "islands" of substance. On the other hand we know that the average density of matter in our Universe is rather low. Looking at it in general, we can neglect presence of occasional "islands" and inhomogeneities, which locally distort it. Hence our space can be generally assumed as a de Sitter space with the constant curvature radius equal to that of the Universe.

Theoretically a de Sitter space may have either a positive curvature $K>0$ or a negative curvature $K<0$. Analysis (see Synge's book) shows that in de Sitter worlds with $K<0$ time-like geodesic lines are closed: a test-particle repeats its motion again and again along the same trajectory. This brings to mind some ideas, which seem to be too "revolutionary" from the viewpoint of today's physics [36]. Consequently, most physicists (Synge, Gliner, Petrov, and others) have left negative curvature de Sitter spaces beyond the scope of their consideration.

As it is known, positive curvature Riemannian spaces are generalization of a regular sphere, while the negative curvature ones are generalization of Lobachewski-Bolyai space, an imaginary-radius sphere. In Poincaré interpretation negative curvature spaces reflect on the inner surface of a sphere. Using the methods of chronometric invariants, Zelmanov showed that in the pseudo-Riemannian space (its metric is sign-alternating) the three-dimensional observable curvature is negative to the Riemannian four-dimensional curvature. Because we perceive our planet as a sphere, the observable curvature is positive in our world. If any hypothetical beings inhabited the "inner" surface of the Earth, they would perceive it as concave and their world will be of negative curvature.

This illustration inspired some researchers for the idea of possible existence of our mirror twin, the *mirror Universe* inhabited by antipodes. Initially it was assumed that once our world has a positive curvature, the mirror Universe must be a negative curvature space. But Synge showed (see [36, Chapter VII]) that in a positive curvature de Sitter space space-like geodesic trajectories are open, while in a negative curvature de Sitter space they are closed. In other words, a negative curvature de Sitter space is not a mirror reflection of its positive curvature counterpart.

On the other hand, in our study [19] (see also §1.3 herein) we found another approach to the concept of the mirror Universe. This study considered motion of free particles with the reverse time flow. As a result it has been obtained that the observable scalar component of a particle's four-dimensional momentum vector is its negative relativistic mass. Noteworthy, the particles of "mirror" masses were obtained as a formal result of projecting their four-dimensional momentum on a regular observer's time line and the result was not related to changing sign of the space curvature: particles with either the direct or reverse flow of time may either exist in positive or negative curvature spaces.

These results obtained by geometric methods of the General Theory of Relativity inevitably affect our view of matter and cosmology of our Universe.

In §5.2 we are going to obtain the energy-momentum tensor of vacuum and in the same time a formula for its observable density. We will also introduce a classification of matter according to the obtained forms of the energy-momentum tensor (namely — *T-classification*). In §5.3 we are going to look at physical properties of vacuum in Einstein spaces of the kind I; in particular, we will discuss physical properties of vacuum in de Sitter space and make conclusions on the global structure of the Universe. Following this approach in §5.4 we will set forth the concept of origin and development of the Universe as a result of the *Inversion Explosion* from the pra-particle that possessed some specific properties. In §5.5 we will obtain a formula for non-Newtonian gravitational inertial force, which is proportional to distance, §5.6 and §5.7 will focus on collapse in a Schwarzschild space (gravitational collapse, a *gravitational collapsar*) and in a de Sitter space (inflational collapse, an *inflanton*). In §5.8 it will be shown that our Universe and the mirror Universe are worlds with mirror time that co-exist in a de Sitter space with four-dimensional negative curvature. Also we will set forth physical conditions, which allow transition through the membrane which separates our world and the mirror Universe.

§ 5.2 The observable density of vacuum. Introducing T-classification of matter

The Einstein equations (the field equations in Einstein's theory of gravitation) are functions which link the space curvature to distribution of matter. Generally they are $R_{\alpha\beta}-\frac{1}{2}\,g_{\alpha\beta}R=-\varkappa T_{\alpha\beta}+\lambda g_{\alpha\beta}$. The left hand side, as it is known, describes geometry of the space, while the right hand side describes matter filled into the space. The sign of the second term depends on that of λ. As we are going to see, the sign of λ, and so the behaviour of Newtonian gravitation (attraction or repulsion) is directly linked to the sign of the vacuum density.

Einstein spaces are defined by the condition $T_{\alpha\beta}\sim g_{\alpha\beta}$, the field equations for them are $R_{\alpha\beta}=kg_{\alpha\beta}$. Such field equations can exist in two cases: a) where $T_{\alpha\beta}\neq 0$, i. e. in a substance; b) where $T_{\alpha\beta}=0$, i. e. in vacuum. But because in Einstein spaces, filled with vacuum, the energy-momentum tensor equals zero, it can not be proportional to the metric tensor that contradicts the definition of Einstein spaces ($T_{\alpha\beta} \sim g_{\alpha\beta}$). So what is the problem here? In the absence of any substance, but in the presence of vacuum, the field equations are $R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = \lambda g_{\alpha\beta}$, so the space curvature is produced by λ-fields (non-Newtonian fields of gravitation) rather than by substances. In the absence of both substances and λ-fields we have $R_{\alpha\beta} = 0$, so the space is empty but generally it is not flat.

As a result we can see that λ-fields and vacuum are practically the same thing, so *vacuum is a non-Newtonian field of gravitation.* We will call this point of the theory the *physical definition of vacuum.* Hence λ-fields are action of vacuum potential.

This means that the term $\lambda g_{\alpha\beta}$ can not be lost in the field equations in vacuum, no matter how small it is, because it describes vacuum, which is one of the reasons that make the space curved. Then the field equations $R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$ can be put down as follows

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa \widetilde{T}_{\alpha\beta}\,, \tag{5.12}$$

on the right hand side of which the tensor

$$\widetilde{T}_{\alpha\beta} = T_{\alpha\beta} + \breve{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{\lambda}{\varkappa}\, g_{\alpha\beta} \tag{5.13}$$

is the energy-momentum tensor which describes matter in general (both substance and vacuum). The first term here is the energy-momentum tensor of the substance. The second term

$$\breve{T}_{\alpha\beta} = -\frac{\lambda}{\varkappa}\, g_{\alpha\beta} \tag{5.14}$$

is analogous to the energy-momentum tensor for vacuum.

Therefore, because Einstein spaces may be filled with vacuum, their mathematical definition is better to be set forth in a more general form to take care of the presence of both substance and vacuum (λ-fields): $\widetilde{T}_{\alpha\beta} \sim g_{\alpha\beta}$. In particular, doing this helps to avoid contradictions when considering Einstein empty spaces.

Note, the obtained formula for the energy-momentum tensor of vacuum (5.14) is a direct consequence of the field equations in general form.

If $\lambda > 0$ (the non-Newtonian forces of gravitation repulsion) the observable density of vacuum is negative

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = -\frac{\lambda}{\varkappa} = -\frac{|\lambda|}{\varkappa} < 0\,, \tag{5.15}$$

while if $\lambda < 0$ (the non-Newtonian forces of gravitation attraction) the observable density of vacuum is, to the contrary, positive

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = -\frac{\lambda}{\varkappa} = \frac{|\lambda|}{\varkappa} > 0\,. \tag{5.16}$$

The latter fact, as we will see in §5.3, is of great importance, because a de Sitter space with $\lambda < 0$, being a constant-negative curvature space* filled with vacuum only (no substance present), best fits our observation data on our Universe in general.

Therefore proceeding from studies by Petrov and Gliner and taking into account our note on existence of the energy-momentum tensor and, hence, physical properties in vacuum (λ-fields), we can set forth a "geometric" classification of states of matter according to its energy-momentum tensor. We will call this *T-classification of matter*:

I) Emptiness: $T_{\alpha\beta} = 0$, $\lambda = 0$ (a space-time without matter), so that the field equations are $R_{\alpha\beta} = 0$;

II) Vacuum: $T_{\alpha\beta} = 0$, $\lambda \neq 0$ (produced by λ-fields), so the field equations are $G_{\alpha\beta} = -\lambda\, g_{\alpha\beta}$;

III) μ-vacuum: $T_{\alpha\beta} = \mu\, g_{\alpha\beta}$, $\mu = const$ (a vacuum-like state of the substance, filled into the space), in this case the field equations are $G_{\alpha\beta} = -\varkappa\mu\, g_{\alpha\beta}$;

IV) Substance: $T_{\alpha\beta} \neq 0$, $T_{\alpha\beta} \not\sim g_{\alpha\beta}$ (this state comprises both a regular substance and electromagnetic fields).

Generally the energy-momentum tensor of substance (kind IV in T-classification) is not proportional to the metric tensor. On the other hand, there are states of substance in which the energy-momentum tensor contains a term proportional to the metric tensor, but because it also contains other terms so it is not μ-vacuum. Such, for instance, is an ideal fluid

$$T_{\alpha\beta} = \left(\rho - \frac{p}{c^2}\right) U_\alpha U_\beta - \frac{p}{c^2}\, g_{\alpha\beta}\,, \tag{5.17}$$

where p is the fluid pressure, and also electromagnetic fields

$$T_{\alpha\beta} = F_{\rho\sigma}F^{\rho\sigma} g_{\alpha\beta} - F_{\alpha\sigma}F_{\beta\cdot}^{\cdot\sigma}\,, \tag{5.18}$$

where $F_{\rho\sigma}F^{\rho\sigma}$ is the first invariant of an electromagnetic field under consideration (3.27), $F_{\alpha\beta}$ is the Maxwell tensor. If $p = \rho c^2$ (a substance inside atomic nuclei) and $p = const$, the energy-momentum tensor of the ideal fluid seems to be proportional to the metric tensor.

*We mean here the Riemannian four-dimensional curvature.

But in the next section, §5.3, we will show that the state equation of μ-vacuum has fully different form $p = -\rho c^2$ (the state of inflation, expansion of the media in the case of its positive density). Hence the pressure and density in atomic nuclei should not be constant as to prevent transition of their inner substance into a vacuum-like state.

Noteworthy, this T-classification, just like the field equations, is only about *distribution of matter* which affects the space curvature, but not about test-particles — material points whose masses and sizes are so small that their effect on the space curvature can be neglected. Therefore the energy-momentum tensor is not defined for particles, and they should be considered beyond this T-classification.

§5.3 The physical properties of vacuum. Cosmology

Einstein spaces are defined by field equations like $R_{\alpha\beta} = k\, g_{\alpha\beta}$, where $k = const$. With $\lambda \neq 0$ and $T_{\alpha\beta} = \mu\, g_{\alpha\beta}$ the space is filled with matter, whose energy-momentum tensor is proportional to the fundamental metric tensor, so this matter is μ-vacuum. As we saw in the previous section, §5.2, for vacuum the energy-momentum tensor is also proportional to the metric tensor. This implies that physical properties of vacuum and those of μ-vacuum are mostly the same, except for a scalar coefficient which defines the composition of the matter (λ-fields or a substance) and the absolute values of the acting forces. Therefore we are going to consider an Einstein space filled with vacuum and μ-vacuum. In this case the field equations become

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\left(\varkappa\mu - \lambda\right) g_{\alpha\beta}\,. \tag{5.19}$$

Putting them down in a mixed form and then contracting we arrive at the scalar curvature

$$R = 4\left(\varkappa\mu - \lambda\right), \tag{5.20}$$

substituting it into the initial equations (5.19) we obtain the field equations in their final form

$$R_{\alpha\beta} = \left(\varkappa\mu - \lambda\right) g_{\alpha\beta}\,, \tag{5.21}$$

where $\varkappa\mu - \lambda = const = k$.

Let us look at physical properties of vacuum and μ-vacuum. We deduce chr.inv.-projections of the energy-momentum tensor: the observable density of matter $\rho = \frac{T_{00}}{g_{00}}$, the observable density of momentum $J^i = \frac{c\, T_0^i}{\sqrt{g_{00}}}$, and the observable strength tensor $U^{ik} = c^2 T^{ik}$.

For the energy-momentum tensor of μ-vacuum $T_{\alpha\beta}=\mu\, g_{\alpha\beta}$ chr.inv.-projections are

$$\rho = \frac{T_{00}}{g_{00}} = \mu\,, \tag{5.22}$$

$$J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}} = 0\,, \tag{5.23}$$

$$U^{ik} = c^2 T^{ik} = -\,\mu c^2 h^{ik} = -\rho c^2 h^{ik}. \tag{5.24}$$

For the energy-momentum tensor $\breve{T}_{\alpha\beta} = -\frac{\lambda}{\varkappa}\, g_{\alpha\beta}$ (5.14), which describes vacuum, chr.inv.-projections are

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = -\frac{\lambda}{\varkappa}\,, \tag{5.25}$$

$$\breve{J}^i = \frac{c\,\breve{T}_0^i}{\sqrt{g_{00}}} = 0\,, \tag{5.26}$$

$$\breve{U}^{ik} = c^2 \breve{T}^{ik} = \frac{\lambda}{\varkappa}\, c^2 h^{ik} = -\breve{\rho} c^2 h^{ik}. \tag{5.27}$$

From here we see that vacuum (λ-fields) and μ-vacuum have a constant density, so these are *uniformly distributed matter*. They are also *non-emitting medias*, because the energy flux $c^2 J^i$ in them is zero

$$c^2 \breve{J}^i = \frac{c^3 \breve{T}_0^i}{\sqrt{g_{00}}} = 0\,, \qquad c^2 J^i = \frac{c^3 T_0^i}{\sqrt{g_{00}}} = 0\,. \tag{5.28}$$

In the reference frame, which accompanies the medium, the strength tensor equals (see Zelmanov's book [9])

$$U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}\,, \tag{5.29}$$

where p_0 is the equilibrium pressure, defined from the state equation, p is the true pressure, α_{ik} is the *viscosity of the 2nd kind* (the viscous strength tensor), and $\beta_{ik} = \alpha_{ik} - \frac{1}{3}\,\alpha h_{ik}$ is its anisotropic part (the *viscosity of the 1st kind*, which reveal itself in anisotropic deformations), where $\alpha = \alpha_i^i$ is the spur of the tensor α_{ik}.

Formulating the strength tensor for μ-vacuum (5.24) in the reference frame, which accompanies μ-vacuum itself, we arrive at

$$U_{ik} = p h_{ik} = -\,\rho c^2 h_{ik}\,, \tag{5.30}$$

and similarly to the strength tensor of vacuum (5.27), we have

$$\breve{U}_{ik} = \breve{p}h_{ik} = -\breve{\rho}c^2 h_{ik}\,. \tag{5.31}$$

This implies that vacuum and μ-vacuum are non-viscous media ($\alpha_{ik}=0$, $\beta_{ik}=0$) whose equations of state* is

$$\breve{p} = -\breve{\rho}c^2, \qquad p = -\rho c^2. \tag{5.32}$$

This state is referred to as *inflation* because at the positive density of the matter the pressure becomes negative, so the media expands.

These are physical properties of vacuum and μ-vacuum: they are homogeneous $\rho = const$, non-viscous $\alpha_{ik}=\beta_{ik}=0$, and non-emitting $J^i=0$ medias filled in the state of inflation.

Having these general physical properties as a base, let us turn to analysis of vacuum, which fills constant curvature spaces, in particular, a de Sitter space, which is the closest approximation of our Universe as a whole.

In constant curvature spaces the Riemann-Christoffel tensor is (see Chapter VII in Synge's book [36])

$$R_{\alpha\beta\gamma\delta} = K\left(g_{\alpha\gamma}\,g_{\beta\delta} - g_{\alpha\delta}\,g_{\beta\gamma}\right), \qquad K = const. \tag{5.33}$$

Having the tensor contracted by two indices, we obtain a formula for the Ricci tensor, which on subsequent contraction allows us to deduce the scalar curvature. As a result we have

$$R_{\alpha\beta} = -3Kg_{\alpha\beta}\,, \qquad R = -12K\,. \tag{5.34}$$

Assuming our Universe is a constant curvature space, we obtain the field equations formulated with the curvature

$$3Kg_{\alpha\beta} = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{5.35}$$

We put them down in Synge's notation as $(\lambda - 3K)\,g_{\alpha\beta} = \varkappa T_{\alpha\beta}$. Then the energy-momentum tensor of a substance in constant curvature spaces is

$$T_{\alpha\beta} = \frac{\lambda - 3K}{\varkappa}\,g_{\alpha\beta}\,. \tag{5.36}$$

*The state equation of a distributed matter is the relationship between its pressure and the density. For instance, $p=0$ is the state equation of a dust media, $p=\rho c^2$ is the state equation of a matter in atomic nuclei, $p=\frac{1}{3}\rho c^2$ is the state equation of an ultra-relativistic gas.

From here we see that in constant curvature space the problem of geometrization of matter solves by itself: the energy-momentum tensor (5.36) contains the metric tensor and constants only.

De Sitter space is a constant curvature space, where $T_{\alpha\beta}=0$ and $\lambda \neq 0$, so it is filled with vacuum (any substance is absent). Then having the energy-momentum tensor of substance (5.36) equated to zero we obtain the same result as Synge's results: in de Sitter spaces $\lambda = 3K$.

Taking into account this relationship, the formula for the observable density of vacuum in de Sitter world becomes

$$\breve{\rho} = -\frac{\lambda}{\varkappa} = -\frac{3K}{\varkappa} = -\frac{3Kc^2}{8\pi G}\,. \tag{5.37}$$

Now we are arriving at the key question about the sign of the four-dimensional curvature in our Universe. The reason to ask this question is not only curiosity. Depending on the answer, the de Sitter world cosmology we have built may fit the available data of observations or may lead to results totally alien to commonly accepted astronomical facts.

As a matter of fact, given that the four-dimensional curvature is positive $K>0$ the vacuum density must be negative and hence the inflational pressure must be greater than zero — vacuum contracts. Then because of $\lambda > 0$, non-Newtonian forces of gravitation are those of repulsion. We will then observe an encounter between two actions: at the positive inflational pressure of vacuum, which tend to compress the space, we will observe repulsion forces of non-Newtonian gravitation. The result will be as follows: at first, because λ-forces are proportional to distance, their expanding effect would grow along with growth of the radius of the Universe and the expansion would accelerate. Secondly, if the Universe has ever been of size less than the distance, at which the contracting pressure of vacuum is equal to the expanding action of λ-forces, the expansion would become impossible.

If to the contrary the four-dimensional curvature is negative $K<0$, the inflational pressure will be less than zero — vacuum expands. Besides, because in this case $\lambda<0$, non-Newtonian forces of gravitation are those of attraction. Then the Universe can keep expanding from nearly a point until the vacuum density becomes so low that its expanding action becomes equal to the compressing action of non-Newtonian λ-forces.

As seen, the question of the curvature sign is the most crucial question for cosmology of our Universe.

But human perception is three-dimensional and a regular observer can not judge anything on sign of the four-dimensional curvature by means of direct observations. What can be done then? The way out of the situation is in the theory of chronometric invariants — a method to define physical observable quantities.

Among the goals that Zelmanov set for himself was to build the curvature tensor of an observer's spatial section (the observable three-dimensional space — inhomogeneous, non-holonomic, deformed, and curved, in general case). The Zelmanov curvature tensor (see formula 5.40 — for the tensor, and 5.41 — for the contractions of it) possesses all properties of the Riemann-Christoffel tensor in the observer's three-dimensional space and also, in the same time, possesses the property of chronometric invariance.

Zelmanov decided to build such a tensor using similarity with the Riemann-Christoffel tensor, which results from non-commutativity of the second derivatives from an arbitrary vector in a given space. Deducing the difference of the second chr.inv.-derivatives from an arbitrary vector, he arrived at the equation

$$^*\nabla_i\, ^*\nabla_k\, Q_l - {}^*\nabla_k\, ^*\nabla_i\, Q_l = \frac{2A_{ik}}{c^2}\frac{^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdots j}\, Q_j\,, \tag{5.38}$$

which contains the chr.inv.-tensor

$$H_{lki\cdot}^{\cdots j} = \frac{^*\partial \Delta_{il}^{j}}{\partial x^k} - \frac{^*\partial \Delta_{kl}^{j}}{\partial x^i} + \Delta_{il}^{m}\Delta_{km}^{j} - \Delta_{kl}^{m}\Delta_{im}^{j}\,, \tag{5.39}$$

which is similar to Schouten's tensor from the theory of non-holonomic manifolds*. But in a general case in the presence of the space rotation ($A_{ik}\neq 0$), the tensor $H_{lki\cdot}^{\cdots j}$ is algebraically different from the Riemann-Christoffel tensor. Therefore Zelmanov introduced a new tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \tag{5.40}$$

which was not only chr.inv.-quantity, but it also possessed all algebraic properties of the Riemann-Christoffel tensor. Therefore C_{lkij} is the curvature tensor of the three-dimensional observable space of an observer, who accompanies his reference body. Having it contracted, we obtain the chr.inv.-quantities

$$C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im}C_{kimj}\,, \qquad C = C_j^j = h^{lj}C_{lj}\,, \tag{5.41}$$

*Schouten had built the theory of non-holonomic manifolds for an arbitrary dimension space, considering an m-dimensional sub-space in an n-dimensional space, where $m<n$ [37]. In the theory of chronometric invariants we actually consider an observer's ($m=3$)-dimensional sub-space in the ($n=4$)-dimensional pseudo-Riemannian space. In the same time the theory of chronometric invariants is applicable to any metric space, in general — see [9].

which also describe the curvature of the three-dimensional space. Because C_{lkij}, C_{kj}, and C are chr.inv.-quantities, they are physical observable quantities for this observer. In particular, the C is the *three-dimensional observable curvature* [9].

Concerning our analysis of physical properties of vacuum and cosmology, we need to know how the observable three-dimensional curvature C is linked to the four-dimensional curvature K in general and in a de Sitter space in particular. We are going to tackle this problem step-by-step.

The Riemann-Christoffel four-dimensional curvature tensor is a tensor of the 4th-rank, hence it has $n^4 = 256$ components, out of which only 20 are significant. The remaining components are either zeroes or identical to each other, because the Riemann-Christoffel tensor is:

a) Symmetric by each pair of its indices $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$;

b) Antisymmetric in respect of transposition inside each pair of the indices $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$, $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$;

c) Its components are constructed with the property $R_{\alpha(\beta\gamma\delta)} = 0$, where round brackets stand for (β, γ, δ)-transpositions.

Significant components of the Riemann-Christoffel tensor produce three chr.inv.-tensors

$$X^{ik} = -c^2 \frac{R_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \quad Y^{ijk} = -c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \quad Z^{ijkl} = c^2 R^{ijkl}. \tag{5.42}$$

The tensor X^{ik} has 6 components, Y^{ijk} has 9 components, while Z^{ijkl} has only 9 due to its symmetry. Components of the second tensor are constructed by the property $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$. Substituting the necessary components of the Riemann-Christoffel tensor [9], and having indices lowered, we obtain

$$X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} - \left(D_i^l + A_{i\cdot}^{\cdot l}\right)(D_{jl} + A_{jl}) + \frac{1}{2}\left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2} F_i F_j\,, \tag{5.43}$$

$$Y_{ijk} = {}^*\nabla_i \left(D_{jk} + A_{jk}\right) - {}^*\nabla_j \left(D_{ik} + A_{ik}\right) + \frac{2}{c^2} A_{ij} F_k\,, \tag{5.44}$$

$$Z_{iklj} = D_{ik} D_{lj} - D_{il} D_{kj} + A_{ik} A_{lj} - A_{il} A_{kj} + 2 A_{ij} A_{kl} - c^2 C_{iklj}\,. \tag{5.45}$$

From these Zelmanov formulae we see that spatial observable components of the Riemann-Christoffel tensor (5.45) are directly linked to the chr.inv.-tensor of the three-dimensional observable curvature C_{iklj}.

Let us deduce a formula for the three-dimensional observable curvature in a constant curvature space.

In such a space the Riemann-Christoffel tensor is as in (5.33), then

$$R_{0i0k} = -K h_{ik} g_{00}\,, \tag{5.46}$$

$$R_{0ijk} = \frac{K}{c}\sqrt{g_{00}}\left(v_j h_{ik} - v_k h_{ij}\right), \tag{5.47}$$

$$\begin{aligned} R_{ijkl} = K\Big[h_{ik}h_{jl} - h_{il}h_{kj} + \frac{1}{c^2}\,v_i\left(v_l h_{kj} - v_k h_{jl}\right) + \\ + \frac{1}{c^2}\,v_j\left(v_k h_{il} - v_l h_{ik}\right)\Big], \end{aligned} \tag{5.48}$$

Having deduced its chr.inv.-projections (5.42), we obtain

$$X^{ik} = c^2 K h^{ik}, \quad Y^{ijk} = 0\,, \quad Z^{ijkl} = c^2 K\left(h^{ik}h^{jl} - h^{il}h^{jk}\right), \tag{5.49}$$

hence the spatial observable components with lower indices are

$$Z_{ijkl} = c^2 K\left(h_{ik}h_{jl} - h_{il}h_{jk}\right). \tag{5.50}$$

Contracting this quantity step-by-step, we obtain

$$Z_{jl} = Z^{i\cdots}_{\cdot jil} = 2c^2 K h_{jl}\,, \qquad Z = Z^{j}_{j} = 6c^2 K\,. \tag{5.51}$$

On the other hand, we know the formula for Z_{ijkl} in an arbitrary curvature space (5.45). The formula contains the three-dimensional observable curvature. Evidently it is true for $K = const$ as well. Then having the general formula (5.45) contracted, we have

$$Z_{il} = D_{ik}D^{k}_{l} - D_{il}D + A_{ik}A_{l\cdot}^{\cdot k} + 2A_{ik}A^{k\cdot}_{\cdot l} - c^2 C_{il}\,, \tag{5.52}$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2 C\,. \tag{5.53}$$

In a constant curvature space we have $Z = 6c^2K$ (5.51), hence in such a space the relationship between the four-dimensional curvature K and the three-dimensional observable curvature C is

$$6c^2 K = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2 C\,. \tag{5.54}$$

We see that in the absence of space rotation and deformation, the four-dimensional curvature has the opposite sign with respect to the three-dimensional observable curvature. In de Sitter spaces (because there is no rotation or deformation) we have

$$K = -\frac{1}{6}\,C\,, \tag{5.55}$$

so there the three-dimensional observable curvature is $C = -6K$.

Now we are able to build a model for development of our Universe relying on two experimental facts: a) the sign of the observable density of matter, and b) the sign of the observable three-dimensional curvature.

At first, our everyday experience shows that the density of matter in our Universe is positive however sparse it may be. Then to ensure that the vacuum density (5.37) is positive, the cosmological term should be negative $\lambda < 0$ (non-Newtonian forces attract) and hence the four-dimensional curvature should be negative $K < 0$.

Secondly, as Ivanenko referred to McVittie's speech [38] in his preface to the Russian edition* of Weber's book [29]:

> "Though the data of cosmological observations are evidently not exact, but, for instance, McVittie maintains that the best results of observation of the Hubble red shift to $H \approx 75$ km/sec Mpc and of average density of matter $\rho \approx 10^{-31}$ gram/cm^3 support the idea of the non-disappearing cosmological term $\lambda < 0$".

As a result, the vacuum density in our Universe is positive and the three-dimensional observable curvature is $C > 0$. Hence the four-dimensional curvature is $K < 0$ and hence the cosmological term is $\lambda < 0$. Then from (5.37) we obtain the observable density of vacuum

$$\breve{\rho} = -\frac{\lambda}{\varkappa} = -\frac{3K}{\varkappa} = \frac{C}{2\varkappa} > 0\,, \tag{5.56}$$

so the inflational pressure of vacuum is negative $\breve{p} = -\breve{\rho}c^2$ (vacuum expands). Because homogeneous distribution of matter in the Universe is among the physical properties of vacuum, the negative inflational pressure of vacuum also implies expansion of the Universe as a whole.

Therefore the observable three-dimensional space of our Universe ($C > 0$) is a three-dimensional expanding sphere, which is a sub-space of the four-dimensional space-time ($K < 0$, a space whose geometry is a generalized case of Lobachewski-Bolyai geometry).

Of course a de Sitter space is merely an approximation of our Universe. Astronomical data say that though "islands" of masses are occasional and hardly affect the global curvature, their effect on the space curvature in their vicinities is significant (deviation of light rays and similar effects). But in our study of the Universe as a whole we can neglect occasional "islands" of substance and local non-uniformities in the curvature. In such cases a de Sitter space with the negative four-dimensional curvature (so, the observable three-dimensional curvature is positive) can be assumed to be the background of our Universe.

*Published by Foreign Literature, Moscow, 1962.

§5.4 The concept of the Inversion Explosion of the Universe

From the previous section, §5.3, we know that in a de Sitter space $\lambda = 3K$, so that according to its physical sense λ-term is approximately the same as the curvature. For a three-dimensional spherical sub-space the observable curvature $C = -6K$ is

$$C = \frac{1}{R^2}\,, \tag{5.57}$$

where R is the observable radius of the curvature (the sphere radius). Then the four-dimensional curvature of the de Sitter space is

$$K = -\frac{1}{6R^2}\,, \tag{5.58}$$

i. e. the larger the radius of the sphere, the smaller the curvature K. According to astronomical estimates, our Universe emerged 10÷20 billion years ago. Hence the distance covered by a photon since it was born at the dawn of the Universe is $R_H \approx 10^{27} \div 10^{28}$ cm. This distance is referred to as the *radius of the horizon of events*. Assuming our Universe as whole to be a de Sitter space with $K < 0$, for the four-dimensional curvature and hence for λ-term $\lambda = 3K$ we have the estimate

$$K = -\frac{1}{6R_H^2} \approx -10^{-56}\ \mathrm{cm}^{-2}. \tag{5.59}$$

On the other hand, similar figures for the event horizon, the curvature and λ-term are available from Roberto di Bartini [39, 40], who studied relationships between physical constants from topological viewpoint. In his works the *space radius of the Universe* is interpreted as the *longest distance*, defined from topological context. According to *di Bartini's inversion relationship*

$$\frac{R\,\rho}{r^2} = 1\,, \tag{5.60}$$

the space radius R (the longest distance) is the inversion image of the gravitational radius of electron $\rho = 1.347 \times 10^{-55}$ cm with respect to the radius of a spherical inversion $r = 2.818 \times 10^{-13}$ cm, which is the same like the classical radius of electron (according to di Bartini — the radius of the spherical inversion). The space radius (the largest radius of the event horizon) equals

$$R = 5.895 \times 10^{29}\,\mathrm{cm}\,. \tag{5.61}$$

From topological context di Bartini defined the *space mass* (the mass within the space radius) and the *space density* as

$$M = 3.986\times10^{57}\,\text{gram}\,, \qquad \rho = 9.87\times10^{-34}\,\text{gram/cm}^3. \tag{5.62}$$

As a matter of fact, studies done by di Bartini say that the space of the Universe (from the classical radius of electron up to the event horizon) is an external inversion image of the inner space of a certain particle with the size of electron (its radius can be estimated within the range from the classical radius of electron up to its gravitational radius). From other viewpoints the particle is different from electron: its mass equals the space mass $M = 3.986\times10^{57}$ gram, while that of electron is $m = 9.11\times10^{-28}$ gram.

The space within the particle can not be represented as a de Sitter space. As a matter of fact, the vacuum density in a de Sitter space with $K < 0$ and the curvature observable radius $r = 2.818\times10^{-13}$ cm is

$$\breve{\rho} = -\frac{3K}{\varkappa} = -\frac{1}{2\varkappa}\,r^2 = 3.39\times10^{51}\,\text{gram/cm}^3, \tag{5.63}$$

while that inside the di Bartini particle is

$$\rho = \frac{M}{2\pi^2 r^3} = 9.03\times10^{93}\,\text{gram/cm}^3. \tag{5.64}$$

On the other hand, an outer space, being the inversion image of the inner space, according to its properties can be assumed as de Sitter space. So forth let us assume that a space with the curvature radius, equal to the di Bartini radius $R = 5.895\times10^{29}$ cm, is a de Sitter space with $K < 0$. Then the four-dimensional curvature and λ-term are

$$K = -\frac{1}{6R^2} = -4.8\times10^{-61}\,\text{cm}^{-2}, \tag{5.65}$$

$$\lambda = 3K = -\frac{1}{2R^2} = -14.4\times10^{-61}\,\text{cm}^{-2}, \tag{5.66}$$

so they are five orders of magnitude less than the observed estimate, which equals $|\lambda| < 10^{-56}$. This can be explained because the Universe keeps on expanding and in a distant future numerical values of the space curvature and the cosmological term will grow down to approach the figures in (5.65, 5.66), calculated for the longest distance (the space radius). The estimated density of vacuum in the de Sitter space within the space radius is

$$\breve{\rho} = -\frac{3K}{\varkappa} = -\frac{3Kc^2}{8\pi G} \approx 7.7\times10^{-34}\,\text{gram/cm}^3, \tag{5.67}$$

so it is also less than the observed average density in the Universe ($5\div10\times10^{-30}$ gram/cm^3) and it is close to the density of matter within the space radius according to di Bartini 9.87×10^{-34} gram/cm^3.

To find out how long our Universe will keep on expanding; we have to define the difference between the observed radius of the event horizon R_H and the curvature radius R. Assuming the maximal radius of the event horizon in the Universe $R_{H(\max)}$ equal to the space radius (the outer inversion distance), which according to di Bartini is $R = R_{H(\max)} = 5.895\times10^{29}$ cm (5.61), and comparing it with the observed radius of the event horizon ($R_H \approx 10^{27}\div10^{28}$ cm), we obtain $\Delta R = R_{H(\max)} - R_H \approx 5.8\times10^{29}$ cm, so the time left for the expansion of our Universe is

$$t = \frac{\Delta R}{c} \approx 600 \text{ billion years.} \tag{5.68}$$

These calculations of the vacuum density and other properties of the de Sitter space pave the way for conclusions on the origin and evolution of our Universe and allow the only interpretation of the di Bartini inversion relationship. We will call it the *cosmological concept of Inversion Explosion.* This concept is based on our analysis of properties of the de Sitter space using geometric methods of the General Theory of Relativity, and the di Bartini inversion relationship as a result of the contemporary knowledge of physical constants. We can set forth the concept as follows:

> In the beginning there existed a single pra-particle with a radius equal to the classical radius of electron and with a mass equal to the mass of the entire Universe.
>
> Then the inversion explosion occurred: a topological transition inverted matter in the pra-particle with respect to its surface into the outer world, which gave birth to our expanding Universe. At present, 10÷20 billion years since the explosion, the Universe is in the early stage of its evolution. The expansion will continue for almost 600 billion years.
>
> At the end of this period the expanding Universe will reach its curvature radius, at which non-Newtonian forces of gravitation, proportional to distance, will be equal to the inflational expanding pressure of vacuum. The expansion will discontinue and stability will be reached, which will last until the next inversion topological transition occurs.

Parameters of matter at stages of the evolution are calculated in Table 5.1 — the pra-particle before the inversion explosion, the stage

Evolution stage	Age, years	Space radius, cm	Density, gram/cm^3	λ-term, cm^{-2}
Pra-particle	0	2.82×10^{-13}	9.03×10^{93}	?
Present time	$10\div20\times10^{9}$	$10^{27}\div10^{28}$	$5\div10\times10^{-30}$	$<10^{-56}$
After expansion	623×10^{9}	5.89×10^{29}	9.87×10^{-34}	1.44×10^{-60}

Table 5.1: Parameters of matter and space at different stages of the evolution of the Universe.

of the inversion expansion at the present time, and the stage after the expansion.

The reasons for this topological transition, which led to the spherical inversion of matter from the pra-particle (after its Inversion Explosion), remain unknown... but so do the reasons for the "emergence" of the Universe in some other contemporary cosmological concepts, for instance, in the concept of Big Bang from a singular point.

§ 5.5 Non-Newtonian gravitational forces

Einstein spaces of the kind I, including constant curvature spaces, besides having occasional "islands of matter" may be either empty or filled with a homogeneous matter. But an empty Einstein space of the kind I (its curvature is $K=0$) is dramatically different from non- empty spaces ($K=const\neq 0$).

To make our discourse more concrete, let us look at the most typical examples of empty and non-empty Einstein spaces of the kind I.

If an island of mass is a ball (spherically symmetric distribution of mass in the island) located in emptiness, then the curvature of such a space is derived from Newtonian field of gravitation, produced by the island, and such a space is not a constant curvature space. At an infinite distance from the island the space becomes flat, i. e. a constant curvature space with $K=0$. A typical example of the field of gravitation, produced by a spherically symmetric island of mass in emptiness is a field described by Schwarzschild's metric

$$ds^2=\left(1-\frac{r_g}{r}\right)c^2dt^2-\frac{dr^2}{1-\frac{r_g}{r}}-r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right), \qquad (5.69)$$

where r is the distance from the island, r_g is the island's gravitational radius.

No space rotation or deformation exist in a Schwarzschild space. Components of the chr.inv.-vector of gravitational inertial force (1.38) in such a space can be deduced as follows. According to the metric (5.69), the component g_{00} is

$$g_{00} = 1 - \frac{r_g}{r}\,, \tag{5.70}$$

then, differentiating the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ with respect to x^i, we obtain

$$\frac{\partial \mathrm{w}}{\partial x^i} = -\frac{c^2}{2\sqrt{g_{00}}}\frac{\partial g_{00}}{\partial x^i}\,. \tag{5.71}$$

Substituting it into the formula for gravitational inertial force (1.38), in the absence of space rotation we have

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1 - \frac{r_g}{r}}\,, \qquad F^1 = -\frac{c^2 r_g}{2r^2}\,. \tag{5.72}$$

Therefore, the vector F^i in a Schwarzschild space describes a Newtonian gravitational force, which is reciprocal to the square of the distance r from the gravitating mass.

If a space is *filled* with a spherically symmetric distribution of vacuum and does not include any island of mass, its curvature will be everywhere the same. An example of such a field is that described by de Sitter's metric*

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{5.73}$$

Note that though any de Sitter space has no islands of mass, it produces Newtonian fields of gravitation. So, in a de Sitter space we can consider motion of small test-particles, whose Newtonian fields are so weak that they can be neglected.

Any de Sitter metric space is a constant curvature, which becomes flat only in the absence of λ-fields. No rotation or deformation exist there, while components of the chr.inv.-vector of gravitational inertial force are

$$F_1 = \frac{\lambda c^2}{3}\frac{r}{1 - \frac{\lambda r^2}{3}}\,, \qquad F^1 = \frac{\lambda c^2}{3}\,r\,, \tag{5.74}$$

*According to the latest studies [41], de Sitter's space metric (5.73) meets the condition of spherical symmetry in only ultimate case where $\lambda = 0$, while in the common case where $\lambda \neq 0$ de Sitter's space can be spherically symmetric only if it has zero volume (i.e. only if de Sitter's space degenerates into a point). This means that an actual de Sitter space (wherein $\lambda \neq 0$, i.e. a space filled by vacuum) shouldn't have the property of spherical symmetry.

so the vector F^i in a de Sitter space describes non-Newtonian gravitational forces, proportional to r: if $\lambda < 0$, these are attraction forces, if $\lambda > 0$ these are repulsion forces. Therefore forces of non-Newtonian gravitation (λ-forces) grow along with distance at which they act.

Therefore we can see the principal difference between empty and non-empty Einstein spaces of the kind I: in empty spaces with an island of mass only Newtonian forces exist, while in the spaces filled with vacuum without islands of mass there are non-Newtonian gravitation forces only. An example of a "mixed" space of the kind I is that with Kottler's metric [42]

$$\left.\begin{aligned} ds^2 &= \left(1+\frac{ar^2}{3}+\frac{b}{r}\right)c^2dt^2-\frac{dr^2}{1+\frac{ar^2}{3}+\frac{b}{r}}-r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right) \\ F_1 &= -\,c^2\,\frac{\frac{ar}{3}-\frac{b}{2r^2}}{1+\frac{ar^2}{3}+\frac{b}{r}}\,, \qquad F^1 = -\,c^2\left(\frac{ar}{3}-\frac{b}{2r^2}\right) \end{aligned}\right\}. \tag{5.75}$$

where both Newtonian and λ-forces exist: it is filled with vacuum and includes islands of mass, the latter which produce Newtonian forces of gravitation. On the other hand, Kottler proposed his metric with two unknown constants a and b to define which additional constraints are required. Hence despite some attractive features of Kottler's metric, only two of its "ultimate" cases are of practical interest to us — Schwarzschild's metric (Newtonian forces of gravitation) and de Sitter's metric (λ-forces — non-Newtonian forces of gravitation).

§5.6 Gravitational collapse

Evidently, representing our Universe as either a de Sitter space (filled with vacuum without islands of mass) or a Schwarzschild space (an island of mass in emptiness) is a certain approximating assumption. The real metric of our world is "something in between". Nevertheless, some problems deal with non-Newtonian gravitation (produced by vacuum), where influence of concentrated masses can be neglected, de Sitter's metric is optimal. And vice versa, in problems with gravitating fields of concentrated masses Schwarzschild's metric is more reasonable. An illustrative example of such a "split" of the models is collapse — a state, where $g_{00} = 0$.

Gravitational potential w for an arbitrary metric is (1.38). Then

$$g_{00} = \left(1-\frac{\mathrm{w}}{c^2}\right)^2 = 1-\frac{2\mathrm{w}}{c^2}+\frac{\mathrm{w}^2}{c^4}\,, \tag{5.76}$$

so collapse $g_{00}=0$ occurs at $\mathrm{w}=c^2$.

Commonly, *gravitational collapse* is considered — compression of an island of mass under action of Newtonian gravitation until the mass reaches a very small size equal to its gravitational radius. Hence "strict" gravitational collapse occurs in a Schwarzschild metric space (5.69), because only Newtonian field of a spherically symmetric island of mass in emptiness is present.

At larger distances from the concentrated mass the gravitational field becomes weak and Newton's law of gravitation becomes true. Hence in a weak field of Newtonian gravitation the field potential is

$$\mathrm{w} = \frac{GM}{r}\,, \tag{5.77}$$

where G is the Gauss gravitational constant, M is the mass of the island, which produced that gravitational field. Because in the weak field the third term in (5.76) is so small that it can be neglected, hence the formula for g_{00} becomes

$$g_{00} = 1 - \frac{2GM}{c^2 r}\,, \tag{5.78}$$

so gravitational collapse in a Schwarzschild space occurs if

$$\frac{2GM}{c^2 r} = 1\,, \tag{5.79}$$

where the quantity

$$r_g = \frac{2GM}{c^2}\,, \tag{5.80}$$

which has the dimension of length, is referred to as the *gravitational radius* of the island of mass. Then g_{00} can be presented as follows

$$g_{00} = 1 - \frac{r_g}{r}\,. \tag{5.81}$$

From here we see that the collapse occurs in a Schwarzschild space at $r=r_g$.

In such a case, all the mass of the spherically symmetric island (the source of the Newtonian field) becomes concentrated within its gravitational radius. Therefore the surface of such an island of mass is referred to as a *Schwarzschild sphere*. Such objects are also-called *gravitational collapsar*, because within the gravitational radius an escape velocity is above that of the light velocity so light can not be emitted from such objects outside.

As it is easy to see from formula (5.69), in a Schwarzschild field of gravitation the three-dimensional space does not rotate ($g_{0i}=0$), hence an interval of observable time (1.25) is

$$d\tau = \sqrt{g_{00}}\, dt = \sqrt{1-\frac{r_g}{r}}\; dt\,, \tag{5.82}$$

so at the distance $r=r_g$ the interval of observable time equals zero $d\tau=0$: from the viewpoint of an external observer the time on the surface of a Schwarzschild sphere stops*. Inside the Schwarzschild sphere the interval of observable time becomes imaginary. We can also be sure that a regular observer who lives on the Earth surface, apparently stays outside its Schwarzschild sphere with radius of 0.443 cm and he can only look at process of gravitational collapse from "outside".

If $r=r_g$ then the quantity

$$g_{11} = -\frac{1}{1-\frac{r_g}{r}} \tag{5.83}$$

grows up to infinity. But the determinant of the metric tensor $g_{\alpha\beta}$ is

$$g = -r^4 \sin^2\theta < 0\,, \tag{5.84}$$

so a space-time area inside a gravitational collapsar is generally not degenerate, though the collapse is also possible in a zero-space.

At this point a note concerning photometric distance and metric observable distance should be taken. The quantity r is not a metric distance along the axis $x^1=r$, because the metric (5.69) has dr^2 with the coefficient $\left(1-\frac{r_g}{r}\right)^{-1}$. The quantity r is a *photometric distance* defined as function of illumination, produced by a stable source of light and reciprocal to the square of distance. In other words, r is the radius of a non-Euclidean sphere with the surface area $4\pi r^2$ [9].

---

*At $g_{00}=0$ (collapse) an interval of observable time (1.25) is $d\tau=-\frac{1}{c^2}v_i\,dx^i$, where $v_i=-c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation (1.37). Only assuming $g_{oi}=0$ and $v_i=0$ the condition of collapse can be defined correctly: for an external observer the observable time flow on the surface of a collapsar stops $d\tau=0$, while a four-dimensional interval is $ds^2=-d\sigma^2=g_{ik}\,dx^i dx^k$. From here a single conclusion can be made: on the surface of a collapsar the space is holonomic, so the collapsar does not rotate.

As it was shown in the study [19], a fully degenerate space-time (so-called zero-space, where $ds=0$, $d\tau=0$, and $d\sigma=0$ are true) collapses if it does not rotate. Here we proved a more general theorem: if $g_{00}=0$ the space is holonomic irrespective of whether it is degenerate ($g=0$, a zero-space) or for it $g<0$ (the space-time of the General Theory of Relativity).

According to the theory of chronometric invariants (physical observable quantities in the General Theory of Relativity), an elementary observable metric distance between two points in a Schwarzschild space is

$$d\sigma = \sqrt{\frac{dr^2}{1-\frac{r_g}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}\,. \tag{5.85}$$

At $\theta = const$ and $\varphi = const$ it is

$$\sigma = \int_{r_1}^{r_2} \sqrt{h_{11}}\, dr = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-\frac{r_g}{r}}} \tag{5.86}$$

and it is not the same as the photometric distance r.

Let us define the space-time metric inside a Schwarzschild sphere. So forth, we formulate the external metric (5.69) for a radius $r < r_g$. As a result we have

$$ds^2 = -\left(\frac{r_g}{r} - 1\right) c^2 dt^2 + \frac{dr^2}{\frac{r_g}{r} - 1} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{5.87}$$

Introducing notations $r = c\tilde{t}$ and $ct = \tilde{r}$ we obtain

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\frac{r_g}{c\tilde{t}} - 1} - \left(\frac{r_g}{c\tilde{t}} - 1\right) d\tilde{r}^2 - c^2 d\tilde{t}^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{5.88}$$

so the space-time metric inside the Schwarzschild sphere is similar to the external metric, provided that the time coordinate and the spatial coordinate r swap their roles: the photometric distance r outside the gravitational collapsar is the coordinate time $c\tilde{t}$ inside, while outside the gravitational collapsar the coordinate time ct is the photometric distance $\tilde{r}$ inside.

From the first term of the Schwarzschild inner metric (5.88) we see that it is not stationary and it exists within a limited period of time

$$\tilde{t} = \frac{r_g}{c}\,. \tag{5.89}$$

For the Sun, whose gravitational radius is 3 km, the life span of such a space would be approximately $< 10^{-5}$ sec. For the Earth, whose gravitational radius is as small as 0.443 cm, the life span of the inner Schwarzschild metric would be even less at 1.5×10^{-11} sec.

Comparison of the metrics inside a gravitational collapsar (5.88) and outside the collapsed body (5.69) implies that:

a) The space of both metrics is holonomic, i. e. it does not rotate ($A_{ik}=0$);

b) The external metric is stationary, the vector of gravitational inertial force is $F^1=-\frac{GM}{r^2}$;

c) The internal metric is non-stationary, the vector of gravitational inertial force is zero.

Let us give more detailed analysis of the external and internal metrics. To make the analysis simpler we assume $\theta=const$ and $\varphi=const$, so that out of all possible spatial directions we limit our study to radial directions only. Then the external metric is

$$ds^2 = -\left(\frac{r_g}{r}-1\right)c^2dt^2 + \frac{dr^2}{\frac{r_g}{r}-1}\,, \tag{5.90}$$

while for the internal metric we have

$$ds^2 = \frac{c^2d\tilde{t}^2}{\frac{r_g}{c\tilde{t}}-1} - \left(\frac{r_g}{c\tilde{t}}-1\right)d\tilde{r}^2. \tag{5.91}$$

Now we will define the physical observable distance (5.86) to the attracting mass (namely — the gravitational collapsar)

$$\sigma = \int\frac{dr}{\sqrt{1-\frac{r_g}{r}}} = \sqrt{r\left(r-r_g\right)} + r_g\ln\left(\sqrt{r}+\sqrt{r-r_g}\right) + const \tag{5.92}$$

along the radial direction r. From here we see: at $r=r_g$ the observable distance is

$$\sigma_g = r_g\ln\!\sqrt{r_g} + const, \tag{5.93}$$

and it is a constant value.

This means that a Schwarzschild sphere, defined by a photometric radius r_g, for an external observer is a sphere with the observable radius $\sigma_g=r_g\ln\!\sqrt{r_g}+const$ (5.93). Therefore for an external observer any gravitational collapsar is a sphere with constant observable radius, on whose surface his observable time stops.

Let us analyze a gravitational collapsar's interiors. An interval of observable time (5.82) inside a Schwarzschild sphere is imaginary for an external observer

$$d\tau = i\sqrt{\frac{r_g}{r}-1}\;dt\,, \tag{5.94}$$

or, in the "interior" coordinates $r = c\tilde{t}$ and $ct = \tilde{r}$ (from viewpoint of an "inner" observer),

$$d\tilde{\tau} = \frac{1}{\sqrt{\frac{r_g}{c\tilde{t}} - 1}}\, d\tilde{t}\,. \tag{5.95}$$

Hence for the external observer the collapsar's internal "imaginary" time (5.94) stops at its surface, while the "inner" observer sees the pace of his observable time on the surface growing infinitely.

So, from viewpoint of the external observer, the physical observable distance inside the collapsar, according to the metric (5.87), is

$$\sigma = \int \frac{dr}{\sqrt{\frac{r_g}{r} - 1}} = -\sqrt{r\,(r - r_g)} + r_g \arctan \sqrt{\frac{r_g}{r} - 1} + const, \tag{5.96}$$

or, from viewpoint of the "inner" observer

$$\tilde{\sigma} = \int \sqrt{\frac{r_g}{c\tilde{t}} - 1}\; dr\,. \tag{5.97}$$

From here we see: at $r = c\tilde{t} = r_g$ for the external observer the observable distance between any two points converges to a constant, while for the "inner" observer the observable distance grows down to zero.

In conclusion we will address the question of what happens to particles, which fall from "outside" on a Schwarzschild sphere along its radial direction. Its external metric is as follows

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \quad d\tau = \left(1 - \frac{r_g}{r}\right) dt\,, \quad d\sigma = \frac{dr}{1 - \frac{r_g}{r}}\,. \tag{5.98}$$

For real-mass particles $ds^2 > 0$, for light-like particles $ds^2 = 0$, for super-light tachyons $ds^2 < 0$ (their masses are imaginary). In radial motion towards the gravitational collapsar these conditions are:

1) Massive real particles: $\left(\frac{d\tau}{dt}\right)^2 < c^2\left(1 - \frac{r_g}{r}\right)^2$;
2) Light-like particles: $\left(\frac{d\tau}{dt}\right)^2 = c^2\left(1 - \frac{r_g}{r}\right)^2$;
3) Imaginary particles-tachyons: $\left(\frac{d\tau}{dt}\right)^2 > c^2\left(1 - \frac{r_g}{r}\right)^2$.

On any Schwarzschild sphere we have $r = r_g$, so $\frac{d\tau}{dt} = 0$ there. Hence any particle, including a light-like particle, will stop there. A four-dimensional interval on the sphere is

$$ds^2 = -\,d\sigma^2 < 0\,, \tag{5.99}$$

so it is space-like. This implies that Schwarzschild spheres (gravitational collapsars) are filled with particles with imaginary rest-mass.

§ 5.7 Inflational collapse

There are no islands of mass in de Sitter spaces, hence fields of Newtonian gravitation are absent as well — gravitational collapse is impossible. Nevertheless, the condition $g_{00} = 0$ is a strictly geometric definition of collapse, not necessarily related to Newtonian fields. Subsequently, we can consider collapse in any arbitrary space.

We are going to look at de Sitter's metric (5.73), which describes a non-Newtonian field of gravitation in a constant curvature space without islands of mass (a de Sitter space). In this case collapse may occur due to non-Newtonian gravitational forces. From de Sitter's metric (5.73) we see that

$$g_{00} = 1 - \frac{\lambda r^2}{3}\,, \tag{5.100}$$

so gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ in a de Sitter space is

$$\mathrm{w} = c^2\left(1 - \sqrt{1 - \frac{\lambda r^2}{3}}\right). \tag{5.101}$$

Because it is a potential of non-Newtonian gravitation, produced by vacuum, we will call it λ-potential. From here we see that the λ-potential is zero, if the de Sitter space is flat so that $\lambda = 3K = 0$.

Because in any de Sitter space $\lambda = 3K$, hence

1) $g_{00} = 1 - Kr^2 > 0$ at distances $r < \frac{1}{\sqrt{K}}$;
2) $g_{00} = 1 - Kr^2 < 0$ at distances $r > \frac{1}{\sqrt{K}}$;
3) $g_{00} = 1 - Kr^2 = 0$ (collapse) at distances $r = \frac{1}{\sqrt{K}}$.

At curvature $K < 0$ the numerical value of $g_{00} = 1 - Kr^2$ is always greater than zero. Hence collapse is only possible in a de Sitter space with $K > 0$.

In §5.3 we showed that the basic space of our Universe as a whole has $K < 0$. But we can assume the presence of local inhomogeneities with $K > 0$, which do not affect the space curvature in general. In particular, on such inhomogeneities collapse may occur. Therefore it is reasonable to consider a de Sitter space with $K > 0$ as a local space in the vicinities of some compact objects.

In de Sitter spaces the three-dimensional observable curvature C is linked to the four-dimensional curvature with relationship $C = -6K$ (5.55). Then assuming the observable three-dimensional space to be a sphere, we obtain $C = \frac{1}{R^2}$ (5.57) and hence $K = -\frac{1}{6R^2}$ (5.58), where R is the observable radius of the curvature. In the case $K < 0$ the value of R is real, at $K > 0$ it becomes imaginary.

Collapse in a de Sitter space is only possible at $K>0$. In this case the observable radius of the curvature is imaginary. We denote $R=iR^*$, where R^* is its absolute value. Then in the de Sitter space with $K>0$ we have

$$K = \frac{1}{6R^{*2}}\,, \tag{5.102}$$

and the collapse condition $g_{00}=1-Kr^2$ can be written as follows

$$r = R^*\sqrt{6}\,. \tag{5.103}$$

So at the distance $r=R^*\sqrt{6}$ in a de Sitter space with $K>0$ the condition $g_{00}=0$ is true, hence the observable time flow stops and collapse occurs.

In other words, an area of a de Sitter space within the radius $r = R^*\sqrt{6}$ stays in collapse. Taking into account that vacuum, which fills any de Sitter space, stays in inflation, we will refer to such a collapse as *inflational collapse* to differentiate it from gravitational collapse (which occurs in Schwarzschild spaces), while the value $r=R^*\sqrt{6}$ (5.77) will be referred to as the *inflational radius* r_{inf}. Then the collapsed area of the de Sitter space within the inflational radius will be referred to as the *inflational collapsar*, or *inflanton*.

Inside an inflanton we have $K>0$, so the three-dimensional observable curvature is $C<0$. In this case the vacuum density is negative (the inflational pressure is positive, vacuum compresses) and $\lambda>0$, so there are non-Newtonian forces of repulsion. This means that the inflational collapsar (inflanton) is filled with vacuum with the negative density and it is in the state of fragile balance between the compressing pressure of vacuum and the expanding forces of non-Newtonian gravitation.

In the de Sitter space with $K>0$ we have

$$d\tau = \sqrt{g_{00}}\,dt = \sqrt{1-Kr^2}\;dt = \sqrt{1-\frac{r^2}{r_{\text{inf}}^2}}\;dt\,, \tag{5.104}$$

so on the surface of the inflational sphere the observable time flow stops $d\tau=0$. The signature we have accepted $(+---)$, i. e. the condition $g_{00}>0$, is true at $r<r_{\text{inf}}$.

Using the term the "inflational radius" we represent de Sitter's metric with $K>0$ as follows

$$ds^2 = \left(1-\frac{r^2}{r_{\text{inf}}^2}\right)c^2dt^2 - \frac{dr^2}{1-\frac{r^2}{r_{\text{inf}}^2}} - r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right), \tag{5.105}$$

then components of the chr.inv.-vector of gravitational inertial force (5.74) are

$$F_1 = \frac{c^2}{1-\frac{r^2}{r_{\text{inf}}^2}}\,\frac{r}{r_{\text{inf}}^2}\,, \qquad F^1 = c^2\,\frac{r}{r_{\text{inf}}^2}\,. \tag{5.106}$$

Let us deduce formulae for observable distances and the observable inflational radius in an inflanton. To make our calculations simpler we assume $\theta = const$ and $\varphi = const$, i. e. out of all spatial directions only the radial direction will be considered. Then an arbitrary three-dimensional observable interval is

$$\sigma = \int \sqrt{h_{11}}\,dr = \int \frac{dr}{\sqrt{1-Kr^2}} = r_{\text{inf}} \arcsin \frac{r}{r_{\text{inf}}} + const, \tag{5.107}$$

so the observable inflational radius is constant

$$\sigma_{\text{inf}} = \int_0^{r_{\text{inf}}} \frac{dr}{\sqrt{1-Kr^2}} = \frac{\pi}{2}\,r_{\text{inf}}\,. \tag{5.108}$$

In a space with Schwarzschild's metric, which we looked at in the previous section, §5.6, a collapsar is a collapsed compact mass, which produces the curvature of the space as a whole. A regular observer of a Schwarzschild space stays outside gravitational collapsar.

In a de Sitter space a collapsar is vacuum, which fills the whole space. Here, a collapse area is comparable to a surface, whose radius equals the radius of the space curvature. So, a regular observer of a de Sitter space stays under the surface of inflational collapsar and he "watches" it from within.

To look beyond an inflational collapsar we present de Sitter's metric with $K > 0$ (5.105) for $r > r_{\text{inf}}$. Considering radial directions, in coordinates of a regular observer ("inner" coordinates of the collapsar) we obtain

$$ds^2 = -\left(\frac{r^2}{r_{\text{inf}}^2} - 1\right)c^2dt^2 + \frac{dr^2}{\frac{r^2}{r_{\text{inf}}^2} - 1}\,, \tag{5.109}$$

or, from viewpoint of an observer, who is located outside the collapsar (in its "external" coordinates $r = c\tilde{t}$ and $ct = \tilde{r}$), we have

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\frac{c^2\tilde{t}^2}{r_{\text{inf}}^2} - 1} - \left(\frac{c^2\tilde{t}^2}{r_{\text{inf}}^2} - 1\right)d\tilde{r}^2. \tag{5.110}$$

§5.8 Conclusions

At low values of density (as observed, $5 \div 10 \times 10^{-30}$ g/cm^3 in the Metagalaxy, i.e. the space is nearly empty) it can be assumed that the energy-momentum tensor $T_{\alpha\beta} \sim g_{\alpha\beta}$. In this case Einstein's equations are $R_{\alpha\beta} = k\, g_{\alpha\beta}$, where $k = const$. This case was studied in details by Petrov [30, 31]. He referred to this kind of spaces as *Einstein spaces*. According to Gliner [33, 34], who studied the algebraic structure of the energy-momentum tensor, a special type of it is outlined: $T_{\alpha\beta} = \mu\, g_{\alpha\beta}$, where $\mu = const$ is density of matter. It characterizes vacuum-like state of matter. He referred to this state as *μ-vacuum*. Gliner also showed that a space filled with μ-vacuum is an Einstein space.

We outlined the meaning of the energy-momentum tensor of vacuum $\breve{T}_{\alpha\beta} = \lambda\, g_{\alpha\beta}$ and that of μ-vacuum $T_{\alpha\beta} = \mu\, g_{\alpha\beta}$, we also found mathematical expressions for the physical observable properties of vacuum and of μ-vacuum such as density, density of momentum, and stess-tensor: vacuum was found as a homogeneous, non-viscous, non-emitting, and inflating (expanding at positive density) medium. Proceeding from Petrov's and Gliner's studies and taking into account the energy-momentum tensor in vacuum (and hence the existence of physical properties of vacuum) we have suggested a "geometrical" classification of matter according to energy-impulse tensor. We referred to it as *T-classification*: *emptiness* (empty space-time) — a condition that occurs when the energy-momentum tensor of matter is zero ($T_{\alpha\beta} = 0$) and non-Newtonian gravitation is not found ($\lambda = 0$); *vacuum* — a condition when no matter ($T_{\alpha\beta} = 0$), but non-Newtonian gravitation is found ($\lambda \neq 0$); *μ-vacuum* $T_{\alpha\beta} = \mu\, g_{\alpha\beta}$, $\mu = const$ (vacuum-like state of matter); *substance* $T_{\alpha\beta} \neq 0$, $T_{\alpha\beta} \nsim g_{\alpha\beta}$ (includes regular matter and electromagnetic field).

Routine experience shows that: density of matter in our Universe is positive. With positive density of vacuum the cosmological term $\lambda < 0$ (non-Newtonian gravitational forces are forces of attraction) and its inflation pressure is negative (vacuum expands).

Studying spaces filled exclusively with vacuum (and no substance inside), such as a de Sitter space, we have found that the collapse condition ($g_{00} = 0$) is realized therein as a collapsed area (object) we referred to as *inflational collapsar*, or *inflanton*. Inside an inflanton $\lambda > 0$, i.e. the density of vacuum is negative, pressure is positive, and non-Newtonian gravitational forces are forces of repulsion making inflanton to exist thereby balancing the compressing pressure of vacuum and the expanding forces of non-Newtonian gravitation.

Chapter 6 The Mirror Universe

§6.1 Introducing the concept of the mirror world

As we mentioned in §5.1, attempts to represent our Universe and the mirror Universe as two spaces with positive and negative curvature failed: even within de Sitter's metric, which is among the simplest space-time metrics; trajectories in a positive curvature space are substantially different from those in its negative curvature twin (see Chapter VII in Synge's book [36]).

On the other hand, numerous researchers, commencing from Dirac, intuitively predicted that the mirror Universe (as the antipode to our Universe) must be sought not in a space with the opposite curvature sign, but rather in a space, where particles have masses and energies with the opposite sign. That is, because masses of particles in our Universe are positive, then those of particles in the mirror Universe must be evidently negative.

Joseph Weber [29] wrote that neither Newton's law of gravitation nor the relativistic theory of gravitation ruled out existence of negative masses; rather, our empirical experience says that they have never been observed. Both Newtonian theory of gravitation and Einstein's General Theory of Relativity predicted behaviour of negative masses, totally different from what electrodynamics prescribes for negative charges. For two bodies, one of which has positive mass and the other has negative mass, but equal in magnitude, it would be expected that positive mass will attract the negative mass, while the negative mass will repulse the positive mass, so that one will chase the other! If motion occurs along a line which links the centres of the two bodies, such a system will move with a constant acceleration. This problem had been studied by Bondi [43]. Assuming the gravitational mass of positron to be negative (observations say that its inertial mass is positive) and using Quantum Electrodynamics' methods, Schiff had obtained that there is a difference between the inertial mass of positron and its gravitational mass. The difference proved to be much greater than the error margin in Eötvös' experiment, which showed equality of gravitational and inertial masses

[44]. As a result, Schiff had concluded that a negative gravitational mass in positron can not exist (see Chapter 1 in Weber's book [29]).

Besides, "co-habitation" of positive and negative masses in the same space-time area would cause ongoing annihilation. Possible consequences of particles of a "mixed" kind, which have both positive and negative masses, were also studied by Terletskii [45, 46].

Therefore this idea of the mirror Universe as a world of negative masses and energies faced two obstacles:

a) The *experimentum crucis*, which would point directly at exchange interactions between our Universe and the mirror Universe;

b) The absence of a theory, which would clearly explain separation of the worlds with positive and negative masses as different space-time locations.

In this section, §6.1, we are going to tackle the second (theoretical) part of the problem.

Let us look at the term "mirror properties" as applied to the space-time metric. To solve this problem we write the square of the space-time interval in chr.inv.-form, namely

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{6.1}$$

where

$$d\sigma^2 = h_{ik}\, dx^i dx^k, \tag{6.2}$$

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i = \left(1 - \frac{\mathrm{w} + v_i u^i}{c^2}\right) dt\,. \tag{6.3}$$

From here we see that an elementary spatial interval (6.2) is a square function of elementary spatial increments dx^i. Spatial coordinates x^i are all equal, so there are no principal differences between translational movement to the right or to the left, up or down. Therefore we will no longer consider mirror reflections with respect to spatial coordinates.

Time is a different thing. Physical observable time τ for a regular observer always flows from the past into the future, so that $d\tau > 0$. But there are two cases where time stops. At first, it is possible in a regular space-time in the state of collapse. Secondly, this happens in a zero-space — the fully degenerate four-dimensional space-time. Therefore the state of an observer, whose own observable time stops, may be regarded to be transitional, i. e. unavailable under regular conditions.

We will consider the problem of the mirror Universe for both $d\tau > 0$ and $d\tau = 0$. In the last case the analysis will be done separately for collapsed areas of the regular space-time and for the zero-space. We start the analysis from a regular case of $d\tau > 0$. From the formula for physical observable time (6.3) it is evident that this condition is true if

$$\mathrm{w} + v_i u^i < c^2. \tag{6.4}$$

In the absence of the space rotation ($v_i = 0$) this formula becomes $\mathrm{w} < c^2$, which corresponds to the space-time structure in the state of collapse.

Then ds^2 (6.1) can be expanded as follows

$$\begin{aligned} ds^2 = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 - 2\left(1 - \frac{\mathrm{w}}{c^2}\right) v_i\, dx^i dt - \\ - h_{ik}\, dx^i dx^k + \frac{1}{c^2}\, v_i v_k\, dx^i dx^k, \end{aligned} \tag{6.5}$$

on the other hand

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right), \qquad \mathrm{v}^2 = h_{ik}\mathrm{v}^i \mathrm{v}^k. \tag{6.6}$$

Let us divide both sides of the formula for ds^2 (6.5) by the next quantities, according to the kind of particle trajectory:

1) $c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right)$ if the space-time interval is real $ds^2 > 0$;

2) $c^2 d\tau^2$ if the space-time interval equals zero $ds^2 = 0$;

3) $-c^2 d\tau^2 \left(\frac{\mathrm{v}^2}{c^2} - 1\right)$ if the interval is imaginary $ds^2 < 0$.

As a result in all cases we obtain the same square equation with respect to the function of the "true coordinate time" t from the observer's measured physical time τ, namely — the equation

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{2 v_i \mathrm{v}^i}{c^2\left(1 - \frac{\mathrm{w}}{c^2}\right)} \frac{dt}{d\tau} + \frac{1}{\left(1 - \frac{\mathrm{w}}{c^2}\right)^2} \left(\frac{1}{c^4}\, v_i v_k \mathrm{v}^i \mathrm{v}^k - 1\right) = 0\,, \tag{6.7}$$

which has two solutions

$$\left(\frac{dt}{d\tau}\right)_1 = \frac{1}{1 - \frac{\mathrm{w}}{c^2}} \left(\frac{1}{c^2}\, v_i \mathrm{v}^i + 1\right), \tag{6.8}$$

$$\left(\frac{dt}{d\tau}\right)_2 = \frac{1}{1 - \frac{\mathrm{w}}{c^2}} \left(\frac{1}{c^2}\, v_i \mathrm{v}^i - 1\right). \tag{6.9}$$

Having t integrated with respect to τ, we obtain

$$t = \frac{1}{c^2}\int \frac{v_i\,dx^i}{1-\frac{\mathrm{w}}{c^2}} \pm \int \frac{d\tau}{1-\frac{\mathrm{w}}{c^2}} + const. \tag{6.10}$$

It can be easily integrated, if the space does not rotate and gravitational potential is $\mathrm{w}=0$. Then the integral is $t=\pm\tau+const.$ Proper choice of the initial conditions can make integration constant zero. In this case we obtain

$$t=\pm\tau\,, \qquad \tau>0\,, \tag{6.11}$$

that graphically represents two beams, which are mirror reflections of each other with respect to $\tau>0$. We can say that here, the observer's own time serves as the mirror membrane, whose mirror surface separates two worlds: one with the direct flow of coordinate time* from the past into the future $t=\tau$, and the other, the mirror case, with the reverse flow of coordinate from the future into the past $t=-\tau$.

Noteworthy, the world with the reverse flow of time is not like a videotape being rewound. Both worlds are quite equal, but for a regular observer the mentioned time coordinate in the mirror Universe is negative. The mirror surface of the membrane in this case only reflects the time flow, but does not affect it.

Now we assume that the space does not rotate $v_i=0$, but gravitational potential is not zero $\mathrm{w}\neq 0$. Then we have

$$t=\pm\int\frac{d\tau}{1-\frac{\mathrm{w}}{c^2}}+const. \tag{6.12}$$

If gravitational potential is weak ($\mathrm{w}\ll c^2$), our integral (6.12) becomes

$$t=\pm\left(\tau+\frac{1}{c^2}\int \mathrm{w}\,d\tau\right)=\pm\left(\tau+\Delta t\right), \tag{6.13}$$

where Δt is a correction that accounts for the field w, which produces acceleration. This quantity w may define any scalar potential field — either a field of Newtonian potential or a field of non-Newtonian gravitation.

If a field of gravitation produced by the potential w is strong, then this integral will take the form of (6.12) and will depend on the potential w: the stronger the field w, the faster the coordinate time flow (6.12).

*Any observer's measured physical time τ everywhere flows from the past into the future, so the condition $d\tau>0$ is true in any observer's reference frame.

In the ultimate case, where $\mathrm{w} = c^2$, we have $t \to \infty$. On the other hand, at $\mathrm{w} = c^2$ collapse occurs $d\tau = 0$. We will look at this case below, but for now we are still assuming that $\mathrm{w} < c^2$.

Let us look at coordinate time in a Schwarzschild space and a de Sitter space. If the potential w describes a Newtonian gravitational field (the space with Schwarzschild's metric), then

$$t = \pm \int \frac{d\tau}{1 - \frac{GM}{c^2 r}} = \pm \int \frac{d\tau}{1 - \frac{r_g}{r}}\,, \tag{6.14}$$

which implies that the closer we approach the gravitational radius of the mass, the greater the difference between coordinate time and the observer's measured time. If w is the potential of a non-Newtonian field of gravitation (the space with de Sitter's metric), then

$$t = \pm \int \frac{d\tau}{\sqrt{1 - \frac{\lambda r^2}{3}}} = \pm \int \frac{d\tau}{\sqrt{1 - \frac{r^2}{r^2_{\mathrm{inf}}}}}\,, \tag{6.15}$$

which implies that the closer the measured photometric distance r to the inflational radius in the space, the faster the coordinate time flow. In the ultimate case at $r \to r_{\mathrm{inf}}$ we have $t \to \infty$.

Therefore, in the absence of the space rotation but in the presence of a gravitational field, the coordinate time flow is faster and the field potential is stronger. This is true both in a Newtonian gravitational field and in a field of non-Newtonian gravitation.

Now we turn to a more general case, when both space rotation and gravitational fields are present. Then the integral for t takes the form (6.10), so coordinate time in a non-holonomic (rotating) space includes:

a) "Rotational" time determined by the presence of the term $v_i dx^i$, which has dimension of rotational momentum divided by unit mass;

b) Regular coordinate time, linked to the pace of the observer's measured time.

From the integral for t (6.10) we see that the "rotational" coordinate time, produced by the space rotation, exists independently from the observer because it does not depend on τ. For an observer who is at rest on the Earth surface (anywhere apart from the poles) it can be interpreted as the time flow determined by rotation of the planet. It always exists irrespective of whether the observer records it at this particular location or not. Regular coordinate time is linked to our presence (it depends on our measured time τ) and to the field, which exists at the point of observation; in particular, to the field of Newtonian potential.

Noteworthy, at $v_i \neq 0$ time coordinate t at the initial moment of observation (when the observer's measured time is $\tau_0 = 0$) is not zero.

Presenting the integral for t (6.10) as

$$t = \int \frac{\frac{1}{c^2}\, v_i\, dx^i \pm d\tau}{1 - \frac{\mathrm{w}}{c^2}}\,, \tag{6.16}$$

we obtain that the quantity under the integral sign is:

1) Positive, if $\frac{1}{c^2}\, v_i\, dx^i > \mp d\tau$;
2) Zero, if $\frac{1}{c^2}\, v_i\, dx^i = \pm d\tau$;
3) Negative, if $\frac{1}{c^2}\, v_i\, dx^i < \mp d\tau$.

Hence coordinate time t for a real observer stops, if the scalar product of the linear velocity of the space rotation and the observable velocity of the object is $v_i \mathrm{v}^i = \pm c^2$. This happens, if numerical values of both velocities equal to that of light, and they are either co-directed or oppositely directed.

An area of the space-time, where the condition $v_i \mathrm{v}^i = \pm c^2$ is true, so that coordinate time stops for a real observer, is the *mirror membrane* separating two areas of positive and negative time coordinate — areas with the direct and reverse flow of time.

It is also evident that no regular observer, who is located in a regular Earth laboratory, can accompany such a space.

We will refer to an area of the space-time, where coordinate time takes negative numerical values as the *mirror space*. Let us analyze properties of particles, which inhabit the mirror space with respect to those of particles located in the regular space, where time coordinate is positive.

The four-dimensional momentum vector of a mass-bearing particle, which has a non-zero rest-mass m_0, is

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}\,, \tag{6.17}$$

whose chr.inv.-projections are

$$\frac{P_0}{\sqrt{g_{00}}} = m \frac{dt}{d\tau} = \pm\, m\,, \qquad P^i = \frac{m}{c}\, \mathrm{v}^i, \tag{6.18}$$

where "plus" stands for the direct flow of coordinate time, while "minus" stands for the reverse time flow with respect to the observer's measured time. The square of the vector P^{α} is

$$P_{\alpha}P^{\alpha} = g_{\alpha\beta}\,P^{\alpha}P^{\beta} = m_0^2\,, \tag{6.19}$$

while its length equals

$$\left|\sqrt{P_{\alpha}P^{\alpha}}\,\right| = m_0\,. \tag{6.20}$$

Therefore any particle with non-zero rest-mass, being a four-dimensional structure, is projected on the observer's time line as a dipole consisting of a positive mass $+m$ and a negative mass $-m$. But in projection of P^{α} on the spatial section, both projections merge into a single one — the particle's three-dimensional observable momentum $p^i = m\mathrm{v}^i$. In other words, each observable particle with a positive relativistic mass has its own *mirror twin* with the same negative mass: the particle and its mirror twin are only different by the sign of mass, while three-dimensional momenta of both particles are positive.

Similarly, for the four-dimensional wave vector

$$K^{\alpha} = \frac{\omega}{c}\frac{dx^{\alpha}}{d\sigma} = k\,\frac{dx^{\alpha}}{d\sigma}\,, \tag{6.21}$$

which describes a massless particle, chr.inv.-projections are

$$\frac{K_0}{\sqrt{g_{00}}} = \pm k\,, \qquad K^i = \frac{k}{c}\,c^i. \tag{6.22}$$

This implies that any massless particle, as a four-dimensional object, also exists in two states: in our world with the direct time flow it is a massless particle with a positive frequency, while in the world with the reverse time flow it is a massless particle with the same negative frequency.

We define the *material Universe* as the four-dimensional space-time, filled with a substance and fields. Then because any particle is a four-dimensional dipole object, we can consider the material Universe to be a combination of the basic space-time and particles and is also a four-dimensional dipole object, which exists in two states: as *our Universe*, inhabited by particles with positive masses and frequencies, and as its mirror twin — the *mirror Universe*, where masses and frequencies of particles are negative, while three-dimensional observable momentum remains positive. On the other hand, our Universe and the mirror Universe have the same background four-dimensional space-time.

For instance, analyzing properties of the Universe as a whole, we neglect action of Newtonian fields, produced by occasional islands of substance, so we assume the basic four-dimensional space of our Universe to be a de Sitter space with the negative four-dimensional curvature, while its three-dimensional observable curvature is positive (see §5.5 herein). Hence we can assume that our Universe as a whole is an area in the de Sitter space with the negative four-dimensional curvature, where the time coordinate is positive as well as masses and frequencies of particles located in the area. Besides, vice versa, the mirror Universe is an area of the same de Sitter space, where the time coordinate is negative as well as masses and frequencies of particles located in it.

The space-time membrane, which separates our Universe and the mirror Universe in the basic space-time, does not allow them to "mix", thus preventing total annihilation. This membrane will be discussed at the end of this section.

Let us turn to the dipole structure of the Universe for $d\tau = 0$, so we will consider collapsed areas of the regular space-time and a fully degenerate space-time area (zero-space).

As we have shown, the condition $d\tau = 0$ is true in a regular (non-degenerate) space-time, where collapse occurs and the space is holonomic (it does not rotate). Then

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt = 0\,. \tag{6.23}$$

This condition is true for collapse of any kind, so for fields of gravitational potential w of any kind, including non-Newtonian potential. At $d\tau = 0$ (6.23) the four-dimensional metric is

$$ds^2 = -d\sigma^2 = -h_{ik}\,dx^i dx^k = g_{ik}\,dx^i dx^k = g_{ik}\,u^i u^k dt^2, \tag{6.24}$$

hence in this case the absolute value of the interval ds equals

$$|ds| = i d\sigma = i\sqrt{h_{ik}u^i u^k}\, dt = iudt\,, \qquad u^2 = h_{ik}\,u^i u^k, \tag{6.25}$$

so that the four-dimensional momentum vector on the surface of a collapsar is

$$P^\alpha = m_0\,\frac{dx^\alpha}{d\sigma}\,, \qquad d\sigma = udt\,. \tag{6.26}$$

Its square is

$$P_\alpha P^\alpha = g_{\alpha\beta}\,P^\alpha P^\beta = -\,m_0^2\,, \tag{6.27}$$

hence the length of the vector P^α (6.26) is imaginary

$$\left|\sqrt{P_\alpha P^\alpha}\,\right| = im_0\,. \tag{6.28}$$

The latter, in particular, implies that the surface of the collapsar is inhabited by particles with imaginary rest-masses. But, at the same time, this does not imply that super-light particles (tachyons) should be found there, because their rest-masses are real (in that time they are regular particles), while their relativistic masses become imaginary only after the particles become super-light tachyons.

On the surface of any collapsar the term "observable velocity" is void, because the observer's measured time stops there $d\tau = 0$.

Components of the four-dimensional momentum vector of a particle found on the surface of a collapsar (6.26), can be formally written as follows

$$P^0 = \frac{m_0 c}{u}\,, \qquad P^i = \frac{m_0}{u}\,u^i. \tag{6.29}$$

But as a matter of facts, we can not observe such a particle, because on the surface of a collapsar our observable time stops. On the other hand, the velocity $u^i = \frac{dx^i}{dt}$, found in this formula, is coordinate and it does not depend on the observer's measured time which stops there. Hence, we can interpret the spatial vector $P^i = \frac{m_0}{u}\,u^i$ as the particle's coordinate momentum, while the quantity $\frac{m_0 c^3}{u}$ can be interpreted as the particle's energy. Here the energy has only one sign, so the surface of any collapsar as a four-dimensional area is not a dipole four-dimensional object, presented by two mirror twins. The surface of any collapsar, irrespective of its Newtonian or non-Newtonian nature, exists in a single state.

On the other hand, the surface of a collapsar ($g_{00} = 0$) can be regarded as a membrane, which separates four-dimensional areas of the space-time before the collapse and after the collapse. Before the collapse we have $g_{00} > 0$, so the observer's measured time τ is real. After the collapse we have $g_{00} < 0$, thus τ becomes imaginary. When the observer, penetrating into the collapsar, crosses the surface then his measured time is subjected to 90° "rotation", swapping roles with his measured spatial coordinates.

The term "light-like particle" has no sense in the surface of a collapsar, because for light-like particles we have $d\sigma = cd\tau$ so on the surface ($d\tau = 0$) for such particles

$$u = \sqrt{h_{ik}\,u^i u^k} = \sqrt{\frac{h_{ik}\,dx^i dx^k}{dt^2}} = \frac{d\sigma}{dt} = \frac{cd\tau}{dt} = 0\,. \tag{6.30}$$

The observer's measured time also stops $d\tau = 0$ in a fully degenerate space-time (zero-space). There, by definition, the conditions $d\tau = 0$ and

$d\sigma = 0$ are true. Thus, as was shown in our study [19], the degeneration conditions can be written as follows

$$\mathrm{w} + v_i u^i = c^2, \qquad g_{ik}\, u^i u^k = c^2 \left(1 - \frac{\mathrm{w}}{c^2}\right)^2. \tag{6.31}$$

Particles found in the degenerate space-time (zero-particles) have zero relativistic mass $m = 0$, but non-zero mass M (1.71) and non-zero constant-sign momentum

$$M = \frac{m}{1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)}\,, \qquad p^i = M u^i. \tag{6.32}$$

Therefore, mirror twins are only found in regular matter — massless and mass-bearing particles, which are not in the state of collapse. Collapsed objects in the regular space-time (including gravitational collapsars), which do not possess the property of mirror dipoles, are common objects for our Universe and the mirror Universe. Zero-space objects, which neither possess the property of mirror dipoles, lay beyond the basic space-time due to total degeneration of the metric. It is possible to enter "neutral zones" on the surfaces of collapsed objects of the regular space and in the zero-space from either our Universe (where coordinate time is positive) or the mirror Universe (where coordinate time is negative).

§6.2 The conditions to move through the membrane, to the mirror world

Now we need to discuss the question of the membrane which separates our Universe and the mirror Universe in the basic space-time, thus preventing total annihilation of all particles with negative and positive masses.

In our Universe $dt > 0$, in the mirror Universe $dt < 0$. Hence the membrane is an area of the space-time, where $dt = 0$ so coordinate time stops. It is an area, where

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{\mathrm{w}}{c^2}} \left(\frac{1}{c^2}\, v_i \mathrm{v}^i \pm 1\right) = 0\,, \tag{6.33}$$

which can also be presented as the physical condition

$$dt = \frac{1}{1 - \frac{\mathrm{w}}{c^2}} \left(\frac{1}{c^2}\, v_i dx^i \pm d\tau\right) = 0\,. \tag{6.34}$$

The latter notation is more versatile, because it is applicable not only to the space-time of the General Theory of Relativity, but also to a generalized space-time, which permits total degeneration of the metric.

Conditions inside the membrane ($t = const$, so that $dt = 0$), in accordance with (6.34) are defined by the formula

$$v_i dx^i \pm c^2 d\tau = 0\,, \tag{6.35}$$

which can be also written in the form

$$v_i \mathrm{v}^i = \pm\, c^2. \tag{6.36}$$

This condition can be presented as follows

$$v_i \mathrm{v}^i = \left|v_i\right|\left|\mathrm{v}^i\right| \cos\left(v_i; \mathrm{v}^i\right) = \pm\, c^2. \tag{6.37}$$

From here we see that it is true, if numerical values of the velocities v_i and v^i equal to that of light and are either co-directed ("plus") or oppositely directed ("minus").

Thus the membrane from the physical viewpoint is a space which experiences translational motion at the light velocity and at the same time rotates at the light velocity, so it is a world of light-like spiral trajectories. It is possible, such a space may be attributed to particles, which possess the spirality property (e. g. massless light-like particles — photons).

Having $dt = 0$ substituted into the formula for ds^2 we obtain the metric inside the membrane

$$ds^2 = g_{ik}\, dx^i dx^k, \tag{6.38}$$

which is the same as that on the surface of a collapsar. Because it is a particular case of a space-time metric with signature $(+---)$, then ds^2 is always positive. This implies that in an area of the four-dimensional space-time, which serves the membrane between our Universe and the mirror Universe, the four-dimensional interval is space-like. The difference from the space-like metric on the surface of a collapsar (6.24) is that there is no rotation of the space so that $g_{ik} = -h_{ik}$, while in this case $g_{ik} = -\,h_{ik} + \frac{1}{c^2}\, v_i v_k$ (1.18). Or, in other words, inside the membrane we have

$$ds^2 = g_{ik}\, dx^i dx^k = -\, h_{ik}\, dx^i dx^k + \frac{1}{c^2}\, v_i v_k\, dx^i dx^k, \tag{6.39}$$

so the four-dimensional metric there becomes space-like due to the space rotation, which makes the condition $v_i dx^i = \pm c^2 d\tau$ true.

As a result a regular mass-bearing particle (irrespective of the sign of its mass) can not in its "natural" form pass through the membrane: this area of the space-time is inhabited by light-like particles which move along light-like spirals.

On the other hand the ultimate case of particles with $m > 0$ or $m < 0$ are particles with zero relativistic mass $m = 0$. From geometric viewpoint the area, where such particles are found, is tangential to areas inhabited by particles with either $m > 0$ or $m < 0$. This implies that zero-mass particles may have exchange interactions with either our-world particles $m > 0$ or mirror-world particles $m < 0$.

Particles with zero relativistic mass, by definition, exist in an area of the space-time where $ds^2 = 0$ and $c^2 d\tau^2 = d\sigma^2 = 0$. Equating ds^2 to zero inside the membrane (6.38) we obtain

$$ds^2 = g_{ik}\, dx^i dx^k = 0\,, \tag{6.40}$$

so this condition may be true in two cases:

1) All values of dx^i are zeroes, so $dx^i = 0$;

2) The three-dimensional metric is degenerate $\tilde{g} = \det \|g_{ik}\| = 0$.

The first case may occur in the regular space-time under the ultimate conditions on the surface of a collapsar: when all the surface shrinks into a point, all $dx^i = 0$ so the metric on the surface according to $ds^2 = -h_{ik}\, dx^i dx^k = g_{ik}\, dx^i dx^k$ (6.24) becomes zero.

The second case occurs on the surface of a collapsar located in the zero-space: because the condition $g_{ik}\, dx^i dx^k = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2$ is true there, then at $\mathrm{w} = c^2$ we have $g_{ik}\, dx^i dx^k = 0$ always.

The first case is asymptotic, because it never occurs in reality. Hence we can expect that "middlemen" in exchanges between our Universe and the mirror Universe are those particles with zero relativistic mass, which inhabit the surfaces of collapsars located in the fully degenerate space-time. In other words, the "middlemen" are those zero-particles, which inhabit the surfaces of collapsars in the zero-space.

§ 6.3 Conclusions

So we have shown that our Universe is the observable area of the basic space-time, where time coordinate is positive so all particles have positive masses and energies. The mirror Universe is an area of the basic space-time, where from the viewpoint of a regular observer time coordinate is negative so all particles have negative masses and energies. From the viewpoint of an our-world observer the mirror Universe is a world with the reverse flow of time, where particles travel from the future into the past with respect to us.

The two worlds are separated with the membrane — an area of the space-time, inhabited by light-like particles which travel along light-like spirals. In the scale of elementary particles such a space can be attributed to particles which possess spirality (e. g. photons). This membrane prevents mixing of positive-mass and negative-mass particles, so it prevents their total annihilation. Exchange interactions between the two worlds can be effected through particles with zero relativistic masses (zero-particles) under physical conditions, which exist on the surfaces of collapsars in the fully degenerate space-time (zero-space).

Appendix A Notations of physical quantities

Theory of chronometric invariants

b^{α}	four-dimensional monad vector
h_{ik}	three-dimensional chr.inv.-metric tensor
τ	physical observable time
$d\sigma$	physical observable spatial interval
v^{i}	three-dimensional chr.inv.-velocity
A_{ik}	three-dimensional antisymmetric chr.inv.-tensor of the space non-holonomity (rotation)
F^{i}	three-dimensional chr.inv.-vector of the gravitational inertial force
w	gravitational potential
v_{i}	three-dimensional linear velocity of the space rotation
c^{i}	three-dimensional chr.inv.-velocity of light
D_{ik}	three-dimensional chr.inv.-tensor of the rate of the space deformations
Δ^{i}_{jk}	chr.inv.-Christoffel symbols of the 2nd kind
$\Delta_{jk,m}$	chr.inv.-Christoffel symbols of the 1nd kind

Motion of particles

u^{α}	four-dimensional velocity
u^{i}	three-dimensional coordinate velocity
P^{α}	four-dimensional momentum vector
p^{i}	three-dimensional momentum vector
K^{α}	four-dimensional wave vector
k^{i}	three-dimensional wave vector
ψ	wave phase (eikonal)
S	action
L	Lagrange function (Lagrangian)
$\hbar^{\alpha\beta}$	four-dimensional antisymmetric Planck tensor
$\hbar^{*\alpha\beta}$	four-dimensional Planck dual pseudotensor

Electromagnetic fields

A^{α}	four-dimensional potential of an electromagnetic field
φ	physical observable scalar potential of an electromagnetic field (time chr.inv.-component of A^{α})
A^{i}	physical observable vector-potential of an electromagnetic field (spatial chr.inv.-components of A^{α})
$F^{\alpha\beta}$	Maxwell tensor of an electromagnetic field
E_{i}	three-dimensional chr.inv.-strength vector of an electric field
E^{*ik}	three-dimensional chr.inv.-strength pseudo-tensor of an electric field
H_{ik}	three-dimensional chr.inv.-strength vector of a magnetic field
H^{*i}	three-dimensional chr.inv.-strength pseudo-tensor of a magnetic field

Riemannian space

x^{α}	four-dimensional coordinates
x^{i}, t	three-dimensional coordinates and time
s	space-time interval
$g_{\alpha\beta}$	four-dimensional fundamental metric tensor
δ^{α}_{β}	four-dimensional unit tensor
J	Jacobi matrix determinant (Jacobian)
$e^{\alpha\beta\mu\nu}$	four-dimensional completely antisymmetric unit tensor
e^{ikm}	three-dimensional completely antisymmetric unit tensor
$E^{\alpha\beta\mu\nu}$	four-dimensional completely antisymmetric tensor
ε^{ikm}	completely antisymmetric chr.inv.-tensor
$\Gamma^{\alpha}_{\mu\nu}$	Christoffel symbols of the 2nd kind
$\Gamma_{\mu\nu,\rho}$	Christoffel symbols of the 1st kind
$R_{\alpha\beta\mu\nu}$	Riemann-Christoffel curvature tensor
$T_{\alpha\beta}$	energy-momentum tensor
ρ	chr.inv.-density of matter
J^{i}	chr.inv.-vector of the density of momentum
U^{ik}	chr.inv.-stress tensor
$R_{\alpha\beta}$	Ricci tensor
K	four-dimensional curvature
C	three-dimensional chr.inv.-curvature
λ	cosmological term (λ-term)

Appendix B Notations of tensor algebra and analysis

Ordinary differential of a vector:

$$dA^{\alpha} = \frac{\partial A^{\alpha}}{\partial x^{\sigma}}\, dx^{\sigma}.$$

Absolute differential of a contravariant vector:

$$\mathrm{D}A^{\alpha} = \nabla_{\beta}\, A^{\alpha} dx^{\beta} = dA^{\alpha} + \Gamma^{\alpha}_{\beta\mu}\, A^{\mu} dx^{\beta}.$$

Absolute differential of a covariant vector:

$$\mathrm{D}A_{\alpha} = \nabla_{\beta}\, A_{\alpha} dx^{\beta} = dA_{\alpha} - \Gamma^{\mu}_{\alpha\beta}\, A_{\mu} dx^{\beta}.$$

Absolute derivative of a contravariant vector:

$$\nabla_{\beta}\, A^{\alpha} = \frac{\mathrm{D}A^{\alpha}}{dx^{\beta}} = \frac{\partial A^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\mu}\, A^{\mu}.$$

Absolute derivative of a covariant vector:

$$\nabla_{\beta}\, A_{\alpha} = \frac{\mathrm{D}A_{\alpha}}{dx^{\beta}} = \frac{\partial A_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\, A_{\mu}\,.$$

Absolute derivative of a 2nd rank contravariant tensor:

$$\nabla_{\beta}\, F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\mu}\, F^{\sigma\mu} + \Gamma^{\sigma}_{\beta\mu}\, F^{\alpha\mu}.$$

Absolute derivative of a 2nd rank covariant tensor:

$$\nabla_{\beta}\, F_{\sigma\alpha} = \frac{\partial F_{\sigma\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\, F_{\sigma\mu} - \Gamma^{\mu}_{\sigma\beta}\, F_{\alpha\mu}\,.$$

Absolute divergence of a vector:

$$\nabla_{\alpha}\, A^{\alpha} = \frac{\partial A^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\alpha\sigma}\, A^{\sigma}.$$

Chr.inv.-divergence of a chr.inv.-vector:

$$^*\nabla_i\, q^i = \frac{^*\partial q^i}{\partial x^i} + q^i\, \frac{^*\partial \ln\sqrt{h}}{\partial x^i} = \frac{^*\partial q^i}{\partial x^i} + q^i\, \Delta^j_{ji}\,.$$

Physical chr.inv.-divergence:

$$^*\widetilde{\nabla}_i\, q^i = {}^*\nabla_i\, q^i - \frac{1}{c^2}\, F_i\, q^i .$$

D'Alembert's general covariant operator:

$$\Box = g^{\alpha\beta}\, \nabla_\alpha \nabla_\beta \,.$$

Laplace's ordinary operator:

$$\Delta = -\, g^{ik}\, \nabla_i \nabla_k \,.$$

Chr.inv.-Laplace operator:

$$^*\Delta = h^{ik}\, {}^*\nabla_i\, {}^*\nabla_k \,.$$

Chr.inv.-derivative with respect to the time coordinate and that with respect to the spatial coordinates:

$$\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\, v_i\, \frac{^*\partial}{\partial t}\,.$$

The square of the physically observable velocity:

$$\mathrm{v}^2 = \mathrm{v}_i\, \mathrm{v}^i = h_{ik}\, \mathrm{v}^i \mathrm{v}^k .$$

The linear velocity of the space rotation:

$$v_i = -\, c\, \frac{g_{0i}}{\sqrt{g_{00}}}\,, \quad v^i = -\, c\, g^{0i}\sqrt{g_{00}}\,, \quad v_i = h_{ik}\, v^k .$$

The square of v_i. This is the proof: because of $g_{\alpha\sigma} g^{\sigma\beta} = g_\alpha^\beta$, then under $\alpha = \beta = 0$ we have $g_{0\sigma} g^{\sigma 0} = \delta_0^0 = 1$, hence $v^2 = v_k v^k = c^2(1 - g_{00}\, g^{00})$, i.e.:

$$v^2 = h_{ik}\, v^i v^k .$$

The determinants of the metric tensors $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are connected as:

$$\sqrt{-\, g} = \sqrt{h}\, \sqrt{g_{00}}\,.$$

Derivative with respect to the physically observable time:

$$\frac{d}{d\tau} = \frac{^*\partial}{\partial t} + \mathrm{v}^k \frac{^*\partial}{\partial x^k}\,.$$

The 1st derivative with respect to the space-time interval:

$$\frac{d}{ds} = \frac{1}{c\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\frac{d}{d\tau}\,.$$

The 2nd derivative with respect to the space-time interval:

$$\frac{d^2}{ds^2}=\frac{1}{c^2-\mathrm{v}^2}\frac{d^2}{d\tau^2}+\frac{1}{\left(c^2-\mathrm{v}^2\right)^2}\left(D_{ik}\mathrm{v}^i\mathrm{v}^k+\mathrm{v}_i\frac{d\mathrm{v}^i}{d\tau}+\frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial x^m}\,\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m\right)\frac{d}{d\tau}\,.$$

The chr.inv.-metric tensor:

$$h_{ik}=-\,g_{ik}+\frac{1}{c^2}\,v_i v_k\,,\quad h^{ik}=-\,g^{ik},\quad h_i^k=\delta_i^k\,.$$

Zelmanov's relations between the Christoffel regular symbols and the chr.inv.-characteristics of the space of reference:

$$D_k^i+A_{k\cdot}^{\cdot i}=\frac{c}{\sqrt{g_{00}}}\left(\Gamma_{0k}^i-\frac{g_{0k}\Gamma_{00}^i}{g_{00}}\right),$$

$$g^{i\alpha}g^{k\beta}\Gamma_{\alpha\beta}^m=h^{iq}h^{ks}\Delta_{qs}^m\,,\quad F^k=-\,\frac{c^2\,\Gamma_{00}^k}{g_{00}}\,.$$

Zelmanov's 1st identity and 2nd identity:

$$\frac{{}^*\partial A_{ik}}{\partial t}+\frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i}-\frac{{}^*\partial F_i}{\partial x^k}\right)=0\,,$$

$$\frac{{}^*\partial A_{km}}{\partial x^i}+\frac{{}^*\partial A_{mi}}{\partial x^k}+\frac{{}^*\partial A_{ik}}{\partial x^m}+\frac{1}{2}\left(F_iA_{km}+F_kA_{mi}+F_mA_{ik}\right)=0\,.$$

Derivative from v^2 with respect to the physically observable time:

$$\frac{d}{d\tau}\left(\mathrm{v}^2\right)=\frac{d}{d\tau}\left(h_{ik}\mathrm{v}^i\mathrm{v}^k\right)=2D_{ik}\mathrm{v}^i\mathrm{v}^k+\frac{{}^*\partial h_{ik}}{\partial x^m}\,\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m+2\,\mathrm{v}_k\frac{d\mathrm{v}^k}{d\tau}\,.$$

The completely antisymmetric chr.inv.-tensor:

$$\varepsilon^{ikm}=\sqrt{g_{00}}\,E^{0ikm}=\frac{e^{0ikm}}{\sqrt{h}}\,,\quad \varepsilon_{ikm}=\frac{E_{0ikm}}{\sqrt{g_{00}}}=e_{0ikm}\sqrt{h}\,.$$

Bibliography

1. Levi-Civita T. Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura Riemanniana. *Rendiconti del Circolo Matematico di Palermo*, 1917, tome 42, 173–205.
2. Tangherlini F. R. The velocity of light in uniformly moving frame. A dissertation. Stanford University, 1958. *The Abraham Zelmanov Journal*, 2009, vol. 2, 44–110.
3. Malykin G. B. and Malykin E. G. Tangherlini's dissertation and its significance for physics of the 21th century. *The Abraham Zelmanov Journal*, 2009, vol. 2, 121–143.
4. Recami E. Classical tachyons and possible applications. *Rivista del Nuovo Cimento*, 1986, vol. 9, 1–178.
5. Liberati S., Sonego S., and Visser M. Faster-than-c signals, special relativity, and causality. *Annals of Physics*, 2002, vol. 298, 151–185.
6. Terletskii Ya. P. The causality principle and the second law of thermodynamics. *Soviet Physics Doklady*, 1961, vol. 5, 782–785 (translated from *Doklady Academii Nauk USSR*, 1960, vol. 133, no. 2, 329–332).
7. Bilaniuk O.-M. P., Deshpande V. K., and Sudarshan E. C. G. "Meta" relativity. *American Journal of Physics*, 1962, vol. 30, no. 10, 718–723.
8. Feinberg G. Possibility of faster-than light particles. *Physical Review*, 1967, vol. 159, no. 5, 1089–1105.
9. Zelmanov A. Chronometric invariants. Dissertation, 1944. American Research Press, Rehoboth (NM), 2006.
10. Landau L. D. and Lifshitz E. M. The classical theory of fields. GITTL, Moscow, 1939. Referred with the 4th edition, Butterworth-Heinemann, 1980 (all these were translated in 1951–1980 by Morton Hammermesh).
11. Zelmanov A. L. Chronometric invariants and accompanying frames of reference in the General Theory of Relativity. *Soviet Physics Doklady*, 1956, vol. 1, 227–230 (translated from *Doklady Academii Nauk USSR*, 1956, vol. 107, no. 6, 815–818).
12. Zelmanov A. L. and Agakov V. G. Elements of the General Theory of Relativity. Nauka, Moscow, 1988 (*in Russian*).
13. Zelmanov A. L. On the relativistic theory of an anisotropic inhomogeneous universe. *The Abraham Zelmanov Journal*, 2008, vol. 1, 33–63 (translated from a manuscript of 1957).

14. Cattaneo C. General Relativity: relative standard mass, momentum, energy, and gravitational field in a general system of reference. *Nuovo Cimento*, 1958, vol. 10, 318–337.
15. Cattaneo C. Conservation laws in General Relativity. *Nuovo Cimento*, 1959, vol. 11, 733–735.
16. Cattaneo C. On the energy equation for a gravitating test particle. *Nuovo Cimento*, 1959, vol. 13, 237–240.
17. Cattaneo C. Problèmes d'interprétation en Relativité Générale. *Colloques Internationaux du Centre National de la Recherche Scientifique*, no. 170 "Fluides et champ gravitationel en Relativité Générale", Éditions du Centre National de la Recherche Scientifique, Paris, 1969, 227–235.
18. Raschewski P. K. Riemannsche Geometrie und Tensoranalysis. Deutscher Verlag der Wissenschaften, Berlin, 1959 (translated by W. Richter); reprinted by Verlag Harri Deutsch, Frankfurt am Main, 1993.
19. Rabounski D. and Borissova L. Particles here and beyond the mirror. 2nd edition (revised and expanded), Svenska fysikarkivet, Stockholm, 2008.
20. Terletskii Ya. P. and Rybakov Yu. P. Electrodynamics. Vishaya Shkola (High School Publishers), Moscow, 1980 (*in Russian*).
21. Papapetrou A. Spinning test-particles in General Relativity. I. *Proceedings of the Royal Society A*, 1951, vol. 209, 248–258.
22. Corinaldesi E. and Papapetrou A. Spinning test-particles in General Relativity. II. *Proceedings of the Royal Society A*, 1951, vol. 209, 259–268.
23. Suhendro I. A four-dimensional continuum theory of space-time and the classical physical fields. *Progress in Physics*, 2007, vol. 4, 34–46.
24. Suhendro I. Spin-curvature and the unification of fields in a twisted space. Svenska fysikarkivet, Stockholm, 2008.
25. Del Prado J. and Pavlov N. V. Private reports to A. L. Zelmanov, 1968.
26. Stanyukovich K. P. Gravitational field and elementary particles. Nauka, Moscow, 1965 (*in Russian*).
27. Stanyukovich K. P. On the problem of the existence of stable particles in the Metagalaxy. *Problemy Teorii Gravitazii i Elementarnykh Chastiz*, vol. 1, Atomizdat, Moscow, 1966, 267–279 (*in Russian*).
28. Stanyukovich K. P. On the problem of the existence of stable particles in the Metagalaxy. *The Abraham Zelmanov Journal*, 2008, vol. 1, 99–110 (translated from a manuscript of 1965).
29. Weber J. General Relativity and gravitational waves. Interscience Publishers, New York, 1961, 200 pages (referred with the reprint by Dover Publications, Mineola, NY, 2004).
30. Petrov A. Z. Einstein spaces. Pergamon Press, Oxford, 1969 (translated by R. F. Kelleher, edited by J. Woodrow).

31. Petrov A. Z. The classification of spaces defining gravitational fields. *The Abraham Zelmanov Journal*, 2008, vol. 1, 81–98 (translated from *Uchenye Zapiski Kazanskogo Universiteta*, 1954, vol. 114, no. 8, 55–69).
32. Grigoreva L. B. Chronometrically invariant representation of the classification of Petrov gravitational fields. *Soviet Physics Doklady*, 1970, vol. 15, 579–582 (translated from *Doklady Academii Nauk USSR*, 1970, vol. 192, no. 6, 1251–1254).
33. Gliner E. B. Algebraic properties of energy-momemtum tensor and vacuum-like states of matter. *Journal of Experimental and Theoretical Physics* (*JETP*), 1966, vol. 22, no. 2, 378–383 (translated from *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, 1966, vol. 49, no. 2, 543–548)
34. Gliner E. B. Vacuum-like state of medium and Friedmann's cosmology. *Soviet Physics Doklady*, 1970, vol. 15, 559–562 (translated from *Doklady Academii Nauk USSR*, 1970, vol. 192, no. 4, 771–774).
35. Sakharov A. D. The initial stage of an expanding Universe and the appearance of a nonuniform distribution of matter. *Journal of Experimental and Theoretical Physics* (*JETP*), 1966, vol. 22, 241–249 (translated from *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, 1966, vol. 49, 345–453).
36. Synge J. L. Relativity: the general theory. North Holland, Amsterdam, 1960 (referred with the 2nd edition, Foreign Literature, Moscow, 1963).
37. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. Noordhoff, Groningen, 1938 (first published in *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19).
38. McVittie G. C. Remarks on cosmology. *Paris Symposium on Radio Astronomy* (IAU Symposium no. 9 and URSI Symposium no. 1, July 30 — August 6, 1958), Stanford University Press, Stanford, 1959, 533–535.
39. Oros di Bartini R. Some relations between physical constants. *Soviet Physics Doklady*, 1965, vol. 10 (translated from *Doklady Academii Nauk USSR*, 1965, vol. 163, no. 4, 861–864).
40. Oros di Bartini R. Relations between physical constants. *Progress in Physics*, 2005, vol. 3, 34–40 (translated from *Problemy Teorii Gravitazii i Elementarnykh Chastiz*, vol. 1, Atomizdat, Moscow, 1966, 249–266).
41. Crothers S. J. On the general solution to Einstein's vacuum field for the point-mass when $\lambda = 0$ and its consequences for relativistic cosmology. *Progress in Physics*, 2005, vol. 3, 7–18.
42. Kottler F. Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie. *Annalen der Physik*, 1918, Bd. 361, Nr. 14, 401–462.
43. Bondi H. Negative mass in General Relativity. *Review of Modern Physics*, 1957, vol. 29, no. 3, 423–428.
44. Schiff L. I. Sign of gravitational mass of a positron. *Physical Review Letters*, 1958, vol. 1, no. 7, 254–255.

45. Terletskii Ya. P. Paradoxes in the theory of relativity. Plenum Press, New York, 1968 (translated from Terletskii Ya. P. Paradoksy teorii otnositelnosti. Patrice Lumumba University Press, Moscow, 1965).
46. Terletskii Ya. P. Paradoxes in the theory of relativity. *American Journal of Physics*, 1969, vol. 37, no. 4, 460–461.

Index

About the authors

Larissa Borissova (b. 1944 in Moscow, Russia) was educated at the Faculty of Astronomy, the Department of Physics of the Moscow State University. Commencing in 1964, she was trained by Dr. Abraham Zelmanov (1913–1987), a famous cosmologist and researcher in General Relativity. She was also trained, commencing in 1968, by Prof. Kyril Stanyukovich (1916–1989), a prominent scientist in gaseous dynamics and General Relativity. In 1975, Larissa Borissova received the "candidate of science" degree on gravitational waves (the Soviet PhD). She has published about 30 scientific papers and 6 books on General Relativity and gravitation. In 2005, Larissa Borissova became a co-founder and Associate Editor of *Progress in Physics*, and is currently continuing her scientific studies as an independent researcher.

Dmitri Rabounski (b. 1965 in Moscow, Russia) was educated at the Moscow High School of Physics. Commencing in 1983, he was trained by Prof. Kyril Stanyukovich (1916–1989), a prominent scientist in gaseous dynamics and General Relativity. He was also trained with Dr. Abraham Zelmanov (1913–1987), the famous cosmologist and researcher in General Relativity. During the 1980's, he was also trained by Dr. Vitaly Bronshten (1918–2004), the well-known expert in the physics of destruction of bodies in atmosphere. Dmitri Rabounski has published about 30 scientific papers and 6 books on General Relativity, gravitation, physics of meteoroids, and astrophysics. In 2005, he started a new American journal on physics, *Progress in Physics*, where he is the Editor-in-Chief, and is currently continuing his scientific studies as an independent researcher.

Svenska fysikarkivet books

A REVISED ELECTROMAGNETIC THEORY WITH FUNDAMENTAL APPLICATIONS — *by Bo Lehnert* — Svenska fysikarkivet, 2008, 158 pages. ISBN 978-91-85917-00-6

SPIN-CURVATURE AND THE UNIFICATION OF FIELDS IN A TWISTED SPACE — *by Indranu Suhendro* — Svenska fysikarkivet, 2008, 78 pages. ISBN 978-91-85917-01-3

PARTICLES HERE AND BEYOND THE MIRROR — *by Dmitri Rabounski and Larissa Borissova* — Svenska fysikarkivet, 2008, 118 pages. ISBN 978-91-85917-03-7

DATA ANALYSIS OF GRAVITATIONAL WAVES — *by Sanjay K. Sahay* — Svenska fysikarkivet, 2008, 118 pages. ISBN 978-91-85917-05-1

UPPER LIMIT IN MENDELEEV'S PERIODIC TABLE — ELEMENT NO.155 — *by Albert Khazan* — Svenska fysikarkivet, 2009, 80 pages. ISBN 978-91-85917-08-2

COSMIC PHYSICAL FACTORS IN RANDOM PROCESSES — *by Simon E. Shnoll* — Svenska fysikarkivet, 2009, 388 pages. ISBN 978-91-85917-07-5 (English edition), ISBN 978-91-85917-06-8 (Russian edition)

Fields, Vacuum, and the Mirror Universe

by Larissa Borissova and Dmitri Rabounski
Svenska fysikarkivet, Stockholm, 2009, 260 p.

In this book, we build the theory of non-geodesic motion of particles in the space-time of General Relativity. Motion of a charged particle in an electromagnetic field is constructed in curved space-time (in contrast to the regular considerations held in Minkowski's space of Special Relativity). Spin particles are explained in the framework of the variational principle: this approach distinctly shows that elementary particles should have masses governed by a special quantum relation. Physical vacuum and forces of non-Newtonian gravitation acting in it are determined through the lambda-term in Einstein's equations. A cosmological concept of the inversion explosion of the Universe from a compact object with the radius of an electron is suggested. Physical conditions inside a membrane that separates space-time regions were the observable time flows into the future and into the past (our world and the mirror world) are examined.

Fält, vakuum och spegeluniversum

av Larissa Borissova och Dmitri Rabounski
Svenska fysikarkivet, Stockholm, 2009, 260 s.

I denna bok föreslår vi en teori som beskriver de icke-geodesiska rörelserna av partiklar i den allmänna relativitetens rumtid. En laddat partikels rörelse i ett elektromagnetiskt fält konstrueras i en krökt rumtid (till skillnad från de vedertagna övervägandena i Minkowskis rymd i Speciella Relativitetsteorin). Spinpartiklar är förklarade inom ramen för variationsprincipen: detta synsätt visar att elementärpartiklarna måste ha en massa, vars storlek kan härledas från en speciell kvantekvation. Fysisk vakuum och icke-Newtonianska gravitationskrafter som verkar inom denna ekvation fastställs genom lambda-termen i Einsteins ekvation. Ett kosmologiskt koncept av en inversionsexplosion av universum från ett kompakt objekt med ett radie av en elektron föreslås. I boken undersöker vi de fysiska förhållandena inuti membranet som separera olika rumtidsregioner där den observerbara tiden flyter mot framtiden och forntiden (vår värld och spegelvärlden).

Printed in the United
States of America

ISBN 978-91-85917-09-9

9 789185 917099 >