A Gravity, $U(4) \times U(4)$ Yang-Mills and Matter Unification in Clifford Spaces

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Abstract

A brief review of a Conformal Gravity and $U(4) \times U(4)$ Yang-Mills Unification model in $four$ dimensions from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) is presented. It based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). The 16 fermions of each generation can be arranged into the 16 entries of a 4×4 matrix associated with the $A = 1, 2, 3, ..., 16$ indices corresponding to the dimensions of the $Cl(4)$ gauge algebra. The Higgs sector is also part of the $Cl(4)$-algebra polyvector valued gauge field in C-space. The Yukawa couplings which furnish masses to the fermions (after symmetry breaking) admit a C-space geometric interpretation as well.

Keywords: C-space Gravity, Clifford Algebras, Grand Unification.

A model of Emergent Gravity with the observed Cosmological Constant from a BF-Chern-Simons-Higgs Model was recently revisited [1] which allowed to show how a Conformal Gravity, Maxwell and $SU(2) \times SU(2) \times U(1) \times U(1)$ Yang-Mills Unification model in $four$ dimensions can be attained from a Clifford Gauge Field Theory in a very natural and geometric fashion. In this first part of this work we will review the results [5] and show how to construct a Complex Conformal Gravity-Maxwell and Yang-Mills Unification based on a Clifford gauge field theory in C-spaces (Clifford spaces). We finalize by showing how the fermions (quarks, leptons), Higgs scalars and Yukawa couplings fit naturally into the picture. The latter construction is new (to our knowledge). A Chern-Simons E_8 Gauge Theory of Gravity in $D = 15$ (the boundary of a $16D$ space) and Grand Unification approach based on a Clifford $Cl(16)$ algebra was proposed in [15]. The advantage of this work is that one does not need to invoke $15, 16$ dimensions.

Let $\eta_{ab} = (+,-,-,-)$, $\epsilon_{0123} = -\epsilon^{0123} = 1$, the Clifford $Cl(1,3)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab}, \quad \Gamma_5 = -i\,\Gamma_0\,\Gamma_1\,\Gamma_2\,\Gamma_3, \quad (\Gamma_5)^2 = 1; \quad \{\Gamma_5, \Gamma_a\} = 0; \quad (1)$$

$$\Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5; \quad \Gamma_{ab} = \frac{1}{2}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a). \quad (2a)$$

$$\Gamma_{abc} = \epsilon_{abcd}\,\Gamma_5\,\Gamma^d; \quad \Gamma_{abcd} = \epsilon_{abcd}\,\Gamma_5. \quad (2b)$$

$$\Gamma_a\,\Gamma_b = \Gamma_{ab} + \eta_{ab}, \quad \Gamma_{ab}\,\Gamma_5 = \frac{1}{2}\epsilon_{abcd}\,\Gamma^{cd}, \quad (2c)$$

$$\Gamma_{ab}\,\Gamma_c = \eta_{bc}\,\Gamma_a - \eta_{ac}\,\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2d)$$

$$\Gamma_c\,\Gamma_{ab} = \eta_{ac}\,\Gamma_b - \eta_{bc}\,\Gamma_a + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2e)$$

$$\Gamma_a\,\Gamma_b\,\Gamma_c = \eta_{ab}\,\Gamma_c + \eta_{bc}\,\Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\,\Gamma_5\,\Gamma^d \quad (2f)$$

$$\Gamma^{ab}\,\Gamma_{cd} = \epsilon^{ab}{}_{cd}\,\Gamma_5 - 4\delta^{[a}_{[c}\,\Gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd}. \quad (2g)$$

$$\delta^{ab}_{cd} = \frac{1}{2}(\delta^a_c\,\delta^b_d - \delta^a_d\,\delta^b_c). \quad (3)$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(1,3)$ algebra exists where the generators $\mathbf{1}, \Gamma_0, \Gamma_5, \Gamma_i\Gamma_5, i = 1,2,3$ are chosen to be Hermitian; while the generators $-i\,\Gamma_0 \equiv \Gamma_4$; Γ_a, Γ_{ab} for $a, b = 1,2,3,4$ are chosen to be anti-Hermitian. For instance, the anti-Hermitian generators Γ_k for $k = 1,2,3$ can be represented by 4×4 matrices, whose block diagonal entries are 0 and the 2×2 block off-diagonal entries are comprised of $\pm\sigma_k$, respectively, where σ_k, are the 3 Pauli's spin Hermitian 2×2 matrices obeying $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. The Hermitian generator Γ_0 has zeros in the main diagonal and $-\mathbf{1}_{2\times2}, -\mathbf{1}_{2\times2}$ in the off-diagonal block so that $-i\,\Gamma_0 = \Gamma_4$ is anti-Hermitian. The Hermitian Γ_5 chirality operator has $\mathbf{1}_{2\times2}, -\mathbf{1}_{2\times2}$ along its main diagonal and zeros in the off-diagonal block. The unit operator $\mathbf{1}_{4\times4}$ has 1 along the diagonal and zeros everywhere else.

Using eqs-(1-3) allows to write the $Cl(1,3)$ algebra-valued one-form as

$$\mathbf{A} = \left(i\,a_\mu\,\mathbf{1} + i\,b_\mu\,\Gamma_5 + e^a_\mu\,\Gamma_a + i\,f^a_\mu\,\Gamma_a\,\Gamma_5 + \frac{1}{4}\omega^{ab}_\mu\,\Gamma_{ab}\right) dx^\mu. \quad (4)$$

The Clifford-valued anti-Hermitian gauge field A_μ transforms according to $A'_\mu = U^{-1}\,A_\mu\,U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The anti-Hermitian Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1}\,F\,U$. Decomposing the anti-Hermitian field strength in terms of the Clifford algebra anti-Hermitian generators gives

$$F_{\mu\nu} = i\,F^1_{\mu\nu}\,\mathbf{1} + i\,F^5_{\mu\nu}\,\Gamma_5 + F^a_{\mu\nu}\,\Gamma_a + i\,F^{a5}_{\mu\nu}\,\Gamma_a\,\Gamma_5 + \frac{1}{4}F^{ab}_{\mu\nu}\,\Gamma_{ab}. \quad (5)$$

where $F = \frac{1}{2}\ F_{\mu\nu}\ dx^\mu \wedge dx^\nu$. The field-strength components are given by

$$F^1_{\mu\nu}\ = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{6a}$$

$$F^5_{\mu\nu}\ =\ \partial_\mu b_\nu - \partial_\nu b_\mu + 2e^a_\mu f_{\nu a} - 2e^a_\nu f_{\mu a} \tag{6b}$$

$$F^a_{\mu\nu}\ =\ \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2f^a_\mu b_\nu - 2f^a_\nu b_\mu \tag{6c}$$

$$F^{a5}_{\mu\nu}\ =\ \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2e^a_\mu b_\nu - 2e^a_\nu b_\mu \tag{6d}$$

$$F^{ab}_{\mu\nu}\ =\ \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \longleftrightarrow \nu. \tag{6e}$$

A Clifford-algebra-valued dimensionless anti-Hermitian scalar field $\Phi(x^\mu) = \Phi^A(x^\mu)\ \Gamma_A$ belonging to a section of the Clifford bundle in $D = 4$ can be expanded as

$$\Phi\ =\ i\ \phi^{(1)}\ \mathbf{1}\ +\ \phi^a\ \Gamma_a\ +\ \phi^{ab}\ \Gamma_{ab}\ + i\ \phi^{a5}\ \Gamma_a\ \Gamma_5\ + i\ \phi^{(5)}\ \Gamma_5 \tag{7}$$

so that the covariant exterior differential is

$$d_A\ \Phi\ =\ (d_A\ \Phi^C)\ \Gamma_C\ =\ \left(\ \partial_\mu\ \Phi^C\ +\ \mathcal{A}^A_\mu\ \Phi^B\ f_{AB}^{\ \ C}\ \right)\ \Gamma_C\ dx^\mu\ . \tag{8}$$

where

$$[\mathcal{A}_\mu,\ \Phi]\ =\ \mathcal{A}^A_\mu\ \Phi^B\ [\Gamma_A,\ \Gamma_B]\ =\ \mathcal{A}^A_\mu\ \Phi^B\ f_{AB}^{\ \ C}\ \Gamma_C\ . \tag{9}$$

The first term in the action is

$$I_1\ =\ \int_{M_4} d^4x\ \ \epsilon^{\mu\nu\rho\sigma}\ <\ \Phi^A\ F^B_{\mu\nu}\ F^C_{\rho\sigma}\ \Gamma_A\ \Gamma_B\ \Gamma_C\ >_0\ . \tag{10}$$

where the operation $< .... >_0$ denotes taking the *scalar* part of the Clifford geometric product of $\Gamma_A\ \Gamma_B\ \Gamma_C$. The scalar part of the Clifford geometric product of the gammas is for example

$$<\ \Gamma_a\ \Gamma_b\ >\ = \delta_{ab},\quad <\ \Gamma_{a_1a_2}\ \Gamma_{b_1b_2}\ >\ = \delta_{a_1b_1}\ \delta_{a_2b_2} - \delta_{a_1b_2}\ \delta_{a_2b_1}$$

$$<\ \Gamma_{a_1}\ \Gamma_{a_2}\ \Gamma_{a_3}\ >\ =\ 0,\quad <\ \Gamma_{a_1a_2a_3}\ \Gamma_{b_1b_2b_3}\ >\ =\ \delta_{a_1b_1}\ \delta_{a_2b_2}\ \delta_{a_3b_3}\ \pm\ ......$$

$$<\ \Gamma_{a_1}\ \Gamma_{a_2}\ \Gamma_{a_3}\ \Gamma_{a_4}\ >\ =\ \delta_{a_1a_2}\ \delta_{a_3a_4} - \delta_{a_1a_3}\ \delta_{a_2a_4} + \delta_{a_2a_3}\ \delta_{a_1a_4},\ etc\ ...... \tag{11}$$

The integrand of (10) is comprised of terms like

$$\begin{gathered} F^{ab} \wedge F^{cd}\ \phi^{(5)}\ \epsilon_{abcd};\quad F^{(1)} \wedge F^{(5)}\ \phi^{(5)};\quad F^a \wedge F^{a5}\ \phi^{(5)}; \\ 2\ F^a_{\ b} \wedge F^b_{\ a}\ \phi^{(1)};\quad F^{(1)} \wedge F^{(1)}\ \phi^{(1)};\quad F^{(5)} \wedge F^{(5)}\ \phi^{(1)}; \\ F^{(1)} \wedge F^{ab}\ \phi_{ab};\quad F^{(1)} \wedge F^{a5}\ \phi_{a5};\quad F^{(1)} \wedge F^a\ \phi_a; \\ F^a \wedge F_a\ \phi^{(1)};\quad F^{a5} \wedge F_{a5}\ \phi^{(1)};\quad F^{ab} \wedge F^c\ (\eta_{bc}\phi_a - \eta_{ac}\phi_b); \\ F^{ab} \wedge F^c\ \phi^{5d}\ \epsilon_{abcd};\quad F^a \wedge F^{b5}\ \phi^{cd}\ \epsilon_{abcd};\ .... \end{gathered} \tag{12}$$

The numerical factors and signs of each one of the above terms is determined from the relations in eqs-(1-2). Due to the fact that $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{\rho\sigma\mu\nu}$ the terms like

$$F^a_{\ b} \wedge F^{bc} \ \phi_{ac} \ = \ F^{bc} \wedge F^a_{\ b} \ \phi_{ac} \ = \ F^{cb} \wedge F_b^{\ a} \ \phi_{ac} \ =$$

$$F^c_{\ b} \wedge F^{ba} \ \phi_{ac} \ = \ - F^a_{\ b} \wedge F^{bc} \ \phi_{ac} \ \Rightarrow \ F^a_{\ b} \wedge F^{bc} \ \phi_{ac} \ = \ 0$$

$$F^a \wedge F^b \ \phi_{ab} \ = \ 0; \quad F^{a5} \wedge F^{b5} \ \phi_{ab} \ = \ 0; \ F^{a5} \wedge F^{b5} \ \phi^{cd} \ \epsilon_{abcd} \ = 0, \ \ldots\ldots.. \qquad (13)$$

vanish. Thus the action (10) is a generalization of the McDowell-Mansouri-Chamseddine-West action [2], [3]. The Clifford-algebra generalization of the Chern-Simons-like terms [1] are

$$I_2 \ = \ \int_{M_4} \ < \ \Phi^E \ d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \ \Gamma_{[E} \ \Gamma_A \ \Gamma_B \ \Gamma_C \ \Gamma_{D]} \ >_0 \ =$$

$$\int_{M_4} \ \Big(\ \phi^{(5)} \ d\phi^a \wedge d\phi^b \wedge d\phi^c \wedge d\phi^d \ \epsilon_{abcd} \ - \ \phi^a \ d\phi^{(5)} \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \ \epsilon_{abcd} \ + \ \ldots\ldots\ldots \Big). \qquad (14)$$

where $d\phi^A$ is the covariant exterior differential $(\partial_\mu \phi^A + [\mathbf{A}_\mu, \phi]^A) dx^\mu$. The Clifford-algebra generalization of the Higgs-like potential is given by

$$I_3 \ = \ -\int_{M_5} \ < \ d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \wedge d\Phi^E \ \Gamma_{[A} \ \Gamma_B \ \Gamma_C \ \Gamma_D \ \Gamma_{E]} \ >_0 \ V(\Phi) \ =$$

$$- \int_{M_5} \ d\Phi^5 \wedge d\Phi^a \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \ \epsilon_{abcd} \ V(\Phi) \ + \ \ldots\ldots.. \qquad (15)$$

where

$$V(\Phi) \ = \ \kappa \ \Big(\ \Phi_A \ \Phi^A \ - \mathbf{v}^2 \ \Big)^2 \qquad (16a)$$

and

$$\Phi_A \ \Phi^A \ = \ \phi^{(1)} \ \phi_{(1)} + \phi^a \ \phi_a + \phi^{ab} \ \phi_{ab} + \phi^{a5} \ \phi_{a5} + \phi^{(5)} \ \phi_{(5)}. \qquad (16b)$$

Vacuum solutions can be found of the form

$$< \phi^{(5)} > \ = \ \mathbf{v}; \ \ < \phi^{(1)} > \ = \ < \phi^a > \ = \ < \phi^{ab} > \ = \ < \phi^{a5} > \ = \ 0. \ (17)$$

A variation of $I_1 + I_2 + I_3$ given by eqs-(10,14,15) w.r.t ϕ^5, and taking into account the VEV (vacuum expectation value) of eq-(17) which minimize the potential (16a) solely *after the variation* w.r.t the scalar fields is taken place, allows to eliminate the scalars on-shell leading to

$$I_1 + I_2 + I_3 \ = \ \frac{4}{5} \ \mathbf{v} \int_M \ d^4x \ \Big(\ F^{ab} \wedge F^{cd} \ \epsilon_{abcd} \ + \ F^{(1)} \wedge F^{(5)} \ + \ F^a \wedge F^{a5} \Big) \ =$$

$$\frac{4}{5} \ \mathbf{v} \ \int_M \ d^4x \ \Big(\ F^{ab}_{\mu\nu} \ F^{cd}_{\rho\sigma} \ \epsilon_{abcd} \ + \ F^{(1)}_{\mu\nu} \ F^{(5)}_{\rho\sigma} \ + \ F^a_{\mu\nu} \ F^{a5}_{\rho\sigma} \ \Big) \ \epsilon^{\mu\nu\rho\sigma}. \qquad (18)$$

where Einstein's summation convention over repeated indices is implied. Despite that one has chosen the VEV conditions (17) on the scalars, one must not forget the equations which result from their variations. Hence, performing a variation of $I_1+I_2+I_3$ w.r.t the remaining scalars $\phi^1, \phi^a, \phi^{ab}, \phi^{a5}$, and taking into account the v.e.v of eq-(2.17) which minimize the potential (16a), yields $F^1 = F^a = 0$ and the action (18) will then reduce to

$$S = \frac{4}{5} \mathbf{v} \int_M d^4x \ \left(F^{ab}_{\mu\nu} F^{cd}_{\rho\sigma} \epsilon_{abcd} \right) \epsilon^{\mu\nu\rho\sigma}. \tag{19}$$

A solution to the the zero torsion condition $F^a = 0$ can be simply found by setting $f^a_\mu = 0$ in eq-(6c), and which in turn, furnishes the Levi-Civita spin connection $\omega^{ab}_\mu(e^a_\mu)$ in terms of the tetrad e^a_μ. Upon doing so, the field strength F^{ab} in eq-(6e) when $f^a_\mu = 0$ and $\omega^{ab}_\mu(e^a_\mu)$ becomes then $F^{ab} = R^{ab}(\omega^{ab}_\mu) + 4e^a \wedge e^b$, where $R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} \ dx^\mu \wedge dx^\nu$ is the standard expression for the Lorentz-curvature two-form in terms of the Levi-Civita spin connection. Finally, the action (19) becomes the Macdowell-Mansouri-Chamseddine-West action

$$S = \frac{4}{5} \mathbf{v} \int d^4x \ (\ R^{ab} \ + 4 \ e^a \wedge e^b \ ) \ \wedge \ (\ R^{cd} \ + 4 \ e^c \wedge e^d \ ) \ \epsilon_{abcd}. \tag{20}$$

comprised of the Gauss-Bonnet term $R \wedge R$; the Einstein-Hilbert term $R \wedge e \wedge e$, and the cosmological constant term $e \wedge e \wedge e \wedge e$. Furthermore, there are *many* other vacuum solutions of the more fundamental action associated with the expressions $I_1+I_2+I_3$ of eqs-(10, 14, 15) and involving *all* of the terms in eq-(12). For example, for *constant* field configurations Φ^A, the inclusion of all the gauge field strengths in eq-(12) *contain* the Euler type terms $F^{ab} \wedge F^{cd} \epsilon_{abcd}$; theta type terms $F^1 \wedge F^1; F^5 \wedge F^5$ corresponding to the Maxwell a_μ and Weyl dilatation b_μ fields, respectively; Pontryagin type terms $F^a{}_b \wedge F^b{}_a$; torsion squared terms $F^a \wedge F^a$, etc ... all in one stroke. The action comprised of $I_1 + I_2 + I_3$ differs fro the action of [7].

A thorough discussion was presented in [5] about how to explicitly construct the $SU(4), SU(3,1)$ algebra generators from the $Cl(4)$ algebra generators. In the most general case one has the following isomorphisms of Lie algebras [8]

$$SO(5,1) \sim SU^*(4) \sim SL(2,H); \quad SO^*(6) \sim SU(3,1); \quad . \tag{21a}$$

$$SO(4,2) \sim SU(2,2); \quad SO(3,3) \sim SL(4,R); \quad SO(6) \sim SU(4). \tag{21b}$$

where the asterisks in $SU^*(4), SO^*(6)$ denote the *noncompact* versions of the compact groups $SU(4), SO(6)$ and $SL(2,H)$ is the special linear Mobius algebra over the field of quaternions H. All these algebras are related to each other via the Weyl unitary trick, therefore they admit an specific realization in terms of the $Cl(4,C)$ generators.

Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have ben studied in [4], [6] where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate

$$\mathbf{X} \;=\; \varphi\,\mathbf{1} \;+\; x_\mu\,\gamma^\mu \;+\; x_{\mu_1\mu_2}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2} \;+\; x_{\mu_1\mu_2\mu_3}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3} \;+\; .... \quad (22)$$

In order to match dimensions in each term of (22) a length scale parameter must be suitably introduced. In [6] we introduced the Planck scale as the expansion parameter in (22). The scalar component φ of the spacetime poly-vector valued coordinate $\mathbf{X}$ was interpreted by [11] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

$\mathcal{A}_M(\mathbf{X}) = A^I_M(\mathbf{X})\,\Gamma_I$ is a poly-vector valued gauge field whose gauge group is based on the Clifford algebra $Cl(5,C) = Cl(4,C) \oplus CL(4,C)$ spanned by $16+16$ generators. The expansion of the poly-vector $\mathcal{A}^I_M$ is also of the form

$$\mathcal{A}^I_M \;=\; \Phi^I\,\mathbf{1} + A^I_\mu\,\gamma^\mu + A^I_{\mu_1\mu_2}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2} + A^I_{\mu_1\mu_2\mu_3}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3} + ... \quad (23)$$

In order to match dimensions in each term of (23) another length scale parameter must be suitably introduced. For example, since $A_{\mu\nu\rho}$ has dimensions of $(length)^{-3}$ and A_μ has dimensions of $(length)^{-1}$ one needs to introduce another length parameter in order to match dimensions. This length parameter does not need to coincide with the Planck scale. The Clifford-algebra-valued gauge field $\mathcal{A}^I_\mu(x^\mu)\Gamma_I$ in ordinary spacetime is naturally embedded into a far richer object $\mathcal{A}^I_M(\mathbf{X})$ in C-spaces. The advantage of recurring to C-spaces associated with the $4D$ spacetime manifold is that one can have a (complex) Conformal Gravity, Maxwell and $U(4)\times U(4)$ Yang-Mills unification in a very geometric fashion.

To briefly illustrate how it can be attained, let us write in $4D$ the several components of the C-space poly-vector valued $Cl(5,C)$ gauge field $\mathbf{A}(\mathbf{X})$ as

$$A^I_0 = \Phi^I;\;\; \mathcal{A}^I_\mu;\;\; \mathcal{A}^I_{\mu\nu};\;\; \mathcal{A}^I_{\mu\nu\rho} \;=\; \epsilon_{\mu\nu\rho\sigma}\,\widetilde{\mathcal{A}}^I_\sigma;\;\; \mathcal{A}^I_{\mu\nu\rho\sigma} \;=\; \epsilon_{\mu\nu\rho\sigma}\,\widetilde{\Phi}^I. \quad (24)$$

where Φ^I and $\widetilde{\Phi}^I$ correspond to the scalar (pseudo-scalars) components of the poly-vector gauge field. Let us freeze all the degrees of freedom of the poly-vector C-space coordinate $\mathbf{X}$ in $\mathbf{A}(\mathbf{X})$ except those of the ordinary spacetime vector coordinates x^μ. As we have shown above, Conformal Gravity and Maxwell are encoded in the components of $\mathcal{A}^A_\mu\,\Gamma_A$ where Γ_A span the 16 basis elements of the $Cl(4,C)$ algebra. The antisymmetric tensorial gauge field of rank three $\mathcal{A}^A_{\mu\nu\rho}$ is dual to the vector $\widetilde{\mathcal{A}}^A_\sigma$ and has 4 independent spacetime components $(\sigma = 1,2,3,4)$, the same number as the vector gauge field $\mathcal{A}^I_\mu$. Therefore, there is another copy of the Conformal Gravity-Maxwell multiplet based on the algebra $U(2,2)$ encoded in the field $\widetilde{\mathcal{A}}^A_\sigma$.

In order to accommodate the Standard Model Group $SU(3)_c \times SU(2)_L \times U(1)_Y$ one must not forget that there is an *additional* $U(4)$ group associated to the *second* factor algebra $Cl(4,C)$ in the decomposition of $Cl(5,C) = CL(4,C) \oplus Cl(4,C)$. Hence, the basis of 32 generators of $Cl(5,C)$ given by Γ_I $(I = 1,2,3.....,32)$ appearing in $\mathcal{A}^I_\mu\,\Gamma_I$, and *in the dual* to the rank 3 antisymmetric tensor in C-space $\mathcal{A}^I_{\mu\nu\rho}\,\Gamma_I = \epsilon_{\mu\nu\rho\sigma}\,\widetilde{A}^I_\sigma\,\Gamma_I$ will provide another copy of the Conformal Gravitational-Maxwell multiplet (based on the algebra $U(2,2)$) and of the $U(4)$ Yang-Mills multiplet.

To conclude, the combination of the fields $\mathcal{A}^I_\mu \ \Gamma_I$ and $\widetilde{A}^I_\mu \ \Gamma_I$, when Γ_I are the 32 generators of the (complex) $Cl(5,C)$ algebra, by *doubling* the number of $Cl(4,C)$ degrees of freedom in the internal group space and doubling the number of degrees of freedom in spacetime, will yield two copies of a Conformal Gravity-Maxwell-like multiplet which can be assembled into a Complex Gravity-Maxwell-like theory and a $U(4)\times U(4)$ Yang-Mills multiplet in $4D$, as required, if one wishes to incorporate the $SU(3)$ and $SU(2)$ groups.

A breaking of $U(4) \times U(4) \longrightarrow SU(2)_L \times SU(2)_R \times SU(4)$ leads to the Pati-Salam GUT group [9] which contains the Standard Model Group, which in turn, breaks down to the ordinary Maxwell Electro-Magnetic (EM) $U(1)_{EM}$ and color (QCD) group $SU(3)_c$ after the following chain of symmetry breaking patterns

$$SU(2)_L \times SU(2)_R \times SU(4) \ \rightarrow \ SU(2)_L \times U(1)_R \times U(1)_{B-L} \times SU(3)_c \ \rightarrow$$
$$SU(2)_L \times U(1)_Y \times SU(3)_c \ \rightarrow \ U(1)_{EM} \times SU(3)_c. \tag{25}$$

where $B-L$ denotes the Baryon minus Lepton number charge; Y = hypercharge and the Maxwell EM charge is $Q = I_3 + (Y/2)$ where I_3 is the third component of the $SU(2)_L$ isospin.

Having explained how one generates the Standard model group and Gravity one must not forget the scalar $\Phi^I, \widetilde{\Phi}^I$ multiplets and the rank two antisymmetric tensor field $A^I_{\mu\nu}$ multiplet. The scalar Φ^I admits the $2^5 = 32$ components $\phi, \ \phi^i, \ \phi^{[ij]}, \phi^{[ijk]}, \phi^{[ijkl]}, \phi^{[ijklm]}$ associated with the $Cl(5,C)$ gauge group. Similar results apply to the $\widetilde{\Phi}^I$ components. The ϕ and $\tilde{\phi}$ fields are gauge-singlets that can be identified with the *dilaton* and *axion* scalar fields in modern Cosmology. The other scalar fields carry gauge charges and some of them can be interpreted as the Higgs scalars that will break the Weyl Conformal symmetry leading to ordinary gravity, and break the $U(4)\times U(4)$ symmetry leading to the Standard Model Group. The rank two antisymmetric tensor field $A^I_{\mu\nu}$ multiplet leads to a *generalized* Yang-Mills theory based on *tensorial* antisymmetric gauge fields of rank *two* [4]. Such antisymmetric fields do appear in the massive spectrum of strings and in the physics of membranes. Therefore, the Clifford gauge field theory in C-spaces presented here yields findings compatible with string/M theory.

Despite the appealing nature of our construction one can improve it. It is more elegant not to have to recur to the algebra $Cl(5,C)$ but instead to stick to the $Cl(4,C)$ algebra associated with the tangent space of a *complexified* $4D$ spacetime (like it occurs in Twistor theory). In this case one has then a $U(4) \times U(4)$ Yang-Mills sector corresponding to $A^A_\mu \ \Gamma_A, \ \widetilde{A}^A_\mu \ \Gamma_A$, respectively, where the Γ_A generators, $A = 1,2,3,....,16$ belong to the 16-dim $Cl(4,C)$ algebra. The key question is now : How do we incorporate gravity into the picture ? The answer to this question lies in the novel physical interpretation behind the anti-symmetric tensor gauge field of rank two $A^A_{\mu\nu} \ \Gamma_A$. It has been shown in [6] when we constructed the generalized gravitational theories in *curved* C-spaces (Clifford spaces) that covariant derivatives in C-spaces of a poly-vector $A_M(\mathbf{X})$ with respect to the area *bivector* coordinate $x^{\mu\nu}$ involves generalized connections (with more indices) in C-space and which are related to the Torsion $T^\rho_{\mu\nu} = T^a_{\mu\nu}\ V^\rho_a$ and Riemannian curvature $R^\sigma_{\mu\nu\rho} = R^{ab}_{\mu\nu}\ V^\sigma_a\ V_{b\rho}$ tensors of the underlying spacetime (V^a_μ is the tetrad/vielbein field). The generalized curvature scalar in *curved* C-spaces [6] admits an expansion in terms of sums of powers of ordinary curvature and torsion tensors; i.e. it looks like a higher derivative theory. Therefore, the components $A^a_{\mu\nu}\ \Gamma_a$ and $A^{ab}_{\mu\nu}\ \Gamma_{ab}$ of the anti-symmetric tensor gauge field of rank two $A^A_{\mu\nu}\ \Gamma_A$ can be identified with the Torsion and Riemannian curvature two-forms as follows

$$(A^a_{\mu\nu}\ \Gamma_a + A^{a5}_{\mu\nu}\ \Gamma_a\ \Gamma_5\ )\ dx^{\mu\nu} \longleftrightarrow (\widetilde{F}^a_{\mu\nu}[P]\ P_a + \widetilde{F}^a_{\mu\nu}[K]\ K_a\ )\ dx^\mu \wedge dx^\nu \quad (26a)$$

$$A^{ab}_{\mu\nu}\ \Gamma_{ab}\ dx^{\mu\nu} \longleftrightarrow R^{ab}_{\mu\nu}\ \Gamma_{ab}\ dx^\mu \wedge dx^\nu. \quad (26b)$$

where the torsion two form is defined in terms of the spin connection $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ and vielbein one forms $V^a = V^a_\mu dx^\mu$ as $\widetilde{F}^a[P] = T^a = dV^a + \omega^a{}_b \wedge V^b$; the curvature two form is defined as $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$. The conformal-boost field strength is $\widetilde{F}^a_{\mu\nu}[K]$.

Therefore, in this more natural fashion by performing the key identifications (26) relating C-space quantities to the curvature and torsion of ordinary spacetime, we may encode gravity as well, in addition to the $U(4) \times U(4)$ Yang-Mills structure *without* having to use the $Cl(5, C)$ algebra which has an intrinsic $5D$ nature, but instead we retain only the $Cl(4, C)$ algebra that is intrinsic to the *complexified* $4D$ spacetime. A *real* slice must be taken in order to extract the real four-dimensional theory from the four complex dimensional one (8 real dimensions) with complex coordinates z_1, z_2, z_3, z_4. A real slice can be taken for instance by setting $z_3 = \bar{z}_1,\ z_4 = \bar{z}_2$. Gravity and $U(4) \times U(4)$ Yang-Mills unification in $4D$ can be obtained from a $Cl(4, C)$ gauge theory in the C-space (Clifford space) comprised of poly-vector valued coordinates $\varphi, x^\mu, x^{\mu\nu}, x^{\mu\nu\rho}$..... and poly-vector valued gauge fields $A_0, A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}$, ... $A_0 = \Phi$ is the Clifford scalar. The only caveat with the C-space/spacetime correspondence of eqs-(26) is that it involves imposing constraints among the $A_{\mu\nu}$ and A_μ components of a poly-vector A_M since the field strengths $F_{\mu\nu}$ are defined in terms of A_μ. In doing so, one needs to verify that no inconsistencies arise in C-space.

Other approaches, for instance, to Grand Unification with Gravity based on spinors, C-spaces and Clifford algebras have been proposed by [12] and [13], respectively. The Gravity-Yang-Mills-Maxwell-Matter GUT model by [13] relies on the $Cl(8)$ algebra in $8D$ leading to the observed three fermion families and their masses, force strengths coupling constants, mixing angles, ... In the model by [12] the 16-dim C-space metric G_{MN} (corresponding to $4D$ Clifford algebra) has enough components in principle to accommodate ordinary gravity and the $SU(3) \times SU(2) \times U(1)$ gauge degrees of freedom in the decomposition of the C-space metric $G_{\mu\nu} = g_{\mu\nu} + A^i_\mu\ A^j_\nu\ g_{ij}$. A geometric basis for the Standard Model in term sof the Clifford algebra $Cl(7)$ was advanced by [10].

If one wishes to incorporate string theory into the picture, one needs to start with the geometrical C-space (Clifford space) corresponding to the 5 complex dimensional spacetime (10 real dimensions) and associated to the complex Clifford algebra $Cl(5,C)$. In this case the $Cl(5,C)$ symmetry is the one associated with the tangent space to the 5-complex dim-spacetime. This is *another* arena where the extended gravitational theory of the C-space belonging to the $Cl(5,C)$ algebra has enough of degrees of freedom to retrieve the physics of the Standard Model and Gravity in four real dimensions. 10 real dimensions is the dimensions of the anomaly-free superstring theory. If one wishes to incorporate F theory the natural setting would be a 6 complex dim space (12 real dimensional) corresponding the $Cl(6,C)$ algebra isomorphic to the 8×8 matrix algebra over the complex numbers.

The 16 fermions (quarks and leptons) of the first generation can be arranged into the 16 entries of the 4×4 matrix associated with the $A = 1,2,3,.....,16$ indices of the $Cl(4)$ gauge algebra as follows

$$\Psi^A\ \Gamma_A\ \equiv\ \begin{pmatrix} e & u_r & u_b & u_g \\ \nu_e & d_r & d_b & d_g \\ e^c & u^c_r & u^c_b & u^c_g \\ \nu^c_e & d^c_r & d^c_b & d^c_g \end{pmatrix} \tag{27}$$

where we have omitted the spacetime spinorial indices $\alpha = 1,2,3,4$ in each one of the entries of the above 4×4 matrix. In particular, e, ν_e denote the electron and its neutrino. The subscripts r, b, g denote the red, blue, green color of the up and down quarks, u, d. The superscript c denotes their anti-particles. The fermionic matter kinetic terms is

$$\mathcal{L}_m\ =\ \sum_{i=1}^{n_f}\ \bar\Psi^A_{\alpha i}\ \Gamma^\mu_{\alpha\beta}\ (\ \delta_{AC}\ \partial_\mu\ +\ f_{ABC}\ \mathcal{A}^B_\mu\ )\ \Psi^C_{\beta i}. \tag{28}$$

where the indices $i = 1,2,3,...n_f$ extend over the number of generations (flavors) and $A,B,C = 1,2,3,.....,16$. f_{ABC} denote the structure constants of the $Cl(4)$ gauge algebra.

The Pati-Salam (PS) $SU(4)\times SU(2)_L\times SU(2)_R$ group arises from the symmetry breaking of *one* of the $SU(4)$ factors in $SU(4)\times SU(4)$ given by $SU(4)\rightarrow SU(2)_L\times SU(2)_R\times U(1)_Z$. This requires taking the following vacuum expectation value (VEV) of the Higgs scalar

$$<\ \Phi\ >\ \equiv\ v_1\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{29}$$

Taking the VEV of the other Higgs scalar

$$<\ \tilde\Phi\ >\ \equiv\ v_2\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{30}$$

leads to a breaking of $SU(4) \rightarrow SU(3)_c \times U(1)_{B-L}$. Therefore, an overall breaking of $SU(4) \times SU(4)$ contains the Patti-Salam (PS) model in the intermediate stage as follows

$$SU(4) \times SU(4) \;\rightarrow\; [SU(4) \times SU(2)_L \times SU(2)_R]_{PS} \times U(1)_Z \rightarrow$$
$$SU(3)_c \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R \times U(1)_Z. \qquad (31)$$

The Higgs Potential $V(\Phi, \tilde{\Phi})$ involving quadratic and quartic powers of the fields is of the form

$$V \;=\; -m_1^2\, Tr(\Phi^2) + \lambda_1\, [Tr(\Phi^2)]^2 + \lambda_2\, Tr(\Phi^4) - m_2^2\, Tr(\tilde{\Phi}^2) + \lambda_3\, [Tr(\tilde{\Phi}^2)]^2 +$$
$$\lambda_4\; Tr(\tilde{\Phi}^4) \;+\; \lambda_5\; Tr(\Phi^2 \tilde{\Phi}^2) \;+\; \lambda_6\; Tr(\Phi \tilde{\Phi} \Phi \tilde{\Phi})\;. \qquad (32)$$

A further symmetry breaking

$$U(1)_{B-L} \times SU(2)_R \times U(1)_Z \;\rightarrow\; U(1)_Y. \qquad (33)$$

requires additional Higgs fields leading to the Standard Model

$$SU(3)_c \times SU(2)_L \times U(1)_Y \;\rightarrow\; SU(3)_c \times U(1)_{EM}. \qquad (34)$$

The Yukawa coupling terms furnishing mass terms for the quarks and leptons are contained in the $f_{ABC} \bar{\Psi}^A \mathcal{A}_0^B \Psi^C, f_{ABC} \bar{\Psi}^A \mathcal{A}_5^B \Psi^C$ pieces (after taking the VEV of the Higgs scalars) associated to the C-space fermionic kinetic terms $\bar{\Psi}_A \Gamma^M (D_M)^{AB} \Psi_B$ due to the fact that the Higgs scalar fields are identified by $\Phi^A = \mathcal{A}_0^A$ and $\epsilon_{\mu\nu\rho\tau} \tilde{\Phi}^A = \mathcal{A}^A_{\mu\nu\rho\tau}$ as shown in (24). The kinetic terms for the Higgs field $(D_\mu \Phi)^\dagger (D^\mu \Phi)$ are contained in the components $F_{0M} F^{0M}$ associated to the $F_{MN} F^{MN}$ terms. Whereas, the kinetic terms for the Higgs field $(D_\mu \tilde{\Phi})^\dagger (D^\mu \tilde{\Phi})$ are contained in the components $F_{5M} F^{5M}$ associated to the $F_{MN} F^{MN}$ terms. Inserting the VEV of the Higgs scalars into their kinetic terms, after redefining the fields such that the new fields have zero VEV, yields the mass terms from the gauge fields associated to the broken gauge symmetries.

There is another symmetry-breaking branch that leads to the Standard Model and which does not contain the PS model. This requires breaking one of the $SU(4)$ factors as

$$SU(4) \times SU(4) \;\rightarrow\; SU(3)_c \times SU(4) \times U(1)_{B-L}. \qquad (2.34)$$

leading to a partial unification model based on $SU(4) \times U(1)_{B-L}$. which can be broken down to the minimal left-right model via the Higgs mechanism [14]. More work remains to be done to verify whether or not this approach to unification is feasible. In particular, a thorough analysis of the parameters involved in the potential $V(\Phi, \tilde{\Phi})$, the gauge couplings g, the expectation values parameters v_1, v_2, ... is warranted.

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