An Exceptional E_8 Gauge Theory of Gravity in $D = 8$, Clifford Spaces and Grand Unification

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314, castro@ctps.cau.edu

September 2008, Revised November 2008

Abstract

A candidate action for an Exceptional E_8 gauge theory of gravity in $8D$ is constructed. It is obtained by recasting the E_8 group as the semi-direct product of $GL(8, R)$ with a deformed Weyl-Heisenberg group associated with canonical-conjugate pairs of vectorial and antisymmetric tensorial generators of rank two and three. Other actions are proposed, like the *quartic* E_8 group-invariant action in $8D$ associated with the Chern-Simons E_8 gauge theory defined on the 7-dim boundary of a $8D$ bulk. To finalize, it is shown how the E_8 gauge theory of gravity can be embedded into a more general extended gravitational theory in Clifford spaces associated with the $Cl(16)$ algebra and providing a solid geometrical program of a grand-unification of gravity with Yang-Mills theories. The key question remains if this novel gravitational model based on gauging the E_8 group may still be renormalizable *without* spoiling unitarity at the quantum level.

Keywords: C-space Gravity, Clifford Algebras, Grand Unification, Exceptional algebras, String Theory.

1 Introduction

Exceptional, Jordan, Division and Clifford algebras are deeply related and essential tools in many aspects in Physics [3], [8], [9]. Ever since the discovery [1] that $11D$ supergravity, when dimensionally reduced to an n-dim torus led to maximal supergravity theories with hidden exceptional symmetries E_n for $n \leq 8$, it has prompted intensive research to explain the higher dimensional origins of these hidden exceptional E_n symmetries [2] . More recently, there has been a lot of interest in the infinite-dim hyperbolic Kac-Moody E_{10} and non-linearly realized E_{11} algebras arising in the asymptotic chaotic oscillatory solutions of Supergravity fields close to cosmological singularities [1], [2].

Grand-Unification models in $4D$ based on the exceptional E_8 Lie algebra have been known for sometime [7]. The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [6] under the assumption that there is a vacuum gluino condensate but this condensate is *not* accompanied by a dynamical generation of a mass gap in the pure E_8 gauge sector. Supersymmetric non-linear σ models of Kahler coset spaces $\frac{E_8}{SO(10)\times SU(3)\times U(1)}$; $\frac{E_7}{SU(5)}$; $\frac{E_6}{SO(10)\times U(1)}$ are known to contain three generations of quarks and leptons as (quasi) Nambu-Goldstone $superfields$ [4] (and references therein). The coset model based on $G = E_8$ gives rise to 3 left-handed generations assigned to the **16** multiplet of $SO(10)$, and 1 right-handed generation assigned to the $\mathbf{16}^*$ multiplet of $SO(10)$. The coset model based on $G = E_7$ gives rise to 3 generations of quarks and leptons assigned to the $\mathbf{5}^* + \mathbf{10}$ multiplets of $SU(5)$, and a Higgsino (the fermionic partner of the scalar Higgs) in the **5** representation of $SU(5)$.

A Chern-Simons E_8 Gauge theory of Gravity proposed in [15] is a unified field theory (at the Planck scale) of a Lanczos-Lovelock Gravitational theory with a E_8 Generalized Yang-Mills (GYM) field theory, and is defined in the $15D$ boundary of a 16D bulk space. The Exceptional E_8 Geometry of the Clifford (16) Superspace Grand-Unification of Conformal Gravity and Yang-Mills was studied by [16]. In particular, it was discussed how an E_8 Yang-Mills in $8D$, after a sequence of symmetry breaking processes $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow SO(8,2)$, leads to a Conformal gravitational theory in $8D$ based on the conformal group $SO(8,2)$ in $8D$. Upon performing a Kaluza-Klein-Batakis [19] compactification on CP^2, involving a nontrivial *torsion*, leads to a Conformal Gravity-Yang-Mills unified theory based on the Standard Model group $SU(3)\times SU(2)\times U(1)$ in $4D$. Batakis [19] has shown that, contrary to the standard lore that it is not possible to obtain the Standard Model group from compactifications of $8D$ to $4D$, the inclusion of a nontrivial *torsion* in the internal $CP^2 = SU(3)/SU(2) \times U(1)$ space permits to do so. Furthermore, it was shown [16] how a conformal (super) gravity and (super) Yang-Mills unified theory in any dimension can be embedded into a (super) Clifford-algebra-valued gauge field theory by choosing the appropriate Clifford group.

The action that defines a Chern-Simons E_8 gauge theory of (*Euclideanized*) Gravity in 15-dim (the boundary of a $16D$ space) was based on the *octic* E_8 invariant constructed by [5] and is defined as [15]

$$S \;=\; \int_{\mathcal{M}^{16}} \;<\; F\; F\; ......\; F\; >_{E_8} \;=$$

$$\int_{\mathcal{M}^{16}} (F^{M_1} \wedge F^{M_2} \wedge ...... \wedge F^{M_8})\, \Upsilon_{M_1 M_2 M_3 .... M_8} \;=\; \int_{\partial\mathcal{M}^{16}} \mathcal{L}^{(15)}_{CS}(\mathbf{A}, \mathbf{F}) \quad (1.1)$$

The E_8 Lie-algebra valued 16-form $< \;\; F^8 \;\; >$ is *closed* : $d\;(< \;\; F^{M_1} T_{M_1} \wedge F^{M_2} T_{M_2} \wedge \; ..... \; \wedge F^{M_8} T_{M_8} \;\; >) = 0$ and *locally* can always be written as an exact form in terms of an E_8-valued Chern-Simons 15-form as $I_{16} = \; d\mathcal{L}^{(15)}_{CS}(\mathbf{A}, \mathbf{F})$.

For instance, when $\mathcal{M}^{16} = S^{16}$ the 15-dim boundary integral (1.1) is evaluated in the two coordinate patches of the equator $S^{15} = \partial\mathcal{M}^{16}$ of S^{16} leading to the integral of $tr(\mathbf{g}^{-1}d\mathbf{g})^{15}$ (up to numerical factors) when the gauge potential $\mathbf{A}$ is written locally as $\mathbf{A} = \mathbf{g}^{-1}d\mathbf{g}$ and $\mathbf{g}$ belongs to the E_8 Lie-algebra. The integral is characterized by the elements of the homotopy group $\pi_{15}(E_8)$. S^{16} can also be represented in terms of quaternionic and octonionic projectives spaces as HP^4, OP^2 respectively.

In order to evaluate the operation $< .... >_{E_8}$ in the action (1.1) it involves the existence of an *octic* E_8 group invariant tensor $\Upsilon_{M_1 M_2 .... M_8}$ that was recently constructed by Cederwall and Palmkvist [5] using the Mathematica package GAMMA based on the full machinery of the Fierz identities. The entire *octic* E_8 invariant contains powers of the $SO(16)$ bivector X^{IJ} and spinorial Y^α generators $X^8, X^6Y^2, X^4Y^4, X^2Y^6, Y^8$. The corresponding number of terms is $6, 11, 12, 5, 2$ respectively giving a total of **36** terms for the octic E_8 invariant involving **36** numerical coefficients multiplying the corresponding powers of the E_8 generators. There is an extra term (giving a total of **37** terms) with an *arbitrary* constant multiplying the fourth power of the E_8 quadratic invariant $I_2 = -\frac{1}{2} tr[\ (F^{IJ}_{\mu\nu} X_J)^2 + (F^\alpha_{\mu\nu} Y_\alpha)^2\ ]$.

Thus, the E_8 invariant action has 37 terms containing : (i) the Lanczos-Lovelock (Euclideanized) Gravitational action associated with the 15-dim boundary $\partial\mathcal{M}^{16}$ of the 16-dim manifold. ; (ii) 5 terms with the same structure as the Pontryagin $p_4(F^{IJ})$ 16-form associated with the $SO(16)$ spin connection Ω^{IJ}_μ and where the indices I, J run from $1, 2, ..., 16$; (iii) the fourth power of the standard quadratic E_8 invariant $[I_2]^4$; (iv) plus 30 additional terms involving powers of the E_8-valued $F^{IJ}_{\mu\nu}$ and $F^\alpha_{\mu\nu}$ field-strength (2-forms). The most salient feature of the action (1.1) is that it furnishes a unification of gravity and E_8 Yang-Mills theory in $16D$.

It is the purpose of this work to explore further the Exceptional theories of Gravity based on gauging the E_8 group in $8D$ and how to embed these theories into generalized theories of gravity in C-spaces (Clifford spaces) providing a solid geometrical program of Grand-Unification of Gravity with the other forces in Nature. Recent approaches to the E_8 group based on Clifford algebras can be found in [11], [29].

2 An E_8 Gauge Theory of Gravity in $D = 8$

We will base our construction of the E_8 Gauge Theory of Gravity in $D = 8$ on Born's deformed reciprocal complex gravitational theory and Noncommutative Gravity in $4D$ [18] which was constructed as a local gauge theory of the $deformed$ Quaplectic group $U(1,3) \times_s \mathcal{H}(4)$ advanced by [10], and that was given [18] by the semi-direct product of $U(1,3)$ with the $deformed$ (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* generators $[Z_a, Z_b] \neq 0$. To achieve our goal we need to show why the E_8 group can be recast as the semi-direct product of $GL(8,R)$ with a deformed Weyl-Heisenberg group involving canonical-conjugate pairs of vectorial and antisymmetric tensorial generators.

The commutation relations of E_8 can be expressed in terms of the 120 $SO(16)$ bivector generators $X^{[IJ]}$ and the 128 $SO(16)$ chiral spinorial generators Y^α as [12] (and references therein)

$$[X^{IJ}, X^{KL}] \;=\; 4\;(\;\delta^{IK}\;X^{LJ} - \delta^{IL}\;X^{KJ} + \delta^{JK}\;X^{IL} - \delta^{JL}\;X^{IK}\;).$$

$$[X^{IJ}, Y_\alpha] \;=\; -\frac{1}{2}\;\Gamma^{[IJ]}_{\alpha\beta}\;Y^\beta; \qquad [Y_\alpha, Y_\beta] \;=\; \frac{1}{4}\;\Gamma^{[IJ]}_{\alpha\beta}\;X_{IJ}. \tag{2.1a}$$

where $X^{IJ} = -X^{JI}$. It is required to choose a representation of the gamma matrices such that $\Gamma^{[IJ]}_{\alpha\beta} = -\Gamma^{[IJ]}_{\beta\alpha}$ since $[Y_\alpha, Y_\beta]$ is antisymmetric under $\alpha \leftrightarrow \beta$. The Jacobi identities among the triplet $[Y_\alpha, [Y_\beta, Y_\gamma]]$ + cyclic permutation are

$$\Gamma^{IJ}_{\alpha\beta}\;\Gamma^{IJ}_{\gamma\delta}\;Y^\delta \;+\; \textit{cyclic permutation among}\;(\alpha,\beta,\gamma) \;=\; 0. \tag{2.2a}$$

the above Jacobi identity can be shown to be satisfied by contracting two of the spinorial indices (α, β) in (2.2a) after multiplying (2.2a) by $\Gamma^{\alpha\beta}_{KL}$ and $\Gamma^{\alpha\beta}_{K_1K_2....K_6}$, respectively, giving

$$\Gamma^{IJ}_{\alpha\beta}\;\Gamma^{\alpha\beta}_{KL}\;\Gamma^{IJ}_{\gamma\delta} \;+\; \Gamma^{IJ}_{\beta\gamma}\;\Gamma^{\alpha\beta}_{KL}\;\Gamma^{IJ}_{\alpha\delta} \;+\; \Gamma^{IJ}_{\gamma\alpha}\;\Gamma^{\alpha\beta}_{KL}\;\Gamma^{IJ}_{\beta\delta} \;=\; 0. \tag{2.2b}$$

and

$$\Gamma^{IJ}_{\alpha\beta}\;\Gamma^{\alpha\beta}_{K_1K_2....K_6}\;\Gamma^{IJ}_{\gamma\delta} + \Gamma^{IJ}_{\beta\gamma}\;\Gamma^{\alpha\beta}_{K_1K_2....K_6}\;\Gamma^{IJ}_{\alpha\delta} + \Gamma^{IJ}_{\gamma\alpha}\;\Gamma^{\alpha\beta}_{K_1K_2....K_6}\;\Gamma^{IJ}_{\beta\delta} = \;0. \tag{2.2c}$$

Eqs-(2.2b, 2.2c) are zero (which implies that eq-(2.2a) is also zero) due to the very special properties of the *chiral* representation of the Clifford gamma matrices in $16D$ and after decomposing the $\frac{1}{2}(128 \times 127) = 8128$ dimensional space of antisymmetric $\Sigma_{[\alpha\beta]}$ matrices into a space involving 120 *antisymmetric* $\Gamma^{IJ}_{\gamma\delta}$ and 8008 $\Gamma^{I_1I_2....I_6}_{\gamma\delta}$ matrices in their chiral spinorial indices $\gamma\delta$ [21] .

The E_8 algebra as a sub-algebra of $Cl(8) \otimes Cl(8)$ is consistent with the $SL(8,R)$ 7-grading decomposition of $E_{8(8)}$ (with 128 noncompact and 120 compact generators) as shown by [12]. Such $SL(8,R)$ 7-grading is based on the diagonal part $[SO(8) \times SO(8)]_{diag} \subset SO(16)$ described in full detail by [12] and can be deduced from the $Cl(8) \otimes Cl(8)$ 7-grading decomposition of E_8 provided by Larsson [11] as follows,

$$[\gamma^\mu_{(1)} \,\oplus\, \gamma^{\mu\nu}_{(1)} \,\oplus\, \gamma^{\mu\nu\rho}_{(1)}\,]\otimes\;\mathbf{1}_{(2)} \;+\; \mathbf{1}_{(1)}\otimes[\gamma^\mu_{(2)} \,\oplus\, \gamma^{\mu\nu}_{(2)} \,\oplus\, \gamma^{\mu\nu\rho}_{(2)}\,] \;+\; \gamma^\mu_{(1)}\otimes\gamma^\nu_{(2)}. \tag{2.3}$$

these tensor products of elements of the two factor $Cl(8)$ algebras, described by the subscripts $(1), (2)$, furnishes the 7 grading of $E_{8(8)}$

$$8 \;+\; 28 \;+\; 56 \;+\; 64 \;+\; 56 \;+\; 28 \;+ 8 \;=\; 248. \tag{2.4}$$

8 corresponds to the $8D$ vector γ^μ; 28 is the $8D$ bivector $\gamma^{\mu\nu}$; 56 is the $8D$ tri-vector $\gamma^{\mu\nu\rho}$, and $64 = 8 \times 8$ corresponds to the tensor product $\gamma^\mu_{(1)} \otimes \gamma^\nu_{(2)}$.

In essence one can rewrite the E_8 algebra in terms of $8+8$ vectors Z^a, Z_a ($a = 1, 2, ...8$); $28 + 28$ bivectors $Z^{[ab]}, Z_{[ab]}$; $56 + 56$ tri-vectors $E^{[abc]}, E_{[abc]}$, and the $SL(8,R)$ generators E_a^b which are expressed in terms of a $8 \times 8 = 64$-component tensor Y^{ab} that can be decomposed into a symmetric part $Y^{(ab)}$ with 36 independent components, and an anti-symmetric part $Y^{[ab]}$ with 28 independent components. Its trace $Y^{cc} = N$ yields an element N of the Cartan subalgebra such that the degrees $-3, -2, -1, 0, 3, 2, 1$ of the 7-grading of $E_{8(8)}$ can be read from [12]

We begin by following very closely [12] by writing the full E_8 commutators in the $SL(8,R)$ basis of [13], after decomposing the $SO(16)$ representations into representations of the subgroup $SO(8) \equiv \big(SO(8) \times SO(8)\big)_{\rm diag} \subset SO(16)$. The indices corresponding to the $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$, respectively, will be denoted by a, α and $\dot\alpha$. After a triality rotation the $SO(8)$ vector and spinor representations decompose as [12]

$$\mathbf{16} \rightarrow \mathbf{8_s} \oplus \mathbf{8_c}. \tag{2.5}$$

$$\mathbf{128_s} \rightarrow (\mathbf{8_s} \otimes \mathbf{8_c}) \oplus (\mathbf{8_v} \otimes \mathbf{8_v}) = \mathbf{8_v} \oplus \mathbf{56_v} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v}. \tag{2.6a}$$

$$\mathbf{128_c} \rightarrow (\mathbf{8_v} \otimes \mathbf{8_s}) \oplus (\mathbf{8_c} \otimes \mathbf{8_v}) = \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{8_c} \oplus \mathbf{56_c}. \tag{2.6b}$$

respectively. We thus have $I = (\alpha, \dot\alpha)$ and $A = (\alpha\dot\beta, ab)$, and the E_8 generators decompose as

$$X^{[IJ]} \rightarrow (X^{[\alpha\beta]},\ X^{[\dot\alpha\dot\beta]},\ X^{\alpha\dot\beta}); \quad Y^A \rightarrow (Y^{\alpha\dot\alpha}, Y^{ab}). \tag{2.7}$$

Next we regroup these generators as follows. The 63 generators

$$E_a^b \;=\; \frac{1}{8}(\ \Gamma^{ab}_{\alpha\beta}\ X^{[\alpha\beta]} \;+\; \Gamma^{ab}_{\dot\alpha\dot\beta}\ X^{[\dot\alpha\dot\beta]}) \;+\; Y^{(ab)} \;-\; \frac{1}{8}\delta^{ab} Y^{cc}. \tag{2.8}$$

for $1 \leq a, b \leq 8$ span an $SL(8,R)$ subalgebra of E_8. The generator given by the trace $N = Y^{cc}$ extends this subalgebra to $GL(8,R)$. $\Gamma^{ab}, \Gamma^{abc}, ..$ are signed sums of antisymmetrized products of gammas. The remainder of the E_8 Lie algebra then decomposes into the following representations of $SL(8,R)$:

$$Z^a \;=\; \frac{1}{4}\Gamma^a_{\alpha\dot\alpha}\ (\ X^{\alpha\dot\alpha} \;+\; Y^{\alpha\dot\alpha}). \tag{2.9a}$$

$$Z_{[ab]} \;=\; Z_{ab} \;=\; \frac{1}{8}\left(\ \Gamma^{ab}_{\alpha\beta}\ X^{[\alpha\beta]} \;-\; \Gamma^{ab}_{\dot\alpha\dot\beta}\ X^{[\dot\alpha\dot\beta]}\ \right) \;+\; Y^{[ab]}. \tag{2.9b}$$

$$E^{[abc]} \;=\; E^{abc} \;=\; -\frac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}\ (\ X^{\alpha\dot\alpha} \;-\; Y^{\alpha\dot\alpha}). \tag{2.9c}$$

and

$$Z_a \;=\; -\frac{1}{4}\Gamma^a_{\alpha\dot\alpha}\ (X^{\alpha\dot\alpha} \;-\; Y^{\alpha\dot\alpha}\ ). \tag{2.10a}$$

$$Z^{[ab]} \;=\; Z^{ab} \;=\; -\frac{1}{8}\left(\;\Gamma^{ab}_{\alpha\beta}\;X^{[\alpha\beta]}\;-\;\Gamma^{ab}_{\dot\alpha\dot\beta}\;X^{[\dot\alpha\dot\beta]}\;\right)\;+\;Y^{[ab]}. \qquad (2.10b)$$

$$E_{[abc]} \;=\; E_{abc} \;=\; -\frac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}\;(\;X^{\alpha\dot\alpha}\;+\;Y^{\alpha\dot\alpha}). \qquad (2.10c)$$

It is important to emphasize that $Z_a \neq \eta_{ab}Z^b$, $Z_{ab} \neq \eta_{ac}\eta_{db}Z^{cd}$, ... and for these reasons one could use the more convenient notation for the generators

$$\mathcal{Z}^a_{\pm} \equiv (Z^a, Z_a); \quad \mathcal{Z}^{ab}_{\pm} \equiv (Z^{ab}, Z_{ab}); \quad \mathcal{Z}^{abc}_{\pm} \equiv (E^{abc}, E_{abc}). \qquad (2.11)$$

which permits to view these *doublets* of generators (2.11) as pairs of "canonically conjugate variables", and which in turn, allows us to view their commutation relations as a defining a generalized deformed Weyl-Heisenberg algebra with noncommuting coordinates and momenta as shown next. One may now define the pairs of complex generators to be used later

$$V^a \;=\; \frac{1}{\sqrt{2}}\;(\mathcal{Z}^a_+\;-i\;\mathcal{Z}^a_-), \quad \bar V^a \;=\; \frac{1}{\sqrt{2}}\;(\mathcal{Z}^a_+\;+i\;\mathcal{Z}^a_-). \qquad (2.12a)$$

$$V^{ab} \;=\; \frac{1}{\sqrt{2}}\;(\mathcal{Z}^{ab}_+\;-i\;\mathcal{Z}^{ab}_-), \quad \bar V^{ab} \;=\; \frac{1}{\sqrt{2}}\;(\mathcal{Z}^{ab}_+\;+i\;\mathcal{Z}^{ab}_-). \qquad (2.12b)$$

$$V^{abc} \;=\; \frac{1}{\sqrt{2}}\;(\mathcal{Z}^{ab}_+\;-i\;\mathcal{Z}^{ab}_-), \quad \bar V^{abc} \;=\; \frac{1}{\sqrt{2}}\;(\mathcal{Z}^{abc}_+\;+i\;\mathcal{Z}^{abc}_-). \qquad (2.12c)$$

The remaining $GL(8,R) = Sl(8,R)\times U(1)$ generators are

$$\mathcal{E}^{ab} \;=\; \mathcal{E}^{(ab)} + \mathcal{E}^{[ab]}. \qquad (2.13)$$

The Cartan subalgebra is spanned by the diagonal elements $E^1_1, ..., E^7_7$ and N, or, equivalently, by $Y^{11}, ..., Y^{88}$. The elements E^b_a for $a < b$ (or $a > b$) together with the elements for $a < b < c$ generate the Borel subalgebra of E_8 associated with the positive (negative) roots of E_8. Furthermore, these generators are graded w.r.t. the number of times the root α_8 (corresponding to the element N in the Cartan subalgebra) appears, such that for any basis generator X we have $[N, X] = deg\ (X)\cdot X$.

The degree can be read off from

$$[N,\;Z^a]\;=\;3Z^a,\quad [N,\;Z_a]\;=\;-3\,Z_a,\quad [N,\;Z_{ab}]\;=\;2Z_{ab};\quad [N,\;Z^{ab}]\;=\;-2Z^{ab}$$

$$[N,\;E^{abc}]\;=\;E^{abc},\quad [N,\;E_{abc}]\;=\;-\,E_{abc};\quad [N,\;E^b_a]\;=\;0. \qquad (2.14)$$

The remaining commutation relations defining the generalized deformed Weyl-Heisenberg algebra involving pairs of canonical conjugate generators are

$$[Z^a, Z^b] = 0; \quad [Z_a, Z_b] = 0; \quad [Z_a, Z^b] = E^b_a - \frac{3}{8}\delta^b_a N. \qquad (2.15)$$

This last commutator between the pairs of conjugate Z_a, Z^b generators (like phase space coordinates) yields the deformed Weyl-Heisenberg algebra. The latter algebra is *deformed* due to the presence of the E_a^b generator in the r.h.s of (2.15) and also because the $\mathcal{N}$ trace generator does *not* commute with Z_a, Z^a as seen in (2.14). Similarly, one has the deformed Weyl-Heisenberg algebra among the pairs of conjugate Z_{ab}, Z^{ab} antisymmetric rank-two tensorial generators (like tensorial phase space coordinates)

$$[Z_{ab}, Z_{cd}] = 0; \quad [Z^{ab}, Z^{cd}] = 0; \quad [Z_{ab}, Z^{cd}] = 4\delta^{[c}_{[a}\ E^{d]}_{b]} \ + \ \frac{1}{2}\delta^{cd}_{ab} N; \tag{2.16}$$

The commutators among the pairs of conjugate and *noncommuting* E_{abc}, E^{abc} antisymmetric rank-three generators (like noncommuting tensorial phase space coordinates) are

$$[E^{abc}, E^{def}] \ = \ -\frac{1}{32}\epsilon^{abcdefgh} Z_{gh} \ \neq \ 0 \quad [E_{abc}, E_{def}] \ = \ \frac{1}{32}\epsilon_{abcdefgh} Z^{gh} \ \neq \ 0 \tag{2.17}$$

$$[E^{abc}, E_{def}] \ = \ -\ \frac{1}{8}\ \delta^{[ab}_{[de} E_{f]}{}^{c]} \ - \frac{3}{4}\ \delta^{abc}_{def}\ N. \tag{2.18}$$

The other commutators among the generalized antisymmetric tensorial generators are

$$[Z_{ab}, Z^c] = 0; \ [Z_{ab}, Z_c] = -E_{abc}; \ [Z^{ab}, Z^c] = -E^{abc}; \ [Z^{ab}, Z_c] = 0. \tag{2.19}$$

$$[E^{abc}, Z^d] = 0; \quad [E_{abc}, Z^d] = 3\delta^d_{[a}\ Z_{bc]}; \quad [E^{abc}, Z_{de}] = -6\delta^{[ab}_{de} Z^{c]}; \quad [E_{abc}, Z_{de}] = 0. \tag{2.20}$$

$$[E^{abc}, Z_d] = 3\delta^{[a}_d\ Z^{bc]}; \quad [E_{abc}, Z_d] = 0; \quad [E^{abc}, Z^{de}] = 0; \quad [E_{abc}, Z^{de}] \ = \ 6\delta^{de}_{[ab} Z_{c]}. \tag{2.21}$$

The homogeneous commutators among the $GL(8,R)$ generators and those belonging to the deformed Weyl-Heisenberg algebra are

$$[E^b_a,\ Z^c] \ = \ -\ \delta^c_a\ Z^b \ + \ \frac{1}{8}\ \delta^b_a\ Z^c; \quad [E^b_a,\ Z_c] \ = \ \delta^b_c\ Z_a \ - \ \frac{1}{8}\ \delta^b_a\ Z_c. \tag{2.22}$$

$$[E^b_a,\ Z_{cd}] \ = \ -\ 2\delta^b_{[c}\ Z_{d]a} \ - \ \frac{1}{4}\ \delta^b_a\ Z_{cd}; \quad [E^b_a,\ Z^{cd}] \ = \ 2\delta^{[c}_a\ Z^{d]b} \ + \ \frac{1}{4}\ \delta^b_a\ Z^{cd}.$$

$$[E^b_a,\ E^{cde}] \ = \ -3\delta^{[c}_a\ E^{de]b} + \frac{3}{8}\ \delta^b_a\ E^{cde}; \quad [E_a{}^b, E_{cde}] \ = \ 3\delta^b_{[c}\ E_{de]a} - \frac{3}{8}\ \delta^b_a\ E_{cde}. \tag{2.23}$$

Finally, the commutators among the $GL(8,R)$ generators are

$$[E_a^b,\ E_c^d] \ = \ \delta_c^b\ E_a^d \ - \ \delta_a^d\ E_c^b. \tag{2.24}$$

The elements $\{Z^a, Z_{ab}\}$ (or equivalently $\{Z_a, Z^{ab}\}$) span the maximal 36-dimensional abelian nilpotent subalgebra of E_8 [12], [13]. Finally, the generators are normalized according to the values of the traces given by

$$Tr\ (NN) \ = \ 60 \cdot 8; \quad Tr\ (Z^a Z_b) \ = \ 60\ \delta_b^a, \quad Tr\ (Z^{ab} Z_{cd}) \ = \ 60 \cdot 2!\ \ \delta_{cd}^{ab}$$

$$Tr\ (E_{abc} E^{def}) \ = \ 60 \cdot 3!\ \delta_{abc}^{def}, \quad Tr\ (E_a^b E_c^d) \ = \ 60\ \delta_a^d \delta_c^b \ - \ \frac{15}{2}\ \delta_a^b\ \delta_c^d. \tag{2.25}$$

with all other traces vanishing.

Using the redefinitions of the generators in eqs-(2.11, 2.12) allows to write the E_8 Hermitian gauge connection associated with the E_8 generators as

$$\mathcal{A}_\mu \ = \ E_\mu^a\ V_a \ + \ \bar{E}^a\ \bar{V}_a \ + \ E_\mu^{ab}\ V_{ab} \ + \ \bar{E}_\mu^{ab}\ \bar{V}_{ab} \ +$$
$$E_\mu^{abc}\ V_{abc} \ + \ \bar{E}_\mu^{abc}\ \bar{V}_{abc} \ + i\ \Omega_\mu^{(ab)}\ \mathcal{E}_{(ab)} \ + \ \Omega_\mu^{[ab]}\ \mathcal{E}_{[ab]} \tag{2.26}$$

where one may set the length scale $L = 1$, scale that is attached to the vielbeins to match the $(length)^{-1}$ dimensions of the connection in (2.26). The $GL(8,R)$ components of the E_8 (Hermitian) gauge connection are the (real-valued symmetric) $\Omega_\mu^{(ab)}$ shear and (real-valued antisymmetric) $\Omega_\mu^{[ab]}$ rotational parts of the $GL(8,R)$ anti-Hermitian gauge connection $i\ (\Omega_\mu^{(ab)} - i\Omega_\mu^{[ab]})$ such that the $GL(8,R)$ Lie-algebra-valued connection $i\ \Omega_\mu^{ab}\ \mathcal{E}_{ab}$ is Hermitian because the $GL(8,R)$ generators $\mathcal{E}_{(ab)}, \mathcal{E}_{[ab]}$, and the remaining ones appearing in the E_8 commutators of eqs-(2.14-2.24), are all chosen to be anti-Hermitian (there are no i factors in the r.h.s of the latter commutators). The (generalized) vielbeins fields are $E_\mu^a, E_\mu^{ab}, E_\mu^{abc}$ plus their complex conjugates. These (generalized) vielbeins fields involving antisymmetric tensorial tangent space indices also appear in generalized gravity in Clifford spaces (C-spaces) where one has polyvector-valued coordinates in the base space and in the tangent space such that the generalized vielbeins are represented by square *and* rectangular matrices [22]. The trace part $\mathcal{N}$ is included in the symmetric shear-like generator $\mathcal{E}_{(ab)}$ of $Gl(8,R)$. The rotational part corresponds to $\mathcal{E}_{[ab]}$.

The E_8 (Hermitian) field strength (in natural units $\hbar = c = 1$) is

$$\mathbf{F}_{\mu\nu} \ = \ i\ [\ D_\mu,\ D_\nu\ ] \ = \ (\ \partial_\mu\ \mathcal{A}_\nu^A \ - \ \partial_\nu\ \mathcal{A}_\mu^A \ + \ i\ f_{BC}^A\ \mathcal{A}_\mu^B\ \mathcal{A}_\nu^C\ )\ L_A. \tag{2.27a}$$

where the indices $A = 1, 2, 3, ...248$ are spanned by the 248 generators L_A of E_8

$$V_a,\ \ \bar{V}_a,\ \ V_{ab},\ \ \bar{V}_{ab},\ \ V_{abc},\ \ \bar{V}_{abc},\ \ \mathcal{E}_{(ab)},\ \ \mathcal{E}_{[ab]}. \tag{2.27b}$$

giving a total of $8 + 8 + 28 + 28 + 56 + 56 + 36 + 28 = 248$, respectively.

The structure constants are determined by the commutators eqs-(2.14-2.24) in terms of the redefinitions of the generators in eqs-(2.11, 2.12). It is the $GL(8,R)$ field strength sector of the E_8 field strength the one which is associated with the Hermitian $GL(8,R)$-valued *curvature* two form

$$\mathbf{R} \ = \ (\ i \ R^{(ab)}_{\mu\nu} \ \mathcal{E}_{(ab)} \ + \ R^{[ab]}_{\mu\nu} \ \mathcal{E}_{[ab]} \ ) \ dx^\mu \wedge dx^\nu \ = \\ i \ (\ R^{(ab)}_{\mu\nu} \ - i \ R^{[ab]}_{\mu\nu} \ ) \ (\ \mathcal{E}_{(ab)} \ + \ \mathcal{E}_{[ab]} \ ) \ dx^\mu \wedge dx^\nu \tag{2.28}$$

and whose components are given by

$$R^{[ab]}_{\mu\nu} \ = \ \partial_\mu \Omega^{[ab]}_\nu \ - \ \partial_\nu \Omega^{[ab]}_\mu \ + \ \Omega^{[ac]}_{[\mu} \ \Omega^{[cb]}_{\nu]} \ - \\ \Omega^{(ac)}_{[\mu} \ \Omega^{(cb)}_{\nu]} \ + \frac{1}{L^2} \ E^a_{[\mu} \ E^b_{\nu]} \ + \frac{1}{L^2} \ \bar{E}^a_{[\mu} \ \bar{E}^b_{\nu]} \ + \ .... \tag{2.29}$$

$$R^{(ab)}_{\mu\nu} \ = \ \partial_\mu \Omega^{(ab)}_\nu \ - \ \partial_\nu \Omega^{(ab)}_\mu \ + \ \Omega^{(ac)}_{[\mu} \ \Omega^{[cb]}_{\nu]} \ + \ \Omega^{(bc)}_{[\mu} \ \Omega^{[ca]}_{\nu]} \ + \\ \frac{1}{L^2} \ E^a_{[\mu} \ \bar{E}^b_{\nu]} \ + \frac{1}{L^2} \ E^b_{[\mu} \ \bar{E}^a_{\nu]} \ + \ .... \tag{2.30}$$

A summation over the repeated c indices is implied and $[\mu\nu]$ denotes the anti-symmetrization of indices with weight one. One may set the length scale $L = 1$ (necessary to match dimensions).

The components of the (generalized) torsion two-form correspond to the field strength associated with the (generalized) vielbeins

$$F^a_{\mu\nu} \ = \ \partial_\mu E^a_\nu \ - \ \partial_\nu E^a_\mu \ + \ \Omega^{[ac]}_{[\mu} \ E^c_{\nu]} \ + \ .... \tag{2.31}$$

$$F^{ab}_{\mu\nu} \ = \ \partial_\mu E^{ab}_\nu \ - \ \partial_\nu E^{ab}_\mu \ + \ \Omega^{[ac]}_{[\mu} \ E^{cb}_{\nu]} \ + \ .... \tag{2.32}$$

$$F^{abc}_{\mu\nu} \ = \ \partial_\mu E^{abc}_\nu \ - \ \partial_\nu E^{abc}_\mu \ + \ \Omega^{[ad]}_{[\mu} \ E^{dbc}_{\nu]} \ + \ ...... \tag{2.33}$$

plus their complex conjugates $\bar{F}^a_{\mu\nu}, \bar{F}^{ab}_{\mu\nu}, \bar{F}^{abc}_{\mu\nu}$.

The *complex* Hermitian metric with symmetric $g_{(\mu\nu)}$ and antisymmetric $g_{[\mu\nu]}$ components (which could play the role of a symplectic structure) associated with the Exceptional E_8 Geometry is defined in terms of the (generalized) complex vielbeins

$$E^a_\mu \ = \ \frac{1}{\sqrt{2}} \ (\ e^a_\mu \ + \ i f^a_\mu \ ); \quad \bar{E}^a_\mu \ = \ \frac{1}{\sqrt{2}} \ (\ e^a_\mu \ - \ i f^a_\mu \ ). \tag{2.34}$$

$$E^{ab}_\mu \ = \ \frac{1}{\sqrt{2}} \ (\ e^{ab}_\mu \ + \ i f^{ab}_\mu \ ); \quad \bar{E}^{ab}_\mu \ = \ \frac{1}{\sqrt{2}} \ (\ e^{ab}_\mu \ - \ i f^{ab}_\mu \ ). \tag{2.35}$$

$$E^{abc}_\mu \ = \ \frac{1}{\sqrt{2}} \ (\ e^{abc}_\mu \ + \ i f^{abc}_\mu \ ); \quad \bar{E}^{abc}_\mu \ = \ \frac{1}{\sqrt{2}} \ (\ e^{abc}_\mu \ - \ i f^{abc}_\mu \ ). \tag{2.36}$$

as

$$g_{\mu\nu} \equiv g_{(\mu\nu)} + i\, g_{[\mu\nu]} \equiv E^a_\mu \, \bar E^b_\nu \, \eta_{ab} + E^{ab}_\mu \, \bar E^{cd}_\nu \, \rho_{abcd} + E^{abc}_\mu \, \bar E^{def}_\nu \, \rho_{abcdef}. \tag{2.37}$$

such that $(g_{\mu\nu})^\dagger = g_{\mu\nu} \Rightarrow (g_{\mu\nu})^* = g_{\nu\mu}$ and where the generalized (tangent space) area and volume metrics are given as

$$\rho_{abcd} = \eta_{ac}\, \eta_{bd} - \eta_{bc}\, \eta_{ad}. \tag{2.38a}$$

$$\rho_{abcdef} = \eta_{ad}\, \eta_{be}\, \eta_{cf} \pm \textit{permutations of } a, b, c \textit{ indices} \tag{2.38b}$$

The complex-valued Hermitian curvature tensor is defined

$$\mathcal{R}_{\mu\nu\rho\lambda} = (R^{(ab)}_{\mu\nu} - i\, R^{[ab]}_{\mu\nu}) (E_{a\rho}\, E_{b\lambda} + \bar E_{a\rho}\, \bar E_{b\lambda}). \tag{2.39a}$$

$$\mathcal{R}^\rho_{\mu\nu\lambda} = (R^{(ab)}_{\mu\nu} - i\, R^{[ab]}_{\mu\nu}) (E^\rho_a\, E_{b\lambda} + \bar E^\rho_a\, \bar E_{b\lambda}). \tag{2.39b}$$

where

$$E_{a\mu} = \eta_{ab}\, \bar E^b_\mu , \ \bar E_{a\mu} = \eta_{ab}\, E^b_\mu , \ E^\rho_a\, E^b_\rho = \delta^b_a, \ \bar E^\rho_a\, \bar E^b_\rho = \delta^b_a. \tag{2.39c}$$

$$E_{\mu ab} = \bar E^{cd}_\mu\, \rho_{abcd}, \ E_{\mu abc} = \bar E^{def}_\mu\, \rho_{abcdef}, \ E^\rho_{ab}\, E^{cd}_\rho = \delta^{cd}_{ab}, \ E^\rho_{abc}\, E^{def}_\rho = \delta^{def}_{abc}. \tag{2.39d}$$

The contraction of spacetime indices of the Hermitian curvature tensor with the complex Hermitian metric $g_{\mu\nu}$ yields *two* different *complex* valued Hermitian Ricci tensors [1] given by

$$\mathcal{R}_{\mu\lambda} = g_{\rho\sigma}\, g^{\sigma\nu}\, R^\rho_{\mu\nu\lambda} = \delta^\nu_\rho\, R^\rho_{\mu\nu\lambda} = R_{(\mu\lambda)} + i\, R_{[\mu\lambda]}; \ (\mathcal{R}_{\mu\lambda})^* = \mathcal{R}_{\lambda\mu} \tag{2.40}$$

and

$$\mathcal{S}_{\mu\lambda} = g_{\sigma\rho}\, g^{\sigma\nu}\, R^\rho_{\mu\nu\lambda} = \mathcal{S}_{(\mu\lambda)} + i\, \mathcal{S}_{[\mu\lambda]}; \ (\mathcal{S}_{\mu\lambda})^* = \mathcal{S}_{\lambda\mu} \tag{2.41}$$

due to the fact that

$$g_{\rho\sigma}\, g^{\sigma\nu} = \delta^\nu_\rho \ \textit{but} \ g_{\sigma\rho}\, g^{\sigma\nu} \neq \delta^\nu_\rho. \tag{2.42}$$

because $g_{\sigma\rho} \neq g_{\rho\sigma}$. The position of the indices is crucial.

A further contraction yields the generalized (real-valued) Ricci scalars

$$\mathcal{R} = g^{\lambda\mu}\, R_{\mu\lambda} = (g^{(\mu\lambda)} - i\, g^{[\mu\lambda]}) (R_{(\mu\lambda)} + i\, R_{[\mu\lambda]}) =$$

$$\mathcal{R} = g^{(\mu\lambda)}\, R_{(\mu\lambda)} + B^{\mu\lambda}\, R_{[\mu\lambda]}. \tag{2.43}$$

[1] There is a third Ricci tensor $Q_{[\mu\nu]} = \mathcal{R}^\rho_{\mu\nu\lambda}\delta^\lambda_\rho$ related to the curl of the *nonmetricity* Weyl vector Q_μ [24]

$$\mathcal{S} \;=\; g^{\lambda\mu}\, S_{\mu\lambda} \;=\; (g^{(\mu\lambda)} \;-i\; g^{[\mu\lambda]})\;(\; S_{(\mu\lambda)} \;+\; i\; S_{[\mu\lambda]}\;) =$$

$$\mathcal{S} \;=\; g^{(\mu\lambda)}\, \mathcal{S}_{(\mu\lambda)} \;+\; B^{\mu\lambda}\, \mathcal{S}_{[\mu\lambda]}. \tag{2.44}$$

The antisymmetric part of the metric $g^{[\mu\lambda]} \equiv B^{\mu\lambda}$ can be identified with a Kalb-Ramond field. The first term $g^{(\mu\lambda)}\, R_{(\mu\lambda)}$ corresponds to the usual scalar curvature of the ordinary Riemannian geometry. The presence of the extra terms $B^{\mu\lambda}\, R_{[\mu\lambda]}$ and $B^{\mu\lambda}\, \mathcal{S}_{[\mu\lambda]}$ due to the anti-symmetric components of the complex metric, the two different types of Ricci tensors and the presence of generalized vielbeins with antisymmetric tensorial tangent space indices in the definition of the complex Hermitian metric (2.37, 2.38) are one of the hallmarks of this Exceptional complex gravity based on E_8. We should notice that the inverse complex metric is

$$g^{(\mu\lambda)} \;+\; ig^{[\mu\lambda]} \;=\; [\; g_{(\mu\nu)} \;+\; ig_{[\mu\nu]}\;]^{-1} \;\neq\; (g_{(\mu\nu)})^{-1} \;+\; (ig_{[\mu\nu]})^{-1}. \tag{2.45}$$

so $g^{(\mu\nu)}$ is now a complicated expression of both $g_{\mu\nu}$ *and* $g_{[\mu\nu]} = B_{\mu\nu}$. The same occurs with $g^{[\mu\nu]} = B^{\mu\nu}$. Rigorously we should have used a different notation for the inverse metric $\tilde{g}^{(\mu\lambda)} \;+\; i\tilde{B}^{[\mu\lambda]}$, but for notational simplicity we chose to drop the tilde symbol.

The generalized real-valued torsion tensor $\mathcal{T}^{\rho}_{\mu\nu}$ is defined in terms of the complex-valued torsion $T^{\rho}_{\mu\nu}, \bar{T}^{\rho}_{\mu\nu}$ quantities as

$$\mathcal{T}^{\rho}_{\mu\nu} \;\equiv\; T^{\rho}_{\mu\nu} \;+\; \bar{T}^{\rho}_{\mu\nu} \;=$$

$$F^{a}_{\mu\nu}\, E^{\rho}_{a} + \bar{F}^{a}_{\mu\nu}\, \bar{E}^{\rho}_{a} + F^{ab}_{\mu\nu}\, E^{\rho}_{ab} + \bar{F}^{ab}_{\mu\nu}\, \bar{E}^{\rho}_{ab} \;+\; F^{abc}_{\mu\nu}\, E^{\rho}_{abc} + \bar{F}^{abc}_{\mu\nu}\, \bar{E}^{\rho}_{abc}. \tag{2.46a}$$

The real-valued torsion vector is

$$\mathcal{T}_{\mu} \;=\; \delta^{\nu}_{\rho}\, \mathcal{T}^{\rho}_{\mu\nu} \;=\; T_{\mu} \;+\; \bar{T}_{\mu} \tag{2.46b}$$

where the complex valued torsion tensors are

$$T_{\mu\nu\rho} \;=\; T^{\sigma}_{\mu\nu}\, g_{\sigma\rho} \;=\; F^{a}_{\mu\nu}\, E_{\rho a} \;+\; F^{ab}_{\mu\nu}\, E_{\rho ab} \;+\; F^{abc}_{\mu\nu}\, E_{\rho abc}. \tag{2.46c}$$

$$\bar{T}_{\mu\nu\rho} \;=\; \bar{T}^{\sigma}_{\mu\nu}\, \bar{g}_{\sigma\rho} \;=\; \bar{F}^{a}_{\mu\nu}\, \bar{E}_{\rho a} \;+\; \bar{F}^{ab}_{\mu\nu}\, \bar{E}_{\rho ab} \;+\; \bar{F}^{abc}_{\mu\nu}\, \bar{E}_{\rho abc}. \tag{2.46d}$$

the complex-valued torsion vectors are

$$T_{\mu} \;=\; T_{\mu\nu\rho}\, g^{\rho\nu} \;=\; \delta^{\nu}_{\sigma}\, T^{\sigma}_{\mu\nu}, \quad \bar{T}_{\mu} \;=\; \bar{T}_{\mu\nu\rho}\, \bar{g}^{\rho\nu} \;=\; \delta^{\nu}_{\sigma}\, \bar{T}^{\sigma}_{\mu\nu}. \tag{2.46e}$$

The inverse vielbeins are defined by

$$E^{\rho}_{a}\, E^{b}_{\rho} \;=\; \delta^{b}_{a}, \quad E^{\rho}_{ab}\, E^{cd}_{\rho} \;=\; \delta^{cd}_{ab}, \quad E^{\rho}_{abc}\, E^{def}_{\rho} \;=\; \delta^{def}_{abc}, \;....... \tag{2.47a}$$

$$E^{\nu}_{a}\, E^{a}_{\mu} \;=\; \delta^{\nu}_{\mu}, \quad E^{\nu}_{ab}\, E^{ab}_{\mu} \;=\; \delta^{\nu}_{\mu}, \quad E^{\nu}_{abc}\, E^{abc}_{\mu} \;=\; \delta^{\nu}_{\mu}, ... \tag{2.47b}$$

and one has the following relationships

$$g_{\sigma\rho}\, E^{\rho}_{a} \;=\; E_{\sigma a}, \quad \bar{g}_{\sigma\rho}\, \bar{E}^{\rho}_{a} \;=\; \bar{E}_{\sigma a}, \quad g_{\sigma\rho}\, E^{\rho}_{ab} \;=\; E_{\sigma ab}, \quad \bar{g}_{\sigma\rho}\, \bar{E}^{\rho}_{ab} \;=\; \bar{E}_{\sigma ab}, ... \tag{2.47c}$$

but

$$g_{\sigma\rho}\ \bar{E}^{\rho}_{a}\ =\ \bar{g}_{\rho\sigma}\ \bar{E}^{\rho}_{a}\ \neq\ \bar{E}_{\sigma a},\quad \bar{g}_{\sigma\rho}\ E^{\rho}_{a}\ =\ g_{\rho\sigma}\ E^{\rho}_{a}\ \neq\ E_{\sigma a},\ \ldots\ldots\ldots\ldots \qquad (2.47d)$$

One should notice the last inequalities because the position of indices is essential as indicated by eq-(2.42). The (real-valued) action, linear in the two (real-valued) Ricci curvature scalars and quadratic in the (real-valued) generalized torsion is of the form

$$\frac{1}{2\kappa^2}\int d^8x\ \sqrt{|\ det\ (g_{(\mu\nu)} + i g_{[\mu\nu]})\ |}\ (\ a_1\ \mathcal{R}\ +\ a_2\ \mathcal{S}\ +\ a_3\ \mathcal{T}^{\rho}_{\mu\nu}\ \mathcal{T}^{\mu\nu}_{\rho}\ +\ a_4\ \mathcal{T}_{\mu}\ \mathcal{T}^{\mu}). \qquad (2.48)$$

The real-valued torsion squared terms can be explictly written as

$$\mathcal{T}^{\rho}_{\mu\nu}\ \mathcal{T}^{\mu\nu}_{\rho}\ =\ T_{\mu\nu\rho}\ T^{\mu\nu\rho}\ +\ \bar{T}_{\mu\nu\rho}\ \bar{T}^{\mu\nu\rho}\ +\ T_{\mu\nu\rho}\ \bar{T}^{\mu\nu\rho}\ +\ \bar{T}_{\mu\nu\rho}\ T^{\mu\nu\rho}. \qquad (2.49a)$$

$$\mathcal{T}_{\mu}\ \mathcal{T}^{\mu}\ =\ T_{\mu}\ T^{\mu}\ +\ \bar{T}_{\mu}\ \bar{T}^{\mu}\ +\ T_{\mu}\ \bar{T}^{\mu}\ +\ \bar{T}_{\mu}\ T^{\mu}. \qquad (2.49b)$$

where κ^2 is the gravitational coupling in $8D$ of dimensions $(length)^6$ and a_1, a_2, a_3, a_4 are suitable parameters which can be constrained if one wishes to avoid the presence of tachyons, ghosts and higher order poles in the quantum propagators, in a similar vein as it occurs in nonsymmetric theories of gravity [24], [25], [26], [28] and ordinary metric affine theories of gravity based in gauging the affine group $GL(8,R)\times_s T_8$ in $8D$ which is given as the semi-direct product of $GL(8,R)$ with the translations group [23].

Curvature squared terms (and higher powers) could be added to the action, and terms involving the nonvanishing nonmetricity tensor as well. For instance, affine theories of gravity are *not* Riemannian and as such have a *nonvanishing* nonmetricity tensor $Q_{\mu\nu\rho} = D_\mu g_{(\nu\rho)} \neq 0$. Since we have now a symmetric and antisymmetric metric $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ it is possible to still have $D_\mu g_{(\nu\rho)} \neq 0$ and $D_\mu g_{\nu\rho} = 0$ simultaneously. Quantum gravity models in $4D$ based on gauging the (covering of the) $GL(4,R)$ group were shown to be *renormalizable* [34], however due to the presence of fourth-derivatives terms in the metric which appeared in the quantum effective action, upon including gauge fixing terms and ghost terms, the prospects of *unitarity* were spoiled. The key question remains if this novel gravitational model based on gauging the E_8 group may still be renormalizable *without* spoiling unitarity at the quantum level.

To sum up, the action (2.48) is a candidate action for an Exceptional E_8 gauge theory of gravity in $8D$ obtained by viewing the E_8 group as the semi-direct product of $GL(8,R)$ with a deformed Weyl-Heisenberg group associated with canonical-conjugate vectorial and antisymmetric tensorial generators of rank two and three. The curvature has a dilational, shear and rotational part corresponding to the one dilation generator, 35 shear $\mathcal{E}^{(ab)}$ and 28 rotational(Lorentz) $\mathcal{E}^{[ab]}$ generators of $GL(8,R)$. The generalized torsion is the field strength corresponding to the remaining vectorial and tensorial generators. The gauge fields associated with the latter generators are the generalized complex vielbeins from which the complex Hemitian metric in eqs-(2.37, 2.38) with symmetric $g_{(\mu\nu)}$ and anti-symmetric $g_{[\mu\nu]}$ (Kalb-Ramond like) components is explicitly defined.

As mentioned in the introduction, a Kaluza-Klein-Batakis [19] compactification of an $8D$ Conformal gravitational theory on an internal four-dim CP^2 space , involving a nontrivial *torsion*, leads to a conformal Gravity-Yang-Mills unified theory based on the Standard Model group $SU(3)\times SU(2)\times U(1)$ in $4D$. For these reasons, the E_8 gauge theory of gravity in $8D$ constructed here is very appealing. We will discuss in the next section how the inherent E_8 Geometry present in the action (2.48) can be seen as a particular case of a more general Clifford space gravity associated with the Clifford algebra $Cl(16)$ in $16D$ [16], [22].

The standard E_8 Yang-Mills action in $8D$ associated with the field strengths in eq-(2.27) and involving the ordinary real symmetric metric $\hat{g}_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \neq g_{(\mu\nu)}$ of the manifold $\mathcal{M}^8$ is

$$I_{YM} \;=\; \frac{1}{4g^2} \int d^8x \,\sqrt{|det\ \hat{g}_{\mu\nu}|}\ Trace\ (\ \mathbf{F}_{\mu\nu}\ \mathbf{F}^{\mu\nu}\ ). \qquad (2.49c)$$

The trace operation is performed in the 248-dim adjoint representation and to evaluate the action (2.49) it is more convenient to use the normalization condition of the 248 original anti-Hermitian generators given by eq-(2.25) in terms of the trace operation. The Yang-Mills coupling g^2 has dimensions of $(length)^4$ in $8D$.

A topological invariant action based on the *quartic* E_8-invariant action in $8D$ is given by

$$I_{(4)} \;=\; \int_{\mathcal{M}^8} \;<\ F\wedge F\wedge F\wedge F\ >_{E_8}. \qquad (2.50)$$

the $<\ ......\ >$ operation involves the existence of a *quartic* E_8 group invariant tensor that allows us to contract group indices and which contains powers of the 120 $SO(16)$ bivectors X_{IJ} and the chiral spinorial Y_α generators. A dimensionless factor in front of the integral can be included. The action (2.50) is *locally* a total derivative and since the E_8 Lie-algebra valued 8-form $<\ F^4\ >$ is *closed* : $d\ (<\ F^{M_1}T_{M_1}\wedge F^{M_2}T_{M_2}\wedge\ .....\ \wedge F^{M_4}T_{M_4}\ >) = 0$, the action *locally* can always be written as an exact form in terms of an E_8-valued Chern-Simons 7-form as $I_8 =\ d\mathcal{L}^{(7)}_{CS}(\mathbf{A},\mathbf{F})$, exactly as we did in the $16D$ case [15].

A generalized Yang-Mills (GYM) action in $8D$ involving quartic powers of the field strength is

$$I_{GYM} \;=\; \int_{M^8} \;<\ (F\wedge F)\ \wedge\ {}^*(F\wedge F)\ >. \qquad (2.51)$$

where the $<\ ......\ >$ operation requires again the use of the quartic E_8 group-invariant tensor in order to contract group indices like the Killing Lie group invariant metric in ordinary quadratic Yang-Mills actions. The Hodge star dual operation is defined in terms of the ordinary real symmetric metric $\hat{g}_{\mu\nu}$ of the manifold $\mathcal{M}^8$. Scalar and spinorial matter fields (minimally coupled to the gauge/geometric fields of the E_8 gauge theory of gravity) can be added to the action (2.48) and their equations of motion can be found, the Noether symmetry currents (associated with conservations laws) can be constructed, etc .... like in the metric affine theories of gravity [23].

3 Affine Theories of Extended Gravity in Clifford-Spaces

We begin this final section by showing how to embed the E_8 gauge theory of gravity and E_8 Yang-Mills theories into more general actions associated with the gauging of the $Cl(16)$ algebra in $16D$. Let us start by constructing the actions associated to the most general Clifford $Cl(16)$ gauge field theory by writing the $Cl(16)$-valued gauge field

$$\mathbf{A}_\mu \ = \ \mathcal{A}_\mu^A \ \Gamma_{\mathcal{A}} \ = \ \mathcal{A}_\mu \mathbf{1} + \mathcal{A}_\mu^a \Gamma_a + \mathcal{A}_\mu^{a_1 a_2} \Gamma_{a_1 a_2} + \mathcal{A}_\mu^{a_1 a_2 a_3} \Gamma_{a_1 a_2 a_3} \ + \ .... + \mathcal{A}_\mu^{a_1 a_2 \ldots a_{16}} \Gamma_{a_1 a_2 \ldots\ldots a_{16}}. \tag{3.1}$$

the $Cl(16)$-algebra-valued field strength (*omitting* numerical coefficients attached to the Γ's) is

$$\begin{gathered}\mathcal{F}_{\mu\nu}^A \ \Gamma_A = \partial_{[\mu} A_{\nu]} \ \mathbf{1} \ + \ [\ \partial_{[\mu} A_{\nu]}^a + A_{[\mu}^{b_2} A_{\nu]}^{b_1 a} \eta_{b_1 b_2} \ + ... \ ] \ \Gamma_a \ + \\ [\ \partial_{[\mu} A_{\nu]}^{ab} + A_{[\mu}^a A_{\nu]}^b - A_{[\mu}^{a_1 a} A_{\nu]}^{b_1 b} \eta_{a_1 b_1} - A_{[\mu}^{a_1 a_2 a} A_{\nu]}^{b_1 b_2 b} \eta_{a_1 b_1 a_2 b_2} \ + ... \ ] \ \Gamma_{ab} \ + \\ [\ \partial_{[\mu} A_{\nu]}^{abc} + A_{[\mu}^{a_1 a} A_{\nu]}^{b_1 bc} \eta_{a_1 b_1} \ +...... \ ] \ \Gamma_{abc} + [\ \partial_{[\mu} A_{\nu]}^{abcd} - A_{[\mu}^{a_1 a} A_{\nu]}^{b_1 bcd} \eta_{a_1 b_1} \ +...... \ ] \ \Gamma_{abcd} \\ +[\ \partial_{[\mu} A_{\nu]}^{a_1 a_2 \ldots a_5 b_1 b_2 \ldots b_5} + A_{[\mu}^{a_1 a_2 \ldots a_5} A_{\nu]}^{b_1 b_2 \ldots b_5} \ + ... \ ] \ \Gamma_{a_1 a_2 \ldots a_5 b_1 b_2 \ldots b_5} + ...\end{gathered} \tag{3.2}$$

and is obtained from the evaluation of the commutators of the Clifford-algebra generators. The most general formulae for all commutators and anti-commutators of $\Gamma^\mu, \Gamma^{\mu_1 \mu_2}, ...$, with the appropriate numerical coefficients, can be found in [20], in general for $pq = odd$ one has

$$[\gamma_{b_1 b_2 \ldots b_p}, \gamma^{a_1 a_2 \ldots\ldots a_q}] \ = \ 2 \gamma_{b_1 b_2 \ldots b_p}^{a_1 a_2 \ldots\ldots a_q} \ - \frac{2 p! q!}{2!(p-2)!(q-2)!} \ \delta_{[b_1 b_2}^{[a_1 a_2} \ \gamma_{b_3 \ldots b_p]}^{a_3 \ldots a_q]} \ + \ \frac{2 p! q!}{4!(p-4)!(q-4)!} \ \delta_{[b_1 \ldots b_4}^{[a_1 \ldots a_4} \ \gamma_{b_5 \ldots b_p]}^{a_5 \ldots a_q]} \ - \ ...... \tag{3.3a}$$

for $pq = even$ one has

$$[\gamma_{b_1 b_2 \ldots b_p}, \gamma^{a_1 a_2 \ldots\ldots a_q}] \ = \ - \ \frac{(-1)^{p-1} 2 p! q!}{1!(p-1)!(q-1)!} \ \delta_{[b_1}^{[a_1} \ \gamma_{b_2 b_3 \ldots b_p]}^{a_2 a_3 \ldots a_q]} \ - \frac{(-1)^{p-1} 2 p! q!}{3!(p-3)!(q-3)!} \ \delta_{[b_1 \ldots b_3}^{[a_1 \ldots a_3} \ \gamma_{b_4 \ldots b_p]}^{a_4 \ldots a_q]} \ + \ ...... \tag{3.3b}$$

The anti-commutators of the gammas can also be found in [20], and one has the reciprocal situation as eqs-(3.3), one has instead that for $pq = even$

$$\{\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\} \ = \ 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} \ - \ ...... \tag{3.4a}$$

for $pq = odd$ one has

$$\{\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\} \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} \ + \ ...... \tag{3.4b}$$

Therefore, one of the most salient features of this work is that the octic E_8-invariant actions can be embedded into a more general octic $Cl(16)$-invariant action involving a large number of terms. The octic $Cl(16)$-invariant action in $16D$ is of the form

$$S = \int \ d^{16}x \ < \ F^{\mathcal{A}_1}_{\mu_1 \nu_1} \ F^{\mathcal{A}_2}_{\mu_2 \nu_2} \ .... \ F^{\mathcal{A}_8}_{\mu_8 \nu_8} \ \Gamma_{\mathcal{A}_1} \Gamma_{\mathcal{A}_2} \ ...... \ \Gamma_{\mathcal{A}_8} \ > \ \epsilon^{\mu_1 \nu_1 \mu_2 \nu_2 .....\mu_8 \nu_8}. \tag{3.5}$$

where $< .... >$ denotes the scalar part of the Clifford geometric product associated with the products of the $Cl(16)$ algebra generators. For instance

$$< \ \Gamma_a \ \Gamma_b \ > \ = \delta_{ab}, \quad < \ \Gamma_{a_1 a_2} \ \Gamma_{b_1 b_2} \ > \ = \delta_{a_1 b_1} \ \delta_{a_2 b_2} - \delta_{a_1 b_2} \ \delta_{a_2 b_1}$$

$$< \ \Gamma_{a_1} \ \Gamma_{a_2} \ \Gamma_{a_3} \ > \ = \ 0, \quad < \ \Gamma_{a_1 a_2 a_3} \ \Gamma_{b_1 b_2 b_3} \ > \ = \ \delta_{a_1 b_1} \ \delta_{a_2 b_2} \ \delta_{a_3 b_3} \ \pm \ ......$$

$$< \ \Gamma_{a_1} \ \Gamma_{a_2} \ \Gamma_{a_3} \ \Gamma_{a_4} \ > \ = \ \delta_{a_1 a_2} \ \delta_{a_3 a_4} - \delta_{a_1 a_3} \ \delta_{a_2 a_4} + \delta_{a_2 a_3} \ \delta_{a_1 a_4}, \ etc \ ...... \tag{3.6}$$

The integrand of the 16-dim action (3.5) is locally a total derivative and upon integration yields the Chern-Simons $Cl(16)$ gauge theory of gravity in $15D$ and which is an extension of the action for the Chern-Simons E_8 gauge theory of gravity in $15D$ described by eq-(1.1) [15].

A $Cl(16)$-invariant Yang-Mills action is

$$S_{YM}[Cl(16)] \ = \ \frac{1}{4g^2} \ \int \ d^{16}x \ \sqrt{g} \ < \ F^{\mathcal{A}}_{\mu\nu} \ F^{\mathcal{B}}_{\rho\tau} \ \Gamma_{\mathcal{A}} \Gamma_{\mathcal{B}} \ >_{scalar} \ g^{\mu\rho} g^{\nu\tau}. \tag{3.7}$$

where $< \Gamma_{\mathcal{A}} \Gamma_{\mathcal{B}} > \ = G_{\mathcal{A}\mathcal{B}} \ \mathbf{1}$ denotes the *scalar* part of the Clifford geometric product of the gammas Γ. There are a total of $2^{16} = 65536$ terms in

$$F^{\mathcal{A}}_{\mu\nu} \ F^{\mathcal{B}}_{\rho\tau} \ G_{\mathcal{A}\mathcal{B}} = F_{\mu\nu} F_{\rho\tau} \ + \ F^a_{\mu\nu} F^a_{\rho\tau} \ + \ F^{a_1 a_2}_{\mu\nu} F^{a_1 a_2}_{\rho\tau} \ + ... +$$

$$F^{a_1 a_2 ....a_{16}}_{\mu\nu} \ F^{a_1 a_2 ......a_{16}}_{\rho\tau}. \tag{3.8}$$

where the indices run as $a = 1, 2, ...16$. The Clifford algebra $Cl(16) = Cl(8) \otimes Cl(8)$ has the graded structure (scalars, bivectors, trivectors,....., pseudoscalar) given by

$$1\ 16\ 120\ 560\ 1820\ 4368\ 8008\ 11440\ 12870$$

$$11440\ 8008\ 4368\ 1820\ 560\ 120\ 16\ 1. \tag{3.9}$$

consistent with the dimension of the $Cl(16)$ algebra $2^{16} = 256 \times 256 = 65536$. The anomaly-free group of the Heterotic string $E_8 \times E_8 \subset Cl(16) \otimes Cl(16) = Cl(32)$, and whose bivector generators can be identified with the $SO(32)$ algebra generators that is consistent with the fact that $SO(32)$ is the anomaly-free group of the open superstring.

Let us extend the above actions to the more general case involving the C-space (Clifford space) associated with the $Cl(16)$ and $Cl(8)$ algebras. In particular , we will focus on the latter where the $2^8 = 256$ components of the $Cl(8)$-space polyvector $\mathbf{X}$ can be expanded as

$$\mathbf{X} \ = \ \sigma\ \mathbf{1} \ + \ x_\mu \Gamma^\mu \ + \ x_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} \ + \ .... \ x_{\mu_1\mu_2\mu_3.......\mu_8}\Gamma^{\mu_1\mu_2....\mu_8}. \tag{3.10}$$

In order to match dimensions in the expansion (3.10) one requires to introduce powers of a length scale [22] which we could set equal to the Planck scale and set it to unity. In Clifford Phase Spaces [30] one needs two length scales parameters, a lower and an upper scale.

The novel affine theories of gravity in the C-space associated with the $Cl(8)$ algebra involves gauging the semi-direct product of the $Cl(8)$ group with the polyvector-valued translation group $\mathcal{T}$ in $2^8 = 256$ dimensions. An extended theory of gravity in C-spaces was presented by [22] based on generalizations of the Poincare group ($SO(D-1,1) \times_s T_D$) to Clifford spaces of dimension 2^D corresponding to polyvector-valued rotations, boosts and translations. The ordinary affine group $GL(D,R) \times_s T_D$ is a further extension of the Poincare group by including the shear transformations in additional to rotations. Therefore, the affine theories of extended gravity in C-spaces proposed here is a further generalization of the C-space gravity results in [22].

The $Cl(8)$ polyvector-valued gauge connection can be decomposed into symmetric and antisymmetric pieces as

$$\Omega_M^{(AB)}\ \{\Gamma_A, \Gamma_B\} + \Omega_M^{[AB]}\ [\Gamma_A, \Gamma_B] \ = \ (\ \Omega_M^{(AB)}\ d_{AB}^C + \Omega_M^{[AB]}\ f_{AB}^C\ )\ \Gamma_C \ \equiv \ \Omega_M^C\ \Gamma_C. \tag{3.11}$$

where the structure constants f_{AB}^C and d_{AB}^C are given by eqs-(3.3, 3.4). Adding the polyvector-valued translations $\mathcal{P}_A$ allows us to construct the affine connection in the $Cl(8)$-space as follows

$$\mathcal{A}_M \ = \ \Omega_M^A\ \Gamma_A \ + \ \mathcal{E}_M^A\ \mathcal{P}_A. \tag{3.12}$$

where Γ_A are the $Cl(8)$ generators corresponding to the generalized Lorentz and shear transformations in C-space, and P_A are the polyvector-valued translation generators. The polyvector-valued gauge connection Ω_M^A has 256×256 components, since the base space index M and tangent space index A span $2^8 = 256$ degrees of freedom. The C-space vielbein $\mathcal{E}_M^A$ which gauges the polyvector-valued translations has also 256×256 components. For instance we can see why now there are square *and* rectangular matrices of the form

$$\mathcal{E}_M^A \;=\; E_\mu^a,\; E_\mu^{a_1 a_2},\; ....,E_\mu^{a_1 a_2 ....a_8},\; E^a_{\mu_1\mu_2},\; ....,\; E^a_{\mu_1\mu_2........\mu_8},$$

$$E^{a_1 a_2}_{\mu_1\mu_2},\; E^{a_1 a_2}_{\mu_1\mu_2\mu_3},\; ......, E^{a_1 a_2}_{\mu_1\mu_2........\mu_8},\; ...... \tag{3.13}$$

we must also include the (pseudo)scalar-(pseudo) scalar components $E,\ E^{a_1 a_2 .....a_8}_{\mu_1\mu_2.....\mu_8}$ as well.

The generalized curvature and torsion two-forms in C-space associated with the $Cl(8)$ gauge connection and vielbein one-forms

$$\mathbf{\Omega}^A \;\equiv\; \Omega_M^A\; dX^M, \quad \mathbf{E}^A \;\equiv\; \mathcal{E}_M^A\; dX^M. \tag{3.14}$$

are

$$\mathcal{R}^A_{MN}\; dX^M \wedge dX^N \;= \mathbf{R}^A \;=\; \mathbf{d}\,\mathbf{\Omega}^A \;+\; f^A_{BC}\; \mathbf{\Omega}^B \;\wedge\; \mathbf{\Omega}^C. \tag{3.15}$$

$$\mathcal{T}^A_{MN}\; dX^M \;\wedge\; dX^N \;=\; \mathbf{T}^A \;=\; \mathbf{d}\,\mathbf{E}^A \;+\; g^A_{BC}\; \mathbf{\Omega}^B \;\wedge\; \mathbf{E}^C. \tag{3.16}$$

To illustrate why $\mathcal{R}^A_{MN}$ is a true generalized curvature in C-space despite the fact that it has 3 polyvector-valued indices it suffices to select A among the 28 *bivector* components $[a_1 a_2]$ of the tangent $Cl(8)$-space and M, N to be the vectorial components of the base $Cl(8)$-space manifold. Hence, $R^{[a_1 a_2]}_{\mu\nu}$ has the correct number of indices corresponding to the $SO(7,1)$-valued curvature two-form $R^{[a_1 a_2]}_{\mu\nu} dx^\mu \wedge dx^\nu$. Care must be taken when working with $Cl(8)$ or $Cl(7,1), Cl(1,7)$ algebras since they are not isomorphic.

Notice that the expressions in eqs-(3.15, 3.16) are just the polyvector valued extensions of the usual Poincare algebra involving the commutators $[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\sigma}]$ and $[\mathcal{M}_{\mu\nu}, P_\rho]$, when the Lorentz algebra generators are realized in terms of Clifford bivectors as $\mathcal{M}_{\mu\nu} \sim [\gamma_\mu, \gamma_\nu] = 2\gamma_{\mu\nu}$. In Clifford affine spaces associated with $Cl(8) \times_s \mathcal{T}$ the commutators involving the *polyvector* generators are

$$[\Gamma_A, \Gamma_B] \;=\; f^C_{AB}\; \Gamma_C, \quad [\Gamma_A, \mathcal{P}_B] \;=\; g^C_{AB}\; \mathcal{P}_C, \quad [\mathcal{P}_A, \mathcal{P}_B] \;=\; 0. \tag{3.17}$$

that permits us to evaluate each one of the (very large number of) components $\mathcal{R}^A_{MN}, \mathcal{T}^A_{MN}$ in eqs-(3.15, 3.16). Further contractions of the curvature and torsion with the inverse vielbeins give

$$\mathcal{R}_M \;=\; \mathcal{R}^A_{MN}\, \mathcal{E}^N_A;\;\; \mathcal{T}_M \;=\; \mathcal{T}^A_{MN}\, \mathcal{E}^N_A;\;\; \mathcal{R}_{MNP} \;=\; \mathcal{R}^A_{MN}\, \mathcal{E}_{AP}, \;\; \mathcal{T}_{MNP} \;=\; \mathcal{T}^A_{MN}\, \mathcal{E}_{AP}, \tag{3.18}$$

where the C-space metric is defined in terms of the vielbein and the 256-dim tangent space metric η_{AB} as

$$G_{MN} \equiv \mathcal{E}^A_M \, \mathcal{E}^B_N \, \eta_{AB}; \quad \eta_{AB} = G_{MN} \, \mathcal{E}^M_A \, \mathcal{E}^N_B; \quad \mathcal{E}_{BN} = \eta_{AB} \, \mathcal{E}^A_N. \tag{3.19a}$$

Two important remarks are in order. Firstly, note that there is in general an *scalar* component in $\mathcal{R}_M$ when the polyvector-valued index M corresponds to the scalar element of the Clifford algebra as described by the first term in the expansion of the polyvector $\mathbf{X}$ in eq-(3.10). Secondly, instead of working with the expressions for the curvature $\mathcal{R}^A_{MN}$ given in terms of the Clifford connection Ω^C_M by eqs-(3.15-3.18), one could work alternatively with the expressions $\mathcal{R}^{(AB)}_{MN}$, $\mathcal{R}^{[AB]}_{MN}$ given in terms of $\Omega^{(AB)}_M$, $\Omega^{[AB]}_M$ which follow explicitly from eq-(3.11). In the latter case one can construct a C-space Ricci curvature and Ricci scalar using the standard contractions

$$\mathcal{R}_{MP} = \mathcal{R}^{AB}_{MN} \, \mathcal{E}^N_B \, \mathcal{E}_{AP}; \quad \mathcal{R} = \mathcal{R}^{AB}_{MN} \, \mathcal{E}^N_B \, \mathcal{E}^M_A. \tag{3.19b}$$

We prefer at the moment to work with eqs-(3.18) where the tangent space polyvector-valued indices are $A, B, C, ...$ and span a $2^8 = 256$ dim space. The base space polyvector-valued indices are $M, N, P, Q, ...$ and also span a $2^8 = 256$ dim space. The inverse vielbein $\mathcal{E}^M_A$ is defined as $\mathcal{E}^M_A \mathcal{E}^B_M = \delta^B_A$ and $\mathcal{E}^M_A \mathcal{E}^A_N = \delta^M_N$.

In the traditional description of C-spaces [22] there is one component of the C-space metric $G^{MN} = G^{scalar\ scalar} = \Phi$ corresponding the scalar element of the Clifford algebra that must be included as well. Such scalar component is a dilaton-like Jordan-Brans-Dicke scalar field. In [31] we were able to show how Weyl-geometry solves the riddle of the cosmological constant within the context of a Robertson-Friedmann-Lemaitre-Walker cosmology by coupling the Weyl scalar curvature to the Jordan-Brans-Dicke scalar ϕ field with a self-interacting potential $V(\phi)$ and kinetic terms $(\mathcal{D}_\mu \phi)(\mathcal{D}^\mu \phi)$. Upon eliminating the Weyl gauge field of dilations A_μ from its algebraic (non-propagating) equations of motion, and fixing the Weyl gauge scalings, by setting the scalar field to a constant ϕ_o such that $\phi_o^2 = 1/16\pi G_N$, where G_N is the present day observed Newtonian constant, we were able to prove that $V(\phi_o) = 3H_o^2/8\pi G$ and which was *precisely* equal to the observed vacuum energy density of the order of $10^{-122} \ M^4_{Planck}$. H_o is the present value of the Hubble scale. One must also include the pseudo-scalar elements of G^{MN} as well when both indices M, N are $[\mu_1 \mu_2 .....\mu_8]$ corresponding to the top grade part of the $Cl(8)$ polyvector. This component of the C-space metric corresponds to another scalar field.

The affine theory of extended gravity in $Cl(8)$-space admits an action

$$S = \frac{1}{2\kappa^2} \int_{\mathcal{M}^{256}} [d^{(256)}\mathbf{X}] \sqrt{|det \ G^{MN}|} \ \mathcal{L}. \tag{3.20}$$

whose Lagrangian density is

$$[\ a_1 \, \mathcal{R}_M \, \mathcal{R}^M + a_2 \, \mathcal{T}_M \, \mathcal{T}^M + a_3 \, \mathcal{R}_{MNP} \, \mathcal{R}^{MNP} + a_4 \, \mathcal{T}_{MNP} \, \mathcal{T}^{MNP}]. \tag{3.21}$$

where the 256-dim measure of integration is defined by

$$[d^{(256)}\mathbf{X}] \;=\; d\sigma \;\prod dx_\mu \;\prod dx_{\mu_1\mu_2} \;\prod dx_{\mu_1\mu_2\mu_3} .... \; dx_{\mu_1\mu_2\mu_3.......\mu_8}. \quad (3.22)$$

in terms of the 256 components of the polyvector $\mathbf{X}$

A generalized Einstein-Hilbert gravity action based on gauging the generalized Poincare group in C-spaces was given by [22] where in very special cases the C-space scalar curvature $\mathcal{R}$ admits an expansion in terms of sums of powers of the ordinary scalar curvature R , Riemann curvature $R_{\mu\nu\rho\sigma}$ and Ricci $R_{\mu\nu}$ tensor of the underlying Riemannian spacetime manifold. The exterior products of the (Clifford-algebra-valued) spin-connection and vielbein one-forms in Clifford-spaces can also be constructed in Clifford-Superspaces by including both orthogonal *and* symplectic Clifford algebras and generalizing the Clifford super-differential exterior calculus in ordinary superspace [33], to the full fledged Clifford-Superspace outlined in [16]. Clifford-Superspace is far richer than ordinary superspace and Clifford-Supergravity involving polyvector-valued extensions of Poincare and (Anti) de Sitter supergravity [27] is far richer than ordinary supergravity. Fermionic matter and scalar-field actions can be constructed in C-spaces in terms of Dirac-Barut-Hestenes spinors as in [22], [32].

To finalize we write down the most general extension of the $Cl(2n)$ Chern-Simons gravitational action in $D = 2n$ in the case when one replaces the ordinary $D = 2n$-dim space with a $Cl(2n)$-space of dimensions $2^D = 2^{2n}$ associated with polyvector-valued $\mathbf{X}$ coordinates instead of ordinary vectors x^μ. The action is of the form

$$I \;=\; \int_{\mathcal{M}^{2^{2n}}} \sum \;<\; F \wedge F \wedge F......... \wedge F \;>. \quad (3.23)$$

where the summands are of the form

$$<\; F^{I_1I_2}_{\mu_1\mu_2}\; F^{I_3I_4}_{\mu_3\mu_4} \;.....\; F^{I_{2^{2n}-1}I_{2^{2n}}}_{\mu_{2^{2n}-1}\mu_{2^{2n}}}\; \Gamma_{I_1I_2}\;\Gamma_{I_3I_4}\;......\;>\; \epsilon^{\mu_1\mu_2..........\mu_{2^{2n}-1}\mu_{2^{2n}}}. \quad (3.25)$$

$$<\; F^{I_1I_2I_3I_4}_{\mu_1\mu_2...\mu_4}\;........\; F^{I_{2^{2n}-3}......I_{2^{2n}}}_{\mu_{2^{2n}-3}........\mu_{2^{2n}}}\;\Gamma_{I_1I_2I_3I_4}\;\Gamma_{I_5I_6I_7I_8}..........\;>\;\epsilon^{\mu_1\mu_2..........\mu_{2^{2n}}}. \quad (3.26)$$

etc .... and where the brackets $< ...... >$ denotes taking the scalar parts of the Clifford geometric product of the gamma factors inside the bracket. The properties of the most general action (3.23) warrants further investigation.

Acknowledgements

We thank Frank (Tony) Smith and Matej Pavsic for discussions and M. Bowers for assistance.

References

[1] E. Cremmer, B. Julia and J. Scherk, *Phys. Letts* **B 76** (1978) 409;

[2] S. de Buyl, M. Henneaux and L. Paulot, Extended E_8 invariance of 11-dim Supergravity, hep-th/0512292; P. West, *Class. Quan. Grav* **18** (2001) 4443; V. A. Belinksy, I. M Khalatnikov and E. M. Lifshitz, *Adv. Phys.* **19** (1970) 525.

[3] I. R. Porteous, *Clifford algebras and Classical Groups* (Cambridge Univ. Press, 1995).

[4] K. Itoh, T. Kugo and H. Kunimoto, Progress of Theoretical Physics **75**, no. 2 (1986) 386.

[5] M. Cederwall and J. Palmkvist, "The octic E_8 invariant", hep-th/0702024.

[6] S. Adler, "Further thoughts on Supersymmetric E_8 as family and grand unification theory", hep-ph/0401212.

[7] I. Bars and M. Gunaydin, *Phys. Rev. Lett* **45** (1980) 859; N. Baaklini, *Phys. Lett* **91 B** (1980) 376; S. Konshtein and E. Fradkin, *Pis'ma Zh. Eksp. Teor. Fiz* **42** (1980) 575; M. Koca, *Phys. Lett* **107 B** (1981) 73; R. Slansky, *Phys. Reports* **79** (1981) 1.

[8] C. H Tze and F. Gursey, *On the role of Divison, Jordan and Related Algebras in Particle Physics* (World Scientific, Singapore 1996); S. Okubo, *Introduction to Octonion and other Nonassociative Algebras in Physics* (Cambridge Univ. Press, 2005); R. Schafer, *An introduction to Nonassociative Algebras* (Academic Press, New York 1966); G. Dixon, *Division Algebras, Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics* (Kluwer, Dordrecht, 1994); G. Dixon, *J. Math. Phys* **45** no 10 (2004) 3678; P. Ramond, Exceptional Groups and Physics, hep-th/0301050; J. Baez, *Bull. Amer. Math. Soc* **39** no. 2 (2002) 145.

[9] P. Jordan, J von Neumann and E. Wigner, *Ann. Math* **35** (1934) 2964; K. MacCrimmon, *A Taste of Jordan Algebras* (Springer Verlag, New York 2003); H. Freudenthal, *Nederl. Akad. Wetensch. Proc. Ser* **57 A** (1954) 218; J. Tits, *Nederl. Akad. Wetensch. Proc. Ser* **65 A** (1962) 530; T. Springer, *Nederl. Akad. Wetensch. Proc. Ser* **65 A** (1962) 259.

[10] S. Low: Jour. Phys **A** Math. Gen **35**, 5711 (2002). J. Math. Phys. **38**, 2197 (1997). S. Low, "Hamilton Relativity Group for Noninertial States In Quantum Mechanics" [arXiv : math-ph/0710.3599]. S. Low, "Reciprocally Relativity of Noninertial Frames : Quantum Mechanics" [arXiv : math-ph/0606015]. J. Govaerts, P. Jarvis, S. Morgan and S. Low, " Worldline Quantization of a Reciprocally Invariant System" arXiv : 0706.3736. P. Jarvis and S. Morgan, "Constraint Quantization of a worldline system invariant under reciprocal Relativity, II " arXiv.org : 0806.4794. P. Jarvis

and S. Morgan, "Born reciprocity and the granularity of spacetime" [arXiv : math-ph/0508041]. R. Delbourgo and D. Lashmar, "Born reciprocity and the $1/r$ potential" hep-th/0709.0776.

[11] F.D. Smith Jr, "The Physics of E_8 and $Cl(16) = Cl(8) \otimes Cl(8)$ " www.tony5m17h.net/E8physicsbook.pdf (Carterville, Georgia, June 2008, 367 pages).

[12] K. Koepsell, H. Nicolai and H. Samtleben, " An Exceptional Geometry of $d = 11$ Supergravity" hep-th/0006034.

[13] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, Nucl. Phys. **B523** (1998) 73.

[14] A. Malcev. Izv. Akad. Nauk SSR, Ser. Mat. **9** (1945) 291.

[15] C. Castro, " A Chern-Simons E_8 Gauge theory of Gravity in $D = 15$, Grand-Unification and Generalized Gravity in Clifford Spaces" IJGMMP **4** , No. 8 (2007) 1239;

[16] C. Castro, "The Exceptional E_8 Geometry of Clifford (16) Superspace and Conformal Gravity Yang-Mills Grand Unification" to appear in the IJGMMP vol **6** , no. 3, May 2009.

[17] C. Castro, Int. J. Mod. Phys **A 23**, (2008) 1487.

[18] C. Castro, "On Born's Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity" Phys Letts **B 668** (2008) 442.

[19] N. Batakis, Class and Quantum Gravity **3** (1986) L 99.

[20] K. Becker, M. Becker and J. Schwarz, *String Theory and M Theory* (Cambride University Press, 2007, pages 543-545).

[21] B. Green, J. Schwarz and E. Witten, *Superstring Theory* vol I, Appendix 6A (Cambridge University Press 1987).

[22] C. Castro and M. Pavsic, Progress in Physics **vol 1** , (2005) 31. *Phys. Letts* **B 559** (2003) 74; *Int. J. Theor. Phys* **42** (2003) 1693.

[23] F. Hehl, J. McCrea, E. Mielke and Y. Ne'eman, Phys. Reports **258** (1995) 1-171.

[24] J. Moffat, J. Math. Phys **36**, no. 10 (1995) 5897. J. Moffat and D. Boal, Phys. Rev **D 11**, 1375 (1975). J. Moffat, "Noncommutative Quantum Gravity" arXiv : hep-th/0007181. N. Poplawski, " On Nonsymmetric purely affine Gravity" gr-qc/0610132, T. Damour, S. Deser and J. McCarthy, Phys. Rev **D 47**, 1541 (1993). T. Jansen and T. Prokopec, " Problems and Hopes in Nonsymmetric Gravity" arXiv : gr-qc/0611005. C. Mantz and T. Prokopec, " Resolving Curvature Singularities in Holomorphic Gravity" arXiv : 0805.4721.

[25] A. Chamseddine, Comm. Math. Phys **218**, 283 (2001). " Gravity in Complex Hermitian Spacetime" arXiv : hep-th/0610099.

[26] A. Einstein, Ann. Math **46**, 578 (1945). A. Einstein and E. Strauss, Ann. Math **47**, 731 (1946).

[27] J. Cornwell, *Group Theory in Physics, vol. 3*, Appendix L, pp. 475-502, (Academic Press, London, 1989).

[28] J. A. Nieto "Towards a Background Independent Quantum Gravity in Eight Dimensions" arXiv : 0704.2769 (hep-th). J. A. Nieto, Class. Quant. Grav. 22, 947 (2005). "Towards an Ashtekar formalism in eight dimensions" arXiv; hep-th/0410260. J. A. Nieto and J. Socorro, "Is nonsymmetric gravity related to string theory?", arXiv: hep-th/9610160.

[29] M. Pavsic, "A Novel view on the Physical Origin of E_8" arXiv : 0806.4365 (hep-th). M.Pavsic, *The Landscape of Theoretical Physics: A Global View, From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle* (Kluwer Academic Publishers, Dordrecht-Boston-London, 2001).

[30] C. Castro, *Foundations of Physics* **35** , no. 6 (2005) 971; *Progress in Physics* **1** (2005) 20.

[31] C. Castro, Foundations of Physics **37**, no. 3 (2007) 366; Mod. Phys. Lett **A 21** , no. 35 (2006) 2685-2701. "Does Weyl Geometry solves the Riddle of Dark Energy " *Quantization Astrophysics, Brownian Motion and Supersymmetry* pp. 88-96 (eds F. Smarandache and V. Christianato, Math. Tiger, Chennai, India 2007).

[32] C. Castro, On Generalized Yang-Mills and Extensions of the Standard Model in Clifford (Tensorial) Spaces, *Annals of Physics* **321** no.4 (2006) 813; *Mod. Phys Letters* **A 19** (2004) 14.

[33] H. De Bie and F. Sommen, " A Clifford analysis to Superspace" arXiv : 0707.2859; " Hermite and Gegenbauer polynomials in superspace using Clifford analysis" arXiv : 0707. 2863.

[34] C.Y Lee, Class. Quan Grav **9** (1992) 2001. C. Y. Lee and Y. Ne'eman, Phys. Letts **B 242** (1990) 59.