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(Anti) de Sitter Relativity, Modified Newtonian Dynamics, Noncommutative Phase Spaces and the Pioneer Anomaly

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Abstract

It is shown how the de-Sitter Relativistic behaviour of the *hyperbolic* trajectory of Pioneer, due to the expansion of the Universe (non-vanishing cosmological constant), *is* the underlying physical reason for the observed anomalous acceleration of the Pioneer spacecraft of the order of $c^2/R_H \sim 8.74 \times 10^{-10}\ m/s^2$, where c is the speed of light and R_H is the present Hubble scale. We display the explicit isomorphism among Yang's Noncommutative space-time algebra, the $4D$ Conformal algebra $SO(4,2)$ and the area-bi-vector-coordinates algebra in Clifford spaces. The former Yang's algebra involves noncommuting coordinates and momenta with a *minimum* Planck scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). It is shown how Modified Newtonian dynamics is also a consequence of Yang's algebra resulting from the modified Poisson brackets. To finalize we study the $deformed$ Kepler and free motion resulting from the modified Newtonian dynamics due to the Leznov-Khruschev noncommutative phase space algebra and which stems also from the Conformal algebra $SO(4,2)$ in four dimensions. Numerical examples are found which yield results close to the experimental observations, but only in very extreme special cases and which seem to be consistent with a Machian view of the Universe.

1 INTRODUCTION

Since Anderson et al (see [12] for an extensive detailed account of the history of the project) announced that the Pioneer 10 and 11 spacecrafts exhibit an unexplained anomalous acceleration of the order of $c^2/R_H \sim 8.74 \times 10^{-10}\ m/s^2$, where c is the speed of light and R_H is the present Hubble scale, numerous articles appeared with many plausible explanations [12], [15], [16]. To our knowledge, the first person who predicted the anomalous acceleration (based on a blueshifting phenomenon due to subquantum kinetics) years *before* Anderson et al observed it, was [14].

In this letter we propose a concise, elegant and geometrical interpretation of the anomalous acceleration based on the de-Sitter Relativistic behaviour of the *hyperbolic* trajectory of Pioneer, due to the expansion of the Universe (non-vanishing cosmological constant). We believe that de Sitter Relativity [11], Relativity based on the de Sitter group $SO(4,1)$ in four-dimensions, *is* the underlying physical reason for the observed anomalous acceleration of the Pioneer spacecraft of the order of $c^2/R_H \sim 8.74 \times 10^{-10}\ m/s^2$. In essence, the origins of the Pioneer anomalous acceleration is that from the de Sitter-Relativity perspective, the sun is *not* an inertial frame of reference. Our derivation is presented in section **2** .

In section **3** we display the explicit isomorphism among Yang's Noncommutative space-time algebra, the $4D$ Conformal algebra $SO(4,2)$ and the area-bi-vector-coordinates algebra in Clifford spaces [5]. The former Yang's algebra involves noncommuting coordinates and momenta with a *minimum* Planck scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). We analyze a simple Quantum Mechanical model for a scalar field in a Noncommutative $4D$ spacetime based on ordinary QM in $D+2$-dim. Finally in section **4**, we study the *deformed* Kepler and free motion resulting from the modified Newtonian dynamics due to the Yang algebra [1] and the Leznov-Khruschev [13] Noncommutative phase space algebra, which stems also from the Conformal algebra $SO(4,2)$ in four dimensions. Numerical examples are found which yield results close to the experimental observations, but only in very extreme special cases and which seem to be consistent with a Machian view of the Universe [4].

2 de SITTER RELATIVITY AND THE PIONEER ANOMALY

The action that describes the motion of a particle of mass m on a hyperboloid is given in terms of the square root of the quadratic Casimir [11] :

$$S \;=\; -\int d\tau\ [\ -\frac{1}{2R^2}\ \Sigma_{AB}\ \Sigma^{AB}]^{\frac{1}{2}}. \tag{2.1}$$

where the angular-momentum-like variables are

$$\Sigma^{AB} \;=\; mc\ (\ Y^A\frac{dY^B}{d\tau}\ -\ Y^B\frac{dY^A}{d\tau}\ ). \tag{2.2a}$$

and the Y^A coordinates are subjected to the $SO(4,1)$-invariant norm condition

$$\eta_{AB}\ Y^A\ Y^B\ =\ -\ R^2,\quad A,B=0,1,2,3,4;\quad \eta_{AB}=(+1,-1,-1,-1,-1).. \tag{2.2b}$$

The indices $a, b = 0, 1, 2, 3$ refer to the coordinates x^0, x^1, x^2, x^3 obtained from a stereographic projection of the 4-dim hyperboloid onto the base manifold (equator of the hyperboloid) represented by a 4-dim Minkowski spacetime. The constraint (2.2b) can be implemented by the addition of a Lagrange multiplier β in the action

$$S\ =\ -\int d\tau\ [\ (\ -\frac{1}{2R^2}\ \Sigma_{AB}\ \Sigma^{AB}\ )^{\frac{1}{2}}\ +\ \beta\ (\eta_{AB}\ Y^A\ Y^B + R^2)\ ]. \tag{2.3}$$

A variation of the action yields the equations of motion [11] :

$$\frac{d^2 Y^A}{d\tau^2}\ -\ \frac{Y^A}{R^2}\ =\ 0. \tag{2.4a}$$

after using the constraints

$$\eta_{AB}\ Y^A\ Y^B\ =\ -\ R^2,\quad \eta_{AB}\ \frac{dY^A}{d\tau}\ \frac{dY^B}{d\tau}\ =\ 1. \tag{2.4b}$$

that fixes the Lagrange multiplier to the value $\beta = mc/R^2$ (the proper time τ has length dimensions, like ct),

The simplest solution of (2.5) is a *hyperbolic* trajectory in the $Y^0 - Y^4$ plane

$$Y^0\ =\ R\ sinh\ (\frac{\tau}{R}),\quad Y^4\ =\ R\ cosh\ (\frac{\tau}{R}),\quad Y^1 = Y^2 = Y^3 = 0. \tag{2.6}$$

one can verify that

$$\eta_{AB}\ Y^A\ Y^B\ =\ (Y^0)^2\ -(Y^4)^2\ =\ R^2\ [sinh^2(\frac{\tau}{R})\ -\ cosh^2(\frac{\tau}{R})\ ]\ =\ -R^2. \tag{2.7}$$

so the components of the acceleration are :

$$\frac{d^2Y^0}{d\tau^2}\ =\ \frac{Y^0}{R^2}\ =\ \frac{1}{R}\ sinh\ (\frac{\tau}{R}),\quad \frac{d^2Y^4}{d\tau^2}\ =\ \frac{Y^4}{R^2}\ =\ \frac{1}{R}\ cosh\ (\frac{\tau}{R}). \tag{2.8}$$

From eq-(2.4b) by a simple differentiation one can infers why the acceleration is *spacelike* when the velocity is *timelike*,

$$a^2\ =\ (\frac{d^2Y^0}{d\tau^2})^2\ -\ (\frac{d^2Y^4}{d\tau^2})^2\ =-\frac{1}{R^2}. \tag{2.9}$$

therefore, the *magnitude* of the acceleration is then

$$| a | = \sqrt{-a^2} = \frac{1}{R}. \tag{2.10}$$

in units of $\hbar = c = 1$. Since the proper time τ has length dimensions (like ct), in order to attain the right units of acceleration in eq-(2.10) one needs to insert the standard numerical value of the speed of light $c \sim 3 \times 10^8$ m/s. After doing so and by setting the de Sitter scale R (one half of the throat size $2R$) to coincide precisely with the Hubble scale $R_H \sim 10^{26}$ m, one recovers the Pioneer acceleration [12]

$$| a | = \frac{c^2}{R_H} \sim 8.74 \times 10^{-8} \frac{cm}{s^2} = 8.74 \times 10^{-10} \frac{m}{s^2} = a_{Pioneer}. \tag{2.11}$$

What is the physical reason behind this ? Since the path traced by the Pioneer spacecraft is indeed a *hyperbola* [12] , one can view the Pioneer spacecraft after it has left the outer edge of the solar system as it were a free particle (devoid of external forces) moving on a 4-dim hyperboloid. Such hyperboloid can be embedded into a 5-dim pseudo-Euclidean space of coordinates Y^0, Y^1, Y^2, Y^3, Y^4 and signature $(+, -, -, -, -)$ associated with the 4-dim de Sitter group $SO(4, 1)$. The 4-dim Anti de Sitter group is $SO(3, 2)$. The reason why a 4-dim hyperboloid is involved in the trajectory is a direct consequence of the de Sitter solutions to Einstein's equations with a $non-vanishing$ cosmological constant equal to $\Lambda = \frac{3}{R^2}$; namely, the present universe is in an accelerated expansion de Sitter phase, and the scaling factor $a(t)$ obeys the condition $\frac{1}{a(t)} \frac{da(t)}{dt} = H$ so that $a(t) = exp\ (\int\ Hdt)$. When $H = H_o = constant$ (Hubble constant observed today $H_o = c/R_H$) one has a scaling factor $a(t) = exp\ (H_o t)$ in a purely de Sitter universe.

Let us answer the poignant question posed in [12] : Why planets revolving around the sun in elliptical orbits $don't$ experience such anomalous acceleration ? ... because the planets are *bound* to the solar system, they are not moving freely along the hyperbolas (geodesics) of the 4-dim hyperboloid. The *geodesics* (hyperbolic paths) from the point of view of the $conformally\ flat$ de Sitter metric are *not* "straight lines" (geodesics) from the point of view a flat Minkowski metric, and hence, this is the physical meaning of the anomalous Pioneer acceleration w.r.t the sun (solar system). The sun (solar system) is not a truly inertial system within the background de Sitter space perspective. The solar system can be seen as $non-expanding$ pennies on an expanding balloon (de Sitter universe).

For example, the path of a falling projectile from a plane seems parabolic (curved) from the point of view of an observer fixed on the ground. Such observer does not constitute an inertial frame of reference because he experiences an upward $force$ on its feet due to the reaction-force of the ground to his own weight. A free falling observer is truly inertial, and from his point of view, the path of the falling projectile from a plane appears as a *straight* line, as a geodesic. This is in essence the origins of the Pioneer anomalous acceleration : from the de Sitter-Relativity perspective the sun is *not* an inertial frame of reference.

Notice that our proposal *is very different* from the *vacuole* proposal of [15] involving Schwarzschild-de Sitter metrics and many other proposals involving blueshifts, dark matter, dark energy, scalar-tensor modified theories of gravity, ... that can be found in [14], [16] and in the references of [12]. Our proposal is more closely related to I. Segal's conformal-algebraic approach to cosmology [21], [22]. The stereographic projections of the 4-dim hyperboloid $\mathcal{H}^4$ (embedded in a 5-dim pseudo-Euclidean space) onto the 4-dim Minkowski spacetime with coordinates $x^a = x^0, x^1, x^2, x^3$, and metric $\eta_{ab} = diag\ (1, -1, -1, -1)$ are given by

$$Y^0 = \frac{x^0}{1 - (x_a x^a/4R^2)}, \quad Y^1 = \frac{x^1}{1 - (x_a x^a/4R^2)},$$
$$Y^2 = \frac{x^2}{1 - (x_a x^a/4R^2)}, \; Y^3 = \frac{x^3}{1 - (x_a x^a/4R^2)}. \tag{2.12a}$$

and

$$Y^4 \;=\; R\,\frac{1 + (x_a x^a/4R^2)}{1 - (x_a x^a/4R^2)}, \quad x_a x^a \;=\; \eta_{ab} x^a x^b,\; a, b = 0, 1, 2, 3. \tag{2.12b}$$

The stereographic projection process of the 4-dim hyperboloid $\mathcal{H}^4$ onto the 4-dim Minkowski spacetime M^4 is tantamount of studying the motion of a particle in the 4-dim Lobatchevsky hyperbolic space, a disc with a conformally flat metric $g_{ab} = \Omega^2\ \eta_{ab}$ where the conformal factor Ω is given by

$$\Omega \;=\; \frac{1}{1 - (x_a x^a/4R^2)}, \quad g_{ab} \;=\; \Omega^2\ \eta_{ab}, \quad a = 0, 1, 2, 3. \tag{2.13}$$

A quantum theory in Lobatchevsky hyperbolic space has been recently been reviewed in [17]. The stereographic projection $\mathcal{H}^4 \to M^4$ can also be used to recast the equations of motion (expressed by the embedding 5-dim pseudo-Euclidean space coordinates Y^A) in terms of the 4-dim Minkowski spacetime coordinates $x^a = x^0, x^1, x^2, x^3$ as follows [11]

$$\frac{dp^a}{d\tau} \;+\; \frac{x^c\ u_c}{R^2\ \Omega}\, p^a \;-\; \frac{mc}{2R^2\Omega}\, x^a \;=\; 0. \tag{2.14}$$

$u^a = (dx^a/d\tau)$ is the four-velocity . The above equations of motion is just the geodesic equation [11]

$$\frac{dp^a}{d\tau} + \Gamma^a_{bc}\, p^b\, u^c = 0. \tag{2.15}$$

associated to the Levi-Civita connection Γ^a_{bc} and corresponding to the de Sitter metric which is conformally flat $g_{ab} = \Omega^2\ \eta_{ab}$. The connection in (2.15) plays the role of a "pseudo-force" which imparts the massive particle with an "acceleration" independent of its mass.

Thus, to conclude, the geodesic *free* motion of a massive particle living on a $4D$ hyperboloid $\mathcal{H}^4$, upon performing the stereographic projection of the points of the hyperboloid onto the points of the $4D$ Minkowski base spacetime, is governed by a geodesic equation described by (2.15) in terms of the x^a, $p^a = m(dx^a/d\tau)$ variables. It is not surprising to see why there are *hyperbolic* trajectories associated with a *constant* acceleration $|a| = c^2/R = c^2/R_H \sim a_{Pioneer}$ described by the above solutions (2.6) of eqs-(2.4a). One finds solutions in terms of the Y^A variables (hyperbolas) and afterwards one performs the stereographic projections onto the base Minkowski spacetime.

Conformal boosts bestow a massive particle with a constant acceleration, whereas ordinary momentum transformations amount to translations. See [11] for further details pertaining the differences between the conformal-boost-momentum versus the ordinary translation momentum. In particular, we can see why the interplay between large (cosmological) and small (solar system) scales stems from the fact that the canonical conjugate variable to the conformal-boost-momentum is the *inverse* of the position coordinate $z^a = \frac{x^a}{x^2}$. Since under space inversions large and small scales are interchanged this explains the interplay between large and small scales.

Finally, to address the issue of the *direction* of the acceleration one would have to take into account the effect of the solar mass on the spacetime geometry in addition to the cosmological constant. The study of the anomalous acceleration within the context of a Schwarzschild-de Sitter metric and the two different scale regimes (solar system versus cosmological scales) has been performed by [15] where it was found the anomalous acceleration points towards the sun.

3 ON NONCOMMUTATIVE PHASE SPACES

The main result of this section is that there is a *subalgebra* of the C-space (Clifford space) operator-valued coordinates which is *isomorphic* to the Noncommutative Yang's spacetime algebra [1] . This, in conjunction to the discrete spectrum of angular momentum, leads to discrete area quantization in multiples of Planck areas. Namely, the $4D$ Yang's Noncommutative space-time (YNST) algebra [1] (written in terms of $8D$ phase-space coordinates) is isomorphic to the 15-dimensional *subalgebra* of the C-space (Cliiford space) operator-valued coordinates associated with the *holographic areas* of C-space [5]. This connection between Yang's algebra and the $6D$ Clifford algebra is possible because the $8D$ phase-space coordinates x^μ, p^μ (associated to a $4D$ spacetime) have a one-to-one correspondence to the $\hat{X}^{\mu 5}; \hat{X}^{\mu 6}$ holographic area-coordinates of the C-space (corresponding to the $6D$ Clifford algebra).

Furhermore, Tanaka [3] has shown that the Yang's algebra [1] (with 15 generators) is related to the $4D$ conformal algebra (15 generators) which in turn is isomorphic to a subalgebra of the $4D$ Clifford algebra because it is known that the 15 generators of the $4D$ conformal algebra $SO(4,2)$ can be explicitly realized in terms of the $4D$ Clifford algebra as [6] :

$$P^\mu = \mathcal{M}^{\mu 5} + \mathcal{M}^{\mu 6} = \gamma^\mu(\mathbf{1}+\gamma^5). \;\; K^\mu = \mathcal{M}^{\mu 5} - \mathcal{M}^{\mu 6} = \gamma^\mu(\mathbf{1}-\gamma^5). \;\; D = \gamma^5. \;\; M^{\mu\nu} = i[\gamma^\mu, \gamma^\nu].. \tag{3.1}$$

where the Clifford algebra generators :

$$\mathbf{1}, \quad and \quad \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma^5. \tag{3.2}$$

account for the extra *two* directions within the C-space associated with the $4D$ Cliffiord-algebra leaving effectively $4 + 2 = 6$ degrees of freedom that match the degrees of freedom of a $6D$ spacetime [6] . The relevance of [6] is that it was not necessary to work directly in $6D$ to find a realization of the $4D$ conformal algebra $SO(4,2)$. It was possible to attain this by recurring solely to the $4D$ Clifford algebra as shown in eq-(3.1) .

One can also view the $4D$ conformal algebra $SO(4,2)$ realized in terms of a 15-dim *subalgebra* of the $6D$ Clifford algebra. The bivector holographic area-coordinates $X^{\mu\nu}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_\nu$. The bivector coordinates $X^{\mu 5}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_5$ where now the Γ^5 is another generator of the $6D$ Clifford algebra and *must not* be confused with the usual γ^5 defined by eq-(3.2). The bivector coordinates $X^{\mu 6}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_6$. The bivector coordinate X^{56} couples to the basis generator $\Gamma_5 \wedge \Gamma_6$.

In view of this fact that these bivector holographic area-coordinates in $6D$ *couple* to the bivectors basis elements $\Gamma_\mu \wedge \Gamma_\nu, ...$, and whose algebra is in turn isomorphic to the $4D$ conformal algebra $SO(4,2)$ via the realization in terms of the $6D$ angular momentum generators (and boosts generators) $\mathcal{M}^{\mu\nu} \sim [\Gamma^\mu, \Gamma^\nu]$, $\mathcal{M}^{\mu 5} \sim [\Gamma^\mu, \Gamma^5]$,.... we shall $define$ the holographic *area coordinates algebra* in C-space as the *dual* algebra to the $SO(4,2)$ conformal algebra (realized in terms of the $6D$ angular momentum, boosts, generators in terms of a $6D$ Clifford algebra generators as shown)

Notice that the conformal boosts K^μ and the translations P^μ in eq-(3.1) do commute $[P^\mu, P^\nu] = [K^\mu, K^\nu] = 0$ and for this reason we shall assign the appropriate correspondence $p^\mu \leftrightarrow X^{\mu 6}$ and $x^\mu \leftrightarrow X^{\mu 5}$, up to numerical factors (lengths) to match dimensions, in order to attain *noncommuting* variables x^μ, p^μ .

Therefore, one has two possible routes to relate Yang's algebra with Clifford algebras. One can relate Yang's algebra with the holographic area-coordinates algebra in the C-space associated to a $6D$ Clifford algebra and/or to the subalgebra of a $4D$ Clifford algebra via the realization of the conformal algebra $SO(4,2)$ in terms of the $4D$ Clifford algebra generators $\mathbf{1}, \gamma^5, \gamma^\mu$ as shown in eq-(3.1). Since the relation between the $4D$ conformal and Yang's algebra and the implications for the AdS/CFT, dS/CFT duality have been discussed before by Tanaka [3], in this section we shall establish the following *correspondence* between the C-space holographic-area coordinates algebra (associated to the $6D$ Clifford algebra) and the Yang's spacetime algebra via the angular momentum generators $\hat{M}^{\mu\nu}$ in $6D$ (after inserting $\hbar$ in the appropriate places) as follows :

$$i\hat{M}^{\mu\nu} = i\hbar\Sigma^{\mu\nu} \leftrightarrow i\frac{\hbar}{\lambda^2}\hat{X}^{\mu\nu}. \qquad (3.3)$$

$$i\hat{M}^{56} = i\hbar\Sigma^{56} \leftrightarrow i\frac{\hbar}{\lambda^2}\hat{X}^{56}. \qquad (3.4)$$

$$i\lambda^2\Sigma^{\mu 5} \leftrightarrow i\lambda\hat{x}^\mu \leftrightarrow i\hat{X}^{\mu 5}. \qquad (3.5)$$

$$i\lambda^2\Sigma^{\mu 6} \leftrightarrow i\lambda^2\frac{R}{\hbar}\hat{p}^\mu \leftrightarrow i\hat{X}^{\mu 6}. \qquad (3.6)$$

Notice that one is establishing a *correspondence* among $\hat{x}^\mu, \hat{p}^\mu$ with $\Sigma^{\mu 5}, \Sigma^{\mu 6}$, respectively, which is *not* to say that x^μ, p^μ are the *same* as angular momentum operators.

With the Hermitian (bivector) operator- coordinates :

$$(\hat{X}^{\mu\nu})^\dagger = \hat{X}^{\mu\nu}. \quad (\hat{X}^{\mu 5})^\dagger = \hat{X}^{\mu 5}. \quad (\hat{X}^{\mu 6})^\dagger = \hat{X}^{\mu 6}. \quad (\hat{X}^{56})^\dagger = \hat{X}^{56}. \qquad (3.7)$$

The algebra generators can be realized as :

$$\hat{X}^{\mu\nu} = i\lambda^2(X^\mu\frac{\partial}{\partial X_\nu} - X^\nu\frac{\partial}{\partial X_\mu}). \qquad (3.8a)$$

$$\hat{X}^{\mu 5} = i\lambda^2(X^\mu\frac{\partial}{\partial X_5} - X^5\frac{\partial}{\partial X_\mu}). \qquad (3.8b)$$

$$\hat{X}^{\mu 6} = i\lambda^2(X^\mu\frac{\partial}{\partial X_6} - X^6\frac{\partial}{\partial X_\mu}). \qquad (3.8c)$$

$$\hat{X}^{56} = i\lambda^2(X^5\frac{\partial}{\partial X_6} - X^6\frac{\partial}{\partial X_5}). \qquad (3.8d)$$

where the angular momentum generators are defined as usual :

$$\hat{M}^{\mu\nu} \equiv \hbar\Sigma^{\mu\nu}. \quad \hat{M}^{\mu 5} \equiv \hbar\Sigma^{\mu 5}. \quad \hat{M}^{\mu 6} \equiv \hbar\Sigma^{\mu 6}. \quad \hat{M}^{56} \equiv \hbar\Sigma^{56}. \tag{3.8e}$$

which have a one-to-one correspondence to the Yang Noncommutative space-time (YNST) algebra generators in $4D$. These generators (angular momentum differential operators) act on the coordinates of a $5D$ hyperboloid AdS_5 space defined by :

$$-(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 - (X^6)^2 = R^2. \tag{3.9a}$$

where R is the *throat* size of the hyperboloid. This introduces an extra and crucial scale in addition to the Planck scale. Notice that $\eta^{55} = +1. \quad \eta^{66} = -1.$ $5D$ de Sitter space dS_5 has the topology of $S^4 \times R^1$. Whereas AdS_5 space has the topology of $R^4 \times S^1$ and its conformal (projective) boundary at infinity has a topology $S^3 \times S^1$. Whereas the *Euclideanized* Anti de Sitter space AdS_5 can be represented geometrically as two disconnected branches (sheets) of a $5D$ hyperboloid embedded in $6D$. The topology of these two disconnected branches is that of a $5D$ disc and the metric is the Lobachevsky one of constant negative curvature. The conformal group $SO(4,2)$ leaves the $4D$ lightcone at infinity invariant.

Thus, *Euclideanized* AdS_5 is defined by a Wick rotation of the x^6 coordinate giving :

$$-(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2 = R^2. \tag{3.9b}$$

whereas de Sitter space dS_5 with the topology of a pseudo-sphere $S^4 \times R^1$, and *positive* constant scalar curvature is defined by :

$$-(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2 = -R^2. \tag{3.9c}$$

(Notice that Tanaka [3] uses $different$ conventions than ours in his definition of the $5D$ hyperboloids. He has a sign change from R^2 to $-R^2$ because he introduces i factors in iR) .

After this discussion and upon a direct use of the correspondence in eqs-(3.3, 3.4, 3.5, 3.6,3.7, 3.8) yields the exchange algebra between the position and momentum coordinates :

$$[\hat{X}^{\mu 6}, \hat{X}^{56}] = -i\lambda^2\eta^{66}\hat{X}^{\mu 5} \leftrightarrow [\frac{\lambda^2 R}{\hbar}\hat{p}^{\mu},\ \lambda^2\Sigma^{56}] = -i\lambda^2\eta^{66}\lambda\hat{x}^{\mu}. \tag{3.10}$$

from which we can deduce that :

$$[\hat{p}^\mu,\ \Sigma^{56}] = -i\eta^{66}\frac{\hbar}{\lambda R}\hat{x}^\mu. \tag{3.11}$$

and after using the definition $\mathcal{N} = (\lambda/R)\Sigma^{56}$ one has the exchange algebra commutator of p^μ and $\mathcal{N}$ of the Yang's spacetime algebra :

$$[\hat{p}^\mu, \mathcal{N}] = -i\eta^{66}\frac{\hbar}{R^2}\hat{x}^\mu. \tag{3.12}$$

The other commutator is :

$$[\hat{X}^{\mu 5}, \hat{X}^{56}] = -[\hat{X}^{\mu 5}, \hat{X}^{65}] = i\eta^{55}\lambda^2\hat{X}^{\mu 6} \leftrightarrow [\lambda\hat{x}^\mu, \lambda^2\Sigma^{56}] = i\eta^{55}\lambda^2\lambda^2\frac{R}{\hbar}\hat{p}^\mu. \tag{3.13}$$

from which we can deduce that :

$$[\hat{x}^\mu, \Sigma^{56}] = i\eta^{55}\frac{\lambda R}{\hbar}\hat{p}^\mu. \tag{3.14}$$

and after using the definition $\mathcal{N} = (\lambda/R)\Sigma^{56}$ one has the exchange algebra commutator of x^μ and $\mathcal{N}$ of the Yang's spacetime algebra :

$$[\hat{x}^\mu, \mathcal{N}] = i\eta^{55}\frac{\lambda^2}{\hbar}\hat{p}^\mu. \tag{3.15}$$

The other relevant holographic area-coordinates commutators in C-space are :

$$[\hat{X}^{\mu 5}, \hat{X}^{\nu 5}] = -i\eta^{55}\lambda^2\hat{X}^{\mu\nu} \leftrightarrow [\hat{x}^\mu, \hat{x}^\nu] = -i\eta^{55}\lambda^2\Sigma^{\mu\nu}. \tag{3.16}$$

after using the representation of the C-space operator holographic area-coordinates :

$$i\hat{X}^{\mu\nu} \leftrightarrow i\lambda^2\frac{1}{\hbar}\mathcal{M}^{\mu\nu} = i\lambda^2\Sigma^{\mu\nu} \quad i\hat{X}^{56} \leftrightarrow i\lambda^2\Sigma^{56}. \tag{3.17}$$

where we appropriately introduced the Planck scale λ as one should to match units.

From the correspondence :

$$\hat{p}^\mu = \frac{\hbar}{R}\Sigma^{\mu 6} \leftrightarrow \frac{\hbar}{R}\frac{1}{\lambda^2}\hat{X}^{\mu 6}. \tag{3.18}$$

one can obtain nonvanishing momentum commutator :

$$[\hat{X}^{\mu 6}, \hat{X}^{\nu 6}] = -i\eta^{66}\lambda^2\hat{X}^{\mu\nu} \leftrightarrow [\hat{p}^\mu, \hat{p}^\nu] = -i\eta^{66}\frac{\hbar^2}{R^2}\Sigma^{\mu\nu}. \tag{3.19}$$

The signatures for AdS_5 space are $\eta^{55} = +1$; $\eta^{66} = -1$ and for the *Euclideanized* AdS_5 space are $\eta^{55} = +1$ and $\eta^{66} = +1$. Yang's space-time algebra corresponds to the latter case.

Finally, the *modified* Weyl-Heisenberg algebra can be read from the following C-space commutators :

$$[\hat{X}^{\mu 5}, \hat{X}^{\nu 6}] = -i\eta^{\mu\nu}\lambda^2\hat{X}^{56} \leftrightarrow$$

$$[\hat{x}^{\mu}, \hat{p}^{\mu}] = -i\hbar\eta^{\mu\nu}\frac{\lambda}{R}\Sigma^{56} = -i\hbar\eta^{\mu\nu}\mathcal{N}. \tag{3.20}$$

There are other commutation relations like

$$[\Sigma^{\mu\nu}, x^{\rho}] \ = \ i\ (\eta^{\nu\rho}x^{\mu} - \eta^{\mu\rho}x^{\nu}), \quad [\Sigma^{\mu\nu}, p^{\rho}] \ = \ i\ (\eta^{\nu\rho}p^{\mu} - \eta^{\mu\rho}p^{\nu}). \tag{3.21}$$

that are just the well known rotations (boosts) of the coordinates and momenta and the standard Lorentz algebra commutators

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] \ = \ i\ (\ \eta^{\mu\sigma}\Sigma^{\nu\rho} \ + \ \eta^{\nu\rho}\Sigma^{\mu\sigma} \ - \ \eta^{\mu\rho}\Sigma^{\nu\sigma} \ - \ \eta^{\nu\sigma}\Sigma^{\mu\rho}\ ). \tag{3.22}$$

Concluding, eqs-(3.12, 3.15, 3.16, 3.19, 3.20, 3.21, 3.22) *are* the defining relations of Yang's Noncommutative $4D$ spacetime algebra (noncommutative phase space) involving the $8D$ phase-space variables. These commutators obey the Jacobi identities.

Notice that if one imposes a *different* correspondence among the coordinates and momenta, like the following

$$\hat{p}^{\mu} \ \leftrightarrow \ \frac{1}{\sqrt{2}}\ \frac{\hbar}{R}\ (\Sigma^{\mu 6} - \Sigma^{\mu 5}), \quad \hat{x}^{\mu} \ \leftrightarrow \ \frac{1}{\sqrt{2}}\ \lambda\ (\Sigma^{\mu 6} + \Sigma^{\mu 5}). \tag{3.23}$$

one gets the following modifications to the Weyl-Heisenberg algebra, when $\eta^{55} = -\eta^{66} = 1$ (the algebra related to $SO(4,2)$)

$$[\hat{x}^{\mu}, \hat{p}^{\nu}] \ = \ -i\hbar\ (\ \eta^{\mu\nu}\frac{\lambda}{R}\Sigma^{56} \ - \ \frac{\Sigma^{\mu\nu}}{S}\ ). \tag{3.24}$$

where the dimensionless parameter S in eq-(3.24) is defined by $S = (R/\lambda)$ and one arrives at similar modifications of the phase space commutator algebra that was proposed by Leznov and Kruschev long ago [13] when one identifies $\frac{\lambda}{R}\Sigma^{56}$ with the "unit" operator I of [13]. The classical limit furnishes the following *deformed* Poisson bracket [13]

$$\{x^{\mu}, p^{\mu}\} \ = \ -\ (\ \eta^{\mu\nu}\frac{\lambda}{R}\Sigma^{56} \ - \ \frac{\Sigma^{\mu\nu}}{S}\ ) \tag{3.25}$$

An immediate consequence of Yang's [1] and the Leznov-Khruschev [13] noncommutative phase space algebra is that now one has modified Heisenberg uncertainty relations [8]. QM in D-dim Noncommutative spaces can be described from QM in ordinary (commuting) $D+2$-dim spaces [5]. The double-scaling limit of Yang's algebra $\lambda \to 0$, $R \to \infty$, in conjunction with the large $n \to \infty$ limit, leads naturally to the *area quantization* condition $\lambda R = L^2 = n\lambda^2$ (in Planck area units) given in terms of the discrete angular-momentum eigenvalues n associated with the rotation operator Σ^{56} [8].

In order to write QM wave equations in non-commuting spacetimes [5], we start with a Hamiltonian written in *dimensionless* variables involving the terms of the relativistic oscillator (let us say oscillations of the center of mass) and the rigid rotor/top terms (rotations about the center of mass)

$$H = (\frac{p_\mu}{(\hbar/R)})^2 + (\frac{x_\mu}{L_P})^2 + (\Sigma^{\mu\nu})^2. \tag{3.26}$$

with the fundamental difference that the coordinates x^μ and momenta p^μ obey the non-commutative Yang's space time algebra. For this reason one *cannot* naively replace p^μ any longer by the differential operator $-i\hbar\partial/\partial x^\mu$ nor write the $\Sigma^{\mu\nu}$ generators as $(1/\hbar)(x^\mu\partial_{x_\nu} - x^\nu\partial_{x_\mu})$. The correct coordinate realization of Yang's noncommutative spacetime algebra requires, for example, embedding the 4-dim space into 6-dim and expressing the coordinates and momenta operators as follows :

$$\frac{p_\mu}{(\hbar/R)} \leftrightarrow \Sigma^{\mu 6} = i\frac{1}{\hbar}(X^\mu\partial_{X_6} - X^6\partial_{X_\mu}). \quad \frac{x_\mu}{L_P} \leftrightarrow \Sigma^{\mu 5} = i\frac{1}{\hbar}(X^\mu\partial_{X_5} - X^5\partial_{X_\mu}).$$

$$\Sigma^{\mu\nu} \leftrightarrow i\frac{1}{\hbar}(X^\mu\partial_{X_\nu} - X^\nu\partial_{X_\mu}). \quad \mathcal{N} = \Sigma^{56} \leftrightarrow i\frac{1}{\hbar}(X^5\partial_{X_6} - X^6\partial_{X_5}). \tag{3.27}$$

this allows to express H in terms of the standard angular momentum operators in 6-dim. The $X^A = X^\mu, X^5, X^6$ coordinates ($\mu = 1,2,3,4$) and $P^A = P^\mu, P^5, P^6$ momentum variables obey the standard commutation relations of ordinary QM in 6-dim

$$[X^A, X^B] = 0. \quad [P^A, P^B] = 0. \quad [X^A, P^B] = i\hbar\eta^{AB}. \tag{3.28}$$

so that the momentum admits the standard realization as $P^A = -i\hbar\partial/\partial X_A$

Therefore, concluding, the Hamiltonian H in eq-(3-26) associated with the non-commuting coordinates x^μ and momenta p^μ in $d-1$-dimensions can be written in terms of the standard angular momentum operators in $(d-1)+2 = d+1$-dim as $H = \mathcal{C}_2 - \mathcal{N}^2$, where $\mathcal{C}_2$ agrees precisely with the quadratic Casimir operator of the $SO(d-1,2)$ algebra in the spin $s=0$ case,

$$C_2 = \Sigma_{AB}\Sigma^{AB} = (X_A\partial_B - X_B\partial_A)(X^A\partial^B - X^B\partial^A). \tag{3.29}$$

One remarkable feature is that $\mathcal{C}_2$ also agrees with the D'Alambertian operator for the Anti de Sitter Space AdS_d of *unit radius* (throat size) $(D_\mu D^\mu)_{AdS_d}$ as it was shown by [18], [19]. The proof requires to show that the D'Alambertian operator for the $d+1$-dim embedding space (expressed in terms of the X^A coordinates) is related to the D'Alambertian operator in AdS_d space of *unit* radius expressed in terms of the $z^1, z^2, ..., z^d$ *bulk intrinsic* coordinates as :

$$(D_\mu D^\mu)_{R^{d+1}} = -\frac{\partial^2}{\partial\rho^2} - \frac{d}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}(D_\mu D^\mu)_{AdS} \Rightarrow$$

$$\mathcal{C}_2 = \rho^2 (D_\mu D^\mu)_{R^{d+1}} + [\ (d-1) + \rho\frac{\partial}{\partial\rho}\ ]\ \rho\frac{\partial}{\partial\rho}\ =\ (D_\mu D^\mu)_{AdS_d}. \tag{3.30}$$

This result is just the hyperbolic-space generalization of the standard decomposition of the Laplace operator in spherical coordinates in terms of the radial derivatives plus a term containing the square of the orbital angular momentum operator L^2/r^2. In the case of nontrivial spin, the Casimir $C_2 = \Sigma_{AB}\Sigma^{AB} + S_{AB}S^{AB}$ has additional terms stemming from the spin operator.

The quantity $\Phi(z^1, z^2, ..., z^d)|_{boundary}$ restricted to the $d-1$-dim projective boundary of the conformally compactified AdS_d space (of unit throat size, whose topology is $S^{d-2}\times S^1$) is the sought-after solution to the Casimir invariant wave equation associated with the non-commutative x^μ coordinates and momenta p^μ of the Yang's algebra ($\mu = 1, 2, ..., d-1$). Pertaining to the boundary of the conformally compactified AdS_d space, there are two radii R_1, R_2 associated with S^{d-2} and S^1, respectively, and which must not be confused with the two scales R, L_P appearing in eq-(3.26). One can choose the units such that the present value of the Hubble scale (taking the Hubble scale as the infrared cutoff) is $R = 1$. In these units the Planck scale L_P will be of the order of $L_P \sim 10^{-60}$. In essence, there has been a trade-off of two scales L_P, R with the two radii R_1, R_2.

Once can parametrize the coordinates of $AdS_d = AdS_{p+2}$ by writing [19]

$$X_0 = R\ cosh(\rho) cos(\tau).\ \ X_{p+1} = R\ cosh(\rho) sin(\tau).\ \ X_i = R\ sinh(\rho)\Omega_i. \tag{3.31a}$$

The metric of $AdS_d = AdS_{p+2}$ space in these coordinates is :

$$ds^2 = R^2[-(cosh^2\rho)d\tau^2 + d\rho^2 + (sinh^2\rho)d\Omega^2]. \tag{3.31b}$$

where $0 \le \rho$ and $0 \le \tau < 2\pi$ are the global coordinates. The topology of this hyperboloid is $S^1 \times R^{p+1}$. To study the causal structure of AdS it is convenient to unwrap the circle S^1 (closed-timelike coordinate τ) to obtain the universal covering of the hyperboloid without closed-timelike curves and take $-\infty \le \tau \le +\infty$. Upon introducing the new coordinate $0 \le \theta < \pi/2$ related to ρ by $tan(\theta) = sinh(\rho)$, the metric in (3-6b) becomes

$$ds^2 = \frac{R^2}{cos^2\theta}[-d\tau^2 + d\theta^2 + (sinh^2\rho)d\Omega^2]. \tag{3.32}$$

It is a conformally-rescaled version of the metric of the Einstein static universe. Namely, $AdS_d = AdS_{p+2}$ can be conformally mapped into one-half of the Einstein static universe, since the coordinate θ takes values $0 \leq \theta < \pi/2$ rather than $0 \leq \theta < \pi$. The boundary of the conformally compactified AdS_{p+2} space has the topology of $S^p \times S^1$ (identical to the conformal compactification of the $p+1$-dim Minkowski space). Therefore, the equator at $\theta = \pi/2$ is a *boundary* of the space with the topology of S^p. Ω_p is the solid angle coordinates corresponding to S^p and τ is the coordinate which parametrizes S^1. For a detailed discussion of AdS spaces and the AdS/CFT duality see [19].

The D'Alambertian in AdS_d space (of radius R, later we shall set $R = 1$) is :

$$D_\mu D^\mu = \frac{1}{\sqrt{g}}\ \partial_\mu\ (\ \sqrt{g}\ g^{\mu\nu}\partial_\nu\ ) =$$

$$\frac{cos^2\theta}{R^2}[\ -\partial_\tau^2 + \frac{1}{(R\ tan\theta)^p}\ \partial_\theta\ (\ (R\ tan\theta)^p\ \partial_\theta\ )\ ]\ +\ \frac{1}{R^2\ tan^2\theta}\ \mathcal{L}^2 \tag{3.33}$$

where $\mathcal{L}^2$ is the Laplacian operator in the p-dim sphere S^p whose eigenvalues are $l(l+p-1)$. The scalar field can be decomposed as $\Phi = e^{\omega R\tau}\ Y_l(\Omega_p)\ G(\theta)$ and the wave equation

$$(D_\mu D^\mu - m^2)\Phi = 0. \tag{3.34}$$

leads to :

$$[\ cos^2\theta\ (\ \omega^2 + \partial_\theta^2 + \frac{p}{tan\theta\ cos^2\theta}\partial_\theta\ )\ +\ \frac{l(l+p-1)}{tan^2\theta} - m^2R^2\ ]\ G(\theta) = 0. \tag{3.35}$$

whose solution is

$$G(\theta) = (sin\theta)^l\ (cos\theta)^{\lambda_\pm}\ {}_2F_1(a,b,c;sin\theta). \tag{3.36}$$

The hypergeometric function is defined

$${}_2F_1(a,b,c,z) = \sum\ \frac{(a)_k(b)_k}{(c)_k k!}\ z^n. \quad |z| < 1. \tag{3.37}$$

$$(\lambda)_o = 1. \quad (\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \lambda(\lambda+1)(\lambda+2).......(\lambda+k-1). \quad k = 1,2,.... \tag{3.38}$$

where

$$a = \frac{1}{2}(l+\lambda_\pm - \omega R). \quad b = \frac{1}{2}(l+\lambda_\pm + \omega R). \quad c = l + \frac{1}{2}(p+1) > 0. \tag{3.39a}$$

$$\lambda_\pm = \frac{1}{2}(p+1) \pm \frac{1}{2}\sqrt{(p+1)^2 + 4(mR)^2}. \tag{3.39b}$$

The analytical continuation of the hypergeometric function for $|z| \geq 1$ is :

$$ {}_2F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt. \quad . \tag{3.40} $$

with $Real(c) > 0$ and $Real(b) > 0$. The boundary value when $\theta = \pi/2$ gives

$$ lim_{z\to 1^-} \; F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.41} $$

This suggests that QM over Yang's Noncommutative Spacetimes could be well defined in terms of ordinary QM in *higher* dimensions, this idea deserves further investigation, see [5], [8] for further details, in particular about a matrix-valued generalization of Planck's constant in order to study QM in Clifford spaces.

4 MODIFIED NEWTONIAN MECHANICS

4.1 The Yang's Deformed Algebra case

The dynamical consequences of the Yang's Noncommutative spacetime algebra can be derived from the quantum/classical correspondence :

$$ \frac{1}{i\hbar}[\hat{A},\hat{B}] \leftrightarrow \{A,B\}_{PB}. \tag{4.1} $$

i.e. commutators correspond to Poisson brackets. More precisely, to Moyal brackets. in Phase Space. In the classical limit $\hbar \to 0$ Moyal brackets reduce to Poisson brackets. Since the coordinates and momenta are no longer commuting variables the classical Newtonian dynamics is going to be modified since the symplectic two-form $\omega^{\mu\nu}$ in Phase Space will have additional non-vanishing elements stemming from these non-commuting coordinates and momenta. In particular, the modified brackets read now :

$$ \{\{A(x,p),B(x,p)\}\} = \partial_\mu A \omega^{\mu\nu}\partial_\nu B = \{A(x,p),B(x,p)\}_{PB}\{x^\mu,p^\nu\} + \\ \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial x^\nu}\{x^\mu,x^\nu\} + \frac{\partial A}{\partial p^\mu}\frac{\partial B}{\partial p^\nu}\{p^\mu,p^\nu\}. \tag{4.2} $$

If the coordinates and momenta were commuting variables the modified bracket will reduce to the first term only :

$$ \{\{A(x,p),B(x,p)\}\} = \{A(x,p),B(x,p)\}_{PB}\{x^\mu,p^\nu\} = [\,\frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p^\nu} - \frac{\partial A}{\partial p^\mu}\frac{\partial B}{\partial x^\nu}\,]\eta^{\mu\nu}\mathcal{N}. \tag{4.3} $$

In the nonrelativistic limit, the modfied dynamical equations are :

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{\partial H}{\partial p^j}\{x^i, p^j\} + \frac{\partial H}{\partial x^j}\{x^i, x^j\}. \tag{4.4}$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = -\frac{\partial H}{\partial x^j}\{x^i, p^j\} + \frac{\partial H}{\partial p^j}\{p^i, p^j\}. \tag{4.5}$$

The non-relativistic Hamiltonian for a central potential $V(r)$ is :

$$H = \frac{p_i p^i}{2m} + V(r). \quad r = [\sum_i x_i x^i]^{1/2} \tag{4.6}$$

Defining the magnitude of the central force by $F = -\frac{\partial V}{\partial r}$ and using $\frac{\partial r}{\partial x^i} = \frac{x_i}{r}$ one has the *modified* dynamical equations of motion associated with the Yang's *deformed* phase space algebra are :

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} \ = \ \frac{p_j \ N}{m} \ \delta^{ij} \ - \ F \ \frac{x_j}{r} \ L_P^2 \ \Sigma^{ij}. \tag{4.7}$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} \ = \ F \ \frac{x_j \ N}{r} \ \delta^{ij} \ + \ \frac{p_j}{m} \ \frac{\Sigma^{ij}}{R^2}. \tag{4.8}$$

where N is the classical counterpart of $\mathcal{N} = (\lambda/R) \ \Sigma^{56}$. The angular momentum bi-vector Σ^{ij} can be written as the *dual* of a vector $\vec{J}$ as follows $\Sigma^{ij} = \epsilon^{ijk} J_k$ so that :

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p^i \ N}{m} \ - \ L_P^2 \ F \ \frac{x_j}{r} \ \epsilon^{ijk} \ J_k. \tag{4.9}$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = \ F \ \frac{x^i \ N}{r} \ + \ \frac{p_j}{m} \ \frac{\epsilon^{ijk} J_k}{R^2}. \tag{4.10}$$

For planar motion (central forces) the cross-product of $\vec{J}$ with $\vec{p}$ and $\vec{x}$ is not zero since $\vec{J}$ points in the perpendicular direction to the plane. Thus, one will have nontrivial corrections to the ordinary Newtonian equations of motion which are induced from the Yang's Noncommutative spacetime algebras in the non-relativistic limit.

Concluding, eqs-(4.9, 4.10) determine the *modified* Newtonian dynamics of a test particle under the influence of a central potential explicitly in terms of the two L_P, R minimal/maximal scales. When $L_P \rightarrow 0$, $R \rightarrow \infty$ and $N = 1$ one recovers the ordinary Newtonian dynamics $v^i = (p^i/m)$ and $F(x^i/r) = m(dv^i/dt)$. The unit vector in the radial direction has for components $\hat{r} = (\vec{r}/r) = (x^1/r, x^2/r, x^3/r)$. The Modified Newtonian dynamics represented by eqs-(4.9, 4.10) should have important astrophysical consequences.

4.2 The Leznov-Khruschev Algebra : Deformed Free Motion

Let us focus at the moment on the modified free motion of a particle in the absence of external forces. The Hamiltonian (Energy) for free motion was chosen by Leznov et al [13] so it will commute with infinitesimal translations $\vec{p}$ and rotations $\vec{J}$:

$$H \;=\; E \;=\; \frac{1}{2m}\;(\;\vec{p}^2\;+\;\frac{J^2}{R^2}\;). \tag{4.11}$$

It follows from the modified brackets relations associated with the Leznov-Khruschev algebra that in the absence of external forces $F = 0$, the $\vec{p}$ and $\vec{J}$ are still constants of the motion and [13] :

$$\frac{d\vec{x}}{dt} \;=\; \frac{I(t)}{m}\;\vec{p}\;-\;\frac{\vec{x}\wedge\vec{J}}{R^2}\;+\;\frac{\vec{p}\wedge\vec{J}}{S}. \tag{4.12}$$

$$\frac{dI(t)}{dt} \;=\; \frac{\vec{p}^2}{m\;S}\;-\;\frac{\vec{p}\;.\;\vec{x}}{m\;R^2}. \tag{4.13}$$

where I is the "unit" operator (rotation involving the extra dimension) associated with the $SO(4,1)$ algebra (and $SO(5), SO(3,2)$ algebras as well). From eqs-(4.12, 4.13) one could view the quantity $m(t) = m/I(t)$ as a time-dependent mass. When $R = S = \infty$ one recovers the standard Newtonian equations of motion in the absence of external forces. Differentiating (4.13) and taking into account that $\vec{p}$ and $\vec{J}$ are conserved quantities one gets as shown by [13]

$$\frac{d^2I}{dt^2}\;+\;\frac{2E}{m\;R^2}\;I\;=\;0\;\Rightarrow\;I(t)\;=\;I_o\;cos\;[\;\sqrt{\frac{2E}{m}}\;\frac{1}{R}(t-t_o)\;]. \tag{4.14}$$

and

$$\vec{x}\;=\;\vec{x}_o+\frac{I_o}{|p|^2}\;R\;\sqrt{2mE}\;sin\;[\;\sqrt{\frac{2E}{m}}\;\frac{1}{R}(t-t_o)\;]\;\vec{p}-\frac{I_o}{|p|^2}\;cos\;[\;\sqrt{\frac{2E}{m}}\;\frac{1}{R}(t-t_o)\;]\;(\vec{J}\wedge\vec{p}). \tag{4.15}$$

A Circular Motion centered at $\vec{x}_o$ (whose projection onto a line is a Harmonic Oscillator oscillating about $\vec{x}_o$) occurs when the radius of the circle ρ obeys the conditions

$$\rho\;=\;\frac{I_o}{|p|}\;R\;\sqrt{2mE}\;=\;\frac{I_o}{|p|}\;|\vec{J}|\;. \tag{4.16a}$$

so the magnitude of the acceleration of the circular motion is

$$a \;=\; \omega^2 \;\rho \;=\; \frac{2E}{mR^2} \;(\; I_o \;|p| \;R \;\sqrt{\frac{1}{2mE}}\;) \;=\; \frac{1}{R}\;\frac{|p|^2}{m^2}. \qquad (4.16b)$$

after inserting in (4.16b) the value of the constant

$$I_o \;=\; \frac{p}{\sqrt{2ME}} \;[\; 1 + 2mE\;(\frac{R^2}{S^2} - \frac{1}{M^2})\;]^{\frac{1}{2}}. \qquad (4.17)$$

in the special case when $S = RM$. The expression for I_o was obtained from normalizing the quadratic Casimir to unity [13]. In the special case when $S = RM$, since $|p| = \sqrt{2mE} \Rightarrow I_o = 1$, then the radius becomes $\rho = R$. Notice that $|p|^2 = \vec{p}.\vec{p} \neq p_\mu p^\mu = m^2$. If we set $\frac{|p|^2}{m^2} = v^2$ in terms of a characteristic velocity v, one can infer from (4.16b) by setting $R = R_H$ and $v = c$, that

$$a \;=\; \frac{1}{R}\;\frac{p^2}{m^2} \;=\; \frac{v^2}{R} \;\sim\; \frac{c^2}{R_H} \;\sim\; a_{Pioneer} \qquad (4.18)$$

however, there is a caveat because when $R = R_H$ and $v = c$, one reaches the purely *relativistic* domain of validity where one cannot longer use the modified (deformed) Newtonian non-relativistic equations of motion. This was the purpose of studying the relativistic case within the context of (Anti) de Sitter Relativity in Cosmology (section **2**) in order to explain the origins of the Pioneer anomalous acceleration.

A purely *linear* motion is obtained when $R = R_H$ = Hubble scale, such that

$$\frac{d^2 I}{dt^2} \;+\; \frac{2E}{m\;R_H^2}\;I \;\sim\; \frac{d^2 I}{dt^2} = 0 \;\Rightarrow\; I(t) \;=\; I_o \;+\; (\frac{|p|^2}{m\;S})\;t. \qquad (4.19)$$

after using eq-(4.13) and neglecting the last term in eq-(4.13). Hence, the time dependence of $\vec{x}$ is now given by

$$\vec{x} \;=\; \vec{x}_o \;+\; \frac{I_o\;t}{m}\;\vec{p} \;+\; \frac{1}{2}\;\frac{|p|^2\;t^2}{m^2\;S}\;\vec{p}. \qquad (4.20)$$

From this last equation one can deduce that the *constant* acceleration, despite the absence of external forces, due to the *modified* (deformed) Newtonian mechanics, is

$$a \;=\; \frac{|p|^2}{m^2\;S}\;|p|. \qquad (4.21)$$

The value of the parameter S which has dimensions of an action can be determined from the experimentally observed anomalous Pioneer acceleration

$$a_p \;\sim\; 8.74 \times 10^{-10}\frac{m}{s^2} \;\sim\; \frac{c^2}{R_H}. \qquad (4.22a)$$

by writing

$$S = R_H \; M_{Planck} \; c. \tag{4.22b}$$

in terms of the Hubble scale and Planck's mass M_{Planck} . Let us equate

$$a = \frac{p^2}{m^2 \; S} \; p = \frac{p^2}{m^2} \; \frac{1}{R_H} \; \frac{p}{M_{Planck} \; c} \sim \frac{c^2}{R_H} \tag{4.22c}$$

We can see by simple inspection that when $m = M_{Planck}$ and $p = m \; c = M_{Planck} \; c$ in (4.22), one would have *an exact equality* between the l.h.s and r.h.s of eq-(4.22) ! , setting aside (of course) for the moment that a momentum of the magnitude $|p| = m \; c$ falls in the purely relativistic regime which is incompatible with the non-relativistic modified Newtonian equations of motion. This was the reason, once again, why we studied the relativistic dynamics and hyperbolic motion in section **2**.

When $m \neq M_{Planck}$, on dimensional grounds one may set $\frac{p}{m} = v$ where v is a characteristic velocity, and instead of setting $S = R_H \; M_{Planck} \; c$, now we will set the action parameter to be $S = mc \; \rho$, and such that eq-(4.22) becomes now

$$a = v^3 \; (\frac{1}{\rho}) \; (\frac{m}{m \; c}) \sim \frac{c^2}{R_H} \Rightarrow (\frac{v}{c})^3 \sim \frac{\rho}{R_H}. \tag{4.23}$$

One can notice that when $\rho = 67$ AU ($1 \; AU = 1.49 \times 10^{11} m$) is inserted in the above equation (4.23) , one deduces a velocity $v = 13.91 \; km/s \sim 12.2 \; km/s$, that indeed is quite *close* to the observed velocity of Pioneer at a distance of $\rho = 67$ AU [12]. Such cubic scaling behaviour (4.23) was found by Kolgomorov in a very different context related to eddies, turbulence, ... in fluid mechanics.

4.3 The Deformed Kepler Problem

The Hamiltonian for the deformed Kepler problem is chosen [13] to commute with the Runge-Lenz vector

$$\vec{A} = \vec{p} \wedge \vec{J} + \alpha \frac{\vec{x}}{\rho}; \quad \rho^2 = |\vec{x}|^2 + \frac{J^2}{M^2}, \quad M = \frac{1}{\lambda}. \tag{4.24}$$

where α is a numerical constant proportional to the solar mass of momentum dimensions and related to the shape of the orbits, and the mass parameter M is related to the inverse of the "minimal" length scale λ. The Poisson bracket relations for the components of $\vec{A}$ are

$$\{A_i, A_j\} = 2 \, (E - \frac{J^2}{mR^2}) \, \epsilon_{ijk} \; J_k, \quad E = \frac{1}{2m} \, (|\vec{p}|^2 + \frac{J^2}{R^2} + \frac{2\alpha}{r} I - \frac{\alpha^2}{M^2 r^2}). \tag{4.25}$$

where I is the "unit" operator associated with one of the $SO(4,1), SO(5), SO(3,2)$ algebras. The quadratic Casimir is

$$\mathcal{C}_2 \;=\; I^2 \;+\; \frac{|\vec{x}|^2}{R^2} \;+\; \frac{|\vec{p}|^2}{M^2} \;-\; 2\,(\frac{\vec{p}.\vec{x}}{S}) \;-\; J^2(\frac{1}{S^2} - \frac{1}{R^2M^2}). \tag{4.26}$$

The most salient feature of the deformed Kepler problem is that when the 3 vectors $\vec{x}, \vec{p}, \vec{A}$ lie in the plane orthogonal to the angular momentum vector $\vec{J}$ the orbits belong to a family of plane *quartics* as first suggested by Cassini in the 17-th century. These curves are known as the Cassini ovals [13], it is the locus of points where the product of their distances from two fixed points is constant.

The most relevant equation to study at the moment is the temporal dependence of the position $\vec{x}$ [13]

$$\frac{d\vec{x}}{dt} \;=\; \frac{1}{J^4}\;(\;|\vec{A}|\;|\vec{x}| - \alpha\rho\;)\;[\;(\frac{\alpha}{m\rho})\;\vec{x}\wedge\vec{J}\;-(\frac{\vec{A}\wedge\vec{J}}{m})\;]\;-\frac{\vec{x}\wedge\vec{J}}{mR^2}\;+\frac{\vec{A}}{S}. \tag{4.27}$$

From eq-(4.27) one can get a numerical estimate of the *corrections* to the acceleration due to the modified Newtonian dynamics resulting from the noncommutative phase space Leznov-Khruschev algebra. If one sets the mass parameter scale $M \rightarrow \infty$, the first term of eq-(4.27) does not generate corrections since $\rho = r$. Hence, a differentiation of eq-(4.27) w.r.t the time variable, due to the fact that the Runge-Lenz vector and angular momentum is a constant of motion, will furnish the *corrections* $\Delta\vec{a}$ to the acceleration given in terms of the *corrections* to the velocity $(\Delta\;\frac{d\vec{x}}{dt})$

$$\Delta\vec{a} \;=\; -\,\frac{1}{mR^2}\;(\Delta\;\frac{d\vec{x}}{dt})\;\wedge\;\vec{J} \;=\; -\,\frac{1}{mR^2}\;(\;-\,\frac{\vec{x}\wedge\vec{J}}{mR^2}\;+\;\frac{\vec{A}}{S}\;)\;\wedge\;\vec{J}. \tag{4.28}$$

When the parameter S is set to ∞ one arrives at

$$\Delta\vec{a} \;=\; \frac{1}{mR^2}\;(\frac{\vec{x}\wedge\vec{J}}{mR^2})\;\wedge\vec{J}. \tag{4.29}$$

For a planar orbit perpendicular to the angular momentum vector, the vector $\Delta\vec{a}$ given by eq-(4.29) points *towards* the *origin*; i.e. towards the sun as it is observed. The magnitude is

$$|\Delta\vec{a}| \;=\; \frac{1}{mR^2}\;(\;\frac{|\vec{x}|\;|\vec{J}|^2}{mR^2}\;). \tag{4.30}$$

The magnitude of the corrections when $R = R_H$ is *extremely small* unless, of course, one has

$$m\;R_H = 1, \quad |\;\vec{x}\;| \;=\; R_H, \quad |\vec{J}| \;= |\vec{p}|\;R_H \;=\; (mc)R_H \;=\; c\;\Rightarrow$$

$$|\Delta\vec{a}| \;=\; \frac{c^2}{R_H} \;\sim\; 8.74 \times 10^{-10}\; m/s^2. \tag{4.31}$$

This is an *exceptional* case which is more related to a Machian view of the Universe [4] where a minimal acceleration c^2/R_H is dual to a maximal acceleration c^2/L_{Planck} when the mass of the Universe M_U is of the order of $M_{Planck}\ (R_H/L_{Planck})$. These results can be derived, simply, by setting the *maximal* proper force associated with a Dual Phase Space Relativity principle (initiated by Max Born [9], [10]) to be given by [4] :

$$Maximal\ Proper\ Force \;=\; M_U\,\frac{c^2}{R_H} \;=\; M_{Planck}\,\frac{c^2}{L_{Planck}}. \tag{4.32}$$

The Machian principle states that the rest mass of a particle is related to the gravitational binding energy associated with its gravitational interaction with all the masses of the Universe

$$mc^2 \;=\; \frac{GM_U m}{R_H} \;\Rightarrow c^2 \;=\; \frac{GM_U}{R_H} \;\Rightarrow \frac{c^2}{R_H} \;=\; \frac{GM_U}{R_H^2} \;\sim\; a_{Pioneer}. \tag{4.33}$$

Another possibility to generate the right order of magnitude is to choose the parameters m, R and $|\vec{J}| = |\vec{x} \wedge \vec{p}|$ in such a way that

$$|\Delta\vec{a}| \;=\; \frac{1}{mR^2}\;(\;\frac{|\vec{x}|\;|\vec{J}|^2}{mR^2}\;) \;\sim\; \frac{c^2}{R_H}. \tag{4.34}$$

A reasonable choice corresponding to the "dry" mass of the Pioneer satellite of $m_s = 223$ Kg; with a velocity of $v_s = 12.2$ Km/s, at a location $R = r_o = |\vec{x}| = 67$ AU [12], and with angular momentum $|\vec{J}| \sim m_s v_s\ r_o$, should give

$$|\Delta\vec{a}| \;=\; \frac{1}{m_s\ r_o^2}\;(\;\frac{r_o\ (m_s v_s\ r_o)^2}{m_s\ r_o^2}\;) \;=\; \frac{v_s^2}{r_o} \tag{4.35}$$

However it is evident by simple inspection that

$$\frac{v_s^2}{r_o} \;<\; \frac{c^2}{R_H}. \tag{4.36}$$

A more appropriate scaling behaviour was found earlier in eq-(4.23) to be

$$\frac{v_s^3}{r_o} \;\sim\; \frac{c^3}{R_H} \;\Rightarrow\; (\frac{v_s}{c})^3 \;\sim\; \frac{r_o}{R_H}. \tag{4.37}$$

Therefore, only in the extreme case scenario (4.31), which invalidates the non-relativistic approximation in the first place, one would have found agreement with observations. For this reason, we are more certain that the de

Sitter Relativistic behaviour of the hyperbolic trajectory of Pioneer, due to the expansion of the Universe (non-vanishing cosmological constant), is the underlying physical reason for the observed anomalous acceleration of the Pioneer spacecraft.

A purely circular motion, in the relativistic limit (in units $\hbar = 1$), such that

$$\omega \ R_H \ = \ c, \quad \omega = m \ \Rightarrow mR_H = 1. \tag{4.38}$$

leads to an acceleration

$$a \ = \ \frac{J^2}{R_H} \ = \ \frac{(mcR_H)^2}{R_H} \ = \ \frac{c^2}{R_H} \ \sim \ a_{Pioneer} \tag{4.39}$$

compatible with Mach's principle once again [20]. A Weyl geometry interpretation of the anomalous acceleration was advanced by [21]. In [7] we were able to prove how a proper use of Weyl's geometry within the context of the Friedman-Lemaitre-Robertson-Walker cosmological models can account for *both* the origins and the *small* value of the observed vacuum energy density of the order of $10^{-123} M^4_{Planck}$. The source of dark energy is just the dilaton-like Brans-Dicke-Jordan scalar field that is required to implement the Weyl symmetry invariance of the gravity-scalar field action. It is warranted to explore the hyperbolic motion of Pioneer in de Sitter space depicted in section **2** within the framework of a Friedman-Lemaitre-Robertson-Walker cosmological model, since de Sitter space is a special case of such cosmological model. In such case the anomalous acceleration would be time dependent.

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