p-Branes as Antisymmetric Nonabelian Tensorial Gauge Field Theories of Diffeomorphisms in $p + 1$ dimensions

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August, 2009

Abstract

Long ago, Bergshoeff, Sezgin, Tanni and Townsend have shown that the light-cone gauge-fixed action of a super p-brane belongs to a *new* kind of supersymmetric gauge theory of p-volume preserving diffeomorphisms (diffs) associated with the p-spatial dimensions of the extended object. These authors conjectured that this new kind of supersymmetric gauge theory must be related to an infinite-dim *nonabelian* antisymmetric gauge theory. It is shown in this work how this new theory should be part of an underlying antisymmetric nonabelian tensorial gauge field theory of $p + 1$-dimensional diffs (upon supersymmetrization) associated with the world volume evolution of the p-brane. We conclude by embedding the latter theory into a more fundamental one based on the Clifford-space geometry of the p-brane configuration space.

Keywords: Antisymmetric Nonabelian Tensor Gauge Theories, Diffeomorphisms, Clifford spaces, strings, branes, Yang-Mills, large N limit, n-ary algebras.

1 Introduction

We begin with an introduction reviewing earlier findings that are indispensable in order to proceed with the main results of this work in section **2**. Section **3** contains a brief introductory review before presenting other novel results pertaining to gauged nonlinear sigma models and Matrix Models based on generalized matrices (hyper-matrices with multi-indices). Finally, we conclude with a thorough discussion of how to extend our findings to the study of generalized branes in Clifford spaces.

1.1 Strings and Branes from the large N limit of $SU(N)$ Yang-Mills, Wilson Loops and Confinement

A novel approach to evaluate the Wilson loops asociated with a $SU(\infty)$ gauge theory in terms of pure string degrees of freedom was presented in [5]. It was based on the Guendelman-Nissimov-Pacheva [10] formulation of composite antisymmetric tensor field theories of area (volume) preserving diffeomorphisms which admit p-brane solutions and which provide a *new* route to scale symmetry breaking and confinement in Yang-Mills theory. The quantum effects were discussed and we evaluated the vacuum expectation values (vev) of the Wilson loops in the large N limit ofthe *quenched* reduced $SU(N)$ Yang-Mills theory in terms of a path integral involving pure string degrees of freedom. The *quenched* approximation is necessary to avoid a crumpling of the string world-sheet giving rise to very large Hausdorff dimensions as pointed out by Olesen [22] The approach was also consistent with the recent results based on the AdS/CFT correspondence [6] and dual QCD models (dual Higgs model with dual Dirac strings) [7].

It has been believed for a long time that QCD confinement is supposed to be a non-perturbative solution to QCD in four dimensions ; i.e to $SU(3)$ Yang-Mills theory [1] . A formal proof of the colour confinement amounts to a derivation of the area-law for a Wilson loop associated, for example, with the world lines of a quark-antiquark pair joined in by a string . The area law in the Euclidean regime is $W(C) \sim exp[-TA]$ where T is the string tension and the (Euclidean) area is $A = lt_E$. The colour-electric potential rises linearly with the length of the string separating the quark-antiquark and blows up in the $l \rightarrow \infty$. This would be a signal of (colour-electric field lines) confinement, an infinite amount energy would be required to separate the quarks.

Many attempts have been explored to solve this problem, in particular those based on the so-called string ansatz [2], [3]

$$W[C] \sim \int_{\Sigma(C)} [DX]\ exp\ (iS_{string}). \tag{1.1}$$

which says that the effective (collective) infrared degrees of QCD at strong coupling are given by string configurations whose worldsheets have for boundary the loop C . The Schwinger-Dyson equations for QCD can be reformulated as an infinite chain of equations for the Wilson loops that simplify *drastically* in the large N limit giving the single equation known as the Makeenko-Migdal loop equation [4] .

In light of the Maldacena AdS/CFT correspondence formulated by many authors [6] as a relation between partition functions, Maldacena and others proposed that the average value of a Wilson loop in the large N limit, for $\mathcal{N} = 4$ $SU(N)$ SYM was given by the partition function of a world-sheet string action which ends along the loop C in the four-dim boundary. Another approach has been based on the dual formulation of QCD [7] (in the infrared limit) given by a $U(1)$ gauge theory adjoined by a dual Higgs model with dual Dirac strings

[8] (where the quarks live at their end-points) . The average value of the Wilson loop in this dual phase obeys the area-law fall-off.

For other approaches to solve the confinement problem based on Skyrmions and others methods see [9]. In [5] we presented a novel approach to evaluate the Wilson loops asociated with a gauge theory of area-preserving diffeomorphisms in terms of the (area) string degress of freedom. It was based on the Guendelman-Nissimov-Pacheva formulation of composite antisymmetric tensor field theories of area (volume) preserving diffeomorphims [10]. Such theories admit p-brane solutions after a dualization procedure [11] . Our results allowed us to recast the Wilson loop, in the large N limit, in terms of pure string degrees of freedom and to implement the new route to spontaneous scale-symmetry breaking and confinement in Yang-Mills theory proposed by [26], followed by the brane/wave duality principle [27] which permits to show how scale-symmetry breaking and confinement occurs in p-branes solutions of composite antisymmetric tensor field theories of area (volume) preserving diffs through the introduction of a preferred scale.

In [12] it was shown how the *quenched* large N limit of $SU(N)$ Yang-Mills theory admits strings, membranes and bag excitations. The *quenched* approximation was necessary to avoid a crumpling of the string world-sheet giving rise to very large Hausdorff dimensions as pointed out by Olesen [22] and the collapse (clustering) of eigenvalues [21]. The large N limit of $4D$ Yang-Mills theoryin the *quenched* approximation, that furnishes the Eguchi-Schild string action [12] was based on the following quenched action reduced to a "point" in $D = 4$

$$S = -\frac{1}{4}(\frac{2\pi}{a})^4 \frac{N}{g^2_{YM}} Tr\ (F_{\mu\nu}F^{\mu\nu}). \tag{1.2}$$

$$Tr\ F_{\mu\nu}F^{\mu\nu} = Tr\ [A_\mu(0), A_\nu(0)][A^\mu(0), A^\nu(0)]. \tag{1.3}$$

Notice that the reduced-quenched action is definedat a " point " $x_o = 0$. This is attained by neglecting the off-diagonal components of the matrices (" fast moving modes ") and absorbing the full space-time dependence of the gauge fields into a unitary translational operator given by a plane-wave diagonal $N \times N$ matrix $U(x) = exp[ip^\mu_a x_\mu]$. The N entries of the plane-wave elements along the diagonal are evaluated in terms of N distinct eigenvectors p^μ_a , $a = 1, 2, 3...N$. Notice that the matrix $U(x)$ is diagonal but is *not* proportional to the unit matrix in general. Thus, one can absorb the space-time dependence of the gauge fields as follows:

$$A_\mu(x) = U^\dagger(x) A_\mu(0) U(x) \Rightarrow F_{\mu\nu}(x) = U^\dagger(x) F_{\mu\nu}(0) U(x). \tag{1.4}$$

Due the cyclic property of the trace and the unitary condition $U^\dagger U = 1$, after simple algebra the trace of the field-strength squared reduces simply to $Tr\ [A_\mu(0), A_\nu(0)]^2$. Therefore, there are no $\partial_\mu A_\nu$ terms since one has reduced the theory to a "point", the origin $x_o = 0$. For simplicity we have omitted the matrix $SU(N)$ indices in eq-(1.4).

The Weyl-Wigner-Groenowold-Moyal (WWGM) quantization establishes a one-to-one correspondence between a linear operator $D_\mu = \partial_\mu + A_\mu$ acting on the Hilbert space $\mathcal{H}$ of square integrable functions in R^D and a smooth c-number function $\mathcal{A}_\mu(x,y)$ which is the Fourier transform of $\mathcal{A}_\mu(q,p)$. The latter quantity is obtained by evaluating the trace of the $D_\mu = \partial_\mu + A_\mu$ operator summing over the diagonal elements with respect to an orthonormal basis in the Hilbert space. Under the WWGM correspondence , in the quenched-reduced approximation, the matrix operator product $A_\mu.A_\nu$ is mapped into the *noncommutative* Moyal star product of their symbols $\mathcal{A}_\mu * \mathcal{A}_\nu$ and the commutators are mapped into their Moyal brackets:

$$[\ A_\mu,\ A_\nu]\ \Rightarrow\ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu\ \}_{MB}. \tag{1.5}$$

Replacing the Trace operation with an integration w.r.t the internal phase space variables $\sigma \equiv q,p$ gives

$$\frac{(2\pi)^4}{N^4}\ Trace\ \rightarrow\ \int\ d^2\sigma. \tag{1.6}$$

The WWGM deformation quantization of the quenched-reduced original actions is

$$S^* = -\frac{1}{4}\ (\frac{2\pi}{a})^4\ \frac{N}{g^2_{YM}}\ \int\ d^2\sigma\ \mathcal{F}_{\mu\nu}(\sigma)\ *\ \mathcal{F}^{\mu\nu}(\sigma). \tag{1.7a}$$

$$\mathcal{F}_{\mu\nu}\ =\ \{\ i\mathcal{A}_\mu,\ i\mathcal{A}_\nu\ \}. \tag{1.7b}$$

By performing the following gauge fields/coordinates correspondence

$$\mathcal{A}_\mu(\sigma)\ \rightarrow\ (\frac{2\pi}{N})^{1/4}\ X_\mu(\sigma) \tag{1.8a}$$

$$\mathcal{F}_{\mu\nu}(\sigma)\ \rightarrow\ (\frac{2\pi}{N})^{1/2}\ \{\ X_\mu(\sigma),\ X_\nu(\sigma)\ \}_{MB}. \tag{1.8b}$$

And, finally, by setting the Moyal deformation parameter "$\hbar$" $= 2\pi/N$ of the WWGM deformed action (1.7), to zero ; i.e by taking the classical $\hbar = 0$ limit , which is tantamount to taking the large $N = \infty$ limit, one can see that the quenched-reduced YM action in the large N limit will become

$$S\ =\ -\ \frac{1}{4g^2_{YM}}\ (\frac{2\pi}{a})^4\ \int\ d^2\sigma\ \{\ X_\mu,\ X_\nu\ \}_{PB}\ \{\ X^\mu,\ X^\nu\ \}_{PB}. \tag{1.9a}$$

due to the fact that the Moyal brackets collapse to the ordinary Poisson brackets in the $\hbar = 2\pi/N = 0$ limit (large N limit). Therefore, one has obtained the same functional form of the Eguchi-Schild string action (1.9), up to a different numerical factors than the string tension, and which is invariant under area-preserving reparametrizations from the quenched-reduced large N Yang-Mills theory via the WWGM quantization procedure.

The large N limit of $4D$ Yang-Mills theory in the *quenched* approximation, and supplemented by a topological *theta* term can be related through a Weyl-Wigner Groenowold Moyal (WWGM) quantization procedure also to a bag model; i.e. to an *open* domain (a bag) of the 3-dim disk D^3 [12] . The bulk $D^3 \times R^1$ is the interior of a hadronic bag and the (lateral) boundary is the world volume $S^2 \times R^1$ of a Chern-Simons-membrane of topology S^2 (a codimension two object). Hence, we have an example where the world-volume of a boundary $S^2 \times R^1$ is the boundary of theworld-volume of an open 3-brane of topology D^3 such $\partial(D^3 \times R^1) = S^2 \times R^1$ (setting asside the points at infinity). The boundary dynamics is *not* trivial despite the fact that there are no transverse bulk dynamics associated with the interior of the bag. This is due to the fact that the 3-brane is spacetime filling : $3+1=4$ and therefore has no transverse physical degrees of freedom.

To obtain the 3-brane (a bag) action with the proper value of the bag tension, from the WWGM quantization of Yang-Mills, one must *enlarge* the two-dim phase space to a four-dim one : q^1, p^1, q^2, p^2 and to repeat the same procedure as before. The trace becomes now an integration w.r.t the four q^i, p^i variables that have a correspondence to the four world-volume σ^a variables. The large N limit of quenched Yang-Mills yields the Dolan-Tchrakian action for a 3-brane (bag) in the *conformal gauge*, once the correspondence $A_\mu(\sigma^a) \rightarrow X^\mu(\sigma^a)$ is made, [12]

$$S = -\frac{1}{4g^2_{YM}} \ (\frac{2\pi}{a})^4 \ \int d^4\sigma \ \{X_\mu, X_\nu\}_{PB}\{X^\mu, X^\nu\}_{PB} \ =$$

$$-\frac{1}{4g^2_{YM}} \ (\frac{2\pi}{a})^4 \ \int d^4\sigma \ \omega^{a_1a_2} \ (\partial_{a_1} \ X_\mu \ \partial_{a_2} \ X_\nu) \ \omega^{b_1b_2} \ (\partial_{b_1} \ X^\mu \ \partial_{b_2} \ X^\nu) \quad (1.9b)$$

where $\omega^{a_1a_2}$ is an antisymmetric 4×4 matrix whose entries are ± 1. The bag constant μ of mass dimension was related to the bag (3-brane) tension T_{bag} of dimensions of $(mass)^4$ as[12]

$$T_{bag} = \mu^4 \sim \frac{1}{a^4 g^2_{YM}}. \quad (1.10)$$

where a was the lattice spacing of the large N quenched, reduced QCD given by $(2\pi/a) = \Lambda_{QCD} = 200$ Mev.

This Moyal deformation approach also furnishes dynamical membrane actions (in the light-cone gauge) as well [12] when one uses the spatial quenching approximation to a *line* (one dimension) , instead of quenching to a *point*. Basically, a Moyal quantization takes the operator $\hat{A}_\mu(x^\mu)$ into $A_\mu(x^\mu; q, p)$ and commutators into Moyal brackets. A dimensional reduction to one temporal dimension (quenching to a line) brings us to functions of the form $A_\mu(t, q, p)$, which precisely corresponds to the membrane coordinates $X_\mu(t, \sigma^1, \sigma^2)$ after identifying the σ^a variables with q, p. The $\hbar = 0$ limit turns the Moyal bracket into a Poisson one. Upon the identification of $\hbar = 2\pi/N$, the classical $\hbar = 0$ limit is tantamount to the $N = \infty$ limit and it is in this fashion how the large

N limit of the $SU(N)$ (noncovariant) matrix model bears a direct relation to the physics of membranes (in the light cone gauge).

To obtain superstrings and supersymmetric branes from the large N limit of $SU(N)$ supersymmetric Yang-Mills theory via deformation quantization requires the supersymmetric extension of the Moyal brackets [25] which is a non-trivial problem. These results were extended to more general p-brane actions (when $p = 4k$) given by Dolan-Tchrakian (Skyrme type actions) starting from Generalized Yang Mills theories in the large N limit ; i.e. branes are roughly speaking Moyal deformations of Generalized Yang-Mills theories [12]. It waslater shown how Nambu-Goto strings can be also be obtained directly from $SU(N)$ Born-Infeld models in the large N limit [15] . The average Wilson loop

$$< W_A[C] >_{vev} \quad = \quad \int [DA] \; W_A(C) \; e^{iS_{YM}[A]}. \tag{1.11}$$

is defined with

$$W_A[C] = \frac{1}{N} \; trace \; Path \; exp \; [i \oint_C A_\mu dx^\mu]. \tag{1.12}$$

In the quenched-reduced approxomation, defined at a "point", the Wilson loop $shrinks$ to zero size$C \rightarrow 0$ and hence the exponential reduces to *unity* since the integral has collapsed to zero and one gets

$$W[C] \rightarrow W[C = 0] = \frac{1}{N} trace \; 1_{N \times N} = \frac{N}{N} = 1. \tag{1.13}$$

Notice how important is the factor of $1/N$ in eq-(1.13) is in order to cancel the N factor stemming from taking the *trace* of the unit $N \times N$ matrix $1_{N \times N}$. As we have shown above in eqs- (1.2-1.9) the quenched-reduced YM action in the large N limit becomes the Eguchi-Schild action for the string after using the $A_\mu(\sigma) \rightarrow X_\mu(\sigma)$ correspondence via the WWGM quantization method. Thus, we have the following result

$$[DA]_{quenched} \rightarrow [DX] \quad\; W(C) \rightarrow W(0) = 1. \quad e^{iS_{YM}}_{quenched} \rightarrow e^{iS_{string}} \tag{1.14}$$

Under the conditions (1.14), the quenched-reduced $SU(N)$ Yang-Mills in the large N limit allows to compute the vacuum expecation values (vev) of the Wilson loop purely in terms of string degrees of freedom using the Eguchi-Schild action for the string (the square of the Poisson brackets) which is area-preserving diffs invariant; i.e. in the quenched-reduced large N Yang-Mills, the spacetime-independent sector (zero modes) of the gauge fields A_μ have a one-to-one correspondence to the string coordinates X_μ as explained earlier in the previous section , leading finally to:

$$< W[C] >_{quenched} \; = \int \; [DA]_{quenched} \, W(0) e^{iS_{YM}(A)} = \int \; [DA]_{quenched} \; e^{iS_{YM}(A)} =$$

$$\int_{\Sigma(C)} [DX]\ e^{iS_{string}} \equiv \Psi_o[C]. \tag{1.15}$$

The state $\Psi_o[C]$ is the vacuum wave functional representing the creation of a string (a loop) from from the vacuum (a point) and sweeping in the processs a world-sheet Σ whose *boundary* is the loop configuration C. The path integral involves a summation over all string embeddings X subject to the condition that the *boundary* of Σ is C. We have not included the bag-action in (1.14) in the evaluation of the Wilson-loop averages in large N limit since the bag action is devoid of bulk transverse degrees of freedom. The bag in 4-dim is spacetime filling giving a trivial action equal to the four-dim spacetime-volume of the bag. For this reason we concentrated solely on the string-degrees of freedom (areas) in (1.15). To sum up, this last expression (1.15) related the vev of the Wilson loop of the large N limit of $SU(N)$ Yang-Mills, in the quenched approximation, in terms of string degrees of freedom [5].

To our knowledge, Eguchi and Kawai [20] were the first to propose a reduction of the dynamical degrees of freedom of Yang-Mills by arguing that Yang-Mills theory on a D-dim spacetime is equivalent, in the large $N = \infty$ limit, to a *reduced* model based on the action

$$S_{EK} \ = \ \frac{1}{2g^2\Lambda^D}\ trace\ [\ A_\mu,\ A_\nu\ ]^2. \tag{1.16}$$

Namely, this model amounts to a dimensional-reduction of the Yang-Mills theory to a "point", where A_μ are a collection of D space-time independent matrices and Λ is a dimensional parameter related to the inverse lattice scale. However, it turned out that strictly speaking the Wilson-loop average

$$< \ \frac{1}{N}\ trace\ Path\ e^{i\oint dx^\mu A_\mu}\ > \ = \ \int\ [\mathcal{D}A_\mu]\ \frac{1}{N}\ trace\ Path\ e^{i\oint dx^\mu A_\mu}\ e^{iS[A]}. \tag{1.17}$$

using the full-fledged Yang-Mills action $S_{YM}[A]$ in the path-integral is *not* equal to the average using the reduced Eguchi-Kawai action, $S_{EK}[A]$, except in $D = 2$ [21], [23] . One reason is that the shift-symmetry invariance of the reduced Eguchi-Kawai action, symmetry were the matrices A_μ^{ij} are shifted by a diagonal matrix $a_\mu\delta^{ij}$, can be broken spontaneously in the large N limit [21]. Such large N limit plays the role of an statistical averaging and phase transitions may occur in perturbation theory of the reduced model in $D \geq 3$. It happens that the path integral measure for $D \geq 3$ is *singular* and the eigenvalues collapse (cluster) leading to a breakdown of the shift-symmetry in perturbation theory.

Related to the stringy picture of the large N Yang-Mills reduced models, another interpretation of the shift-symmetry breakdown phenomenon has been given by Olesen [22] by showing that the dynamics of the reduced Eguchi-Kawai action is not able to *suppress* the higher-modes thereby resulting in a very crumpled (fractal-like) world-sheet with a very high Hausdorff dimension. A smooth string world-sheet would require a dominace of the lower-modes in the path integral. For this to occur one could introduce a *quenching* procedure in the Eguchi-Kawai action to be able to suppress the higher-modes. It has been shown that the equivalence between the large N limit of Yang-Mills theory on a whole space and the reduced Eguchi-Kawai model is valid provided a *quenching* prescription is introduced [24] such that the D-dim planar graphs associated with the large N Yang-Mills theory are truly reproduced by the reduced, quenched Eguchi-Kawai model.

The physical meaning of this relation (1.15) can be envisaged as follows. As we shrink the Wilson loop to a point, the subsequent large N limit procedure amounts to introducing an extra dependence on the phase space variables (q, p) , that later are identified as the string coordinates. The $SU(N)$ fiber "sitting" at the point P becomes the area world-sheet of the string in the large N limit. Hence the Wilson loop which had initially shrunk to a point re-emerges as an internal loop living in the $SU(N)$ fiber that was sitting at the point P. This is compatible with the area-preserving diffs invariant nature of the Eguchi-Schild action. Roughly speaking, since areas are preserved, as we shrink the Wilson loop to a point (to zero) it must re-emerged along the fibers in order to preserve the area.

Hence we have obtained an exact result consistent with those given in the literature since (by definition) the vacuum wave-functional $\Psi_o[C]$, appearing in the r.h.s, is defined by a path integral over all world-sheets whose boundary is C. The latter is the quantum amplitude for a closed string to emerge from the vacuum (a " point ") and sweep a world-sheet whose boundary is C. The topology is given by a disc. A perturbative evaluation of the path integral requires summing over surfaces of all genera. For more general actions one must restrict the measure of integration modulo the volume of the world-sheet diffs group and the group of Weyl diffs for Polyakov-Howe-Tucker type of actions .

1.2 Branes as Composite Antisymmetric Tensor Field theories of Diffeomorphisms, Strings and Wilson Loops

The construction of p'-brane solutions to the rank $p+1$ composite antisymmetric tensor field theories developed by Guendelman, Nissimov and Pacheva [10] when the condition $D = p+p'+2$ is satisfied was provided in [11]. These field theories display an infinite-dimensional group of volume-preserving diffeomorphisms of the target space of the scalar primitive field constituents. The role of local gauge symmetry is traded over to an infinite-dimensional *global* Noether symmetry of volume-preserving diffs. The study of the Ward identities for this infinite-dim global Noether symmetry to obtain non-perturbative information in the mini-QED models (the composite form of QED) was analysed in [10] .

The starting Lagrangian is defined

$$L \;=\; -\; \frac{1}{g^2}\; F_{\mu_1\mu_2\ldots\mu_{p+1}}\; F^{\mu_1\mu_2\ldots\mu_{p+1}}$$

$$F \;=\; dA \;=\; \epsilon_{a_1 a_2 \ldots. a_{p+1}}\; \partial_{\mu_1}\phi^{a_1}\; \partial_{\mu_2}\phi^{a_2}\; \ldots\ldots\; \partial_{\mu_{p+1}}\; \phi^{a_{p+1}}. \tag{1.18}$$

the rank $p+1$ composite field strength is given in terms of $p+1$ scalar fields $\phi^1(x), \phi^2(x)....\phi^{p+1}(x)$. An Euler variation w.r.t the ϕ^a fields yields the following field equations, after pre-multiplying by a factor of $\partial_{\mu_{p+2}}\phi^{a_1}$ and using the Bianchi identity $dF = 0$

$$\partial_{\mu_1} \left(\frac{\delta L}{\delta(\partial_{\mu_1}\phi^{p+2})} \right) = 0 \Rightarrow F_{\mu_{p+2}\mu_2...\mu_{p+1}} \partial_{\mu_1} F^{\mu_1\mu_2...\mu_{p+1}} = 0. \quad (1.19)$$

Notice that despite the Abelian-looking form $F = dA$ the infinite-dimensional (global) symmetry of volume-preserving diffs is *not* Abelian. The theory we are describing is *not* the standard YM type . We are going to find now p'-brane solutions to eq-(1.18) , where $D = p + p' + 2$. These brane solutions obeyed the classical analogs of S and T-duality [11] . Ordinary EM duality for branes requires $D = p + p' + 4$. The latter condition is more closely related to the EM duality among two Chern-Simons p, p'-branes which are embeddings of a p, p'-dimensional object into $p + 2; p' + 2$ dimensions. These co-dimension two objects are nothing but high-dimensional Knots .

A special class of (non-Maxwellian) extended- solutions to eqs-(1.19) requires a *dualization* procedure [11]

$$G =^* F \Rightarrow G^{\nu_1\nu_2...\nu_{p'+1}}(\tilde{\phi}(x)) = \epsilon^{\mu_1\mu_2....\mu_{p+1}\nu_1\nu_2....\nu_{p'+1}} F_{\mu_1\mu_2....\mu_{p+1}}(\phi(x)). \quad (1.20)$$

After this dualization procedure the eqs-(1.19) are recast in the form:

$$G^{\mu_1\nu_2...\nu_{p'+1}} \partial_{\mu_1} G_{\nu_2\nu_3....\nu_{p'+2}}(\tilde{\phi}(x)) = 0. \quad (1.21)$$

The dualized equations (1.21) have a different form than eqs-(1.19) due to the position of the indices (the index contraction differs in both cases). Extended p'-brane solutions to eqs-(1.21) exist based on solutions to the Aurilia-Smailagic-Spallucci local gauge field theory reformulation of extended objects given in [13]. These solutions are

$$G^{\nu_1\nu_2...\nu_{p'+1}}(\tilde{\phi}(x))|_{x=X} =$$

$$T \frac{\{ X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p'+1}} \}}{\sqrt{-\frac{1}{(p'+1)!} [\{X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p'+1}}\}] [\{X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p'+1}}\}]}}. \quad (1.22)$$

where T is the p'-brane tension and the Nambu-Poisson bracket w.r.t the $p'+1$ world-volume variables is defined as the ordinary determinant /Jacobian

$$\{ X^{\nu_1}, X^{\nu_2}, X^{\nu_3}, ..., X^{\nu_{p'+1}} \}_{NPB} =$$

$$\epsilon^{\sigma^1\sigma^2\sigma^3....\sigma^{p'+1}} \partial_{\sigma^1} X^{\nu_1} \partial_{\sigma^2} X^{\nu_2} .... \partial_{\sigma^{p'+1}} X^{\nu_{p'+1}}. \quad (1.23)$$

All quantities are *evaluated* on the $p' + 1$-dim world-volume *support* of the p'-brane; i.e. one must *restrict* the dual-scalar solutions $\tilde{\phi}^a(x)$ to those points in the D-dimensional spacetime which have support on the brane given by $x =$

$X(\sigma^1, \sigma^2, ...)$. Solutions to *all* of the D-dim spacetime region can be extended simply by using delta functionals $\delta(x - X(\sigma))$.

String solutions ($p' = 1$) to the rank two ($p + 1 = 2$) composite antisymmetric tensor field theories of area-preserving diffs in $D = 4 = p + p' + 2 = 2 + 2$. The Wilson loop associated with the composite gauge field is defined:

$$exp \left[i \oint_C A_\mu(\phi^a) \ dx^\mu \right], \quad A_\mu(\phi) \equiv \epsilon_{ab} \ \phi^a(x) \ \partial_\mu \ \phi^b(x). \tag{1.24}$$

Due to the Abelian-loooking form of the composite field strength (as we said earlier, the algebra of volume-preserving diffs is *not* abelian) one *can* nevertheless use Stokes law

$$F \ = \ dA \ \Rightarrow \ F_{\mu\nu}(\phi) \ \equiv \ \{ \ \phi^1, \ \phi^2 \}_{PB} \ = \ \epsilon_{ab} \ \partial_\mu \ \phi^a \ \partial_\nu \phi^b, \quad a, b = 1, 2. \tag{1.25}$$

after using Stokes law the exponential can be written as

$$exp \left[i \int \int_{\Sigma(C)} \ F_{\mu\nu}(\phi^a) \ dx^\mu \wedge dx^\nu \right]. \tag{1.26}$$

where the flux is evaluated through a surface $\Sigma(C)$ whose boundary is C. If one evaluates all these quantities along the points x whose *support* lie on the string-world sheet $x = X$ one may use the string solutions above to the composite antisymmetric tensor field theory given by the previous equations

$$G(\tilde{\phi}) \ = \ \Pi \ = \ {}^*F(\phi) \Rightarrow$$

$$G^{\nu_1\nu_2}(\tilde{\phi})|_{x=X} = \Pi^{\nu_1\nu_2}(X) = \frac{T \ \{X^{\nu_1}, X^{\nu_2}\}}{\sqrt{-\frac{1}{2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}}} = \epsilon^{\nu_1\nu_2\mu_1\mu_2} F_{\mu_1\mu_2}(\phi)|_{x=X}. \tag{1.27}$$

where T is the string's tension and one is using now ordinary Poisson brackets.

The quantity $\Pi^{\mu\nu}$ is the area-conjugate momentum of the string obeying the Hamilton-Jacobi equation for the string analog of a point particle momentum. Hamilton-Jacobi equations for strings and branes have been given in [13] . Using these relations above allows one to rewrite the flux (after inserting the product of two spacetime epsilon tensors $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$) as

$$\frac{1}{4!} \ \epsilon^{\mu_1\mu_2\mu_3\mu_4} \ F_{\mu_1\mu_2}(\phi) \ \epsilon_{\mu_1\mu_2\mu_3\mu_4} \ dx^{\mu_1} \wedge dx^{\mu_2} \ = \ G^{\mu_3\mu_4}(\tilde{\phi}) \ d\tilde{\Sigma}_{\mu_3\mu_4}. \tag{1.28}$$

For those self dual string configurations , the following relations among the Poisson brackets are obeyed

$$Self \ Dual \ Strings \ \Rightarrow \ d\Sigma \ = \ {}^*d\Sigma \ \Rightarrow$$

$$\{ \ X_{\mu_3}, \ X_{\mu_4} \ \}_{PB} \ = \ \epsilon_{\mu_1\mu_2,\mu_3,\mu_4} \ \{ \ X^{\mu_1}, \ X^{\mu_2} \ \}_{PB}. \tag{1.29}$$

Self dual strings automatically obey the string equations of motion as a result of the Jacobi identities for the Poisson brackets:

$$\{ X^\nu, \{ X_\mu, X_\nu \} \} = \epsilon_{\mu\nu\rho\tau} \{ X^\nu, \{ X^\rho, X^\tau \} \} = 0. \quad (1.30)$$

The vanishing of the second term of the last equation is due to the Jacobi identities of the Poisson bracket.

Upon evaluation of the flux through the (self-dual) string world sheet , whose boundary is C , and restricting to self dual string configurations allows finally to yield the explicit relationship between the Wilson loop for the field $A_\mu(\phi)$ and the Dirac-Nambu-Goto string action, in terms of the string coordinates $X^\mu(\sigma,\tau)$, and whose worldsheet boundary is C

$$W(C) = exp \left[i \oint_C A_\mu(\phi) dx^\mu \right] |_{x=X} =$$

$$exp \left[iT \int \int_{\Sigma(C)} d\sigma d\tau \sqrt{ -\{X^\mu, X^\nu\} \{X_\mu, X_\nu\} } \right]. \quad (1.31)$$

since the determinant of the induced worldsheet metric as a result of the string's embedding onto the (flat) target spacetime is:

$$det\ (h_{ab}) = det\ (\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu) = \{ X^\mu, X^\nu \} \{ X_\mu, X_\nu \}. \quad (1.32)$$

Therefore, we have proven, $on-shell$, that the Wilson loop associated with the composite antisymmetric tensor field theory of area-preserving diffeomorphisms , after using Stokes law, equals the exponential of the self-dual string action (an area) whose worldsheet boundary is C.

2 Branes as Antisymmetric Nonabelian Tensorial Gauge Field Theories of Diffeomorphisms

2.1 Gravity as Gauge Theory of Diffs and Holography

Some time ago Park [28] showed that $4D$ Self Dual Gravity is equivalent to a WZNW model based on the group $SU(\infty)$. Namely, $4D$ Self Dual Gravity is the non-linear sigma model based in $2D$ whose target space is the "group manifold" of area-preserving diffs of another $2D$-dim manifold. Roughly speaking, this means that the effective $D = 4$ manifold, where Self Dual Gravity is defined, is "spliced" into two $2D$-submanifolds: one submanifold is the original $2D$ base manifold where the non-linear sigma model is defined. The other $2D$ submanifold is the target group manifold of area-preserving diffs of a two-dim sphere S^2.

The authors [29] went further and generalized this particular Self Dual Gravity case to the full fledged gravity in $D = 2 + 2 = 4$ dimensions, and in general, to *any* combinations of $m + n$-dimensions. Their main result is that $m + n$-dim Einstein gravity can be identified with an m-dimensional generally invariant gauge theory of $Diffs\ \mathcal{N}$, where $\mathcal{N}$ is an n-dim manifold. Locally the $m + n$-dim space can be written as $\Sigma = \mathcal{M} \times \mathcal{N}$ and the metric G_{AB} decomposes as:

$$G_{AB} = \begin{pmatrix} g_{\mu\nu}(x,y) \ + \ e^2 g_{ab}(x,y)\ A^a_\mu(x,y)\ A^b_\nu(x,y) & eA^a_\mu(x,y)\ g_{ab}(x,y) \\ eA^a_\mu(x,y)\ g_{ab}(x,y) & g_{ab}(x,y) \end{pmatrix}, \tag{2.1}$$

The connection $A^a_\mu(x,y)$ is the *nonlinear* connection of Lagrange-Finsler and Hamilton-Cartan spaces [30], [31]. The decomposition (2.1) must *not* be confused with the Kaluza-Klein reduction where one imposes an isometry restriction on the G_{AB} that turns A^a_μ into a gauge connection associated with the gauge group G generated by isometry. Dropping the isometry restrictions allows *all* the fields to depend on *all* the coordinates x, y. Nevertheless $A^a_\mu(x,y)$ can still be identified as a connection associated with the infinite-dim gauge group of $Diffs\ \mathcal{N}$. The gauge transformations are now given in terms of the Lie derivatives w.r.t the *internal* space indices y^a as follows

$$\delta A^a_\mu \ = \ -\frac{1}{e}\ D_\mu \xi^a \ = \ -\frac{1}{e}\ (\ \partial_\mu\ \xi^a \ - e\ [\ A_\mu,\ \xi]^a\ ). \tag{2.2a}$$

The Lie bracket is defined as

$$A_\mu \ \equiv \ A^a_\mu\ \partial_a,\ \xi \ \equiv \ \xi^a\ \partial_a \ \Rightarrow \ \mathcal{L}_{A_\mu}\ \xi \ = \ [\ A_\mu,\ \xi]^a \ = \ A^b_\mu\ \partial_b\ \xi^a \ - \ \xi^b\ \partial_b\ A^a_\mu. \tag{2.2b}$$

$$\delta g_{ab} \ = \ \mathcal{L}_\xi\ g_{ab} \ = \ [\xi,\ g]_{ab} \ = \ \xi^c\ \partial_c\ g_{ab} \ + \ g_{ac}\ \partial_b\ \xi^c \ + \ g_{cb}\ \partial_a\ \xi^c. \tag{2.2c}$$

$$\delta g_{\mu\nu} \ = \ \mathcal{L}_\xi\ g_{\mu\nu} \ = \ [\xi,\ g_{\mu\nu}] \ = \ \xi^a\ \partial_a\ g_{\mu\nu}. \tag{2.2d}$$

$$[\ A_\mu,\ A_\nu\ ]^a \ = \ A^c_\mu\ \partial_c\ A^a_\nu \ - \ A^c_\nu\ \partial_c\ A^a_\mu. \tag{2.2e}$$

In general, the Lie derivative $\mathcal{L}_X \mathbf{T}$ along the vector $X = X^a\ \partial_a$ of the mixed tensor $\mathbf{T}$ in the internal space is defined by [32]

$$\mathcal{L}_X\ T^{a_1 a_2 \ldots\ldots a_n}_{b_1 b_2 \ldots\ldots b_m} \ = \ (\ X^c\ \partial_c\ T^{a_1 a_2 \ldots\ldots a_n}_{b_1 b_2 \ldots\ldots b_m}\ ) \ + \ \sum_{i=1}^m\ (\partial_{b_i}\ X^c)\ T^{a_1 a_2 \ldots\ldots a_n}_{b_1 b_2 \ldots\ldots c \ldots\ldots b_m} \ - \\ \sum_{i=1}^n\ (\partial_c\ X^{a_i})\ T^{a_1 a_2 \ldots\ldots c \ldots\ldots a_n}_{b_1 b_2 \ldots\ldots\ldots\ldots b_m}. \tag{2.2f}$$

there is a key *minus* sign in the last term of (2.2f) relative to the first two terms.

Using eq-(2.1) the authors [29] have shown that the curvature scalar $R^{(m+n)}$ in $m+n$-dim decomposes into

$$R^{(m+n)} \ = \ g^{\mu\nu} \ R^{(m)}_{\mu\nu} \ + \ \frac{e^2}{4} \ g_{ab} \ F^a_{\mu\nu} \ F^b_{\rho\tau} \ g^{\mu\rho} \ g^{\nu\tau} \ + \ g^{ab} \ R^{(n)}_{ab} \ +$$

$$\frac{1}{4} g^{\mu\nu} \ g^{ab} \ g^{cd} \ [\ (D_\mu \ g_{ac}) \ (D_\nu \ g_{bd}) \ - \ (D_\mu \ g_{ab})(D_\nu \ g_{cd}) \ ] +$$

$$\frac{1}{4} \ g^{ab} \ g^{\mu\nu} \ g^{\rho\tau} \ [\ \partial_a \ g_{\mu\rho} \ \partial_b g_{\nu\tau} \ - \ \partial_a \ g_{\mu\nu} \ \partial_b \ g_{\rho\tau} \ ] \tag{2.3}$$

plus total derivative terms given by

$$\partial_\mu(\ \sqrt{|det \ g_{\mu\nu}|} \ \sqrt{|det \ g_{ab}|} \ J^\mu \ ) \ - \partial_a(\ \sqrt{|det \ g_{\mu\nu}|} \ \sqrt{|det \ g_{ab}|} \ eA^a_\mu J^\mu) \ +$$

$$\partial_a(\ \sqrt{|det \ g_{\mu\nu}|} \ \sqrt{|det \ g_{ab}|} \ \ J^a \ ), \tag{2.4}$$

with the currents:

$$J^\mu \ = \ g^{\mu\nu} \ g^{ab} \ D_\nu \ g_{ab}, \quad J^a \ = \ g^{ab} \ g^{\mu\nu} \ \partial_b \ g_{\mu\nu}. \tag{2.5}$$

$$S \ = \ \frac{1}{2\kappa^2} \int \ d^m x \ d^n y \ \sqrt{|det(g_{\mu\nu})|} \ \sqrt{|det(g_{ab})|} \ R^{(m+n)}(x,y). \tag{2.6}$$

Therefore, Einstein gravity in $m+n$-dim describes an m-dim generally invariant field theory under the gauge transformations or Diffs $\mathcal{N}$. Notice how A^a_μ couples to the graviton $g_{\mu\nu}$, meaning that the graviton is charged /gauged in this theory and also to the g_{ab} fields. The "metric" g_{ab} on $\mathcal{N}$ can be identified as a non-linear sigma field whose self interaction potential term is given by: $g^{ab}R^{(n)}_{ab}$. The currents J^μ, J^a are functions of $g_{\mu\nu}, A_\mu, g_{ab}$. Their contribution to the action is essential when there are boundaries involved; i.e. like in the AdS/CFT correspondence.

When the internal manifold $\mathcal{N}$ is a homogeneous compact space one can perform a harmonic expansion of the fields w.r.t the internal y coordinates, and after integrating w.r.t these y coordinates, one will generate an infinite-component field theory on the m-dimensional space. A reduction of the Diffs $\mathcal{N}$, via the inner automorphims of a subgroup G of the Diffs $\mathcal{N}$, yields the usual Einstein-Yang-Mills theory interacting with a nonlinear sigma field. But in general, the theory described in (2.3) is by far *richer* than the latter theory. A crucial fact of the decomposition in (2.3) is that *each* single term is by itself independently invariant under Diffs $\mathcal{N}$.

In the special case when $g_{\mu\nu}(x)$ depends solely on x and $g_{ab}(y)$ depends on y then the spacetime gauged "Ricci scalar" coincides with the ordinary Ricci scalar $g^{\mu\nu}(x) \ R^{(m)}_{\mu\nu}(x)$ and the internal space "Ricci scalar" becomes the true Ricci scalar of the internal space. However, the gauge field $A_\mu(x,y)$ still retains its full dependence on both variables x, y.

We have shown [33] that in this particular case the $D = m + n$ dimensional gravitational action restricted to $AdS_m \times S^n$ backgrounds admits a *holographic* reduction to a lower $d = m$-dimensional Yang-Mills-like gauge theory of diffs of S^n, interacting with a charged/gauged nonlinear sigma model plus boundary terms, by a simple tuning of the radius of S^n and the size of the throat of the AdS_m space. Namely, in the case of $AdS_5 \times S^5$, the holographic reduction occurs if, and only if, the size of the AdS_5 throat *coincides* precisely with the radius of S^5 ensuring a *cancellation* of the scalar curvatures $g^{\mu\nu} R^{(m)}_{\mu\nu}$ and $g^{ab} R^{(n)}_{ab}$ in eq-(2.3) [33] such that the scalar curvature (Einstein-Hilbert Lagrangian) in $D = 10$ becomes

$$R^{(10)} \;=\; \frac{e^2}{4}\; g_{ab}(y)\; F^a_{\mu\nu}(x,y)\; F^b_{\rho\tau}(x,y)\; g^{\mu\rho}(x)\; g^{\nu\tau}(x) \;+$$

$$\frac{1}{4}\; g^{\mu\nu}(x)\; g^{ab}(y)\; g^{cd}(y)\; [\; (D_\mu g_{ac})\; (D_\nu g_{bd}) - \; (D_\mu g_{ab})\; (D_\nu g_{cd})\;]. \qquad (2.7)$$

plus total derivative terms (boundary terms)

$$D_\mu\; g_{ab} \;=\; \partial_\mu g_{ab} \;+\; [\; A_\mu,\; g_{ab}\;].$$

where the Lie-bracket above is

$$[\; A_\mu\;,\; g_{ab}\;] \;=\; (\partial_a\; A^c_\mu(x^\mu, y^a))\; g_{cb}(x^\mu, y^a) \;+\; (\partial_b\; A^c_\mu(x^\mu, y^a))\; g_{ac}(x^\mu, y^a) \;+$$

$$A^c_\mu(x^\mu, y^a)\; \partial_c\; g_{ab}(x^\mu, y^a). \qquad (2.8)$$

and the Yang-Mills like field strength is

$$F^a_{\mu\nu} \;=\; \partial_\mu\; A^a_\nu \;-\partial_\nu\; A^a_\mu \;-\; [\; A_\mu,\; A_\nu\;]^a \;=$$

$$\partial_\mu\; A^a_\nu \;-\partial_\nu\; A^a_\mu \;-\; A^c_\mu\; \partial_c\; A^a_\nu \;+\; A^c_\nu\; \partial_c\; A^a_\mu. \qquad (2.9)$$

Eq-(2.7) is nothing but the *holographic* reduction of the $D = 10$-dim pure gravitational Einstein-Hilbert action to a 5-dim Yang-Mills-like action (of diffeomorphisms of the internal S^5 space) interacting with a charged nonlinear sigma model (involving the g_{ab} field) plus boundary terms. The previous argument can also be generalized to gravitational actions restricted to de Sitter spaces, like $dS_m \times H^n$ backgrounds as well, where H^n is an internal hyperbolic noncompact space of constant negative curvarture, and dS_m is a de Sitter space of positive constant scalar curvature.

2.2 Antisymmetric Nonabelian Tensorial Gauge Field Theories of $p+1$-dim Diffeomorphisms

Given the spacetime vectors and antisymmetric tensor gauge fields in d-dimensions of rank $\leq d$ and associated with the diffeomorphisms of an internal $p+1$-dim space

$$A_\rho \;=\; A^a_\rho\;(x^\mu, y^a)\;\frac{\partial}{\partial y^a}, \quad A_{\mu\nu} \;=\; A^a_{\mu\nu}\;(x^\mu, y^a)\;\frac{\partial}{\partial y^a}$$

$$A_{\mu\nu\rho} \;=\; A^a_{\mu\nu\rho}\;(x^\mu, y^a)\;\frac{\partial}{\partial y^a},\; etc\; .... \quad a = 1, 2, 3, ..., p+1. \tag{2.10}$$

the Lie-Brackets are defined as follows

$$[A_\mu, A_\rho] \;=\; [A_\mu, A_\rho]^a\;\frac{\partial}{\partial y^a} \;=\; \left(A^b_\mu\;\partial_b\;A^a_\rho \;-\; A^b_\rho\;\partial_b\;A^a_\mu\right)\;\frac{\partial}{\partial y^a} \tag{2.11a}$$

$$[A_{\mu\nu}, A_\rho] \;=\; [A_{\mu\nu}, A_\rho]^a\;\frac{\partial}{\partial y^a} \;=\; \left(A^b_{\mu\nu}\;\partial_b\;A^a_\rho \;-\; A^b_\rho\;\partial_b\;A^a_{\mu\nu}\right)\;\frac{\partial}{\partial y^a}. \tag{2.11b}$$

$$[A_{\mu\nu\rho}, A_\tau] \;=\; [A_{\mu\nu\rho}, A_\tau]^a\;\frac{\partial}{\partial y^a} \;=\; \left(A^b_{\mu\nu\rho}\;\partial_b\;A^a_\tau \;-\; A^b_\tau\;\partial_b\;A^a_{\mu\nu\rho}\right)\;\frac{\partial}{\partial y^a},\; etc\, ... \tag{2.11c}$$

The antisymmetric field strengths components are

$$F^a_{\mu\nu} \;=\; \partial_\nu\;A^a_\mu \;-\; \partial_\mu\;A^a_\nu \;-\; [A_\mu,\; A_\nu]^a. \tag{2.12a}$$

$$F^a_{\mu\nu\rho} \;=\; \partial_\rho\;A^a_{\mu\nu} + \partial_\mu\;A^a_{\nu\rho} + \partial_\nu\;A^a_{\rho\mu} - [A_{\mu\nu},\; A_\rho]^a - [A_{\nu\rho},\; A_\mu]^a - [A_{\rho\mu},\; A_\nu]^a. \tag{2.12b}$$

$$F^a_{\mu\nu\rho\tau} \;=\; \partial_\tau\;A^a_{\mu\nu\rho} \;-\; \partial_\mu\;A^a_{\nu\rho\tau} \;+\; \partial_\nu\;A^a_{\rho\tau\mu} \;-\partial_\rho\;A^a_{\tau\mu\nu} \;-$$
$$[A_{\mu\nu\rho},\; A_\tau]^a \;+\; [A_{\nu\rho\tau},\; A_\mu]^a \;-\; [A_{\rho\tau\mu},\; A_\nu]^a \;+\; [A_{\tau\mu\nu},\; A_\rho]^a,\; etc\; ..... \tag{2.12c}$$

The gauge transformations are

$$\delta\;A^a_\mu \;=\; \partial_\mu\;\xi^a \;+\; [A_\mu,\; \xi]^a. \tag{2.13a}$$

$$\delta\;A^a_{\mu\nu} \;=\; \partial_\mu\;\xi^a_\nu \;-\; \partial_\nu\;\xi^a_\mu \;+\; [A_{\mu\nu},\; \xi]^a. \tag{2.13b}$$

$$\delta\;A^a_{\mu\nu\rho} \;=\; \partial_\rho\;\xi^a_{[\mu\nu]} \;+\; \partial_\mu\;\xi^a_{[\nu\rho]} \;+\; \partial_\nu\;\xi^a_{[\rho\mu]} \;+\; [A_{\mu\nu\rho},\; \xi]^a,\; etc\; .... \tag{2.13c}$$

There is a residual symmetry of the gauge parameter ξ^a_μ given by

$$\xi^a_\mu \;\to\; (\xi^a_\mu)' \;=\; \xi^a_\mu \;+\; \partial_\mu\;\Lambda^a. \tag{2.14}$$

such that it leaves invariant the transformation $\delta A^a_{\mu\nu}$ due to the fact that $\partial_{[\mu}\;\partial_{\nu]}\;\Lambda^a = 0$. There is a residual symmetry of the gauge parameter $\xi^a_{[\mu\nu]}$

$$\xi^a_{[\mu\nu]} \;\to\; (\xi^a_{[\mu\nu]})' \;=\; \xi^a_{[\mu\nu]} \;+\; \partial_{[\mu}\;\Lambda^a_{\nu]}. \tag{2.15}$$

such that it leaves invariant the transformation $\delta A^a_{\mu\nu\rho}$ due to the fact that

$$\partial_\rho\ \partial_{[\mu}\ \Lambda^a_{\nu]}\ +\ \partial_\mu\ \partial_{[\nu}\ \Lambda^a_{\rho]}\ +\ \partial_\nu\ \partial_{[\rho}\ \Lambda^a_{\mu]}\ =\ 0. \tag{2.16}$$

etc.....

One can fix these residual symmetries of the gauge parameters by setting

$$\xi^a_\mu\ =\ \xi^a_{[\mu\nu]}\ =\ \xi^a_{[\mu\nu\rho]}\ =\ ....\ =\ 0. \tag{2.17}$$

such that the gauge transformations become

$$\delta\ A^a_\mu\ =\ \partial_\mu\ \xi^a + [A_\mu,\ \xi]^a,\quad \delta\ A^a_{\mu\nu}\ =\ [A_{\mu\nu},\ \xi]^a,\quad \delta\ A^a_{\mu\nu\rho}\ =\ [A_{\mu\nu\rho},\ \xi]^a,\ etc\ ...... \tag{2.18}$$

Under these gauge transformations (2.18) the field strengths transform covariantly as

$$\delta\ F^a_{\mu\nu}\ =\ [F_{\mu\nu},\ \xi]^a,\quad \delta\ F^a_{\mu\nu\rho}\ =\ [F_{\mu\nu\rho},\ \xi]^a,\quad \delta\ F^a_{\mu\nu\rho\tau}\ =\ [F_{\mu\nu\rho\tau},\ \xi]^a,\ etc\ ...... \tag{2.19a}$$

The tracelike scalar terms $\mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ F_a^{\mu_1\mu_2.....\mu_n}$ transform as

$$\delta\ \left(\ \mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ F_a^{\mu_1\mu_2.....\mu_n}\ \right)\ =\ \delta\ \left(\ \mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ \right)\ F_a^{\mu_1\mu_2.....\mu_n}\ +$$

$$\mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ \delta\left(\ F_a^{\mu_1\mu_2.....\mu_n}\ \right)\ =\ \xi^b\ \partial_b\ (\ \mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ F_a^{\mu_1\mu_2.....\mu_n}\ ). \tag{2.19b}$$

resulting from the transformations

$$\delta\ \mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ =\ [\ \mathcal{F}_{\mu_1\mu_2.....\mu_n},\ \xi\ ]^a\ =\ \xi^c\ \partial_c\ \mathcal{F}^a_{\mu_1\mu_2.....\mu_n}\ -\ \mathcal{F}^c_{\mu_1\mu_2.....\mu_n}\ \partial_c\ \xi^a$$

$$\delta\ \mathcal{F}_a^{\mu_1\mu_2.....\mu_n}\ =\ [\ \mathcal{F}^{\mu_1\mu_2.....\mu_n},\ \xi\ ]_a\ =\ \xi^c\ \partial_c\ \mathcal{F}_a^{\mu_1\mu_2.....\mu_n}\ +\ \mathcal{F}_c^{\mu_1\mu_2.....\mu_n}\ \partial_a\ \xi^c. \tag{2.19c}$$

after relabeling indices and which follow from the definitions of the Lie derivatives of a mixed tensor along a vector field $\xi^a\partial_a$ given by eq-(2.2f) . A scalar *density* $\mathcal{L}$ of weight *one* under gauge transformations behaves as

$$\delta\mathcal{L}\ =\ [\ \mathcal{L},\ \xi\ ]\ =\ \xi^a\ \partial_a\ \mathcal{L}\ +\ \mathcal{L}\ \partial_a\ \xi^a\ =\ \partial_a\ (\xi^a\ \mathcal{L}) \tag{2.20}$$

There is the extra term $\mathcal{L}\ \partial_a\ \xi^a$ in r.h.s of eq-(2.20) when $\mathcal{L}$ is a scalar density of weight one. Volume-preserving diffs require $\partial_a\ \xi^a = 0$ so that the closure of the algebra of volume-preserving diffs is

$$[\ \delta_{\xi_1},\ \delta_{\xi_2}\ ]\ =\ \delta_{[\xi_1,\xi_2]}\ =\ \delta_{\xi_3}. \tag{2.21a}$$

where one can write

$$\xi^a_3\ =\ [\xi_1,\xi_2]^a\ =\ \xi^b_1\ \partial_b\ \xi^a_2\ -\ \xi^b_2\ \partial_b\ \xi^a_1\ =\ \partial_b\ (\xi^b_1\ \xi^a_2\ -\ \xi^b_2\ \xi^a_1). \tag{2.21b}$$

resulting from the conditions $\partial_b\ \xi^b_1 = \partial_b\ \xi^b_2 = 0$, so that

$$\partial_a\ \xi_3^a\ =\ \partial_a\ \partial_b\ (\ \xi_1^b\ \xi_2^a\ -\ \xi_2^b\ \xi_1^a\ )\ =\ 0. \tag{2.22}$$

and the algebra closes, since the term inside the parenthesis is antisymmetric under the exchange of the $a \leftrightarrow b$ indices while $\partial_a\partial_b = \partial_b\partial_a$ is symmetric.

The covariant Bianchi identities are

$$D_\rho\ F^a_{\mu\nu}\ +\ D_\mu\ F^a_{\nu\rho}\ +\ D_\nu\ F^a_{\rho\mu}\ =\ 0$$
$$D_\tau\ F^a_{\mu\nu\rho}\ -\ D_\mu\ F^a_{\nu\rho\tau}\ +\ D_\nu\ F^a_{\rho\tau\mu}\ -\ D_\rho\ F^a_{\tau\mu\nu}\ =\ 0$$
$$D_{\mu_{n+1}}\ F^a_{\mu_1\mu_2.......\mu_n}\ +\ signed\ cyclic\ permutations\ =\ 0. \tag{2.23a}$$

where

$$D_{\mu_{n+1}}\ F^a_{\mu_1\mu_2.......\mu_n}\ =\ \partial_{\mu_{n+1}}\ F^a_{\mu_1\mu_2.......\mu_n}\ +\ [\ A_{\mu_{n+1}},\ F_{\mu_1\mu_2.......\mu_n}\ ]^a. \tag{2.23b}$$

$$F^a_{\mu_1\mu_2.......\mu_n}\ =\ \partial_{\mu_n}\ A^a_{\mu_1\mu_2.......\mu_{n-1}}\ +\ [\ A_{\mu_n},\ A_{\mu_1\mu_2.......\mu_{n-1}}\ ]^a\ +\ signed$$
$$cyclic\ permutations\ of\ indices. \tag{2.23c}$$

In the quenched approximation one has

$$\partial_\mu\ A_\nu\ (y^a) = 0, \quad \partial_\rho\ A_{\mu\nu}(y^a)\ = 0, \quad \partial_\mu\ A_{\nu\rho\tau}(y^a)\ =\ 0,\ ....\ . \tag{2.24}$$

since the fields don't have a dependence on the spacetime coordinates x^μ, there is only a dependence on the internal coordinates y^a . Therefore, in the quenched approximation one has

$$F^a_{\mu\nu}\ \frac{\partial}{\partial y^a}\ =\ (\ [A_\mu, A_\nu]^a\ )\ \frac{\partial}{\partial y^a} \tag{2.25}$$

$$F^a_{\mu\nu\rho}\ \frac{\partial}{\partial y^a}\ =\ -\ (\ [A_{\mu\nu}, A_\rho]^a\ +\ [A_{\nu\rho}, A_\mu]^a\ +\ [A_{\rho\mu}, A_\nu]^a\ )\ \frac{\partial}{\partial y^a} \tag{2.26}$$

$$F^a_{\mu\nu\rho\tau}\ \frac{\partial}{\partial y^a}\ =\ (\ -\ [A_{\mu\nu\rho}, A_\tau]^a\ +\ [A_{\nu\rho\tau}, A_\mu]^a\ -\ [A_{\rho\tau\mu}, A_\nu]^a\ +\ [A_{\tau\mu\nu}, A_\rho]^a\ )\ \frac{\partial}{\partial y^a} \tag{2.27}$$

etc .....

2.3 p-Branes Actions from Antisymmetric Nonabelian Tensorial Gauge Theories of Diffs in $p+1$-dim

In section **1** we discussed the results of [12] about Poisson brackets and their deformation into Moyal brackets; how the Weyl-Wigner-Groneowold-Moyal (WWGM) correspondence maps operators into c-functions in phase space and how the commutators of operators in a Hilbert space are mapped into the Moyal brackets of their corresponding symbols. In particular, in order to implement the WWGM procedure for gauge theories, one needs to find a representation of the gauge group in terms of (linear) operators in a Hilbert space and map them, via the WWGM correspondence, into c-functions in phase space. The unitary irreducible representations of the infinite-dim group of diffeomorphisms is not known. To tackle this problem one can propose a morphism (correspondence) in *the quenched approximation* among the differential operators (on one side) and c-functions (on the other side) as follows

For $y^a = y^1, y^2$ and $\sigma^i = \sigma^1, \sigma^2$

$$F^a_{\mu\nu}(y^a)\ \frac{\partial}{\partial y^a}\ \Leftrightarrow\ \mathcal{F}_{\mu\nu}\ =\ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu\ \}\ =\ \epsilon^{ij}\ \frac{\partial \mathcal{A}_\mu(\sigma)}{\partial \sigma^i}\ \frac{\partial \mathcal{A}_\nu(\sigma)}{\partial \sigma^j} \tag{2.28a}$$

Under an infinitesimal gauge transformation in the quenched approximation

$$\delta\ \mathcal{F}_{\mu\nu}\ =\ \{\ \mathcal{F}_{\mu\nu},\ \Lambda\ \}\ =\ \{\ \{\mathcal{A}_\mu,\ \mathcal{A}_\nu\},\ \Lambda\ \}\ =\ \epsilon^{ij}\ \frac{\partial\{\mathcal{A}_\mu,\ \mathcal{A}_\nu\}}{\partial \sigma^i}\ \frac{\partial \Lambda(\sigma)}{\partial \sigma^j} \tag{2.28b}$$

The third rank antisymmetric tensor correspondence is

$$F^a_{\mu\nu\rho}(y^a)\ \frac{\partial}{\partial y^a}\ \Leftrightarrow\ \{\ \mathcal{A}_{\mu\nu},\ \mathcal{A}_\rho\ \} + \{\ \mathcal{A}_{\nu\rho},\ \mathcal{A}_\mu\ \} + \{\ \mathcal{A}_{\rho\mu},\ \mathcal{A}_\nu\ \},\ etc\ ...... \tag{2.29}$$

Notice the presence of the second-rank anti-symmetric tensor $\mathcal{A}_{\mu\nu}(\sigma)$ in eq-(2.29).

For $y^a = y^1, y^2, y^3$ and $\sigma^i = \sigma^1, \sigma^2, \sigma^3$

$$F^a_{\mu\nu}(y^a)\ \frac{\partial}{\partial y^a}\ \Leftrightarrow\ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu\ \}\ =\ \omega^{i_1 i_2}\ \frac{\partial \mathcal{A}_\mu}{\partial \sigma^{i_1}}\ \frac{\partial \mathcal{A}_\nu}{\partial \sigma^{i_2}} \tag{2.30a}$$

where $\omega^{i_1 i_2}$ is an antisymmetric tensor (not to be confused with a symplectic form in even dimensional phase spaces) and the indices $i_1, i_2 \subset I = 1, 2, 3$. In section **2.4** will be devoted to the construction of the antisymmetric tensors $\omega^{i_1 i_2}, \omega^{i_1 i_2 i_3}, ...$ required to build brackets like those in eq-(2.30a) when the rank of the field strength does *not* match the dimensions of the internal space.

Under a gauge transformation in the quenched approximation

$$\delta\ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu\ \}\ =\ \{\ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu\ \},\ \Lambda\ \}\ =$$

$$\omega^{i_1 i_2} \ (\partial_{i_1} \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu \ \}) \ (\partial_{i_2} \ \Lambda). \qquad (2.30b)$$

where Λ is the gauge parameter . The third rank antisymmetric tensor correspondence is

$$F^a_{\mu\nu\rho}(y^a) \ \frac{\partial}{\partial y^a} \ \Leftrightarrow \ \mathcal{F}_{\mu\nu\rho} \ = \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu, \ \mathcal{A}_\rho \ \} \ = \ \epsilon^{ijk} \ \frac{\partial \mathcal{A}_\mu}{\partial \sigma^i} \ \frac{\partial \mathcal{A}_\nu}{\partial \sigma^j} \ \ \frac{\partial \mathcal{A}_\rho}{\partial \sigma^k}. \ (2.31a)$$

Under a gauge transformation in the quenched approximation

$$\delta \ \mathcal{F}_{\mu\nu\rho} \ = \ \{ \ \mathcal{F}_{\mu\nu\rho}, \ \Lambda_1, \ \Lambda_2 \ \} \ = \ \{ \ \{\mathcal{A}_\mu, \ \mathcal{A}_\nu, \ \mathcal{A}_\rho\}, \ \Lambda_1, \ \Lambda_2 \ \} \ . \qquad (2.31b)$$

where now Λ_1, Λ_2 are two gauge parameters needed to saturate the entries of the NPB brackets in (2.31a). The fourth rank antisymmetric tensor correspondence is

$$F^a_{\mu\nu\rho\tau}(y^a) \ \frac{\partial}{\partial y^a} \ \Leftrightarrow \ \ - \ \{ \ \mathcal{A}_{\mu\nu\rho}, \ \mathcal{A}_\tau \ \} \ + \ \{ \ \mathcal{A}_{\nu\rho\tau}, \ \mathcal{A}_\mu \ \} \ -$$

$$\{ \ \mathcal{A}_{\rho\tau\mu}, \ \mathcal{A}_\nu \ \} \ + \ \{ \ \mathcal{A}_{\tau\mu\nu}, \ \mathcal{A}_\rho \ \}, \ etc \ .... \qquad (2.32a)$$

where

$$\{ \ \mathcal{A}_{\mu\nu\rho}, \ \mathcal{A}_\tau \ \} \ = \ \omega^{i_1 i_2} \ (\partial_{i_1} \ \mathcal{A}_{\mu\nu\rho}) \ (\partial_{i_2} \ \mathcal{A}_\tau), \ etc... \qquad (2.32b)$$

Notice the presence of the third-rank anti-symmetric tensors $\mathcal{A}_{\mu\nu\rho}(\sigma), ...$ in eqs-(2.32a, 2.32b).

For $y^a = y^1, y^2, y^3, y^4$ and $\sigma^i = \sigma^1, \sigma^2, \sigma^3, \sigma^4$ one has the mappings

$$F^a_{\mu\nu}(y^a) \ \frac{\partial}{\partial y^a} \ \Leftrightarrow \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu \ \} \ = \ \omega^{i_1 i_2} \ \frac{\partial \mathcal{A}_\mu}{\partial \sigma^{i_1}} \ \frac{\partial \mathcal{A}_\nu}{\partial \sigma^{i_2}} \qquad (2.33a)$$

where the indices run now over $i_1, i_2 \subset I = 1, 2, 3, 4$ and $\omega^{i_1 i_2}$ is an antisymmetric tensor.

Under a gauge transformation in the quenched approximation

$$\delta \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu \ \} \ = \ \{ \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu \ \}, \ \Lambda \ \} \qquad (2.33b)$$

The third rank antisymmetric tensor correspondence is

$$F^a_{\mu\nu\rho}(y^a) \ \frac{\partial}{\partial y^a} \ \Leftrightarrow \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu, \ \mathcal{A}_\rho \ \} \ = \ \omega^{i_1 i_2 i_3} \ \frac{\partial \mathcal{A}_\mu}{\partial \sigma^{i_1}} \ \frac{\partial \mathcal{A}_\nu}{\partial \sigma^{i_2}} \ \ \frac{\partial \mathcal{A}_\rho}{\partial \sigma^{i_3}} \qquad (2.34a)$$

where $\omega^{i_1 i_2 i_3}$ is an antisymmetric tensor of *third* rank and the indices $i_1, i_2, i_3 \subset I = 1, 2, 3, 4$

Under a gauge transformation in the quenched approximation

$$\delta \ \{ \ \mathcal{A}_\mu, \ \mathcal{A}_\nu, \ \mathcal{A}_\rho \ \} \ = \ \{ \ \{\mathcal{A}_\mu, \ \mathcal{A}_\nu, \ \mathcal{A}_\rho \ \}, \ \Lambda_1, \ \Lambda_2 \ \}. \qquad (2.34b)$$

The fourth rank antisymmetric tensor correspondence is

$$F^a_{\mu\nu\rho\tau}(y^a)\ \frac{\partial}{\partial y^a} \Leftrightarrow \ \mathcal{F}_{\mu\nu\rho\tau} \ = \ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu,\ \mathcal{A}_\rho,\ \mathcal{A}_\tau\} \ =$$

$$\epsilon^{ijkl}\ \frac{\partial \mathcal{A}_\mu}{\partial \sigma^i}\ \frac{\partial \mathcal{A}_\nu}{\partial \sigma^j}\ \ \frac{\partial \mathcal{A}_\rho}{\partial \sigma^k}\ \frac{\partial \mathcal{A}_\tau}{\partial \sigma^l}\ . \qquad (2.35a)$$

Under a gauge transformation in the quenched approximation

$$\delta\, \mathcal{F}_{\mu\nu\rho\tau} \ = \ \{\ \mathcal{F}_{\mu\nu\rho\tau},\ \Lambda_1,\ \Lambda_2,\ \Lambda_3\ \} \ = \ \{\ \{\ \mathcal{A}_\mu,\ \mathcal{A}_\nu,\ \mathcal{A}_\rho,\ \mathcal{A}_\tau\},\ \Lambda_1,\ \Lambda_2,\ \Lambda_3\ \},\ etc\ .... \qquad (2.35b)$$

In general one has three cases to consider. In the quenched approximation, when the rank is $n = p + 1$, the combination of Lie brackets given by

$$[A_{\mu_1},\ A_{\mu_2\mu_3.......\mu_n}]^a\ \partial_{y^a}\ +\ signed\ cyclic\ permutations\ of\ indices \qquad (2.36a)$$

will be mapped to the Nambu brackets

$$\{\ \mathcal{A}_{\mu_1},\ \mathcal{A}_{\mu_2},\ \mathcal{A}_{\mu_3}, ..., \mathcal{A}_{\mu_{p+1}}\ \} \ = \ \epsilon^{i_1 i_2......i_{p+1}}\ \partial_{i_1}\ \mathcal{A}_{\mu_1}\ \partial_{i_2}\ \mathcal{A}_{\mu_2}\ ....\ \partial_{i_{p+1}}\ \mathcal{A}_{\mu_{p+1}}. \qquad (2.36b)$$

when $n < p + 1$, the Lie brackets are mapped to

$$\{\ \mathcal{A}_{\mu_1},\ \mathcal{A}_{\mu_2},\ \mathcal{A}_{\mu_3}, ..., \mathcal{A}_{\mu_n}\ \} \ = \ \omega^{i_1 i_2......i_n}\ \partial_{i_1}\ \mathcal{A}_{\mu_1}\ \partial_{i_2}\ \mathcal{A}_{\mu_2}\ ....\ \partial_{i_n}\ \mathcal{A}_{\mu_n}. \qquad (2.36c)$$

where $\omega^{i_1 i_2......i_n}$ is an antisymmetric tensor of rank $n < p + 1$ which is no longer equal to the epsilon symbol. The bracket (2.36c) must obey the Nambu-Filippov special condition (the so-called fundamental "identity") and which will restrict the values of $\omega^{i_1 i_2......i_n}$ as we shall see in the next section. The indices $i_1, i_2, ..., i_n \subset I = 1, 2, 3, ..., p + 1$. And when $n > p + 1$, the Lie brackets are mapped to

$$\{\ \mathcal{A}_{\mu_n},\ \mathcal{A}_{\mu_1\mu_2..........\mu_{n-1}}\ \}\ +\ signed\ cyclic\ permutations. \qquad (2.36d)$$

where the bracket involving both vectors and antisymmetric tensors is

$$\{\ \mathcal{A}_{\mu_n},\ \mathcal{A}_{\mu_1\mu_2..........\mu_{n-1}}\ \} \ = \ \omega^{i_1 i_2}\ (\partial_{i_1}\ \mathcal{A}_{\mu_n})\ (\partial_{i_2}\ \mathcal{A}_{\mu_1\mu_2..........\mu_{n-1}}). \qquad (2.36e)$$

Having established the one-to-one correspondences in the above equations (2.28-2.36) among the differential operators in the left hand side and the c-functions in the right hand side (in terms of brackets) and after replacing the gauge fields for the coordinates $\mathcal{A}_\mu(\sigma) \leftrightarrow X_\mu(\sigma)$; $\mathcal{A}_{\mu\nu}(\sigma) \leftrightarrow X_{\mu\nu}(\sigma)$ (a bi-vector); $\mathcal{A}_{\mu\nu\rho}(\sigma) \leftrightarrow X_{\mu\nu\rho}(\sigma)$ (a tri-vector); etc .... one can write the actions associated with the propagation of strings, membranes, p-branes in C-space target backgrounds characterized by Clifford poly-vector valued coordinates and which shall be discussed at the end of this work.

In the meantime, we arrive now at one of the main results of this work : given an antisymmetric nonabelian tensorial gauge field theory of diffeomorphisms of an internal $p+1$-dim space one may establish the correspondence with the reparametrization invariant p-brane action, via a *covariant* trace operation, given in terms of a dynamical measure $\mathcal{J} \ = \ \{ \ \phi^1, \phi^2, ..., \phi^{p+1} \}$ which is a function of $p+1$ auxiliary scalars $\phi^1(\sigma), \phi^2(\sigma), ..., \phi^{p+1}(\sigma)$ as follows :

The relevant gauge invariant action in a d dimensional base spacetime of Lorentzian signature associated with an antisymmetric nonabelian tensor gauge field strength corresponding to the diffeomorphisms of an internal $p+1$-dim space is

$$S \ = \ - \frac{1}{g^2} \int d^d x \ d^{p+1} y \ \sqrt{|g|} \ \sqrt{|h|} \ h_{ab} \ F^a_{\mu_1 \mu_2 .... \ \mu_{p+1}} \ F^b_{\nu_1 \nu_2 ...... \ \nu_{p+1}} \ \times$$

$$g^{\mu_1 \nu_1} \ g^{\mu_2 \nu_2} \ .... \ g^{\mu_{p+1} \nu_{p+1}}. \qquad (2.37)$$

we can omit factorials $1/(p+1)!$ in (2.36) by imposing the ordering prescription $\mu_1 \ < \ \mu_2 \ < \ \mu_3 \ < \ .... \ < \ \mu_{p+1}$. The coupling g^{-2} in (2.37) has dimensions of l^{p+1-d} where l is a suitable length scale to ensure that the integral is dimensionless. The dimension of $\mathbf{F}_{\mu_1 \mu_2 .....\mu_{p+1}}$ is l^{-p-1}. As shown in eq-(2.1), the metric components $g_{\mu\nu}(x^\mu, y^a)$; $h_{ab}(x^\mu, y^a)$ and the antisymmetric field strengths $F^a_{\mu_1 \mu_2 .... \ \mu_{p+1}}(x^\mu, y^a)$ in eq-(2.37) depend in general on the base spacetime coordinates x^μ *and* on the internal space coordinates y^a as well.

As mentioned earlier, the Lie bracket of a scalar Lagrangian *density* $\mathcal{L} = \sqrt{|g|} \ \sqrt{|h|} \ ||\mathbf{F}^2||$ of weight one and the vector field $\xi = \xi^a \partial_a$ is defined as

$$[\ \xi, \ \mathcal{L} \ ] \ = \ \xi^a \ \partial_a \ \mathcal{L} \ + \ \mathcal{L} \ \partial_a \ \xi^a \ = \ \partial_a \ (\xi^a \ \mathcal{L}) \qquad (2.38a)$$

there is a second extra term in the r.h.s of (2.38a), so that under infinitesimal gauge transformations acting on the fields (and *not* on the coordinates) given by eqs-(2.18, 2.19), the action S transforms as

$$\delta_\xi \ S \ = \ \int d^d x \ d^{p+1} y \ [\ \xi, \ \mathcal{L} \ ] \ = \ \int d^d x \ d^{p+1} y \ \partial_a \ (\xi^a \ \mathcal{L}) \ = \ 0. \qquad (2.38b)$$

the integral of a total derivative

$$\delta_\xi \ S \ = \ \int d^d x \ \int_{-\infty}^{+\infty} d^{p+1} y \ \partial_a \ (\ \xi^a \ \mathcal{L} \ ) \ = \ 0. \qquad (2.38d)$$

vanishes if $\xi^a \mathcal{L}$ vanishes at $y^a \ = \ \pm\infty$ and/or there are no boundaries in the integration domain of the y^a variables (the integration domain has compact support). Hence, the action (2.37) is gauge invariant.

In the very special case of the quenched approximation there is *no* dependence on the x^μ variables so eq-(2.37) can be written in terms of a covariant trace operation as

$$S \ = \ - \frac{1}{g^2} \ \left[\ \int d^d x \ \right] \ Trace \ \left[\ \mathbf{F}_{\mu_1 \mu_2 .....\mu_{p+1}} \ \mathbf{F}^{\mu_1 \mu_2 ......\mu_{p+1}} \ \right]. \qquad (2.39)$$

one can pull out the volume V_d factor in d-dim ouside the integral. By recurring to the fields/coordinates correspondence $\mathcal{A}_\mu \leftrightarrow X_\mu$, $\mathcal{A}_{\mu\nu} \leftrightarrow X_{\mu\nu}$ (bi-vectors) ...... in eqs-(2.28-2.36), the above covariant trace of the square of the antisymmetric nonabelian tensor gauge field strength, associated with the diffeomorphisms of the internal $p+1$-dim space in the quenched approximation, has a correspondence with the covariant p-brane action given by

$$-\frac{V_d}{g^2}\ Trace\ \left[\ \mathbf{F}_{\mu_1\mu_2.....\mu_{p+1}}\ \mathbf{F}^{\mu_1\mu_2......\mu_{p+1}}\ \right]\ =$$

$$-\frac{V_d}{g^2}\int d^{p+1}y\ \sqrt{|g|}\ \sqrt{|h|}\ h_{ab}\ F^a_{\mu_1\mu_2.......\ \mu_{p+1}}\ F^b_{\nu_1\nu_2......\ \nu_{p+1}}\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_{p+1}\nu_{p+1}}$$

$$\Leftrightarrow\ -\frac{1}{l^{p+1}}\int d^{p+1}\sigma\ \left(\{\ \phi^1,\phi^2,.......,\phi^{p+1}\ \}\right)^{-1}\ \mathcal{F}_{\mu_1\mu_2.......\ \mu_{p+1}}\ \mathcal{F}_{\nu_1\nu_2......\ \nu_{p+1}}\ \times$$

$$G^{\mu_1\nu_1}(X)\ G^{\mu_2\nu_2}(X)\ ....\ G^{\mu_{p+1}\nu_{p+1}}(X)\ =$$

$$-\frac{1}{l^{p+1}}\int d^{p+1}\sigma\ \left(\{\ \phi^1,\phi^2,.......,\phi^{p+1}\ \}\right)^{-1}\ \{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}\ \times$$

$$\{\ X_{\nu_1},\ X_{\nu_2},\ ....,\ X_{\nu_{p+1}}\ \}\ G^{\mu_1\nu_1}(X)\ G^{\mu_2\nu_2}(X)\ ....\ G^{\mu_{p+1}\nu_{p+1}}(X).\quad (2.40)$$

The trace becomes an integral w.r.t the p-brane coordinates. Notice that one must *not* confuse the x^μ coordinates in eq-(2.36) with $X^\mu(\sigma)$. Also relevant, besides the $\mathcal{A}_\mu \leftrightarrow X_\mu$ correspondence, is $g_{\mu\nu}(y^a) \leftrightarrow G_{\mu\nu}(X^\rho(\sigma^a))$. In flat target spacetime backgrounds $G^{\mu\nu}(X^\rho(\sigma^a)) = \eta^{\mu\nu}$ one can pull the target spacetime metric *inside* the brackets such that the action (2.40) can be rewritten as

$$-\frac{1}{l^{p+1}}\int d^{p+1}\sigma\ \left(\{\ \phi^1,\phi^2,.......,\phi^{p+1}\ \}\right)^{-1}\ \{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}^2\ .\quad (2.41)$$

One can eliminate the $p+1$ auxiliary scalars fields $\phi^i(\sigma)$ in eq-(2.41) which comprise the dynamical measure field $\mathcal{J} \equiv \{\ \phi^1,\phi^2,.......,\phi^{p+1}\ \}$ after performing an Euler variation of the action (2.41) w.r.t the scalars $\phi^i(\sigma)$. Hence, an Euler variation w.r.t the $\phi^1(\sigma)$ field gives

$$-\partial_{a_1}\ \frac{\delta S}{\delta(\partial_{a_1}\phi^1)}\ =\ \epsilon^{a_1a_2a_3.....a_{p+1}}\ (\partial_{a_2}\phi^2)\ (\partial_{a_3}\phi^3)\ ....(\partial_{a_{p+1}}\phi^{p+1})\ \times$$

$$\partial_{a_1}\ \left(\ \mathcal{J}^{-2}\ \{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}^2\ \right)\ =\ 0.\quad (2.42a)$$

since

$$\partial_{a_1}\ \left[\ \epsilon^{a_1a_2a_3.....a_{p+1}}\ (\partial_{a_2}\phi^2)\ (\partial_{a_3}\phi^3)\ ....\ (\partial_{a_{p+1}}\phi^{p+1})\ \right] =\ 0.\quad (2.42b)$$

Similar equations as (2.42) are obtained from the Euler variation w.r.t the $\phi^2(\sigma), ..., \phi^{p+1}$ fields by replacing the derivative ∂_{a_1} in eqs-(2.42) with the derivatives $\partial_{a_2},\ \partial_{a_2},\ ....\partial_{a_{p+1}}$ Therefore, from the $p+1$ equations of the form (2.42) one learns that

$$2\,\frac{\partial_a\ \mathcal{J}}{\mathcal{J}}\ =\ \frac{\partial_a\ \{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}^2}{\{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}^2},\ a=1,2,....,p{+}1,\ integrating\ \Rightarrow$$

$$\mathcal{J}\ =\ \{\ \phi^1,\phi^2,.......,\phi^{p+1}\ \}\ =\ \sqrt{\ \{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}^2}. \qquad (2.43)$$

Substituting the measure field (2.43) into the action (2.40) furnishes the Dirac-Nambu-Goto reparametrization-invariant p-brane action under $\sigma^a \to \tilde{\sigma}^a$ of the form

$$S\ =\ -\frac{1}{l^{p+1}}\int d^{p+1}\sigma\ \sqrt{\ |\{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}\{\ X^{\mu_1},\ X^{\mu_2},\ ....,\ X^{\mu_{p+1}}\ \}|}. \qquad (2.44)$$

where we have omitted factorials $1/(p+1)!$ inside the square root of (2.44) by imposing the ordering prescription $\mu_1\ <\ \mu_2\ <\ \mu_3\ <\ ....\ <\ \mu_{p+1}$ and taken the absolute sign to ensure that the contribution of the NPB terms inside the square root is positive-definite. The NPB terms inside the square root coincide with the absolute value of the determinant of the $(p+1)\times(p+1)$ induced metric g_{ab} resulting from the embedding of the $p+1$-dim world volume of the p-brane onto the flat target spacetime D-dim background

$$g_{ab}\ =\ (\partial_a X^\mu)\ (\partial_b X^\nu)\ \eta_{\mu\nu},\quad |det\ g_{ab}\ |\ =\ |\{\ X_{\mu_1},\ X_{\mu_2},\ ....,\ X_{\mu_{p+1}}\ \}^2|\ \Rightarrow$$

$$S\ =\ -\,T_p\int d^{p+1}\sigma\ \sqrt{|\ det\ (\partial_a X^\mu\ \partial_b X^\nu\ \eta_{\mu\nu})\ |} \qquad (2.45)$$

where we have identified the p-brane tension T_p with l^{-p-1} in (2.45). It is important to remark that the world volume coordinates $\sigma^0,\sigma^1,......,\sigma^p$ involve one temporal coordinate σ^0 (the p-brane's clock) and p spatial coordinates. The d-dim base spacetime has a temporal coordinate x^0 and $d-1$ spatial ones $x^1,x^2,....,x^{d-1}$.

There are other actions that do not correspond to the standard p-brane actions. In general one must have that d must be greater or equal to the *rank* of the antisymmetric field strength, and $d\geq p{+}1$, resulting from the gauge fields /p-brane coordinates correspondence $\mathcal{A}_\mu \leftrightarrow X_\mu$. There are two other cases to consider : when the rank is greater or less than $p+1$. For example, in the case that $a=1,2,3$, in a flat target background, and setting the coupling $g^2=1$,

$$\int d^3y\ \sqrt{|g|}\ \sqrt{|h|}\ h_{ab}\ \ F^a_{\mu_1\mu_2}\ F^b_{\nu_1\nu_2}\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ \Leftrightarrow$$

$$\int d^3\sigma\ \mathcal{J}(\phi^i)^{-1}\ \{\ X_{\mu_1},\ X_{\mu_2}\ \}^2 \qquad (2.46a)$$

where

$$\{ \ X_{\mu_1}, \ X_{\mu_2} \ \} \ = \ \omega^{i_1 i_2} \ (\partial_{i_1} \ X_{\mu_1}) \ (\partial_{i_2} \ X_{\mu_2}), \quad i_1, i_2 \ \subset \ I \ = \ 1,2,3. \quad (2.46b)$$

once again we have the antisymmetric tensor $\omega^{i_1 i_2}$ that must not be confused with a symplectic form in an even-dim phase space as we shall see in the next section. After eliminating the auxiliary scalars ϕ^i in eq-(2.46a) it leads to a "string-membrane" hybrid action of the form

$$\int d^3\sigma \ \sqrt{ \{ \ X_{\mu_1}, \ X_{\mu_2} \ \}^2 } \tag{2.47}$$

where the spacetime coordinates depend on the three membrane coordinates $X^\mu(\sigma^0, \sigma^2, \sigma^3)$; $\mu = 1, 2, 3, ...d$ and $d \geq 3$.

In the case when $a = 1, 2$, in a flat target background, setting the coupling $g^2 = 1$,

$$\int d^2y \ \sqrt{|g|} \ \sqrt{|h|} \ h_{ab} \ \ F^a_{\mu_1\mu_2\mu_3} \ F^b_{\nu_1\nu_2\nu_3} \ g^{\mu_1\nu_1} \ g^{\mu_2\nu_2} \ g^{\mu_3\nu_3} \Leftrightarrow$$

$$\int d^2\sigma \ \mathcal{J}(\phi^i)^{-1} \ (\ \{ \ X_{\mu_1\mu_2}, \ X_{\mu_3} \ \} + cyclic \ ) \ \ (\ \{ \ X^{\mu_1\mu_2}, \ X^{\mu_3} \ \} + cyclic \ ) \, . \tag{2.49}$$

where $X_{\mu_1\mu_2}(\sigma)$ are the bi-vectors coordinates of the C-space associated with the Clifford algebra of the target spacetime. By "cyclic" in (2.49) one means a cyclic permutation of the indices. After eliminating the auxiliary scalars ϕ^i in (2.49) it leads to a Dirac-Born-Infeld-like action of the form

$$\int d^2\sigma \ \sqrt{ \ (\ \{ \ X_{\mu_1\mu_2}, \ X_{\mu_3} \ \} + cyclic \ ) \ \ (\ \{ \ X^{\mu_1\mu_2}, \ X^{\mu_3} \ \} + cyclic \ ) } \tag{2.50}$$

The Eguchi-Schild p-brane action (setting $T_p = 1$)

$$\int d^{p+1}\sigma \ \{ \ X_{\mu_1}, \ X_{\mu_2}, \ ...., \ X_{\mu_{p+1}} \ \} \ \{ \ X^{\mu_1}, \ X^{\mu_2}, \ ...., \ X^{\mu_{p+1}} \ \} \, . \tag{2.51}$$

is *not* fully reparametrization invariant under $\sigma^a \rightarrow \tilde{\sigma}^a$, but only invariant under $p+1$-volume-preserving diffs leaving the NPB invariant. In order to obtain a fully reparametrization invariant p-brane action one needs to have a *covariant* trace operation defined in terms of a dynamical measure $\mathcal{J} \ = \ \{ \ \phi^1, \phi^2, ..., \phi^{p+1} \}$ expressed in terms of $p+1$ auxiliary scalars $\phi^1(\sigma), \phi^2(\sigma), ..., \phi^{p+1}(\sigma)$ as shown above.

The reparametrization-invariant action ($T_p \ = \ 1$) involving the auxiliary scalar-density $e(\sigma)$

$$\int d^{p+1}\sigma \ \frac{1}{e(\sigma)} \{ \ X_{\mu_1}, \ X_{\mu_2}, \ ...., \ X_{\mu_{p+1}} \ \} \ \{ \ X^{\mu_1}, \ X^{\mu_2}, \ ...., \ X^{\mu_{p+1}} \ \} \, . \tag{2.52}$$

is problematic because a variation w.r.t the auxiliary scalar-density $e(\sigma)$ field leads to a zero action. The proper reparametrization-invariant Eguchi-Schild p-brane action has a cosmological constant term of the form

$$\int d^{p+1}\sigma \left[\frac{1}{e(\sigma)} \frac{1}{(p+1)!} \{ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}} \}^2 - e(\sigma) \right]. \quad (2.53)$$

eliminating $e(\sigma)$ via its equations of motion leads once again to the Nambu-Goto-Dirac p-brane action. The *advantage* of introducing the dynamical measure $\mathcal{J} = \{\phi^1, \phi^2, ..., \phi^{p+1}\}$ in the action (2.40) is that one can eliminate $\mathcal{J}$, via the equations of motion of the scalars $\phi^i(\sigma)$, *without* constraining the action to zero nor introducing a cosmological constant term. A dynamical measure $\mathcal{J} = \{\phi^1, \phi^2, ..., \phi^{p+1}\}$ was originally introduced by [10] and has many applications in many realms of Physics. Scalar fields have also been introduced by [53] in his formulation of the so-called "wiggly" branes.

2.4 N-ary Algebras, Kalb-Ramond couplings, Chern-Simons Branes and Dualities

This section would not be complete without a discussion of **n**-ary brackets $\{X^1, X^2, ..., X^n\}$ for the case that $n \le p+1 \le d$. The brackets should obey the Nambu-Filippov special condition (fundamental "identity") that can be written as a derivation with respect vectors and "poly-vectors" as

$$\mathcal{D}_X \{Y^1, Y^2\} = \{ X, \{Y^1, Y^2\} \} = \{ \{X, Y^1\}, Y^2 \} + \{ Y^1, \{X, Y^2\} \}. \quad (2.54)$$

$$\mathcal{D}_{\{X^1,X^2\}} \{Y^1, Y^2, Y^3\} = \{ X^1, X^2, \{Y^1, Y^2, Y^3\} \} =$$
$$\{ \{X^1, X^2, Y^1\}, Y^2, Y^3 \} + \{ Y^1, \{X^1, X^2, Y^2\}, Y^3 \} + \{ Y^1, Y^2, \{X^1, X^2, Y^3\} \}. \quad (2.55)$$

$$\mathcal{D}_{\{X^1,X^2,X^3\}} \{Y^1, Y^2, Y^3, Y^4\} = \{ X^1, X^2, X^3, \{Y^1, Y^2, Y^3, Y^4\} \} =$$
$$\{ \{X^1, X^2, X^3, Y^1\}, Y^2, Y^3, Y^4 \} + \{ Y^1, \{X^1, X^2, X^3, Y^2\}, Y^3, Y^4 \} +$$
$$\{ Y^1, Y^2, \{X^1, X^2, X^3, Y^3\}, Y^4 \} + \{ Y^1, Y^2, Y^3, \{X^1, X^2, X^3, Y^4\} \} \quad (2.56)$$

etc ..... For n-ary brackets, Nambu showed that the Jacobian (the classical Nambu bracket)

$$\{X^1, X^2, ..., X^n\} = \epsilon^{i_1 i_2 ..... i_n} \partial_{i_1} X^1 \partial_{i_2} X^2 .... \partial_{i_n} X^n. \quad (2.57)$$

satisfies the Nambu-Filippov special conditions, [38], [40]. It is highly nontrivial to satisfy these conditions for brackets involving different structure functions (constants) than the epsilon symbols and comprised of antisymmetric tensors $\omega^{i_1 i_2}, \omega^{i_1 i_2 i_3}, ...$ For instance, given $n < p+1$, the brackets are defined by

$$\{X^1,\ X^2, ...,\ X^n\} \ = \ \omega^{i_1 i_2 .....i_n}\ \partial_{i_1} X^1\ \partial_{i_2} X^2\ ....\ \partial_{i_n} X^n. \qquad (2.58)$$

where the indices are $i_1, i_2, ...i_n \subset I = 1, 2, 3, ..., p+1$ such that one has $(p+1)!/(p+1-n)!n!$ different sets of combinations of indices with respect to which one can take the derivatives in (2.58).

Given $n \leq p+1 \leq d$ and after recurring to the ansatz

$$\{\ \sigma^{i_1},\ \sigma^{i_2}\ \} \ = \ \omega^{i_1 i_2}(\sigma) \ = $$
$$f^{[i_1 i_2]} \ + \ f^{[i_1 i_2]}_{j_1}\ \sigma^{j_1} \ + \ f^{[i_1 i_2]}_{j_1 j_2}\ \sigma^{j_1}\ \sigma^{j_2} \ + \ .... \qquad (2.59a)$$

$$\{\ \sigma^{i_1},\ \sigma^{i_2},\ \sigma^{i_3}\ \} \ = \ \omega^{i_1 i_2 i_3}(\sigma) \ = $$
$$f^{[i_1 i_2 i_3]} \ + \ f^{[i_1 i_2 i_3]}_{j_1}\ \sigma^{j_1} \ + \ f^{[i_1 i_2 i_3]}_{j_1 j_2}\ \sigma^{j_1}\ \sigma^{j_2} \ + \ .... \qquad (2.59b)$$

$$\{\ \sigma^{i_1},\ \sigma^{i_2},\ ....,\ \sigma^{i_n}\ \} \ = \ \omega^{i_1 i_2 ....i_n}(\sigma) \ = $$
$$f^{[i_1 i_2 ....i_n]} \ + \ f^{[i_1 i_2 ......i_n]}_{j_1}\ \sigma^{j_1} \ + \ f^{[i_1 i_2 ......i_n]}_{j_1 j_2}\ \sigma^{j_1}\ \sigma^{j_2} \ + \ .... \qquad (2.59c)$$

and finally

$$\{\ \sigma^{i_1},\ \sigma^{i_2},\ ....,\ \sigma^{i_{p+1}}\ \} \ = \ \epsilon^{i_1 i_2 ....i_{p+1}}. \qquad (2.59d)$$

leads to a general family of structure functions (constants) that one can use in the definitions of the brackets. The constants $f^{[i_1 i_2 ....i_n]}$ are antisymmetric in the i indices. The constants $f^{[i_1 i_2 ......i_n]}_{j_1 j_2 ....j_m}$ are antisymmetric in the upper i indices but *symmetric* in the lower j indices. The Nambu-Filippov special conditions will *restrict* the form of the plausible structure functions (constants) of (2.59) in a very stringent fashion [40]. The particular terms $f^{[i_1 i_2 ......i_n]}_{j_1}\ \sigma^{j_1}$ (linear in the σ's) in the r.h.s of eqs-(2.59) correspond to an **n**-ary Filippov-Lie algebraic structure. Hence, if one constrains all the terms in the r.h.s of eqs-(2.59) to zero, *except* the former ones, the Nambu-Filippov special conditions on the structure constants of the Filippov-Lie algebra become [36] (and references therein)

$$f^{[i_1 i_2 ......i_n]}_{c}\ f^{[k_1 k_2 ......k_n]}_{i_n} \ = \ \sum_{j=1}^{j=n}\ f^{[i_1 i_2 ......i_{n-1} k_j]}_{d}\ f^{[k_1 k_2 ....k_{j-1}\ d\ k_{j+1} ....k_n]}_{c}. \qquad (2.59e)$$

Despite the difficulties in finding solutions for the structure constants obeying eq-(2.59e), for all practical purposes, the most relevant case to consider is the one which leads to p-brane actions based on (2.37); i.e. the case when $n = p+1$, such that the relevant brackets coincide precisely with the Nambu-Poisson brackets involving the epsilon symbol which obey the Nambu-Filippov special conditions (fundamental "identity").

We finalize this section by pointing out an interesting relation between Kalb-Ramond couplings to branes, Chern-Simons branes and the duality among fields variables and branes coordinates. Let us begin with the integral given by the following Kalb-Ramond coupling

$$\int_{\partial M^n} A_{i_1 i_2 ...... i_{n-1}}(X) \ dX^{i_1} \wedge dX^{i_2} \wedge .... \wedge dX^{i_{n-1}} =$$

$$\int_{M^n} F_{i_1 i_2 ...... i_{n-1} i_n}(X) \ dX^{i_1} \ \wedge \ dX^{i_2} \ \wedge .... \wedge \ dX^{i_{n-1}} \ \wedge \ dX^{i_n}. \quad (2.60)$$

where the *Abelian* field strength which allowed us to use Stokes theorem is $\mathbf{F} = \mathbf{dA}$. In the particular case when the Kalb-Ramond field is of the form

$$A_{i_1 i_2 ...... i_{n-1}}(X) = \epsilon_{i_1 i_2 .... i_{n-1} i_n} \ X^{i_n}. \quad (2.61)$$

one has for the expressions in eq-(2.60)

$$\int_{\partial M^n} \epsilon_{i_1 i_2 .... i_{n-1} i_n} \ X^n \ dX^{i_1} \wedge dX^{i_2} \wedge .... \wedge dX^{i_{n-1}} =$$

$$\int_{M^n} \epsilon_{i_1 i_2 ...... i_n}(X) \ dX^{i_1} \wedge dX^{i_2} \wedge .... \wedge dX^{i_{n-1}} \wedge dX^{i_n}. \quad (2.62)$$

since the indices $i_1, i_2, ..., i_n \subset I = 1, 2, 3, ..., n$, the last integral (2.62) is just the n-dim volume of a bulk region M^n whose $n-1$-dim boundary ∂M^n is the world-volume of a Chern-Simons $n-2$-brane; i.e. the first integral of (2.62) is the action of a Chern-Simons p-brane such that its $p+1$-dim world volume is the boundary of a $p+2$-dim bulk region. In our case $p = n-2$. In one wishes to preserve translational invariance in the target spacetime for the Chern-Simons action, the coordinates X should be defined w.r.t to a reference point $X(\sigma) - X(\sigma = 0)$, see [35] for details.

The loop transform (section **1**) in the *Abelian* case is just a duality among $A_{\mu_1 \mu_2 ...... \mu_{n-1}}$ and $X_{\mu_1 \mu_2 ..... \mu_{n-1}}$. Using the gauge fields /brane coordinates correspondence in (2.60) one infers that

$$A_{i_1 i_2 ...... i_{n-1}}(X) \leftrightarrow X_{i_1 i_2 ..... i_{n-1}} \Rightarrow X_{i_1 i_2 ..... i_{n-1}} \leftrightarrow \epsilon_{i_1 i_2 .... i_{n-1} i_n} \ X^{i_n}. \quad (2.63)$$

hence, eq-(2.63) states how one has a *Hodge duality* among the poly-vector valued coordinate $X^{\mu_1 \mu_2 ..... \mu_{n-1}}$ with the vector valued X^{μ_n} coordinate in n-dim. In section **1** we discussed the gauge fields/coordinates correspondence via a "Fourier"-like Wilson-Loop transform. These dualities are important for our discussion of branes in C-spaces in section **4**, in particular how to extend the discussion above to the *nonabelian* case.

3 Super p-branes, Covariant Matrix Models and Gauged Nonlinear σ Models

These results for the bosonic covariant p-brane actions should apply to the covariant super p-brane actions [34]

$$S = -T_p \int d^{p+1}\sigma \left[\sqrt{- \, det \, (\Pi_i^\mu \, \Pi_j^\nu \, \eta_{\mu\nu})} + \frac{2}{(p+1)!} \epsilon^{i_1 i_2 \ldots\ldots i_{p+1}} \, B_{i_1 i_2 \ldots\ldots i_{p+1}} \right]. \tag{3.1}$$

which is an integral over the $p+1$-dim world volume of the p-brane moving in a target superspace of local coordinates $Z^M = X^\mu, \theta^\alpha$ (a Majorana spinor); $\eta_{\mu\nu}$ is the metric of the D-dim Minkowski spacetime; the second term in (3.1) is the Wess-Zumino term where the coefficients $B_{i_1 i_2 \ldots\ldots i_{p+1}}$ are

$$B_{i_1 i_2 \ldots\ldots i_{p+1}} = - \frac{\eta}{(p+1)!} \epsilon^{i_1 i_2 \ldots i_{p+1}} \, \bar\theta \, \Gamma_{\mu_1 \mu_2 \ldots\ldots \mu_p} \, \partial_{i_{p+1}} \, \theta \, \times$$

$$\sum_{r=0}^{p} i^{r+1} \, C_{r+1}^{p+1} \, (\bar\theta \Gamma^{\mu_1} \partial_{i_1} \theta) \, \ldots\ldots \, (\bar\theta \Gamma^{\mu_r} \partial_{i_r} \theta) \, \Pi_{i_{r+1}}^{\mu_{r+1}} \, \ldots\ldots\ldots \, \Pi_{i_p}^{\mu_p}. \tag{3.2}$$

C_{r+1}^{p+1} are the binomial coefficients $(p+1)!/(r+1)!(p-r)!$, $\eta = (-1)^{(p-1)(p+6)/4}$ and the super-vielbeins

$$\Pi_i^\mu = \partial_i \, X^\mu - i\bar\theta \, \Gamma^\mu \, \partial_i \, \theta. \tag{3.3}$$

are the components of the pullback to the world volume of the supersymetric invariant one-form $\Pi^\mu = dX^\mu - i\bar\theta \Gamma^\mu d\theta$. The determinant can also be rewritten as

$$det \, (\Pi_i^\mu \, \Pi_j^\nu \, \eta_{\mu\nu}) = < \Pi^{\mu_1}, \, \Pi^{\mu_2}, \ldots\ldots, \, \Pi^{\mu_{p+1}} > \; < \Pi_{\mu_1}, \, \Pi_{\mu_2}, \ldots\ldots, \, \Pi_{\mu_{p+1}} > . \tag{3.4}$$

where the analog of the bracket among the Π_i^μ variables is defined as [36]

$$< \Pi^{\mu_1}, \, \Pi^{\mu_2}, \, \Pi^{\mu_3}, \ldots\ldots, \, \Pi^{\mu_{p+1}} > = \epsilon^{i_1 i_2 \ldots\ldots i_{p+1}} \, \Pi_{i_1}^{\mu_{i_1}} \, \Pi_{i_2}^{\mu_{i_2}} \, \ldots\ldots\ldots \, \Pi_{i_{p+1}}^{\mu_{i_{p+1}}} =$$

$$\sum_{n=0}^{p+1} C_n^{p+1} \, i^n \, (-1)^{n(n+1)/2} \, \bar\theta_{\alpha_1} \bar\theta_{\alpha_2} \, \ldots\ldots \, \bar\theta_{\alpha_n} \, \times$$

$$\{ \, (\Gamma^{\mu_1}\theta)^{\alpha_1}, \, (\Gamma^{\mu_2}\theta)^{\alpha_2}, \ldots\ldots, (\Gamma^{\mu_n}\theta)^{\alpha_n}, \, X^{\mu_{n+1}}, \, X^{\mu_{n+2}}, \, \ldots\ldots, X^{\mu_{p+1}} \, \}_{NPB}. \tag{3.5}$$

C_n^{p+1} are the binomial coefficients $(p+1)!/n!(p+1-n)!$.

The Fierz identity required for the construction of supersymmetic invariant actions restricts the possible values of p and the target spacetime dimension D. The light-cone gauge-fixed action of the super p-brane action for the restricted values of p and D was constructed by [34]. For the membrane case $p = 2$ (in $D = 4,5,7,11$) the gauge-fixed action (excluding the zero mode sector) is equivalent to that of a one-dim super Yang-Mills theory of the area-preserving diffs group of the membrane. For $p > 2$, the authors [34] have shown that the light-cone gauge-fixed action is that of a *new* kind of supersymmetric gauge theory of p-volume preserving diffs associated with the p *spatial* dimensions of the extended object. Therefore, after gauge fixing, the original $p+1$-dim diffs associated with the $p+1$-dim world volume of the p-brane is *reduced* to the p-volume-preserving diffs associated with the p *spatial* dimensions of the p-brane. The authors [34] conjectured that the *new* kind of supersymmetric gauge theory is related to an infinite-dim *nonabelian* anti-symmetric gauge theory. Hence, we propose that this new theory should be part of (the supersymmetric version) of the antisymmetric nonabelian tensor gauge field theory of $p+1$-dim diffs in d-dim constructed in section **2**.

To illustrate this, let us begin with the light-cone gauge-fixed action of the super p-brane action for $p = 1,2,3,4,5$ (excluding the zero modes sector)

$$I \;=\; \frac{1}{2}\int d^{p+1}\sigma\;\Big[\;(\mathcal{D}_o X^I)^2 \;+\; i\bar{S}\;\mathcal{D}_o S \;-\; det\;(\;h_{ab}\;)\;-\; i^p\bar{S}\;\gamma^a\gamma\;\partial_a S\;\Big]. \tag{3.6}$$

The world volume clock τ is defined as $X^+ = \frac{1}{\sqrt{2}}(X^0+X^{d-1}) = x^+ + p^+\tau$; x^+, p^+ are the center of mass position and momentum of the p-brane, respectively. X^I, for $I = 1,2,3,......,d-2$, are the transverse bosonic coordinates; S is a spinor of the $SO(d-2)$ transverse Lorentz group. The determinant is

$$det\;(\;h_{ab}\;)\;=\;\{\;X^{I_1},\;X^{I_2},\;.......,X^{I_p}\;\}\;\{\;X_{I_1},\;X_{I_2},\;.......,X_{I_p}\;\}. \tag{3.7}$$

the reduced γ-matrices satisfy

$$\{\gamma^I,\;\gamma^J\}\;=\;2\;\delta^{IJ},\;tr\;(\;\gamma^I\gamma^J\;)\;=\;(D-p-1)\;\delta^{IJ},\quad \gamma^2\;=\;det\;(\;h_{ab}\;). \tag{3.8}$$

the pull-back to the world volume of the anti-symmetrized products of the *gamma*'s is

$$\gamma_{a_1a_2.....a_n}\;=\;\partial_{a_1}\;X^{I_1}\;\partial_{a_2}\;X^{I_2}.......\partial_{a_n}\;X^{I_n}\;\gamma_{I_1I_2......I_n}. \tag{3.9}$$

$$\gamma\;=\;\frac{\eta}{p!}\;\epsilon^{a_1a_2.....a_p}\;\partial_{a_1}\;X^{I_1}\;\partial_{a_2}\;X^{I_2}.......\partial_{a_p}\;X^{I_p}\;\gamma_{I_1I_2......I_p},\;\;\eta = (-1)^{(p-1)(p+6)/4}. \tag{3.10}$$

The "covariant" time derivative is

$$\mathcal{D}_o\;X^I\;=\;\frac{\partial X^I}{\partial\tau}\;+\;\mathcal{A}^a_o\;\partial_a\;X^I(\tau;\sigma^1,\sigma^2,......,\sigma^p). \tag{3.11}$$

the gauge field $\mathcal{A}^a_o$ is required to satisfy

$$\partial_a\;\mathcal{A}^a_o\;=\;0\;\Rightarrow\;\mathcal{A}^a_o\;=\;\epsilon^{aa_1a_2......a_{p-1}}\;\partial_{a_1}A^1\;\partial_{a_2}A^2.......\partial_{a_{p-1}}A^{p-1}. \tag{3.12}$$

where $A^1, A^2, ..., A^{p-1}$ are auxiliary functions so that the derivative can be rewritten as

$$\mathcal{D}_o\ X^I\ =\ \frac{\partial X^I}{\partial \tau}\ +\ \mathcal{A}^a_o\ \partial_a\ X^I\ =\ \frac{\partial X^I}{\partial \tau}\ +\ \{\ A^1,\ A^2,\ ....,A^{p-1},\ X^I\ \}_{NPB}. \tag{3.13}$$

The action (3.6) has the following gauge invariance (for $p \geq 2$)

$$\delta X^I\ =\ \lambda^a\ \partial_a\ X^I,\ \ \delta S\ =\ \lambda^a\ \partial_a\ S$$

$$\delta \mathcal{A}^a_o\ =\ \partial_\tau\ \lambda^a\ +\ [\ \mathcal{A}_o,\ \lambda\ ]^a\ =\ \partial_\tau\ \lambda^a\ +\ \mathcal{A}^b_o\ \partial_b\ \lambda^a\ -\ \lambda^b\ \partial_b\ \mathcal{A}^a_o. \tag{3.14}$$

where the gauge parameter is required to obey

$$\partial_a\ \lambda^a\ =\ 0\ \Rightarrow\ \lambda^a\ =\ \epsilon^{aa_1a_2......a_{p-1}}\ \partial_{a_1}\Lambda^1\ \partial_{a_2}\Lambda^2.......\partial_{a_{p_1}}\Lambda^{p-1}. \tag{3.15}$$

where $\Lambda^1, \Lambda^2, ..., \Lambda^{p-1}$ are auxiliary functions so that eqs-(3.14) can be rewritten as

$$\delta X^I\ =\ \{\ \Lambda^1,\ \Lambda^2,\ ....,\Lambda^{p-1},\ X^I\ \}_{NPB}$$

$$\delta S\ =\ \{\ \Lambda^1,\ \Lambda^2,\ ....,\Lambda^{p-1},\ S\ \}_{NPB}$$

$$\delta \mathcal{A}^a_o\ =\ \partial_\tau\ \lambda^a + \{\ A^1,\ A^2,\ ....,A^{p-1},\ \lambda^a\ \} - \{\ \Lambda^1,\ \Lambda^2,\ ....,\Lambda^{p-1},\ \mathcal{A}^a_o\ \}. \tag{3.16}$$

For $p \geq 3$ there is an extra symmetry obtained by rewriting $\mathcal{A}^a_o = \partial_b\ \Upsilon^{[ab]}$, where

$$\Upsilon^{[ab]}\ =\ \epsilon^{a_1a_2.......a_{p-2}ba}\ A_{p-1}\ \partial_{a_1}A^1\ ....\ \partial_{a_{p-2}}A^{p-2}\ . \tag{3.17}$$

such that $\mathcal{A}^a_o$ is invariant under the transformation which is *not* of the Yang-Mills type

$$\Upsilon^{ab}\ \rightarrow\ \Upsilon^{ab}\ +\ \partial_c\ \Lambda^{[abc]}. \tag{3.18}$$

where $\Lambda^{[abc]}(\sigma^a)$ is an arbitrary function of σ^a.

After this brief review of super p-branes and the light-cone gauge one can return to the results of the previous section where (by dropping the measure factors for convenience) the dimensional reduction of the d-dim nonabelian gauge theory of 2-dim diffs of the internal space down to one temporal dimension has for action

$$\int\ dt\ Trace\ \mathcal{F}^a_{\mu\nu}\ \mathcal{F}^{\mu\nu}_a\ \Leftrightarrow$$

$$\int\ dt\ \int\ d^2\sigma\ \left(\ \partial_t\ X^I(t;\sigma^0,\sigma^1)\ +\ \{\ \mathcal{A}_t,\ X^I\ \}\right)^2\ +\ \{\ X^I,\ X^J\ \}\ \{\ X_I,\ X_J\ \}\ \ . \tag{3.19}$$

$I, J\ =\ 1, 2, 3, ..., d-1$. Despite that (3.19) has the same functional form as the light-cone gauge-fixed membrane action, the former action differs from the light-cone membrane action in several aspects : (i) The range of the target spacetime indices is now $1, 2, 3, ...d-1$ instead of the $1, 2, 3, ...d-2$ transverse coordinates in the light-cone gauge. (ii) the spacetime temporal variable t is no longer the same as the the world volume clock parameter τ of the light-cone gauge membrane action. (iii) σ^0, σ^1 are the two world sheet coordinates (one temporal and one spatial) associated with the motion of a string through the d-dim target spacetime, and *not* the two *spatial* coordinates of the 3-dim world volume of a membrane. (iv) there is no constraint imposed on the $\mathcal{A}^a_0$ gauge field like $\partial_a \ \mathcal{A}^a_0 = 0$.

The dimensional reduction of the d-dim rank-three antisymmetric nonabelian tensorial gauge theory of 3-dim diffs of the internal space down to one temporal dimension involves the action

$$\int dt \ Trace \ \mathcal{F}^a_{\mu\nu\rho} \ \mathcal{F}_a^{\mu\nu\rho} \ \Leftrightarrow$$

$$\int dt \int d^3\sigma \ \left(\partial_t \ X^{IJ} \ + \ \{ \ \mathcal{A}_t, \ X^I, X^J \ \}\right)^2 + \{ \ X^I, \ X^J, \ X^K \ \} \{ \ X_I, \ X_J, X_K \ \}. \tag{3.20}$$

The action (3.20) clearly does *not* correspond to the light-cone gauge-fixed action of a 3-brane since it contains the bi-vectors X^{IJ} and vector X^I coordinates. In general, after dropping measure factors, the dimensional reduction of the d-dim rank-$p+1$ antisymmetric nonabelian tensorial gauge theory of $p+1$-dim diffs of the internal space down to one temporal dimension involves the action

$$\int dt \ Trace \ \mathcal{F}^a_{\mu_1\mu_2....\mu_{p+1}} \ \mathcal{F}_a^{\mu_1\mu_2....\mu_{p+1}} \ \Leftrightarrow$$

$$\int dt \int d^{p+1}\sigma \ \left(\partial_t \ X^{I_1 I_2.....I_p} \ + \ \{ \ \mathcal{A}_t, \ X^{I_1}, \ X^{I_2}, \ ..... \ X^{I_p} \ \} \ \right)^2 \ +$$
$$\{ \ X^{I_1}, \ X^{I_2}, ..., \ X^{I_{p+1}} \ \} \{ \ X_{I_1}, \ X_{I_2}, ..., \ X_{I_{p+1}} \ \}. \tag{3.21}$$

The action (3.21) clearly does *not* correspond to the light-cone gauge-fixed action of a $p' = p+1$-brane since it contains the poly-vectors $X^{I_1 I_2....I_p}$ in addition to the vector X^I coordinates. Nevertheless, the action (3.21) can be recast in the same functional form as a gauged nonlinear sigma model plus the square of NPB terms, when

$$\mathcal{A}^{I_1 I_2.....I_p} \ \Leftrightarrow \ X^{I_1 I_2.....I_p} \ = \ \epsilon^{I_1 I_2.....I_{p+1}} \ X_{I_{p+1}}. \tag{3.22a}$$

after the mapping

$$[\ A_t, \ A^{I_1 I_2.....I_p} \ ]^a \ \partial_{y^a} \ \Leftrightarrow \ \{ \ \mathcal{A}_t, \ \mathcal{A}^{I_1 I_2.....I_p} \ \} \ \Leftrightarrow \ \epsilon^{I_1 I_2.....I_{p+1}} \ \{ \ \mathcal{A}_t, \ X_{I_{p+1}} \ \}. \tag{3.22b}$$

so that

$$\partial_t \ X^{I_1 I_2.....I_p} + \{ \ \mathcal{A}_t, \ X^{I_1 I_2.....I_p} \ \} \ = \ \epsilon^{I_1 I_2.....I_{p+1}} \ \left(\partial_t \ X_{I_{p+1}} \ + \ \{ \ \mathcal{A}_t, \ X_{I_{p+1}} \ \} \ \right). \tag{3.23}$$

upon squaring (3.23), dividing by $(p+1)!$ and adding the square of NPB terms, one arrives at

$$\int dt \ d^{p+1}\sigma \ [\left(\partial_t \ X_{I_{p+1}} \ + \ \{ \ \mathcal{A}_t, \ X_{I_{p+1}} \ \}\right)^2 \ + \{ \ X^{I_1}, \ X^{I_2}, ..., \ X^{I_{p+1}} \ \}^2]. \tag{3.24}$$

Upon rewriting

$$\{ \ \mathcal{A}_t, \ X_{I_{p+1}} \ \} \ = \ \omega^{ab} \ (\partial_a \ \mathcal{A}_t) \ (\partial_b \ X_{I_{p+1}}) \ = \ \tilde{A}_t^b \ \partial_b \ X_{I_{p+1}}, \quad \tilde{A}_t^b \ \equiv \ \omega^{ab} \ \partial_a \ \mathcal{A}_t, \tag{3.25}$$

one can see that the action in eq-(3.24) has the same functional form as the action of a one-dim *gauged* nonlinear sigma model based on the gauge group of diffs in $p+1$-dim, whose target spacetime variables are $X^I(t; \ \sigma^0, \sigma^1, ..., \sigma^p)$, the one-dim gauge field is $\tilde{A}_t^b(t; \ \sigma^0, \sigma^1, ..., \sigma^p)$; plus the NPB squared terms which play the role of the *potential* terms. The authors [36] have shown that when a p'-brane is extended over two topologically different spaces : over a compact d_1-dim and a non-compact d_2-dim region such that $p'+1 = d_1 + d_2$, the p'-brane action can be written as a d_2-dim *gauged* non-linear sigma model based on the group of diffs in d_1-dim, plus Nambu-brackets squared terms and a cosmological constant term involving an auxiliary field ω

$$\int \ d^{d_2} z \ \int \ d^{d_1} y \ \sqrt{|g|} \ g^{\alpha\beta} \ [\ (D_\alpha X^I) \ (D_\beta X_I) \ +$$

$$\frac{\omega^{d_2-1}}{4(d_1)!} \ \{X^{I_1}, X^{I_2}, ..., X^{I_{d_1}}\}^2_{NB} \ - \ (d_2 - 1) \ \omega \ ]. \tag{3.26a}$$

$$D_\alpha X^I \ = \ \partial_\alpha \ X^I \ + \ \tilde{A}^a_\alpha \ \partial_a \ X^I; \quad g_{\alpha\beta}(z^\alpha; \ y^a)$$

$$\alpha, \beta = 1, 2, ..., d_2; \quad a, b = 1, 2, ..., d_1; \quad X^I \ (z^\alpha; y^a); \quad \tilde{A}^a_\alpha \ (z^\beta; y^b). \tag{3.26b}$$

The physical interpretation of (3.26) is that of a p'-brane whose $p'+1$-dim world volume ($p'+1 = d_1 + d_2$) can be seen as a "condensate" of lower dimensional p''-branes such that $p''+1 = d_1$. When $d_2 = 1$, one can see that the action (3.26) has the same functional form as (3.24) after using eq-(3.25) and setting $g_{\alpha\beta} = \eta_{\alpha\beta}, \ p+1 = p''+1 = d_1 = p'$. When $d_2 = 0$, the action (3.26) reduces to the covariant form of the Eguchi-Schild p'-brane action ($p'+1 = d_1$) involving the auxiliary field ω.

To finalize this section we discuss why the large N limit of covariant Matrix Models based on generalized n-th power matrices [39] $\mathbf{X}_{i_1 i_2 ......i_n}$, that are extensions of square, cubic, quartic, ... matrices, should bear a relationship to Eguchi-Schild p-brane actions for $p+1 = n$. The range of indices is $i_1, i_2, ..., i_n \subset I = 1, 2, ...N$. By analogy to the results of section **1** for ordinary square matrices, the n-ary commutator of n generalized n-th power matrices in the large $N \rightarrow \infty$ limit should have a correspondence with the Nambu-brackets

$$[\ \mathbf{X}^1, \ \mathbf{X}^2, \ ...., \ \mathbf{X}^n \ ]_{i_1 i_2 ......i_n} \ \rightarrow \ \{ \ X^1, \ X^2, \ ...., \ X^n \ \}_{NB}. \tag{3.27}$$

by replacing $\mathbf{X}_{i_1 i_2 ......i_n}$ for the c-function of n-variables $X(\sigma^1, \sigma^2, ..., \sigma^n)$. The trace operation in the large N limit has a correspondence with the integral $\int d^n\sigma$ so that

$$Trace \ \left(\ [\ \mathbf{X}^1, \ \mathbf{X}^2, \ ...., \ \mathbf{X}^n \ ]^2 \ \right) \ \rightarrow \ \int \ d^n\sigma \ \{ \ X^1, \ X^2, \ ...., \ X^n \ \}^2_{NB}, \ n = p+1. \tag{3.28}$$

recovering in this fashion the Eguchi-Schild p-brane actions for $p+1=n$. The fermionic version of (3.28) is

$$\int d^n\sigma \; \bar{\Psi} \; \Gamma_{12....n-1} \; \{ \; X^1, \; X^2, \; ...., \; X^{n-1} \; , \Psi\}, \quad n = p+1. \tag{3.29}$$

The functional form of the terms (3.28, 3.29) is the same as that of the last two terms appearing in (3.6). Missing in (3.28, 3.29) is the corresponding kinetic terms of (3.6). A different approach to obtain super-p-brane actions than taking the large N limit of n-ary commutators of generalized matrices can be found in [36].

The deformation quantization of Nambu brackets for arbitrary values of n is not known at the present. For even n one could decompose the Nambu bracket into sums of products of ordinary Poisson brackets, then deform them into Moyal brackets and their products into Moyal products. For *odd* values of n this procedure does not work [40]. A deformation quantization of Nambu Classical Mechanics (brackets) to furnish Nambu Quantum Mechanics by constructing the n-ary analog of the Moyal bracket and the Weyl-Wigner-Groenowold-Moyal map among operators and c-functions remains an open problem.

4 Generalized p-brane actions in Clifford Spaces

The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [43]. For a comprehensive review we refer to [42] . A natural generalization of the notion of a space-time interval in Minkowski space to C-space is

$$dX^2 \;=\; dX_0 \; dX^0 \;+\; dx_\mu \; dx^\mu \;+\; dx_{\mu\nu} \; dx^{\mu\nu} \;+\; ...... \tag{4.1}$$

The Clifford valued poly-vector is defined by

$$X \;=\; X^M \; E_M \;=\; X^0 \; \mathbf{1} + x^\mu \; \gamma_\mu + x^{\mu\nu} \; \gamma_\mu \wedge \gamma_\nu + ... \; x^{\mu_1\mu_2....\mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \; .... \wedge \gamma_{\mu_D}. \tag{4.2}$$

denotes the position of a polyparticle in a manifold, called Clifford space or C-space. The series of terms in (4.2) terminates at a $finite$ value depending on the dimension D. A Clifford algebra $Cl(r,q)$ with $r+q=D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime

$$\frac{1}{2} \; \{\gamma^\mu, \gamma^\nu\} \;=\; \eta^{\mu\nu} \; \mathbf{1}. \tag{4.3}$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$. The multi-graded basis elements E_M of the Clifford-valued poly-vectors are

$$E_M \equiv \mathbf{1}, \quad \gamma^\mu, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3}, \quad \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \wedge ..... \wedge \gamma^{\mu_D}. \quad (4.4)$$

It is convenient to order the collective M indices as $\mu_1 < \mu_2 < \mu_3 < ...... < \mu_D$.

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x_{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x_{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetimr background. This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, ...D - 2$. The $p = 0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}....$ have been *coined* in the string-brane literature [46] as the *holographic* areas, volumes, ...projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes.

If we take the differential dX and compute the scalar product among two polyvectors $< dX^\dagger dX >_{scalar}$ [44] , [45] , [47] we obtain the C-space extension of the particles proper time in Minkowski space. The symbol $X^\dagger$ denotes the *reversion* operation and involves *reversing* the order of all the basis γ^μ elements in the expansion of X . It is the analog of the transpose (Hermitian) conjugation $(\gamma^\mu \wedge \gamma^\nu)^\dagger = \gamma^\nu \wedge \gamma^\mu$, etc... Therefore, the inner product can be rewritten as the scalar part of the geometric product as $< X^\dagger X >_{scalar}$. The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$< X'^\dagger \ X' >_{scalar} \ = \ < (R^{-1})^\dagger \ X^\dagger \ R^\dagger \ R \ X \ R^{-1} >_{scalar} =$$

$$< R \ X^\dagger \ X \ R^{-1} >_{scalar} \ = \ < X^\dagger \ X >_{scalar} =$$

$$(X^0)^2 + \Lambda^{2D-2} \ (x_\mu x^\mu) + \Lambda^{2D-4} \ (x_{\mu\nu} x^{\mu\nu}) + ... + (x_{\mu_1\mu_2.....\mu_D})(x^{\mu_1\mu_2.....\mu_D}) \quad (4.5)$$

we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to match units. The Planck scale can be set to unity for convenience.

This condition $R^\dagger = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R in eq-(2.5) because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p})^\dagger = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} ..... \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} ..... \wedge \gamma_{\mu_p} \quad (4.6)$$

Hence only those terms that *change* sign (under the reversal operation) are permitted in the exponential defining $R = exp[\theta^A E_A]$. For example, in $D = 4$, in order to satisfy the condition $R^\dagger = R^{-1}$, one must have from the behavior under the reversal operation expressed in eq-(4.6) that

$$R = exp\ [\ \theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2} + \theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}\ ]. \tag{4.7}$$

such that

$$R^\dagger = exp\ [\ \theta^{\mu_1\mu_2}(\gamma_{\mu_1} \wedge \gamma_{\mu_2})^\dagger + \theta^{\mu_1\mu_2\mu_3}(\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3})^\dagger\ ] =$$
$$exp\ [\ -\theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2} - \theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}\ ] = R^{-1}. \tag{4.8}$$

These transformations are the analog of Lorentz transformations in C-spaces which transform a poly-vector X into another poly-vector X' given by $X' = RXR^{-1}$. The theta parameters $\theta^{\mu_1\mu_2}, \theta^{\mu_1\mu_2\mu_3}$ are the C-space version of the Lorentz rotations/boosts parameters. The ordinary Lorentz rotation/boosts involves only the $\theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2}$ terms, because the Lorentz algebra generator can be represented as $\mathcal{M}^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$. The $\theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$ are the C-space corrections to the ordinary Lorentz transformations when $D = 4$.

The C-space invariant proper time associated with a polyparticle motion is then :

$$< dX^\dagger dX >_{scalar} = d\Sigma^2 = dX_0\, dX^0 + \Lambda^{2D-2} dx_\mu dx^\mu + \Lambda^{2D-4} dx_{\mu\nu} dx^{\mu\nu} + .. \tag{4.9}$$

Here we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops,..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkowski space interval: $ds^2 = c^2 dt^2 - dx_i dx^i$, a similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [43].

The author [44] has shown why the derivatives of the area-bivector coordinates $(dx^{\mu\nu}/ds)$ with respect to the ordinary spacetime proper time parameter $s = c\tau \neq ct$ (where $s \neq \Sigma$) can be identified with the spin $S^{\mu\nu}$ (per unit mass) and such that the poly-geodesic equation of a poly-particle leads to the terms of the Papapetrou equation coupling the curvature Riemann tensor to the spin $R^\rho_{\mu_1\mu_2\mu_3}\ S^{\mu_1\mu_2}\ (dx^{\mu_3}/ds)$. The introduction of generalized gravity in *curved* C-spaces involves area, volume, hypervolume metrics and leads to a higher derivative Gravity with Torsion. Area metrics were first introduced by Cartan long ago. A thorough discussion of superluminal behavior in ordinary spacetime while *not* being superluminal in C-space can be found in [42] and why there is no Einstein-Podolski-Rosen paradox in Clifford spaces can be seen in [49]. The analog of photons in C-space are *tensionless* branes. See [42] for further details about the Extended Relativity Theory in curved Clifford spaces and Grand Unification [50], [51]. References about Clifford algebras can be found in [48].

A Unified Theory of all p-Branes in C-Spaces can be constructed as follows. The generalization to C-spaces of string and p-brane actions as embeddings of world-manifolds onto target spacetime backgrounds involves the embeddings of polyvector-valued world-manifolds (of dimensions 2^d) onto polyvector-valued target spaces (of dimensions 2^D), given by the Clifford-valued maps $X = X(\Sigma)$ (see [45]). These are maps from the Clifford-valued world-manifold, parametrized by the polyvector-valued variables Σ, onto the Clifford-valued target space parametrized by the polyvector-valued coordinates X. Physically one envisions these maps as taking an n-dimensional simplicial cell (n-loop) of the world-manifold onto an m-dimensional simplicial cell (m-loop) of the target C-space manifold ; i.e. maps from n-dim objects onto m-dim objects generalizing the old maps of taking points onto points. One is basically dealing with a dimension-category of objects. The size of the simplicial cells (p-loops), upon quantization of a generalized harmonic oscillator, for example, are given by multiples of the Planck scale, in area, volume, hypervolume units or Clifford-bits.

In compact multi-index notation $X = X^M \Gamma_M$ one denotes for each one of the components of the target space polyvector X:

$$X^M \equiv X^{\mu_1 \mu_2 \ldots \mu_r}, \ \mu_1 < \mu_2 < \ldots < \mu_r \tag{4.10}$$

and for the world-manifold polyvector $\Sigma = \Sigma^A E_A$:

$$\Sigma^A \equiv \xi^{a_1 a_2 \ldots a_s}, \ a_1 < a_2 < \ldots\ldots < a_s. \tag{4.11}$$

where $\Gamma_M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, \ldots)$ and $E_A = (\underline{1}, e_a, e_{ab}, \ldots)$ form the basis of the target manifold and world manifold Clifford algebra, respectively. It is very important to order the indices within each multi-index M and A as shown above. The above Clifford-valued coordinates X^M, Σ^A correspond to antisymmetric tensors of ranks r, s in the target spacetime background and in the world-manifold, respectively.

There are many different ways to construct C-space brane actions which are on-shell equivalent to the analogs of the Dirac-Nambu-Goto action for extended objects and that are given by the world-volume spanned by the branes in their motion through the target spacetime background.

One of these actions is the Polyakov-Howe-Tucker action

$$I = \frac{T}{2} \int [D\Sigma] \sqrt{|H|} \left[H^{AB} (\partial_A X^M) (\partial_B X^N) G_{MN} + (2 - 2^d) \right]. \tag{4.12}$$

with the 2^d-dim world-manifold measure:

$$[D\Sigma] = (d\xi) (d\xi^a) (d\xi^{a_1 a_2}) (d\xi^{a_1 a_2 a_3}) \ldots.. \tag{4.13}$$

Upon the algebraic elimination of the auxiliary world-manifold metric H^{AB} from the action (4.12), via the equations of motion, yields for its on-shell solution the pullback of the target C-space metric onto the C-space world-manifold

$$H_{AB}(on - shell) = G_{AB} = \partial_A X^M \partial_B X^N G_{MN}. \tag{4.14}$$

upon inserting back the on-shell solutions (4.14) into (4.12) gives the Dirac-Nambu-Goto action for the C-space branes directly in terms of the C-space determinant, or measure, of the induced C-space world-manifold metric G_{AB}, as a result of the embedding

$$I = T \int [D\Sigma] \sqrt{\text{Det}\ (\partial_A\ X^M\ \partial_B\ X^N\ G_{MN})}. \tag{4.15}$$

However in C-space, the Polyakov-Howe-Tucker action admits an even further generalization that is comprised of two terms $S_1 + S_2$. The first term is [45]

$$S_1 = \int [D\Sigma]\ |E|\ E^A\ E^B (\partial_A\ X^M)\ (\partial_B\ X^N)\ \Gamma_M\ \Gamma_N. \tag{4.16}$$

Notice that this is a generalized action which is written in terms of the C-space coordinates $X^M(\Sigma)$ and the C-space analog of the target-spacetime vielbein/frame one-forms $e^m = e^m{}_\mu dx^\mu$ given by the Γ^M variables. The auxiliary world-manifold vielbein variables e^a, are given now by the Clifford-valued frame E^A variables.

In the conventional Polyakov-Howe-Tucker action, the auxiliary world-manifold metric h^{ab} associated with the standard p-brane actions is given by the usual scalar product of the frame vectors $e^a.e^b = e^a_\mu e^b_\nu g^{\mu\nu} = h^{ab}$. Hence, the C-space world-manifold metric H^{AB} appearing in (4.12) is given by scalar product $< (E^A)^\dagger\ E^B >_0\ = H^{AB}$, where $(E^A)^\dagger$ denotes the reversal operation of E^A which requires reversing the orderering of the vectors present in the Clifford aggregate E^A.

Notice, however, that the form of the action (4.16) is far more general than the action in (4.12). In particular, the S_1 itself can be decomposed futher into two additional pieces by rewriting the Clifford product of two basis elements into a symmetric plus an antisymmetric piece, respectively

$$E^A\ E^B = \frac{1}{2}\{\ E^A,\ E^B\ \} + \frac{1}{2}\ [\ E^A,\ E^B\ ]. \tag{4.17}$$

$$\Gamma_M\ \Gamma_N = \frac{1}{2}\ \{\ \Gamma_M,\ \Gamma_N\ \} + \frac{1}{2}\ [\Gamma_M,\ \Gamma_N\ ]. \tag{4.18}$$

In this fashion, the S_1 component has *two* kinds of terms. The first term containing the symmetric combination is just the analog of the standard non-linear sigma model action, and the second term is a Wess-Zumino-like term, containing the antisymmetric combination . To extract the non-linear sigma model part of the generalized action above, we may simply take the scalar product of the vielbein-variables as follows

$$(S_1)_{sigma} = \frac{T}{2} \int [D\Sigma]\ |E|\ <\ (E^A \partial_A\ X^M\ \Gamma_M)^\dagger\ (E^B\ \partial_B\ X^N\ \Gamma_N)\ >_0 . \tag{4.19}$$

where once again we have made use of the reversal operation (the analog of the hermitian adjoint) before contracting multi-indices. In this fashion we recover again the Clifford-scalar valued action given by [45]. Actions like the ones presented here in terms of derivatives with respect to quantities with multi-indices can be mapped to actions involving *higher* derivatives, in the same fashion that the C-space scalar curvature, the analog of the Einstein-Hilbert action, could be recast as a higher derivative gravity with torsion [42].

The S_2 (scalar) component of the C-space brane action is the usual cosmological constant term given by the C-space determinant $|E| = \det(H^{AB})$ based on the scalar part of the geometric product $< (E^A)^\dagger E^B >_0 = H^{AB}$

$$S_2 \;=\; \frac{T}{2} \int \; [D\Sigma] \; |E| \; (2 - 2^d) \qquad (4.20)$$

where the C-space determinant $|E| = \sqrt{|\det(H^{AB})|}$ of the $2^d \times 2^d$ generalized world-manifold metric H^{AB} is given by

$$\det(H^{AB}) \;=\; \frac{1}{(2^d)!} \; \epsilon_{A_1 A_2 .... A_{2^d}} \; \epsilon_{B_1 B_2 .... B_{2^d}} \; H^{A_1 B_1} \; H^{A_2 B_2} \; .... \; H^{A_{2^d} B_{2^d}}. \qquad (4.21)$$

The $\epsilon_{A_1 A_2 .... A_{2^d}}$ is the totally antisymmetric tensor density in C-space.

Lie algebra-valued differential form in C-space is

$$\mathbf{A} \;=\; \mathbf{A}_M \; dX^M \;=\; (\mathbf{A}^i_M T_i) \; dX^M. \qquad (4.25)$$

where T_i are the Lie algebra generators of the group with structure constants $[T_i, T_j] = c_{ij}{}^k \; T_k$. E_8 Yang-Mills theories based on Clifford-algebra-valued poly-vector gauge theories have been studied by [41]. When the gauge group is the diffeomorphisms of an internal $p{+}1$ dim space, on has for Clifford-algebra valued field strength

$$\mathbf{F}^a_{MN} \;=\; \partial_M \; \mathbf{A}^a_N \;-\; \partial_N \; \mathbf{A}^a_M \;+\; [\; \mathbf{A}_M, \; \mathbf{A}_N \;]^a. \qquad (4.26a)$$

where the poly-vector derivatives are

$$\partial_M \;=\; \partial_{X^0}, \; \partial_{x^\mu}, \; \partial_{x^{\mu\nu}}, \; \partial_{x^{\mu\nu\rho}}, ... \qquad (4.26b)$$

For instance, there is a very particular component that is relevant to the physics of p-branes

$$\mathbf{F}^a_{M0} \;=\; \mathbf{F}^a_{\mu_1 \mu_2 ...... \mu_m \; 0} \;=\; \partial_M \; \mathbf{A}^a_0 \;-\; \partial_0 \; \mathbf{A}^a_M \;+\; [\; \mathbf{A}_M, \; \mathbf{A}_0 \;]^a. \qquad (4.27)$$

where X^0 is the scalar component "direction" of the Clifford-polyvector. Despite that $\mathbf{F}^a_{\mu_1 \mu_2 ...... \mu_m \; 0} \; (\mathbf{X}; y^a)$ in C-space has the same index structure as an antisymmetric $F_{\mu_1 \mu_2 ...... \mu_m}(x; y^a)$ tensor in ordinary spacetime, there is a fundamental difference because the former has a dependence on the $\mathbf{X}$ Clifford poly-vector coordinates rather than on the ordinary spacetime x^μ coordinates. In the quenched approximation one freezes the dependence on the poly-vector degrees of freedom and truncates them by restricting the C-space field strength to depend *solely* on the coordinates y^a of the internal space. Therefore, one can perform the following map in the quenched case

$$[\ \mathbf{A}_{\mu_1\mu_2......\mu_m},\ \mathbf{A}_0\ ]^a\ \partial_a\ \Leftrightarrow\ F_{\mu_1\mu_2......\mu_m}\ =\ \{\ \mathcal{A}_{\mu_1},\ \mathcal{A}_{\mu_2},\ ....\ ,\mathcal{A}_{\mu_m}\ \}. \quad (4.28)$$

when the range of values of $a = 1, 2, 3, ..., n = p + 1$ coincides with the rank m of the field strength, $n = m$. The NPB are taken w.r.t the $p + 1$ coordinates $\sigma^0, \sigma^1, ..., \sigma^p$ of the p-brane.

There is yet another way to find a *different* correspondence than eq-(4.28) such that

$$[\ \mathbf{A}_M,\ \mathbf{A}_N\ ]^a\ \partial_a\ \Leftrightarrow\ \{\ X_{\mu_1\mu_2\ ....\mu_m},\ X_{\mu_1\mu_2\ ....\mu_n}\ \}\ =$$

$$\omega^{i_1 i_2}\ \frac{\partial X_{\mu_1\mu_2\ ....\mu_m}}{\partial\sigma^{i_1}}\ \frac{\partial X_{\mu_1\mu_2\ ....\mu_n}}{\partial\sigma^{i_2}};\quad i_1, i_2 \subset I\ =\ 1, 2, 3, ..., p + 1. \quad (4.29)$$

There are $(p + 1)p/2$ different combinations of indices in the evaluation of the bracket in (4.29). Using the generalized brackets defined in eq-(4.29) one has the interesting cases

$$[\ \mathbf{A}_M,\ \mathbf{A}_0\ ]^a\ \partial_a\ \Leftrightarrow\ \{\ X_{\mu_1\mu_2\ ....\mu_m},\ X_0\ \}\ =$$

$$\omega^{i_1 i_2}\ \frac{\partial X_{\mu_1\mu_2\ ....\mu_m}}{\partial\sigma^{i_1}}\ \frac{\partial X_0}{\partial\sigma^{i_2}}. \quad (4.30a)$$

$$[\ \mathbf{A}_M,\ \mathbf{A}_\mu\ ]^a\ \partial_a\ \Leftrightarrow\ \{\ X_{\mu_1\mu_2\ ....\mu_m},\ X_\mu\ \}\ =$$

$$\omega^{i_1 i_2}\ \frac{\partial X_{\mu_1\mu_2\ ....\mu_m}}{\partial\sigma^{i_1}}\ \frac{\partial X_\mu}{\partial\sigma^{i_2}}. \quad (4.30b)$$

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_0\ ]^a\ \partial_a\ \Leftrightarrow\ \{\ X_\mu,\ X_0\ \}\ =$$

$$\omega^{i_1 i_2}\ \frac{\partial X_\mu}{\partial\sigma^{i_1}}\ \frac{\partial X_0}{\partial\sigma^{i_2}}. \quad (4.30c)$$

And, in general, the generalized **n**-ary brackets among poly-vector valued coordinates are

$$\{\ X^{M_1},\ X^{M_2}, ..., \ X^{M_n}\ \}\ =\ \omega^{i_1 i_2 .... i_n}\ \frac{\partial X^{M_1}}{\partial\sigma^{i_1}}\ \frac{\partial X^{M_2}}{\partial\sigma^{i_2}}\ ......\ \frac{\partial X^{M_n}}{\partial\sigma^{i_n}}. \quad (4.31)$$

with $i_1, i_2, ..., i_n \subset I = 1, 2, 3, ...p + 1$ such that $n \leq p + 1$. Therefore the brackets terms in eq-(4.31) consist of a *hierarchy* of brackets of the form

$$\{\ X^{M_1},\ X^{M_2}\ \},\quad \{\ X^{M_1},\ X^{M_2},\ X^{M_3}\ \},\ ....\ ,\{\ X^{M_1},\ X^{M_2}, ..., \ X^{M_{p+1}}\ \}. \quad (4.32)$$

The poly-vectors $X^{M_1}, X^{M_2}, ..., X^{M_{p+1}}$ belong to the 2^D components of the Clifford poly-vector-valued coordinates $\mathbf{X}$ associated to the target 2^D-dim C-space, and involve stringy-like, membrane-like, ... , and actual p-brane terms (involving actual Nambu-Poisson brackets for the last term only) in on stroke.

Armed with the generalized brackets (4.31), the analog of the Eguchi-Schild p-brane action is now

$$S = \int d^{p+1}\sigma \; [\; \{ \; X^{M_1}, \; X^{M_2} \; \}^2 \; + \; \{ \; X^{M_1}, \; X^{M_2}, \; X^{M_3} \; \}^2 \; + \; .... \; + \; \{ \; X^{M_1}, \; X^{M_2}, ..., \; X^{M_{p+1}} \; \}^2 \;]. \tag{4.33}$$

Notice that the action in eq-(4.33) contains the *sum* of stringy-like, membrane-like, ... , and *actual* p-brane terms (the last term of (4.33)).

One can extend the construction of this work to the most general case of Lie derivatives along poly-vectors and related to the diffeomorphisms of the 2^{p+1}-dim Clifford space associated with the $p+1$-dim internal space. Instead of having generators of the form $\mathbf{A}^a_M \; \partial_a; \; a = 1, 2, ..., p+1$, one has now $\mathbf{A}^A_M \; \partial_A$ with A spanning $A = 1, 2, ..., 2^{p+1}$ corresponding to the 2^{p+1} dimensions of the C-space associated with the internal $p+1$-dim space. In this case, one has now a *hierarchy* of brackets of the form

$$\{ \; X^{M_1}, \; X^{M_2}, ..., \; X^{M_n} \; \} \; = \; \Omega^{A_1 A_2 ....A_n} \; \frac{\partial X^{M_1}}{\partial \sigma^{A_1}} \; \frac{\partial X^{M_2}}{\partial \sigma^{A_2}} \; ...... \; \frac{\partial X^{M_n}}{\partial \sigma^{A_n}}. \tag{4.34}$$

with poly-vector valued indices $A_1, A_2, ..., A_n \subset A = 1, 2, 3, ..., 2^{p+1}$, and $n \le 2^{p+1} \; \le 2^D$. The poly-vector-valued coordinates $X^{M_1}, X^{M_2}, ..., X^{M_{2^{p+1}}}$ of the target 2^D-dim C-space are now functions of the poly-vector-valued coordinates $\sigma^A = \sigma, \sigma^i, \sigma^{i_1 i_2}, \sigma^{i_1 i_2 i_3}, ...$ associated to the 2^{p+1}-dim C-space corresponding to the $p+1$-dim internal space. Hence, to sum up, $X^M(\sigma^A)$ represent maps of the 2^{p+1}-dim world manifold of the C-brane onto the target 2^D-dim C-space.

The analog of the Eguchi-Schild p-brane action has a similar form as eq-(4.33) with the main difference that the domain of integration is the 2^{p+1} dimensional measure of the C-brane, the brackets are given by eq-(4.34), where the last term is now given by $\{ \; X^{M_1}, X^{M_2}, ..., X^{M_{2^{p+1}}} \}^2$ involving Nambu-Poisson brackets and corresponds to an *actual* C-space brane (2^{p+1}-dimensional) embedded in a target 2^D-spacetime. This last term, by itself, contains the sum of $C^{2^D}_{2^{p+1}}$ terms, the latter being the binomial coefficient $(2^D)!/(2^{p+1})! \; (2^D - 2^{p+1})!$.

The author [52] has proposed a geometrical approach to strings and branes based on the Clifford Geometry of the configuration spaces of strings and branes. Gauge fields are encoded in the metric of Clifford space without to recur to the Kaluza-Klein program. It turns out that amongst the latter gauge fields there also exist higher grade, antisymmetric fields of the Kalb-Ramond type, and their non-Abelian generalization. All those fields are naturally coupled to the generalized branes, whose dynamics is given by a generalized Howe-Tucker action in curved C-space given by eq-(4.12). Having these results by [52] one can generalize the construction of [29] in section **2.1**, starting with the Clifford space extension of eq-(2.1) and ending with the Clifford space version of the decomposition of the scalar curvature in eq-(2.3). In this way one would be able to generate the required antisymmetric nonabelian tensorial gauge field strengths to build the action (2.40) (leading to p-brane actions) directly from a Clifford-space gravitational action.

To conclude, the physical content of the action (4.33) involving a *hierarchy* of generalized brackets (4.34), is richer than the Eguchi-Schild p-brane action, since the latter action is *contained* in the former. The same applies to the C-space branes. The impending project is to derive a hierarchy of p-brane actions directly from a C-space gravitational theory following a similar decomposition as eqs-(2.1,2.3).

Acknowledgments

We thank M. Bowers for her assistance and J.F. Gonzalez for discussions.

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