Formalistics of Generalization*

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Abstract

In the framework of ZF formally considered generalizations, such as whole numbers generalizing natural number, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc. The formal consideration of this may be especially useful for computer proof assistants.

Keywords: ZF, ZFC, generalization, formalistics, bijection, injection

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The rationale and an example

In mathematics it is often encountered that a small set S naturally bijectively corresponds to a subset R of a larger set B. (In other words, there is specified an injection from S to B.) It is a widespread practice to equate S with R.

Remark 1. I denote the first set S from the first letter of the word "small" and the second set B from the first letter of the word "big", because S is intuitively considered as smaller than B. (However we do not require $\operatorname{card} S < \operatorname{card} B$.)

For example take S the set of whole numbers and B the set of rational numbers.

Through this example we see that B can be considered a generalization of S.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as "without proof".

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M: B \rightarrow B'$ such that $S \subseteq B'$. (I call this bijection M from the first letter of the word "move" which signifies the move from the old set B to a new set B').

The formalistic

Let S and B are sets. Let E is an injection from S to B. Let $R = \operatorname{im} E$.

Let $t = \mathscr{P} \bigcup \bigcup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t; x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t; x) = \{\{t\}, \{t, x\}\}$ by definition.

Theorem 2. $(t; x) \notin S$.

Proof. Suppose $(t; x) \in S$. Then $\{\{t\}, \{t, x\}\} \in S$. Consequently $\{t\} \in \bigcup S$; $t \subseteq \bigcup \bigcup S$; $t \in \mathscr{P} \bigcup \bigcup S$; $t \in t$ what contradicts to the axiom of foundation (aka axiom of regularity). □

*. This document has been written using the GNU TeXmacs text editor (see `www.texmacs.org`).

Definition 3. Let $B' = \operatorname{im} M$.

Theorem 4. $S \subseteq B'$

Proof. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$; $x \in \operatorname{im} M = B'$. $\square$

Obvious 5. E is a bijection from S to R.

Theorem 6. M is a bijection from B to B'.

Proof. Surjectivity of M is obvious. Let's prove injectivity.

Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases:

- $\boldsymbol{a, b \in R}$**.** $M(a) = E^{-1}a$; $M(b) = E^{-1}b$; $E^{-1}a = E^{-1}b$. Thus $a = b$ because E^{-1} is a bijection.
- $\boldsymbol{a \in R}$**,** $\boldsymbol{b \notin R}$**.** $M(a) = E^{-1}a$; $M(b) = (t; b)$; $M(a) \in S$; $M(b) \notin S$. Thus $M(a) \neq M(b)$.
- $\boldsymbol{a \notin R}$**,** $\boldsymbol{b \in R}$**.** Analogous.
- $\boldsymbol{a, b \notin R}$**.** $M(a) = (t; a)$; $M(b) = (t; b)$. Thus $M(a) = M(b)$ implies $a = b$. $\square$

Theorem 7. $M \circ E = \mathrm{id}_S$.

Proof. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$. $\square$

Obvious 8. $E = M^{-1}|_S$.