TGD AS A GENERALIZED NUMBER THEORY

Ph.D Matti Pitkänen

TGD AS A GENERALIZED NUMBER THEORY

Matti Pitkänen

Köydenpunojankatu D 11, 10900, Hanko, Finland

Preface

This book belongs to a series of online books summarizing the recent state Topological Geometrodynamics (TGD) and its applications. TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students at seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 37 years of my life to this enterprise and am still unable to write The Rules.

If I remember correctly, I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1977, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory. This required also the understanding of the relationship to General Relativity.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of imbedding space is 4-dimensional. During last year it became clear that 4-D Minkowski space and 4-D complex projective space CP_2 are completely unique in the sense that they allow twistor space with Kähler structure.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space (CP_2) providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, main stream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to same multiplet of the gauge group implying instability of proton.

There have been also longstanding problems.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. Furthermore, the imbeddings of Robertson-Walker cosmologies turned out to be vacuum extremals with respect to the inertial energy. About 25 years was needed to realize that the sign of the inertial energy can be also negative and in cosmological scales the density of inertial energy vanishes: physically acceptable universes are creatable from vacuum. Eventually this led to the notion of zero energy ontology (ZEO) which deviates dramatically from the standard ontology being however consistent with the crossing symmetry of quantum field theories. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of so called causal diamonds defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent.

  Equivalence Principle as it is expressed by Einstein's equations follows from Poincare invariance once it is realized that GRT space-time is obtained from the many-sheeted space-time of TGD by lumping together the space-time sheets to a regionof Minkowski space and endowing it with an effective metric given as a sum of Minkowski metric and deviations of the metrices of space-time sheets from Minkowski metric. Similar description relates classical gauge potentials identified as components of induced spinor connection to Yang-Mills gauge potentials in GRT space-time. Various topological inhomogenities below resolution scale identified as particles are described using energy momentum tensor and gauge currents.

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields.

  It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to dark matter. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem, and 32 years was needed to discover that what I call weak form of electric-magnetic duality gives a satisfactory solution of the problem and provides also surprisingly powerful insights to the mathematical structure of quantum TGD.

  The latest development was the realization that the well- definedness of electromagnetic charge as quantum number for the modes of the induced spinors field requires that the CP_2 projection of the region in which they are non-vanishing carries vanishing W boson field and is 2-D. This implies in the generic case their localization to 2-D surfaces: string world sheets and possibly also partonic 2-surfaces. This localization applies to all modes except covariantly constant right handed neutrino generating supersymmetry and mplies that string model in 4-D space-time is part of TGD. Localization is possible only for Kähler-Dirac assigned with Kähler action defining the dynamics of space-time surfaces. One must however leave open the question whether W field might vanish for the space-time of GRT if related to many-sheeted space-time in the proposed manner even when they do not vanish for space-time sheets.

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but the first discovery made already during first year of TGD was that these formalisms might be useless due to the extreme non-linearity and enormous vacuum degeneracy of the theory. This turned out to be the case.

- It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as 3-dimensional surfaces. Still few years had to pass before I understood that general coordinate invariance leads to a more or less unique solution of the problem and in positive energyontology implies that space-time surfaces are analogous to Bohr orbits. This in positive energy ontology in which space-like 3-surface is basic object. It is not clear whether Bohr orbitology is necessary also in ZEO in which space-time surfaces connect space-like 3-surfaces at the light-like boundaries of causal diamond CD obtained as intersection of future and past directed light-cones (with CP_2 factor included). The reason is that the pair of 3-surfaces replaces the boundary conditions at single 3-surface involving also time derivatives. If one assumes Bohr orbitology then strong correlations between the 3-surfaces at the ends of CD follow. Still a couple of years and I discovered that quantum states of the Universe can be identified as classical spinor fields in WCW. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

- TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. The somewhat cryptic statement "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly.

The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

If one requires consistency of Negentropy Mazimization Pronciple with standard measurement theory, negentropic entanglement defined in terms of number theoretic negentropy is necessarily associated with a density matrix proportional to unit matrix and is maximal and is characterized by the dimension n of the unit matrix. Negentropy is positive and maximal for a p-adic unique prime dividing n.

- One of the latest threads in the evolution of ideas is not more than nine years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. Second motivation for the hierarchy of Planck constants comes from bio-electromagnetism suggesting that in living systems Planck constant could have large values making macroscopic quantum coherence possible. The interpretation of dark matter as a hierarchy of phases of ordinary matter characterized by the value of Planck constant is very natural.

  During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant $h_{eff} = n \times h$ coming as a multiple of minimal value of Planck constant. Quite recently it became clear that the non-determinism of Kähler action is indeed the fundamental justification for the hierarchy: the integer n can be also interpreted as the integer characterizing the dimension of unit matrix characterizing negentropic entanglement made possible by the many-sheeted character of the space-time surface.

  Due to conformal invariance acting as gauge symmetry the n degenerate space-time sheets must be replaced with conformal equivalence classes of space-time sheets and conformal transformations correspond to quantum critical deformations leaving the ends of space-time surfaces invariant. Conformal invariance would be broken: only the sub-algebra for which conformal weights are divisible by n act as gauge symmetries. Thus deep connections between conformal invariance related to quantum criticality, hierarchy of Planck constants, negentropic entanglement, effective p-adic topology, and non-determinism of Kähler action perhaps reflecting p-adic non-determinism emerges.

  The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. It has indeed turned out that the dream about explicit formula is unrealistic before one has understood what happens in quantum jump. Symmetries and general physical principles have turned out to be the proper guide line here. To give some impressions about what is required some highlights are in order.

- With the emergence of ZEO the notion of S-matrix was replaced with M-matrix defined between positive and negative energy parts of zero energy states. M-matrix can be interpreted as a complex square root of density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in ZEO can be said to define a square root of thermodynamics at least formally. M-matrices in turn bombine to form the rows of unitary U-matrix defined between zero energy states.

- A decisive step was the strengthening of the General Coordinate Invariance to the requirement that the formulations of the theory in terms of light-like 3-surfaces identified as 3-surfaces at which the induced metric of space-time surfaces changes its signature and in terms of space-like 3-surfaces are equivalent. This means effective 2-dimensionality in the sense that partonic 2-surfaces defined as intersections of these two kinds of surfaces plus 4-D tangent space data at partonic 2-surfaces code for the physics. Quantum classical correspondence requires the coding of the quantum numbers characterizing quantum states assigned to the partonic 2-surfaces to the geometry of space-time surface. This is achieved by adding to the modified Dirac action a measurement interaction term assigned with light-like 3-surfaces.

- The replacement of strings with light-like 3-surfaces equivalent to space-like 3-surfaces means enormous generalization of the super conformal symmetries of string models. A further generalization of these symmetries to non-local Yangian symmetries generalizing the recently discovered Yangian symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is highly suggestive. Here the replacement of point like particles with partonic 2-surfaces means the replacement of conformal symmetry of Minkowski space with infinite-dimensional superconformal algebras. Yangian symmetry provides also a further refinement to the notion of conserved quantum numbers allowing to define them for bound states using non-local energy conserved currents.

- A further attractive idea is that quantum TGD reduces to almost topological quantum field theory. This is possible if the Kähler action for the preferred extremals defining WCW Kähler function reduces to a 3-D boundary term. This takes place if the conserved currents are so called Beltrami fields with the defining property that the coordinates associated with flow lines extend to single global coordinate variable. This ansatz together with the weak form of electric-magnetic duality reduces the Kähler action to Chern-Simons term with the condition that the 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric magnetic duality. It is the latter constraint which prevents the trivialization of the theory to a topological quantum field theory. Also the identification of the Kähler function of WCW as Dirac determinant finds support as well as the description of the scattering amplitudes in terms of braids with interpretation in terms of finite measurement resolution coded to the basic structure of the solutions of field equations.

- In standard QFT Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

  In TGD framework this generalization of Feynman diagrams indeed emerges unavoidably. Light-like 3-surfaces replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic 2-surfaces. Zero energy ontology and the interpretation of parton orbits as light-like "wormhole throats" suggests that virtual particle do not differ from on mass shell particles only in that the four- and three- momenta of wormhole throats fail to be parallel. The two throats of the wormhole contact defining virtual particle would contact carry on mass shell quantum numbers but for virtual particles the four-momenta need not be parallel and can also have opposite signs of energy.

The localization of the nodes of induced spinor fields to 2-D string world sheets (and possibly also to partonic 2-surfaces) implies a stringy formulation of the theory analogous to stringy variant of twistor formalism with string world sheets having interpretation as 2-braids. In TGD framework fermionic variant of twistor Grassmann formalism leads to a stringy variant of twistor diagrammatics in which basic fermions can be said to be on mass-shell but carry non-physical helicities in the internal lines. This suggests the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras.

What I have said above is strongly biased view about the recent situation in quantum TGD. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 32 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks.

Matti Pitkänen

Hanko,
September 16, 2014

Acknowledgements

Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 32 years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss about my work. I have had also stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Pekka Rapinoja has offered his help in this respect and I am especially grateful for him for my Python skills. Also Matti Vallinkoski has helped me in computer related problems.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation to CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. In particualr, I am grateful for Mark McWilliams and Ulla Matfolk for providing links to possibly interesting web sites and articles. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at least in principle leak to the publicity through the iron wall of the academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as individual. Homepage and blog are however not enough since only the formally published

result is a result in recent day science. Publishing is however impossible without a direct support from power holders- even in archives like arXiv.org.

Situation changed for five years ago as Andrew Adamatsky proposed the writing of a book about TGD when I had already got used to the thought that my work would not be published during my life time. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loop holes. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy. And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his sixty year birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from the society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During last few years when the right wing has held the political power this trend has been steadily strengthening. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

Matti Pitkänen

Hanko,
September 15, 2010

Contents

List of Figures

Chapter 1

Introduction

1.1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged 37 years ago - would emerge now it would be seen as an attempt trying to solve the difficulties of these approaches to unification.

The basic physical picture behind TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. The CMAP files at my homepage provide an overview about ideas and evolution of TGD and make easier to understand what TGD and its applications are about (`http://www.tgdtheory.fi/cmaphtml.html` [L18]).

1.1.1 Basic vision very briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K1].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional imbedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections ofCP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface.

3. Geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and

lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. Imbedding space H has a number theoretic interpretation as 8-D space allowing octonionic tangent space structure. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of imbedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particle in space-time can be identified as a topological inhomogenuity in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distance of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which standard model and general relativity follow as a topological simplification however forcing to increase dramatically the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The resolution of problem is implied by the condition that the modes of the induced spinor fields have well-defined electromagnetic charge. This forces their localization to 2-D string world sheets in the generic case having vanishing weak gauge fields so that parity breaking effects emerge just as they do in standard model. Also string model like picture emerges from TGD and one ends up with a rather concrete view about generalized Feynman diagrammatics.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last thirty seven years for the realization of this dream and this has resulted in eight online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

1.1.2 Two manners to see TGD and their fusion

As already mentioned, TGD can be interpreted both as a modification of general relativity and generalization of string models.

TGD as a Poincare invariant theory of gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A210, A164, A191, A154].

The identification of the space-time as a sub-manifold [A138, A207] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

TGD as a generalization of the hadronic string model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Fusion of the two approaches via a generalization of the space-time concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possibly existence vapour phase.

What one obtains is what I have christened as many-sheeted space-time (see fig. `http://www.tgdtheory.fi/appfigures/manysheeted.jpg` or fig. 9 in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory system does not possess this kind of field identity. The notion of magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian and interpreted as lines of generalized Feynman diagrams. Also the Euclidian 4-D regions would have similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about strong form of holography.

1.1.3 Basic objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four imbedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particle topologically condenses to several space-time sheets simultaneously and experiences the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the imbeddability to 8-D imbedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation allows to understand the relationship to GRT space-time and how Equivalence Principle (EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrices of space-time sheets from Minkowski metric. Poincare invariance suggests strongly classical EP for the GRT limit in long length scales at least. One can consider also other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

Topological field quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.1.4 p-Adic variants of space-time surfaces

There is a further generalization of the space-time concept inspired by p-adic physics forcing a generalization of the number concept through the fusion of real numbers and various p-adic number fields. Also the hierarchy of Planck constants forces a generalization of the notion of space-time but this generalization can be understood in terms of the failure of strict determinism for Kähler action defining the fundamental variational principle behind the dynamics of space-time surfaces.

A very concise manner to express how TGD differs from Special and General Relativities could be following. Relativity Principle (Poincare Invariance), General Coordinate Invariance, and Equivalence Principle remain true. What is new is the notion of sub-manifold geometry: this allows to realize Poincare Invariance and geometrize gravitation simultaneously. This notion also allows a geometrization of known fundamental interactions and is an essential element of all applications of TGD ranging from Planck length to cosmological scales. Sub-manifold geometry is also crucial in the applications of TGD to biology and consciousness theory.

1.1.5 The threads in the development of quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

The theoretical framework involves several threads.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name 'TGD as a generalized number theory'. The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and extremely fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to four.

TGD forces the generalization of physics to a quantum theory of consciousness, and represent TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations. The eight online books [K77, K59, K52, K92, K69, K91, K90, K67] about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology [K73, K13, K55, K12, K32, K37, K40, K66, K88] are warmly recommended to the interested reader.

Quantum TGD as spinor geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones:

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude in the configuration space CH ("world of classical worlds",WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included. Particle reactions are identified as topology changes [A188, A213, A215]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B + C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

2. During years this naive and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects unexpected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

3. WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operator, appearing in the field equations of the theory [1]. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operator of WCW so that this classical free field theory would dictate M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. Given M-matrix in turn would decompose to a product of a hermitian density matrix and unitary S-matrix.

   M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the complex square roots of density matrices commuting with S-matrix means that they span infinite-dimensional Lie algebra acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in well-defined sense: its own symmetries would define the symmetries of the theory. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian

[1] There are four kinds of Dirac operators in TGD. WCW Dirac operator appearing in Super-Virasoro conditions, imbedding space Dirac operator whose modes define the ground states of Super-Virasoro representations, Kähler-Dirac operator at space-time surfaces, and the algebraic variant of M^4 Dirac operator appearing in propagators

square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible.

4. By quantum classical correspondence the construction of WCW spinor structure reduces to the second quantization of the induced spinor fields at space-time surface. The basic action is so called modified Dirac action (or Kähler-Dirac action) in which gamma matrices are replaced with the modified (Kähler-Dirac) gamma matrices defined as contractions of the canonical momentum currents with the imbedding space gamma matrices. In this manner one achieves super-conformal symmetry and conservation of fermionic currents among other things and consistent Dirac equation. The modified gamma matrices define as anti-commutators effective metric, which might provide geometrization for some basic observables of condensed matter physics. One might also talk about bosonic emergence in accordance with the prediction that the gauge bosons and graviton are expressible in terms of bound states of fermion and anti-fermion.

5. An important result relates to the notion of induced spinor connection. If one requires that spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrino generating super-symmetries forms an exception. The vanishing of also Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization simplifies enormously the mathematics and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces at which the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes one can pose the condition that the algebraic analog of massless Dirac equation is satisfied by the nodes so that Kähler-Dirac action gives massless Dirac propagator localizable at the boundaries of the string world sheets.

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The manner to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak

form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this manner almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worldss", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name 'TGD as a generalized number theory'. It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of hyper-counterparts of classical number fields identified as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product, and the notion of infinite prime.

1. p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired 'Universe as Computer' vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.

2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that

clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

The notion of p-adic manifold [K95] identified as p-adic space-time surface solving p-adic analogs of field equations and having real space-time sheets as chart maps provides a possible solution of the basic challenge. One can also speak of real space-time surfaces having p-adic space-time surfaces as chart maps (cognitive maps, "thought bubbles"). Discretization required having interpretation in terms of finite measurement resolution is unavoidable in this approach.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structures. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of imbedding space and space-time concept and one can speak about real and p-adic space-time sheets. The quantum dynamics should be such that it allows quantum transitions transforming space-time sheets belonging to different number fields to each other. The space-time sheets in the intersection of real and p-adic worlds are of special interest and the hypothesis is that living matter resides in this intersection. This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see fig. `http://www.tgdtheory.fi/appfigures/cat.jpg` or fig. 21 in the appendix of this book).

The basic principle is number theoretic universality stating roughly that the physics in various number fields can be obtained as completion of rational number based physics to various number fields. Rational number based physics would in turn describe physics in finite measurement resolution and cognitive resolution. The notion of finite measurement resolution has become one of the basic principles of quantum TGD and leads to the notions of braids as representatives of 3-surfaces and inclusions of hyper-finite factors as a representation for finite measurement resolution. The braids actually co-emerge with string world sheets implied by the condition that em charge is well-defined for spinor modes.

2. The role of classical number fields

The vision about the physical role of the classical number fields relies on certain speculative questions inspired by the idea that space-time dynamics could be reduced to associativity or co-associativity condition. Associativity means here associativity of tangent spaces of space-time region and co-associativity associativity of normal spaces of space-time region.

1. Could space-time surfaces X^4 be regarded as associative or co-associative ("quaternionic" is equivalent with "associative") surfaces of H endowed with octonionic structure in the sense that tangent space of space-time surface would be associative (co-associative with normal space associative) sub-space of octonions at each point of X^4 [K72]. This is certainly possible and an interesting conjecture is that the preferred extremals of Kähler action include associative and co-associative space-time regions.

2. Could the notion of compactification generalize to that of number theoretic compactification in the sense that one can map associative (co-associative) surfaces of M^8 regarded as octonionic linear space to surfaces in $M^4 \times CP_2$ [K72]? This conjecture - $M^8 - H$ duality - would give for $M^4 \times CP_2$ deep number theoretic meaning. CP_2 would parametrize associative planes of octonion space containing fixed complex plane $M^2 \subset M^8$ and CP_2 point would thus characterize the tangent space of $X^4 \subset M^8$. The point of M^4 would be obtained

by projecting the point of $X^4 \subset M^8$ to a point of M^4 identified as tangent space of X^4. This would guarantee that the dimension of space-time surface in H would be four. The conjecture is that the preferred extremals of Kähler action include these surfaces.

3. $M^8 - H$ duality can be generalized to a duality $H \rightarrow H$ if the images of the associative surface in M^8 is associative surface in H. One can start from associative surface of H and assume that it contains the preferred M^2 tangent plane in 8-D tangent space of H or integrable distribution $M^2(x)$ of them, and its points to H by mapping M^4 projection of H point to itself and associative tangent space to CP_2 point. This point need not be the original one! If the resulting surface is also associative, one can iterate the process indefinitely. WCW would be a category with one object.

4. G_2 defines the automorphism group of octonions, and one might hope that the maps of octonions to octonions such that the action of Jacobian in the tangent space of associative or co-associative surface reduces to that of G_2 could produce new associative/co-associative surfaces. The action of G_2 would be analogous to that of gauge group.

5. One can also ask whether the notions of commutativity and co-commutativity could have physical meaning. The well-definedness of em charge as quantum number for the modes of the induced spinor field requires their localization to 2-D surfaces (right-handed neutrino is an exception) - string world sheets and partonic 2-surfaces. This can be possible only for Kähler action and could have commutativity and co-commutativity as a number theoretic counterpart. The basic vision would be that the dynamics of Kähler action realizes number theoretical geometrical notions like associativity and commutativity and their co-notions.

The notion of number theoretic compactification stating that space-time surfaces can be regarded as surfaces of either M^8 or $M^4 \times CP_2$. As surfaces of M^8 identifiable as space of hyper-octonions they are hyper-quaternionic or co-hyper-quaternionic- and thus maximally associative or co-associative. This means that their tangent space is either hyper-quaternionic plane of M^8 or an orthogonal complement of such a plane. These surface can be mapped in natural manner to surfaces in $M^4 \times CP_2$ [K72] provided one can assign to each point of tangent space a hyper-complex plane $M^2(x) \subset M^4 \subset M^8$. One can also speak about $M^8 - H$ duality.

This vision has very strong predictive power. It predicts that the preferred extremals of Kähler action correspond to either hyper-quaternionic or co-hyper-quaternionic surfaces such that one can assign to tangent space at each point of space-time surface a hyper-complex plane $M^2(x) \subset M^4$. As a consequence, the M^4 projection of space-time surface at each point contains $M^2(x)$ and its orthogonal complement. These distributions are integrable implying that space-time surface allows dual slicings defined by string world sheets Y^2 and partonic 2-surfaces X^2. The existence of this kind of slicing was earlier deduced from the study of extremals of Kähler action and christened as Hamilton-Jacobi structure. The physical interpretation of $M^2(x)$ is as the space of non-physical polarizations and the plane of local 4-momentum.

Number theoretical compactification has inspired large number of conjectures. This includes dual formulations of TGD as Minkowskian and Euclidian string model type theories, the precise identification of preferred extremals of Kähler action as extremals for which second variation vanishes (at least for deformations representing dynamical symmetries) and thus providing space-time correlate for quantum criticality, the notion of number theoretic braid implied by the basic dynamics of Kähler action and crucial for precise construction of quantum TGD as almost-topological QFT, the construction of WCW metric and spinor structure in terms of second quantized induced spinor fields with modified Dirac action defined by Kähler action realizing the notion of finite measurement resolution and a connection with inclusions of hyper-finite factors of type II_1 about which Clifford algebra of WCW represents an example.

The two most important number theoretic conjectures relate to the preferred extremals of Kähler action. The general idea is that classical dynamics for the preferred extremals of Kähler action should reduce to number theory: space-time surfaces should be either associative or co-associative in some sense.

Associativity (co-associativity) would be that tangent (normal) spaces of space-time surfaces associative (co-associative) in some sense and thus quaternionic (co-quaternionic). This can be formulated in two manners.

1. One can introduce octonionic tangent space basis by assigning to the "free" gamma matrices octonion basis or in terms of octonionic representation of the imbedding space gamma matrices possible in dimension $D = 8$.

2. Associativity (quaternionicity) would state that the projections of octonionic basic vectors or induced gamma matrices basis to the space-time surface generates associative (quaternionic) sub-algebra at each space-time point. Co-associativity is defined in analogous manner and can be expressed in terms of the components of second fundamental form.

3. For gamma matrix option induced rather than modified gamma matrices must be in question since modified gamma matrices can span lower than 4-dimensional space and are not parallel to the space-time surfaces as imbedding space vectors.

3. Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of rational (hyper-)quaternions and (hyper-)octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

What is especially interesting is that p-adic and real regions of the space-time surface might aso emerge automatically as solutions of the field equations. In the space-time regions where the solutions of field equations give rise to in-admissible complex values of the imbedding space coordinates, p-adic solution can exist for some values of the p-adic prime. The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to 'mind stuff', the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

1.1.6 Hierarchy of Planck constants and dark matter hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark matter as large $\hbar$ phases

D. Da Rocha and Laurent Nottale [E3] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K64] .

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Hierarchy of Planck constants from the anomalies of neuroscience and biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K56, K57, K87]) support the view that dark matter might be a key player in living matter.

Does the hierarchy of Planck constants reduce to the vacuum degeneracy of Kähler action?

This starting point led gradually to the recent picture in which the hierarchy of Planck constants is postulated to come as integer multiples of the standard value of Planck constant. Given integer multiple $\hbar = n\hbar_0$ of the ordinary Planck constant $\hbar_0$ is assigned with a multiple singular covering of the imbedding space [K25]. One ends up to an identification of dark matter as phases with non-standard value of Planck constant having geometric interpretation in terms of these coverings providing generalized imbedding space with a book like structure with pages labelled by Planck constants or integers characterizing Planck constant. The phase transitions changing the value of Planck constant would correspond to leakage between different sectors of the extended imbedding space. The question is whether these coverings must be postulated separately or whether they are only a convenient auxiliary tool.

The simplest option is that the hierarchy of coverings of imbedding space is only effective. Many-sheeted coverings of the imbedding space indeed emerge naturally in TGD framework. The huge vacuum degeneracy of Kähler action implies that the relationship between gradients of the imbedding space coordinates and canonical momentum currents is many-to-one: this was the very fact forcing to give up all the standard quantization recipes and leading to the idea about physics as geometry of the "world of classical worlds". If one allows space-time surfaces for which all sheets corresponding to the same values of the canonical momentum currents are present, one obtains effectively many-sheeted covering of the imbedding space and the contributions from sheets to the Kähler action are identical. If all sheets are treated effectively as one and the same sheet, the value of Planck constant is an integer multiple of the ordinary one. A natural boundary condition would be that at the ends of space-time at future and past boundaries of causal diamond containing the space-time surface, various branches co-incide. This would raise the ends of space-time surface in special physical role.

A more precise formulation is in terms of presence of large number of space-time sheets connecting given space-like 3-surfaces at the opposite boundaries of causal diamond. Quantum criticality presence of vanishing second variations of Kähler action and identified in terms of conformal invariance broken down to to sub-algebras of super-conformal algebras with conformal weights divisible by integer n is highly suggestive notion and would imply that n sheets of the effective covering are actually conformal equivalence classes of space-time sheets with same Kähler action and same values of conserved classical charges (see fig. `http://www.tgdtheory.fi/appfigures/planckhierarchy.jpg`, which is also in the appendix of this book). n would naturally correspond the value of h_{eff} and its factors negentropic entanglement with unit density matrix would be between the n sheets of two coverings of this kind. p-Adic prime would be largest prime power factor of n.

Dark matter as a source of long ranged weak and color fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of biochemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.2 Bird's eye of view about the topics of the book

The focus of this book is the number theoretical vision about physics. This vision involves three loosely related parts.

1. The fusion of real physic and various p-adic physics to a single larger whole by generalizing the number concept by fusing real numbers and various p-adic number fields along common rationals. Extensions of p-adic number fields can be introduced by gluing them along common algebraic numbers to reals. Algebraic continuation of the physics from rationals and their their extensions to various number fields (completion of rational physics to physics in various number fields) is the key idea and the challenge is to understand whether how one could achieve this dream. A very profound implication is that purely local p-adic physics codes for the p-adic fractality of long length length scale real physics and vice versa. As a consequence

one can understand the origins of p-adic length scale hypothesis and ends up with a very concrete view about space-time correlates of cognition and intentionality.

2. Second part of the vision involves what I call hyper counterparts of the classical number fields defined as subspaces of their complexifications with Minkowskian signature of the metric. The hypothesis is that allowed space-time surfaces correspond to what might be called hyper-quaternionic sub-manifolds of a hyper-octonionic space. Second hypothesis is that space-time surfaces can be also regarded hyper-quaternionic sub-manifolds of the hyper-octonionic imbedding space. This means that one can assign to each point of space-time surface a hyper-quanternionic 4-plane which is the plane defined by the modified gamma matrices and co-incides with tangent plane only for action defined by the metric determinant. Hence the basic variational principle of TGD would have deep number theoretic content. Reduction to a closed form would also mean that classical TGD would define a generalized topological field theory with Noether charges defining topological invariants.

3. The third part of the vision involves infinite primes, which can be identified in terms of an infinite hierarchy of second quantized arithmetic quantum fields theories on one hand, and as having representations as space-time surfaces analogous to zero surfaces of polynomials on the other hand. In this framework space-time surface would represent an infinite number. This vision leads also the conclusion that single point of space-time has an infinitely complex structure since real unity can be represented as a ratio of infinite numbers in infinitely many manners each having its own number theoretic anatomy. Thus single space-time point is in principle able to represent in its structure the quantum state of the entire universe. This number theoretic variant of Brahman=Atman identity also means that Universe is an algebraic hologram.

   Besides this holy trinity I will discuss also loosely related topics. Included are the possible applications of the category theory in TGD and in TGD inspired theory of consciousness; various TGD inspired considerations related to Riemann hypothesis - in particular, a strategy for proving Riemann hypothesis using a modification of Hilbert-Polya conjecture replacing quantum states with coherent states of a unique conformally invariant physical system; topological quantum computation in TGD Universe; and TGD inspired approach to Langlands program.

1.3 Sources

The eight online books about TGD [K77, K59, K92, K69, K52, K91, K90, K67] and nine online books about TGD inspired theory of consciousness and quantum biology [K73, K13, K55, K12, K32, K37, K40, K66, K88] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (`http://www.tgdtheory.com/curri.html`) contains a lot of material about TGD. In particular, there is summary about TGD and its applications using CMAP representation serving also as a TGD glossary [L18, L19] (see `http://www.tgdtheory.fi/cmaphtml.html` and `http://www.tgdtheory.fi/tgdglossary.pdf`).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (`http://journals.sfu.ca/jnonlocality/index.php/jnonlocality` founded by Lian Sidorov and in *Prespacetime Journal* (`http://prespacetime.com`), *Journal of Consciousness Research and Exploration* (`https://www.createspace.com/4185546`), and *DNA Decipher Journal* (`http://dnadecipher.com`), all of them founded by Huping Hu. One can find the list about the articles published at `http://www.tgdtheory.com/curri.html`. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

1.4 The contents of the book

1.4.1 PART I: Number theoretical vision

TGD as a Generalized Number Theory I: p-Adicization Program

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation.

The first part of the 3-part chapter is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

1. Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the imbedding space coordinates are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics could emerge from the basic equations of the theory. One could interpret the resulting rational-adic or p-adic regions as geometrical correlates for 'mind stuff'.

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2. The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

3. Number theoretical Universality and number theoretical criticality

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix (generalization of S-matrix) so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field - number theoretical criticality - becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough.

4. p-Adicization by algebraic continuation

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. It must be however emphasized that for weaker form of number theoretical universality this restriction applies only at number theoretical quantum criticality. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality. The basic stumbling block of this program is integration and algebraic continuation should allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution. Zero energy ontology provides a natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

5. Number theoretic democracy

The interpretation allows all finite-dimensional extensions of p-adic number fields and perhaps even infinite-P p-adics. The notion arithmetic quantum theory generalizes to include Gaussian and Eisenstein variants of infinite primes and corresponding arithmetic quantum field theories. Also the notion of p-adicity generalizes: it seems that one can indeed assign to Gaussian and Eisenstein primes what might be called G-adic and E-adic numbers.

p-Adicization by algebraic continuation gives hopes of continuing quantum TGD from reals to various p-adic number fields. The existence of this continuation poses extremely strong constraints on theory.

TGD as a Generalized Number Theory II: Quaternions, Octonions and their Hyper Counterparts

This chapter is second one in a multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme is the role of classical number fields in quantum TGD. A central notion is $M^8 - H$ duality which might be also called number theoretic compactification. This duality allows to identify imbedding space equivalently either as M^8 or $M^4 \times CP_2$ and explains the symmetries of standard model number theoretically. These number theoretical symmetries induce also the symmetries dictaging the geometry of the "world of classical worlds" (WCW) as a union of symmetric spaces. This infinite-dimensional Kähler geometry is expected to be highly unique from the mere requirement of its existence requiring infinite-dimensional symmetries provided by the generalized conformal symmetries of the light-cone boundary $\delta M^4_+ \times S$ and of light-like 3-surfaces and the answer to the question what makes 8-D imbedding space and $S = CP_2$ so unique would be the reduction of these symmetries to number theory.

Zero energy ontology has become the corner stone of both quantum TGD and number theoretical vision. In zero energy ontology either light-like or space-like 3-surfaces can be identified as the fundamental dynamical objects, and the extension of general coordinate invariance leads to effective 2-dimensionality (strong form of holography) in the sense that the data associated with partonic 2-surfaces and the distribution of 4-D tangent spaces at them located at the light-like boundaries of causal diamonds (CDs) defined as intersections of future and past directed light-cones code for quantum physics and the geometry of WCW.

The basic number theoretical structures are complex numbers, quaternions and octonions, and their complexifications obtained by introducing additional commuting imaginary unit $\sqrt{-1}$. Hyper-octonionic (-quaternionic,-complex) sub-spaces for which octonionic imaginary units are multiplied by commuting $\sqrt{-1}$ have naturally Minkowskian signature of metric. The question is whether and how the hyper-structures could allow to understand quantum TGD in terms of classical number fields. The answer which looks the most convincing one relies on the existence of octonionic representation of 8-D gamma matrix algebra.

1. The first guess is that associativity condition for the sub-algebras of the local Clifford algebra defined in this manner could select 4-D surfaces as associative (hyper-quaternionic) sub-spaces of this algebra and define WCW purely number theoretically. The associative sub-spaces in question would be spanned by the modified gamma matrices defined by the modified Dirac action fixed by the variational principle (Kähler action) selecting space-time surfaces as preferred extremals.

2. This condition is quite not enough: one must strengthen it with the condition that a preferred commutative and thus hyper-complex sub-algebra is contained in the tangent space of the space-time surface. This condition actually generalizes somewhat since one can introduce a family of so called Hamilton-Jacobi coordinates for M^4 allowing an integrable distribution of decompositions of tangent space to the space of non-physical and physical polarizations. The physical interpretation is as a number theoretic realization of gauge invariance selecting a preferred local commutative plane of non-physical polarizations.

3. Even this is not yet the whole story: one can define also the notions of co-associatitivy and co-commutativity applying in the regions of space-time surface with Euclidian signature of the induced metric. The basic unproven conjecture is that the decomposition of space-time surfaces to associative and co-associative regions containing preferred commutative *resp.* co-commutative 2-plane in the 4-D tangent plane is equivalent with the preferred extremal property of Kähler action and the hypothesis that space-time surface allows a slicing by string world sheets and by partonic 2-surfaces.

TGD as a Generalized Number Theory III: Infinite Primes

Infinite primes are besides p-adicization and the representation of space-time surface as a hyper-quaternionic sub-manifold of hyper-octonionic space the basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible.

1. Why infinite primes are unavoidable

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at a lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

2. Two views about the role of infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1. The first view is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D hyper-octonions make this correspondence very concrete since 8-D hyper-octonions have interpretation as 8-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to p-adic space-time sheets should be coded by infinite hyper-octonionic primes. Infinite primes could even have a representation as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic imbedding space.

2. The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

3. Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

It turns out that associativity constraint allows only rational infinite primes. One can however replace classical associativity with quantum associativity for quantum states assigned with infinite prime. One can also decompose rational infinite primes to hyper-octonionic infinite primes at lower level of the hierarchy. Physically this would mean that the number theoretic 8-momenta have only time-component. This decomposition is completely analogous to the decomposition of hadrons to its colored constituents and might be even interpreted in terms of color confinement. The interpretation of the decomposition of rational primes to primes in the algebraic extensions of rationals, hyper-quaternions, and hyper-octonions would have an interpretation as an increase of number theoretical resolution and the principle of number theoretic confinement could be seen as a fundamental physical principle implied by associativity condition.

4. Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. Note that this conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

The representation of space-time surfaces as algebraic surfaces in M^8 is however too naive idea and the attempt to map hyper-octonionic infinite primes to algebraic surfaces seems has not led to any concrete progress.

The endless updating of quantum TGD might be blamed to be a waste of time. The interaction of new ideas with old ones has however again and again turned out to be an extremely fruitful process leading to rather precise view about how infinite hyper-octonionic rationals can be mapped to space-time surfaces without ad hoc assumptions. The progress in quantum TGD during the second half of the first decade of the new millenium led to several new and quite convincing ideas. Mention only zero energy ontology, the generalization of the imbedding space concept realizing the hierarchy of Planck constants, hyper-finite factors and their inclusions, and in particular, the realization of quantum classical correspondence in terms of measurement interaction term associated with the modified Dirac action.

The crucial observation is that quantum classical correspondence allows to map quantum numbers of configuration space spinor fields to space-time geometry. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries.

5. Generalization of ordinary number fields: infinite primes and cognition

Both fermions and p-adic space-time sheets are identified as correlates of cognition in TGD Universe. The attempt to relate these two identifications leads to a rather concrete model for how bosonic generators of super-algebras correspond to either real or p-adic space-time sheets (actions and intentions) and fermionic generators to pairs of real space-time sheets and their p-adic variants obtained by algebraic continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real and also more general units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is completely invisible and is relevant only for the physics of cognition. On the other hand, one can consider the possibility of mapping the configuration space and configuration space spinor fields to the number theoretical anatomies of a single point of imbedding space so that the structure of this point would code for the world of classical worlds and for the quantum states of the Universe. Quantum jumps would induce changes of configuration space spinor fields interpreted as wave functions in the set of number theoretical anatomies of single point of imbedding space in the ordinary sense of the word, and evolution would reduce to the evolution of the structure of a typical space-time point in the system. Physics would reduce to space-time level but in a generalized sense. Universe would be an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

Infinite rationals are in one-one correspondence with quantum states and in zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

1.4.2 PART II: TGD and p-Adic Numbers

p-Adic Numbers and Generalization of Number Concept

In this chapter the general TGD inspired mathematical ideas related to p-adic numbers are discussed. The extensions of the p-adic numbers including extensions containing transcendentals, the correspondences between p-adic and real numbers, p-adic differential and integral calculus, and p-adic symmetries and Fourier analysis belong the topics of the chapter.

The basic hypothesis is that p-adic space-time regions correspond to cognitive representations for the real physics appearing already at the elementary particle level. The interpretation of the p-adic physics as a physics of cognition is justified by the inherent p-adic non-determinism of the p-adic differential equations making possible the extreme flexibility of imagination.

p-Adic canonical identification and the identification of reals and p-adics by common rationals are the two basic identification maps between p-adics and reals and can be interpreted as two basic types of cognitive maps. The concept of p-adic fractality is defined and p-adic fractality is the basic property of the cognitive maps mapping real world to the p-adic internal world. Canonical identification is not general coordinate invariant and at the fundamental level it is applied only to map p-adic probabilities and predictions of p-adic thermodynamics to real numbers. The correspondence via common rationals is general coordinate invariant correspondence when general coordinate transformations are restricted to rational or extended rational maps: this has interpretation in terms of fundamental length scale unit provided by CP_2 length.

A natural outcome is the generalization of the notion of number. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and configuration space are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

p-Adic differential calculus obeys the same rules as real one and an interesting outcome are p-adic fractals involving canonical identification. Perhaps the most crucial ingredient concerning the practical formulation of the p-adic physics is the concept of the p-adic valued definite integral. Quite generally, all general coordinate invariant definitions are based on algebraic continuation by common rationals. Integral functions can be defined using just the rules of ordinary calculus and the ordering of the integration limits is provided by the correspondence via common rationals. Residy calculus generalizes to p-adic context and also free Gaussian functional integral generalizes to p-adic context and is expected to play key role in quantum TGD at configuration space level.

The special features of p-adic Lie-groups are briefly discussed: the most important of them being an infinite fractal hierarchy of nested groups. Various versions of the p-adic Fourier analysis are proposed: ordinary Fourier analysis generalizes naturally only if finite-dimensional extensions of p-adic numbers are allowed and this has interpretation in terms of p-adic length scale cutoff. Also p-adic Fourier analysis provides a possible definition of the definite integral in the p-adic context by using algebraic continuation.

p-Adic Physics: Physical Ideas

The most important p-adic concepts and ideas are p-adic fractality, spin glass analogy, p-adic length scale hypothesis, p-adic realization of the Slaving Principle, p-adic criticality, and the non-determinism of the p-adic differential equations justifying the interpretation of the p-adic space-time regions as cognitive representations. These ideas are discussed in this chapter in a more concrete level than in previous chapters in the hope that this might help the reader to assimilate the material more easily. Some of the considerations might be a little bit out of date since the chapter is written much earlier than the preceding chapters.

a) The criticality of quantum TGD and the need to generalize conformal invariance to the 4-dimensional context were the original motivations of the p-adic approach. It however turned out that quaternion conformal invariance, rather than p-adic conformal invariance for the space-time surface regarded as an algebraic extension of p-adics, is the correct manner to realize conformal invariance. In TGD as a generalized number theory approach p-adic space-time regions emerge completely naturally and have interpretation as cognitive representations of the real physics. If this occurs already at the level of elementary particles, one can understand p-adic physics as a model for a cognitive model about physics provided by Nature itself. The basic motivation for this assumption is the p-adic non-determinism of the p-adic field equations making them ideal for the simulation purposes. The p-adic–real phase transitions are the second basic concept allowing to understand how intention is transformed to action and vice versa: the occurrence of this process even at elementary particle level explains why p-adic length scale hypothesis works. This picture is consistent with the idea about evolution occurring already at the level of elementary particles and allowing the survival of the systems with largest cognitive resources.

b) Spin glass analogy, which was the original motivation for p-adicization before the discovery that p-adic regions of space-time emerge automatically from TGD as a generalized number theory approach, is discussed at configuration space level. The basic idea is that the maximum (several of them are possible) of the exponential of the Kähler function with respect to the fiber degrees of freedom as function of zero modes is p-adic fractal. This together with spin glass analogy suggest p-adic ultra-metricity of the reduced configuration space CH_{red}, the TGD counterpart of the energy landscape.

c) Slaving Principle states that there exists a hierarchy of dynamics with increasing characteristic length (time) scales and the dynamical variables of a given length scale obey dynamics, where the dynamical variables of the longer length (time) scale serve as "masters" that is effectively as external parameters or integration constants. The dynamics of the "slave" corresponds to a rapid adaptation to the conditions posed by the "master". p-Adic length scale hypothesis allows a concrete quantification of this principle predicting a hierarchy of preferred length, time, energy and frequency scales.

d) Critical systems are fractals and the natural guess is that p-adic topology serves also as an effective topology of real space-time sheets in some length scale range and that real non-determinism of Kähler action mimics p-adic non-determinism for some value of prime p. This motivates some qualitative p-adic ideas about criticality.

e) The properties of the CP_2 type extremals providing TGD based model for elementary particles and topological sum contacts, are discussed in detail. CP_2 type extremals are for TGD what black holes are for General Relativity. Black hole elementary particle analogy is discussed in detail and the generalization of the Hawking-Bekenstein formula is shown to lead to a prediction for the radius of the elementary particle horizon and to a justification for the p-adic length scale hypothesis. A deeper justification for the p-adic length scale hypothesis comes from the assumption that systems with maximal cognitive resources are winners in the fight for survival even in elementary particle length scales.

f) Quantum criticality allows the dependence of the Kähler coupling strength on zero modes. It would be nice if α_K were RG invariant in strong sense but the expression for gravitational coupling constant implies that it increases rapidly as a function of p-adic length scale in this case. This led to the hypothesis that G is RG invariant. The hypothesis fixes the p-adic evolution of α_K completely and implies logarithmic dependence of α_K on p-adic length scale. It has however turned out that the RG invariance might after all be possible and is actually strongly favored by different physical arguments. The point is that M_{127} is the largest Mersenne prime for which p-adic length scale is non-super-astronomical. If gravitational interaction is mediated by space-time sheets labelled by Mersenne prime, gravitational constant is effective RG invariant even if α_K is RG invariant in strong sense. This option is also ideal concerning the p-adicization of the theory.

Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

The mathematical aspects of p-adicization of quantum TGD are discussed. In a well-defined sense Nature itself performs the p-adicization and p-adic physics can be regarded as physics of cognitive regions of space-time which in turn provide representations of real space-time regions. Cognitive representations presumably involve the p-adicization of the geometry at the level of the space-time and imbedding space by a mapping of a real space time region to a p-adic one. One can differentiate between two kinds of maps: the identification induced by the common rationals of real and p-adic space time region and the representations of the external real world to internal p-adic world induced by a canonical identification type maps.

Only the identification by common rationals respects general coordinate invariance, and it leads to a generalization of the number concept. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and configuration space are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

The program makes sense only if also extensions containing transcendentals are allowed: the p-dimensional extension containing powers of e is perhaps the most important transcendental extension involved. Entire cognitive hierarchy of extension emerges and the dimension of extension can be regarded as a measure for the cognitive resolution and the higher the dimension the shorter the length scale of resolution. Cognitive resolution provides also number theoretical counterpart for the notion of length scale cutoff unavoidable in quantum field theories: now the length scale cutoffs are part of the physics of cognition rather than reflecting the practical limitations of theory building.

There is a lot of p-adicizing to do.

a) The p-adic variant of classical TGD must be constructed. Field equations make indeed sense also in the p-adic context. The strongest assumption is that real space time sheets have the same functional form as real space-time sheet so that there is non-uniqueness only due to the hierarchy of dimensions of extensions.

b) Probability theory must be generalized. Canonical identification playing central role in p-adic mass calculations using p-adic thermodynamics maps genuinely p-adic probabilities to their real counterparts. p-Adic entropy can be defined and one can distinguish between three kinds of entropies: real entropy, p-adic entropy mapped to its real counterpart by canonical identification, and number theoretical entropies applying when probabilities are in finite-dimensional extension of rationals. Number theoretic entropies can be negative and provide genuine information measures, and it turns that bound states should correspond in TGD framework to entanglement coefficients which belong to a finite-dimensional extension of rationals and have negative number theoretic entanglement entropy. These information measures generalize by quantum-classical correspondence to space-time level.

c) p-Adic quantum mechanics must be constructed. p-Adic unitarity differs in some respects from its real counterpart: in particular, p-adic cohomology allows unitary S-matrices $S = 1 + T$

such that T is hermitian and nilpotent matrix. p-Adic quantum measurement theory based on Negentropy Maximization Principle (NMP) leads to the notion of monitoring, which might have relevance for the physics of cognition.

d) Generalized quantum mechanics results as fusion of quantum mechanics in various number fields using algebraic continuation from the field of rational as a basic guiding principle. It seems possible to generalize the notion of unitary process in such a manner that unitary matrix leads from rational Hilbert space H_Q to a formal superposition of states in all Hilbert spaces H_F, where F runs over number fields. If this is accepted, state function reduction is a pure number theoretical necessity and involves a reduction to a particular number field followed by state function reduction and state preparation leading ultimately to a state containing only entanglement which is rational or finitely-extended rational and because of its negative number theoretic entanglement entropy identifiable as bound state entanglement stable against NMP.

e) Generalization of the configuration space and related concepts is also necessary and again gluing along common rationals and algebraic continuation is the basic guide line also now. Configuration space is a union of symmetric spaces and this allows an algebraic construction of the configuration space Kähler metric and spinor structure, whose definition reduces to the super canonical algebra defined by the function basis at the light cone boundary. Hence the algebraic continuation is relatively straightforward. Even configuration space functional integral could allow algebraic continuation. The reason is that symmetric space structure together with Duistermaat Hecke theorem suggests strongly that configuration space integration with the constraints posed by infinite-dimensional symmetries on physical states is effectively equivalent to Gaussian functional integration in free field theory around the unique maximum of Kähler function using contravariant configuration space metric as a propagator. Algebraic continuation is possible for a subset of rational valued zero modes if Kähler action and Kähler function are rational functions of configuration space coordinates for rational values of zero modes.

What p-adic icosahedron could mean? And what about p-adic manifold?

The original focus of this chapter was p-adic icosahedron. The discussion of attempt to define this notion however leads to the challenge of defining the concept of p-adic sphere, and more generally, that of p-adic manifold, and this problem soon became the main target of attention since it is one of the key challenges of also TGD.

There exists two basic philosophies concerning the construction of both real and p-adic manifolds: algebraic and topological approach. Also in TGD these approaches have been competing: algebraic approach relates real and p-adic space-time points by identifying the common rationals. Finite pinary cutoff is however required to avoid totally wild fluctuations and has interpretation in terms of finite measurement resolution. Canonical identification maps p-adics to reals and vice versa in a continuous manner but is not consistent with p-adic analyticity nor field equations unless one poses a pinary cutoff. It seems that pinary cutoff reflecting the notion of finite measurement resolution is necessary in both approaches. This represents a new notion from the point of view of mathematics.

1. One can try to generalize the theory of real manifolds to p-adic context. The basic problem is that p-adic balls are either disjoint or nested so that the usual construction by gluing partially overlapping spheres fails. One attempt to solve the problem relies on the notion of Berkovich disk obtained as a completion of p-adic disk having path connected topology (non-ultrametric) and containing p-adic disk as a dense subset. This plus the complexity of the construction is heavy price to be paid for path-connectedness. A related notion is Bruhat-Tits tree defining kind of skeleton making p-adic manifold path connected. The notion makes sense for the p-adic counterparts of projective spaces, which suggests that p-adic projective spaces (S^2 and CP_2 in TGD framework) are physically very special.

2. Second approach is algebraic and restricts the consideration to algebraic varieties for which also topological invariants have algebraic counterparts. This approach looks very natural in TGD framework - at least for imbedding space. Preferred extremals of Kähler action can be characterized purely algebraically - even in a manner independent of the action principle - so that they might make sense also p-adically.

Number theoretical universality is central element of TGD. Physical considerations force to generalize the number concept by gluing reals and various p-adic number fields along rationals and possible common algebraic numbers. This idea makes sense also at the level of space-time and of "world of classical worlds" (WCW).

Algebraic continuation between different number fields is the key notion. Algebraic continuation between real and p-adic sectors takes place along their intersection , which at the level of WCW ("world of classical worlds") correspond to surfaces allowing interpretation both as real and p-adic surfaces for some value(s) of prime p. The algebraic continuation from the intersection of real and p-adic WCWs is not possible for all p-adic number fields. For instance, real integrals as functions of parameters need not make sense for all p-adic number fields. This apparent mathematical weakness can be however turned to physical strength: real space-time surfaces assignable to elementary particles can correspond only some particular p-adic primes. This would explain why elementary particles are characterized by preferred p-adic primes. The p-adic prime determining the mass scale of the elementary particle could be fixed number theoretically rather than by some dynamical principle formulated in real context (number theoretic anatomy of rational number does not depend smoothly on its real magnitude!).

Although Berkovich construction of p-adic disk does not look promising in TGD framework, it suggests that the difficulty posed by the total disconnectedness of p-adic topology is real. TGD in turn suggests that the difficulty could be overcome without the completion to a non-ultrametric topology. Two approaches emerge, which ought to be equivalent.

1. The TGD inspired solution to the construction of path connected effective p-adic topology is based on the notion of canonical identification mapping reals to p-adics and vice versa in a continuous manner. The trivial but striking observation was that canonical identification satisfies triangle inequality and thus defines an Archimedean norm allowing to induce real topology to p-adic context. Canonical identification with finite measurement resolution defines chart maps from p-adics to reals and vice versa and preferred extremal property allows to complete the discrete image to hopefully space-time surface unique within finite measurement resolution so that topological and algebraic approach are combined. Finite resolution would become part of the manifold theory. p-Adic manifold theory would also have interpretation in terms of cognitive representations as maps between realities and p-adicities.

2. One can ask whether the physical content of path connectedness could be also formulated as a quantum physical rather than primarily topological notion, and could boil down to the non-triviality of correlation functions for second quantized induced spinor fields essential for the formulation of WCW spinor structure. Fermion fields and their n-point functions could become part of a number theoretically universal definition of manifold in accordance with the TGD inspired vision that WCW geometry - and perhaps even space-time geometry - allow a formulation in terms of fermions. This option is a mere conjecture whereas the first one is on rigorous basis.

1.4.3 PART III: Miscellaneous topics

Riemann hypothesis and physics

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the critical line $Re(s) = 1/2$. Since Riemann zeta function allows a formal interpretation as thermodynamical partition function for a quantum field theoretical system consisting of bosons labeled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. The complex value of temperature is not however consistent with thermodynamics. In zero energy ontology one obtains quantum theory as a square root of thermodynamics and this objection can be circumvented and a nice argument allowing to interpret RH physically emerges.

Conformal invariance leads to a beautiful generalization of Hilbert-Polya conjecture allowing to interpret RH in terms of coherent states rather than energy eigenstates of a Hamiltonian. In zero energy ontology the interpretation is that the coherent states in question represent Bose-Einstein condensation at criticality. Zeros of zeta correspond to coherent states orthogonal to the coherent state characterized by $s = 0$, which has finite norm, and therefore does not represent Bose-Einstein condensation.

Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true. These considerations are however highly speculative and are represented at the end of the chapter.

1. Super-conformal invariance and generalization of Hilbert-Polya hypothesis

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

2. Zero energy ontology and RH

A further approach to RH is based on zero energy ontology and is consistent with the approach based on the notion of coherent state. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

3. Miscellaneous ideas

During years I have also considered several ideas about Riemann hypothesis which I would not call miscellaneous. I have moved them to the end of the chapter because of the highly speculative nature.

Category theory, quantum TGD and TGD inspired theory of consciousness

Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. Category theory might provide the desired systematic approach to fuse together the bundles of general ideas related to the construction of quantum TGD proper. Category theory might also have natural applications in the general theory of consciousness and the theory of cognitive representations.

a) The ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences could be expressed elegantly using the language of the category theory. Quantum classical correspondence might allow a mathematical formulation in terms of structure respecting functors mapping the categories associated with the three kinds of existences to each other. Basic results are following.

i) Self hierarchy has indeed functorial map to the hierarchy of space-time sheets and also configuration space spinor fields reflect it. Thus the self referentiality of conscious experience has a functorial formulation (it is possible to be conscious about what one *was* conscious).

ii) The inherent logic for category defined by Heyting algebra must be modified in TGD context.

Set theoretic inclusion is replaced with the topological condensation. The resulting logic is two-valued but since same space-time sheet can simultaneously condense at two disjoint space-time sheets the classical counterpart of quantum superposition has a space-time correlate so that also quantum jump should have space-time correlate in many-sheeted space-time.

iii) The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

iv) The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

b) Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience.

c) Categories possess inherent generalized logic based on set theoretic inclusion which in TGD framework is naturally replaced with topological condensation: the outcome is quantum variants for the notions of sieve, topos, and logic. This suggests the possibility of geometrizing the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through three-valued logic. Also the right-wrong logic of moral rules and beautiful-ugly logic of aesthetics seem to be too naive and might be replaced with a more general quantum logic.

TGD and Non-Standard Numbers

The chapter represents a comparison of ultrapower fields (loosely surreals, hyper-reals, long line) and number fields generated by infinite primes having a physical interpretation in Topological Geometrodynamics. Ultrapower fields are discussed in very physicist friendly manner in the articles of Elemer Rosinger and these articles are taken as a convenient starting point. The physical interpretations and principles proposed by Rosinger are considered against the background provided by TGD. The construction of ultrapower fields is associated with physics using the close analogies with gauge theories, gauge invariance, and with the singularities of classical fields. Non-standard numbers are compared with the numbers generated by infinite primes and it is found that the construction of infinite primes, integers, and rationals has a close similarity with construction of the generalized scalars. The construction replaces at the lowest level the index set $\Lambda = \mathbb{N}$ of natural numbers with algebraic numbers $\mathbb{A}$, Frechet filter of $\mathbb{N}$ with that of $\mathbb{A}$, and $\mathbb{R}$ with unit circle S^1 represented as complex numbers of unit magnitude. At higher levels of the hierarchy generalized -possibly infinite and infinitesimal- algebraic numbers emerge. This correspondence maps a given set in the dual of Frechet filter of $\mathbb{A}$ to a phase factor characterizing infinite rational algebraically so that correspondence is like representation of algebra. The basic difference between two approaches to infinite numbers is that the counterpart of infinitesimals is infinitude of real units with complex number theoretic anatomy: one might loosely say that these real units are exponentials of infinitesimals.

Infinite Primes and Motives

In this chapter the goal is to find whether the general mathematical structures associated with twistor approach, superstring models and M-theory could have a generalization or a modification in TGD framework. The contents of the chapter is an outcome of a rather spontaneous process, and represents rather unexpected new insights about TGD resulting as outcome of the comparisons.

1. Infinite primes, Galois groups, algebraic geometry, and TGD

In algebraic geometry the notion of variety defined by algebraic equation is very general: all number fields are allowed. One of the challenges is to define the counterparts of homology and cohomology groups for them. The notion of cohomology giving rise also to homology if Poincare duality holds true is central. The number of various cohomology theories has inflated and one of the basic challenges to find a sufficiently general approach allowing to interpret various cohomology theories as variations of the same motive as Grothendieck, who is the pioneer of the field responsible for many of the basic notions and visions, expressed it.

Cohomology requires a definition of integral for forms for all number fields. In p-adic context the lack of well-ordering of p-adic numbers implies difficulties both in homology and cohomology since the notion of boundary does not exist in topological sense. The notion of definite integral is problematic for the same reason. This has led to a proposal of reducing integration to Fourier analysis working for symmetric spaces but requiring algebraic extensions of p-adic numbers and an appropriate definition of the p-adic symmetric space. The definition is not unique and the interpretation is in terms of the varying measurement resolution.

The notion of infinite has gradually turned out to be more and more important for quantum TGD. Infinite primes, integers, and rationals form a hierarchy completely analogous to a hierarchy of second quantization for a super-symmetric arithmetic quantum field theory. The simplest infinite primes representing elementary particles at given level are in one-one correspondence with many-particle states of the previous level. More complex infinite primes have interpretation in terms of bound states.

1. What makes infinite primes interesting from the point of view of algebraic geometry is that infinite primes, integers and rationals at the n:th level of the hierarchy are in 1-1 correspondence with rational functions of n arguments. One can solve the roots of associated polynomials and perform a root decomposition of infinite primes at various levels of the hierarchy and assign to them Galois groups acting as automorphisms of the field extensions of polynomials defined by the roots coming as restrictions of the basic polynomial to planes $x_n = 0$, $x_n = x_{n-1} = 0$, etc...

2. These Galois groups are suggested to define non-commutative generalization of homotopy and homology theories and non-linear boundary operation for which a geometric interpretation in terms of the restriction to lower-dimensional plane is proposed. The Galois group G_k would be analogous to the relative homology group relative to the plane $x_{k-1} = 0$ representing boundary and makes sense for all number fields also geometrically. One can ask whether the invariance of the complex of groups under the permutations of the orders of variables in the reduction process is necessary. Physical interpretation suggests that this is not the case and that all the groups obtained by the permutations are needed for a full description.

3. The algebraic counterpart of boundary map would map the elements of G_k identified as analog of homotopy group to the commutator group $[G_{k-2}, G_{k-2}]$ and therefore to the unit element of the abelianized group defining cohomology group. In order to obtains something analogous to the ordinary homology and cohomology groups one must however replaces Galois groups by their group algebras with values in some field or ring. This allows to define the analogs of homotopy and homology groups as their abelianizations. Cohomotopy, and cohomology would emerge as duals of homotopy and homology in the dual of the group algebra.

4. That the algebraic representation of the boundary operation is not expected to be unique turns into blessing when on keeps the TGD as almost topological QFT vision as the guide line. One can include all boundary homomorphisms subject to the condition that the anticommutator $\delta^i_k\delta^j_{k-1} + \delta^j_k\delta^i_{k-1}$ maps to the group algebra of the commutator group $[G_{k-2}, G_{k-2}]$. By adding dual generators one obtains what looks like a generalization of anticommutative fermionic algebra and what comes in mind is the spectrum of quantum states of a SUSY algebra spanned by bosonic states realized as group algebra elements and fermionic states realized in terms of homotopy and cohomotopy and in abelianized version in terms of homology and cohomology. Galois group action allows to organize quantum states into multiplets of Galois groups acting as symmetry groups of physics. Poincare duality would map the analogs of fermionic creation operators to annihilation operators and vice versa and the counterpart of pairing of k:th and $n-k$:th homology groups would be inner product analogous to that given by Grassmann integration. The interpretation in terms of fermions turns however to be wrong and the more appropriate interpretation is in terms of Dolbeault cohomology applying to forms with homomorphic and antiholomorphic indices.

5. The intuitive idea that the Galois group is analogous to 1-D homotopy group which is the only non-commutative homotopy group, the structure of infinite primes analogous to the

braids of braids of braids of ... structure, the fact that Galois group is a subgroup of permutation group, and the possibility to lift permutation group to a braid group suggests a representation as flows of 2-D plane with punctures giving a direct connection with topological quantum field theories for braids, knots and links. The natural assumption is that the flows are induced from transformations of the symplectic group acting on $\delta M^2_\pm \times CP_2$ representing quantum fluctuating degrees of freedom associated with WCW ("world of classical worlds"). Discretization of WCW and cutoff in the number of modes would be due to the finite measurement resolution. The outcome would be rather far reaching: finite measurement resolution would allow to construct WCW spinor fields explicitly using the machinery of number theory and algebraic geometry.

6. A connection with operads is highly suggestive. What is nice from TGD perspective is that the non-commutative generalization homology and homotopy has direct connection to the basic structure of quantum TGD almost topological quantum theory where braids are basic objects and also to hyper-finite factors of type II_1. This notion of Galois group makes sense only for the algebraic varieties for which coefficient field is algebraic extension of some number field. Braid group approach however allows to generalize the approach to completely general polynomials since the braid group make sense also when the ends points for the braid are not algebraic points (roots of the polynomial).

This construction would realize the number theoretical, algebraic geometrical, and topological content in the construction of quantum states in TGD framework in accordance with TGD as almost TQFT philosophy, TGD as infinite-D geometry, and TGD as generalized number theory visions.

2. p-Adic integration and cohomology

This picture leads also to a proposal how p-adic integrals could be defined in TGD framework.

1. The calculation of twistorial amplitudes reduces to multi-dimensional residue calculus. Motivic integration gives excellent hopes for the p-adic existence of this calculus and braid representation would give space-time representation for the residue integrals in terms of the braid points representing poles of the integrand: this would conform with quantum classical correspondence. The power of 2π appearing in multiple residue integral is problematic unless it disappears from scattering amplitudes. Otherwise one must allow an extension of p-adic numbers to a ring containing powers of 2π.

2. Weak form of electric-magnetic duality and the general solution ansatz for preferred extremals reduce the Kähler action defining the Kähler function for WCW to the integral of Chern-Simons 3-form. Hence the reduction to cohomology takes places at space-time level and since p-adic cohomology exists there are excellent hopes about the existence of p-adic variant of Kähler action. The existence of the exponent of Kähler gives additional powerful constraints on the value of the Kähler fuction in the intersection of real and p-adic worlds consisting of algebraic partonic 2-surfaces and allows to guess the general form of the Kähler action in p-adic context.

3. One also should define p-adic integration for vacuum functional at the level of WCW. p-Adic thermodynamics serves as a guideline leading to the condition that in p-adic sector exponent of Kähler action is of form $(m/n)^r$, where m/n is divisible by a positive power of p-adic prime p. This implies that one has sum over contributions coming as powers of p and the challenge is to calculate the integral for K= constant surfaces using the integration measure defined by an infinite power of Kähler form of WCW reducing the integral to cohomology which should make sense also p-adically. The p-adicization of the WCW integrals has been discussed already earlier using an approach based on harmonic analysis in symmetric spaces and these two approaches should be equivalent. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $n + p - 2$ powers of $e^{1/n}$ can be considered.

4. If the Galois group algebras indeed define a representation for WCW spinor fields in finite measurement resolution, also WCW integration would reduce to summations over the Galois groups involved so that integrals would be well-defined in all number fields.

3. Floer homology, Gromov-Witten invariants, and TGD

Floer homology defines a generalization of Morse theory allowing to deduce symplectic homology groups by studying Morse theory in loop space of the symplectic manifold. Since the symplectic transformations of the boundary of $\delta M^4_\pm \times CP_2$ define isometry group of WCW, it is very natural to expect that Kähler action defines a generalization of the Floer homology allowing to understand the symplectic aspects of quantum TGD. The hierarchy of Planck constants implied by the one-to-many correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates leads naturally to singular coverings of the imbedding space and the resulting symplectic Morse theory could characterize the homology of these coverings.

One ends up to a more precise definition of vacuum functional: Kähler action reduces Chern-Simons terms (imaginary in Minkowskian regions and real in Euclidian regions) so that it has both phase and real exponent which makes the functional integral well-defined. Both the phase factor and its conjugate must be allowed and the resulting degeneracy of ground state could allow to understand qualitatively the delicacies of CP breaking and its sensitivity to the parameters of the system. The critical points with respect to zero modes correspond to those for Kähler function. The critical points with respect to complex coordinates associated with quantum fluctuating degrees of freedom are not allowed by the positive definiteness of Kähler metric of WCW. One can say that Kähler and Morse functions define the real and imaginary parts of the exponent of vacuum functional.

The generalization of Floer homology inspires several new insights. In particular, space-time surface as hyper-quaternionic surface could define the 4-D counterpart for pseudo-holomorphic 2-surfaces in Floer homology. Holomorphic partonic 2-surfaces could in turn correspond to the extrema of Kähler function with respect to zero modes and holomorphy would be accompanied by super-symmetry.

Gromov-Witten invariants appear in Floer homology and topological string theories and this inspires the attempt to build an overall view about their role in TGD. Generalization of topological string theories of type A and B to TGD framework is proposed. The TGD counterpart of the mirror symmetry would be the equivalence of formulations of TGD in $H = M^4 \times CP_2$ and in $CP_3 \times CP_3$ with space-time surfaces replaced with 6-D sphere bundles.

4. K-theory, branes, and TGD

K-theory and its generalizations play a fundamental role in super-string models and M-theory since they allow a topological classification of branes. After representing some physical objections against the notion of brane more technical problems of this approach are discussed briefly and it is proposed how TGD allows to overcome these problems. A more precise formulation of the weak form of electric-magnetic duality emerges: the original formulation was not quite correct for space-time regions with Euclidian signature of the induced metric. The question about possible TGD counterparts of R-R and NS-NS fields and S, T, and U dualities is discussed.

5. p-Adic space-time sheets as correlates for Boolean cognition

p-Adic physics is interpreted as physical correlate for cognition. The so called Stone spaces are in one-one correspondence with Boolean algebras and have typically 2-adic topologies. A generalization to p-adic case with the interpretation of p pinary digits as physically representable Boolean statements of a Boolean algebra with $2^n > p > p^{n-1}$ statements is encouraged by p-adic length scale hypothesis. Stone spaces are synonymous with profinite spaces about which both finite and infinite Galois groups represent basic examples. This provides a strong support for the connection between Boolean cognition and p-adic space-time physics. The Stone space character of Galois groups suggests also a deep connection between number theory and cognition and some arguments providing support for this vision are discussed.

Langlands Program and TGD

Number theoretic Langlands program can be seen as an attempt to unify number theory on one hand and theory of representations of reductive Lie groups on the other hand. So called automorphic functions to which various zeta functions are closely related define the common denominator. Geometric Langlands program tries to achieve a similar conceptual unification in the case of function fields. This program has caught the interest of physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional Kähler geometry and spinor structure of the "world of classical worlds" (WCW). Since TGD ce be regarded also as a generalized number theory, it is difficult to escape the idea that the interaction of Langlands program with TGD could be fruitful.

More concretely, TGD leads to a generalization of number concept based on the fusion of reals and various p-adic number fields and their extensions implying also generalization of manifold concept, which inspires the notion of number theoretic braid crucial for the formulation of quantum TGD. TGD leads also naturally to the notion of infinite primes and rationals. The identification of Clifford algebra of WCW as a hyper-finite factors of type II_1 in turn inspires further generalization of the notion of imbedding space and the idea that quantum TGD as a whole emerges from number theory. The ensuing generalization of the notion of imbedding space predicts a hierarchy of macroscopic quantum phases characterized by finite subgroups of $SU(2)$ and by quantized Planck constant. All these new elements serve as potential sources of fresh insights.

1. The Galois group for the algebraic closure of rationals as infinite symmetric group?

The naive identification of the Galois groups for the algebraic closure of rationals would be as infinite symmetric group S_∞ consisting of finite permutations of the roots of a polynomial of infinite degree having infinite number of roots. What puts bells ringing is that the corresponding group algebra is nothing but the hyper-finite factor of type II_1 (HFF). One of the many avatars of this algebra is infinite-dimensional Clifford algebra playing key role in Quantum TGD. The projective representations of this algebra can be interpreted as representations of braid algebra B_∞ meaning a connection with the notion of number theoretical braid.

2. Representations of finite subgroups of S_∞ as outer automorphisms of HFFs

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial for Langlands program. Apart from one-dimensional representations complex finite-dimensional representations are not possible if S_∞ identification is accepted (there might exist finite-dimensional l-adic representations). This suggests that the finite-dimensional representations correspond to those for finite Galois groups and result through some kind of spontaneous breaking of S_∞ symmetry.

a) Sub-factors determined by finite groups G can be interpreted as representations of Galois groups or, rather infinite diagonal imbeddings of Galois groups to an infinite Cartesian power of S_n acting as outer automorphisms in HFF. These transformations are counterparts of global gauge transformations and determine the measured quantum numbers of gauge multiplets and thus measurement resolution. All the finite approximations of the representations are inner automorphisms but the limit does not belong to S_∞ and is therefore outer. An analogous picture applies in the case of infinite-dimensional Clifford algebra.

b) The physical interpretation is as a spontaneous breaking of S_∞ to a finite Galois group. One decomposes infinite braid to a series of n-braids such that finite Galois group acts in each n-braid in identical manner. Finite value of n corresponds to IR cutoff in physics in the sense that longer wave length quantum fluctuations are cut off. Finite measurement resolution is crucial. Now it applies to braid and corresponds in the language of new quantum measurement theory to a sub-factor $\mathcal{N} \subset \mathcal{M}$ determined by the finite Galois group G implying non-commutative physics with complex rays replaced by $\mathcal{N}$ rays. Braids give a connection to topological quantum field theories, conformal field theories (TGD is almost topological quantum field theory at parton level), knots, etc..

c) TGD based space-time correlate for the action of finite Galois groups on braids and for the cutoff is in terms of the number theoretic braids obtained as the intersection of real partonic 2-surface and its p-adic counterpart. The value of the p-adic prime p associated with the parton is fixed by the scaling of the eigenvalue spectrum of the modified Dirac operator (note that renormalization group evolution of coupling constants is characterized at the level free theory since p-adic prime characterizes the p-adic length scale). The roots of the polynomial would determine the positions of braid strands so that Galois group emerges naturally. As a matter fact, partonic 2-surface decomposes into regions, one for each braid transforming independently under its own Galois group. Entire quantum state is modular invariant, which brings in additional constraints.

Braiding brings in homotopy group aspect crucial for geometric Langlands program. Another global aspect is related to the modular degrees of freedom of the partonic 2-surface, or more precisely to the regions of partonic 2-surface associated with braids. $Sp(2g,R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

3. Correspondence between finite groups and Lie groups

The correspondence between finite and Lie group is a basic aspect of Langlands.

a) Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_∞ rather than being induced from S_∞ outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_∞, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

b) There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G \subset SU(2)$ to ADE type groups. Similar correspondence is found for Jones inclusions with index $\mathcal{M} : \mathcal{N} \leq 4$. The challenge is to understand this correspondence.

i) The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G \subset SU(2)$ to those of $SU(2)$.

ii) The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with $\mathcal{N}$ rays, where $\mathcal{N}$ is the sub-factor.

iii) The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H = M^4_\pm \times CP_2 \to H/G_a \times G_b$, $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2,C) \times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_∞ would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s)) \times (s-1)/s)$, where $\xi(s) = \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s-s_n)/(1-s+s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

5. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

Quantum Arithmetics and the Relationship between Real and p-Adic Physics

This chapter considers possible answers to the basic questions of the p-adicization program, which are following.

Some of the basic questions of the p-adicization program are following.

1. Is there a duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

   Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this)?

   This chapter studies some ideas but does not provide a clearcut answer to these questions. The notion of quantum arithmetics obtained is central in this approach.

The starting point of quantum arithmetics is the map $n \rightarrow n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \rightarrow p_q$. The map takes prime first to finite field $G(p,1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map

for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p,1)$ there are no primes.

One can distinguish between two basic options concerning the definition of quantum integers.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

   The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n) p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

1. The quantum counterparts of special linear groups $SL(n,F)$ exists always. For the covering group $SL(2,C)$ of $SO(3,1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.

2. For the quantum counterparts of $SO(3)$ ($SU(2)/$ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k+7)$). The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \to 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.

The findings about quantum $SO(3)$ suggest a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p) = 0$ approximation) do not support this idea in the case of Mersenne primes.

2. An alternative idea is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

Quantum Adeles

Quantum arithmetics provides a possible resolution of a long-lasting challenge of finding a mathematical justification for the canonical identification mapping p-adics to reals playing a key role in TGD - in particular in p-adic mass calculations. p-Adic numbers have p-adic pinary expansions $\sum a_n p^n$ satisfying $a_n < p$. of powers p^n to be products of primes $p_1 < p$ satisfying $a_n < p$ for ordinary p-adic numbers. One could map this expansion to its quantum counterpart by replacing a_n with their counterpart and by canonical identification map $p \rightarrow 1/p$ the expansion to real number. This definition might be criticized as being essentially equivalent with ordinary p-adic numbers since one can argue that the map of coefficients a_n to their quantum counterparts takes place only in the canonical identification map to reals.

One could however modify this recipe. Represent integer n as a product of primes l and allow for l all expansions for which the coefficients a_n consist of primes $p_1 < p$ but give up the condition $a_n < p$. This would give 1-to-many correspondence between ordinary p-adic numbers and their quantum counterparts.

It took time to realize that $l < p$ condition might be necessary in which case the quantization in this sense - if present at all - could be associated with the canonical identification map to reals. It would correspond only to the process taking into account finite measurement resolution rather than replacement of p-adic number field with something new, hopefully a field. At this step one might perhaps allow $l > p$ so that one would obtain several real images under canonical identification.

One can however imagine a third generalization of number concept. One can replace integer n with n-dimensional Hilbert space and sum $+$ and product $\times$ with direct sum $\oplus$ and tensor product $\otimes$ and introduce their co-operations, the definition of which is highly non-trivial. This procedure yields also Hilbert space variants of rationals, algebraic numbers, p-adic number fields, and even complex, quaternionic and octonionic algebraics. Also adeles can be replaced with their Hilbert space counterparts. Even more, one can replace the points of Hilbert spaces with Hilbert spaces and repeat this process, which is very similar to the construction of infinite primes having interpretation in terms of repeated second quantization. This process could be the counterpart for construction of n^{th} order logics and one might speak of Hilbert or quantum mathematics. The construction would also generalize the notion of algebraic holography and provide self-referential cognitive representation of mathematics.

This vision emerged from the connections with generalized Feynman diagrams, braids, and with the hierarchy of Planck constants realized in terms of coverings of the imbedding space. Hilbert space generalization of number concept seems to be extremely well suited for the purposes of TGD. For instance, generalized Feynman diagrams could be identifiable as arithmetic Feynman diagrams describing sequences of arithmetic operations and their co-operations. One could interpret $\times_q$ and $+_q$ and their co-algebra operations as 3-vertices for number theoretical Feynman diagrams describing algebraic identities $X = Y$ having natural interpretation in zero energy ontology. The two vertices have direct counterparts as two kinds of basic topological vertices in quantum TGD (stringy vertices and vertices of Feynman diagrams). The definition of co-operations would characterize quantum dynamics. Physical states would correspond to the Hilbert space states assignable to numbers. One prediction is that all loops can be eliminated from generalized Feynman diagrams and diagrams are in projective sense invariant under permutations of incoming (outgoing legs).

About Absolute Galois Group

Absolute Galois Group defined as Galois group of algebraic numbers regarded as extension of rationals is very difficult concept to define. The goal of classical Langlands program is to understand the Galois group of algebraic numbers as algebraic extension of rationals - Absolute Galois Group

(AGG) - through its representations. Invertible adeles -ideles - define Gl_1 which can be shown to be isomorphic with the Galois group of maximal Abelian extension of rationals (MAGG) and the Langlands conjecture is that the representations for algebraic groups with matrix elements replaced with adeles provide information about AGG and algebraic geometry.

I have asked already earlier whether AGG could act is symmetries of quantum TGD. The basis idea was that AGG could be identified as a permutation group for a braid having infinite number of strands. The notion of quantum adele leads to the interpretation of the analog of Galois group for quantum adeles in terms of permutation groups assignable to finite l braids. One can also assign to infinite primes braid structures and Galois groups have lift to braid groups.

Objects known as dessins d'enfant provide a geometric representation for AGG in terms of action on algebraic Riemann surfaces allowing interpretation also as algebraic surfaces in finite fields. This representation would make sense for algebraic partonic 2-surfaces, and could be important in the intersection of real and p-adic worlds assigned with living matter in TGD inspired quantum biology, and would allow to regard the quantum states of living matter as representations of AGG. Adeles would make these representations very concrete by bringing in cognition represented in terms of p-adics and there is also a generalization to Hilbert adeles.

Part I

NUMBER THEORETICAL VISION

Chapter 2

TGD as a Generalized Number Theory I: p-Adicization Program

2.1 Introduction

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation. The original chapter representing the number theoretic vision as a consistent narrative grew so massive that I decided to divide it to three parts.

The first part is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

Second part focuses on the idea that the tangent spaces of space-time and imbedding space can be regarded as 4- *resp.* 8-dimensional algebras such that space-time tangent space defines subalgebra of imbedding space. The basic candidates for the pair of algebras are hyper-quaternions and hyper-octonions.

The great idea is that space-time surfaces X^4 correspond to hyper-quaternionic or co-hyper-quaternionic sub-manifolds of $HO = M^8$. The possibility to assign to X^4 a surface in $M^4 \times CP_2$ means a number theoretic analog for spontaneous compactification. Of course, nothing dynamical is involved and dual relation between totally different descriptions of the physical world would be in question.

The third part is devoted to infinite primes. Infinite primes are in one-one correspondence with the states of super-symmetric arithmetic quantum field theories. The infinite-primes associated with hyper-quaternionic and hyper-octonionic numbers are the most natural ones physically because of the underlying Lorentz invariance, and the possibility to interpret them as momenta with mass squared equal to prime. Most importantly, the polynomials associated with hyper-octonionic infinite primes have automatically space-time surfaces as representatives so that space-time geometry becomes a representative for the quantum states.

2.1.1 The painting is the landscape

The work with TGD inspired theory of consciousness has led to a vision about the relationship of mathematics and physics. Physics is not in this view a model of reality but objective reality itself: painting is the landscape. One can also equate mathematics and physics in a well defined sense and the often implicitly assumed Cartesian theory-world division disappears. Physical realities are mathematical ideas represented by configuration space spinor fields (quantum histories) and quantum jumps between quantum histories give rise to consciousness and to the subjective existence of mathematician.

The concrete realization for the notion algebraic hologram based on the notion of infinite prime is a second new element. The notion of infinite rationals leads to the generalization of also the notion of finite number since infinite-dimensional space of real units obtained from finite rational valued ratios q of infinite integers divided by q. These units are not units in p-adic sense. The generalization to the quaternionic and octonionic context means that ordinary space-time points become infinitely structured and space-time point is able to represent even the quantum physical state of the Universe in its algebraic structure. Single space-time point becomes the Platonia not visible at the level of real physics but essential for mathematical cognition.

In this view evolution becomes also evolution of mathematical structures, which become more and more self-conscious quantum jump by quantum jump. The notion of p-adic evolution is indeed a basic prediction of quantum TGD but even this vision might be generalized by allowing rational-adic topologies for which topology is defined by a ring with unit rather than number field.

2.1.2 Real and p-adic regions of the space-time as geometric correlates of matter and mind

One could end up with p-adic space-time sheets via field equations. The solutions of the equations determining space-time surfaces are restricted by the requirement that the coordinates are real. When this is not the case, one might apply instead of a real completion with some p-adic completion. It however seems that p-adicity is present at deeper level and automatically present via the generalization of the number concept obtained by fusing reals and p-adics along rationals and common algebraics.

p-Adic non-determinism due to the presence of non-constant functions with vanishing derivative implies extreme flexibility and therefore suggests the identification of the p-adic regions as seats of cognitive representations. Unlike the completion of reals to complex numbers, the completions of p-adic numbers preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2.1.3 The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

2.1.4 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts zero energy ontology [K19, K18] .

Zero energy ontology classically

In TGD inspired cosmology [K65] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [K74] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

Zero energy ontology at quantum level

Also the construction of S-matrix [K18] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state have interpretation as M-matrix identifiable as a "complex square root" of density matrix expressible as a product of positive diagonal square root of the density matrix and of a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

The collection of M-matrices defines an orthonormal state basis for zero energy states and together they define unitary U-matrix charactering transition amplitudes between zero energy states. This matrix would not be however the counterpart of the usual S-matrix. Rather the unitary matrix phase of a given M-matrix would define the S-matrix measured in laboratory. U-matrix would also characterize the transitions between different number fields possible in the intersection of rel and p-adic worlds and having interpretation in terms of intention and cognition.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by the time scale of the causal diamond (CD) and the rational (perhaps integer) characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also. CDs are indeed important also in TGD inspired cosmology [K65] .

Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [K80] . This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the "world of classical worlds", WCW briefly) is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [K80] . $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [K18] . The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

2.1.5 What number theoretical universality might mean?

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic

physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough. It is however clear that number theoretical criticality could provide important insights to quantum TGd. p-Adic thermodynamics [K46] is an excellent example of this. In consciousness theory the transitions transforming intentions to actions and actions to cognitions would be key applications. Needless to say, zero energy ontology is absolutely essential: otherwise this kind of transitions would not make sense.

2.1.6 p-Adicization by algebraic continuation

The basic challenges of the p-adicization program are following.

1. The first problem -the conceptual one- is the identification of preferred coordinates in which functions are algebraic and for which algebraic values of coordinates are in preferred position. This problem is encountered both at the level of space-time, imbedding space, and configuration space. Here the group theoretical considerations play decisive role and the selection of preferred coordinates relates closely to the selection of quantization axes. This selection has direct physical correlates at the level of imbedding space and the hierarchy of Planck constants has interpretation as a correlate for the selection of quantization axes [K25] .

   Algebraization does not necessarily mean discretization at space-time level: for instance, the coordinates characterizing partonic 2-surface can be algebraic so that algebraic point of the configuration space results and surface is not discretized. If this kind of function spaces are finite-dimensional, it is possible to fix X^2 completely data for a finite number of points only.

2. Local physics generalizes as such to p-adic context (field equations, etc...). The basic stumbling block of this program is integration already at space-time (Kähler action etc..). The problem becomes really horrible looking at configuration space level (functional integral). Algebraic continuation could allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics. Also the existence of the Kähler geometry does this and the solution to the constraint is that WCW is a union of symmetric spaces. In the case of symmetric spaces Fourier analysis generalizes to harmonics analysis and one can reduces integration to summation for functions allowing Fourier decomposition. In p-adic context the existence of plane waves requires an algebraic extension allowing roots of unity characterizing the measurement accuracy of angle like variables. This leads in the case of symmetric spaces to a general p-adicization recipe. One starts from a discrete variantnof the symmetric space defined for which points correspond to roots of unity and replaces each discrete point with is p-adic completion representing the p-adic variant of the symmetric space. There is infinite hierarchy of p-adicizations corresponding to measurement resolutions and to the choice of preferred coordinates and the interpretation is in terms of cognitive representations and refined view about General Coordinate Invariance taking into account the fact that cognition is also part of the quantum state.

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane.

1. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is also such a function.

2. For instance, residue calculus essential in the construction of N-point functions of conformal field theory might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values

in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

3. Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality.

4. For instance, the idea that number theoretically critical partonic 2-surfaces are expressible in terms of rational functions with rational or algebraic coefficients so that also p-adic variants of these surfaces make sense, is very attractive.

5. Finite sums and products respect algebraic number property and the condition of finiteness is coded naturally by the notion of finite measurement resolution in terms of the notion of (number theoretic) braid. This simplifies dramatically the algebraic continuation since configuration space reduces to a finite-dimensional space and the space of configuration space spinor fields reduces to finite-dimensional function space.

The real configuration space can well contain sectors for which p-adicization does not make sense. For instance, if the exponent of Kähler function and Kähler function are not expressible in terms of algebraic functions with rational or at most algebraic functions or more general functions making sense p-adically, the continuation is not possible. p-Adic non-determinism in p-adic sectors makes also impossible the continuation to real sector. All this is consistent with vision about rational and algebraic physics as as analog of rational and algebraic numbers allowing completion to various continuous number fields.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. For instance, most points p-adic space-time sheets reside at infinity in real sense and p-adically infinitesimal is infinite in real sense. Two types of cutoffs are predicted p-adic length scale cutoff and a cutoff due to phase resolution related to the hierarchy of Planck constants. Zero energy ontology provides natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions. These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

2.1.7 For the reader

Most of this chapter has been written for about decade before the above discussion of number theoretical universality and criticality. Therefore the chapter in its original form reflects the first violent burst of ideas of an innocent novice rather than the recent more balanced vision about the role of number theory in quantum TGD. For instance, in the original view about number theoretic universality is the strong one and is un-necessarily restricting. Although I have done my best to update the sections, the details of the representation may still reflect in many aspects quantum TGD as I understood it for a decade ago and the recent vision differs dramatically from this view.

The plan of the chapter is following. In the first one half I describe general ideas as they emerged years ago in a rather free flowing "Alice in the Wonderland" mood. I also describe phenomenological applications, such as conjectures about number theoretic anatomy of coupling constants which are now at rather firm basis. The chapter titled "The recent view about Quantum TGD" represents kind of turning point and introduces quantum TGD in its recent formulation in the real context. The remaining chapters are devoted to the challenge of understanding p-adic counterpart of this general theory.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- p-Adic number fields [L26]
- p-Adic physics [L27]
- p-Adic number fields and cognition and intention [L25]
- p-Adic length scale hypothesis [L28]
- p-Adic manifold [L23]

2.2 How p-adic numbers emerge from algebraic physics?

The new algebraic vision leads to several generalization of the p-adic philosophy. Besides p-adic topologies more general rational-adic topologies are possible. Topology is purely dynamically determined and ..-adic topologies are quite 'real'. There is a physics oriented review article by Brekke and Freund [A123] . The books of Gouvêa [A158] and Khrennikov [A179] give a more mathematics-oriented views about p-adics.

This section is written before the discovery that it is possible to generalize the notion of the number field by the fusion reals and various p-adic numbers fields and their extensions together along common rationals (and also common algebraic numbers) to form a book like structure. The interpretation of p-adic physics as physics of intention and cognition removes interpretational problems. This vision provides immediately an answer to many questions raised in the text. In particular, it leads naturally to a complete algebraic democracy. The introduction of infinite primes, which are discussed in next chapter, extends the algebraic democracy even further and gives hopes of describing mathematically also mathematical cognition.

2.2.1 Basic ideas and questions

It is good to list the basic ideas and pose the basic question before more detailed considerations.

Topology is dynamical

The dynamical emergence of p-adicity is strongly supported both by the applications of p-adic and algebraic physics. The solutions of polynomial equations involving more than one variable involve roots of polynomials. Only roots in the real algebraic extensions of rationals are allowed since the components of quaternions must be real numbers. When the root is complex in real topology, one can however introduce p-adic topology such that the root exists as a number in a real extension of p-adics. In p-adic context only a finite-dimensional algebraic extension of rational numbers is needed. The solutions of the derivative conditions guaranteeing Lagrange manifold property involve p-adic pseudo constants so that the p-adic solutions are non-deterministic. The interpretation is that real roots of polynomials correspond to geometric correlates of matter whereas p-adic regions are geometric correlates of mind in consistency with the p-adic non-determinism.

Does this picture imply the physically attractive working hypothesis stating that the decomposition of infinite prime into primes of lower level corresponds to a decomposition of the space-time surface to various p-adic regions appearing in the definition of the infinite prime? Generating infinite primes correspond to quaternionic rationals and these rationals contain powers of quaternionic primes defining the infinite prime. The convergence of the power series solution of the polynomial equations defining space-time surface might depend crucially on the norms of these rationals in the p-adic topology used. This could actually force in a given space-time region p-adic topology associated with some prime involved in the expansion. This is in complete accordance with the idea that p-adic topologies are topologies of sensory experience and real topology is the topology of reality.

Various generalizations of p-adic topologies

p-Adicized quaternions is not a number field anymore. One could allow also rational-adic extensions [A179] for which pinary expansions are replaced by expansions in powers of rational. These extensions give rise to rings with unit but not to number fields. In this approach p-adic, or more generally rational-adic, topology determined by the algebraic number field on a given space-time sheet would be absolutely 'real' rather than mere effective topology. Space-time surface decomposes into regions which look like fractal dust when seen by an observer characterized by different number field unless the observer uses some resolution.

This approach suggests even further generalizations. The original observation stimulated by the work with Riemann hypothesis was that the primes associated with the algebraic extensions of rationals, in particular Gaussian primes and Eisenstein primes, have very attractive physical interpretation. Quaternionic primes and rationals might in turn define what might be regarded as non-commutative generalization of the p-adic and rational-adic topology.

...-Adic topology measures the complexity of the quantum state

The higher the degree of the polynomial, and thus the number of particles in the physical state and its complexity, the higher the algebraic dimension of the rational quaternions. A complete algebraic and quaternion and octonion-dimensional democracy would prevail. Accordingly, space-time topology would be completely dynamical in the sense that space-time contains both rational-adic, p-adic regions, and real regions. Physical evolution could be seen as evolution of mathematical structures in this framework: p-adic topologies would be obviously winners over rational-adic topologies and p-adic length scale hypothesis would select the surviving p-adic topologies. For instance, Gaussian-adic and Eisenstein-adic topologies would in turn be higher level survivors possibly associated with biological systems.

Dimensional democracy would be realized in the sense that one can regard the space-time sheets defining n-sheeted topological condensate also as a $4n$-dimensional surface in H^n. This hypothesis fixes the interactions associated with the topological condensation, and the hierarchical structure of the topological condensate conforms with the hierarchical ordering of the quaternionic arguments of the polynomials to which infinite primes are mapped. Polynomials (infinite integers) at a given level of hierarchy in turn can be interpreted in terms of formation of bound states by the formation of join along boundaries bonds.

Is adelic principle consistent with the dynamical topology?

There is competing, and as it seems, almost diametrically opposite view. Just like adelic formula allows to express the norm of a rational number as product of its p-adic norms, various algebraic number fields and even more general structures such as quaternions allowing the notion of prime, provide a collection of incomplete but hopefully calculable views about physics. The net description gives rise to quantum TGD formulated using real numbers. These descriptions would be like summary over all experiences about world of conscious experiencers characterized by p-adic completions of various four-dimensional algebraic number rationals. What is important is that the descriptions using algebraic number fields or their generalization might be calculable. This view need not be conflict with the dynamical view and one could indeed claim that the p-adic physics associated with various algebraic extensions of rational quaternions provide a model about physics constructed by various conscious observers. For a given quantum state there would be however minimal algebraic extension containing all points of the space-time surface in it.

2.2.2 Are more general adics indeed needed?

The considerations related to Riemann hypothesis inspired the notion of G- and E-adic numbers in which rational prime p is replaced with Gaussian or Eisenstein prime. The notion of Eisenstein prime is so attractive because it makes possible to circumvent the complexification of p-adic numbers for $p \ mod \ 4 = 1$ for which $\sqrt{-1}$ exists as a p-adic number. What forces to take the notion of G-adics very seriously is that Gaussian Mersennes correspond to the p-adic length scale of atomic nucleus and to important biological length scales in the range between 10 nanometers and few micrometers. Also the key role of Golden Mean τ in biology and self-organizing systems could be understood if $Q(\tau, i)$ defines D-adic topology. Thus there is great temptation to believe that the notion of p-adic number generalizes in these sense that any irreducible associated with real or complex algebraic extension defines generalization of p-adic numbers and that these extensions appear in the algebraic extensions of quaternions.

Thus one must consider seriously also generalized p-adic numbers, D-adics as they were called in [K61] . D-adics would correspond to powers series of a prime belonging to a complex algebraic extension of rationals. Quaternions decompose naturally in longitudinal and transversal part and transversal part can be interpreted as a complex algebraic extension of rationals in the case of both M^4 and CP_2. Thus some irreducibles of this complex extension could define a generalization of p-adic numbers used to define the algebraic extension of rational quaternions reduced to a pair of complex coordinates.

Perhaps one could go even further: quaternion-adics defined as power series of quaternionic primes of norm p suggest themselves. What would be nice that this prime could perhaps be interpreted as a representation for the momentum of corresponding space-time sheets. The components of the prime belong to algebraic extension of rationals and would even code information about external world if the proposed interpretations are correct. One can also ask whether quaternionic primes could define what might be called quaternion-adic algebras and whether these algebras might be a basic element of algebraic physics.

This would mean that space-time topology would code information about the quantum numbers of a physical state. Rings with unit rather than number fields are in question since the p-adic counterparts of quaternionic integers in general fail to have inverse. It must be emphasized that the field property might not be absolutely essential. For instance 'rational-adics' [A179] , for which prime p is replaced with a rational q such that norm comes as a power of q, exists as rings with unit and define topology. Rational-adic topologies could have also quaternionic counterparts.

The idea of q-rational topologies is supported by the physical picture about the correspondence between Fock states and space-time sheets. Single 3-surface can in principle carry arbitrarily high fermion and boson numbers but is unstable to a topological decay to 3-surfaces carrying single fermion and boson states. The translation of this statement to ...-adic context would be that the Fock states associated with infinite primes which correspond to rational-adic quaternionic topologies are unstable against decay to states described by polynomial primes in which each factor corresponds to prime (bosons) or its inverse (fermions) in algebraic extension of quaternions. This tendency to evolve to prime-adic topologies could be seen also as a manifestation of p-adic evolution and self-organization. Rational-adic topologies would be simply losers in the fight for survival against topologies defining number fields. Since also quaternion-adic topologies fail to define number fields they are expected to be losers in the fight for survival. Winners would be ...-adic topologies defining number fields. At the level of Fock states this would mean the instability of states which contain more than one prime: that this is indeed the case, is one of the basic assumptions of quantum TGD forced by the experimental fact that elementary particles correspond to simplest Fock states associated with WCW spinor s.

2.2.3 Why completion to p-adics necessarily occurs?

There is rather convincing argument in favor of ...-adic physics. Typically one must find zeros of rational functions of several variables. Simplifying somewhat, at the first level one must find zeros of polynomials $P(x_1, x_2)$. Newton's theorem states that the monic polynomial $P_n(y, x) = y^n + a_{n-1}x^{n-1} + ..$ allows a factorization in an algebraically closed number field

$$P(y, x^m) = \prod_k (y - f_k(x)) \quad . \tag{2.2.1}$$

Here f_k are polynomials and m is integer which divides n and equals to n for an irreducible polynomial P. Since the multiplication of x by m:th root of unity (ζ_m) leaves left hand side invariant it must permute the factors on right hand side. Thus one can express the formula also as

$$P(y,x) = \prod_{k=1,..m} (y - f_k(\zeta_m^k x^{1/m})) \ . \tag{2.2.2}$$

When number field is not algebraically closed this means that one must introduce an algebraic extension by m:th roots of all rationals.

The problem is that these roots are not real in general and one cannot solve the problem by using a completion to complex numbers since only real extensions for the components of quaternion are possible. Only in the region where some of the roots of the polynomial are real, this is possible. The only manner to achieve consistency with the reality requirement is to allow p-adic topology or possibly rational-adic topology: in this case also the algebraic extension allowing m:th roots is always finite-dimensional. For instance, for $m = 2$ p-adic extension of rationals would be 4-dimensional for $p > 2$. The situation is similar for rational-adic topology.

If this argument is correct, one can conclude that real topology is possible only in the regions where real roots of the polynomial equation are possible: in the regions where all roots are complex, p-adicization gives rise to roots in the algebraic extension of p-adics and p-adic topology emerges naturally. This picture provides a precise view about how the space-time surface defined by the polynomial of quaternions decomposes to real and p-adic regions. Also a connection with catastrophe theory [A219] emerges: the boundaries of the catastrophe regions where some roots coincide, serve also as boundaries between ...-adic and real regions.

2.2.4 Decomposition of space-time to ...-adic regions

Number-theoretic constraints are important in determining which ...-adic topologies are possible in a given space-time region. There is no hope of building any unique vision unless one poses some general principles. Complete algebraic and topological democracy and the generalization of the notion of p-adic evolution to what might be called rational-adic evolution allow to build plausible and sufficiently general working hypothesis not requiring too much ad hoc assumptions and allowing at least mathematical testing. A further natural principle states that the topology for a given region is such that complex extension of rationals is not needed and that the series defining the normal quaternionic coordinate as function of the space-time quaternionic coordinate converges and gives rise to a smooth surface.

The power series defining solutions of polynomial equations must converge in some topology

The roots of polynomials of several variables can be expressed as Taylor series. When the root is complex, real topology is not possible and some p-adic topology must be considered. This suggests a very attractive dynamical mechanism of p-adicization. In the regions where the root belongs to a complex extension of rationals in the real topology, one could find those values of p for which the series converges p-adically. The rational numbers characterizing the polynomials associated with the generating infinite primes certainly determine the convergence and the primes for which p-adic convergence occurs are certainly functions of these rationals. Hence it could occur that the p-adic topologies for which convergence occurs correspond to the primes appearing as factors in these rationals.

In this approach topology is a result of dynamics. Note that also the notion of symmetry depends on the region of space-time. Contrary to the basic working hypothesis, ...-adic topology of a given space-time sheet is its 'real' topology rather than being only an effective topology and the topology of space-time is completely dynamical being dictated by algebraic physics and smoothness requirement.

It is also possible that convergence does not occur with respect to any ...-adic topology and in this case the topology would be discrete. This situation would correspond to primordial chaos but still the algebraic formulation and Fock space description of the theory would make sense.

Space-time surfaces must be smooth in the completion

The completion must give rise to a smooth or at least continuous ....-adic or real surface defining a critical extremal of Kähler action in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes. This requirement might allow only finite number of...-adic topologies for a given space-time region. If the completion involves functions expandable in powers of a (possibly quaternionic) rational $q = m/n$, then the prime factors of m define natural p-adic number fields for which completion is possible. Also q itself could define rational-adic topology. Since the space-time surface decomposes into regions labeled by rationals in an algebraic extension of rationals q_1, there is interesting possibility that q_1 as such defines the rational-adic topology so that there would be no need to understand why the space-time region labeled by q decomposes into space-time sheets labeled by the prime factors of q.

Whatever the details of the coding are, the coding would mean that the quantum numbers associated with the space-time sheet would determine the generalized ...-adic topology associated with it. The information about quantum systems would be mapped to space-time physics and the coding of quantum numbers to ...-adic topology would solve at a general level the problem how the information about quantum state is coded into the structure of space-time.

2.2.5 Universe as an algebraic hologram?

Quaternionic primes have a natural indentification as four-momenta. If the Minkowski norm for the quaternion is defined using the algebraic norm of the real extension of rationals involved with the state, mass squared is integer-valued as in super-conformal theories. The use of the algebraic norm means a loss of information carried by the units of the real algebraic extension $K(\theta)$ (see the appendix of this chapter). Hence one can say that besides ordinary elementary particle quantum numbers there are algebraic quantum numbers which presumably carry algebraic information. Very effective coding of information about quantum numbers becomes possible and these quantum numbers commute with ordinary quantum numbers. This information does not become manifest for matter-like regions where a real completion of rationals are used. In p-adic regions representing geometric correlates of mind the situation is different since p-adic number field in question is a finite algebraic extension of rationals.

Almost every calculation is approximation and completion to reals or p-adics makes possible to measure how good the approximation is. Real numbers are extremely practical in this respect but the failure of the real number based physics is that it reduces number to a mere quantity having a definite size but no number-theoretical properties. This is practical from the point of view of numerics but means huge loss of capacity for information storage and representation. In algebraic number theory number contains representation for its construction recipe. It seems that the correct manner to see numbers is as elements of the state space provided by the algebraic extension. p-Adic physics using p-adic versions of the algebraic extensions does not lead to a loss of this information unlike real physics. Thus the basic topology of the space-time sheet could code the quantum numbers associated with it.

Since the algebraic extension of rationals, and hence also of p-adics, depends on the number of particles present in the Fock state coded by the infinite prime, the only possible interpretation is that the additional quantum numbers code information about the many-particle state. Hence the idea about 'cognitive representation' of the fractal quantum numbers of particles of the external world suggests itself naturally. In particular, the degree of the minimal polynomial for the real extension $Q(\theta)$ is n, where n is the number of particles in the Fock state in the case the resulting state represents infinite prime. This means that there are $n-1$ quantum numbers represented by fractal scalings (see Appendix for Dirichlet's unit theorem). The interpretation as a representation for the fractal quantum numbers representing information about states of other particles in the system suggests itself. One cannot exclude the possibility that the fractal quantum numbers represent momenta or some other quantum numbers of other particles.

If this rather un-orthodox interpretation is correct, then cognitive representations are present already at the elementary particle level in p-adic regions associated with particles and are realized as algebraic holograms. Universe as a Computer consisting of sub-computers mimicking each other would be realized already at the elementary particle level. This view is consistent with the TGD inspired theory of consciousness. Algebraic physics would also make possible kind of a Gödelian loop by providing a representation for how the information about the structure of a physical system is coded into its properties.

This view has also immediate implications for complexity theory. The dimension of the minimal algebraic extension containing the algebraic number is a unique measure for its complexity. More concretely: the degree of the minimal polynomial measures the complexity. Everyone can solve second order polynomial but very few of us remembers formulas for the roots of fourth order polynomials. For higher orders quadratures do not even exist. Of course, numbers represent typically coordinates and this is consistent with the general coordinate invariance only if some preferred coordinates exist. In TGD based physics these coordinates exist: imbedding space allows (apart from isometries) unique coordinates in which the components of the metric tensor are rational functions of the coordinates.

Similar realization is fundamental in the second almost-proof of Riemann hypothesis described in [K61] . In this case ζ is interpreted as an element in an infinite-dimensional algebraic extension of rationals allowing all roots of rationals. The vanishing of ζ requires that all components of this infinite-dimensional vector contain a common rational factor which vanishes. This is possible only if an infinite number of partition functions in the product representation of the modulus squared of ζ are rational and their product vanishes. This implies Riemann hypothesis. The assumption that only square roots of rationals are needed is very probably wrong and must be replaced with the assumption that p^{iy} is algebraic numbers when $z = 1/2 + iy$ is zero of ζ for any prime p. It is quite possible that the almost-proof survives this generalization.

The notion of Platonia discussed already in the introduction adds cognition to this picture and allows to understand where all those mathematical structures continually invented by mathematicians but not realized physically in the conventional sense of the word reside. This notion takes also the notion of algebraic hologram to its extreme by making space-time points infinitely structured.

2.2.6 How to assign a p-adic prime to a given real space-time sheet?

p-Adic mass calculations force to assign p-adic prime also to the real space-time sheets and the longstanding problem is how this p-adic prime, or possibly many of them, are determined. Number theoretic view about information concept provides a possible solution of this long-standing problem.

Number theoretic information concept

The notion of information in TGD framework differs in some respects from the standard notion.

1. The definition of the entropy in p-adic context is based on the notion p-adic logarithm depending on the p-adic norm of the argument only ($Log_p(x) = Log_p(|x|_p) = n$) [K42] . For rational- and even algebraic number valued probabilities this entropy can be regarded as a real number. The entanglement entropy defined in this manner can be negative so that the entanglement can carry genuine positive information. Rationally/algebraically entangled p-adic system has a positive information content only if the number of the entangled state pairs is proportional to a positive power of the p-adic prime p.

2. This kind of definition of entropy works also in the real-rational/algebraic case and makes always sense for finite ensembles. This would have deep implications. For ordinary definition of the entropy NMP [K42] states that entanglement is minimized in the state preparation process. For the number theoretic definition of entropy entanglement could be generated during state preparation for both p-adic and real sub-systems, and NMP forces the emergence of p-adicity (say the number of entangled state is power of prime). The fragility of quantum coherence is the basic problem of quantum computations and the good news would be that Nature itself (according to TGD) tends to stabilize quantum coherence both in the real and p-adic contexts.

3. Quantum-classical correspondence suggests that the notion of information is well defined also at the space-time level. In the presence of the classical non-determinism of Kähler action and p-adic non-determinism one can indeed define ensembles, and therefore also probability distributions and entropies. For a given space-time sheet the natural ensemble consists of the deterministic pieces of the space-time sheet regarded as different states of the same system.

Are living systems in the intersection of real and p-adic world?

NMP combined with number theoretic entropies leads to an important exception to the rule that the generation of bound state entanglement between system and its environment during U process leads to a loss of consciousness. When entanglement probabilities are rational (or even algebraic) numbers, the entanglement entropy defined as a number theoretic variant of Shannon entropy can be negative so that entanglement carries information. NMP favors the generation of algebraic entanglement. The attractive interpretation is that the generation of algebraic entanglement leads to an expansion of consciousness ("fusion into the ocean of consciousness") instead of its loss. Rational and even algebraic entanglement coefficients make sense in the intersection of real and p-adic words, which suggests that life and conscious intelligence reside in the intersection of the real and p-adic worlds. Life would represent number theoretically criticality so that the quantum criticality of TGD Universe would allow to understand also life.

1. To be in the intersection of real and p-adic worlds means that partonic 2-surfaces and their 4-D tangent planes representing the information about space-time sheet (holography) have a mathematical representation allowing an interpretation either as a real or p-adic surface (just like rationals can be regarded as being common to reals and p-adic numbers). Number theoretical criticality makes also possible the transformation of intentions to actions as transformations of a p-adic 2-surfaces to a real 2-surfaces via leakage through this common intersection. This process makes sense only in zero energy ontology. This would generalize the observation that rationals and algebraics in a well-defined sense represent islands of order in the seas of chaos defined by real and p-adic continua.

2. A more concrete interpretation for the intersection of real and p-adic worlds would be as the intersection of real and p-adic variants of space-time surface allowing interpretation in both number fields. This intersection is discrete set containing besides rational points also algebraic points common to reals and algebraic extension of p-adics involved.

3. These two interpretations for the intersection of real and p-adic worlds need not be independent. The absence of definite integral in p-adic number fields suggests that the transition amplitudes between p-adic and real sectors must be expressible using only the data associated with rational and common algebraic points (in the algebraic extension of p-adic numbers used) of imbedding space. This intersection is discrete and could even consist of a finite number of points. For instance, Fermat's last theorem tells that the surface $x^n + y^n = z^n$ contains only origin as rational point for $n = 3, 4, ...$ whereas for $n = 2$ it contains all rational multiples of integer valued points defining Pythagorean triangles: this is due to the homogenuity of the polynomial in question. Therefore p-adic-to real transition amplitudes would have a purely number theoretical interpretation. One could speak of number theoretical field theory as an analogy for topological field theory.

Does space-time sheet represent integer and its prime factorization?

A long-standing problem of quantum TGD is how to associate to a given real space-time sheet a (not necessarily) unique p-adic prime as required by the p-adic length scale hypothesis. One could achieve this by requiring that for this prime the negentropy associated with the ensemble is maximal. The simplest hypothesis is that a real space-time sheet consisting of N deterministic pieces corresponds to p-adic prime defining the largest factor of N. One could also consider a more general possibility. If N contains p^n as a factor, then the real fractality above n-ary p-adic length scale $L_p(n) = p^{(n-1)/2}L_p$ corresponds to smoothness in the p-adic topology. This option is more attractive since it predicts that the fundamental p-adic length scale L_p for a given p can be effectively replaced by any integer multiple NL_p, such that N is not divisible by p. There is indeed a considerable evidence for small p p-adicity in long length scales. For instance, genetic code and the appearance of binary pairs like cell membrane consisting of liquid layers suggests 2-adicity in nano length scales. This view means that the fractal structure of a given real space-time sheet represents both an integer N and its decomposition to prime factors physically. This obviously conforms with the physics as a generalized number theory vision.

Quantum-classical correspondence suggests that quantum computation processes might have counterparts at the level of space-time. An especially interesting process of this kind is the factorization of integers to prime factors. The classical cryptography relies on the fact that the factorization of large integers to prime factors is a very slow process using classical computation: the time needed to factor 100 digit number using modern computer would take more than the recent age of the universe. For quantum computers the factorization is achieved very rapidly using the famous Shor's algorithm. Does the factorization process indeed have a space-time counterpart?

Suppose that one can map the integer N to be factored to a real space-time sheet with N deterministic pieces. If one can measure the powers $p_i^{n_i}$ of primes p_i for which the fractality above the appropriate p-adic length scale looks smoothness in the p-adic topology, it is possible to deduce the factorization of N by direct physical measurements of the p-adic length scales characterizing the representative space-time sheet (say from the resonance frequencies of the radiation associated with the space-time sheet). If only the p-adic topology corresponding to the largest prime p_1 is realized in this manner, one can deduce first it, and repeat the process for N/p_1^n, and so on, until the full factorization is achieved. A possible test is to generate resonant radiation in a wave guide of having length which is an integer multiple of the fundamental p-adic length scale and to see whether frequencies which correspond to the factors of N appear spontaneously.

2.2.7 Gaussian and Eistenstein primes and physics

Gaussian and Eisenstein primes could give rise to what might be called G- and E-adicities and also these -adicities might be of physical interest.

Gaussian and Eisenstein primes and elementary particle quantum numbers

The properties of Gaussian and Eisenstein primes have intriguing parallels with quantum TGD at the level of elementary particle quantum numbers.

1. The lengths of the complex vectors defined by the non-degenerate Gaussian and Eisenstein primes are square roots of primes as are also the preferred p-adic length scales L_p: this suggests a direct connection with quantum TGD.

2. Each non-degenerate (purely real or imaginary) Gaussian prime of given norm p corresponds to 8 different complex numbers $G = \pm r \pm is$ and $G = \pm s \pm ir$. This is the number of different spin states for the imbedding space spinors and also for the color states of massless gluons (note that in TGD quark color is not spin like quantum number but is analogous to orbital angular momentum). Complex conjugation might be interpreted as a representation of charge conjugation and multiplication by $\pm 1, \pm i$ could give rise to different spin states. The 4-fold degeneracy associated with the $p\ mod\ 4 = 3$ Gaussian primes could correspond to the quartet of massless electro-weak gauge bosons with a given helicity $[(\gamma, Z^0) \leftrightarrow \pm p)$ and $(W^+, W^-) \leftrightarrow \pm ip]$.

3. For Eisenstein prime E_{p_1} the multiplication by $\pm i$ does not respect the rationality of the real part of $|Z_{p_1}|^2$ and the number of states is reduced to four. Eisenstein primes $r + isw$ and $s + irw$ have however the same norm squared so that also now the 8-fold degeneracy is present. When p_1^{iy} is of the general form $r + i\sqrt{k}s$ this degeneracy is not present.

4. The basic character of the quark color is triality realized as phases w which are third roots of unity. The fact that the phases are associated with the Eisenstein primes suggests that they might provide a representation of quark color. One can indeed multiply any Eisenstein prime in the product decomposition by factor 1, w or $\overline{w}$ and the interpretation is that the three primes represent three color states of quark. The obvious interpretation is that each factor Z_{p_1} with $p_1\ mod\ 4 = 1$ could represent 8 possible leptonic states. Each factor Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 1$ conditions simultaneously would correspond to a product of Eisenstein prime with Eisenstein phase and each prime p_i associated with Eisenstein phase would correspond to one color state of quark. Even a number theoretical counterpart of color confinement could be imagined.

   There is also a further interesting analogy supporting the idea about number theoretical counterpart of the quark color. ζ decomposes into a product $\zeta_1 \times \zeta_3$, such that ζ_1 is the product of $p \ mod \ 4 = 1$ partition functions and ζ_3 the product of $p \ mod \ 4 = 3$ partition functions. This decomposition reminds of the leptonic color singlets and color triplet of quarks. Rather interestingly, leptons and quarks correspond to Ramond and Neveu-Schwartz type super Virasoro representations and the fields of N-S representation indeed contain square roots of complex variable existing p-adically for $p \ mod \ 4 = 3$.

5. What about the most general factors $r+is\sqrt{k}$? Can one assign some kind of color degeneracy also with these factors? It seems that this is the case. One can always find phase factors of type $U_\pm = (r \pm is\sqrt{k})/n$ with minimal values of n ($r^2+s^2k = n^2$). The factors $1, U_\pm$ clearly give rise to a 3-fold degeneracy analogous to color degeneracy.

6. What about interpretation of the components of the complex integers? For Super Virasoro representations basic quantum numbers of this kind correspond to energy and longitudinal momentum. This suggests the interpretation of r^2+s^2k as energy, r^2-s^2k as mass, and $2rs\sqrt{k}$ as momentum. For the squares $r^2-s^2+(2rs-s^2)w$ of Eisenstein primes $r^2-s^2/2-rs$ corresponds to mass, r^2+s^2-rs to energy, and $(2rs-s^2)\sqrt{3}/2$ to momentum. Note that the sign of mass changes for Gaussian primes in the interchange $r \leftrightarrow s$. The fact that the hexagonal lattice defined by Eisenstein integers correspond to the root lattice of $SU(3)$ group means that energy, momentum and mass corresponds to the sides of the triangles in the root lattice of color group.

The following argument suggests that finite Gaussian and Eistenstein primes might be forced by zero energy ontology (ZEO)

1. In ZEO M-matrix is in a well-defined sense "complex" square root of density matrix reducing to a product of Hermitian square root of density matrix multiplied by unitary S-matrix. A natural guess is that p-adic thermodynamics possesses this kind of square root or better to say: is modulus squared for it.

2. For fermions the value of p-adic temperature is however $T=1$ and thus maximal. It is not possible to construct real square root by simply taking the square root of thermodynamical probabilities for various conformal weights. One manner to solve the problem is to assume that one has quadratic algebraic extension of p-adic numbers in which the p-adic prime splits as $p = \pi\overline{\pi}$, $\pi = m+\sqrt{-k}n$. For $k=1$ primes $p \ \ mod \ \ 4 = 1$ indeed allow a representation as product of Gaussian prime and its conjugate.

3. For primes $p \ \ mod \ \ 4 = 3$ this is not the case and Mersenne primes are important examples of these primes. Eisenstein primes provide the simplest extension of rationals splitting Mersenne primes. For Eisenstein primes one has $k=3$ and all ordinary primes satisfying either $p=3$ or $p \ \ mod \ \ 3 = 1$ (true for Mersenne primes) allows this splitting. For the square root of p-adic thermodynamics the complex square roots of probabilities would be given by $\pi^{L_0/T}/\sqrt{Z}$, and the moduli squared would give thermodynamical probabilities as $p^{L_0/T}/Z$. Here Z is the partition function.

4. An interesting question is whether $T=1$ for fermions means that complex square of thermodynamics is indeed complex and whether $T=1/2$ for bosons means that the square root is actually real valued.

G-adic, E-adic and even more general fractals?

Still one line of thoughts relates to the possibility to generalize the notion of p-adicity so that could speak about G-adic and E-adic number fields. The properties of the Gaussian and Eisenstein primes indeed strongly suggest a generalization for the notion of p-adic numbers to include what might be called G-adic or E-adic numbers. In fact, the argument generalizes to the case of all nine $\sqrt{-d}$ type extensions of rationals allowing a unique prime decomposition so that one might perhaps speak about D-adics.

1. Consider for definiteness Gaussian primes. The basic point is that the decomposition into a product of prime factors is unique. For a given Gaussian prime one could consider the

representation of the algebraic extension involved (complex integers in the case of Gaussian primes) as a ring formed by the formal power series

$$G = \sum_n z_n G_p^n \ . \tag{2.2.3}$$

Here z_n is Gaussian integer with norm smaller than $|G_p|$, which equals to p for $p\ mod\ 4 = 3$ and $\sqrt{p}$ for $p\ mod\ 4 = 1$.

2. If any Gaussian integer z has a unique expansion in powers of G_p such that coefficients have norm squared smaller than p, modulo G arithmetics makes sense and one can construct the inverse of G and number field results. This is the case if Gaussian integers behave with respect to modulo G_p arithmetics like finite field $G(p,2)$. For $p\ mod\ 4 = 1$ the extension of the p-adic numbers by introducing $\sqrt{-1}$ as a unit is not possible since $\sqrt{-1}$ exists as a p-adic number: the proposed structure might perhaps provide the counterpart of the p-adic complex numbers in the case$p\ mod\ 4 = 1$. Thus the question is whether one could regard Gaussian p-adic numbers as a natural complexification of p-adics for $p\ mod\ 4 = 1$, perhaps some kind of square root of R_p, and if they indeed form a number field, do they reduce to some known algebraic extension of R_p?

3. In the case of Eisenstein numbers one can identify the coefficients z_n in the formal power series $E = \sum z_n E_p^n$ as Eisenstein numbers having modulus square smaller than p associated with E_p and similar argument works also in this case.

4. As already noticed, in the case of complex extensions of form $r + \sqrt{-d}s$ a unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [A140] . The poor man's argument above does not distinguish between G- and E-adics ($d = 1, 3$) and these extensions.One might perhaps call this extensions generally D-adics. This suggests that generalized p-adics could exist also in this case. In fact, the generalization p-adics could make sense also for higher-dimensional algebraic extensions allowing unique prime decomposition. For $d = 2$ complex algebraic primes are of form $r + s\sqrt{-2}$ satisfying the condition $r^2 + 2s^2 = p$. For $d > 2$ complex algebraic primes are of form $(r + s\sqrt{-d})/2$ such that both r and s are even or odd. Quite generally, the condition $p\ mod\ d = k^2$ must be satisfied. $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$ so that the powers of a complex primes in this case have a finite number of possible phase angles: this might make G- and E-adics physically special.

TGD suggests rather interesting physical applications of D-adics.

1. What is interesting from the physics point of view is that for $p\ mod\ 4 = 1$ the points D_p^n are on the logarithmic spiral $z_n = p^{n/2}exp(in\phi_0/2)$, where ϕ is the phase associated with D_p^2. The logarithmic spiral can be written also as $\rho = exp(nlog(p)\phi/\phi_0)$. This reminds strongly of the logarithmic spirals, which are fractal structures frequently encountered in self-organizing systems: D-adics might provide the mathematics for the modelling of these structures.

2. p-Adic length scale hypothesis should hold true also for Gaussian primes, in particular, Gaussian Mersennes of form $(1 \pm i)^k - 1$ should be especially interesting from TGD point of view.

   (a) The integers k associated with the lowest Gaussian Mersennes are following: $2, 3, 5, 7, 11, 19, 29, 47, 73, 79, 113$. $k = 113$ corresponds to the p-adic length scale associated with the atomic nucleus and muon. Thus all known charged leptons, rather than only e and τ, as well as nuclear physics length scale, correspond to Mersenne primes in the generalized sense.

   (b) The primes $k = 151, 157, 163, 167$ define perhaps the most fundamental biological length scales: $k = 151$ corresponds to the thickness of the cell membrane of about ten nanometers and $k = 167$ to cell size about 2.56 μm. This strongly suggests that cellular organisms have evolved to their present form through four basic stages.

(c) $k = 239, 241, 283, 353, 367, 379, 457$ associated with the next Gaussian Mersennes define astronomical length scales. $k = 239$ and $k = 241$ correspond to the p-adic time scales .55 ms and 1.1 ms: basic time scales associated with nerve pulse transmission are in question. $k = 283$ corresponds to the time scale of 38.6 min. An interesting question is whether this period could define a fundamental biological rhythm. The length scale $L(353)$ corresponds to about 2.6×10^6 light years, roughly the size scale of galaxies. The length scale $L(367) \simeq \times 3.3 \times 10^8$ light years is of same order of magnitude as the size scale of the large voids containing galaxies on their boundaries (note the analogy with cells). $T(379) \simeq 2.1 \times 10^{10}$ years corresponds to the lower bound for the order of the age of the Universe. $T(457) \sim 10^{22}$ years defines a completely super-astronomical time and length scale.

3. Eisenstein integers form a hexagonal lattice equivalent with the root lattice of the color group $SU(3)$. Microtubular surface defines a hexagonal lattice on the surface of a cylinder which suggests an interpretation in terms of E-adicity. Also the patterns of neural activity form often hexagonal lattices.

Gaussian and Eisenstein versions of infinite primes

The vision about quantum TGD as a generalized number theory generates a further line of thoughts.

1. As has been found, the zeros of ζ code for the physical states of a super-symmetric arithmetic quantum field theory. As a matter fact, the arithmetic quantum field theory in question can be identified as arithmetic quantum field theory in which single particle states are labeled by Gaussian primes. The properties of the Gaussian primes imply that the single particle states of this theory have 8-fold degeneracy plus the four-fold degeneracy related to the $\pm i$ or ± 1-factor which could be interpreted as a phase factor associated with any $p \ mod \ 4 = 3$ type Gaussian prime. Also Eisenstein primes could allow the construction of a similar arithmetic quantum field theory.

2. The construction of the infinite primes reduces to a repeated second quantization of an arithmetic quantum field theory. A straightforward generalization of the procedure of the previous chapter allows to define also the notion of infinite Gaussian and Eisenstein primes. Since each infinite prime is in a well-defined sense a composite of finite primes playing the role of elementary particles, this would mean that each composite prime in the expansion of an infinite prime has either four-fold degeneracy or eight-fold degeneracy. The interpretation of infinite primes could thus literally be as many-particle states of quantum TGD.

2.2.8 p-Adic length scale hypothesis and quaternionic primality

p-Adic length scale hypothesis states that fundamental length scales correspond to the so called p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives strong support for this hypothesis. Elementary particles correspond to the so called CP_2 type extremals in TGD. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [A206] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one can identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime.

The crucial point from the TGD point of view is the appearance of the *square* of the norm instead of the norm. One can even define the product of spheres $R^2 = n_1$ and $R^2 = n_2$ by defining the product sphere with norm squared $R^2 = n_1 n_2$ to consist of the quaternions, which are products of quaternions with norms squared n_1 and n_2 respectively. Prime spheres correspond to $n = p$. The powers of sphere p correspond to a multiplicatively closed structure consisting of powers p^n of the sphere p. It is also possible to speak about the multiplication of balls and prime balls in the case of integer quaternions.

p-Adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the p-adic QFT limit of TGD as also in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k:th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface. This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the criticality of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

2.3 Scaling hierarchies and physics as a generalized number theory

The scaling hierarchies defined by powers of Φ and primes p probably reflect something very profound. Mueller has proposed also a scaling law in powers of e [B3] . This scaling law can be however questioned since $\Phi^2 = 2.6180..$ is rather near to $e = 2.7183...$ Note that powers of e define p-dimensional extension of R_p since e^p exists as a p-adic number in this case.

The interpretation of the p-adic as physics of cognition and the vision about reduction of physics to rational physics continuable algebraically to various extensions of rationals and p-adic number fields is an attractive general framework allowing to understand how p-adic fractality could emerge in real physics. In this section it will be found that this vision provides a concrete tool in principle allowing to construct global solutions of field equations by reducing long length scale real physics to short length scale p-adic physics. Also p-adic length scale hypothesis can be understood and the notion of multi-p p-fractality can be formulated in precise sense in this framework. This vision leads also to a concrete quantum model for how intentions are transformed to actions and the S-matrix for the process has the same general form as the ordinary S-matrix.

The fractal hierarchy associated with Golden mean cannot be understood in a manner analogous to p-adic fractal hierarchies. Rather, the understanding of Golden Mean and Fibonacci series could reduce to the hypothesis that space-time surfaces, and thus the geometry of physical systems, provide a representations for the hierarchy of Fibonacci numbers characterizing the Jones inclusions of infinite-dimensional Clifford sub-algebras of WCW spinors identifiable as infinite-dimensional von Neumann algebras known as hyper-finite factors of type II_1 (not that WCW corresponds here to the "world of classical worlds"). The emergence of powers of e has been discussed in [K61] and will not be discussed here.

2.3.1 p-Adic physics and the construction of solutions of field equations

The number theoretic vision about physics relies on the idea that physics or, rather what we can know about it, is basically rational number based. One interpretation would be that space-time surfaces, the induced spinors at space-time surfaces, WCW spinor fields, S-matrix, etc..., can be obtained by algebraically continuing their values in a discrete subset of rational variant of the geometric structure considered to appropriate completion of rationals (real or p-adic). The existence of the algebraic continuation poses very strong additional constraints on physics but has not provided any practical means to solve quantum TGD.

In the following it is however demonstrated that this view leads to a very powerful iterative method of constructing global solutions of classical field equations from local data and at the same time gives justification for the notion of p-adic fractality, which has provided very successful approach not only to elementary particle physics but also physics at longer scales. The basic idea is that mere p-adic continuity and smoothness imply fractal long range correlations between rational points which are very close p-adically but far from each other in the real sense and vice versa.

The emergence of a rational cutoff

For a given p-adic continuation only a subset of rational points is acceptable since the simultaneous requirements of real and p-adic continuity can be satisfied only if one introduces ultraviolet cutoff length scale. This means that the distances between subset of rational points fixing the dynamics of the quantities involved are above some cutoff length scale, which is expected to depend on the p-adic number field R_p as well as a particular solution of field equations. The continued quantities coincide only in this subset of rationals but not in shorter length scales.

The presence of the rational cutoff implies that the dynamics at short scales becomes effectively discrete. Reality is however not discrete: discreteness and rationality only characterize the inherent limitations of our knowledge about reality. This conforms with the fact that our numerical calculations are always discrete and involve finite set of points.

The intersection points of various p-adic continuations with real space-time surface should code for all actual information that a particular p-adic physics can give about real physics in classical sense. There are reasons to believe that real space-time sheets are in the general case characterized by integers n decomposing into products of powers of primes p_i. One can expect that for p_i-adic continuations the sets of intersection points are especially large and that these p-adic space-time surfaces can be said to provide a good discrete cognitive mimicry of the real space-time surface.

Adelic formula represents real number as product of inverse of its p-adic norms. This raises the hope that taken together these intersections could allow to determine the real surface and thus classical physics to a high degree. This idea generalizes to quantum context too.

The actual construction of the algebraic continuation from a subset of rational points is of course something which cannot be done in practice and this is not even necessary since much more elegant approach is possible.

Hierarchy of algebraic physics

One of the basic hypothesis of quantum TGD is that it is possible to define exponent of Kähler action in terms of fermionic determinants associated with the modified Dirac operator derivable from a Dirac action related super-symmetrically to the Kähler action.

If this is true, a very elegant manner to define hierarchy of physics in various algebraic extensions of rational numbers and p-adic numbers becomes possible. The observation is that the continuation to various p-adic numbers fields and their extensions for the fermionic determinant can be simply done by allowing only the eigenvalues which belong to the extension of rationals involved and solve field equations for the resulting Kähler function. Hence a hierarchy of fermionic determinants results. The value of the dynamical Planck constant characterizes in this approach the scale factor of the M^4 metric in various number theoretical variants of the imbedding space $H = M^4 \times CP_2$ glued together along subsets of rational points of H. The values of $\hbar$ are determined from the requirement of quantum criticality [K80] meaning that Kähler coupling strength is analogous to critical temperature.

In this approach there is no need to restrict the imbedding space points to the algebraic extension of rationals and to try to formulate the counterparts of field equations in these discrete imbedding spaces.

p-Adic short range physics codes for long range real physics and vice versa

One should be able to construct global solutions of field equations numerically or by engineering them from the large repertoire of known exact solutions [K10] . This challenge looks formidable since the field equations are extremely non-linear and the failure of the strict non-determinism seems to make even in principle the construction of global solutions impossible as a boundary value problem or initial value problem.

The hope is that short distance physics might somehow code for long distance physics. If this kind of coding is possible at all, p-adicity should be crucial for achieving it. This suggests that one must articulate the question more precisely by characterizing what we mean with the phrases "short distance" and "long distance". The notion of short distance in p-adic physics is completely different from that in real physics, where rationals very close to each other can be arbitrary far away in the real sense, and vice versa. Could it be that in the statement "Short length scale physics codes for long length scale physics" the attribute "short"/"long" could refer to p-adic/real norm, real/p-adic norm, or both depending on the situation?

The point is that rational imbedding space points very near to each other in the real sense are in general at arbitrarily large distances in p-adic sense and vice versa. This observation leads to an elegant method of constructing solutions of field equations.

1. Select a rational point of the imbedding space and solve field equations in the real sense in an arbitrary small neighborhood U of this point. This can be done with an arbitrary accuracy by chosing U to be sufficiently small. It is possible to solve the linearized field equations or use a piece of an exact solution going through the point in question.

2. Select a subset of rational points in U and interpret them as points of p-adic imbedding space and space-time surface. In the p-adic sense these points are in general at arbitrary large distances from each and real continuity and smoothness alone imply p-adic long range correlations. Solve now p-adic field equations in p-adically small neighborhoods of these points. Again the accuracy can be arbitrarily high if the neighborhoods are choose small enough. The use of exact solutions of course allows to overcome the numerical restrictions.

3. Restrict the solutions in these small p-adic neighborhoods to rational points and interpret these points as real points having arbitrarily large distances. p-Adic smoothness and continuity alone imply fractal long range correlations between rational points which are arbitrary distant in the real sense. Return to 1) and continue the loop indefinitely.

In this manner one obtains even in numerical approach more and more small neighborhoods representing almost exact p-adic and real solutions and the process can be continued indefinitely.

Some comments about the construction are in order.

1. Essentially two different field equations are in question: real field equations fix the local behavior of the real solutions and p-adic field equations fix the long range behavior of real solutions. Real/p-adic global behavior is transformed to local p-adic/real behavior. This might be the deepest reason why for the hierarchy of p-adic physics.

2. The failure of the strict determinism for the dynamics dictated by Kähler action and p-adic non-determinism due to the existence of p-adic pseudo constants give good hopes that the construction indeed makes it possible to glue together the (not necessarily) small pieces of space-time surfaces inside which solutions are very precise or exact.

3. Although the full solution might be impossible to achieve, the predicted long range correlations implied by the p-adic fractality at the real space-time surface are a testable prediction for which p-adic mass calculations and applications of TGD to biology provide support.

4. It is also possible to generalize the procedure by changing the value of p at some rational points and in this manner construct real space-time sheets characterized by different p-adic primes.

5. One can consider also the possibility that several p-adic solutions are constructed at given rational point and the rational points associated with p-adic space-time sheets labeled by $p_1, ..., p_n$ belong to the real surface. This would mean that real surface would be multi-p p-adic fractal.

I have earlier suggested that even elementary particles are indeed characterized by integers and that only particles for which the integers have common prime factors interact by exchanging particles characterized by common prime factors. In particular, the primes $p = 2, 3, ..., 23$ would be common to the known elementary particles and appear in the expression of the gravitational constant. Multi-p p-fractality leads also to an explanation for the weakness of the gravitational constant. The construction recipe for the solutions would give a concrete meaning for these heuristic proposals.

This approach is not restricted to space-time dynamics but is expected to apply also at the level of say S-matrix and all mathematical object having physical relevance. For instance, p-adic four-momenta appear as parameters of S-matrix elements. p-Adic four-momenta very near to each other p-adically restricted to rational momenta define real momenta which are not close to each other and the mere p-adic continuity and smoothness imply fractal long range correlations in the real momentum space and vice versa.

p-Adic length scale hypothesis

Approximate p_1-adicity implies also approximate p_2-adicity of the space-time surface for primes $p \simeq p_1^k$. p-Adic length scale hypothesis indeed states that primes $p \simeq 2^k$ are favored and this might be due to simultaneous $p \simeq 2^k$- and 2-adicity. The long range fractal correlations in real space-time implied by 2-adicity would indeed resemble those implied by $p \simeq 2^k$ and both $p \simeq 2^k$-adic and 2-adic space-time sheets have larger number of common points with the real space-time sheet.

If the scaling factor λ of $\hbar$ appearing in the dark matter hierarchy is in good approximation $\lambda = 2^{11}$ also dark matter hierarchy comes into play in a resonant manner and dark space-time sheets at various levels of the hierarchy tend to have many intersection points with each other.

There is however a problem involved with the understanding of the origin of the p-adic length scale hypothesis if the correspondence via common rationals is assumed.

1. The mass calculations based on p-adic thermodynamics for Virasoro generator L_0 predict that mass squared is proportional to $1/p$ and Uncertainty Principle implies that L_p is proportional to $\sqrt{p}$ rather than p, which looks more natural if common rationals define the correspondence between real and p-adic physics.

2. It would seem that length $d_p \simeq pR$, R or order CP_2 length, in the induced space-time metric must correspond to a length $L_p \simeq \sqrt{p}R$ in M^4. This could be understood if space-like geodesic lines at real space-time sheet obeying effective p-adic topology are like orbits of a particle performing Brownian motion so that the space-like geodesic connecting points with M^4 distance r_{M^4} has a length $r_{X^4} \propto r_{M^4}^2$. Geodesic random walk with randomness associated with the motion in CP_2 degrees of freedom could be in question. The effective p-adic topology indeed induces a strong local wiggling in CP_2 degrees of freedom so that r_{X^4} increases and can depend non-linearly on r_{M^4}.

3. If the size of the space-time sheet associated with the particle has size $d_p \sim pR$ in the induced metric, the corresponding M^4 size would be about $L_p \propto \sqrt{p}R$ and p-adic length scale hypothesis results.

4. The strongly non-perturbative and chaotic behavior $r_{X^4} \propto r_{M^4}^2$ is assumed to continue only up to L_p. At longer length scales the space-time distance d_p associated with L_p becomes the unit of space-time distance and geodesic distance r_{X^4} is in a good approximation given by

$$r_{X^4} = \frac{r_{M^4}}{L_p} d_p \propto \sqrt{p} \times r_{M^4} \ , \tag{2.3.1}$$

and is thus linear in M^4 distance r_{M^4}.

Does cognition automatically solve real field equations in long length scales?

In TGD inspired theory of consciousness p-adic space-time sheets are identified as space-time correlates of cognition. Therefore our thoughts would have literally infinite size in the real topology if p-adics and reals correspond to each other via common rationals (also other correspondence based on the separate canonical identification of integers m and n in $q = m/n$ with p-adic numbers).

The cognitive solution of field equations in very small p-adic region would solve field equations in real sense in a discrete point set in very long real length scales. This would allow to understand why the notions of Universe and infinity are a natural part of our conscious experience although our sensory input is about an infinitesimally small region in the scale of universe.

The idea about Universe performing mimicry at all possible levels is one of the basic ideas of TGD inspired theory of consciousness. Universe could indeed understand and represent the long length scale real dynamics using local p-adic physics. The challenge would be to make quantum jumps generating p-adic surfaces having large number of common points with the real space-time surface. We are used to call this activity theorizing and the progress of science towards smaller real length scales means progress towards longer length scales in p-adic sense. Also real physics can represent p-adic physics: written language and computer represent examples of this mimicry.

2.3.2 A more detailed view about how local p-adic physics codes for p-adic fractal long range correlations of the real physics

The vision just described gives only a rough heuristic view about how the local p-adic physics could code for the p-adic fractality of long range real physics. There are highly non-trivial details related to the treatment of M^4 and CP_2 coordinates and to the mapping of p-adic H-coordinates to their real counterparts and vice versa.

How real and p-adic space-time regions are glued together?

The first task is to visualize how real and p-adic space-time regions relate to each other. It is convenient to start with the extension of real axis to contain also p-adic points. For finite rationals $q = m/n$, m and n have finite power expansions in powers of p and one can always write $q = p^k \times r/s$ such that r and s are not divisible by p and thus have pinary expansion of in powers of p as $x = x_0 + \sum_1^N x_n p^n$, $x_i \in \{0, p\}$, $x_0 \neq 0$.

One can always express p-adic number as $x = p^n y$ where y has p-adic norm 1 and has expansion in non-negative powers of p. When x is rational but not integer the expansion contains infinite number of terms but is periodic. If the expansion is infinite and non-periodic, one can speak about *strictly p-adic* number having infinite value as a real number.

In the same manner real number x can be written as $x = p^n y$, where y is either rational or has infinite non-periodic expansion $y = r_0 + \sum_{n>0} r_n p^{-n}$ in negative powers of p. As a p-adic number y is infinite. In this case one can speak about strictly real numbers.

This gives a visual idea about what the solution of field equations locally in various number fields could mean and how these solutions are glued together along common rationals (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also). In the following I shall be somewhat sloppy and treat the rational points of the imbedding space as if they were points of real axis in order to avoid clumsy formulas.

1. The p-adic variants of field equations can be solved in the strictly p-adic realm and by p-adic smoothness these solutions are well defined also in as subset of rational points. The strictly p-adic points in a neighborhood of a given rational point correspond as real points to infinitely distant points of M^4. The possibility of p-adic pseudo constants means that for rational points of M^4 having sufficiently large p-adic norm, the values of CP_2 coordinates or induced spinor fields can be chosen more or less freely.

2. One can solve the p-adic field equations in any p-adic neighborhood $U_n(q) = \{x = q + p^n y\}$ of a rational point q of M^4, where y has a unit p-adic norm and select the values of fields at different points q_1 and q_2 freely as long as the spheres $U_n(q_1)$ and $U_n(q_2)$ are disjoint (these spheres are either identical or disjoint by p-adic ultra-metricity).

The points in the p-adic continuum part of these solutions are at an infinite distance from q in M^4. The points which are well-defined in real sense form a discrete subset of rational points of M^4. The p-adic space-time surface constructed in this manner defines a discrete fractal hierarchy of rational space-time points besides the original points inside the p-adic spheres. In real sense the rational points have finite distances and could belong to disjoint real space-time sheets. The failure of the strict non-determinism for the field equations in the real sense gives hopes for gluing these sheets partially together (say in particle reactions with particles represented as 3-surfaces).

3. All rational points q of the p-adic space-time sheet can be interpreted as real rational points and one can solve the field equations in the real sense in the neighborhoods $U_n(q) = \{x = q+p^n y\}$ corresponding to real numbers in the the range $p^n \leq x \leq p^{n+1}$. Real smoothness and continuity fix the solutions at finite rational points inside $U_n(q)$ and by the phenomenon of p-adic pseudo constants these values can be consistent with p-adic field equations. Obviously one can can continue the construction process indefinitely.

p-Adic scalings act only in M^4 degrees of freedom

p-Adic fractality suggests that finite real space-time sheets around points $x + p^n$, $x = 0$, are obtained as by just scaling of the M^4 coordinates having origin at $x = 0$ by p^n of the solution defined in a neighborhood of x and leaving CP_2 coordinates as such. The known extremals of Kähler action indeed allow M^4 scalings as dynamical symmetries.

One can understand why no scaling should appear in CP_2 degrees of freedom. CP_2 is complex projective space for which points can be regarded as complex planes and for these p-adic scalings act trivially. It is worth of emphasizing that here could lie a further deep number theoretic reason for why the space S in $H = M^4 \times S$ must be a projective space.

What p-adic fractality for real space-time surfaces really means?

The identification of p-adic and real M^4 coordinates of rational points as such is crucial for p-adic fractality. On the other hand, the identification rational real and p-adic CP_2 coordinates as such would not be consistent with the idea that p-adic smoothness and continuity imply p-adic fractality manifested as long range correlations for real space-time sheets

The point is that p-adic fractality is not stable against small p-adic deformations of CP_2 coordinates as function of M^4 coordinates for solutions representable as maps $M^4 \rightarrow CP_2$. Indeed, if the rational valued p-adic CP_2 coordinates are mapped as such to real coordinates, the addition of large power p^n to CP_2 coordinate implies small modification in p-adic sense but large change in the real sense so that correlations of CP_2 at p-adically scaled M^4 points would be completely lost.

The situation changes if the map of p-adic CP_2 coordinates to real ones is continuous so that p-adically small deformations of the p-adic space-time points are mapped to small real deformations of the real space-time points.

1. Canonical identification $I : x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ satisfies continuity constraint but does not map rationals to rationals.

2. The modification of the canonical identification given by

$$I(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{2.3.2}$$

is uniquely defined for rational points, maps rationals to rationals, has a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$.

3. The form of this map is not general coordinate invariant nor invariant under color isometries. The natural requirement is that the map should respect the symmetries of CP_2 maximally.

Therefore the complex coordinates transforming linearly under U(2) subgroup of SU(3) defining the projective coordinates of CP_2 are a natural choice. The map in question would map the real components of complex coordinates to their p-adic variants and vice versa. The residual $U(2)$ symmetries correspond to rational unitary 2×2-matrices for which matrix elements are of form $U_{ij} = p^k r/s$, $r < p, s < p$. It would seem that these transformations must form a finite subgroup if they define a subgroup at all. In case of $U(1)$ Pythagorean phases define rational phases but sufficiently high powers fail to satisfy the conditions $r < p, s < p$. Also algebraic extensions of p-adic numbers can be considered.

4. The possibility of pseudo constant allows to modify canonical identification further so that it reduces to the direct identification of real and p-adic rationals if the highest powers of p in r and s ($q = p^n r/s$) are not higher than p^N. Write $x = \sum_{n\geq 0} x_n p^n = x^{N)} + p^{N+1}y$ with $x^{N)} = \sum_{n=0}^{N} x_n p^n$, $x_0 \neq 0$, $y_0 \neq 0$, and define $I_N(x) = x^{N)} + p^{N+1}I(y)$. For $q = p^n r/s$ define $I_N(q) = p^n I_N(r)/I_N(s)$. This map reduces to the direct identification of real and p-adic rationals for $y = 0$.

5. There is no need to introduce the imaginary unit explicitly. In case of spinors imaginary unit can be represented by the antisymmetric 2×2-matrix ϵ_{ij} satisfying $\epsilon_{12} = 1$. As a matter fact, the introduction of imaginary unit as number would lead to problems since for $p \ mod\ 4 = 3$ imaginary unit should be introduced as an algebraic extension and CP_2 in this sense would be an algebraic extension of RP_2. The fact that the algebraic extension of p-adic numbers by $\sqrt{-1}$ is equivalent with an extension introducing $\sqrt{p-1}$ supports the view that algebraic imaginary unit has nothing to do with the geometric imaginary unit defined by Kähler form of CP_2. For $p \ mod\ 4 = 1$ $\sqrt{-1}$ exists as a p-adic number but is infinite as a real number so that the notion of finite complex rational would not make sense.

Preferred CP_2 coordinates as a space-time correlate for the selection of quantization axis

Complex CP_2 coordinates are fixed only apart from the choice of the quantization directions of color isospin and hyper charge axis in SU(3) Lie algebra. Hence the selection of quantization axes seems to emerge at the level of the generalized space-time geometry as quantum classical correspondence indeed requires.

In a well-defined sense the choice of the quantization axis and a special coordinate system implies the breaking of color symmetry and general coordinate invariance. This breaking is induced by the presence of p-adic space-time sheets identified as correlates for cognition and intentionality. One could perhaps say that the cognition affects real physics via the imbedding space points shared by real and p-adic space-time sheets and that these common points define discrete coordinatization of the real space-time surface analogous to discretization resulting in any numerical computation.

Relationship between real and p-adic induced spinor fields

Besides imbedding space coordinates also induced spinor fields are fundamental variables in TGD. The free second quantized induced spinor fields define the fermionic oscillator operators in terms of which the gamma matrices giving rise to spinor structure of the "world of classical worlds" can be expressed.

p-Adic fractal long range correlations must hold true also for the induced spinor fields and they are in exactly the same role as CP_2 coordinates so that the variant of canonical identification mapping rationals to rationals should map the real and imaginary parts of real induced spinor fields to their p-adic counterparts and vice versa at the rational space-time points common to p-adic and real space-time sheets.

Could quantum jumps transforming intentions to actions really occur?

The idea that intentional action corresponds to a quantum jump in which p-adic space-time sheet is transformed to a real one traversing through rational points common to p-adic and real space-time sheet is consistent with the conservation laws since the sign of the conserved inertial energy can be also negative in TGD framework and the density of inertial energy vanishes in cosmological length scales [K65] . Also the non-diagonal transitions $p_1 \rightarrow p_2$ are in principle possible and would correspond to intersections of p-adic space-time sheets having a common subset of rational points. Kind of phase transitions changing the character of intention or cognition would be in question.

1. Realization of intention as a scattering process

The first question concerns the interpretation of this process and possibility to find some familiar counterpart for it in quantum field theory framework. The general framework of quantum TGD suggests that the points common to real and p-adic space-time sheets could perhaps be regarded as arguments of an n-point function determining the transition amplitudes for p-adic to real transition or $p_1 \rightarrow p_2$-adic transitions. The scattering event transforming an p-adic surface (infinitely distant real surface in real M^4) to a real finite sized surface (infinitely distant p-adic surface in p-adic M^4) would be in question.

2. Could S-matrix for realizations of intentions have the same general form as the ordinary S-matrix?

One might hope that the realization of intention as a number theoretic scattering process could be characterized by an S-matrix, which one might hope of being unitary in some sense. These S-matrix elements could be interpreted at fundamental level as probability amplitudes between intentions to prepare a define initial state and the state resulting in the process.

Super-conformal invariance is a basic symmetry of quantum TGD which suggests that the S-matrix in question should be constructible in terms of n-point functions of a conformal field theory restricted to a subset of rational points shared by real and p-adic space-time surfaces or their causal determinants. According to the general vision discussed in [K19] , the construction of n-point functions effectively reduces to that at 2-dimensional sections of light-like causal determinants of space-time surfaces identified as partonic space-time sheets.

The idea that physics in various number fields results by algebraic continuation of rational physics serves as a valuable guideline and suggests that the form of the S-matrices between different number fields (call them non-diagonal S-matrices) could be essentially the same as that of diagonal S-matrices. If this picture is correct then the basic differences to ordinary real S-matrix would be following.

1. Intentional action could transform p-adic space-time surface to a real one only if the exponent of Kähler function for both is rational valued (or belongs to algebraic extension of rationals).

2. The points appearing as arguments of n-point function associated with the non-diagonal S-matrix are a subset of rational points of imbedding space whereas in the real case, where the integration over these points is well defined, all values of arguments can be allowed. Thus the difference between ordinary S-matrix and more general S-matrices would be that a continuous Fourier transform of n-point function in space-time domain is not possible in the latter case. The inherent nature of cognition would be that it favors localization in the position space.

3. Objection and its resolution

Exponent of Kähler function is the key piece of the configuration space spinor field. There is a strong counter argument against the existence of the Kähler function in the p-adic context. The basic problem is that the definite integral defining the Kähler action is not p-adically well-defined except in the special cases when it can be done algebraically. Algebraic integration is however very tricky and numerically completely unstable.

The definition of the exponent of Kähler function in terms of Dirac determinants or, perhaps equivalently, as a result of normal ordering of the modified Dirac action for second quantized induced spinors might however lead to an elegant resolution of this problem. This approach is discussed in detail in [K15, K10] . The idea is that Dirac determinant can be defined as a product of eigenvalues of the modified Dirac operator and one ends up to a hierarchy of theories based on the restriction of the eigenvalues to various algebraic extensions of rationals identified as a hierarchy associated with corresponding algebraic extensions of p-adic numbers. This hierarchy corresponds to a hierarchy of theories (and also physics!) based on varying values of Planck constant. The elegance of this approach is that no discretization at space-time level would be neededeverything reduces to the generalized eigenvalue spectrum of the modified Dirac operator.

4. A more detailed view

Consider the proposed approach in more detail.

1. Fermionic oscillator operators are assigned with the generalized eigenvectors of the modified Dirac operator defined at the light-like causal determinants:

$$\begin{aligned} \Psi &= \sum_n \Psi_n b_n \ , \\ D\Psi_n &= \Gamma^\alpha D_\alpha \Psi_n = \lambda_n O \Psi_n \ , \ \ O \equiv n_\alpha \Gamma^\alpha \ . \end{aligned} \tag{2.3.3}$$

   Here $\Gamma^\alpha = T^{\alpha k}\Gamma_k$ denote so called modified gamma matrices expressible in terms of the energy momentum current $T^{\alpha k}$ assignable to Kähler action [K15] . The replacement of the ordinary gamma matrices with modified ones is forced by the requirement that the super-symmetries of the modified Dirac action are consistent with the property of being an extremal of Kähler action. n_α is a light like vector assignable to the light-like causal determinant and $O = n_\alpha \Gamma^\alpha$ must be rational and have the same value at real and p-adic side at rational points. The integer n labels the eigenvalues λ_n of the modified Dirac operator, and b_n corresponds to the corresponding fermionic oscillator operator.

2. The condition that the p-adic and real variants Ψ if the Ψ are identical at common rational points of real and p-adic space-time surface (the same applies to 4-surfaces corresponding to different p-adic number fields) poses a strong constraint on the algebraic continuation from rationals to p-adics and gives hopes of deriving implications of this approach.

3. Ordinary fermionic anti-commutation relations do not refer specifically to any number field. Super Virasoro (anti-)commutation relations involve only rationals. This suggest that fermionic Fock space spanned by the oscillator operators b_n is universal and same for reals and p-adic numbers and can be regarded as rational. Same would apply to Super Virasoro representations. Also the possibility to interpret WCW spinor fields as quantum superpositions of Boolean statements supports this kind of universality. This gives good hopes that the contribution of the inner produces between Fock states to the S-matrix elements are number field independent.

4. Dirac determinant can be defined as the product of the eigenvalues λ_n restricted to a given algebraic extension of rationals. The solutions of the modified Dirac equation correspond to vanishing eigen values and define zero modes generating conformal super-symmetries and are not of course included.

5. Only those operators b_n for which λ_n belongs to the algebraic extension of rationals in question are used to construct physical states for a given algebraic extension of rationals. This might mean an enormous simplification of the formalism in accordance with the fact that WCW Clifford algebra corresponds as a von Neumann algebra to a hyper-finite factor of type II_1 for which finite truncations by definition allow excellent approximations [K80] . One can even ask whether this hierarchy of algebraic extensions of rationals could in fact define a hierarchy of finite-dimensional Clifford algebras. If so then the general theory of hyper-finite factors of type II_1 would provide an extremely powerful tool.

2.3.3 Cognition, logic, and p-adicity

There seems to be a nice connection between logic aspects of cognition and p-adicity. In particular, p-valued logic for $p = 2^k - n$ has interpretation in terms of ordinary Boolean logic with n "taboos" so that p-valued logic does not conflict with common sense in this case. Also an interpretation of projections of p-adic space-time sheets to an integer lattice of real Minkowski space M^4 in terms of generalized Boolean functions emerges naturally so that M^4 projections of p-adic space-time would represent Boolean functions for a logic with n taboos.

2-adic valued functions of 2-adic variable and Boolean functions

The binary coefficients f_{nk} in the 2-adic expansions of terms $f_n x^n$ in the 2-adic Taylor expansion $f(x) = \sum_{n=0} f_n x^n$, assign a sequence of truth values to a 2-adic integer valued argument $x \in \{0, 1, ..., 2^N\}$ defining a sequence of N bits. Hence $f(x)$ assigns to each bit of this sequence a sequence of truth values which are ordered in the sense that the truth values corresponding to bits are not so important p-adically: much like higher decimals in decimal expansion. If a binary cutoff in N:th bit of $f(x)$ is introduced, B^M-valued function in B^N results, where B denotes Boolean algebra fo 2 elements. The formal generalization to p-adic case is trivial: 2 possible truth values are only replaced by p truth values representable as $0, ..., p-1$.

p-Adic valued functions of p-adic variable as generalized Boolean functions

One can speak of a generalized Boolean function mapping finite sequences of p-valued Boolean arguments to finite sequences of p-valued Boolean arguments. The restriction to a subset $x = kp^n$, $k = 0, ..., p-1$ and the replacement of the function $f(x)$ with its lowest pinary digit gives a generalized Boolean function of a single p-valued argument. If $f(x)$ is invariant under the scalings by powers of p^k, one obtains a hologram like representation of the generalized Boolean function with same function represented in infinitely many length scales. This guarantees the robustness of the representation.

The special role of 2-adicity explaining p-adic length scale hypothesis $p \simeq 2^k$, k integer, in terms of multi-p-acidic fractality would correlate with the special role of 2-valued logic in the world order. The fact that all generalizations of 2-valued logic ultimately involve 2-adic logic at the highest level, where the generalization is formulated would be analog of p-adic length scale hypothesis.

$p = 2^k - n$-adicity and Boolean functions with taboos

It is difficult to assign any reasonable interpretation to $p > 2$-valued logic. Also the generalization of logical connectives AND and OR is far from obvious. In the case $p = 2^k - n$ favored by the p-adic length scale hypothesis situation is however different. In this case one has interpretation in terms B^k with n Boolean statements dropped out so that one obtains what might be called $\hat{b^k}$. Since n is odd this set is not invariant under Boolean conjugation so that there is at least one statement, which is identically true and could be called taboo, axiom, or dogma: depending on taste. The allowed Boolean functions would be constructed in this case using standard Boolean functions AND and OR with the constraint that taboos are respected in other words, both the inputs and values of functions belong to $\hat{b^k}$.

A unique manner to define the logic with taboos is to require that the number of taboos is maximal so that if statement is dropped its negation remains in the logic. This implies $n > B^k/2$.

The projections of p-adic space-time sheets to real imbedding space as representations of Boolean functions

Quantum classical correspondence suggests that generalized Boolean functions should have space-time correlates. Since Boolean cognition involves free will, it should be possible to construct space-time representations of arbitrary Boolean functions with finite number of arguments freely. The non-determinism of p-adic differential equations guarantees this freedom.

p-Adic space-time sheets and p-adic non-determinism make possible to represent generalization of Boolean functions of four Boolean variables obtained by replacing both argument and function with p-valued pinary digit instead of bit. These representations result as discrete projections of p-adic space-time sheets to integer valued points of real Minkowski space M^4. The interpretation would be in terms of 4 sequences of truth values of p-valued logic associated with a finite 4-D integer lattice whose lattice points can be identified as sequences of truth values of a p-valued logic with a set of p-valued truth value at each point so that in the 2-adic case one has map $B^{4M} \to B^{4N}$. Here the number of lattice points in a given coordinate direction of M^4 is M and N is the number of bits allowed by binary cutoff for CP_2 coordinates. For $p = 2^k - n$ representing Boolean algebra with n taboos, the maps can be interpreted as maps $\hat{b}^{4M} \to \hat{b}^{4N}$.

These lattices can be seen as subsets of rational shadows of p-adic space-time sheets to Minkowski space. The condensed matter analog would be a lattice with a a sequence of p-valued dynamical variables (sequence of bits/spins for $p = 2$) at each lattice point. At a fixed spatial point of M^4 the lowest bits define a time evolution of a generalized Boolean function: $B \rightarrow B$.

These observations support the view that intentionality and logic related cognition could perhaps be regarded as 2-adic aspects of consciousness. The special role of primes $p = 2^k - n$ could also be understood as special role of Boolean logic among p-valued logics and $p = 2^k - n$ logic would correspond to B^k with n axioms representing logic respecting a belief system with n beliefs. Recall that multi-p p-adic fractality involving 2-adic fractality is possible for the solutions of field equations and explains p-adic length scale hypothesis.

Most points of the p-adic space-time sheets correspond to real points which are literally infinite as real points. Therefore cognition would be in quite literal sense outside the real cosmos. Perhaps this is a direct correlate for the basic experience that mind is looking the material world from outside.

Connection with the theory of computational complexity?

There are interesting questions concerning the interpretation of four generalized Boolean arguments. TGD explains the number $D = 4$ for space-time dimensions and also the dimension of imbedding space. Could one also find explanation why $d = 4$ defines special value for the number of generalized Boolean inputs and outputs?

1. Could the general theory of computational complexity allow to understand $d = 4$ as a maximum number of inputs and outputs allowing the computation of something related to these functions in polynomial time? For instance, complexity theorist could probably immediately answer following questions. Could the computation of the 2-adic values of CP_2 coordinates as a function of 2-adic M^4 coordinates expressed in terms of fundamental logical connectives take a time which is polynomial as a function of the number of N^4 pinary digits of M^4 coordinates and N^4 pinary digits of CP_2 coordinates? Is this time non-polynomial for M^d and S_d, S_d d-dimensional internal space, $d > 4$. Unfortunately I do not possess the needed complexity theoretic knowhow to answer these questions.

2. The same question could make sense also for $p > 2$ if the notion of the logical connectives and functions generalizes as it indeed does for $p = 2^k - n$. Therefore the question would be whether p-adic length scale hypothesis and dimensions of imbedding space and space-time are implied by a polynomial computation time? This could be the case since essentially a restriction of values and arguments of Boolean functions to a subset of B^k is in question.

Some calculational details

In the following the details of p-adic non-determinism are described for a differential equation of single p-adic variable and some comments about the generalization to the realistic case are given.

1. One-dimensional case

To understand the essentials consider for simplicity a solution of a p-adic differential equation giving function $y = f(x)$ of one independent variable $x = \sum_{n \geq n_0} x_n p^n$.

1. p-Adic non-determinism means that the initial values $f(x)$ of the solution can be fixed arbitrarily up to $N + 1$:th pinary digit. In other words, $f(x_N)$, where $x_N = \sum_{n_0 \leq n \geq N} x_n p^n$ is a rational obtained by dropping all pinary digits higher than N in $x = \sum_{n \geq n_0} x_n p^n$ can be chosen arbitrarily.

2. Consider the projection of $f(x)$ to the set of rationals assumed to be common to reals and p-adics.

   i) Genuinely p-adic numbers have infinite number of positive pinary digits in their non-periodic expansion (non-periodicity guarantees non-rationality) and are strictly infinite as real numbers. In this regime p-adic differential equation fixes completely the solution. This is the case also at rational points $q = m/n$ having infinite number of pinary digits in their pinary expansion.

ii) The projection of p-adic x-axis to real axis consists of rationals. The set in which solution of p-adic differential equations is non-vanishing can be chosen rather freely. For instance, p-adic ball of radius p^{-n} consisting of points $x = p^M y$, $y \neq 0$, $|y|_p \leq 1$, can be considered. Assume $N > M$. p-Adic nondeterminism implies that $f(q)$ for $q = \sum_{M \leq n \leq N} x_n p^n$, can be chosen arbitrarily. For $M \geq 0$ q is always integer valued and the scaling of x by a suitable power of p always allows to get a finite integer lattice at x-axis.

iii) The lowest pinary digit in the expansion of $f(q)$ in powers of p in defines a pinary digit. These pinary digits would define a representation for a sequence of truth values of p-logic. $p = 2$ gives the ordinary Boolean logic. It is also interpret this pinary function as a function of pinary argument giving Boolean function of one variable in 2-adic case.

2. Generalization to the space-time level

This picture generalizes to space-time level in a rather straight forward manner. y is replaced with CP_2 coordinates, x is replaced with M^4 coordinates, and differential equation with field equations deducible from the Kähler action. The essential point is that p-adic space-time sheets have projection to real Minkowski space which consists of a discrete subset of integers when suitable scaling of M^4 coordinates is allowed. The restriction of 4 CP_2 coordinates to a finite integer lattice of M^4 defines 4 Boolean functions of four Boolean arguments or their generalizations for $p > 2$. Also the modes of the induce spinor field define a similar representation.

2.3.4 Fibonacci numbers, Golden Mean, and Jones inclusions

The picture discussed above does not apply in the case of Golden Mean since powers of Φ do not have any special role for the algebraic extension of rationals by $\sqrt{5}$. It is however possible to understand the emergence of Fibonacci numbers and Golden Mean using quantum classical correspondence and the fact that the Clifford algebra and its sub-algebras associated with configuration space spinors corresponds to the so called hyper-finite factor of type II_1 (WCW refers to the "world of classical worlds").

Infinite braids as representations of Jones inclusions

The appearance of hyper-finite factor of type II_1 at the level of basic quantum tGD justifies the expectation that Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of these factors play a key role in TGD Universe. For instance, subsystem system inclusions could induce Jones inclusions.

For the Jones inclusion $\mathcal{N} \subset \mathcal{M}$ $\mathcal{M}$ can be regarded as an $\mathcal{N}$-module with fractal dimension given by Beraha number $B_n = 4cos^2(\pi/n)$, $n \geq 3$ or equivalently by the quantum group phases $exp(i\pi/n)$. B_5 satisfies $B_5 = 4cos^2(\pi/5) = \Phi^2 = \Phi + 1$ so that the special role of $n = 5$ inclusion could explain the special role of Golden Mean in Nature.

Hecke algebras H_n, which are also characterized by quantum phase $q = exp(i\pi/n)$ or the corresponding Beraha number $B_n = 4cos^2(\pi/n)$, characterize the anyonic quantum statistics of n-braid system. Braids are understood as threads which can get linked and define in this manner braiding. Braid group describes these braidings. Like any algebra, Hecke algebra H_n can be decomposed into a direct sum of matrix algebras. Fibonacci numbers characterize the dimensions of these matrix algebras for $n = 5$. Interestingly, topological quantum computation is based on the idea that computer programs can be coded into braidings. What is remarkable is that $n = 5$ characterizes the simplest universal quantum computer so that Golden Mean could indeed have very deep roots to quantum information processing.

The so called Bratteli diagrams characterize the inclusions of various direct summands of H_k to direct summands H_{k+1} in the sequence $H_3 \subset H_4 \subset ... \subset H_k \subset ...$ of Hecke algebras. Essentially the reduction of the representations of H_{k+1} to those of H_k is in question. The same Bratteli diagrams characterize also the Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors of type II_1 with index n as a limit of a finite-dimensional inclusion. Thus Jones inclusion can be visualized as a system consisting of infinite number of braids. In TGD framework the braids could be represented by magnetic flux lines or flux tubes.

Logarithmic spirals as representations of Jones inclusions

The inclusion sequence for Hecke algebras has a representations as a logarithmic spiral. The angle $\pi/5$ can be identified as a limit for angles ϕ_n with $cos(\phi_n) = F_{n+1}/2F_n$ assignable to orthogonal triangle with hypothenuse $2F_n$ and short side F_{n+1} and $\sqrt{4F_n^2 - F_{n+1}^2}$. Fibonacci sequence defines via this prescription a logarithmic spiral as a symbolic representation of the $n = 5$ Jones inclusion representable also in terms of infinite number of braids.

DNA as a topological quantum computer?

Quantum classical correspondence encourages to think that space-time geometry could define a correlate for Jones inclusions of hyper-finite factors of Clifford sub-algebras associated with Clifford algebra of WCW spinor s. The appearance of Fibonacci series in living systems could represent one example of this correspondence. The angle $\pi/10$ closely related to Golden Mean characterizes the winding of DNA double strand. Could this mean that DNA allows to realize topological quantum computer programs as braidings? A possible realization would be based on the notion of super-genes [K38] , which are like pages of a book identified as magnetic flux sheets containing genomes of sequences of cell nuclei as text lines. These text lines would represent line through which magnetic flux lines traverse.

The braiding of magnetic flux lines (or possibly flux sheets regarded as flattened tubes) would define the braiding and the particles involved would be anyons obeying dynamics having quantum group $SU(2)_q$, $q = exp(i\pi/5)$, as its symmetries. The anyons could be assigned with DNA nucleotides or triplets.

TGD predicts also different kind of new physics to DNA double strand. So called H_N-atoms consist of ordinary proton an N dark electrons at space-time sheet which is λ-fold covering of space-time sheet of ordinary hydrogen atom. The effective charge of H_N-atom is $1 - N/\lambda$ since the fine structure constant for dark electrons is scaled down by $1/\lambda$. H_λ-atoms have full electron shell and are therefore exceptionally stable. The proposal is that H_λ-atoms could replace ordinary hydrogen atoms in hydrogen bonds [K38, K28] . Single base pair corresponds to 2 or 3 hydrogen bonds. The question is whether λ-hydrogen atom might somehow relate to the anyons involved with topological quantum computation.

Anyons could be dark protons resulting in the formation dark hydrogen bond in the fusion of H_N atom and its conjugate H_{N_c}, $N_c = \lambda - N$. Neutron scattering and electron diffraction suggest

2.4 The recent view about quantum TGD

Before detailed discussion of what p-adicization of quantum TGD could mean, it is good to have an overall view about what quantum TGD in real context is.

2.4.1 Basic notions

The notions of imbedding space, 3-surface (and 4-surface), and WCW (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [K72, K70] .

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The

generalized imbedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [K15, K19] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [K51] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K25] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed. The mathematical formulation of this vision is however highly nontrivial challenge: is it due to analogs of gauge symmetries or should effective 2-dimensionality formulated explicitly as assumed until 2014 when stringy formulation of WCW geometry emerged.

3. Rather recently came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

1. The obvious guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This choice had some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

   The identification of $X^3(X^3)$ as absolute minimum is however not consistent with quantum criticality suggesting in zero energy ontology (ZEO) a large number of space-time sheets associated with same 3-surface at the ends of causal diamond CD, and having same value of Kähler function. More technically, the Kähler action would have degenerate Hessian as a functional of X^4 with fixed ends X^3).

2. Much later number theoretical vision led to the conclusion that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes a connected component of the light-like 3-surfaces X^3_l, contain in their 4-D tangent space $T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in M^2 degrees of freedom.

   In number theoretical framework M^2_i has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

3. The next step [K15] was the realization that the construction of WCW geometry in terms of modified Dirac action strengthens the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X^3_l. One could say that at X^3_l situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. The weakest form of number theoretic compactification [K72] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, , where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

The notion of WCW ("world of classical worlds")

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of CD.

   The original long-held belief was that the second conformal symmetry corresponds to local imbedding space isometries for light-like 3-surfaces X^3_l, which are either boundaries of X^4 (probably not: it seems that boundary conditions cannot be satisfied so that space-time surfaces must consists of regions defining at least double coverings of M^4) or light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry would be identifiable as the counterpart of the Kac Moody symmetry of string models.

   It has turned out that one can assume Kac-Moody algebra to be sub-algebra of symplectic algebra consisting of the symplectic isometries of $\delta M^4_\pm \times CP_2$. This super Kac-Moody algebra is generated by super-currents assignable to the modes of induced spinor fields other than right-handed neutrino and localized at string world sheets. The symplectic algebra would correspond to right-handed neutrino and one would have direct sum of these two. The beauty of this option is that localization would be inherent property of both algebras and with respect to the light-like coordinate of light-cone boundary rather than forced by hand. The issues related to diffeo-invariance would be avoided in this manner.

   Strong form of holography implied by strong form of GCI suggests the duality between space-like 3-surfaces at the end of CD and light-like 3-surfaces. By parallel translating the boundary of CD one can indeed define the action of symplectic algebra at the light-like 3-surfaces. Therefore also the symplectic and Kac-Moody algebras associated with these surfaces could be used to generate zero energy states, and one would have effective 2-dimensionality in the sense that only the partonic 2-surfaces defined by the intersections of space-like and light-like 3-surfaces and their 4-D tangent space data would code for quantum physics.

A rather plausible conclusion is that WCW (WCW) is a union of sub-WCWs associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta} J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.

2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub-WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras. WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with subgroup leaving the 3-surface invariant for a preferred 3-surface which could be chosen maximum/minimum of Kähler function. Equivalently, the local coset space associated with $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naive!

4. The original construction of WCW metric was in terms of flow Hamiltonians induced by those of $\delta M^4_\pm \times CP_2$. Matrix elements of WCW metric were constructed as anti-commutators of super-Hamiltonians having interpretation also as WCW Hamiltonians.

   The construction had problematic aspects. Flux Hamiltonians were strictly 2-D objects and also the fact that they contained very little explicit information about the dynamics of the modified Dirac action. The realization of super-Hamiltonians in terms of conserved symplectic super-charges of modified Dirac action labelled by the modes of the modified Dirac operator cures the situation and the construction becomes 3-dimensional although effective 2-dimensionality still holds true. Anti-commutations are fixed completely and the construction works for dimension $D = 8$ of imbedding space only. The stringy picture forced by the solutions of the modified Dirac operator becomes very explicit at the level of WCW. 8-D imbedding space only [K96].

5. Generalized coset construction and coset space structure have very deep physical meaning. Symmetric space structure requires involution and it corresponds to inversion in light-like radial coordinate r_M of $\delta M^4_\pm$ (determined only up to Lorentz transformation). Super Virasoro algebra realizes quantum criticality, and one obtains hierarchy of criticalities represented by the hierarchy of sub-algebras of Super Virasoro algebra.

Head aches from Equivalent Principle

Equivalence Princple (EP) has been continual source of headaches during years. It is not even clear whether the uncritical assumption that there gravitational and inertial masses exist at separate notions creates the problem as a pseudoproblem. Stringy description of graviton mediated scattering predicted also by TGD indeed suggests this.

1. A longstanding conjecture has been that coset representations could Equivalence Principle (EP) at quantum level: the identity of Super Virasoro generators for super-symplectic and super Kac-Moody algebras was proposed to imply that inertial and gravitational four-momenta are identical. This conjecture is probably wrong.

2. The equivalence of classical Noether momentum associated with Kähler action with eigenvalues of the corresponding quantal momentum for modified Dirac action certainly realizes quantum classical correspondence. It could also realized EP. Zero energy ontology suggests an alternative formulation for the same idea.

3. A further option is that EP reduces to the identification of the four momenta assignable to Super Virasoro representations assignable to space-like and light-like 3-surfaces and therefore become part of strong form of holography and quantum classical correspondence (QCC).

So it seems that EP might reduce to holography, GCI, or QCC and it might well be that it is trivially true! At classical level the understanding of the relationship between TGD and GRT led to the final break through in the understanding of EP. The recent view is that EP at quantum level reduces to Quantum Classical Correpondence (QCC) in the sense that Cartan algebra Noether charges assignable to 3-surface in case of Kähler action (inertial charges) are identical with eigenvalues of the quantal variants of Noether charges for Kähler-Dirac action (gravitational charges). The well-definedness of the latter charges is due to the conformal invariance assignable to 2-D surfaces (string world sheets and possibly partonic 2-surfaces) at which the spinor modes are localized in generic case. This localization follows from the condition that em charge has well defined value for the spinor modes. The localization is possibly only for the Kähler-Dirac action and key role is played by the modification of gamma matrices to Kähler-Dirac gamma matrices. The gravitational four-momentum is thus completely analogous to stringy four-momentum.

At classical level EP follows from the interpretation of GRT space-time as effective space-time obtained by replacing many-sheeted space-time (see fig. http://www.tgdtheory.fi/appfigures/manysheeted.jpg or fig. 9 in the appendix of this book) with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrices of space-time sheets from Minkowski metric. Poincare invariance suggests strongly classical EP for the GRT limit in long length scales at least. One can consider also other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define anatural starting point and CP_2 indeed defines a gravitational instanton with very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

2.4.2 The most recent vision about zero energy ontology

The generalization of the number concept obtained by fusing real and p-adics along rationals and common algebraics is the basic philosophy behind p-adicization. This however requires that it is possible to speak about rational points of the imbedding space and the basic objection against the notion of rational points of imbedding space common to real and various p-adic variants of the imbedding space is the necessity to fix some special coordinates in turn implying the loss of a manifest general coordinate invariance. The isometries of the imbedding space could save the situation provided one can identify some special coordinate system in which isometry group reduces to its discrete subgroup. The loss of the full isometry group could be compensated by assuming that WCW is union over sub-WCW:s obtained by applying isometries on basic sub-WCW with discrete subgroup of isometries.

The combination of zero energy ontology realized in terms of a hierarchy causal diamonds and hierarchy of Planck constants providing a description of dark matter and leading to a generalization of the notion of imbedding space suggests that it is possible to realize this dream. The article [L10] provides a brief summary about recent state of quantum TGD helping to understand the big picture behind the following considerations.

Zero energy ontology briefly

1. The basic construct in the zero energy ontology is the space $CD \times CP_2$, where the causal diamond CD is defined as an intersection of future and past directed light-cones with time-like separation between their tips regarded as points of the underlying universal Minkowski space M^4. In zero energy ontology physical states correspond to pairs of positive and negative energy states located at the boundaries of the future and past directed light-cones of a particular CD. CD:s form a fractal hierarchy and one can glue smaller CD:s within larger CD along the upper light-cone boundary along a radial light-like ray: this construction recipe allows to understand the asymmetry between positive and negative energies and why the arrow of experienced time corresponds to the arrow of geometric time and also why the contents of sensory experience is located to so narrow interval of geometric time. One can imagine evolution to occur as quantum leaps in which the size of the largest CD in the hierarchy of personal CD:s increases in such a manner that it becomes sub-CD of a larger CD. p-Adic length scale hypothesis follows if the values of temporal distance T between tips of CD come in powers of 2^n: a weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs. All conserved quantum numbers for zero energy states have vanishing net values. The interpretation of zero energy states in the framework of positive energy ontology is as physical events, say scattering events with positive and negative energy parts of the state interpreted as initial and final states of the event.

2. In the realization of the hierarchy of Planck constants $CD \times CP_2$ is replaced with a Cartesian product of book like structures formed by almost copies of CD:s and CP_2:s defined by singular coverings and factors spaces of CD and CP_2 with singularities corresponding to intersection

$M^2 \cap CD$ and homologically trivial geodesic sphere S^2 of CP_2 for which the induced Kähler form vanishes. The coverings and factor spaces of CD:s are glued together along common $M^2 \cap CD$. The coverings and factors spaces of CP_2 are glued together along common homologically non-trivial geodesic sphere S^2. The choice of preferred M^2 as subspace of tangent space of X^4 at all its points and having interpretation as space of non-physical polarizations, brings M^2 into the theory also in different manner. S^2 in turn defines a subspace of the much larger space of vacuum extremals as surfaces inside $M^4 \times S^2$.

3. WCW (the world of classical worlds, WCW) decomposes into a union of sub-WCW:s corresponding to different choices of M^2 and S^2 and also to different choices of the quantization axes of spin and energy and and color isospin and hyper-charge for each choice of this kind. This means breaking down of the isometries to a subgroup. This can be compensated by the fact that the union can be taken over the different choices of this subgroup.

4. p-Adicization requires a further breakdown to discrete subgroups of the resulting sub-groups of the isometry groups but again a union over sub-WCW:s corresponding to different choices of the discrete subgroup can be assumed. Discretization relates also naturally to the notion of number theoretic braid.

Consider now the critical questions.

1. Very naively one could think that center of mass wave functions in the union of sectors could give rise to representations of Poincare group. This does not conform with zero energy ontology, where energy-momentum should be assignable to say positive energy part of the state and where these degrees of freedom are expected to be pure gauge degrees of freedom. If zero energy ontology makes sense, then the states in the union over the various copies corresponding to different choices of M^2 and S^2 would give rise to wave functions having no dynamical meaning. This would bring in nothing new so that one could fix the gauge by choosing preferred M^2 and S^2 without losing anything. This picture is favored by the interpretation of M^2 as the space of longitudinal polarizations.

2. The crucial question is whether it is really possible to speak about zero energy states for a given sector defined by generalized imbedding space with fixed M^2 and S^2. Classically this is possible and conserved quantities are well defined. In quantal situation the presence of the light-cone boundaries breaks full Poincare invariance although the infinitesimal version of this invariance is preserved. Note that the basic dynamical objects are 3-D light-like "legs" of the generalized Feynman diagrams.

Definition of energy in zero energy ontology

Can one then define the notion of energy for positive and negative energy parts of the state? There are two alternative approaches depending on whether one allows or does not allow wave-functions for the positions of tips of light-cones.

Consider first the naive option for which four momenta are assigned to the wave functions assigned to the tips of CD:s.

1. The condition that the tips are at time-like distance does not allow separation to a product but only following kind of wave functions

$$\Psi = exp[ip\cdot(m_+ - m_-)]\Theta(T^2)\Theta(m_+^0 - m_-^0)\Phi(p) \ , \ T^2 = (m_+ - m_-)^2 \ . \tag{2.4.1}$$

Here m_+ and m_- denote the positions of the light-cones and Θ denotes step function. Φ denotes WCW spinor field in internal degrees of freedom of 3-surface. One can introduce also the decomposition into particles by introducing sub-CD:s glued to the upper light-cone boundary of CD.

2. The first criticism is that only a local eigen state of 4-momentum operators $p_\pm = \hbar\nabla/i$ is in question everywhere except at boundaries and at the tips of the CD with exact translational invariance broken by the two step functions having a natural classical interpretation. The second criticism is that the quantization of the temporal distance between the tips to $T = 2^k T_0$ is in conflict with translational invariance and reduces it to a discrete scaling invariance.

The less naive approach relying of super conformal structures of quantum TGD assumes fixed value of T and therefore allows the crucial quantization condition $T = 2^k T_0$.

1. Since light-like 3-surfaces assignable to incoming and outgoing legs of the generalized Feynman diagrams are the basic objects, can hope of having enough translational invariance to define the notion of energy. If translations are restricted to time-like translations acting in the direction of the future (past) then one has local translation invariance of dynamics for classical field equations inside $\delta M^4_\pm$ as a kind of semigroup. Also the M^4 translations leading to interior of X^4 from the light-like 2-surfaces surfaces act as translations. Classically these restrictions correspond to non-tachyonic momenta defining the allowed directions of translations realizable as particle motions. These two kinds of translations have been assigned to super-symplectic conformal symmetries at $\delta M^4_\pm \times CP_2$ and and super Kac-Moody type conformal symmetries at light-like 3-surfaces.

2. The condition selecting preferred extremals of Kähler action is induced by a global selection of M^2 as a plane belonging to the tangent space of X^4 at all its points [K19] . The M^4 translations of X^4 as a whole in general respect the form of this condition in the interior. Furthermore, if M^4 translations are restricted to M^2, also the condition itself - rather than only its general form - is respected. This observation, the earlier experience with the p-adic mass calculations, and also the treatment of quarks and gluons in QCD encourage to consider the possibility that translational invariance should be restricted to M^2 translations so that mass squared, longitudinal momentum and transversal mass squared would be well defined quantum numbers. This would be enough to realize zero energy ontology. Encouragingly, M^2 appears also in the generalization of the causal diamond to a book-like structure forced by the realization of the hierarchy of Planck constant at the level of the imbedding space.

3. That the cm degrees of freedom for CD would be gauge like degrees of freedom sounds strange. The paradoxical feeling disappears as one realizes that this is not the case for sub-CD:s, which indeed can have non-trivial correlation functions with either upper or lower tip of the CD playing a role analogous to that of an argument of n-point function in QFT description. One can also say that largest CD in the hierarchy defines infrared cutoff.

2.4.3 The geometry of "world of classical worlds" (WCW)

The reader not familiar with the basic ideas related to the construction of the WCW (WCW) geometry and spinor structure is warmly encouraged to read [K33, K16, K15] . The number theoretic ideas as all other ideas have evolved through un-necessarily strong conjectures. One of them was the idea that conformal weights are complex and given by the zeros of Riemann zeta. Some numerical accidents motivated this idea but it soon lead to non-plausible conjectures about the number theoretic anatomy for the zeros of zeta and many of them turned out to be wrong. The idea about the role of zeta function was not however completely wrong. It turned out that one can assign to the eigenvalues of the modified Dirac operator what might be called Dirac zeta and ζ_D is expressible in terms of gamma functions and Riemann Zeta with shifted argument but do not satisfy Riemann Hypothesis.

WCW as a union of symmetric spaces

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ , \quad [t,t] \subset h \ , \quad [h,t] \subset t \ . \tag{2.4.2}$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough. $[t, t] \subset h$ condition is highly nontrivial and equivalent with the existence of involution. Inversion in the light-like radial coordinate of δM^4 is a natural guess for this involution and induces complex conjugation in super-conformal algebras mapping positive and negative conformal weights to each other.

WCW geometry allows two super-conformal symmetries. The first one corresponds to super-symplectic transformations acting at the level of imbedding space. The second one corresponds to super Kac-Moody symmetry. The original identification of Kac-Moody was in terms of deformations of light-like 3-surfaces respecting their light-likeness. This not wrong as such: also entire symplectic algebra can be assigned with light-like surfaces and the theory can be constructed using also these conformal algebras.

The more plausible identification is as the sub-algebra of symplectic algebra realized as isometries of δCD so that localization is inherent and in terms of the radial light-like coordinate of light-like boundary. This identification is made possible by the wisdom gained from the solutions of the modified Dirac equations predicting the localization of its modes (except right-handed neutrino) to string world sheets.

1. These super-conformal algebra representations form a direct sum: it is also possible to assume that super Kac-Moody generators vanish at partonic 2-surfaces. p-Adic mass calculations require five super-conformal tensor factors and the number of tensor factors would be indeed this.

2. This algebra has as sub-algebra the algebra for which generators leave 3-surface invariant - in other words, induce its diffeomorphism.

3. Quantum states correspond to the coset representations for entire algebra and this algebra so that differences of the corresponding super-Virasoro generators annihilate physical states. This obviously generalizes Goddard-Olive-Kent construction [A209]. It seems now clear that coset representation does not imply EP: the four-momentum simply does not appear in the representation of the isotropy sub-algebra since translations lead out of CD boundary.

4. The experience with finite-dimensional coset spaces would suggest that the generators of h leave the points of WCW analogous to the origin of say CP_2 invariant and vanish at this point. The maxima of Kähler function could correspond to this kind of points and could play also an essential role in the integration over WCW by generalizing the Gaussian integration of free quantum field theories.

Symmetric space property provides a strong condition on the construction. The generators of t should commute to generators of h. This is equivalent with the existence of involution of the symplectic algebra changing the sign of the elements of t and leaving elements of h invariant. This involution would naturally correspond to inversion with respect to the light-like radial coordinate r_M of $\delta M^4_{\pm}$. It induces change of the sign for the conformal weights of generators of various super-conformal algebras.

Zero modes

Zero modes are by definition those degrees of freedom which do not correspond to the complex coordinates of WCW contributing to the metric.

1. J as function of X^2 coordinates defines the fundamental collection of zero modes and its extrema at the points of braid defines subset of zero modes. There are also other zero modes labeled by symplectic invariants described in [K16] . The size and shape of the 3-surface and classical Kähler field correspond to these zero modes. In particular, the induced Kähler form is purely symplectic invariant from which one can deduce this kind of non-local invariants. Especially interesting local symplectic and diffeo-invariants are the extrema of $J = \epsilon^{\mu\nu} J_{\mu\nu}$. Both CP_2 and $\delta M^4_{\pm}$ Kähler form define this kind of invariants. These appear in the construction of symplectic fusion algebras [K14] .

2. Zero modes decompose to symplectic covariants and invariants. The symplectic transformations are generated by the function basis of $M^4_+ \times CP_2$ consist of complexified Hamiltonians labeled by the label -call it n - assignable to the functions $f_n(J)$ and by the labels of Hamiltonians of $\delta M^4_\pm \times CP_2$. If Hamiltonian is real it corresponds to zero mode. The most obvious candidates for zero modes are Hamiltonians which do not depend neither on the radial coordinate of $\delta M^4_\pm$ nor on J.

3. Since the values of the induced Kähler form represent local zero modes, the quantum fluctuating degrees of freedom are parameterized by the symplectic transformations of $\delta M^\pm \times CP_2$ [K18] . From the point of view of quantum theory WCW decomposes into slices characterized by the induced Kähler form at partonic 2-surfaces and functional integral reduces to that over the symplectic group. Induced Kähler form is genuinely classical field and only the induced metric quantum fluctuates so that TGD in a well-defined sense reduces to quantum gravity in the quantum fluctuating WCW degrees of freedom.

How to construct the super-symplectic algebra?

The WCW of 3-surfaces Y^3 as a union of infinite-dimensional symmetric spaces labeled by zero modes obeying real topology and having metric and spinor structure determined solely by supersymmetry, is the basic intuitive picture about WCW geometry.

Algebraic physics vision suggests that the representation of the generators of the symplectic transformations of the light-like 7-surface $\delta M^4_\pm \times CP_2$ must be expressible in terms of rational functions. In the case that Hamiltonians correspond to irreducible representations of $SU(3)$, they are products of rational functions of preferred CP_2 coordinates with functions depending on coordinates of X^3_l. If the Hamiltonians transform according to an irreducible representation of the rotation group leaving $r_M = constant$ sphere S^2 invariant, they are rational functions of the complex coordinates of S^2. The remaining problems relate to the 3-integrals appearing in the definition of WCW Hamiltonians. The solution of these problems comes in terms of (number theoretic) braids, which are now a basic notion of quantum TGD. Integrals are simply replaced by sums making sense also p-adically.

The modified Dirac action allows to deduce explicit expressions for the super generators. This allows to extend the formulas for the WCW Hamiltonians in terms of the classical symplectic charges associated with the Kähler action to the formulas for super-symplectic charges. WCW metric, being numerically equal to the Kähler form in complex coordinates, in turn relates directly to the symplectic charges. A natural expectation is that gamma matrices can be related by an analogous formula to the expressions for the super-symplectic charges.

2.4.4 The identification of number theoretic braids

To specify number theoretical criticality one must specify some physically preferred coordinates for $M^4 \times CP_2$ or at least $\delta M^4_\pm \times CP_2$. Number theoretical criticality requires that braid belongs to the algebraic intersection of real and p-adic variants of the partonic 2-surface so that number theoretical criticality reduces to a finite number of conditions. This is however not strong enough condition and one must specify further physical conditions.

What are the preferred coordinates for H?

What are the preferred coordinates of M^4 and CP_2 in which algebraicity of the points is required is not completely clear. The isometries of these spaces must be involved in the identification as well as the choice of quantization axes for given CD. In [K47] I have discussed the natural preferred coordinates of M^4 and CP_2.

1. For M^4 linear M^4 coordinates chosen in such manner that $M^2 \times E^2$ decomposition fixing quantization axes is respected are very natural. This restricts the allowed Lorentz transformations to Lorentz boosts in M^2 and rotations in E^2 and the identification of M^2 as hyper-complex plane fixes time coordinate uniquely. E^2 coordinates are fixed apart from the action of $SO(2)$ rotation. The rationalization of trigonometric functions of angle variables allows angles associated with Pythagorean triangles as number theoretically simplest ones.

2. The case of CP_2 is not so easy. The most obvious guess in the case of CP_2 the coordinates corresponds to complex coordinates of CP_2 transforming linearly under $U(2)$. The condition that color isospin rotations act as phase multiplications fixes the complex coordinates uniquely. Also the complex coordinates transforming linearly under $SO(3)$ rotations are natural choice for S^2 ($r_M = constant$ sphere at $\delta M^4_\pm$).

3. Another manner to deal with CP_2 is to apply number M^8-H duality. In M^8 CP_2 corresponds to E^4 and the situation reduces to linear one and $SO(4)$ isometries help to fix preferred coordinate axis by decomposing E^4 as $E^4 = E^2 \times E^2$. Coordinates are fixed apart the action of the commuting $SO(2)$ sub-groups acting in the planes E^2. It is not clear whether the images of algebraic points of E^4 at space-time surface are mapped to algebraic points of CP_2.

The notion of number theoretical braid

Braids -not necessary number theoretical- provide a realization discretization as a space-time correlate for the finite measurement resolution. The notion of braid was inspired by the idea about quantum TGD as almost topological quantum field theory. Although the original form of this idea has been buried, the notion of braid has survived: in the decomposition of space-time sheets to string world sheets, the ends of strings define representatives for braid strands at light-like 3-surfaces.

The notion of number theoretic universality inspired the much more restrictive notion of number theoretic braid requiring that the points in the intersection of the braid with the partonic 2-surface correspond to rational or at most algebraic points of H in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid or at least of the end points of the braid. The following consideration suggest that the number theoretic braids are not a useful notion in the generic case but make sense and are needed in the intersection of real and p-adic worlds which is in crucial role in TGD based vision about living matter [K42] .

It is only the braiding that matters in topological quantum field theories used to classify braids. Hence braid should require only the fixing of the end points of the braids at the intersection of the braid at the light-like boundaries of CDs and the braiding equivalence class of the braid itself. Therefore it is enough is to specify the topology of the braid and the end points of the braid in accordance with the attribute "number theoretic". Of course, the condition that all points of the strand of the number theoretic braid are algebraic is impossible to satisfy.

The situation in which the equations defining X^2 make sense both in real sense and p-adic sense using appropriate algebraic extension of p-adic number field is central in the TGD based vision about living matter [K42] . The reason is that in this case the notion of number entanglement theoretic entropy having negative values makes sense and entanglement becomes information carrying. This motivates the identification of life as something in the intersection of real and p-adic worlds. In this situation the identification of the ends of the number theoretic braid as points belonging to the intersection of real and p-adic worlds is natural. These points -call them briefly algebraic points- belong to the algebraic extension of rationals needed to define the algebraic extension of p-adic numbers. This definition however makes sense also when the equations defining the partonic 2-surfaces fail to make sense in both real and p-adic sense. In the generic case the set of points satisfying the conditions is discrete. For instance, according to Fermat's theorem the set of rational points satisfying $X^n + Y^n = Z^n$ reduces to the point $(0, 0, 0)$ for $n = 3, 4, ...$ Hence the constraint might be quite enough in the intersection of real and p-adic worlds where the choice of the algebraic extension is unique.

One can however criticize this proposal.

1. One must fix the the number of points of the braid and outside the intersection and the non-uniquencess of the algebraic extension makes the situation problematic. Physical intuition suggests that the points of braid define carriers of quantum numbers assignable to second quantized induced spinor fields so that the total number of fermions anti-fermions would define the number of braids. In the intersection the highly non-trivial implication is that this number cannot exceed the number of algebraic points.

2. In the generic case one expects that even the smallest deformation of the partonic 2-surface can change the number of algebraic points and also the character of the algebraic extension of rational numbers needed. The restriction to rational points is not expected to help in the generic case. If the notion of number theoretical braid is meant to be practical, must be able to decompose WCW to open sets inside which the numbers of algebraic points of braid at its ends are constant. For real topology this is expected to be impossible and it does not make sense to use p-adic topology for WCW whose points do not allow interpretation as p-adic partonic surfaces.

3. In the intersection of real and p-adic worlds which corresponds to a discrete subset of WCW, the situation is different. Since the coefficients of polynomials involved with the definition of the partonic 2-surface must be rational or at most algebraic, continuous deformations are not possible so that one avoids the problem.

4. This forces to ask the reason why for the number theoretic braids. In the generic case they seem to produce only troubles. In the intersection of real and p-adic worlds they could however allow the construction of the elements of M-matrix describing quantum transitions changing p-adic to real surfaces and vice versa as realizations of intentions and generation of cognitions. In this the case it is natural that only the data from the intersection of the two worlds are used. In [K42] I have sketched the idea about number theoretic quantum field theory as a description of intentional action and cognition.

There is also the the problem of fixing the interior points of the braid modulo deformations not affecting the topology of the braid.

1. Infinite number of non-equivalent braidings are possible. Should one allow all possible braidings for a fixed light-like 3-surface and say that their existence is what makes the dynamics essentially three-dimensional even in the topological sense? In this case there would be no problems with the condition that the points at both ends of braid are algebraic.

2. Or should one try to characterize the braiding uniquely for a given partonic 2-surfaces and corresponding 4-D tangent space distributions? The slicing of the space-time sheet by partonic 2-surfaces and string word sheets suggests that the ends of string world sheets could define the braid strands in the generic context when there is no algebraicity condition involved. This could be taken as a very natural manner to fix the topology of braid but leave the freedom to choose the representative for the braid. In the intersection of real and p-adic worlds there is no good reason for the end points of strands in this case to be algebraic at both ends of the string world sheet. One can however start from the braid defined by the end points of string world sheets, restrict the end points to be algebraic at the end with a smaller number of algebraic points and and then perform a topologically non-trivial deformation of the braid so that also the points at the other end are algebraic? Non-trivial deformations need not be possible for all possible choices of algebraic braid points at the other end of braid and different choices of the set of algebraic points would give rise to different braidings. A further constraint is that only the algebraic points at which one has assign fermion or anti-fermion are used so that the number of braid points is not always maximal.

3. One can also ask whether one should perform the gauge fixing for the strands of the number theoretic braid using algebraic functions making sense both in real and p-adic context. This question does not seem terribly relevant since since it is only the topology of the braid that matters.

Symplectic triangulations and braids

The identification of the edges of the symplectic triangulation as the end points of the braid is favored by conceptual economy. The nodes of the symplectic triangulation would naturally correspond to the points in the intersection of the braid with the light-like boundaries of CD carrying fermion or anti-fermion number. The number of these points could be arbitrarily large in the generic case but in the intersection of real and p-adic worlds these points correspond to subset of algebraic points belonging to the algebraic extension of rationals associated with the definition of partonic 2-surfaces so that the sum of fermion and anti-fermion numbers would be bounded above. The presence of fermions in the nodes would be the physical prerequisite for measuring the phase factors defined by the magnetic fluxes. This could be understood in terms of gauge invariance forcing to assign to a pair of points of triangulation the non-integrable phase factor defined by the Kähler gauge potential.

The remaining problem is how uniquely the edges of the triangulation can be determined.

1. The allowance of all possible choices for edges would bring in an infinite number of degrees of freedom. These curves would be analogous to freely vibrating strings. This option is not attractive. One should be able to pose conditions on edges and whatever the manner to specify the edges might be, it must make sense also in the intersection of real and p-adic worlds. In this case the total phase factor must be a root of unity in the algebraic extension of rationals involved and this poses quantization rules analogous to those for magnetic flux. The strongest condition is that the edges are such that the non-integrable phase factor is a root of unity for each edge. It will be found that similar quantization is implied also by the associativity conditions and this justifies the interpretation of phase factors defining the fusion algebra in terms of the Kähler magnetic fluxes. This would pose strong constraints on the choice of edges but would not fix completely the phase factors, and it seems that one must allow all possible triangulations consistent with this condition and the associativity conditions so that physical state is a quantum superposition over all possible symplectic triangulations characterized by the fusion algebras.

2. In the real context one would have an infinite hierarchy of symplectic triangulations and fusion algebras satisfying the associativity conditions with the number of edges equal to the total number N of fermions and anti-fermions. Encouragingly, this hierarchy corresponds also to a hierarchy of $\mathcal{N} = N$ SUSY algebras [K27] (large values of $\mathcal{N}$ are not a catastrophe in TGD framework since the physical content of SUSY symmetry is not the same as that in the standard approach). In the intersection of real and p-adic worlds the value of $\mathcal{N}$ would be bounded by the total number of algebraic points. Hence the notion of finite measurement resolution, cutoff in $\mathcal{N}$ and bound on the total fermion number would make physics very simple in the intersection of real and p-adic worlds.

Two kinds of symplectic triangulations are possible since one can use the symplectic forms associated with CP_2 and $r_M = constant$ sphere S^2 of light-cone boundary. For a given collection of nodes the choices of edges could be different for these two kinds of triangulations. Physical state would be proportional to the product of the phase factors assigned to these triangulations.

2.4.5 Finite measurement resolution and reduced configuration space

Finite measurement resolution implies the notion of braid which is now central part of construction of M-matric [K15] . The notion of braid in turn leads to the notion of reduced configuration space.

1. 3-surface reduces effectively to a set of points defined by the intersection of $\delta M^4_\pm \times CP_2$ projection of the partonic 2-surface X^2 with light-like radial geodesic or the intersection of its CP_2 projection with the geodesic sphere S^2_i, $i = I, II$.

2. Second kind of braid corresponds to the extrema of $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at X^2. Here the induced Kähler forms of both $\delta M^4_\pm$ and CP_2 can be considered. Also this option defines the braid physically and the number of points is finite in the generic situation.

Number theoretic braids reduce WCW to a finite-dimensional space defined as a coset space of symplectic group of $\delta M^4_\pm \times CP_2$ obtained by dividing with the sub-group of the symplectic group leaving the braid points invariant. The resulting space is $(\delta M^4_\pm \times CP_2)^n/S_n$, where n is the number of braid points. If the proposed criteria define the braid, n and measurement resolution is characterized by the geometry of X^2.

This raises issues about the metric of the reduced WCW as deduced from the spectrum of the modified Dirac operator.

1. Kac-Moody symmetry would suggest that the finite number of $n = 0$ modes determine the Kähler function and metric exactly. Also the metric of the coset space determined by measurement resolution could naturally determined as derivatives of the logarithms of the eigen values with respect to the complex coordinates of $(S^2 \times CP_2)^n$. In principle, it would be possible to deduced the metric numerically. If one allows arbitrary number of braid points then $n \to \infty$ limit could give rise to the continuum formulation of WCW Hamiltonians and metric.

2. The simplest option would be that the metric reduces apart from a scaling factor to a direct sum of the metrics assignable to the factors of the Cartesian power. Even if this happens, the scaling factor must be non-trivial and carry dependence on the induced Kähler form which is constant along the symplectic orbit and defines the fundamental zero modes. This expectation is probably wrong. Kähler function codes correlations even between different components of partonic 2-surfaces and it would be surprising if there were no correlations between points of the same partonic 2-surface. A new element as compared to general relativity would be geometrization of n-particle system in terms of the metric of the reduced WCW.

2.4.6 Does reduced WCW allow TGD Universe to act as a universal math machine?

The title relates only the very loosely to the main topic of the chapter. The excuse for including this material is that TGD inspired theory of consciousness allows to interpret the notions of zero energy state and reduced WCW in terms of mathematical cognition.

The questions which lead to the arguments represented below were represented in different context [K35] related to the TGD inspired ideas about number theoretic Langlands correspondence. TGD inspired theory of consciousness - in particular the question about the physical correlates of Boolean statements and conscious mathematical deductions- is second definer of context.

The questions are following. Could one find a representations of both Lie groups and their linear and non-linear representation spaces -and even more - of any manifold representable as a sub-manifold of some linear space in terms of braid points at partonic 2-surfaces X^2? What about various kinds of projective spaces and coset spaces? Can one construct representations of corresponding function spaces in terms of WCW spinor fields? Can one build representations of parameter groups of Lie groups as braided representations defined by the orbits of braid points in X^3_l?

A professional mathematician - if she still continues reading - might regard the following argument as rather pathetic poor man's argument but I want to be honest and demonstrate my stupidity openly.

1. The n braid points represent points of $\delta H = \delta M^4_\pm \times CP_2$ so that braid points represent a point of $7n$-dimensional space $\delta H^n/S_n$. $\delta M^4_\pm$ corresponds to E^3 with origin removed but $E^{2n}/S_n = C^n/S_n$ can be represented as a sub-manifold of $\delta M^4_\pm$. This allows to almost-represent both real and complex linear spaces. E^2 has a unique identification based on $M^4 = M^2 \times E_2$ decomposition required by the choice of quantization axis. One can also represent the spaces $(CP_2)^n/S_n$ in this manner.

2. The first - and really serious - problem is caused by the identification of the points obtained by permuting the n coordinates: this is of course what makes possible the braiding since braid group is the fundamental group of $(X^2)^n$. Could the quantum numbers at the braid points act as markers distinguishing between them so that one would effectively have E^{2n}? Could the fact that the representing points are those of imbedding space rather than X^2 be of significance? Second - less serious - problem is that the finite size of CD allows to represent only a finite region of E^2. On the other hand, ideal mathematician is a non-existing species and even non-ideal mathematician can imagine the limit at which the size of CD becomes infinite.

3. Matrix groups can be represented as sub-manifolds of linear spaces defined by the general linear group $Gl(n,R)$ and $Gl(n,C)$. In the p-adic pages of the imbedding space one can

realize also the p-adic variants of general linear groups. Hence it is possible to imbed any real (complex) Lie group to E^{2n} (C^n), if n is chosen large enough.

4. WCW spinor fields restricted to the linear representations spaces or to the group itself represented in this manner would allow to realize as a special case various function spaces, in particular groups algebras. If WCW spinor fields satisfy additional symmetries, projective spaces and various coset spaces can be realized as effective spaces. For instance CP_2 could be realized effectively as $SU(3)/U(2)$ by requiring $U(2)$ invariance of the WCW spinor fields in $SU(3)$ or as C^3/Z by requiring that WCW spinor field is scale invariant. Projective spaces might be also realized more concretely as imbeddings to $(CP_2)^n$.

5. The action of group element $g = exp(Xt)$ belonging to a one-parameter sub-group of a non-compact linear group in a real (complex) linear representation space of dimension m could be realized in a subspace of E^{2n}, $m < 2n$ (C^n, $m \leq n$), as a flow in X^3_l taking the initial configuration of points of representation space to the final configuration. Braid strands - the orbits of points p_i defining the point p of the representation manifold under the action of one-parameter subgroup- would correspond to the points $exp(Xu)(p)$, $0 \leq u \leq t$. Similar representation would work also in the group itself represented in a similar manner.

6. Braiding in X^3_l would induce a braided representation for the action of the one parameter subgroup. This representation is not quite the same thing as the automorphic representation since braiding is involved. Also trivial braid group representation is possible if the representation can be selected freely rather than being determined by the transformation properties of fermionic oscillator operator basis in the braiding.

7. An important prerequisite for math machine property is that the wave function in the space of light-like 3-surfaces with fixed ends can be chosen freely. This is the case since the degrees of freedom associate with the interior of light-like 3-surface X^3_l correspond to zero modes assignable to Kac-Moody symmetries [K16] . Dicretization seems however necessary since functional integral in these degrees of freedom is not-well defined even in the real sense and even less so p-adically. This conforms with the fact that real world mathematical representations are always discrete. Quantum classical correspondence suggests the dynamics represented by X^3_l correlates with the quantum numbers assigned with X^2 so that Boolean statements represented in terms of Fermionic Fock states would be in one-one correspondence with these wave functions.

Besides representing mathematical structures this kind of math machine would be able to perform mathematical deductions. The fermionic part of the state zero energy state could be interpreted as a quantum super-position of Boolean statement $A_i \rightarrow B_i$ representing various instances of the general rule $A \rightarrow B$. Only the statements consistent with fundamental conservation laws would be possible. Quantum measurements performed for both positive and negative energy parts of the state would produce statements. Performing the measurement of the observable $O(A \rightarrow B)$ would produce from a given state a zero energy state representing statement $A \rightarrow B$. If the measurement of observable $O(C \rightarrow D)$ affects this state then the statement $(A \rightarrow B) \rightarrow (C \rightarrow D)$ cannot hold true. For $A = B$ the situation reduces to simpler logic where one tests truth value of statements of form $A \rightarrow B$. By increasing the number of instances in the quantum states generalizations of the rule can be tested.

2.5 p-Adicization at the level of imbedding space and space-time

In this section p-adicization program at the level if imbedding space and space-time is discussed. The general problems of p-adicization, namely the selection of preferred coordinates and the problems caused by the non-existence of p-adic definite integral and algebraic continuation a solution of these problems has been discussed in the introduction.

2.5.1 p-Adic variants of the imbedding space

Consider now the construction of p-adic variants of the imbedding space.

1. Rational values of p-adic coordinates are non-negative so that light-cone proper time $a_{4,+} = \sqrt{t^2 - z^2 - x^2 - y^2}$ is the unique Lorentz invariant choice for the p-adic time coordinate near the lower tip of CD. For the upper tip the identification of a_4 would be $a_{4,-} = \sqrt{(t-T)^2 - z^2 - x^2 - y^2}$. In the p-adic context the simultaneous existence of both square roots would pose additional conditions on T. For 2-adic numbers $T = 2^n T_0$, $n \geq 0$ (or more generally $T = \sum_{k \geq n_0} b_k 2^k$), would allow to satisfy these conditions and this would be one additional reason for $T = 2^n T_0$ implying p-adic length scale hypothesis. Note however that also $T_p = pT_0$, p prime, can be considered. The remaining coordinates of CD are naturally hyperbolic cosines and sines of the hyperbolic angle $\eta_{\pm,4}$ and cosines and sines of the spherical coordinates θ and ϕ.

2. The existence of the preferred plane M^2 of un-physical polarizations would suggest that the 2-D light-cone proper times $a_{2,+} = \sqrt{t^2 - z^2}$ $a_{2,-} = \sqrt{(t-T)^2 - z^2}$ can be also considered. The remaining coordinates would be naturally $\eta_{\pm,2}$ and cylindrical coordinates (ρ, ϕ).

3. The transcendental values of a_4 and a_2 are literally infinite as real numbers and could be visualized as points in infinitely distant geometric future so that the arrow of time might be said to emerge number theoretically. For M^2 option p-adic transcendental values of ρ are infinite as real numbers so that also spatial infinity could be said to emerge p-adically.

4. The selection of the preferred quantization axes of energy and angular momentum unique apart from a Lorentz transformation of M^2 would have purely number theoretic meaning in both cases. One must allow a union over sub-WCWs labeled by points of $SO(1,1)$. This suggests a deep connection between number theory, quantum theory, quantum measurement theory, and even quantum theory of mathematical consciousness.

5. In the case of CP_2 there are three real coordinate patches involved [K11] . The compactness of CP_2 allows to use cosines and sines of the preferred angle variable for a given coordinate patch.

$$
\begin{aligned}
\xi^1 &= tan(u)exp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\
\xi^2 &= tan(u)exp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ .
\end{aligned} \tag{2.5.1}
$$

   The ranges of the variables u, Θ, Φ, Ψ are $[0, \pi/2], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively. Note that u has naturally only the positive values in the allowed range. S^2 corresponds to the values $\Phi = \Psi = 0$ of the angle coordinates.

6. The rational values of the (hyperbolic) cosine and sine correspond to Pythagorean triangles having sides of integer length and thus satisfying $m^2 = n^2 + r^2$ ($m^2 = n^2 - r^2$). These conditions are equivalent and allow the well-known explicit solution [A72] . One can construct a p-adic completion for the set of Pythagorean triangles by allowing p-adic integers which are infinite as real integers as solutions of the conditions $m^2 = r^2 \pm s^2$. These angles correspond to genuinely p-adic directions having no real counterpart. Hence one obtains p-adic continuum also in the angle degrees of freedom. Algebraic extensions of the p-adic numbers bringing in cosines and sines of the angles π/n lead to a hierarchy increasingly refined algebraic extensions of the generalized imbedding space. Since the different sectors of WCW directly correspond to correlates of selves this means direct correlation with the evolution of the mathematical consciousness. Trigonometric identities allow to construct points which in the real context correspond to sums and differences of angles.

7. Negative rational values of the cosines and sines correspond as p-adic integers to infinite real numbers and it seems that one use several coordinate patches obtained as copies of the octant $(x \geq 0, y \geq 0, z \geq 0,)$. An analogous picture applies in CP_2 degrees of freedom.

8. The expression of the metric tensor and spinor connection of the imbedding in the proposed coordinates makes sense as a p-adic numbers in the algebraic extension considered. The induction of the metric and spinor connection and curvature makes sense provided that the gradients of coordinates with respect to the internal coordinates of the space-time surface belong to the extensions. The most natural choice of the space-time coordinates is as subset of imbedding space-coordinates in a given coordinate patch. If the remaining imbedding space coordinates can be chosen to be rational functions of these preferred coordinates with coefficients in the algebraic extension of p-adic numbers considered for the preferred extremals of Kähler action, then also the gradients satisfy this condition. This is highly non-trivial condition on the extremals and if it works might fix completely the space of exact solutions of field equations. Space-time surfaces are also conjectured to be hyper-quaternionic [K72] , this condition might relate to the simultaneous hyper-quaternionicity and Kähler extremal property. Note also that this picture would provide a partial explanation for the decomposition of the imbedding space to sectors dictated also by quantum measurement theory and hierarchy of Planck constants.

2.5.2 p-Adicization at the level of space-time

Number theoretical Universality in weak sense does not seem to pose problems. The field equations defining the preferred extremals of Kähler action make sense also p-adically if the preferred extremals correspond to critical space-time sheets for which the second variation of Kähler action vanishes for some deformations [K15] : this guarantees that the Noether currents associated with the modified Dirac action are conserved. In this case the matrix determined by second variations has rank which is not maximal. The interpretation is in terms of a generalized catastrophe theory: space-time surfaces are critical with respect to the variation of Kähler action. These conditions are algebraic and make sense also p-adically. Also the conditions implied by number theoretical compactification make sense p-adically. Therefore one can construct the p-adic variants of preferred extremals of Kähler action. The new element is the possibility of p-adic pseudo constants depending on finite number of pinary digits only.

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see fig. `http://www.tgdtheory.fi/appfigures/planckhierarchy.jpg`, which is also in the appendix of this book).

At number theoretical criticality it should be possible to assign to the real partonic 2-surface a unique p-adic counterpart. This might be true also for X^3_l and even for the space-time sheet $X^4(X^3_l)$. This is possible if the objects in question are defined by algebraic equations making sense also p-adically. Also trigonometric functions and exponential functions can be considered. Obviously p-adic pseudo constants are genuine constants for the geometric objects being shared in algebraic sense by the worlds defined by different number fields.

1. The starting point are the algebraic equations defining light-like partonic 3-surfaces X^3_l via the condition that the determinant of the induced metric vanishes. If the coordinate functions appearing in the determinant are algebraic functions with algebraic coefficients, p-adicization should make sense.

2. General Coordinate Invariance would suggest that this true also for the light-like 3-surfaces parallel to X^3_l appearing in the slicing of $X^4(X^3_l)$ assumed in the quantization of induced spinor fields and suggested by the properties of known extremals.

3. If the 4-dimensional real space-time sheet is expressible as a hyper-quaternionic surface of hyper-octonionic variant M^8 of the imbedding space as number-theoretic vision suggests [K72] , it might be possible to construct also the p-adic variant of the space-time sheet by algebraic continuation in the case that the functions appearing in the definition of the space-time sheet are algebraic.

Some preferred space-time coordinates are necessary.

1. Standard Minkowski coordinates associated with $M^2 \times E^2$ decomposition are implied by the selection of quantization axes also also preferred CP_2 coordinates and preferred coordinates for geodesic sphere S^2_i , $i = I$ or II. These coordinates could be used to define coordinates also for X^4. Which combination of coordinate variables is good would be determined by the dimensions of projections to M^4 and CP_2.

2. The construction of solutions of field equations leads to the so called Hamilton-Jacobi coordinates for M^4, when the induced metric has Minkowski signature [K10] . These coordinates define a slicing of $X^4(X^3_l)$ by string world sheets and their partonic duals required also by the number theoretic compactification. For 4-D M^4 projection these coordinates could be used also as X^4 coordinates. The light-like coordinates u, v assigned with the string world sheets *resp.* complex coordinate w associated with the partonic 2-surface would give a candidate for preferred coordinates fixed apart from hyper-conformal *resp.* conformal transformations.

3. A good candidate for preferred coordinates for $X^2(v)$ is defined by the fluxes $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ and their canonical conjugates assignable to partonic 2-surfaces X^2 and their translates $X^2(v)$ along $X^3_l(X^2)$. Here J could correspond to either S^2 or CP_2 Kähler form. These coordinates are discussed in detail in the section about number theoretic braids.

4. For u, v coordinates the basic condition is that v varies along $X^3_l(u)$ and u labels these slices. This condition allows only scalings as hyper-complex analytic transformations and one might hope of fixing this scaling uniquely.

2.5.3 p-Adicization of second quantized induced spinor fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at light-like 3-surfaces and satisfy modified Dirac action associated with Kähler action possibly complexified by addition imaginary CP breaking instanton term. The modified Dirac equation makes sense also p-adically as also the anti-commutation relations of the induced spinor fields at different points of the (number theoretic) braid. Here discreteness is essential since delta functions are not easy to define in p-adic context.

Possible difficulties relate to the definition of p-adic variants of plane wave factors appearing in the construction and being defined with respect to the variable u labeling the slices in the slicing of $X^4(X^3_l)$ by light-like 3-surfaces $X^3_l(v)$ "parallel" to X^3_l. Exponent function as such is well-defined in p-adic context if the argument has p-adic norm smaller than one. It however fails to have the basic properties of its real variant failing to be periodic and having fixed unit p-adic norm for all values of its argument. Periodicity does not however seem to be essential for the formulation of quantum TGD in its recent form. The exponential functions involved are of form $exp(i\sqrt{n}u)$, and are not periodic even in real sense. The p-adic existence requires $u \ mod \ p = 0$ unless one introduces e and possibly also some roots of e to the extension of p-adics used (e^p exists so that the extension would be finite-dimensional).

These observations raise the hope that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure at the level of principle.

2.6 p-Adicization at the level of WCW

This section is not a distilled final answer to the challenges involved with the p-adicization of WCW geometry and spinor structure. There are several questions. What is the precise meaning of concepts like number theoretical universality and criticality? What does p-adicization mean and is it needed/possible? Is algebraic continuation the manner to achieve it?

The notion of reduced WCW implied by the notion of finite measurement resolution is what gives hopes about performing this continuation in practice.

1. The weaker notion of reduced WCW emerges from finite measurement resolution and for given induced Kähler form at partonic 2-surfaces reduces WCW to a finite-dimensional space $(\delta M^4_\pm \times CP_2)^n/S_n$ for given number of points of number theoretic braid. The metric and Kähler structure of this space is determined dynamically in terms of the spectrum of the modified Dirac operator.

2. The stronger notion of reduced WCW identified as the space of the maxima of Kähler function in quantum fluctuating degrees of freedom labeled by symplectic group is second key notion and suggests strongly discretization. The points of reduced configuration space with rational of algebraic coordinates would correspond to those 3-surfaces through which leakage between different sectors of WCW is possible. Reduced configuration space in this sense is the direct counterpart of the spin glass landscape known to obey ultra-metric topology naturally. This approach is reasonably concrete and relies heavily on the most recent, admittedly still speculative, view about quantum TGD.

2.6.1 Generalizing the construction of WCW geometry to the p-adic context

A problematics analogous to that related with the entanglement between real and p-adic number fields is encountered also in the construction of WCW geometry. The original construction was performed in the real context. What is needed are Kähler geometry and spinor structure for the WCW, and a construction of the WCW spinor fields. What might solve these immense architectural challenges are the equally immense symmetries of WCW and algebraic continuation as the method of p-adicization.

What one can hope that everything of physical interest reduces to the level of algebra (rational or algebraic numbers) and that topology (be it real or p-adic) disappears totally at the level of the matrix elements of the metric and of U-matrix mediating transitions between sectors of WCW corresponding to different number fields. It is not necessary to require this to happen for M-matrix identified as time-like entanglement coefficients between positive and negative energy parts of zero energy states.

The notions of number theoretical universality and number theoretical criticality

An essential question is however what one means with the notions of number theoretical universality and criticality.

1. The weak form of the number theoretical universality means that there are sub-WCWs which can be regarded as real, those which are genuinely p-adic, and those which are algebraic in the sense that the representation of partonic 2-surface, perhaps also 3-surface, and perhaps even space-time surface is in terms of rational/algebraic functions allows the interpretation in terms of both real and p-adic numbers. These surfaces would be like rational and algebraic numbers common for the continua formed by reals and p-adics. This poses conditions on the representations of surfaces and typically rational functions with rational coefficients would represent these surfaces.

   For these surfaces - and only for these- physics should be expressible in terms of algebraic numbers and define as a completion the physics in real and p-adic number fields. This would allow p-adic non-determinism. Book analogy is convenient here: the physics corresponding to various number fields would be like pages of books glued together along rational and algebraic physics. If the transitions between states in different number field taking place via a leakage between different pages of the book are allowed, one can regard the algebraic sectors of the WCW as critical. This number theoretic criticality could be interpreted in terms of intentionality and cognition, and living matter would represent a school example about number theoretically critical phase. For this option it is not at all obvious whether it makes sense to speak about WCW geometry. The construction of WCW spinor structure reducing exponent of Kähler function to determinant is what gives some hopes.

2. A much stronger condition - which I adopted originally - is that all 3-surfaces allow interpretation as as both real and p-adic surfaces: in this case p-adic non-determinism would be

excluded. The objection is that this kind of number theoretical universality might reduce to a purely algebraic physics. This condition has interpretation in terms of number theoretical criticality if the weaker notion of universality is adopted.

Generalizing the construction for WCW metric

It is not enough to generalize this construction to the p-adic context. 3-surfaces contain both real and p-adic regions and should be able to perform the construction for this kind of objects.

1. Very naively, one could start from the Riemannian construction of the line element which tells the length squared between infinitesimally close points at each point of the Riemann manifold. The notion of line element involves the notion of nearness and one obviously cannot do without topology here. The line element makes formally sense sense for real and p-adic contexts but since p-adic definite integral does not exist, the notions of p-adic length and volume do not exist naturally. Of course, p-adic norm defines very rough measure of distance in number theoretic sense. The notion of line-element is not needed in the quantum theory at WCW level since only the matrix elements of the WCW metric matter.

2. WCW metric can be constructed in terms if Dirac determinant identified as exponent of Kähler function and the formula for matrix elements is expressible in terms of derivatives of logarithms of the eigen values of the modified Dirac operator with respect to complex coordinates of WCW. This means enormous simplification if the number of eigenvalues is finite as implied by finite measurement resolution realized in terms of braids defined by physical conditions. If eigenvalues are algebraic functions of complex coordinates of WCW then also the exponent of Kähler function and WCW covariant metric defining as its inverse as propagator in WCW degrees of freedom are algebraic functions.

I have also proposed a formula for the matrix elements of configuration space metric and Kähler form between the Killing vector fields of isometry generators. Isometries are identified as X^2 local symplectic symmetries. These expressions can be given also in terms of WCW Hamiltonians as "half Poisson brackets" in complex coordinates. Also the construction of quantum states involves WCW Hamiltonians and their super counterparts.

1. The definition of WCWs Hamiltonians involves definite integrals of corresponding complexified Hamiltonians of $(\delta M^4_\pm \times CP_2)^n$ over X^2. Definite integrals are problematic in the p-adic context, as is clear from the fact that in-numerable number of definitions of definite integral have been proposed.

2. Finite measurement resolution would reduce integrals to sums since WCW reduces to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD. Furthermore, only the Hamiltonians corresponding to triplet *resp.* octet representations of $SO(3)$ *resp.* $SU(3)$ would be needed to coordinatize $S^2 \times CP_2$ part of the reduced WCW.

3. Without number theoretic braids the definition of these integrals seems really difficult in p-adic context. Residue calculus might give some hopes but One might however hope that one could reduce the construction in the real case to that for the representations of super-conformal and symplectic symmetries, and analytically continue the construction from the real context to the p-adic contexts by *defining* the matrix elements of the metric to be what the symmetry respecting analytical continuation gives.

WCW integration should be also continued algebraically to the p-adic context. Quantum criticality realized as the vanishing of loop corrections associated with the WCW integral, would reduce WCW integration to purely algebraic process much like in free field theory and this would give could hopes about p-adicization. Matrix elements would be proportional to the exponent of Kähler function at its maximum plus matrix elements expressible as correlation functions of conformal field theory: the recent state of construction is considered in [K18] . This encourages further the hopes about complete algebraization of the theory so that the independence of the basic formulation on number field could be raised to a principle analogous to general coordinate invariance.

Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in WCW inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the WCW would be possible in the entire WCW. Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also WCW spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X_l^3 to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the WCW. This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub-WCWs induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [K15] . The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_K are algebraic numbers for 3-surfaces X_l^3 for which the coefficients characterizing the rational functions defining X_l^3 are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics and physics of cognition and intention support also this conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \ . \tag{2.6.1}$$

An expression of same form but with sum over eigenvalues of the modified Dirac operator with F replaced with eigenvalue results if exponent of Kähler function is expressible as Dirac determinant:

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} \lambda_k}{\lambda_k} - \frac{\partial_K \lambda \partial_{\overline{L}} \lambda_k}{\lambda_k^2} \ . \tag{2.6.2}$$

What is important that this formula in principles relates WCW geometry directly to quantum physics as represented by the modified Dirac operator.

Generalizing the notion of WCW spinor field

One must also construct spinor structure. Also this construction relies crucially super Kac-Moody and super-symplectic symmetries. Spinors at a given point of WCW correspond to the Fock space spanned by fermionic oscillator operators and again one might hope that super-symmetries would allow algebraization of the whole procedure.

The identification of WCW gamma matrices as super Hamiltonians of WCW. The generators of various super-algebras are also needed in order to construction configuration space spinors at given point of WCW. In ideal measurement resolution these algebra elements are expressible as integrals of Hamiltonians and super-Hamiltonians of $\delta M^4_\pm \times CP_2$ and this leads to difficulties in p-adic context. It might be that finite measurement resolution which seems to be coded by the classical dynamics provides the only possible solution of these difficulties. In the case of reduced WCW the construction of orthonormalized based of WCW spinor fields looks a rather reasonable challenge and the continuation of this procedure to p-adic context might make sense.

2.6.2 WCW functional integral

One can make some general statements about WCW functional integral.

1. If only braid points are specified, there is a functional integral over a huge number of 2-surfaces meaning sum of perturbative contributions from very large number of partonic 2-surfaces selected as maxima of Kähler function or by stationary phase approximation. This kind of non-perturbative contribution makes it very difficult to understand what is involved so that it seems that some restrictions must be posed. Also all information about crucial vacuum degeneracy of Kähler action would be lost as a non-local information.

2. Induced Kähler form represents perhaps the most fundamental zero modes since it remains invariant under symplectic transformations acting as isometries of WCW. Therefore it seems natural organize WCW integral in such a manner that each choice of the induced Kähler form represents its own quantized theory and functional integral is only over deformations leaving induced Kähler form invariant. The deformations of the partonic 2-surfaces would leave invariant both the induced areas and magnetic fluxes. The symplectic orbits of the partonic 2-surfaces (and 3-surfaces) would therefore define a slicing of WCW with separate quantization for each slice.

3. The functional integral would be over the symplectic group of CP_2 and over M^4 degrees of freedom -perhaps also in this case over the symplectic group of δM^4_+ - a rather well-defined mathematical structure. Symplectic transformations of CP_2 affect only the CP_2 part of the induced metric so that a nice separation of degrees of freedom results and the functional integral can be assigned solely to the gravitational degrees of freedom in accordance with the idea that fundamental quantum fluctuating bosonic degrees of freedom are gravitational.

4. WCW integration around a partonic 2-surface for which the Kähler function is maximum with respect to quantum fluctuating degrees of freedom should give only tree diagrams with propagator factors proportional to g_K^2 if loop corrections to the WCW integral vanish. One could hope that there exist preferred S^2 and CP_2 coordinates such that vertex factors involving finite polynomials of S^2 and CP_2 coordinates reduce to a finite number of diagrams just as in free field theory.

If WCW functional integral algebraizes by the vanishing of loop corrections, one has hopes that even p-adic variant of WCW functional integral might make sense. The exponent of Kähler function appears and if given by the Dirac determinant it would reduce to a finite product of eigenvalues of modified Dirac operator which makes sense also p-adically.

Algebraization of WCW functional integral

WCW is a union of infinite-dimensional symmetric spaces labeled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case.

1. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K\partial_L K$ and $\partial_{\overline{K}}\partial_{\overline{L}}K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

2. If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each WCW spinor field would define a vertex from which lines representing the propagators defined by the contravariant WCW metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic WCW integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the WCW integrals would exist also in the p-adic sense.

Should one p-adicize only the reduced configuration space?

An attractive approach to p-adicization might be characterized as minimalism and would involve geometrization of only the reduced WCW consisting of the maxima of Kähler function in quantum fluctuating degrees of freedom. A further reduction results from the finite measurement resolution replacing WCW effectively with $(\delta M^4_\pm \times CP_2)^n/S_n$. In zero modes discretization realizing quantum classical correspondence is attractive possibility.

1. If Duistermaat-Heckman theorem [A133] holds true in TGD context, one could express real WCW functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function in quantum fluctuating degrees of freedom defining what might be called reduced WCW CH_{red}. The exponent of Kähler function and propagator identified as contravariant metric of WCW could be deduced from the spectrum of the modified Dirac operator.

2. The huge super-conformal symmetries raise the hope that the rest of M-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of M-matrix.

3. A possible problem of this reduction is that the number of degrees of freedom in functional integral is still infinite, which might mean problems in terms of algebraization. For instance, the inverse of covariant metric identified as algebraic function need not represent algebraic object. Finite measurement resolution improves the situation in this respect. Finite measurement resolution realized in terms of number theoretic braids would reduce WCW to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD and this would reduce the situation to a finite dimensional one and maxima of Kähler function would form a discrete set, possibly only single point of $(\delta M^4_\pm \times CP_2)^n/S_n$. Also in this case exponent of Kähler function and the spectrum of modified Dirac operator are needed. Also the values of $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic braids labeled by $\delta M^4_\pm \times CP_2/S_n$ are needed.

Zero modes pose a further problem.

1. The absence of functional integral measure in zero modes would suggest that states depend on finite number of zero modes only and that there is localization in this degrees of freedom. Finite measurement resolution suggests the same. The extrema of the quantity $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic represent finite set of values of fundamental zero modes assignable to X^2 forming a finite-dimensional space naturally. Non-local isometry invariants can be defined as Kähler magnetic fluxes if it is possible to define symplectic triangulation of X^2 with vertices identifiable naturally as points of number theoretic braid corresponding to the extrema of J. The notion of symplectic fusion algebra based on this kind of triangulation is discussed in [K14] .

2. Kac-Moody group parameterizes zero modes assignable to X_l^3 and a correlation between these zero modes and the quantum numbers of quantum state is natural and could result by stationary phase approximation if finite-dimensional variant of functional integral can be defined. If there is localization in zero modes, this correspondence could be discrete and implied by classical equations of motion for braid points. A unique selection of preferred quantization axis would be made possible by the hierarchy of Planck constants selecting $M^2 \subset M^4$ and $S_i^2 \subset CP_2$ as critical manifolds with respect to the change of Planck constant.

What other difficulties can one imagine?

1. The optimal situation would be that M-matrix elements in real case are algebraic functions or at least functions continuable to the p-adic context in a form having sensible physical interpretation.

2. If one starts directly from Fourier transforms in p-adic context, difficulties are caused by trigonometric functions and exponent function whose p-adic counterparts do not behave in physically acceptable manner. It seems that it is phase factors defined by plane waves which should should restricted to roots of unity and continued to the p-adic realm as such. In p-adic context either momentum or position makes sense as p-adic number unless one introduces infinite-dimensional extension containing logarithms and π. Maybe the only manner to avoid problems is to accept discretization and algebraization of the phase factors.

Concerning number field changing transitions at number theoretical criticality possibly relevant for U-matrix some comments are in order. For real$\leftrightarrow$ p-adic transitions only the algebraic points of number theoretic braid common to both real and p-adic variant of partonic 2-surface are relevant and situation reduces to algebraic braid points in $(\delta M_\pm^4 \times CP_2)/S_n$. Algebraic points in a given extension of rationals would be common to real and p-adic surfaces. It could happen that there are very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$. The integral over reduced WCW should reduce to a sum over possible values of coordinates for these points. If only maxima of Kähler function an analytic continuation of real M-matrix to p-adic-real M-matrix could make sense.

If this picture is correct, the p-adicization of WCW would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. Finite measurement resolution simplifies the situation dramatically. If CH_{red} is a discrete subset of CH or its finite-dimensional variant, ultra-metric topology induced from finite-p p-adic norm is indeed natural for it. 'Discrete set in CH' need not mean a discrete set in the usual sense and the reduced WCW could be even finite-dimensional continuum. p-Adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

2.6.3 Number theoretic constraints on M-matrix

Assume that U-matrix assignable to quantum jump between zero energy states exists simultaneously in all number fields and perhaps even between different number fields at number theoretical quantum criticality (allowing finite-dimensional extensions of p-adics). If so the immediate question is whether also the construction procedure of the M-matrix defined as time-like entanglement coefficients between positive and negative energy parts of zero energy state could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. The identification of M-matrices as building blocks of U-matrix in the manner discussed in [K18] supports affirmative answer to the first question. Not only the WCW but also Kähler function and its exponent, Kähler metric, and WCW functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

Number theoretical Universality and M-matrix

Number theoretic constraints on M-matrix are non-trivial even for the weaker notion of number theoretical universality. Number theoretical criticality (or number theoretical universality in strong sense) requires that M-matrix elements are algebraic numbers. This is achieved naturally if the definition of M-matrix elements involves only the data associated with the number theoretic braid. Note that this data is non-local since it involves information about tangent space of X^4 at the point so that discretization happens in geometric sense but not in information theoretic sense. Note also that for algebraic surfaces finite number of points of surface allows to deduce the parameters of the polynomials involved and thus to deduce the entire surface.

If quantum version of WCW is adopted one must perform quantization for $E^2 \subset M^4$ coordinates of points S_i^2 braid and CP_2 coordinates of M^2 braid. In this kind of situation it becomes unclear whether one can speak about braiding anymore. This might make sense if each braid topology corresponds to its own quantization containing information about the fact that deformations of X_l^3 respect the braiding topology.

The partonic vertices appearing in M-matrix elements should be expressible in terms of N-point functions of some rational super-conformal field theory but with the p-adically questionable N-fold integrals over string appearing in the definition of n-point functions. The most elegant manner to proceed is to replace them with their explicit expressions if they are algebraic functions- quite generally or at number theoretical criticality. Spin chain type string discretization is an alternative, not so elegant option.

Propagators, that is correlations between partonic 2-surfaces, would be due to the interior dynamics of space-time sheets which means a deviation from super string theory. Another function of interior degrees of freedom is to provide zero modes of metric of WCW identifiable as classical degrees of freedom of quantum measurement theory entangling with quantal degrees of freedom at partonic 3-surfaces.

Number theoretical criticality and M-matrix

Number theoretical criticality poses very strong conditions on the theory.

1. The p-adic variants of 4-D field equations associated with Kähler action make sense. Also the notion of preferred extremal makes sense in p-adic context if it corresponds to quantum criticality in the sense that second variation of Kähler action vanishes for dynamical symmetries. A natural further condition is that the surface is representable in terms of algebraic equations involving only rational or algebraic coefficients and thus making sense both in real and p-adic sense. In this case also Kähler action and classical charges could exist in some algebraic extension of p-adic numbers.

2. Also modified Dirac equation makes sense p-adically. The exponent of Kähler function defining vacuum functional is well-defined notion p-adically if the identification as product of finite number of eigenvalues of the modified Dirac operator is accepted and eigenvalues are algebraic. Also the notion of WCW metric expressible in terms of derivatives of the eigenvalues with respect to complex coordinates of WCW makes sense.

3. The functional integral over WCW can be defined only as an algebraic extension of real functional integral around maximum of Kähler function if the theory is integrable and gives as a result an algebraic number. One might hope that algebraic p-adicization makes sense for the vacuum function at points corresponding to the maxima of Kähler function with respect to quantum fluctuating degrees of freedom (assuming they exist) and with respect to zero modes. As discussed already earlier, in the case of zero modes quantum classical correspondence allows to select preferred value of zero modes even if functional integral in zero modes does not make sense. The basic requirement is that the inverse of the matrix defined by the Kähler metric defining propagator is algebraic function of the complex coordinate of WCW. If the eigen-values of the modified Dirac operator satisfy this condition this is indeed the case.

4. Ordinary perturbation series based on Feynman diagrams makes sense also in p-adic sense since the presence of cutoff for the size of CD implies that the number of terms if finite.

One must be however cautious with momentum integrations which should reduce to finite sum due to the presence of both IR and UV cutoff implied by the finite size of CD. The formulation in terms of number theoretic braids whose intersections with partonic 2-surfaces consist of finite number of points supports the possibility of number theoretic universality.

There are hopes that M-matrix make sense p-adically. As far M-matrix is considered, The most plausible interpretation relies on the weaker form of number theoretic universality so that genuinely p-adic M-matrices should exist.

1. Dirac determinant exists for any p-adic 3-surfaces since the eigenvalues of modified Dirac operator represent a purely local notion sensible also in p-adic context. The reason is that finite measurement resolution - now deducible from the vacuum degeneracy of Kähler action-implies that the number of eigenvalues is finite. Preferred extremals of Kähler action obey quantum criticality condition meaning that the second variation of Kähler action vanishes. This condition makes sense also p-adically.

2. If loops vanish, WCW integration gives only contractions with propagator expressible as the contravariant WCW Kähler metric expressible in terms of derivatives of the Kähler function with respect to the preferred complex coordinates of WCW. If this function is algebraic function, it allows algebraic continuation to p-adic context and all that is needed for calculation of M-matrix elements makes sense p-adically. The crucial question is whether the Kähler metric is algebraic function in preferred coordinates.

3. N-point functions involve also symplectically invariant multiplicative factors discussed in [K14] in terms of symplectic fusion algebras. For them algebraic universality holds true. N-point functions of conformal field theory associated with the generalized vertices should also be algebraic functions.

4. Finite measurement resolution realized in terms of braids for given $J = \epsilon^{\alpha\beta} J_{\alpha\beta}$ means a reduction of a given sector of WCW in quantum fluctuating degrees of freedom to finite-dimensional space $\delta M^4_\pm \times CP_2/S_n$ associated with the boundaries of CD. For instance, configuration space Hamiltonians reduce apart from J factor to those assignable naturally to the reduced WCW. Finite-dimensionality gives hopes of algebraic continuation of M-matrix defined in terms of general Feynman diagrams in real context using finite purely algebraic operations due to the cutoff in the size of CDs. In zero modes the simplest option would be that quantum states correspond to sums over different values of zero modes, in particular J as function in X^2.

Also number theoretical criticality is consistent with this picture.

1. If partonic 2-surface X^2 is determined by algebraic equations involving only rational coefficients, same equations define real and p-adic variants of X^2.

2. Number theoretic criticality for braids means that their points are algebraic and common to real and p-adic partonic 2-surfaces. The extrema of J -determined by algebraic conditions-must be algebraic numbers.

3. At quantum criticality Dirac determinant is algebraic number if the number of eigenvalues is finite (implied by finite measurement resolution) and if they are algebraic numbers. If the p-adic counterpart of X^3_l exists, this allows to assign to the p-adic counterpart of X^3_l the exponent of Kähler function as Dirac determinant although Kähler action remains ill-defined p-adically.

The relationship between U-matrix and M-matrix

The following represents the latest result concerning the relationship between the notions of U-matrix and M-matrix and probably provides answer to some of the questions posed in the chapter. What is highly satisfactory that U-matrix dictates M-matrix completely via unitarity conditions. A more detailed discussion can be [K42] discussing Negentropy Maximization Principle, which is the basic dynamical principle of TGD inspired theory of consciousness and states that the information content of conscious experience is maximal.

If state function reduction associated with time-like entanglement leads always to a product of positive and negative energy states (so that there is no counterpart of bound state entanglement and negentropic entanglement possible for zero energy states: these notions are discussed below) U-matrix and can be regarded as a collection of M-matrices

$$U_{m_+n_-,r_+,s_-} = M(m_+,n_-)_{r_+,s_-} \tag{2.6.3}$$

labeled by the pairs (m_+,n_-) labelling zero energy states assumed to reduced to pairs of positive and negative energy states. M-matrix element is the counterpart of S-matrix element $S_{r,s}$ in positive energy ontology. Unitarity conditions for U-matrix read as

$$\begin{aligned}
(UU^\dagger)_{m_+n_-,r_+s_-} &= \sum_{k_+,l_-} M(m_+,n_-)_{k_+,l_-}\overline{M}(r_+,s_-)_{k_+,l_-} = \delta_{m+r_+,n_-s_-} \ , \\
(U^\dagger U)_{m_+n_-,r_+s_-} &= \sum_{k_+,l_-} \overline{M}(k_+,l_-)_{m_+,n_-}M(k_+,l_-)_{r_+,s_-} = \delta_{m+r_+,n_-s_-} \ .
\end{aligned} \tag{2.6.4}$$

The conditions state that the zero energy states associated with different labels are orthogonal as zero energy states and also that the zero energy states defined by the dual M-matrix

$$M^\dagger(m_+,n_-)_{k_+,l_-} \equiv \overline{M}(k_+l_-)_{m_+,n_-} \tag{2.6.5}$$

-perhaps identifiable as phase conjugate states- define an orthonormal basis of zero energy states.

When time-like binding and negentropic entanglement are allowed also zero energy states with a label not implying a decomposition to a product state are involved with the unitarity condition but this does not affect the situation dramatically. As a matter fact, the situation is mathematically the same as for ordinary S-matrix in the presence of bound states. Here time-like bound states are analogous to space-like bound states and by definition are unable to decay to product states (free states). Negentropic entanglement makes sense only for entanglement probabilities, which are rationals or belong to their algebraic extensions. This is possible in what might be called the intersection of real and p-adic worlds (partonic surfaces in question have representation making sense for both real and p-adic numbers). Number theoretic entropy is obtained by replacing in the Shannon entropy the logarithms of probabilities with the logarithms of their p-adic norms. They satisfy the same defining conditions as ordinary Shannon entropy but can be also negative. One can always find prime p for which the entropy is maximally negative. The interpretation of negentropic entanglement is in terms of formations of rule or association. Schrödinger cat knows that it is better to not open the bottle: open bottle-dead cat, closed bottle-living cat and negentropic entanglement measures this information.

2.7 Weak form electric-magnetic duality and its implications

The notion of electric-magnetic duality [B7] was proposed first by Olive and Montonen and is central in $\mathcal{N}=4$ supersymmetric gauge theories. It states that magnetic monopoles and ordinary particles are two different phases of theory and that the description in terms of monopoles can be applied at the limit when the running gauge coupling constant becomes very large and perturbation theory fails to converge. The notion of electric-magnetic self-duality is more natural since for CP_2 geometry Kähler form is self-dual and Kähler magnetic monopoles are also Kähler electric monopoles and Kähler coupling strength is by quantum criticality renormalization group invariant rather than running coupling constant. The notion of electric-magnetic (self-)duality emerged already two decades ago in the attempts to formulate the Kähler geometric of world of classical worlds. Quite recently a considerable step of progress took place in the understanding of this notion [K16] . What seems to be essential is that one adopts a weaker form of the self-duality applying at partonic 2-surfaces. What this means will be discussed in the sequel.

Every new idea must be of course taken with a grain of salt but the good sign is that this concept leads to precise predictions. The point is that elementary particles do not generate monopole fields in macroscopic length scales: at least when one considers visible matter. The first question is whether elementary particles could have vanishing magnetic charges: this turns out to be impossible. The next question is how the screening of the magnetic charges could take place and leads to an identification of the physical particles as string like objects identified as pairs magnetic charged wormhole throats connected by magnetic flux tubes.

1. The first implication is a new view about electro-weak massivation reducing it to weak confinement in TGD framework. The second end of the string contains particle having electroweak isospin neutralizing that of elementary fermion and the size scale of the string is electro-weak scale would be in question. Hence the screening of electro-weak force takes place via weak confinement realized in terms of magnetic confinement.

2. This picture generalizes to the case of color confinement. Also quarks correspond to pairs of magnetic monopoles but the charges need not vanish now. Rather, valence quarks would be connected by flux tubes of length of order hadron size such that magnetic charges sum up to zero. For instance, for baryonic valence quarks these charges could be $(2, -1, -1)$ and could be proportional to color hyper charge.

3. The highly non-trivial prediction making more precise the earlier stringy vision is that elementary particles are string like objects: this could become manifest at LHC energies.

4. The weak form electric-magnetic duality together with Beltrami flow property of Kähler leads to the reduction of Kähler action to Chern-Simons action so that TGD reduces to almost topological QFT and that Kähler function is explicitly calculable. This has enormous impact concerning practical calculability of the theory.

5. One ends up also to a general solution ansatz for field equations from the condition that the theory reduces to almost topological QFT. The solution ansatz is inspired by the idea that all isometry currents are proportional to Kähler current which is integrable in the sense that the flow parameter associated with its flow lines defines a global coordinate. The proposed solution ansatz would describe a hydrodynamical flow with the property that isometry charges are conserved along the flow lines (Beltrami flow). A general ansatz satisfying the integrability conditions is found.

   The strongest form of the solution ansatz states that various classical and quantum currents flow along flow lines of the Beltrami flow defined by Kähler current (Kähler magnetic field associated with Chern-Simons action). Intuitively this picture is attractive. A more general ansatz would allow several Beltrami flows meaning multi-hydrodynamics. The integrability conditions boil down to two scalar functions: the first one satisfies massless d'Alembert equation in the induced metric and the the gradients of the scalar functions are orthogonal. The interpretation in terms of momentum and polarization directions is natural. Also Chern-Simons Dirac equation implies the localization of solutions to flow lines, and this is consistent with the localization solutions of Kähler-Dirac equation to string world sheets.

2.7.1 Could a weak form of electric-magnetic duality hold true?

Holography means that the initial data at the partonic 2-surfaces should fix the WCW metric. A weak form of this condition allows only the partonic 2-surfaces defined by the wormhole throats at which the signature of the induced metric changes. A stronger condition allows all partonic 2-surfaces in the slicing of space-time sheet to partonic 2-surfaces and string world sheets. Number theoretical vision suggests that hyper-quaternionicity *resp.* co-hyperquaternionicity constraint could be enough to fix the initial values of time derivatives of the imbedding space coordinates in the space-time regions with Minkowskian *resp.* Euclidian signature of the induced metric. This is a condition on modified gamma matrices and hyper-quaternionicity states that they span a hyper-quaternionic sub-space.

Definition of the weak form of electric-magnetic duality

One can also consider alternative conditions possibly equivalent with this condition. The argument goes as follows.

1. The expression of the matrix elements of the metric and Kähler form of WCW in terms of the Kähler fluxes weighted by Hamiltonians of $\delta M^4_\pm$ at the partonic 2-surface X^2 looks very attractive. These expressions however carry no information about the 4-D tangent space of the partonic 2-surfaces so that the theory would reduce to a genuinely 2-dimensional theory, which cannot hold true. One would like to code to the WCW metric also information about the electric part of the induced Kähler form assignable to the complement of the tangent space of $X^2 \subset X^4$.

2. Electric-magnetic duality of the theory looks a highly attractive symmetry. The trivial manner to get electric magnetic duality at the level of the full theory would be via the identification of the flux Hamiltonians as sums of of the magnetic and electric fluxes. The presence of the induced metric is however troublesome since the presence of the induced metric means that the simple transformation properties of flux Hamiltonians under symplectic transformations -in particular color rotations- are lost.

3. A less trivial formulation of electric-magnetic duality would be as an initial condition which eliminates the induced metric from the electric flux. In the Euclidian version of 4-D YM theory this duality allows to solve field equations exactly in terms of instantons. This approach involves also quaternions. These arguments suggest that the duality in some form might work. The full electric magnetic duality is certainly too strong and implies that space-time surface at the partonic 2-surface corresponds to piece of CP_2 type vacuum extremal and can hold only in the deep interior of the region with Euclidian signature. In the region surrounding wormhole throat at both sides the condition must be replaced with a weaker condition.

4. To formulate a weaker form of the condition let us introduce coordinates (x^0, x^3, x^1, x^2) such (x^1, x^2) define coordinates for the partonic 2-surface and (x^0, x^3) define coordinates labeling partonic 2-surfaces in the slicing of the space-time surface by partonic 2-surfaces and string world sheets making sense in the regions of space-time sheet with Minkowskian signature. The assumption about the slicing allows to preserve general coordinate invariance. The weakest condition is that the generalized Kähler electric fluxes are apart from constant proportional to Kähler magnetic fluxes. This requires the condition

$$J^{03}\sqrt{g_4} \quad = \quad KJ_{12} \ . \tag{2.7.1}$$

A more general form of this duality is suggested by the considerations of [K33] reducing the hierarchy of Planck constants to basic quantum TGD and also reducing Kähler function for preferred extremals to Chern-Simons terms [B2] at the boundaries of CD and at light-like wormhole throats. This form is following

$$J^{n\beta}\sqrt{g_4} \quad = \quad K\epsilon \times \epsilon^{n\beta\gamma\delta}J_{\gamma\delta}\sqrt{g_4} \ . \tag{2.7.2}$$

Here the index n refers to a normal coordinate for the space-like 3-surface at either boundary of CD or for light-like wormhole throat. ϵ is a sign factor which is opposite for the two ends of CD. It could be also opposite of opposite at the opposite sides of the wormhole throat. Note that the dependence on induced metric disappears at the right hand side and this condition eliminates the potentials singularity due to the reduction of the rank of the induced metric at wormhole throat.

5. Information about the tangent space of the space-time surface can be coded to the WCW metric with loosing the nice transformation properties of the magnetic flux Hamiltonians if Kähler electric fluxes or sum of magnetic flux and electric flux satisfying this condition are used and K is symplectic invariant. Using the sum

$$J_e + J_m = (1+K)J_{12} \ , \tag{2.7.3}$$

where J denotes the Kähler magnetic flux, , makes it possible to have a non-trivial WCW metric even for $K=0$, which could correspond to the ends of a cosmic string like solution carrying only Kähler magnetic fields. This condition suggests that it can depend only on Kähler magnetic flux and other symplectic invariants. Whether local symplectic coordinate invariants are possible at all is far from obvious, If the slicing itself is symplectic invariant then K could be a non-constant function of X^2 depending on string world sheet coordinates. The light-like radial coordinate of the light-cone boundary indeed defines a symplectically invariant slicing and this slicing could be shifted along the time axis defined by the tips of CD.

Electric-magnetic duality physically

What could the weak duality condition mean physically? For instance, what constraints are obtained if one assumes that the quantization of electro-weak charges reduces to this condition at classical level?

1. The first thing to notice is that the flux of J over the partonic 2-surface is analogous to magnetic flux

$$Q_m = \frac{e}{\hbar}\oint BdS = n \ .$$

n is non-vanishing only if the surface is homologically non-trivial and gives the homology charge of the partonic 2-surface.

2. The expressions of classical electromagnetic and Z^0 fields in terms of Kähler form [L2] , [L2] read as

$$\begin{aligned} \gamma &= \frac{eF_{em}}{\hbar} = 3J - sin^2(\theta_W)R_{03} \ , \\ Z^0 &= \frac{g_Z F_Z}{\hbar} = 2R_{03} \ . \end{aligned} \tag{2.7.4}$$

Here R_{03} is one of the components of the curvature tensor in vielbein representation and F_{em} and F_Z correspond to the standard field tensors. From this expression one can deduce

$$J = \frac{e}{3\hbar}F_{em} + sin^2(\theta_W)\frac{g_Z}{6\hbar}F_Z \ . \tag{2.7.5}$$

3. The weak duality condition when integrated over X^2 implies

$$\begin{aligned} \frac{e^2}{3\hbar}Q_{em} + \frac{g_Z^2 p}{6}Q_{Z,V} &= K\oint J = Kn \ , \\ Q_{Z,V} &= \frac{I_V^3}{2} - Q_{em} \ , \ \ p = sin^2(\theta_W) \ . \end{aligned} \tag{2.7.6}$$

Here the vectorial part of the Z^0 charge rather than as full Z^0 charge $Q_Z = I_L^3 + sin^2(\theta_W)Q_{em}$ appears. The reason is that only the vectorial isospin is same for left and right handed components of fermion which are in general mixed for the massive states.

The coefficients are dimensionless and expressible in terms of the gauge coupling strengths and using $\hbar = r\hbar_0$ one can write

$$\begin{aligned} \alpha_{em}Q_{em} &+ p\frac{\alpha_Z}{2}Q_{Z,V} = \frac{3}{4\pi} \times rnK \ , \\ \alpha_{em} &= \frac{e^2}{4\pi\hbar_0} \ , \ \alpha_Z = \frac{g_Z^2}{4\pi\hbar_0} = \frac{\alpha_{em}}{p(1-p)} \ . \end{aligned} \tag{2.7.7}$$

4. There is a great temptation to assume that the values of Q_{em} and Q_Z correspond to their quantized values and therefore depend on the quantum state assigned to the partonic 2-surface. The linear coupling of the modified Dirac operator to conserved charges implies correlation between the geometry of space-time sheet and quantum numbers assigned to the partonic 2-surface. The assumption of standard quantized values for Q_{em} and Q_Z would be also seen as the identification of the fine structure constants α_{em} and α_Z. This however requires weak isospin invariance.

The value of K from classical quantization of Kähler electric charge

The value of K can be deduced by requiring classical quantization of Kähler electric charge.

1. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge g_K would give the condition $K = g_K^2/\hbar$, where g_K is Kähler coupling constant which should invariant under coupling constant evolution by quantum criticality. Within experimental uncertainties one has $\alpha_K = g_K^2/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where α_{em} is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

2. The quantization of Planck constants makes the condition highly non-trivial. The most general quantization of r is as rationals but there are good arguments favoring the quantization as integers corresponding to the allowance of only singular coverings of CD andn CP_2. The point is that in this case a given value of Planck constant corresponds to a finite number pages of the "Big Book". The quantization of the Planck constant implies a further quantization of K and would suggest that K scales as $1/r$ unless the spectrum of values of Q_{em} and Q_Z allowed by the quantization condition scales as r. This is quite possible and the interpretation would be that each of the r sheets of the covering carries (possibly same) elementary charge. Kind of discrete variant of a full Fermi sphere would be in question. The interpretation in terms of anyonic phases [K54] supports this interpretation.

3. The identification of J as a counterpart of $eB/\hbar$ means that Kähler action and thus also Kähler function is proportional to $1/\alpha_K$ and therefore to $\hbar$. This implies that for large values of $\hbar$ Kähler coupling strength $g_K^2/4\pi$ becomes very small and large fluctuations are suppressed in the functional integral. The basic motivation for introducing the hierarchy of Planck constants was indeed that the scaling $\alpha \to \alpha/r$ allows to achieve the convergence of perturbation theory: Nature itself would solve the problems of the theoretician. This of course does not mean that the physical states would remain as such and the replacement of single particles with anyonic states in order to satisfy the condition for K would realize this concretely.

4. The condition $K = g_K^2/\hbar$ implies that the Kähler magnetic charge is always accompanied by Kähler electric charge. A more general condition would read as

$$K = n \times \frac{g_K^2}{\hbar}, n \in Z \ . \tag{2.7.8}$$

This would apply in the case of cosmic strings and would allow vanishing Kähler charge possible when the partonic 2-surface has opposite fermion and anti-fermion numbers (for both leptons and quarks) so that Kähler electric charge should vanish. For instance, for neutrinos the vanishing of electric charge strongly suggests $n = 0$ besides the condition that abelian Z^0 flux contributing to em charge vanishes.

It took a year to realize that this value of K is natural at the Minkowskian side of the wormhole throat. At the Euclidian side much more natural condition is

$$K = \frac{1}{hbar} \ . \tag{2.7.9}$$

In fact, the self-duality of CP_2 Kähler form favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for CP_2 type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of CP_2 radius and α_K the effective replacement $g_K^2 \to 1$ would spoil the argument.

The boundary condition $J_E = J_B$ for the electric and magnetic parts of Kählwer form at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded CP_2 is such that in CP_2 coordinates for the Euclidian region the tensor $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$ remains invariant. This is certainly the case for CP_2 type vacuum extremals since by the light-likeness of M^4 projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole.

Reduction of the quantization of Kähler electric charge to that of electromagnetic charge

The best manner to learn more is to challenge the form of the weak electric-magnetic duality based on the induced Kähler form.

1. Physically it would seem more sensible to pose the duality on electromagnetic charge rather than Kähler charge. This would replace induced Kähler form with electromagnetic field, which is a linear combination of induced Kahler field and classical Z^0 field

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{2.7.10}$$

   Here $Z_0 = 2R_{03}$ is the appropriate component of CP_2 curvature form [L2]. For a vanishing Weinberg angle the condition reduces to that for Kähler form.

2. For the Euclidian space-time regions having interpretation as lines of generalized Feynman diagrams Weinberg angle should be non-vanishing. In Minkowskian regions Weinberg angle could however vanish. If so, the condition guaranteeing that electromagnetic charge of the partonic 2-surfaces equals to the above condition stating that the em charge assignable to the fermion content of the partonic 2-surfaces reduces to the classical Kähler electric flux at the Minkowskian side of the wormhole throat. One can argue that Weinberg angle must increase smoothly from a vanishing value at both sides of wormhole throat to its value in the deep interior of the Euclidian region.

3. The vanishing of the Weinberg angle in Minkowskian regions conforms with the physical intuition. Above elementary particle length scales one sees only the classical electric field

reducing to the induced Kähler form and classical Z^0 fields and color gauge fields are effectively absent. Only in phases with a large value of Planck constant classical Z^0 field and other classical weak fields and color gauge field could make themselves visible. Cell membrane could be one such system [K58]. This conforms with the general picture about color confinement and weak massivation.

The GRT limit of TGD suggests a further reason for why Weinberg angle should vanish in Minkowskian regions.

1. The value of the Kähler coupling strength mut be very near to the value of the fine structure constant in electron length scale and these constants can be assumed to be equal.

2. GRT limit of TGD with space-time surfaces replaced with abstract 4-geometries would naturally correspond to Einstein-Maxwell theory with cosmological constant which is non-vanishing only in Euclidian regions of space-time so that both Reissner-Nordström metric and CP_2 are allowed as simplest possible solutions of field equations [K74]. The extremely small value of the observed cosmological constant needed in GRT type cosmology could be equal to the large cosmological constant associated with CP_2 metric multiplied with the 3-volume fraction of Euclidian regions.

3. Also at GRT limit quantum theory would reduce to almost topological QFT since Einstein-Maxwell action reduces to 3-D term by field equations implying the vanishing of the Maxwell current and of the curvature scalar in Minkowskian regions and curvature scalar + cosmological constant term in Euclidian regions. The weak form of electric-magnetic duality would guarantee also now the preferred extremal property and prevent the reduction to a mere topological QFT.

4. GRT limit would make sense only for a vanishing Weinberg angle in Minkowskian regions. A non-vanishing Weinberg angle would make sense in the deep interior of the Euclidian regions where the approximation as a small deformation of CP_2 makes sense.

The weak form of electric-magnetic duality has surprisingly strong implications for the basic view about quantum TGD as following considerations show.

2.7.2 Magnetic confinement, the short range of weak forces, and color confinement

The weak form of electric-magnetic duality has surprisingly strong implications if one combines it with some very general empirical facts such as the non-existence of magnetic monopole fields in macroscopic length scales.

How can one avoid macroscopic magnetic monopole fields?

Monopole fields are experimentally absent in length scales above order weak boson length scale and one should have a mechanism neutralizing the monopole charge. How electroweak interactions become short ranged in TGD framework is still a poorly understood problem. What suggests itself is the neutralization of the weak isospin above the intermediate gauge boson Compton length by neutral Higgs bosons. Could the two neutralization mechanisms be combined to single one?

1. In the case of fermions and their super partners the opposite magnetic monopole would be a wormhole throat. If the magnetically charged wormhole contact is electromagnetically neutral but has vectorial weak isospin neutralizing the weak vectorial isospin of the fermion only the electromagnetic charge of the fermion is visible on longer length scales. The distance of this wormhole throat from the fermionic one should be of the order weak boson Compton length. An interpretation as a bound state of fermion and a wormhole throat state with the quantum numbers of a neutral Higgs boson would therefore make sense. The neutralizing throat would have quantum numbers of $X_{-1/2} = \nu_L\overline{\nu}_R$ or $X_{1/2} = \overline{\nu}_L\nu_R$. $\nu_L\overline{\nu}_R$ would not be neutral Higgs boson (which should correspond to a wormhole contact) but a superpartner of left-handed neutrino obtained by adding a right handed neutrino. This mechanism

would apply separately to the fermionic and anti-fermionic throats of the gauge bosons and corresponding space-time sheets and leave only electromagnetic interaction as a long ranged interaction.

2. One can of course wonder what is the situation situation for the bosonic wormhole throats feeding gauge fluxes between space-time sheets. It would seem that these wormhole throats must always appear as pairs such that for the second member of the pair monopole charges and I_V^3 cancel each other at both space-time sheets involved so that one obtains at both space-time sheets magnetic dipoles of size of weak boson Compton length. The proposed magnetic character of fundamental particles should become visible at TeV energies so that LHC might have surprises in store!

Well-definedness of electromagnetic charge implies stringiness

Well-definedness of electromagnetic charged at string world sheets carrying spinor modes is very natural constraint and not trivially satisfied because classical W boson fields are present. As a matter fact, all weak fields should be effectively absent above weak scale. How this is possible classical weak fields identified as induced gauge fields are certainly present.

The condition that em charge is well defined for spinor modes implies that the space-time region in which spinor mode is non-vanishing has 2-D CP_2 projection such that the induced W boson fields are vanishing. The vanishing of classical Z^0 field can be poses as additional condition - at least in scales above weak scale. In the generic case this requires that the spinor mode is restricted to 2-D surface: string world sheet or possibly also partonic 2-surface. This implies that TGD reduces to string model in fermionic sector. Even for preferred extremals with 2-D projecting the modes are expected to allow restriction to 2-surfaces. This localization is possible only for Kähler-Dirac action.

A word of warning is however in order. The GRT limit or rather limit of TGD as Einstein Yang-Mills theory replaces the sheets of many-sheeted space-time with Minkowski space with effective metric obtained by summing to Minkowski metric the deviations of the induced metrics of space-time sheets from Minkowski metric. For gauge potentials a similar identification applies. YM-Einstein equations coupled with matter and with non-vanishing cosmological constant are expected on basis of Poincare invariance. One cannot exclude the possibility that the sums of weak gauge potentials from different space-time sheet tend to vanish above weak scale and that well-definedness of em charge at classical level follows from the effective absence of classical weak gauge fields.

Magnetic confinement and color confinement

Magnetic confinement generalizes also to the case of color interactions. One can consider also the situation in which the magnetic charges of quarks (more generally, of color excited leptons and quarks) do not vanish and they form color and magnetic singles in the hadronic length scale. This would mean that magnetic charges of the state $q_{\pm 1/2} - X_{\mp 1/2}$ representing the physical quark would not vanish and magnetic confinement would accompany also color confinement. This would explain why free quarks are not observed. To how degree then quark confinement corresponds to magnetic confinement is an interesting question.

For quark and antiquark of meson the magnetic charges of quark and antiquark would be opposite and meson would correspond to a Kähler magnetic flux so that a stringy view about meson emerges. For valence quarks of baryon the vanishing of the net magnetic charge takes place provided that the magnetic net charges are $(\pm 2, \mp 1, \mp 1)$. This brings in mind the spectrum of color hyper charges coming as $(\pm 2, \mp 1, \mp 1)/3$ and one can indeed ask whether color hypercharge correlates with the Kähler magnetic charge. The geometric picture would be three strings connected to single vertex. Amusingly, the idea that color hypercharge could be proportional to color hyper charge popped up during the first year of TGD when I had not yet discovered CP_2 and believed on $M^4 \times S^2$.

p-Adic length scale hypothesis and hierarchy of Planck constants defining a hierarchy of dark variants of particles suggest the existence of scaled up copies of QCD type physics and weak physics. For p-adically scaled up variants the mass scales would be scaled by a power of $\sqrt{2}$ in the most general case. The dark variants of the particle would have the same mass as the original one. In particular, Mersenne primes $M_k = 2^k - 1$ and Gaussian Mersennes $M_{G,k} = (1+i)^k - 1$ has been proposed to define zoomed copies of these physics. At the level of magnetic confinement this would mean hierarchy of length scales for the magnetic confinement.

One particular proposal is that the Mersenne prime M_{89} should define a scaled up variant of the ordinary hadron physics with mass scaled up roughly by a factor $2^{(107-89)/2} = 512$. The size scale of color confinement for this physics would be same as the weal length scale. It would look more natural that the weak confinement for the quarks of M_{89} physics takes place in some shorter scale and M_{61} is the first Mersenne prime to be considered. The mass scale of M_{61} weak bosons would be by a factor $2^{(89-61)/2} = 2^{14}$ higher and about 1.6×10^4 TeV. M_{89} quarks would have virtually no weak interactions but would possess color interactions with weak confinement length scale reflecting themselves as new kind of jets at collisions above TeV energies.

In the biologically especially important length scale range 10 nm -2500 nm there are as many as four scaled up electron Compton lengths $L_e(k) = \sqrt{5}L(k)$: they are associated with Gaussian Mersennes $M_{G,k}$, $k = 151, 157, 163, 167$. This would suggest that the existence of scaled up scales of magnetic-, weak- and color confinement. An especially interesting possibly testable prediction is the existence of magnetic monopole pairs with the size scale in this range. There are recent claims about experimental evidence for magnetic monopole pairs [D2] .

Magnetic confinement and stringy picture in TGD sense

The connection between magnetic confinement and weak confinement is rather natural if one recalls that electric-magnetic duality in super-symmetric quantum field theories means that the descriptions in terms of particles and monopoles are in some sense dual descriptions. Fermions would be replaced by string like objects defined by the magnetic flux tubes and bosons as pairs of wormhole contacts would correspond to pairs of the flux tubes. Therefore the sharp distinction between gravitons and physical particles would disappear.

The reason why gravitons are necessarily stringy objects formed by a pair of wormhole contacts is that one cannot construct spin two objects using only single fermion states at wormhole throats. Of course, also super partners of these states with higher spin obtained by adding fermions and anti-fermions at the wormhole throat but these do not give rise to graviton like states [K27] . The upper and lower wormhole throat pairs would be quantum superpositions of fermion anti-fermion pairs with sum over all fermions. The reason is that otherwise one cannot realize graviton emission in terms of joining of the ends of light-like 3-surfaces together. Also now magnetic monopole charges are necessary but now there is no need to assign the entities $X_{\pm}$ with gravitons.

Graviton string is characterized by some p-adic length scale and one can argue that below this length scale the charges of the fermions become visible. Mersenne hypothesis suggests that some Mersenne prime is in question. One proposal is that gravitonic size scale is given by electronic Mersenne prime M_{127}. It is however difficult to test whether graviton has a structure visible below this length scale.

What happens to the generalized Feynman diagrams is an interesting question. It is not at all clear how closely they relate to ordinary Feynman diagrams. All depends on what one is ready to assume about what happens in the vertices. One could of course hope that zero energy ontology could allow some very simple description allowing perhaps to get rid of the problematic aspects of Feynman diagrams.

1. Consider first the recent view about generalized Feynman diagrams which relies zero energy ontology. A highly attractive assumption is that the particles appearing at wormhole throats are on mass shell particles. For incoming and outgoing elementary bosons and their super partners they would be positive it resp. negative energy states with parallel on mass shell momenta. For virtual bosons they the wormhole throats would have opposite sign of energy and the sum of on mass shell states would give virtual net momenta. This would make possible twistor description of virtual particles allowing only massless particles (in 4-D sense usually and in 8-D sense in TGD framework). The notion of virtual fermion makes sense only if one assumes in the interaction region a topological condensation creating another wormhole throat having no fermionic quantum numbers.

2. The addition of the particles $X^{\pm}$ replaces generalized Feynman diagrams with the analogs of stringy diagrams with lines replaced by pairs of lines corresponding to fermion and $X_{\pm 1/2}$.

The members of these pairs would correspond to 3-D light-like surfaces glued together at the vertices of generalized Feynman diagrams. The analog of 3-vertex would not be splitting of the string to form shorter strings but the replication of the entire string to form two strings with same length or fusion of two strings to single string along all their points rather than along ends to form a longer string. It is not clear whether the duality symmetry of stringy diagrams can hold true for the TGD variants of stringy diagrams.

3. How should one describe the bound state formed by the fermion and $X^{\pm}$? Should one describe the state as superposition of non-parallel on mass shell states so that the composite state would be automatically massive? The description as superposition of on mass shell states does not conform with the idea that bound state formation requires binding energy. In TGD framework the notion of negentropic entanglement has been suggested to make possible the analogs of bound states consisting of on mass shell states so that the binding energy is zero [K42] . If this kind of states are in question the description of virtual states in terms of on mass shell states is not lost. Of course, one cannot exclude the possibility that there is infinite number of this kind of states serving as analogs for the excitations of string like object.

4. What happens to the states formed by fermions and $X_{\pm 1/2}$ in the internal lines of the Feynman diagram? Twistor philosophy suggests that only the higher on mass shell excitations are possible. If this picture is correct, the situation would not change in an essential manner from the earlier one.

The highly non-trivial prediction of the magnetic confinement is that elementary particles should have stringy character in electro-weak length scales and could behaving to become manifest at LHC energies. This adds one further item to the list of non-trivial predictions of TGD about physics at LHC energies [K43] .

2.7.3 Could Quantum TGD reduce to almost topological QFT?

There seems to be a profound connection with the earlier unrealistic proposal that TGD reduces to almost topological quantum theory in the sense that the counterpart of Chern-Simons action assigned with the wormhole throats somehow dictates the dynamics. This proposal can be formulated also for the modified Dirac action action. I gave up this proposal but the following argument shows that Kähler action with weak form of electric-magnetic duality effectively reduces to Chern-Simons action plus Coulomb term.

1. Kähler action density can be written as a 4-dimensional integral of the Coulomb term $j_K^{\alpha} A_{\alpha}$ plus and integral of the boundary term $J^{n\beta} A_{\beta}\sqrt{g_4}$ over the wormhole throats and of the quantity $J^{0\beta} A_{\beta}\sqrt{g_4}$ over the ends of the 3-surface.

2. If the self-duality conditions generalize to $J^{n\beta} = 4\pi\alpha_K \epsilon^{n\beta\gamma\delta} J_{\gamma\delta}$ at throats and to $J^{0\beta} = 4\pi\alpha_K \epsilon^{0\beta\gamma\delta} J_{\gamma\delta}$ at the ends, the Kähler function reduces to the counterpart of Chern-Simons action evaluated at the ends and throats. It would have same value for each branch and the replacement $\hbar_0 \rightarrow r\hbar_0$ would effectively describe this. Boundary conditions would however give $1/r$ factor so that $\hbar$ would disappear from the Kähler function! The original attempt to realize quantum TGD as an almost topological QFT was in terms of Chern-Simons action but was given up. It is somewhat surprising that Kähler action gives Chern-Simons action in the vacuum sector defined as sector for which Kähler current is light-like or vanishes.

Holography encourages to ask whether also the Coulomb interaction terms could vanish. This kind of dimensional reduction would mean an enormous simplification since TGD would reduce to an almost topological QFT. The attribute "almost" would come from the fact that one has non-vanishing classical Noether charges defined by Kähler action and non-trivial quantum dynamics in M^4 degrees of freedom. One could also assign to space-time surfaces conserved four-momenta which is not possible in topological QFTs. For this reason the conditions guaranteeing the vanishing of Coulomb interaction term deserve a detailed analysis.

1. For the known extremals j_K^α either vanishes or is light-like ("massless extremals" for which weak self-duality condition does not make sense [K10]) so that the Coulomb term vanishes identically in the gauge used. The addition of a gradient to A induces terms located at the ends and wormhole throats of the space-time surface but this term must be cancelled by the other boundary terms by gauge invariance of Kähler action. This implies that the M^4 part of WCW metric vanishes in this case. Therefore massless extremals as such are not physically realistic: wormhole throats representing particles are needed.

2. The original naive conclusion was that since Chern-Simons action depends on CP_2 coordinates only, its variation with respect to Minkowski coordinates must vanish so that the WCW metric would be trivial in M^4 degrees of freedom. This conclusion is in conflict with quantum classical correspondence and was indeed too hasty. The point is that the allowed variations of Kähler function must respect the weak electro-magnetic duality which relates Kähler electric field depending on the induced 4-metric at 3-surface to the Kähler magnetic field. Therefore the dependence on M^4 coordinates creeps via a Lagrange multiplier term

$$\int \Lambda_\alpha (J^{n\alpha} - K\epsilon^{n\alpha\beta\gamma}J_{\beta\ gamma})\sqrt{g_4}d^3x \ . \tag{2.7.11}$$

The (1,1) part of second variation contributing to M^4 metric comes from this term.

3. This erratic conclusion about the vanishing of M^4 part WCW metric raised the question about how to achieve a non-trivial metric in M^4 degrees of freedom. The proposal was a modification of the weak form of electric-magnetic duality. Besides CP_2 Kähler form there would be the Kähler form assignable to the light-cone boundary reducing to that for $r_M = constant$ sphere - call it J^1. The generalization of the weak form of self-duality would be $J^{n\beta} = \epsilon^{n\beta\gamma\delta}K(J_{\gamma\delta} + \epsilon J^1_{\gamma\delta})$. This form implies that the boundary term gives a non-trivial contribution to the M^4 part of the WCW metric even without the constraint from electric-magnetic duality. Kähler charge is not affected unless the partonic 2-surface contains the tip of CD in its interior. In this case the value of Kähler charge is shifted by a topological contribution. Whether this term can survive depends on whether the resulting vacuum extremals are consistent with the basic facts about classical gravitation.

4. The Coulombic interaction term is not invariant under gauge transformations. The good news is that this might allow to find a gauge in which the Coulomb term vanishes. The vanishing condition fixing the gauge transformation ϕ is

$$j_K^\alpha \partial_\alpha \phi = -j^\alpha A_\alpha \ . \tag{2.7.12}$$

This differential equation can be reduced to an ordinary differential equation along the flow lines j_K by using $dx^\alpha/dt = j_K^\alpha$. Global solution is obtained only if one can combine the flow parameter t with three other coordinates- say those at the either end of CD to form space-time coordinates. The condition is that the parameter defining the coordinate differential is proportional to the covariant form of Kähler current: $dt = \phi j_K$. This condition in turn implies $d^2t = d(\phi j_K) = d(\phi j_K) = d\phi \wedge j_K + \phi dj_K = 0$ implying $j_K \wedge dj_K = 0$ or more concretely,

$$\epsilon^{\alpha\beta\gamma\delta}j^K_\beta \partial_\gamma j^K_{delta} = 0 \ . \tag{2.7.13}$$

j_K is a four-dimensional counterpart of Beltrami field [B30] and could be called generalized Beltrami field.

The integrability conditions follow also from the construction of the extremals of Kähler action [K10] . The conjecture was that for the extremals the 4-dimensional Lorentz force vanishes (no dissipation): this requires $j_K \wedge J = 0$. One manner to guarantee this is the topologization of the Kähler current meaning that it is proportional to the instanton current: $j_K = \phi j_I$, where $j_I = {}^*(J \wedge A)$ is the instanton current, which is not conserved for 4-D CP_2 projection. The conservation of j_K implies the condition $j_I^\alpha \partial_\alpha \phi = \partial_\alpha j^\alpha \phi$ and from this ϕ can be integrated if the integrability condition $j_I \wedge dj_I = 0$ holds true implying the same condition for j_K. By introducing at least 3 or CP_2 coordinates as space-time coordinates, one finds that the contravariant form of j_I is purely topological so that the integrability condition fixes the dependence on M^4 coordinates and this selection is coded into the scalar function ϕ. These functions define families of conserved currents $j_K^\alpha \phi$ and $j_I^\alpha \phi$ and could be also interpreted as conserved currents associated with the critical deformations of the space-time surface.

5. There are gauge transformations respecting the vanishing of the Coulomb term. The vanishing condition for the Coulomb term is gauge invariant only under the gauge transformations $A \to A + \nabla\phi$ for which the scalar function the integral $\int j_K^\alpha \partial_\alpha \phi$ reduces to a total divergence a giving an integral over various 3-surfaces at the ends of CD and at throats vanishes. This is satisfied if the allowed gauge transformations define conserved currents

$$D_\alpha (j^\alpha \phi) \quad = \quad 0 \ . \tag{2.7.14}$$

As a consequence Coulomb term reduces to a difference of the conserved charges $Q^e_\phi = \int j^0 \phi \sqrt{g_4} d^3x$ at the ends of the CD vanishing identically. The change of the Chern-Simons type term is trivial if the total weighted Kähler magnetic flux $Q^m_\phi = \sum \int J\phi dA$ over wormhole throats is conserved. The existence of an infinite number of conserved weighted magnetic fluxes is in accordance with the electric-magnetic duality. How these fluxes relate to the flux Hamiltonians central for WCW geometry is not quite clear.

6. The gauge transformations respecting the reduction to almost topological QFT should have some special physical meaning. The measurement interaction term in the modified Dirac interaction corresponds to a critical deformation of the space-time sheet and is realized as an addition of a gauge part to the Kähler gauge potential of CP_2. It would be natural to identify this gauge transformation giving rise to a conserved charge so that the conserved charges would provide a representation for the charges associated with the infinitesimal critical deformations not affecting Kähler action. The gauge transformed Kähler gauge potential couples to the modified Dirac equation and its effect could be visible in the value of Kähler function and therefore also in the properties of the preferred extremal. The effect on WCW metric would however vanish since K would transform only by an addition of a real part of a holomorphic function.

7. A first guess for the explicit realization of the quantum classical correspondence between quantum numbers and space-time geometry is that the deformation of the preferred extremal due to the addition of the measurement interaction term is induced by a $U(1)$ gauge transformation induced by a transformation of $\delta CD \times CP_2$ generating the gauge transformation represented by ϕ. This interpretation makes sense if the fluxes defined by Q^m_ϕ and corresponding Hamiltonians affect only zero modes rather than quantum fluctuating degrees of freedom.

8. Later a simpler proposal assuming Kähler action with Chern-Simons term at partonic orbits and Kähler-Dirac action with Chern-Simons Dirac term at partonic orbits emerged. Measurement interaction terms would correspond to Lagrange multiplier terms at the ends of space-time surface fixing the values of classical conserved charges to their quantum values. Super-symmetry requires the assignment of this kind of term also to modified Dirac action as boundary term.

   Kähler-Dirac equation gives rise to a boundary condition at space-like ends of the space-time surface stating that the action of the Kähler-Dirac gamma matrix in normal direction annihilates the spinor modes. The normal vector would be light-like and the value of the incoming on mass shell four-momentum would be coded to the geometry of the space-time surface and string world sheet.

One can assign to partonic orbits Chern-Simons Dirac action and now the condition would be that the action of C-S-D operator equals to that of massless M^4 Dirac operator. C-S-D Dirac action would give rise to massless Dirac propagator. Twistor Grassmann approach suggests that also the virtual fermions reduce effectively to massless on-shell states but have non-physical helicity.

2.7.4 About the notion of measurement interaction

The notion of measurement has been central notion in quantum TGD but the precise definition of this notion is far from clear. In the following two possibly equivalent formulations are considered. The first formulation relies on the gauge transformations leaving Coulomb term of Kähler action unchanged and the second one to the interpretation of TGD as a square root of thermodynamics allowing to fix the values of conserved classical charges for zero energy energy state using Lagrange multipliers analogous to chemical potentials.

1. There are gauge transformations respecting the vanishing of the Coulomb term. The vanishing condition for the Coulomb term is gauge invariant only under the gauge transformations $A \rightarrow A + \nabla\phi$ for which the scalar function the integral $\int j_K^\alpha \partial_\alpha \phi$ reduces to a total divergence a giving an integral over various 3-surfaces at the ends of CD and at throats vanishes. This is satisfied if the allowed gauge transformations define conserved currents

$$D_\alpha(j^\alpha\phi) \;=\; 0 \; . \tag{2.7.15}$$

As a consequence Coulomb term reduces to a difference of the conserved charges $Q^e_\phi = \int j^0\phi\sqrt{g_4}d^3x$ at the ends of the CD vanishing identically. The change of the Chern-Simons type term is trivial if the total weighted Kähler magnetic flux $Q^m_\phi = \sum\int J\phi dA$ over wormhole throats is conserved. The existence of an infinite number of conserved weighted magnetic fluxes is in accordance with the electric-magnetic duality. How these fluxes relate to the flux Hamiltonians central for WCW geometry is not quite clear.

2. The gauge transformations respecting the reduction to almost topological QFT should have some special physical meaning. The measurement interaction term in the modified Dirac interaction corresponds to a critical deformation of the space-time sheet and is realized as an addition of a gauge part to the Kähler gauge potential of CP_2. It would be natural to identify this gauge transformation giving rise to a conserved charge so that the conserved charges would provide a representation for the charges associated with the infinitesimal critical deformations not affecting Kähler action.

   The gauge transformed Kähler potential couples to the modified Dirac equation and its effect could be visible in the value of Kähler function and therefore also in the properties of the preferred extremal. The effect on WCW metric would however vanish since K would transform only by an addition of a real part of a holomorphic function. Kähler function is identified as a Dirac determinant of Chern-Simons Dirac operator (after many turns and twists) and the spectrum of this operator should not be invariant under these gauge transformations if this picture is correct. This is is achieved if the gauge transformation is carried only in the Dirac action corresponding to instanton term: this assumption is motivated by the breaking of time reversal invariance induced by quantum measurements. The modification of Kähler action can be guessed to correspond just to the Chern-Simons contribution from the instanton term.

3. A reasonable looking guess for the explicit realization of the quantum classical correspondence between quantum numbers and space-time geometry is that the deformation of the preferred extremal due to the addition of the measurement interaction term is induced by a $U(1)$ gauge transformation induced by a transformation of $\delta CD \times CP_2$ generating the gauge transformation represented by ϕ. This interpretation makes sense if the fluxes defined by

Q^m_ϕ and corresponding Hamiltonians affect only zero modes rather than quantum fluctuating degrees of freedom.

In zero energy ontology (ZEO) TGD can be seen as square root of thermodynamics and this suggests an alternative manner to define what measurement interaction term means.

1. The condition that the space-time sheets appearing in superposition of space-time surfaces with given quantum numbers in Cartan algebra have same classical quantum numbers associated with Kähler action can be realized in terms of Lagrange multipliers in standard manner. These kind of terms would be analogous to various chemical potential terms in the partition function. One could call them measurement interaction terms. Measurement interaction terms would code the values of quantum charges to the space-time geometry.

   Kähler action contains also Chern-Simons term at partonic orbits compensating the Chern-Simons terms coming from Kähler action when weak form of electric-magnetic duality is assumed. This guarantees that Kähler action for preferred extremals reduces to Chern-Simons terms at the space-like ends of the spacetime surface and one obtains almost topological QFT.

2. If Kähler-Dirac action is constructed from Kähler action in super-symmetric manner by defining the modified gamma matrices in terms of canonical momentum densities one obtains also the fermionic counterparts of the Lagrange multiplier terms at partonic orbits and could call also them measurement interaction terms. Besides this one has also the Chern-Simons Dirac terms associated with the partonic orbits giving ordinary massless Dirac propagator. In presence of measurement interaction terms at the space-like ends of the space-time surface the boundary conditions $\Gamma^n\Psi = 0$ at the ends would be modified by the addition of term coming from the modified gamma matrix associated with the Lagrange multiplier terms. The original generalized massless generalized eigenvalue spectrum $p^k\gamma_k$ of Γ^n would be modified to massive spectrum given by the condition

$$(\Gamma^n + \sum_i \lambda_i \Gamma^\alpha_{Q_i} D_\alpha)\Psi = 0 \ ,$$

   where Q_i refers to i:th conserved charge.

An interesting question is whether these two manners to introduce measurement interaction terms are actually equivalent.

To sum up, one could understand the basic properties of WCW metric in this framework. Effective 2-dimensionality would result from the existence of an infinite number of conserved charges in two different time directions (genuine conservation laws plus gauge fixing). The infinite-dimensional symmetric space for given values of zero modes corresponds to the Cartesian product of the WCWs associated with the partonic 2-surfaces at both ends of CD and the generalized Chern-Simons term decomposes into a sum of terms from the ends giving single particle Kähler functions and to the terms from light-like wormhole throats giving interaction term between positive and negative energy parts of the state. Hence Kähler function could be calculated without any knowledge about the interior of the space-time sheets and TGD would reduce to almost topological QFT as speculated earlier. Needless to say this would have immense boost to the program of constructing WCW Kähler geometry.

2.8 How to define generalized Feynman diagrams?

S-matrix codes to a high degree the predictions of quantum theories. The longstanding challenge of TGD has been to construct or at least demonstrate the mathematical existence of S-matrix- or actually M-matrix which generalizes this notion in zero energy ontology (ZEO) [K60] . This work has led to the notion of generalized Feynman diagram and the challenge is to give a precise mathematical meaning for this object. The attempt to understand the counterpart of twistors in TGD framework [K79] has inspired several key ideas in this respect but it turned out that twistors themselves need not be absolutely necessary in TGD framework.

1. The notion of generalized Feyman diagram defined by replacing lines of ordinary Feynman diagram with light-like 3-surfaces (elementary particle sized wormhole contacts with throats carrying quantum numbers) and vertices identified as their 2-D ends - I call them partonic 2-surfaces is central. Speaking somewhat loosely, generalized Feynman diagrams (plus background space-time sheets) define the "world of classical worlds" (WCW). These diagrams involve the analogs of stringy diagrams but the interpretation is different: the analogs of stringy loop diagrams have interpretation in terms of particle propagating via two different routes simultaneously (as in the classical double slit experiment) rather than as a decay of particle to two particles. For stringy diagrams the counterparts of vertices are singular as manifolds whereas the entire diagrams are smooth. For generalized Feynman diagrams vertices are smooth but entire diagrams represent singular manifolds just like ordinary Feynman diagrams do. String like objects however emerge in TGD and even ordinary elementary particles are predicted to be magnetic flux tubes of length of order weak gauge boson Compton length with monopoles at their ends as shown in accompanying article. This stringy character should become visible at LHC energies.

2. Zero energy ontology (ZEO) and causal diamonds (intersections of future and past directed light-cones) define second key ingredient. The crucial observation is that in ZEO it is possible to identify off mass shell particles as pairs of on mass shell fermions at throats of wormhole contact since both positive and negative signs of energy are possible and one obtains also space-like total momenta for wormhole contact behaving as a boson. The localization of fermions to string world sheets and the fact that super-conformal generator G carries fermion number combined with twistorial consideration support the view that the propagators at fermionic lines are of form $(1/G)ip^k\gamma_k(1/G^\dagger + h.c.$ and thus hermitian. In strong models $1/G$ would serve as a propagator and this requires Majorana condition fixing the dimension of the target space to 10 or 11.

3. A powerful constraint is number theoretic universality requiring the existence of Feynman amplitudes in all number fields when one allows suitable algebraic extensions: roots of unity are certainly required in order to realize p-adic counterparts of plane waves. Also imbedding space, partonic 2-surfaces and WCW must exist in all number fields and their extensions. These constraints are enormously powerful and the attempts to realize this vision have dominated quantum TGD for last two decades.

4. Representation of 8-D gamma matrices in terms of octonionic units and 2-D sigma matrices is a further important element as far as twistors are considered [K79] . Modified gamma matrices at space-time surfaces are quaternionic/associative and allow a genuine matrix representation. As a matter fact, TGD and WCW could be formulated as study of associative local sub-algebras of the local Clifford algebra of 8-D imbedding space parameterized by quaternionic space-time surfaces.

5. A central conjecture has been that associative (co-associative) 4-surfaces correspond to preferred extremals of Kähler action [K15]. It took long time to realize that in zero energy ontology the notion of preferred extremal might be un-necessary! The reason is that 3-surfaces are now pairs of 3-surfaces at boundaries of causal diamonds and for deterministic dynamics the space-time surface connecting them is expected to be more or less unique. Now the action principle is non-deterministic but the non-determinism would give rise to additional discrete dynamical degrees of freedom naturally assignable to the hierarchy of Planck constants $h_{eff} = n \times h$, n the number of space-time surface with same fixed ends at boundaries of CD and with same values of Kähler action and of conserved quantities. One must be however cautions: this leaves the possibility that there is a gauge symmetry present so that the n sheets correspond to gauge equivalence classes of sheets. Conformal invariance is associated with criticality and is expected to be present also now.

   One can of course also ask whether one can assume that the pairs of 3-surfaces at the ends of CD are totally un-correlated. If this assumption is not made then preferred extremal property would make sense also in ZEO and imply additional correlation between the members of these pairs. This kind of correlations would correspond to the Bohr orbit property, which is very attractive space-time correlate for quantum states. This kind of correlates are also expected as space-time counterpart for the correlations between initial and final state in quantum dynamics.

6. A further conjecture has been that preferred extremals are in some sense critical (second variation of Kähler action could vanish for infinite number of deformations defining a super-conformal algebra). The non-determinism of Kähler action implies this property for $n > 0$ in $h_{eff} = nh$. If the criticality is present, it could correspond to conformal gauge invariance defined by sub-algebras of conformal algebra with conformal weights coming as multiples of n and isomorphic to the conformal algebra itself.

7. As far as twistors are considered, the first key element is the reduction of the octonionic twistor structure to quaternionic one at space-time surfaces and giving effectively 4-D spinor and twistor structure for quaternionic surfaces.

Quite recently quite a dramatic progress took place in this approach [K26, K79] .

1. The progress was stimulated by the simple observation that on mass shell property puts enormously strong kinematic restrictions on the loop integrations. With mild restrictions on the number of parallel fermion lines appearing in vertices (there can be several since fermionic oscillator operator algebra defining SUSY algebra generates the parton states)- all loops are manifestly finite and if particles has always mass -say small p-adic thermal mass also in case of massless particles and due to IR cutoff due to the presence largest CD- the number of diagrams is finite. Unitarity reduces to Cutkosky rules [B15] automatically satisfied as in the case of ordinary Feynman diagrams.

2. Ironically, twistors which stimulated all these development do not seem to be absolutely necessary in this approach although they are of course possible. Situation changes if one does not assume small p-adically thermal mass due to the presence of massless particles and one must sum infinite number of diagrams. Here a potential problem is whether the infinite sum respects the algebraic extension in question.

This is about fermionic and momentum space aspects of Feynman diagrams but not yet about the functional (not path-) integral over small deformations of the partonic 2-surfaces. The basic challenges are following.

1. One should perform the functional integral over WCW degrees of freedom for fixed values of on mass shell momenta appearing in the internal lines. After this one must perform integral or summation over loop momenta. Note that the order is important since the space-time surface assigned to the line carries information about the quantum numbers associated with the line by quantum classical correspondence realized in terms of modified Dirac operator.

2. One must define the functional integral also in the p-adic context. p-Adic Fourier analysis relying on algebraic continuation raises hopes in this respect. p-Adicity suggests strongly that the loop momenta are discretized and ZEO predicts this kind of discretization naturally.

It indeed seems that the functional integrals over WCW could be carried out at general level both in real and p-adic context. This is due to the symmetric space property (maximal number of isometries) of WCW required by the mere mathematical existence of Kähler geometry [K33] in infinite-dimensional context already in the case of much simpler loop spaces [A145] .

1. The p-adic generalization of Fourier analysis allows to algebraize integration- the horrible looking technical challenge of p-adic physics- for symmetric spaces for functions allowing the analog of discrete Fourier decomposition. Symmetric space property is indeed essential also for the existence of Kähler geometry for infinite-D spaces as was learned already from the case of loop spaces. Plane waves and exponential functions expressible as roots of unity and powers of p multiplied by the direct analogs of corresponding exponent functions are the basic building bricks and key functions in harmonic analysis in symmetric spaces. The physically unavoidable finite measurement resolution corresponds to algebraically unavoidable finite algebraic dimension of algebraic extension of p-adics (at least some roots of unity are needed).

The cutoff in roots of unity is very reminiscent to that occurring for the representations of quantum groups and is certainly very closely related to these as also to the inclusions of hyper-finite factors of type II¡sub¿1¡/sub¿ defining the finite measurement resolution.

2. WCW geometrization reduces to that for a single line of the generalized Feynman diagram defining the basic building brick for WCW. Kähler function decomposes to a sum of "kinetic" terms associated with its ends and interaction term associated with the line itself. p-Adicization boils down to the condition that Kähler function, matrix elements of Kähler form, WCW Hamiltonians and their super counterparts, are rational functions of complex WCW coordinates just as they are for those symmetric spaces that I know of. This would allow a continuation to p-adic context.

In the following this vision about generalized Feynman diagrams is discussed in more detail.

2.8.1 Questions

The goal is a proposal for how to perform the integral over WCW for generalized Feynman digrams and the best manner to proceed to to this goal is by making questions.

What does finite measurement resolution mean?

The first question is what finite measurement resolution means.

1. One expects that the algebraic continuation makes sense only for a finite measurement resolution in which case one obtains only finite sums of what one might hope to be algebraic functions. The finiteness of the algebraic extension would be in fact equivalent with the finite measurement resolution.

2. Finite measurement resolution means a discretization in terms of number theoretic braids. p-Adicization condition suggests that that one must allow only the number theoretic braids. For these the ends of braid at boundary of CD are algebraic points of the imbedding space. This would be true at least in the intersection of real and p-adic worlds.

3. The question is whether one can localize the points of the braid. The necessity to use momentum eigenstates to achieve quantum classical correspondence in the modified Dirac action [K15] suggests however a de-localization of braid points, that is wave function in space of braid points. In real context one could allow all possible choices for braid points but in p-adic context only algebraic points are possible if one wants to replace integrals with sums. This implies finite measurement resolution analogous to that in lattice. This is also the only possibility in the intersection of real and p-adic worlds.

   A non-trivial prediction giving a strong correlation between the geometry of the partonic 2-surface and quantum numbers is that the total number $n_F + n_{\overline{F}}$ of fermions and anti-fermions is bounded above by the number n_{alg} of algebraic points for a given partonic 2-surface: $n_F + n_{\overline{F}} \leq n_{alg}$. Outside the intersection of real and p-adic worlds the problematic aspect of this definition is that small deformations of the partonic 2-surface can radically change the number of algebraic points unless one assumes that the finite measurement resolution means restriction of WCW to a sub-space of algebraic partonic surfaces.

4. Braids defining propagator lines for fundamental fermions (to be distinguished from observer particles) emerges naturally. Braid strands correspond to the boundaries of string world sheets at which the modes of induced spinor fields are localized from the condition that em charge is well-defined: induced W field and above weak scale also Z^0 field vanish at them.

   In order to obtain non-trivial fermion propagator one must add to Kähler-Dirac action Chern-Simons Dirac term located at partonic orbits at which the signature of the induced metric changes. The modes of induced spinor field can be required to be generalized eigenmodes of C-S-D operator with generalized eigenvalue $p^k\gamma_k$ with p^k identified as virtual momentum so that massless Dirac propagator is obtained. p^k is discretized by periodic boundary conditions at opposite boundaries of CD and has IR and UV cutoffs due to the finite size of CD and finite lower limit for the size of sub-CDs.

One has also discretization of the relative position of the second tip of CD at the hyperboloid isometric with mass shell. Only the number of braid points and their momenta would matter, not their positions.

By super-symmetry one must add to Kähler action Chern-Simons term located at partonic orbits and this term must cancel the Chern-Simons term coming from Kähler action by weak form of electric-magnetic duality so that Kähler action reduces to the terms associated with space-like ends of the space-time surface. These terms reduce to Chern-Simons terms if one poses weak form of electric magnetic duality also here. The boundary condition for Kähler-Dirac equations states $\Gamma^n\Psi = 0$ so that incoming fundamental fermions are massless and there is a strong temptation to pose the additional condition $\Gamma^n\Psi = p^k\gamma_k\Psi = 0$

The quantum numbers characterizing positive and negative energy parts of zero energy states couple directly to space-time geometry via the measurement interaction terms in Kähler action expressing the equality of classical conserved charges in Cartan algebra with their quantal counterparts for space-time surfaces in quantum superposition. This makes sense if classical charges parametrize zero modes. The localization in zero modes in state function reduction would be the WCW counterpart of state function collapse.

How to define integration in WCW degrees of freedom?

The basic question is how to define the integration over WCW degrees of freedom.

1. What comes mind first is Gaussian perturbation theory around the maxima of Kähler function. Gaussian and metric determinants cancel each other and only algebraic expressions remain. Finiteness is not a problem since the Kähler function is non-local functional of 3-surface so that no local interaction vertices are present. One should however assume the vanishing of loops required also by algebraic universality and this assumption look unrealistic when one considers more general functional integrals than that of vacuum functional since free field theory is not in question. The construction of the inverse of the WCW metric defining the propagator is also a very difficult challenge. Duistermaat-Hecke theorem states that something like this known as localization might be possible and one can also argue that something analogous to localization results from a generalization of mean value theorem.

2. Symmetric space property is more promising since it might reduce the integrations to group theory using the generalization of Fourier analysis for group representations so that there would be no need for perturbation theory in the proposed sense. In finite measurement resolution the symmetric spaces involved would be finite-dimensional. Symmetric space structure of WCW could also allow to define p-adic integration in terms of p-adic Fourier analysis for symmetric spaces. Essentially algebraic continuation of the integration from the real case would be in question with additional constraints coming from the fact that only phase factors corresponding to finite algebraic extensions of rationals are used. Cutoff would emerge automatically from the cutoff for the dimension of the algebraic extension.

How to define generalized Feynman diagrams?

Integration in symmetric spaces could serve as a model at the level of WCW and allow both the understanding of WCW integration and p-adicization as algebraic continuation. In order to get a more realistic view about the problem one must define more precisely what the calculation of the generalized Feynman diagrams means.

1. WCW integration must be carried out separately for all values of the momenta associated with the internal lines. The reason is that the spectrum of eigenvalues λ_i of the modified Dirac operator D depends on the momentum of line and momentum conservation in vertices translates to a correlation of the spectra of D at internal lines.

2. For tree diagrams algebraic continuation to the p-adic context if the expression involves only the replacement of the generalized eigenvalues of D as functions of momenta with their p-adic counterparts besides vertices. If these functions are algebraically universal and expressible in terms of harmonics of symmetric space , there should be no problems.

3. If loops are involved, one must integrate/sum over loop momenta. In p-adic context difficulties are encountered if the spectrum of the momenta is continuous. The integration over on mass shell loop momenta is analogous to the integration over sub-CDs, which suggests that internal line corresponds to a $sub - CD$ in which it is at rest. There are excellent reasons to believe that the moduli space for the positions of the upper tip is a discrete subset of hyperboloid of future light-cone. If this is the case, the loop integration indeed reduces to a sum over discrete positions of the tip. p-Adizication would thus give a further good reason why for zero energy ontology.

4. Propagator is expressible in terms of the inverse of generalized eigenvalue and there is a sum over these for each propagator line. At vertices one has products of WCW harmonics assignable to the incoming lines. The product must have vanishing quantum numbers associated with the phase angle variables of WCW. Non-trivial quantum numbers of the WCW harmonic correspond to WCW quantum numbers assignable to excitations of ordinary elementary particles. WCW harmonics are products of functions depending on the "radial" coordinates and phase factors and the integral over the angles leaves the product of the first ones analogous to Legendre polynomials $P_{l,m}$, These functions are expected to be rational functions or at least algebraic functions involving only square roots.

5. In ordinary QFT incoming and outgoing lines correspond to propagator poles. In the recent case this would mean that incoming stringy lines at the ends of CD correspond to fermions satisfying the stringy mass formula serving as a generalization of masslessness condition.

2.8.2 Generalized Feynman diagrams at fermionic and momentum space level

Negative energy ontology has already led to the idea of interpreting the virtual particles as pairs of positive and negative energy wormhole throats. Hitherto I have taken it as granted that ordinary Feynman diagrammatics generalizes more or less as such. It is however far from clear what really happens in the verties of the generalized Feynman diagrams. The safest approach relies on the requirement that unitarity realized in terms of Cutkosky rules in ordinary Feynman diagrammatics allows a generalization. This requires loop diagrams. In particular, photon-photon scattering can take place only via a fermionic square loop so that it seems that loops must be present at least in the topological sense.

One must be however ready for the possibility that something unexpectedly simple might emerge. For instance, the vision about algebraic physics allows naturally only finite sums for diagrams and does not favor infinite perturbative expansions. Hence the true believer on algebraic physics might dream about finite number of diagrams for a given reaction type. For simplicity generalized Feynman diagrams without the complications brought by the magnetic confinement since by the previous arguments the generalization need not bring in anything essentially new.

The basic idea of duality in early hadronic models was that the lines of the dual diagram representing particles are only re-arranged in the vertices. This however does not allow to get rid of off mass shell momenta. Zero energy ontology encourages to consider a stronger form of this principle in the sense that the virtual momenta of particles could correspond to pairs of on mass shell momenta of particles. If also interacting fermions are pairs of positive and negative energy throats in the interaction region the idea about reducing the construction of Feynman diagrams to some kind of lego rules might work.

Virtual particles as pairs of on mass shell particles in ZEO

The first thing is to try to define more precisely what generalized Feynman diagrams are. The direct generalization of Feynman diagrams implies that both wormhole throats and wormhole contacts join at vertices.

1. A simple intuitive picture about what happens is provided by diagrams obtained by replacing the points of Feynman diagrams (wormhole contacts) with short lines and imagining that the throats correspond to the ends of the line. At vertices where the lines meet the incoming on mass shell quantum numbers would sum up to zero. This approach leads to a straightforward

generalization of Feynman diagrams with virtual particles replaced with pairs of on mass shell throat states of type $++$, $--$, and $+-$. Incoming lines correspond to $++$ type lines and outgoing ones to $--$ type lines. The first two line pairs allow only time like net momenta whereas $+-$ line pairs allow also space-like virtual momenta. The sign assigned to a given throat is dictated by the the sign of the on mass shell momentum on the line. The condition that Cutkosky rules generalize as such requires $++$ and $--$ type virtual lines since the cut of the diagram in Cutkosky rules corresponds to on mass shell outgoing or incoming states and must therefore correspond to $++$ or $--$ type lines.

2. The basic difference as compared to the ordinary Feynman diagrammatics is that loop integrals are integrals over mass shell momenta and that all throats carry on mass shell momenta. In each vertex of the loop mass incoming on mass shell momenta must sum up to on mass shell momentum. These constraints improve the behavior of loop integrals dramatically and give excellent hopes about finiteness. It does not however seem that only a finite number of diagrams contribute to the scattering amplitude besides tree diagrams. The point is that if a the reactions $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$,, where N_i denote particle numbers, are possible in a common kinematical region for N_2-particle states then also the diagrams $N_1 \rightarrow N_2 \rightarrow N_2 \rightarrow N_3$ are possible. The virtual states N_2 include all all states in the intersection of kinematically allow regions for $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$. Hence the dream about finite number possible diagrams is not fulfilled if one allows massless particles. If all particles are massive then the particle number N_2 for given N_1 is limited from above and the dream is realized.

3. For instance, loops are not possible in the massless case or are highly singular (bringing in mind twistor diagrams) since the conservation laws at vertices imply that the momenta are parallel. In the massive case and allowing mass spectrum the situation is not so simple. As a first example one can consider a loop with three vertices and thus three internal lines. Three on mass shell conditions are present so that the four-momentum can vary in 1-D subspace only. For a loop involving four vertices there are four internal lines and four mass shell conditions so that loop integrals would reduce to discrete sums. Loops involving more than four vertices are expected to be impossible.

4. The proposed replacement of the elementary fermions with bound states of elementary fermions and monopoles $X_\pm$ brings in the analog of stringy diagrammatics. The 2-particle wave functions in the momentum degrees of freedom of fermion and $X_\pm$ might allow more flexibility and allow more loops. Note however that there are excellent hopes about the finiteness of the theory also in this case.

Loop integrals are manifestly finite

One can make also more detailed observations about loops.

1. The simplest situation is obtained if only 3-vertices are allowed. In this case conservation of momentum however allows only collinear momenta although the signs of energy need not be the same. Particle creation and annihilation is possible and momentum exchange is possible but is always light-like in the massless case. The scattering matrices of supersymmetric YM theories would suggest something less trivial and this raises the question whether something is missing. Magnetic monopoles are an essential element of also these theories as also massivation and symmetry breaking and this encourages to think that the formation of massive states as fermion $X_\pm$ pairs is needed. Of course, in TGD framework one has also high mass excitations of the massless states making the scattering matrix non-trivial.

2. In YM theories on mass shell lines would be singular. In TGD framework this is not the case since the propagator is defined as the inverse of the 3-D dimensional reduction of the modified Dirac operator D containing also coupling to four-momentum (this is required by quantum classical correspondence and guarantees stringy propagators),

$$\begin{aligned} D &= i\hat{\Gamma}^\alpha p_\alpha + \hat{\Gamma}^\alpha D_\alpha \ , \\ p_\alpha &= p_k \partial_\alpha h^k \ . \end{aligned} \tag{2.8.1}$$

The propagator does not diverge for on mass shell massless momenta and the propagator lines are well-defined. This is of course of essential importance also in general case. Only for the incoming lines one can consider the possibility that 3-D Dirac operator annihilates the induced spinor fields. All lines correspond to generalized eigenstates of the propagator in the sense that one has $D_3\Psi = \lambda\gamma\Psi$, where γ is modified gamma matrix in the direction of the stringy coordinate emanating from light-like surface and D_3 is the 3-dimensional dimensional reduction of the 4-D modified Dirac operator. The eigenvalue λ is analogous to energy. Note that the eigenvalue spectrum depends on 4-momentum as a parameter.

3. Massless incoming momenta can decay to massless momenta with both signs of energy. The integration measure $d^2k/2E$ reduces to dx/x where $x \geq 0$ is the scaling factor of massless momentum. Only light-like momentum exchanges are however possible and scattering matrix is essentially trivial. The loop integrals are finite apart from the possible delicacies related to poles since the loop integrands for given massless wormhole contact are proportional to dx/x^3 for large values of x.

4. Irrespective of whether the particles are massless or not, the divergences are obtained only if one allows too high vertices as self energy loops for which the number of momentum degrees of freedom is $3N-4$ for N-vertex. The construction of SUSY limit of TGD in [K27] led to the conclusion that the parallelly propagating N fermions for given wormhole throat correspond to a product of N fermion propagators with same four-momentum so that for fermions and ordinary bosons one has the standard behavior but for $N > 2$ non-standard so that these excitations are not seen as ordinary particles. Higher vertices are finite only if the total number N_F of fermions propagating in the loop satisfies $N_F > 3N - 4$. For instance, a 4-vertex from which $N = 2$ states emanate is finite.

Taking into account magnetic confinement

What has been said above is not quite enough. The weak form of electric-magnetic duality [B7] leads to the picture about elementary particles as pairs of magnetic monopoles inspiring the notions of weak confinement based on magnetic monopole force. Also color confinement would have magnetic counterpart. This means that elementary particles would behave like string like objects in weak boson length scale. Therefore one must also consider the stringy case with wormhole throats replaced with fermion-$X_\pm$ pairs ($X_\pm$ is electromagnetically neutral and $\pm$ refers to the sign of the weak isospin opposite to that of fermion) and their super partners.

1. The simplest assumption in the stringy case is that fermion-$X_\pm$ pairs behave as coherent objects, that is scatter elastically. In more general case only their higher excitations identifiable in terms of stringy degrees of freedom would be created in vertices. The massivation of these states makes possible non-collinear vertices. An open question is how the massivation fermion-$X_\pm$ pairs relates to the existing TGD based description of massivation in terms of Higgs mechanism and modified Dirac operator.

2. Mass renormalization could come from self energy loops with negative energy lines as also vertex normalization. By very general arguments supersymmetry implies the cancellation of the self energy loops but would allow non-trivial vertex renormalization [K27] .

3. If only 3-vertices are allowed, the loops containing only positive energy lines are possible if on mass shell fermion-$X_\pm$ pair (or its superpartner) can decay to a pair of positive energy pair particles of same kind. Whether this is possible depends on the masses involved. For ordinary particles these decays are not kinematically possible below intermediate boson mass scale (the decays $F_1 \to F_2 + \gamma$ are forbidden kinematically or by the absence of flavor changing neutral currents whereas intermediate gauge bosons can decay to on mass shell fermion-anti-fermion pair).

4. The introduction of IR cutoff for 3-momentum in the rest system associated with the largest CD (causal diamond) looks natural as scale parameter of coupling constant evolution and p-adic length scale hypothesis favors the inverse of the size scale of CD coming in powers of two. This parameter would define the momentum resolution as a discrete parameter of the

p-adic coupling constant evolution. This scale does not have any counterpart in standard physics. For electron, d quark, and u quark the proper time distance between the tips of CD corresponds to frequency of 10 Hz, 1280 Hz, and 160 Hz: all these frequencies define fundamental bio-rhythms [K22] .

These considerations have left completely untouched one important aspect of generalized Feynman diagrams: the necessity to perform a functional integral over the deformations of the partonic 2-surfaces at the ends of the lines- that is integration over WCW. Number theoretical universality requires that WCW and these integrals make sense also p-adically and in the following these aspects of generalized Feynman diagrams are discussed.

2.8.3 Harmonic analysis in WCW as a manner to calculate WCW functional integrals

Previous examples suggest that symmetric space property, Kähler and symplectic structure and the use of symplectic coordinates consisting of canonically conjugate pairs of phase angles and corresponding "radial" coordinates are essential for WCW integration and p-adicization. Kähler function, the components of the metric, and therefore also metric determinant and Kähler function depend on the "radial" coordinates only and the possible generalization involves the identification the counterparts of the "radial" coordinates in the case of WCW.

Conditions guaranteeing the reduction to harmonic analysis

The basic idea is that harmonic analysis in symmetric space allows to calculate the functional integral over WCW.

1. Each propagator line corresponds to a symmetric space defined as a coset space G/H of the symplectic group and Kac-Moody group and one might hope that the proposed p-adicization works for it- at least when one considers the hierarchy of measurement resolutions forced by the finiteness of algebraic extensions. This coset space is as a manifold Cartesian product $(G/H) \times (G/H)$ of symmetric spaces G/H associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. Effective 2-dimensionality and ZEO allow to treat the ends of the propagator line independently. This means an enormous simplification. Each line contributes besides propagator a piece to the exponent of Kähler action identifiable as interaction term in action and depending on the propagator momentum. This contribution should be expressible in terms of generalized spherical harmonics. Essentially a sum over the products of pairs of harmonics associated with the ends of the line multiplied by coefficients analogous to $1/(p^2-m^2)$ in the case of the ordinary propagator would be in question. The optimal situation is that the pairs are harmonics and their conjugates appear so that one has invariance under G analogous to momentum conservation for the lines of ordinary Feynman diagrams.

3. Momentum conservation correlates the eigenvalue spectra of the modified Dirac operator D at propagator lines [K15] . G-invariance at vertex dictates the vertex as the singlet part of the product of WCW harmonics associated with the vertex and one sums over the harmonics for each internal line. p-Adicization means only the algebraic continuation to real formulas to p-adic context.

4. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)} + c.c \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)} + c.c \ , i=1,2 \ . \end{aligned} \tag{2.8.2}$$

Here $K_{kin,i}$ define "kinetic" terms and K_{int} defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories.

Symmetric space property -that is isometry invariance- suggests that one has

$$f_{i,n} = f_{2,n} \equiv f_n \ , \ g_{1,n} = g_{2,n} \equiv g_n \tag{2.8.3}$$

such that the products are invariant under the group H appearing in G/H and therefore have opposite H quantum numbers. The exponent of Kähler function does not factorize although the terms in its Taylor expansion factorize to products whose factors are products of holomorphic and antiholomorphic functions.

5. If one assumes that the exponent of Kähler function reduces to a product of eigenvalues of the modified Dirac operator eigenvalues must have the decomposition

$$\lambda_k = \prod_{i=1,2} exp\left[\sum_n c_{k,n} g_n(Z_i)\overline{g_n(Z_i)} + c.c\right] \times exp\left[\sum_n d_{k,n} g_n(Z_1)\overline{g_n(Z_2)} + c.c\right] \tag{2.8.4}$$

Hence also the eigenvalues coming from the Dirac propagators have also expansion in terms of G/H harmonics so that in principle WCW integration would reduce to Fourier analysis in symmetric space.

Generalization of WCW Hamiltonians

This picture requires a generalization of the view about configuration space Hamiltonians since also the interaction term between the ends of the line is present not taken into account in the previous approach.

1. The proposed representation of WCW Hamiltonians as flux Hamiltonians [K16, K15]

$$\begin{aligned} Q(H_A) &= \int H_A(1+K)Jd^2x \ , \\ J &= \epsilon^{\alpha\beta}J_{\alpha\beta} \ , \ J^{03}\sqrt{g_4} = KJ_{12} \ . \end{aligned} \tag{2.8.5}$$

works for the kinetic terms only since J cannot be the same at the ends of the line. The formula defining K assumes weak form of self-duality (^{03} refers to the coordinates in the complement of X^2 tangent plane in the 4-D tangent plane). K is assumed to be symplectic invariant and constant for given X^2. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge g_K gives the condition $K = g_K^2/\hbar$, where g_K is Kähler coupling constant. Within experimental uncertainties one has $\alpha_K = g_K^2/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where α_{em} is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

The assumption that Poisson bracket of WCW Hamiltonians reduces to the level of imbedding space - in other words $\{Q(H_A), Q(H_B\} = Q(\{H_A, H_B\})$ - can be justified. One starts from the representation in terms of say flux Hamiltonians $Q(H_A)$ and *defines* $J_{A,B}$ as $J_{A,B} \equiv Q(\{H_A, H_B\})$. One has $\partial H_A/\partial t_B = \{H_B, H_A\}$, where t_B is the parameter associated with the exponentiation of H_B. The inverse J^{AB} of $J_{A,B} = \partial H_B/\partial t_A$ is expressible as $J^{A,B} = \partial t_A/\partial H_B$. From these formulas one can deduce by using chain rule that the bracket $\{Q(H_A), Q(H_B\} = \partial t_C Q(H_A) J^{CD} \partial t_D Q(H_B)$ of flux Hamiltonians equals to the flux Hamiltonian $Q(\{H_A, H_B\})$.

2. One should be able to assign to WCW Hamiltonians also a part corresponding to the interaction term. The symplectic conjugation associated with the interaction term permutes the WCW coordinates assignable to the ends of the line. One should reduce this apparently non-local symplectic conjugation (if one thinks the ends of line as separate objects) to a non-local symplectic conjugation for $\delta CD \times CP_2$ by identifying the points of lower and upper end of CD related by time reflection and assuming that conjugation corresponds to time reflection. Formally this gives a well defined generalization of the local Poisson brackets between time reflected points at the boundaries of CD. The connection of Hermitian conjugation and time reflection in quantum field theories is is in accordance with this picture.

3. The only manner to proceed is to assign to the flux Hamiltonian also a part obtained by the replacement of the flux integral over X^2 with an integral over the projection of X^2 to a sphere S^2 assignable to the light-cone boundary or to a geodesic sphere of CP_2, which come as two varieties corresponding to homologically trivial and non-trivial spheres. The projection is defined as by the geodesic line orthogonal to S^2 and going through the point of X^2. The hierarchy of Planck constants assigns to CD a preferred geodesic sphere of CP_2 as well as a unique sphere S^2 as a sphere for which the radial coordinate r_M or the light-cone boundary defined uniquely is constant: this radial coordinate corresponds to spherical coordinate in the rest system defined by the time-like vector connecting the tips of CD. Either spheres or possibly both of them could be relevant.

   Recall that also the construction of number theoretic braids and symplectic QFT [K18] led to the proposal that braid diagrams and symplectic triangulations could be defined in terms of projections of braid strands to one of these spheres. One could also consider a weakening for the condition that the points of the number theoretic braid are algebraic by requiring only that the S^2 coordinates of the projection are algebraic and that these coordinates correspond to the discretization of S^2 in terms of the phase angles associated with θ and ϕ.

   This gives for the corresponding contribution of the WCW Hamiltonian the expression

$$Q(H_A)_{int} = \int_{S^2_\pm} H_A X \delta^2(s_+, s_-) d^2 s_\pm = \int_{P(X^2_+)\cap P(X^2_-)} \frac{\partial(s^1, s^2)}{\partial(x^1_\pm, x^2_\pm)} d^2 x_\pm \ . \qquad (2.8.6)$$

   Here the Poisson brackets between ends of the line using the rules involve delta function $\delta^2(s_+, s_-)$ at S^2 and the resulting Hamiltonians can be expressed as a similar integral of $H_{[A,B]}$ over the upper or lower end since the integral is over the intersection of S^2 projections.

   The expression must vanish when the induced Kähler form vanishes for either end. This is achieved by identifying the scalar X in the following manner:

$$\begin{aligned} X &= J_+^{kl} J^-_{kl} \ , \\ J^{kl}_\pm &= (1 + K_\pm)\partial_\alpha s^k \partial_\beta s^l J^{\alpha\beta}_\pm \ . \end{aligned} \qquad (2.8.7)$$

   The tensors are lifts of the induced Kähler form of $X^2_\pm$ to S^2 (not CP_2).

4. One could of course ask why these Hamiltonians could not contribute also to the kinetic terms and why the brackets with flux Hamiltonians should vanish. This relate to how one *defines* the Kähler form. It was shown above that in case of flux Hamiltonians the definition of Kähler form as brackets gives the basic formula $\{Q(H_A), Q(H_B)\} = Q(\{H_A, H_B\}$ and same

should hold true now. In the recent case $J_{A,B}$ would contain an interaction term defined in terms of flux Hamiltonians and the previous argument should go through also now by identifying Hamiltonians as sums of two contributions and by introducing the doubling of the coordinates t_A.

5. The quantization of the modified Dirac operator must be reconsidered. It would seem that one must add to the super-Hamiltonian completely analogous term obtained by replacing $(1+K)J$ with $X\partial(s^1,s^2)/\partial(x^1_{\pm},x^2_{\pm})$. Besides the anti-commutation relations defining correct anti-commutators to flux Hamiltonians, one should pose anti-commutation relations consistent with the anti-commutation relations of super Hamiltonians. In these anti-commutation relations $(1+K)J\delta^2(x,y)$ would be replaced with $X\delta^2(s^+,s^-)$. This would guarantee that the oscillator operators at the ends of the line are not independent and that the resulting Hamiltonian reduces to integral over either end for $H_{[A,B]}$.

6. In the case of CP_2 the Hamiltonians generating isometries are rational functions. This should hold true also now so that p-adic variants of Hamiltonians as functions in WCW would make sense. This in turn would imply that the components of the WCW Kähler form are rational functions. Also the exponentiation of Hamiltonians make sense p-adically if one allows the exponents of group parameters to be functions $Exp_p(t)$.

Does the expansion in terms of partial harmonics converge?

The individual terms in the partial wave expansion seem to be finite but it is not at all clear whether the expansion in powers of K actually converges.

1. In the proposed scenario one performs the expansion of the vacuum functional $exp(K)$ in powers of K and therefore in negative powers of α_K. In principle an infinite number of terms can be present. This is analogous to the perturbative expansion based on using magnetic monopoles as basic objects whereas the expansion using the contravariant Kähler metric as a propagator would be in positive powers of α_K and analogous to the expansion in terms of magnetically bound states of wormhole throats with vanishing net value of magnetic charge. At this moment one can only suggest various approaches to how one could understand the situation.

2. Weak form of self-duality and magnetic confinement could change the situation. Performing the perturbation around magnetic flux tubes together with the assumed slicing of the space-time sheet by stringy world sheets and partonic 2-surfaces could mean that the perturbation corresponds to the action assignable to the electric part of Kähler form proportional to α_K by the weak self-duality. Hence by $K = 4\pi\alpha_K$ relating Kähler electric field to Kähler magnetic field the expansion would come in powers of a term containing sum of terms proportional to α_K^0 and α_K. This would leave to the scattering amplitudes the exponents of Kähler function at the maximum of Kähler function so that the non-analytic dependence on α_K would not disappear.

A further reason to be worried about is that the expansion containing infinite number of terms proportional to α_K^0 could fail to converge.

1. This could be also seen as a reason for why magnetic singlets are unavoidable except perhaps for $\hbar < \hbar_0$. By the holomorphic factorization the powers of the interaction part of Kähler action in powers of $1/\alpha_K$ would naturally correspond to increasing and opposite net values of the quantum numbers assignable to the WCW phase coordinates at the ends of the propagator line. The magnetic bound states could have similar expansion in powers of α_K as pairs of states with arbitrarily high but opposite values of quantum numbers. In the functional integral these quantum numbers would compensate each other. The functional integral would leave only an expansion containing powers of α_K starting from some finite possibly negative (unless one assumes the weak form of self-duality) power. Various gauge coupling strengths are expected to be proportional to α_K and these expansions should reduce to those in powers of α_K.

2. Since the number of terms in the fermionic propagator expansion is finite, one might hope on basis of super-symmetry that the same is true in the case of the functional integral expansion. By the holomorphic factorization the expansion in powers of K means the appearance of terms with increasingly higher quantum numbers. Quantum number conservation at vertices would leave only a finite number of terms to tree diagrams. In the case of loop diagrams pairs of particles with opposite and arbitrarily high values of quantum numbers could be generated at the vertex and magnetic confinement might be necessary to guarantee the convergence. Also super-symmetry could imply cancellations in loops.

Could one do without flux Hamiltonians?

The fact that the Kähler functions associated with the propagator lines can be regarded as interaction terms inspires the question whether the Kähler function could contain only the interaction terms so that Kähler form and Kähler metric would have components only between the ends of the lines.

1. The basic objection is that flux Hamiltonians too beautiful objects to be left without any role in the theory. One could also argue that the WCW metric would not be positive definite if only the non-diagonal interaction term is present. The simplest example is Hermitian 2×2-matrix with vanishing diagonal for which eigenvalues are real but of opposite sign.

2. One could of course argue that the expansions of $exp(K)$ and λ_k give in the general powers $(f_n\overline{f_n})^m$ analogous to diverging tadpole diagrams of quantum field theories due to local interaction vertices. These terms do not produce divergences now but the possibility that the exponential series of this kind of terms could diverge cannot be excluded. The absence of the kinetic terms would allow to get rid of these terms and might be argued to be the symmetric space counterpart for the vanishing of loops in WCW integral.

3. In zero energy ontology this idea does not look completely non-sensical since physical states are pairs of positive and negative energy states. Note also that in quantum theory only creation operators are used to create positive energy states. The manifest non-locality of the interaction terms and absence of the counterparts of kinetic terms would provide a trivial manner to get rid of infinities due to the presence of local interactions. The safest option is however to keep both terms.

Summary

The discussion suggests that one must treat the entire Feynman graph as single geometric object with Kähler geometry in which the symmetric space is defined as product of what could be regarded as analogs of symmetric spaces with interaction terms of the metric coming from the propagator lines. The exponent of Kähler function would be the product of exponents associated with all lines and contributions to lines depend on quantum numbers (momentum and color quantum numbers) propagating in line via the coupling to the modified Dirac operator. The conformal factorization would allow the reduction of integrations to Fourier analysis in symmetric space. What is of decisive importance is that the entire Feynman diagrammatics at WCW level would reduce to the construction of WCW geometry for a single propagator line as a function of quantum numbers propagating on the line.

2.9 Appendix: Basic facts about algebraic numbers, quaternions and octonions

To understand the detailed connection between infinite primes, polynomial primes and Fock states, some basic concepts of algebraic number theory related to the generalization of prime and prime factorization [A140, A137, A112] (the first reference is warmly recommended for a physicist because it teaches the basic facts through exercises; also second book is highly enjoyable reading because of its non-Bourbakian style of representation).

2.9.1 Generalizing the notion of prime

Algebraic numbers are defined as roots of polynomial equations with rational coefficients. Algebraic integers are identified as roots of monic polynomials (highest coefficient equals to one) with integer coefficients. Algebraic number fields correspond to algebraic extensions of rationals and can have any dimension as linear spaces over rationals. The notion of prime is extremely general and involves rather abctract mathematics in general case.

Quite generally, commutative ring R called integral domain, if the product ab vanishes only if a or b vanishes. To a given integral domain one can assign a number field by essentially the same construction by which one assigns the field of rationals to ordinary integers. The integer valued function $a \rightarrow N(a)$ in R is called norm if it has the properties $N(ab) = N(a)N(b)$ and $N(1) = 1$. For instance, for the algebraic extension $Q(\sqrt{-D})$ of rationals consisting of points $z = r + \sqrt{-D}s$, the function $N(z) = r^2 + Ds^2$ defines norm. More generally, the determinant of the linear map defined by the action of z in algebraic number field defines norm function. This determinant reduces to the product of all conjugates of z in K and is n:th order polynomial with respect to the components of z when K is n-dimensional.

Irreducible elements (almost the counterparts of primes) can be defined as elements P of integral domain having the property that if one has $P = bc$, then either b or c has unit norm. Elements with unit norm are called units and elements differing by a multiplication with unit are called associates. Note that in the case of p-adics all p-adic numbers with unit norm are units.

2.9.2 UFDs, PIDs and EDs

If the elements of R allow a unique factorization to irreducible elements, R is said to be unique factorization domain (UFD). Ordinary integers are obviously UFD. The field $Z(\sqrt{-5})$ is not UFd for instance, one has $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. The fact that prime factorization is not unique forces to generalize the notion of primeness such that ideals in the ring of algebraic integers take the role of integers. The counterparts of primes can be identified as irreducible elements, which generate prime ideals containing one and only one rational prime. Irreducible elements, such as $1 \pm \sqrt{-5}$ in $Z(\sqrt{-5})$, are not primes in this sense.

Principal ideal domain (PID) is defined as an integral domain for which all ideals are principal, that is are generated as powers of single element. In the case of ordinary integers powers of integers define PID.

Euclidian domain (ED) is integral domain with the property that for any pair a and b one can find pair (q, r) such that $a = bq + r$ with $N(r) < N(a)$. This guarantees that the Euclidian algorithm used in the division of rationals converges. Integers form an Euclidian domain but polynomials with integer coefficients do not (elements 2 and x do not allow decomposition $2 = q(x)x + r$). It can be shown that EDs are PIDs in turn are UFDs. For instance, for complex quadratic extensions of integers $Z(\sqrt{-d})$ there are only 9 UFDs and they correspond to $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For extensions of type $Z(\sqrt{d})$ the number of UFD:s is infinite. There are not too many quadratic extensions which are ED:s and the possible values of d are $d = -1, \pm 2, \pm 3, 5, 6, \pm 7, \pm 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73$.

Any algebraic number field K is representable always as a polynomial ring $Q[\theta]$ obtained from the polynomial ring $Q[x]$ by replacing x with an algebraic number θ, which is a root of an irreducible polynomial with rational coefficients. This field has dimension n over rationals, where n is the degree of the polynomial in question.

2.9.3 The notion of prime ideal

As already noticed, a general algebraic number field K does not allow a unique factorization into irreducibles and one must generalize the notion of prime number and integer in order to achieve a unique factorization. The ideals of the ring O_K of algebraic integers in K take the role of integers whereas prime ideals take the role of primes. The factorization of an ideal to a product of prime ideals is unique and each prime ideal contains single rational prime characterizing it. One can assign to an ideal norm which orders the ideals: $N(a) < N(b) \leftrightarrow b \subset a$. The smaller the integer generating ideal, the larger the ideal is and the ideals generated by primes are maximal ones in PID. The equivalence classes of the ideals of O_K under equivalence defined by integer multiplication form a group. The number of classes is a characteristic of an algebraic number field. For class-one algebraic number fields prime factorization of ideals is equivalent with the factorization to irreducibles in K. $Z(\sqrt{-5})$, which is not UFD, allows two classes of prime ideals. Cyclotomic number fields $Q(\zeta_m)$, where ζ_m is m:th root of unity have class number one for $3 \leq m \leq 10$. In particular, the four-dimensional algebraic number fields $Q(\zeta_8)$ and $Q(\zeta_5) = Q(\zeta_{10})$ are ED and thus UFD.

Basic facts about primality for polynomial rings

The notion of primality can be abstracted to the level of polynomial algebras in field K and these polynomial algebras seem to be more or less identical with the algebra formed by infinite integers. The following two results are crucial for the argument demonstrating that this is indeed the case.

Polynomial ring associated with any number field is UFD

The elements in the ring $K[x_1, ..., x_n]$ formed by the polynomials having coefficients in *any* field K and x_i having values in K, allow a unique decomposition into prime factors. This means that things are much simpler at the next abstraction level, since there is no need for refined class theories needed in the case of algebraic number fields.

The number field K appearing as a coefficient field of polynomials could correspond to finite fields (Galois fields), rationals, any algebraic number field obtained as an extension of rational, p-adic numbers, reals or complex numbers. For $Q[x]$, where Q denotes rationals, the simplest prime factors are monomials of form $x - q$, q rational number. More complicated prime factors correspond to minimal polynomials having algebraic number α and its conjugates as their roots. In the case of complex number field only monomomials $x - z$, z complex number are the only prime polynomials. Clearly, the primes at the higher level of abstraction are generalized rationals of previous level plus numbers which are algebraic with respect to the generalized rationals.

The polynomial rings associated with any UFD are UFD

If R is a unique factorization domain (UFD), then also $R[x]$ is UFD: this holds also for $R[x_1, ..., x_n]$. Hence one obtains an infinite hierarchy of UFDs by a repeated abstraction process by starting from a given algebraic number field K. At the first step one obtains the ring $K[x]$ of polynomials in K. At the next step one obtains the ring of polynomials $K^{2)}[y]$ having as coefficient ring the ring $K[x] \equiv K^{1)}[x]$ of polynomials. At the next step one obtains $K^{2)}[z]$, etc.. Note that $O_K[x]$ is not ED in general and need not be UFD neither unless O_K is UFD. $O_K[x]$ is not however interesting from the viewpoint of TGD.

An element of $K^{2)}(y)$ corresponds to a polynomial $P(y, x)$ of y such that its coefficients are K-rational functions of x. A polynomial in $K^{3)}(z)$ corresponds to a polynomial of $P(z, y, z)$ such that the coefficients of z are K-rational functions of functions of y with coefficients which are K-rational functions of z. Note that as a special case, polynomials of all n variables result. Note also the hierarchical ordering of the variables. Thus the hierarchy of polynomials gives rise to a hierarchy of functions having increasingly number of independent variables.

2.9.4 Examples of two-dimensional algebraic number fields

The general two-dimensional (in algebraic sense) algebraic extension of rationals corresponds to $K(\theta)$, where $\theta = (-b \pm \sqrt{b^2 - 4c})/2$ is root of second order irreducible polynomial $x^2 + bx + c$. Depending on whether the discriminant $D = b^2 - 4c$ is positive or negative, one obtains real and complex extensions. θ and its conjugate generate equivalent extensions and all extensions can be obtained as extensions of form $Q(\sqrt{\pm d})$.

For $Q(\sqrt{d})$, d square-free integer, units correspond to powers of $x = \pm(p_{n-1} + q_{n-1}\sqrt{d})$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k:th convergent in the continued fraction expansion. For $Q(\sqrt{-d})$, $d > 1$ units form group Z_2. For $d = 1$ the group is Z_2^2 and for $Q(w)$ where $w = -1/2 + \sqrt{3}/2$ is the third root of unity ($w^3 = 1$), this group is $Z_2 \times Z^3$ (note that in this case the minimal polynomial is $(x^3 - 1)/(x - 1)$.

$Z(w)$ and $Z(i)$ are exceptional in the sense that the group of the roots of unity is exceptionally large. $Z(i)$ and $Z(w)$ allow a unique factorization of their elements into products of irreducibles.

The primes π of $Z(w)$ consist of rational primes p, $p\ mod\ 4 = 3$ and complex Gaussian primes satisfying $N(\pi) = \pi\overline{\pi} = p$, $p\ mod\ 4 = 1$. Squares of the Gaussian primes generate as their product complex numbers giving rise to Pythagorean phases. The primes π of $Z(w)$ consist of rational primes p, $p\ mod\ 3 = 2$ and complex Eisenstein primes satisfing $N(\pi) = \pi\overline{\pi} = p$, $p\ mod\ 3 = 1$.

2.9.5 Cyclotomic number fields as examples of four-dimensional algebraic number fields

By the 'theorem of primitive element' all algebraic number fields are obtained by replacing the polynomial algebra $Q[x]$, by $Q[\theta]$, where θ is a root of an irreducible minimal polynomial which is of fourth order. One can readily calculate the extensions associated with a given irreducible polynomial by using quadratures for 4:th order polynomials. These polynomials are of general form $P_4(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and by a substitution $x = y - a_3/4)$ which does not change the nature of algbebraic number field, they can be reduced to a canonical form $P_4(x) = x^4 + a_2x^2 + a_1x + a_0$. Thus a very rough view is that three rationals parametrize the 4-dimensional algebraic number fields.

A second manner to represent extensions is in form $K(\theta_1, \theta, ..)$ such that the units θ_i have no common factors different from one. In this case the dimension of the extension is 2^n, where n is the number of units. Examples of four-dimensional extensions are the algebraic extensions $Q(\sqrt{\pm d_1}, \sqrt{\pm d_2})$ of rationals, where d_i are square-free integers, reduce to form $Q(\theta)$. The cyclic extension of rationals by the powers of the m:th root of unity with $m = 5, 8, 12$ are four-dimensional extensions called cyclotomic number fields. Also the extensions $Q((\pm)d)^{1/4})$ are simple four-dimensional extensions. These extensions allow completion to a corresponding p-adic algebraic extension for some p-adic primes.

Quite generally, cyclotomic number fields $Q(\zeta_m)$ are obtained from polynomial algebra $Q[x]$ by replacing x with the m:th primitive root of unity denoted by ζ_m and thus satisfying $\zeta_m^m = 1$. There are three cyclic extensions of dimension 4 and they correspond to $Q(\zeta_5) = Q(\zeta_{10})$, $Q(\zeta_8)$ and $Q(\zeta_{12})$. Cyclotomic extensions are higly symmetric since the roots of unity act as symmetries of the norm.

The units of cyclotomic field $Q(\zeta_m)$ form group $Z_2 \times Z_m \times Z$. Z corresponds to the powers of units for $Q(\zeta_m + 1/\zeta_m)$. These powers have unit norm only with respect to the norm of $Q(\zeta_m)$ whereas with respect to the ordinary complex norm they correspond to fractal scalings. What looks fractal obtained by repeated scalings of the same structure with respect to the real norm looks like a lattice when algebraic norm is used.

1. $Q(\zeta_8)$

The cyclotomic number field $Q(\zeta_8)$, $\zeta_8 = exp(i\pi/4)$ satisfying $\zeta_8^8 = 1$, consists of numbers of form $k = m + in + \sqrt{i}(r + is)$. All roots ($\pm i^{1/2}$ and $\pm i^{3/2}$) are complex. The group of units is $Z_2^4 \times Z$. Z corresponds in real topology to the fractal scalings generated by $L = 1 + \sqrt{2}$. The integer multiples of $log(L)$ could be interpreted as a quantized momentum. $Q(\zeta_8)$ can be generated by $\pm\zeta_8$ and $\pm i\zeta_8$. This means additional Z_2^2 Galois symmetry which does not define multiplicative quantum number.

2. $Q(\zeta_{12})$

The extension $Q(\sqrt{-1}, w)$, $w = \zeta_3$, can be regarded as a cyclic extension $Q(iw) = Q(\zeta_{12})$ as is clear from the fact that the six lowest powers of iw come as $iw, -w^2, -i, w = -1 - w^2, iw^2 = -iw - i, -1$. $Z(iw)$ is especially interesting because it contains $Q(i)$ and $Q(w)$ for which primes correspond to Gaussian and Eisenstein primes. A unique factorization to a product of irreducibles is possible only for $Q(\zeta_m)$ $m \leq 10$: thus the algebraic integers in $Z(iw)$ do not always allow a unique decomposition into irreducibles. The most obvious candidates for primes not allowing unique factorization are primes satisfying simultaneously the conditions $p\ mod\ 4 = 3 = 1$ implying decomposition into a product of Gaussian prime and its conjugate and $p\ mod\ 3 = 1$ guaranteeing the decomposition into a product of Eisenstein prime and its conjugate.

The group of units reduces to $Z_2^2 \times Z_3 \times Z$ might have something to do with the group of discrete quantum numbers C,P and $SU(3)$ triality telling the number of quarks modulo 3 in the state. For the extensions $Q(\sqrt{-1}, \sqrt{d})$ the roots of unity form the group Z_2^2: these extensions could correspond to gauge bosons and the quantum numbers would correspond to C and P. For real extensions the group of the roots of unity reduces to Z_2: in this case the interpretation inters of parity suggests itself.

The lattice defined by Z corresponds to the scalings by powers of $\sqrt{3}+2$. It could be also interpreted also as the lattice of longitudinal momenta for hadronic quarks which move collinearly inside space-time sheet which might be identified as a massless extremal (ME) for which longitudinal direction is a preferred spatial direction.

$Q(\zeta_{12})$ can be generated by $\pm iw, \pm iw^2$ and the replacement of iw with these alternatives generates Z_2^2 symmetry not realizable as a multiplication with units.

3. $Q(\zeta_5)$ and biology

$Q(\zeta_5)$ indeed gives 4-dimensional extension of rationals since one has $1+\zeta_5+...\zeta_5^4=0$ implying that $\zeta_5^4 = 1/\zeta_5$ is expressible as rational combination of other units. Both $Q(\zeta_5)$ and $Q(\zeta_8)$ allows a unique decomposition of rational integers into prime factors. The primes in $Q(\zeta_5)$ allow decomposition to a product of $r = 1, 2$ or 4 primes of $Q(\zeta_5)$ [A137] . The value of r for a given p is fixed by the requirement that $f = 4/r$ is the smallest natural number for which $p^f - 1 \ mod \ p = 0$ holds true. For instance, $p = 2, 3$ correspond to $f = 4$ and are primes of $Q(\zeta_5)$, $p = 11$ has decomposition into a product of four primes of $Q(\zeta_5)$, and $p = 19$ has decomposition into two primes of $Q(\zeta_5)$).

What makes this extension interesting is that the phase angle associated with ζ_5 corresponds to the angle of 72 degrees closely related with Golden Mean $\tau = (1+\sqrt{5})/2$ satisfying the equation $\tau^2 - \tau - 1 = 0$. The phase of the fifth root is given by $\zeta_5 = (\tau - 1 + i\sqrt{2+\tau})2$. The group of units is $Z_2 \times Z_5 \times Z$. Z corresponds to the fractal scalings by $\tau = (1+\sqrt{5})/2$. The conjugations $\zeta_5 \rightarrow \zeta_5^k$, $k = 1, 2, 3, 4$ leave the norm invariant and generate group Z_2^5.

Fractal scalings by Golden Mean and the closely related Fibonacci numbers are closely related with the fractal structures associated with living systems (botany is full of logarithmic spirals involving Golden Mean and the phase angle 36 is involved even with DNA). Of course, the very fact that Golden Mean emerges in biological length scales provides strongest evidence for its dynamical origin in algebraic framework.

$Q(\zeta_5)$ cannot be realized as an algebraic extension $K(\theta, i)$ naturally associated with the transversal part of quaternionic primes but can appear only as a subfield of the 8-dimensional extension $K(i, cos(2\pi/5), sin(2\pi/5)$ containing also 20:th root of unity as $\zeta_{20} = i\zeta_5$. In [K78] it is indeed found that Golden Mean plays a fundamental role in topological quantum computation and is indeed a fundamental constant in TGD Universe.

Fractal scalings

By Dirichlet's unit theorem the group of units quite generally reduces to $Z_m \times Z^r$, where Z_m is cyclic group of roots of unity and Z^r can be regarded as an r-dimensional lattice with latticed units determined by the extension. For real extensions Z_m reduces to Z_2 since the only real roots of unity are $\{\pm 1\}$. All components of four-momentum represented by a quaternionic prime can be multiplied by separate real units of $Q(\theta)$. For a given quaternionic prime, one can always factor out the common factor of the units of $Q(\theta)$ or $Q(\theta, i)$.

The units generate nontrivial transformations at the level of single quaternionic prime. If the dimension of the real extension is n, the transformations form an $n-1$-dimensional lattice of scalings. Alternative but less plausible interpretation is that the logarithms of the scalings represent $n-1$-dimensional momentum lattice. Particle would be like a part of an algebraic hologram carrying information about external world in accordance with the ideas about fractality. Of course, units represent fractal scalings only with respect to ordinary real norm, with respect to number theoretical norm they act like phase factors.

For instance, in the case of $Q(\sqrt{5})$ the units correspond to scalings by powers of Golden Mean $\tau = (1+\sqrt{5})/2$ having number theoretic norm equal to one. Bio-systems are indeed full of fractals with scaling symmetry. For $K = Q(\sqrt{3})$ the scalings correspond to powers of $L = 2+\sqrt{3}$. An interesting possibility is that hadron physics might reveal fractality in powers of L. More generally, for $Q(\sqrt{d})$, d square-free integer, the basic fractal scaling is $L = p_{n-1} + q_{n-1}\sqrt{d}$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k:th convergent in the continued fraction expansion.

Four-dimensional algebraic extensions are very interesting for several reasons. First, algebraic dimension four is a borderline in complexity in the sense that for higher-dimensional irreducible algebraic extensions there is no general quadratures analogous to the formulas associated with second order polynomials giving the roots of the polynomial. Secondly, in transversal degrees of freedom the minimal dimension for $K(\theta, i)$ is four. The units of K which are algebraic integers having a unit norm in K. Quite generally, the group of units is a product $Z_{2k} \times Z_r$ of two groups. $Z_{2k} = Z_2 \times Z_k$ is the cyclic group generated by k:th root of unity. For real extensions one has $k = 1$. In transversal degrees of freedom one can have $k > 1$ since extension is $Q(\theta, i)$. The roots of unity possible in four-dimensional case correspond to $k = 2, 4, 6, 8, 10, 12$. Corresponding cyclic groups are products of Z_2^i, Z_3 and Z_5. Z_2, Z_2 and Z_3 and act as symmetries of the root lattices of Cartan algebras.

Z_3 gives rise to the Cartan algebra of $SU(3)$ and an interesting question is whether color symmetry is generated dynamically or whether it can be regarded as a basic symmetry with the lattice of integer quaternions providing scaled-up version for the root lattice of color group. Note that in TGD quark color is not spin like quantum number but corresponds to CP_2 partial waves for quark like spinors.

Permutations of the real roots of the minimal polynomial of θ

The replacements of the primitive element θ of $K(\theta)$ with a new one obtained by acting in it with the elements of Galois group of the minimal polynomial of θ generate different internal states of number theoretic fermions and bosons. The subgroup G_1 of Galois group permuting the real roots of the minimal polynomial with each other acts also as a symmetry. The number of equivalent primitive elements is $n_1 = n - 2r_1$, where r_2 is the number of complex root pairs. For instance, for 2-dimensional extensions these symmetries permute the real roots of a second order polynomial irreducible in the set of rationals. Since the entire polynomial has rational coefficients, kind of G_1-confinement is realized. One could say that kind of algebraically confined n-color is in question.

2.9.6 Quaternionic primes

Primeness makes sense for quaternions and octonions. The following considerations are however restricted to quaternionic primes but can be easily generalized to the octonionic case. Quaternionic primes have Euclidian norm squared equal to a rational prime. The number $N(p)$ of primes associated with a given rational p depends on p and each p allows at least two primes. Quaternionic primes correspond to points of 3-sphere with prime-valued radius squared. Prime-valued radius squared is consistent with p-adic length scale hypothesis, and one can indeed reduce p-adic length scale hypothesis to the assumption that the Euclidian region associated with CP_2 type extremal has prime-valued radius squared.

It is interesting to count the number of quaternionic primes with same prime valued length squared.

1. In the case of algebraic extensions the first definition of quaternionic norm is by using number theoretic norm either for entire quaternion squared or for each component of quaternion separately. The construction of infinite primes suggests that the first definition is more appropriate. Both definitions of norm are natural for four-momentum squared since they give integer valued mass squared spectrum associated with super-conformally invariant systems. One could also decompose quaternion to two parts as $q = (q_0 + Iq_1) + J(q_2 + Iq_3)$ and define number theoretic norm with respect to the algebraic extension $Q(\theta, I)$.

2. Quaternionic primes with the same norm are related by $SO(4)$ rotation plus a change of sign of the real component of quaternion. The components of integer quaternion are analogous to components of four-momentum.

3. There are 2^4 quaternionic $\pm E_i$ and multiplication by these units defines symmetries. Non-commutativity of the quaternionic multiplication makes the interpretation of units as parity like quantum numbers somewhat problematic since the net parity associated with a product of primes representing physical particles associated with the infinite primes depends on the order of quaternionic primes. For real algebraic extensions $K = Q(\theta)$ there is also the units

defining a 'momentum' lattice with dimension $n-1$, where n is the degree of the minimal polynomial $P(\theta)$.

4. Quaternionic primes cannot be real so that a given quaternionic prime with $k \geq 2$ components has 2^k conjugates obtained by changing the signs of the components of quaternion. Basic conjugation changes the signs of imagy components of quaternion. This corresponds to group $Z_2^k \subset Z_2^4$, $2 \leq k \leq 4$.

5. The group S_4 of $4! = 24$ permutations of four objects preserves the norm of a prime quaternion: these permutations are representable as a multiplication with non-prime quaternion and thus identifiable as subgroup of $SO(4)$ and also as a subgroup of $SO(3)$ (invariance group of tetrahedron). In degenerate cases (say when some components of q are identical), some subgroup of S_4 leaves quaternionic prime invariant and the rotational degeneracy reduces from $D = 24$ to some smaller number which is some factor of 24 and equals to $4, 6$ or 12 as is easy to see. There are 16 quaternionic conjugations corresponding to change of sign of any quaternion unit but all these conjugations are obtained from single quaternionic conjugation changing the sign of the imaginary part of quaternion by combining them with a multiplication with unit and its inverse. Thus the restricted group of symmetries is $S_4 \times Z_2$.

6. It is possible to find for every prime p at least two quaternionic (primes with norm squared equal to p. For a given prime p there are in general several quaternionic primes not obtainable from each other by transformations of S_4. There must exist some discrete subgroup of $SO(4)$ relating these quaternionic primes to each other.

7. The maximal number of quaternionic primes generated by $S_4 \times Z_2$ is 24×2. In non-commutative situation it is not clear whether units can be regarded as parity type quantum numbers. In any case, one can divide the entire group with Z_2^4 to obtain Z_3. This group corresponds to cyclic permutations of imaginary quaternion units.

$D = 24$ is the number of physical dimensions in bosonic string model. In TGD framework a possible interpretation is based on the observation that infinite primes constructed from rational primes the product of all primes contains the first power of each prime having interpretation as a representation for a single filled state of the fermionic sea. In the case of quaternions the Fock vacuum defined as a product of all quaternionic primes gives rise to a vacuum state

$$X = \prod_p p^{N(p)/2} \quad ,$$

since each prime and its quaternionic conjugate contribute one power of p.

2.9.7 Imbedding space metric and vielbein must involve only rational functions

Algebraization requires that imbedding space exists in the algebraic sense containing only points for which preferred coordinate variables have values in some algebraic extension of rationals. Imbedding space metric at the algebraic level can be defined as a quadratic form without any reference to metric concepts like line element or distance. The metric tensors of both M_+^4 and CP_2 are indeed represented by algebraic functions in the preferred coordinates dictated by the symmetries of these spaces.

One should also construct spinor structure and this requires the introduction of an algebraic extension containing square roots since vielbein vectors appearing in the definition of the gamma matrices involve square roots of the components of the metric. In CP_2 degrees of freedom this forces the introduction of square root function, and thus all square roots, unless one restricts the values of the radial CP_2 coordinate appearing in the vielbein in such a manner that rationals result. What is interesting is that all components of spinor curvature and Kähler form of CP_2 are quadratic with respect to vierbein and algebraic functions of CP_2 complex coordinates. Also the square root of the determinant of the induce metric appears only as a multiplicative factor in the Euler-Lagrange equations so that one can get rid of the square roots.

Induced spinor structure and Dirac equation relies on the notion of the induced gamma matrices and here the projections of the vierbein of CP_2 containing square roots are unavoidable. In complex coordinates the components of CP_2 vielbein in complex coordinates ξ_1, ξ_2, in which the action of $U(2)$ is linear holomorphic transformation, involve the square roots $r = \sqrt{|\xi|^2 + |\xi_2|^2}$ and $\sqrt{1+r^2}$ (for detailed formulas see Appendix at the end of the book). If one has $r = m/n$, the requirement that $\sqrt{1+r^2}$ is rational, implies $m^2 + n^2 = k^2$ so that (m,n) defines Pythagorean square. Thus induced Dirac equation is rationalized if the allowed values of r correspond to Pythagorean phases. The notion of the phase preserving canonical identification [K29] , crucial for the earlier formulation of TGD, is consistent with this assumption. The metric of $S^2 = CP_1$ is a simplified example of what happens. One can write the metric as $g_{z\bar{z}=r^2} = \frac{1}{1+r^2}$ and vielbein component is proportional to $1/\sqrt{1+r^2}$, this exists for $r = m/n$ as rational number if one has $m^2 + n^2 = k^2$, which indeed defines Pythagorean triangle.

The restriction of the phases associated with the CP_2 coordinates to Pythagorean ones has deeper coordinate-invariant meaning. Rational CP_2 can be defined as a coset space $SU_Q(3)/U_Q(2)$ of rational groups $SU_Q(3)$ and $U_Q(2)$: rationality is required in the linear matrix representation of these groups.

Chapter 3

TGD as a Generalized Number Theory II: Quaternions, Octonions, and their Hyper Counterparts

3.1 Introduction

This chapter is second one in a multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme is the role of classical number fields in quantum TGD. A central notion is $M^8 - H$ duality which might be also called number theoretic compactification. This duality allows to identify imbedding space equivalently either as M^8 or $M^4 \times CP_2$ and explains the symmetries of standard model number theoretically. These number theoretical symmetries induce also the symmetries dictating the geometry of the "world of classical worlds" (WCW) as a union of symmetric spaces. This infinite-dimensional Kähler geometry is expected to be highly unique from the mere requirement of its existence requiring infinite-dimensional symmetries provided by the generalized conformal symmetries of the light-cone boundary $\delta M^4_+ \times S$ and of light-like 3-surfaces and the answer to the question what makes 8-D imbedding space and $S = CP_2$ so unique would be the reduction of these symmetries to number theory.

Zero energy ontology has become the corner stone of both quantum TGD and number theoretical vision. In zero energy ontology either light-like or space-like 3-surfaces can be identified as the fundamental dynamical objects, and the extension of general coordinate invariance leads to effective 2-dimensionality (strong form of holography) in the sense that the data associated with partonic 2-surfaces and the distribution of 4-D tangent spaces at them located at the light-like boundaries of causal diamonds (CDs) defined as intersections of future and past directed light-cones code for quantum physics and the geometry of WCW.

The basic number theoretical structures are complex numbers, quaternions and octonions, and their complexifications obtained by introducing additional commuting imaginary unit $\sqrt{-1}$. Hyper-octonionic (-quaternionic,-complex) sub-spaces for which octonionic imaginary units are multiplied by commuting $\sqrt{-1}$ have naturally Minkowskian signature of metric. The question is whether and how the hyper-structures could allow to understand quantum TGD in terms of classical number fields. The answer which looks the most convincing one relies on the existence of octonionic representation of 8-D gamma matrix algebra.

1. The first guess is that associativity condition for the sub-algebras of the local Clifford algebra defined in this manner could select 4-D surfaces as surface having as tangent spaces associative (co-associative) sub-spaces of this algebra and define WCW purely number theoretically. The associative sub-spaces in question would be spanned by space-time tangent vectors spanning associative (co-associative) sub-algebra of complexified octonions generated by imbedding space tangent vectors. A more concrete representation of vectors of complexified tangent

space as imbedding space gamma matrices is not necessary. One can consider also octonionic representation of imbedding space gamma matrices but whether it has any physical content, remains an open question. The answer to the question whether octonions could correspond to the modified (Kähler Dirac) gamma matrices associated with Kähler-Dirac action turned out to be 'No'.

2. This condition is quite not enough: one must strengthen it with the condition that a preferred commutative and thus hyper-complex sub-algebra is contained in the tangent space of the space-time surface. This condition actually generalizes somewhat since one can introduce a family of so called Hamilton-Jacobi coordinates for M^4 allowing an integrable distribution of decompositions of tangent space to the space of non-physical and physical polarizations [K10] . The physical interpretation is as a number theoretic realization of gauge invariance selecting a preferred local commutative plane of non-physical polarizations.

3. Even this is not yet the whole story: one can define also the notions of co-associativity and co-commutativity applying in the regions of space-time surface with Euclidian signature of the induced metric. The basic unproven conjecture is that the decomposition of space-time surfaces to associative and co-associative regions containing preferred commutative *resp.* co-commutative 2-plane in the 4-D tangent plane is equivalent with the preferred extremal property of Kähler action and the hypothesis that space-time surface allows a slicing by string world sheets and by partonic 2-surfaces [K26] .

3.1.1 Hyper-octonions and hyper-quaternions

The discussions for years ago with Tony Smith [A206] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity. The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

At the end of the chapter it will be found that it might be possible to do without the hyper-variants of classical number fields (not of course number fields!). The idea is obvious already from string model context.

1. For strings in Minkowskian target space the target space coordinates as function of string world sheet coordinates are analytic with respect to hyper-complex coordinate. Quantum theory is however constructed by performing first a Wick rotation to Euclidian target space, calculating the n-point functions using ordinary Euclidian theory, and performing the reverse of Wick rotation.

2. One could generalize the procedure in TGD framework so that octonionic variant of conformal field theory results by algebraic continuation from complex number field to octonionic realm. Octonionic real-analytic functions $f(o)$ are expressible as $f(o) = q_1 + Iq_2$, where q_i are quaternion valued functions and I is octonionic imaginary unit anti-commuting with quaternionic imaginary units. They map the Euclidian variant of $H = M^4 \times CP_2$ to itself. Space-time surfaces can be identified as quaternionic (co-quaternionic) 4-surfaces defined as surfaces for which the imaginary (real) part of an octonion real-analytic function vanishes.The

reversal of Wick rotation maps these Euclidian surfaces to space-time surfaces. One could also see the this process as a complexification in of octonions in which real-analytic functions of complexified octonions are restricted to octonionic and hyper-octonionic sectors. Therefore the two views should be more or less equivalent.

Note that hyper-variants of number fields make also sense p-adically unlike the notions of number fields themselves unless restricted to be algebraic extensions of rational variants of number fields. What deserves separate emphasis is that the basic structure of the standard model would reduce to number theory.

3.1.2 Number theoretical compactification and $M^8 - H$ duality

The notions of associative and hyper-octonionic manifold make sense and one could endow the tangent space of $H = M^4 \times CP_2$ with hyper-octonionic manifold structure. Situation becomes very simple if H is replaced with hyper-octonionic M^8. Suppose that $X^4 \subset M^8$ consists of associative and co-associative regions. The basic observation is that the associative sub-spaces of M^8 with a fixed hyper-complex structure (containing in their tangent space a fixed hyper-complex subspace M^2 or at least one of the light-like lines of M^2) are labeled by points of CP_2. Hence each associative and co-associative four-surface of M^8 defines a 4-surface of $M^4 \times CP_2$. One can loosely say that the number-theoretic analog of spontaneous compactification occurs: this of course has nothing to do with dynamics.

This picture was still too naive and it became clear that not all known extremals of Kähler action contain fixed $M^2 \subset M^4$ or light-like line of M^2 in their tangent space.

1. The first option represents the minimal form of number theoretical compactification. M^8 is interpreted as the tangent space of H. Only the 4-D tangent spaces of light-like 3-surfaces X_l^3 (wormhole throats or boundaries) are assumed to be associative or co-associative and contain fixed M^2 or its light-like line in their tangent space. Hyper-quaternionic regions would naturally correspond to space-time regions with Minkowskian signature of the induced metric and their co-counterparts to the regions for which the signature is Euclidian. What is of special importance is that this assumption solves the problem of identifying the boundary conditions fixing the preferred extremals of Kähler action since in the generic case the intersection of M^2 with the 3-D tangent space of X_l^3 is 1-dimensional. The surfaces $X^4(X_l^3) \subset M^8$ would be associative or co-associative but would not allow a local mapping between the 4-surfaces of M^8 and H.

2. One can also consider a more local map of $X^4(X_l^3) \subset H$ to $X^4(X_l^3) \subset M^8$. The idea is to allow $M^2 \subset M^4 \subset M^8$ to vary from point to point so that $S^2 = SO(3)/SO(2)$ characterizes the local choice of M^2 in the interior of X^4. This leads to a quite nice view about strong geometric form of $M^8 - H$ duality in which M^8 is interpreted as tangent space of H and $X^4(X_l^3) \subset M^8$ has interpretation as tangent for a curve defined by light-like 3-surfaces at X_l^3 and represented by $X^4(X_l^3) \subset H$. Space-time surfaces $X^4(X_l^3) \subset M^8$ consisting of associative and co-associative regions would naturally represent a preferred extremal of E^4 Kähler action. The value of the action would be same as CP_2 Kähler action. $M^8 - H$ duality would apply also at the induced spinor field and at the level of configuration space.

3. Strong form of $M^8 - H$ duality satisfies all the needed constraints if it represents Kähler isometry between $X^4(X_l^3) \subset M^8$ and $X^4(X_l^3) \subset H$. This implies that light-like 3-surface is mapped to light-like 3-surface and induced metrics and Kähler forms are identical so that also Kähler action and field equations are identical. The only differences appear at the level of induced spinor fields at the light-like boundaries since due to the fact that gauge potentials are not identical.

4. The map of $X_l^3 \subset H \to X_l^3 \subset M^8$ would be crucial for the realization of the number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as those preferred coordinates in which the points of imbedding space are rational/algebraic. Thus the point of $X^4 \subset H$ is algebraic if it is mapped to algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not

have practical meaning. Number theoretical compactication could thus be motivated by the number theoretical universality.

5. The possibility to use either M^8 or H picture might be extremely useful for calculational purposes. In particular, M^8 picture based on $SO(4)$ gluons rather than $SU(3)$ gluons could perturbative description of low energy hadron physics. The strong $SO(4)$ symmetry of low energy hadron physics can be indeed seen direct experimental support for the M^8-H duality.

One can imagine an interesting generalization of the M^8-H duality to $H-H$ duality. One can assign to an associative (co-associative) 4-surface of H a surface of H by the same rule as in the case of M^8-H duality. If the outcome is also associative (co-associative) surface one can iterate this map and get infinite number of associative (co-associative) surfaces serving as candidates for preferred extremals and obviously forming a category.

3.1.3 Romantic stuff

Octonions and quaternions have generated a lot of romantic speculations and my only defence is that I did not know! Combined with free speculation about dualities this generated a lot of non-sense which has been dropped from this version of the chapter.

1. A long standing romantic speculation was that conformal invariance could somehow extend to 4-D context. Conformal invariance indeed extends to 3-D situation in the case of light-like 3-surfaces and they indeed are the basic dynamical objects of quantum TGD. It seems however un-necessary to extend the conformal invariance to 4-D context except by slicing $X^4(X_l^3)$ by 3-D light-like slices possessing the 3-D conformal invariance.

2. The triality between 8-D spinors, their conjugates, and vectors has generated a lot of speculative literature and this triality is indeed important in super string models. If M^8 has hyper-octonionic structure, one can ask whether also the spinors of M^8 could be regarded as complexified octonions. Complexified octonions provide also a representation of 8-D gamma matrices which is not a matrix representation. In this framework the Clifford algebra defined by gamma matrices degenerates to algebra of complexified octonions identifiable as the algebra of octonionic spinors and coordinates of M_c^8. One can make all kinds of questions. For instance, could it be that hyper-octonionic triality for hyper-octonionic spinor fields could allow construction of N-point functions in interaction vertices? One cannot exclude the possibility that trialities are important but the recent formulation of M-matrix elements does quite well without them.

3. The $1+\overline{1}+3+\overline{3}$ decomposition of complexified octonion units with respect to group $SU(3)\subset G_2$ acting as automorphisms of octonions inspired the idea that hyper-octonion spinor field could represent leptons, antileptons, quarks and antiquarks. This proposal is problematic. Hyper-octonionic coordinates would carry color and generic hyper-octonionic spinor is superposition of spinor components which correspond to quarks, leptons and and their anti-fermions and a lot of super-selection rules would be needed. The motivations behind these speculations was that in H picture color would correspond to CP_2 partial waves and spin and ew quantum numbers to spin like quantum numbers whereas in M^8 picture color would correspond to spin like quantum number and spin and electro-weak quantum numbers to E^4 partial waves.

3.1.4 About literature

The reader not familiar with the basic algebra of quaternions and octonions is encouraged to study some background material: the home page of Tony Smith provides among other things an excellent introduction to quaternions and octonions [A206] . String model builders are beginning to grasp the potential importance of octonions and quaternions and the articles about possible applications of octonions [A131, A203, A163] provide an introduction to octonions using the language of physicist.

Personally I found quite frustrating to realize that I had neglected totally learning of the basic ideas of algebraic geometry, despite its obvious potential importance for TGD and its applications in string models. This kind of losses are the price one must pay for working outside the scientific community. It is not easy for a physicist to find readable texts about algebraic geometry and algebraic number theory from the bookshelves of mathematical libraries. The book "Algebraic Geometry for Scientists and Engineers" by Abhyankar [A112] , which is not so elementary as the name would suggest, introduces in enjoyable manner the basic concepts of algebraic geometry and binds the basic ideas with the more recent developments in the field. "Problems in Algebraic Number Theory" by Esmonde and Murty [A140] in turn teaches algebraic number theory through exercises which concretize the abstract ideas. The book "Invitation to Algebraic Geometry" by K. E. Smith. L. Kahanpää, P. Kekäläinen and W. Traves is perhaps the easiest and most enjoyable introduction to the topic for a novice. It also contains references to the latest physics inspired work in the field.

3.1.5 Notations

Some notational conventions are in order before continuing. The fields of quaternions *resp.* octonions having dimension 4 *resp.* 8 and will be denoted by Q and O. Their complexified variants will be denoted by Q_C and O_C. The sub-spaces of hyper-quaternions HQ and hyper-octonions HO are obtained by multiplying the quaternionic and octonionic imaginary units by $\sqrt{-1}$. These sub-spaces are very intimately related with the corresponding algebras, and can be seen as Euclidian and Minkowkian variants of the same basic structure. Also the Abelianized versions of the associative and -octonionic sub-spaces can be consideredthese algebras have a representation in the space of spinors of imbedding space $H = M^4 \times CP_2$.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- TGD and classical number fields [L32]
- $M^8 - H$ duality [L22]
- Basic notions behind $M^8 - H$ duality [L20]
- Quaternionic planes of octonions [L31]

3.2 Quaternion and octonion structures and their hyper counterparts

In this introductory section the notions of quaternion and octonion structures and their hyper counterparts are introduced with strong emphasis on the physical interpretation. Literature contains several variants of these structures (Hyper Kähler structure and quaternion Kähler structure possed also by CP_2 [A120]). The notion introduced here is inspired by the physical motivations coming from TGD. As usual the first proposal based on the notions of (hyper-)quaternion and (hyper-)octonion analyticity was not the correct one. Much later a local variant of the notion based on tangent space emerged.

3.2.1 Octonions and quaternions

In the following only the basic definitions relating to octonions and quaterions are given. There is an excellent article by John Baez [A117] describing octonions and their relations to the rest of mathematics and physics.

Octonions can be expressed as real linear combinations $\sum_k x^k I_k$ of the octonionic real unit $I_0 = 1$ (counterpart of the unit matrix) and imaginary units I_a, $a = 1, ..., 7$ satisfying

$$
\begin{aligned}
I_0^2 &= I_0 \equiv 1 \ , \\
I_a^2 &= -I_0 = -1 \ , \\
I_0 I_a &= I_a \ .
\end{aligned}
\tag{3.2.1}
$$

Octonions are closed with respect to the ordinary sum of the 8-dimensional vector space and with respect to the octonionic multiplication, which is neither commutative ($ab \neq ba$ in general) nor associative ($a(bc) \neq (ab)c$ in general).

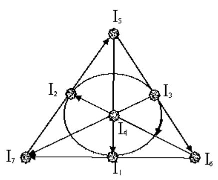

Figure 3.1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

A concise manner to summarize octonionic multiplication is by using octonionic triangle. Each line (6 altogether) containing 3 octonionic imaginary units forms an associative triple which together with $I_0 = 1$ generate a division algebra of quaternions. Also the circle spanned by the 3 imaginary units at the middle of the sides of the triangle is associative triple. The multiplication rules for each associative triple are simple:

$$
I_a I_b = \epsilon_{abc} I_c \ , \tag{3.2.2}
$$

where ϵ_{abc} is 3-dimensional permutation symbol. $\epsilon_{abc} = 1$ for the clockwise sequence of vertices (the direction of the arrow along the circumference of the triangle and circle). As a special case this rule gives the multiplication table of quaternions. A crucial observation for what follows is that any pair of imaginary units belongs to one associative triple.

The non-vanishing structure constants $d_{ab}{}^{c}$ of the octonionic algebra can be read directly from the octonionic triangle. For a given pair I_a, I_b one has

$$
\begin{aligned}
I_aI_b &= d_{ab}{}^cI_c \ , \\
d_{ab} \quad c &= \epsilon_{ab}{}^c \ , \\
I_a^2 &= d_{aa}{}^0I_0 = -I_0 \ , \\
I_0^2 &= d_{00}{}^0I_0 \ , \\
I_0I_a &= d_{0a}{}^aI_a = I_a \ .
\end{aligned} \tag{3.2.3}
$$

For ϵ_{abc} c belongs to the same associative triple as ab.

Non-associativity means that is not possible to represent octonions as matrices since matrix product is associative. Quaternions can be represented and the structure constants provide the defining representation as $I_a \rightarrow d_{abc}$, where b and c are regarded as matrix indices of 4×4 matrix. The algebra automorphisms of octonions form 14-dimensional group G_2, one of the so called exceptional Lie-groups. The isotropy group of imaginary octonion unit is the group $SU(3)$. The Euclidian inner product of the two octonions is defined as the real part of the product $\overline{x}y$

$$
\begin{aligned}
(x,y) &= Re(\overline{x}y) = \sum_{k=0,..7} x_ky_k \ , \\
\overline{x} &= x^0I_0 - \sum_{i=1,..,7} x^kI_k \ ,
\end{aligned} \tag{3.2.4}
$$

and is just the Euclidian norm of the 8-dimensional space.

3.2.2 Hyper-octonions and hyper-quaternions

The Euclidicity of the quaternion norm suggests that octonions are not a sensible concept in TGD context. One can imagine two manners to circumvent this conclusion.

1. Minkowskian metric for octonions and quaternions is obtained by identifying Minkowski inner product xy as the real counterpart of the product

   $$
   x \cdot y \equiv Re(xy) = x^0y^0 - \sum_k x^ky^k \ . \tag{3.2.5}
   $$

   $SO(1,7)$ ($SO(1,3)$ in quaternionic case) Lorentz invariance appears completely naturally as the symmetry of the real part of the octonion (quaternion) product and hence of octonions/quaternions and there is no need to perform the complexification of the octonion algebra. Furthermore, only the signature $(1,7)$ ($(1,3)$ in the quaternionic case) is possible and this would raise $M^4_+ \times CP_2$ in a preferred position.

   This norm does not give rise to a number theoretic norm defining a homomorphism to real numbers. Indeed, the number theoretic norm defined by the determinant of the linear mapping defined by the multiplication with quaternion or octonion, is inherently Euclidian. This is in conflict with the idea that quaternionic and octonionic primes and their infinite variants should have key role in TGD [K70] .

2. Hyper-octonions and hyper-quaternions provide a possible solution to these problems. These are obtained by multiplying imaginary units by commutative and associative $\sqrt{-1}$. These numbers form a sub-space of complexified octonions/quaternions and the cross product of imaginary parts leads out from this sub-space. In this case number theoretic norm induced from Q_C/O_C gives the fourth/eighth power of Minkowski length and Lorentz group acts as its symmetries. Light-like hyper-quaternions and -octonions causing the failure of the number field property have also a clear physical interpretation.

   A criticism against the notion of hyper-quaternionic and octonionic primeness is that the tangent space as an algebra property is lost and the notion of primeness is inherited from Q_C/O_C. Also non-commutativity and non-associativity could cause difficulties.

Zero energy ontology leads to a possible physical interpretation of complexified octonions. The moduli space for causal diamonds corresponds to a Cartesian product of $M^4 \times CP_2$ whose points label the position of either tip of $CD \times CP_2$ and space I whose points label the relative positive of the second tip with respect to the first one. p-Adic length scale hypothesis results if one assumes that the proper time distance between the tips comes in powers of two so that one has union of hyperboloids $H_n \times CP_2$, $H_n = \{m \in M^4_+ | a = 2^n a_0)\}$. A further quantization of hyperboloids H_n is obtained by replacing it with a lattice like structure is highly suggestive and would correspond to an orbit of a point of H_n under a subgroup of $SL(2, Q_C)$ or $SL(2, Z_C)$ acting as Lorentz transformations in standard manner. Also algebraic extensions of Q_C and Z_C can be considered. Also in the case of CP_2 discretization is highly suggestive so that one would have an orbit of a point of CP_2 under a discrete subgroup of $SU(3, Q)$.

The outcome could be interpreted by saying that the moduli space in question is $H \times I$ such that H corresponds to hyper-octonions and I to a discretized version of $\sqrt{-1}H$ and thus a subspace of complexified octonions. An open question whether the quantization has some deeper mathematical meaning.

3.2.3 Basic constraints

Before going to details it is useful to make clear the constraints on the concept of the hyper-octonionic structure implied by TGD view about physics.

$M^4 \times CP_2$ cannot certainly be regarded as having any global octonionic structure (for instance in the sense that it could be regarded as a coset space associated with some exceptional group). There are however clear indications for the importance of the hyper-quaternionic and -octonionic structures.

1. $SU(3)$ is the only simple 8-dimensional Lie-group and acts as the group of isometries of CP_2: if $SU(3)$ had some kind of octonionic structure, CP_2 would become unique candidate for the space S. The decomposition $SU(3) = h + t$ to $U(2)$ subalgebra and its complement corresponds rather closely to the decomposition of (hyper-)octonions to (hyper-)quaternionic sub-space and its complement. The electro-weak $U(2)$ algebra has a natural 1+3 decomposition and generators allow natural hyper-quaternionic structure. Hyper Kähler structure with three covariantly constant quaternionic imaginary units represented Kähler forms is however not possible. The components of the Weyl tensor of CP_2 behave with respect to multiplication like quaternionic imaginary units but only one of them is covariantly constant so that hyper-Kähler structure is not possible. These tensors and metric tensor however define quaternionic structure.

2. M^4_+ has a natural 1+3 decomposition and a unique cosmic time coordinate defined as the light cone proper time. Hyper-quaternionic structure is consistent with the Minkowskian signature of the inner product and hyper quaternion units have a natural representation in terms of covariantly constant self-dual symplectic forms and their contractions with sigma matrices. It is not however clear whether this representation is physically intereting.

3.2.4 How to define hyper-quaternionic and hyper-octonionic structures?

I have considered several proposals for how to define quaternionic and octonionic structures and their hyper-counterparts.

1. (Hyper-)octonionic manifolds would obtained by gluing together coordinate patches using (hyper-)octonion analytic functions with real Laurent coefficients (this guarantees associativity and commutativity). This definition does not yet involve metric or any other structures (such as Kähler structure). This approach does not seem to be physically realistic.

2. Second option is based on the idea of representing quaternionic and octonionic imaginary units as antisymmetric tensors. This option makes sense for quaternionic manifolds and CP_2 indeed represents an example represents of this kind of manifold. The problem with the octonionic structure is that antisymmetric tensors cannot define non-associative product.

3. If the manifold is endowed with metric, octonionic structure should be defined as a local tangent space structure analogous to eight-bein structure and local gauge algebra structures. This can be achieved by contracting octo-bein vectors with the standard octonionic basis to get octonion form I_k. Each vector field a^k defines naturally octonion field $A = a^k I_k$. The product of two vector fields can be defined by the octonionic multiplication and this leads to the introduction of a tensor field d_{klm} of these structure constants obtained as the contraction of the octobein vectors with the octonionic structure constants d_{abc}. Hyper-octonion structure can defined in a completely analogous manner.

   It is possible to induce octonionic structure to any 4-dimensional space-time surface by forming the projection of I_k to the space-time surface and redefining the products of I_k:s by dropping away that part of the product, which is orthogonal to the space-time surface. This means that the structure constants of the new 4-dimensional algebra are the projections of d_{klm} to the space-time surface. One can also define similar induced algebra in the 4-dimensional normal space of the space-time surface. The hypothesis would be that the induced tangential is associative or hyper-quaternionic algebra. Also co-associativity defined as associativity of the normal space algebra is possible. This property would give for the 4-dimensionality of the space-time surface quite special algebraic meaning. The problem is now that there is no direct connection with quantum TGD proper- in particular the connection with the classical dynamics defined by Kähler action is lacking.

4. 8-dimensional gamma matrices allow a representation in terms of tensor products of octonions and 2×2 matrices. Genuine matrices are of course not in question since the product of the gamma matrices fails to be associative. An associative representation is obtained by restricting the matrices to a quaternionic plane of complex octonions. If the space-time surface is hyper-quaternionic in the sense that induced gamma matrices define a quaternionic plane of complexified octonions at each point of space-time surface the resulting local Clifford algebra is associative and structure constants define a matrix representation for the induced gamma matrices.

   A more general definition allows gamma matrices to be modified gamma matrices defined by Kähler action appearing in the modified Dirac action and forced both by internal consistency and super-conformal symmetry [K15, K26] . The modified gamma matrices associated with Kähler action do not in general define tangent space of the space-time surface as the induced gamma matrices do. Also co-associativity can be considered if one can identify a preferred imaginary unit such that the multiplication of the modified gamma matrices with this unit gives a quaternionic basis. This condition makes sense only if the preferred extremals of the action are hyper-quaternionic surfaces in the sense defined by the action. That this is true for Kähler action at least is an is an unproven conjecture.

In the sequel only the fourh option will be considered.

3.2.5 How to end up to quantum TGD from number theory?

An interesting possibility is that quantum TGD could emerge from a condition that a local version of hyper-finite factor of type II_1 represented as a local version of infinite-dimensional Clifford algebra exists. The conditions are that "center or mass" degrees of freedom characterizing the position of CD separate uniquely from the "vibrational" degrees of freedom being represented in terms of octonions and that for physical states associativity holds true. The resulting local Clifford algebra would be identifiable as the local Clifford algebra of WCW (being an analog of local gauge groups and conformal fields).

The uniqueness of M^8 and $M^4 \times CP_2$ as well as the role of hyper-quaternionic space-time surfaces as fundamental dynamical objects indeed follow from rather weak conditions if one restricts the consideration to gamma matrices and spinors instead of assuming that M^8 coordinates are hyper-octonionic as was done in the first attempts.

1. The unique feature of M^8 and any 8-dimensional space with Minkowski signature of metric is that it is possible to have an octonionic representation of the complexified gamma matrices [K15, K19] and of spinors. This does not require octonionic coordinates for M^8.

The restriction to a quaternionic plane for both gamma matrices and spinors guarantees the associativity.

2. One can also consider a local variant of the octonionic Clifford algebra in M^8. This algebra contains associative subalgebras for which one can assign to each point of M^8 a hyper-quaternionic plane. It is natural to assume that this plane is either a tangent plane of 4-D manifold defined naturally by the induced gamma matrices defining a basis of tangent space or more generally, by modified gamma matrices defined by a variational principle (these gamma matrices do not define tangent space in general). Kähler action defines a unique candidate for the variational principle in question. Associativity condition would automatically select sub-algebras associated with 4-D hyper-quaternionic space-time surfaces.

3. This vision bears a very concrete connection to quantum TGD. In [K19] the octonionic formulation of the modified Dirac equation is studied and shown to lead to a highly unique general solution ansatz for the equation working also for the matrix representation of the Clifford algebra. An open question is whether the resulting solution as such defined also solutions of the modified Dirac equation for the matrix representation of gammas. Also a possible identification for 8-dimensional counterparts of twistors as octo-twistors follows: associativity implies that these twistors are very closely related to the ordinary twistors. In TGD framework octo-twistors provide an attractive manner to get rid of the difficulties posed by massive particles for the ordinary twistor formalism.

4. Associativity implies hyperquaternionic space-time surfaces (in a more general sense as usual) and this leads naturally to the notion of WCW and local Clifford algebra in this space. Number theoretic arguments imply $M^8 - H$ duality. The resulting infinite-dimensional Clifford algebra would differ from von Neumann algebras in that the Clifford algebra and spinors assignable to the center of mass degrees of freedom of causal diamond CD would be expressed in terms of octonionic units although they are associative at space-time surfaces. One can therefore say that quantum TGD follows by assuming that the tangent space of the imbedding space corresponds to a classical number field with maximal dimension.

5. The slicing of the Minkowskian space-time surface inside CD by stringy world sheets and by partonic 2-surfaces inspires the question whether the modified gamma matrices associated with the stringy world sheets *resp.* partonic 2-surfaces could be could commutative *resp.* co-commutative. Commutativity would also be seen as the justification for why the fundamental objects are effectively 2-dimensional.

This formulation is undeniably the most convincing one found hitherto since the notion of hyper-quaternionic structure is local and has elegant formulation in terms of modified gamma matrices.

3.2.6 p-Adic length scale hypothesis and quaternionic and hyper-quaternionic primes

p-Adic length scale hypothesis [K51] states that fundamental length scales correspond to the p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives a partial theoretical justification for this hypothesis [K51] . A strong empirical support for the hypothesis comes from p-adic mass calculations [K39, K48, K49, K43] .

Elementary particles correspond to the so called CP_2 type extremals in TGD Universe [K10, K51] . Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [A206] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one could identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime, with integer value E^4 coordinates. The worries are of course raised by the Euclidian signature of the number theoretical norm of quaternions.

Hyper-quaternionic and -octonionic primes and effective 2-dimensionality

The notion of prime generalizes to hyper-quaternionic and -octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic primes can be represented as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. The interpretation of hyper-quaternionic primes (or integers) as four-momenta suggests itself. Note that it is not possible to find a rest system for a massive particle unless the energy is allowed to be a square root of integer.

The notion of "irreducible" (see Appendix of [K71]) is defined as the equivalence class of primes related by a multiplication with a unit (integer with unit norm) and is more fundamental than that of prime. All Lorentz boosts of a hyper prime obtained by multiplication with units labeling $SO(D-1)$ cosets of $SO(D-1,1)$, $D = 4, 8$ to a hyper prime, combine to form a hyper irreducible. Note that the units cannot correspond to real particles in the arithmetic quantum field theory in which primes correspond to D-momenta labeling the physical states.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the hyper-complex case. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality.

Hyper-complex numbers H_2 define the maximal sub-algebra of HQ and HO. In the case of H_2 the failure of the number field property is due to the existence of light-like hyper-complex numbers with vanishing norm. The light-likeness of hyper-quaternions and -octonions is expected to have a deep physical significance and could define a number theoretic analog of propagator pole and light-like 3-D and 7-D causal determinants.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes. Note that the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

The situation becomes more complex if also space-like hyper primes with negative norm squared $n_0^2 - n_1^2 - ... = -p$ are allowed. Gaussian primes with $p \ mod 4 \ = 1$ would be representable as primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$. If all quaternionic primes allow a representation as a quaternionic integer with three non-vanishing components, they can be identified as space-like hyper-quaternionic primes. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers. By their tachyonity space-like primes are not physically favored.

Hyper-quaternionic hyperboloids and p-adic length scale hypothesis

In the hyper-quaternionic case the 3-dimensional sphere $R^2 = p$ is replaced with Lobatchevski space (hyperboloid of M^4 with points having integer valued M^4 coordinates. Hence integer valued hyper-quaternions allow interpretation as quantized four-momenta.

Prime mass hyperboloids correspond to $n = p$. It is not possible to multiply hyperboloids since the cross product leads out of hyper sub-space. It is however possible to multiply the 2-dimensional hyperboloids and act on these by units to get full 3-D hyperboloids. The powers of hyperboloid p correspond to a multiplicatively closed structure consisting of powers p^n of the hyperboloid p. At space-time level the hyper-quaternionic lattice gives rise to a one-dimensional lattices of hyperboloidal lattices labeled by powers p^n, and the values of light-cone proper time $a \propto \sqrt{p}$ are expected to define fundamental p-adic time scales.

Also the space-like hyperboloids $R^2 = -n$ are possible and the notion of primeness makes sense also in this case. The space-like hyperboloids define one-dimensional lattice of space-like hyper-quaternionic lattices and an explanation for the spatial variant of the p-adic length scale hypothesis stating that p-adic length scales are proportional to $\sqrt{p}$ emerges in this manner naturally.

Euclidian version of the p-adic length scale hypothesis

Hyper-octonionic integers have a decomposition into hyper-quaternion and a product of $\sqrt{-1}K$ with quaternion so that quaternionic primes can be identified as hyper-octonionic space-like primes. The Euclidian version of the p-adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k:th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface.

This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the absolute minimization of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

3.3 Quantum TGD in nutshell

This section provides a summary about quantum TGD, which is essential for understanding the recent developments related to $M^8 - H$ duality. The discussions are based on the general vision that quantum states of the Universe correspond to the modes of classical spinor fields in the "world of the classical worlds" identified as the infinite-dimensional WCW of light-like 3-surfaces of $H = M^4 \times CP_2$ (more or less-equivalently, the corresponding 4-surfaces defining generalized Bohr orbits).

3.3.1 Geometric ideas

TGD relies heavily on geometric ideas, which have gradually generalized during the years. Symmetries play a key role as one might expect on basis of general definition of geometry as a structure characterized by a given symmetry.

Physics as infinite-dimensional Kähler geometry

1. The basic idea is that it is possible to reduce quantum theory to WCW geometry and spinor structure. The geometrization of loop spaces inspires the idea that the mere existence of Riemann connection fixes WCW Kähler geometry uniquely. Accordingly, WCW can be regarded as a union of infinite-dimensional symmetric spaces labeled by zero modes labeling classical non-quantum fluctuating degrees of freedom.

The huge symmetries of WCW geometry deriving from the light-likeness of 3-surfaces and from the special conformal properties of the boundary of 4-D light-cone would guarantee the maximal isometry group necessary for the symmetric space property. Quantum criticality is the fundamental hypothesis allowing to fix the Kähler function and thus dynamics of TGD uniquely. Quantum criticality leads to surprisingly strong predictions about the evolution of coupling constants.

2. WCW spinors correspond to Fock states and anti-commutation relations for fermionic oscillator operators correspond to anti-commutation relations for the gamma matrices of the WCW. WCW gamma matrices contracted with Killing vector fields give rise to a super-algebra which together with Hamiltonians of WCW forms what I have used to called super-symplectic algebra.

   Super-symplectic degrees of freedom represent completely new degrees of freedom and have no electroweak couplings. In the case of hadrons super-symplectic quanta correspond to what has been identified as non-perturbative sector of QCD they define TGD correlate for the degrees of freedom assignable to hadronic strings. They are responsible for the most of the mass of hadron and resolve spin puzzle of proton.

   Besides super-symplectic symmetries there are Super-Kac Moody symmetries assignable to light-like 3-surfaces and together these algebras extend the conformal symmetries of string models to dynamical conformal symmetries instead of mere gauge symmetries. The construction of the representations of these symmetries is one of the main challenges of quantum TGD. The assumption that the commutator algebra of these super-symplectic and super Kac-Moody algebras annihilates physical states gives rise to Super Virasoro conditions which could be regarded as analogs of WCW Dirac equation.

   Modular invariance is one aspect of conformal symmetries and plays a key role in the understanding of elementary particle vacuum functionals and the description of family replication phenomenon in terms of the topology of partonic 2-surfaces.

3. WCW spinors define a von Neumann algebra known as hyper-finite factor of type II_1 (HFFs). This realization has led also to a profound generalization of quantum TGD through a generalization of the notion of imbedding space to characterize quantum criticality. The resulting space has a book like structure with various almost-copies of imbedding space representing the pages of the book meeting at quantum critical sub-manifolds. The outcome of this approach is that the exponents of Kähler function and Chern-Simons action are not fundamental objects but reduce to the Dirac determinant associated with the modified Dirac operator assigned to the light-like 3-surfaces.

p-Adic physics as physics of cognition and intentionality

p-Adic mass calculations relying on p-adic length scale hypothesis led to an understanding of elementary particle masses using only super-conformal symmetries and p-adic thermodynamics. The need to fuse real physics and various p-adic physics to single coherent whole led to a generalization of the notion of number obtained by gluing together reals and p-adics together along common rationals and algebraics (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also). The interpretation of p-adic space-time sheets is as correlates for cognition and intentionality. p-Adic and real space-time sheets intersect along common rationals and algebraics and the subset of these points defines what I call number theoretic braid in terms of which both WCW geometry and S-matrix elements should be expressible. Thus one would obtain number theoretical discretization which involves no ad hoc elements and is inherent to the physics of TGD.

Perhaps the most dramatic implication relates to the fact that points, which are p-adically infinitesimally close to each other, are infinitely distant in the real sense (recall that real and p-adic imbedding spaces are glued together along rational imbedding space points). This means that any open set of p-adic space-time sheet is discrete and of infinite extension in the real sense. This means that cognition is a cosmic phenomenon and involves always discretization from the point of view of the real topology. The testable physical implication of effective p-adic topology of real space-time sheets is p-adic fractality meaning characteristic long range correlations combined with short range chaos.

Also a given real space-time sheets should correspond to a well-defined prime or possibly several of them. The classical non-determinism of Kähler action should correspond to p-adic non-determinism for some prime(s) p in the sense that the effective topology of the real space-time sheet is p-adic in some length scale range. p-Adic space-time sheets with same prime should have many common rational points with the real space-time and be easily transformable to the real space-time sheet in quantum jump representing intention-to-action transformation. The concrete model for the transformation of intention to action leads to a series of highly non-trivial number theoretical conjectures assuming that the extensions of p-adics involved are finite-dimensional and can contain also transcendentals.

An ideal realization of the space-time sheet as a cognitive representation results if the CP_2 coordinates as functions of M^4_+ coordinates have the same functional form for reals and various p-adic number fields and that these surfaces have discrete subset of rational numbers with upper and lower length scale cutoffs as common. The hierarchical structure of cognition inspires the idea that S-matrices form a hierarchy labeled by primes p and the dimensions of algebraic extensions.

The number-theoretic hierarchy of extensions of rationals appears also at the level of WCW spinor fields and allows to replace the notion of entanglement entropy based on Shannon entropy with its number theoretic counterpart having also negative values in which case one can speak about genuine information. In this case case entanglement is stable against Negentropy Maximization Principle stating that entanglement entropy is minimized in the self measurement and can be regarded as bound state entanglement. Bound state entanglement makes possible macro-temporal quantum coherence. One can say that rationals and their finite-dimensional extensions define islands of order in the chaos of continua and that life and intelligence correspond to these islands.

TGD inspired theory of consciousness and number theoretic considerations inspired for years ago the notion of infinite primes [K70] . It came as a surprise, that this notion might have direct relevance for the understanding of mathematical cognition. The ideas is very simple. There is infinite hierarchy of infinite rationals having real norm one but different but finite p-adic norms. Thus single real number (complex number, (hyper-)quaternion, (hyper-)octonion) corresponds to an algebraically infinite-dimensional space of numbers equivalent in the sense of real topology. Space-time and imbedding space points ((hyper-)quaternions, (hyper-)octonions) become infinitely structured and single space-time point would represent the Platonia of mathematical ideas. This structure would be completely invisible at the level of real physics but would be crucial for mathematical cognition and explain why we are able to imagine also those mathematical structures which do not exist physically. Space-time could be also regarded as an algebraic hologram. The connection with Brahman=Atman idea is also obvious.

Hierarchy of Planck constants and dark matter hierarchy

The work with hyper-finite factors of type II_1 (HFFs) combined with experimental input led to the notion of hierarchy of Planck constants interpreted in terms of dark matter [K25] . The hierarchy is realized via a generalization of the notion of imbedding space obtained by gluing infinite number of its variants along common lower-dimensional quantum critical sub-manifolds. These variants of imbedding space are characterized by discrete subgroups of SU(2) acting in M^4 and CP_2 degrees of freedom as either symmetry groups or homotopy groups of covering. Among other things this picture implies a general model of fractional quantum Hall effect.

What is especially remarkable is that the construction gives also the 4-D space-time sheets associated with the light-like orbits of the partonic 2-surfaces: it remains to be shown whether they correspond to preferred extremals of Kähler action. It is clear that the hierarchy of Planck constants has become an essential part of the construction of quantum TGD and of mathematical realization of the notion of quantum criticality rather than a possible generalization of TGD.

Number theoretical symmetries

TGD as a generalized number theory vision leads to the idea that also number theoretical symmetries are important for physics.

1. There are good reasons to believe that the strands of number theoretical braids can be assigned with the roots of a polynomial with suggests the interpretation corresponding Galois groups as purely number theoretical symmetries of quantum TGD. Galois groups are subgroups of the permutation group S_∞ of infinitely manner objects acting as the Galois group of algebraic numbers. The group algebra of S_∞ is HFF which can be mapped to the HFF defined by WCW spinor s. This picture suggest a number theoretical gauge invariance stating that S_∞ acts as a gauge group of the theory and that global gauge transformations in its completion correspond to the elements of finite Galois groups represented as diagonal groups of $G \times G \times ....$ of the completion of S_∞. The groups G should relate closely to finite groups defining inclusions of HFFs.

2. HFFs inspire also an idea about how entire TGD emerges from classical number fields, actually their complexifications. In particular, SU(3) acts as subgroup of octonion automorphisms leaving invariant preferred imaginary unit and $M^4 \times CP_2$ can be interpreted as a structure related to hyper-octonions which is a subspace of complexified octonions for which metric has naturally Minkowski signature. This would mean that TGD could be seen also as a generalized number theory. This conjecture predicts the existence of two dual formulations of TGD based on the identification space-times as 4-surfaces in hyper-octonionic space M^8 *resp.* $M^4 \times CP_2$.

3. The vision about TGD as a generalized number theory involves also the notion of infinite primes. This notion leads to a further generalization of the ideas about geometry: this time the notion of space-time point generalizes so that it has an infinitely complex number theoretical anatomy not visible in real topology.

3.3.2 The notions of imbedding space, 3-surface, and configuration space

The notions of imbedding space, 3-surface (and 4-surface), and WCW (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably.

The notion of imbedding space

Two generalizations of the notion of imbedding space were forced by number theoretical vision [K71, K72, K70] .

1. p-Adicization forced to generalize the notion of imbedding space by gluing real and p-adic variants of imbedding space together along rationals and common algebraic numbers. The generalized imbedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [K15, K19] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized in power-of-two multiples of CP_2 length, p-adic length scale hypothesis [K51] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K25] led to a further generalization of the notion of imbedding space. Generalized imbedding space is obtained by gluing together

Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized imbedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of CP_2. Kähler gauge potential must have what one might call pure gauge parts in M^4 in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in CP_2- play key role in the model of anyons, charge fractionization, and quantum Hall effect [K54] .

The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to equivalence believed to be implied by General Coordinate Invariance. There was a problem related to the realization of equivalence since it was not at all obvious why the preferred extremal (assumed to be absolute minimum) $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed.

3. Rather recently came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

1. The obvious guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This choice has some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

2. Much later number theoretical vision led to the conclusion that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes a connected component of the light-like 3-surfaces X^3_l, contain in their 4-D tangent space

$T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW. This is as it must be since complexification does not make sense in M^2 degrees of freedom.

In number theoretical framework M^2_i has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

3. The next step [K15] was the realization that the construction of WCW geometry in terms of modified Dirac action strengthens the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X^3_l. One could say that at X^3_l situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. The weakest form of number theoretic compactification states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$. A second interesting conjecture is that the hyper-quaternionic surfaces correspond to Kähler calibrations giving rise to absolute minima or maxima of Kähler action for M^8.

The notion of WCW

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the imbedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that WCW (WCW) is a union of WCWs associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

3.3.3 The construction of M-matrix

3.3.4 The construction of M-matrix

The construction of S-matrix involves several ideas that have emerged during last years and involve symmetries in an essential manner.

Zero energy ontology

Zero energy ontology motivated originally by TGD inspired cosmology means that physical states have vanishing conserved net quantum numbers and are decomposable to positive and negative energy parts separated by a temporal distance characterizing the system as a space-time sheet of finite size in time direction. The particle physics interpretation is as initial and final states of a particle reaction. Obviously a profound modification of existing views about realization of symmetries is in question.

S-matrix and density matrix are unified to the notion of M-matrix defining time-like entanglement and expressible as a product of square root of density matrix and of unitary S-matrix. Thermodynamics becomes therefore a part of quantum theory. One must distinguish M-matrix from U-matrix defined between zero energy states and analogous to S-matrix and characterizing the unitary process associated with quantum jump. U-matrix is most naturally related to the description of intentional action since in a well-defined sense it has elements between physical systems corresponding to different number fields.

Quantum TGD as almost topological QFT

Light-likeness of the basic fundamental objects suggests that TGD is almost topological QFT so that the formulation in terms of category theoretical notions is expected to work. The original proposal that Chern-Simons action for light-like 3-surfaces defined by the regions at which the signature of the induced metric changes its sign however failed and one must use Kähler action and corresponding modified Dirac action with measurement term to define the fundamental theory. At the limit when the momenta of particles vanish, the theory reduces to topological QFT defined by Kähler action and corresponding modified Dirac action. The imaginary exponent of the instanton term associated with the induced Kähler form defines the counterpart of Chern-Simons action as a phase of the vacuum functional and contributes also to modified Dirac equation.

M-matrices form in a natural manner a functor from the category of cobordisms to the category of pairs of Hilbert spaces and this gives additional strong constraints on the theory. Super-conformal symmetries implied by the light-likeness pose very strong constraints on both state construction and on M-matrix and U-matrix. The notions of n-category and n-groupoid which represents a generalization of the notion of group could be very relevant to this view about M-matrix.

Quantum measurement theory with finite measurement resolution

The notion of measurement resolution represented in terms of inclusions $\mathcal{N} \subset \mathcal{M}$ of HFFs is an essential element of the picture. Measurement resolution corresponds to the action of the included sub-algebra creating zero energy states in time scales shorter than the cutoff scale. This means that complex rays of state space are effectively replaced with $\mathcal{N}$ rays. The condition that the action of $\mathcal{N}$ commutes with the M-matrix is a powerful symmetry and implies that the time-like entanglement characterized by M-matrix is consistent with Connes tensor product. This does not fix the M-matrix as was the original belief but only realizes mathematically the notion of finite measurement resolution. Together with super-conformal symmetries this constraint should fix possible M-matrices to a very high degree if one assumes the existence of universal M-matrix from which M-matrices with finite measurement resolution are obtained.

The notion of number theoretical braid realizes the notion of finite measurement resolution at space-time level and gives a direct connection to topological QFTs describing braids. The connection with quantum groups is highly suggestive since already the inclusions of HFFs involve these groups. Effective non-commutative geometry for the quantum critical sub-manifolds $M^2 \subset M^4$ and $S^2 \subset CP_2$ might provide an alternative notion for the reduction of stringy anti-commutation relations for induced spinor fields to anti-commutations at the points of braids.

Generalization of Feynman diagrams

An essential difference between TGD and string models is the replacement of stringy diagrams with generalized Feynman diagrams obtained by gluing 3-D light-like surfaces (instead of lines) together at their ends represented as partonic 2-surfaces. This makes the construction of vertices very simple. The notion of number theoretic braid in turn implies discretization having also interpretation in terms of non-commutativity due to finite measurement resolution replacing anti-commutativity along stringy curves with anti-commutativity at points of braids. Braids can replicate at vertices which suggests an interpretation in terms of topological quantum computation combined with non-faithful copying and communication of information. The analogs of stringy diagrams have quite different interpretation in TGD for instance, photons traveling via two different paths in double slit experiment are represented in terms of stringy branching of the photonic 2-surface.

Symplectic variant of QFT as basic building block of construction

The latest discovery related to the construction of M-matrix was the realization that a symplectic variant of conformal field theories might be a further key element in the concrete construction of n-point functions and M-matrix in zero energy ontology. Although I have known super-symplectic (super-symplectic) symmetries to be fundamental symmetries of quantum TGD for almost two decades, I failed for some reason to realize the existence of symplectic QFT, and discovered it while trying to understand quite different problem - the fluctuations of cosmic microwave background! The symplectic contribution to the n-point function satisfies fusion rules and involves only variables which are symplectic invariants constructed using geodesic polygons assignable to the sub-polygons of n-polygon defined by the arguments of n-point function. Fusion rules lead to a concrete recursive formula for n-point functions and M-matrix in contrast to the iterative construction of n-point functions used in perturbative QFT.

Bosonic emergence, extended space-time supersymmetry, and generalized twistors

During year 2009 several new ideas emerged and give hopes about a concrete construction of M-matrix.

1. The notion of bosonic emergence [K53] follows from the fact that gauge bosons are identifiable as pairs of fermion and anti-fermion at opposite light-like throats of wormhole contact. As a consequence, bosonic propagators and vertices are generated radiatively from a fundamental action for fermions and their super partners. At QFT limit without super-symmetry this means that Dirac action coupled to gauge bosons is the fundamental action and the counterpart of YM action is generated radiatively. All coupling constants follow as predictions as they indeed must do on basis of the general structure of quantum TGD.

2. Whether the counterparts of space-time supersymmetries are possible in TGD Universe has remained a long-standing open question and my cautious belief has been that the super partners do not exist. The resolution of the problem came with the increased understanading of the dynamics of the Kähler-Dirac action [K26, K27]. In particular, the localization of the electroweakly charged modes at 2-D surfaces - string world sheets and possibly also partonic 2-surfaces- meant an enormous simplification since the solutions of the Kähler-Dirac equation are conformal spinor modes. Chern-Simons-Dirac term associated with with boundaries of string world sheet allows to construct Dirac propagator and together with its bosonic counterpart it explains the breaking of CP and T symmetries.

The oscillator operators associated with the modes of the induced spinor field satisfy the anti-commutation relations defining the generalization of space-time super-symmetry algebra and these oscillator operators serve as the building blocks of various super-conformal algebras. The number of super-symmetry generators is very large, perhaps even infinite. This forces a generalization of the standard super field concept. The action for chiral super-fields emerges as a generalization of the Dirac action to include all possible super-partners. The huge super-symmetry gives excellent hopes about cancelation of UV divergences. The counterpart of super-symmetric YM action emerges radiatively. This formalism works at the QFT limit. The generalization of the formalism to quantum TGD proper is yet to be carried out.

Covariantly constant right-handed neutrino defines a very special solution of Kähler-Dirac equation in that it is need not be localized to 2-D string world sheets. The reason it that it does not have electroweak couplings. This mode is excellent candidate for defining $\mathcal{N}=2$ SUSY, which however differs from the standard SUSY. SUSY breaking would be simple. p-Adic thermodynamics would predict same mass formulas for particles and their superpartners but the p-adic prime characterizing correspond space-time sheets could be different and thus also the p-adic mass scale.

3. Twistor program has become one of the most promising approaches to gauge theories. This inspired the question whether TGD could allow twistorialization [K79] . Massive states -both real and virtual- are the basic problem of twistor approach. In TGD framework the obvious idea is that massive on mass shell states can be interpreted as massless states in 8-D sense. Massive off-mass shell states in turn could be regarded as pairs of positive and negative on mass shell states. This means opening of the black box of virtual state attempted already in the model for bosonic propagators inspired by the bosonic emergence [K53] , and one can even hope that individual loop integrals are finite and that Wick rotation is not needed. The third observation is that 8-dimensional gamma matrices allow a representation in terms of octonions (matrices are not in question anymore). If the modified gamma "matrices" associated with space-time surface define a quaternionic sub-algebra of the complexified octonion algebra, they allow a matrix representation defined by octonionic structure constants. This holds true for are hyper-quaternionic space-time surfaces so that a connection with number theoretic vision emerges. This would more or less reduce the notion of twistor to its 4-dimensional counterpart.

3.4 Number theoretic compactification and M^8-H duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to associativity or co-associativity. Originally M^8-H duality was introduced as a number theoretic explanation for $H=M^4\times CP_2$. Much later it turned out that the completely exceptional twistorial properties of M^4 and CP_2 are enough to justify $X^4\subset H$ hypothesis. Skeptic could therefore criticize the introduction of M^8 (actually its complexification) as an un-necessary mathematical complication producing only unproven conjectures and bundle of new statements to be formulated precisely. However, if quaternionicity can be realized in terms of M^8_c using O_c-real analytic functions and if quaternionicity is equivalent with preferred extremal property, a huge simplification results and one can say that field equations are exactly solvable.

One can question the feasibility of M^8-H duality if the dynamics is purely number theoretic at the level of M^8 and determined by Kähler action at the level of H. Situation becomes more democratic if Kähler action defines the dynamics in both M^8 and H: this might mean that associativity could imply field equations for preferred extremals or vice versa or there might be equivalence between two. This means the introduction Kähler structure at the level of M^8, and motivates also the coupling of Kähler gauge potential to M^8 spinors characterized by Kähler charge or em charge. One could call this form of duality strong form of M^8-H duality.

The strong form M^8-H duality boils down to the assumption that space-time surfaces can be regarded either as 4-surfaces of H or as surfaces of M^8 or even M^8_c composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric. They have the same induced metric and Kähler form and WCW associated with H should be essentially the same as that associated with M^8. Associativity corresponds to hyper-quaterniocity at the level of tangent space and co-associativity to co-hyper-quaternionicity - that is associativity/hyper-quaternionicity of the normal space. Both are needed to cope with known extremals. Since in Minkowskian context precise language would force to introduce clumsy terms like hyper-quaternionicity and co-hyper-quaternionicity, it is better to speak just about associativity or co-associativity.

Remark: The original assumption was that space-times could be regarded as surfaces in M^8 rather than in its complexification M_c^8 identifiable as complexified octonions. This assumption is un-necessarily strong and if one assumes that octonion-real analytic functions characterize these surfaces M_c^8 must be assumed.

For the octonionic spinor fields the octonionic analogs of electroweak couplings reduce to mere K̈hler or electromagnetic coupling and the solutions reduce to those for spinor d'Alembertian in 4-D harmonic potential breaking $SO(4)$ symmetry. Due to the enhanced symmetry of harmonic oscillator, one expects that partial waves are classified by $SU(4)$ and by reduction to $SU(3)\times U(1)$ by em charge and color quantum numbers just as for CP_2 - at least formally.

Harmonic oscillator potential defined by self-dual em field splits M^8 to $M^4 \times E^4$ and implies Gaussian localization of the spinor modes near origin so that E^4 effectively compactifies. The The resulting physics brings strongly in mind low energy physics, where only electromagnetic interaction is visible directly, and one cannot avoid associations with low energy hadron physics. These are some of the reasons for considering $M^8 - H$ duality as something more than a mere mathematical curiosity.

Remark: The Minkowskian signatures of M^8 and M^4 produce technical nuisance. One could overcome them by Wick rotation, which is however somewhat questionable trick. $M_c^8 = O_c$ provides the proper formulation.

1. The proper formulation is in terms of complexified octonions and quaternions involving the introduction of commuting imaginary unit j. If complexified quaternions are used for H, Minkowskian signature requires the introduction of two commuting imaginary units j and i meaning double complexification.

2. Hyper-quaternions/octonions define as subspace of complexified quaternions/octonions spanned by real unit and jI_k, where I_k are quaternionic units. These spaces are obviously not closed under multiplication. One can however however define the notion of associativity for the subspace of M^8 by requiring that the products and sums of the tangent space vectors generate complexified quaternions.

3. Ordinary quaternions Q are expressible as $q = q_0 + q^k I_k$. Hyper-quaternions are expressible as $q = q_0 + jq^k I_k$ and form a subspace of complexified quaternions $Q_c = Q \oplus jQ$. Similar formula applies to octonions and their hyper counterparts which can be regarded as subspaces of complexified octonions $O \oplus jO$. Tangent space vectors of H correspond hyper-quaternions $q_H = q_0 + jq^k I_k + jiq_2$ defining a subspace of doubly complexified quaternions: note the appearance of two imaginary units.

The recent definitions of associativity and M^8 duality has evolved slowly from in-accurate characterizations and there are still open questions.

1. Kähler form for M^8 non-trivial only in $E^4 \subset M^8$ implies unique decomposition $M^8 = M^4 \times E^4$ needed to define $M^8 - H$ duality uniquely. This applies also to M_c^8. This forces to introduce also Kähler action, induced metric and induced Kähler form. Could strong form of duality meant that the space-time surfaces in M^8 and H have same induced metric and induced Kähler form? Could the WCWs associated with M^8 and H be identical with this assumption so that duality would provide different interpretations for the same physics?

2. One can formulate associativity in M^8 (or M_c^8) by introducing octonionic structure in tangent spaces or in terms of the octonionic representation for the induced gamma matrices. Does the notion have counterpart at the level of H as one might expect if Kähler action is involved in both cases? The analog of this formulation in H might be as quaternionic "reality" since tangent space of H corresponds to complexified quaternions: I have however found no acceptable definition for this notion.

The earlier formulation is in terms of octonionic flat space gamma matrices replacing the ordinary gamma matrices so that the formulation reduces to that in M^8 tangent space. This formulation is enough to define what associativity means although one can protest. Somehow H is already complex quaternionic and thus associative. Perhaps this just what is needed since dynamics has two levels: *imbedding space level* and *space-time level*. One must have imbedding space spinor harmonics assignable to the ground states of super-conformal representations and quaternionicity and octonionicity of H tangent space would make sense at the level of space-time surfaces.

3. Whether the associativity using induced gamma matrices works is not clear for massless extremals (MEs) and vacuum extremals with the dimension of CP_2 projection not larger than 2.

4. What makes this notion of associativity so fascinating is that it would allow to iterate duality as a sequence $M^8 \to H \to H...$ by mapping the space-time surface to $M^4 \times CP_2$ by the same recipe as in case of M^8. This brings in mind the functional composition of O_c-real analytic functions (O_c denotes complexified octonions: complexification is forced by Minkowskian signature) suggested to produced associative or co-associative surfaces. The associative (co-associative) surfaces in M^8 would correspond to loci for vanishing of imaginary (real) part of octonion-real-analytic function.

It might be possible to define associativity in H also in terms of modified gamma matrices defined by Kähler action (certainly not M^8).

1. All known extremals are associative or co-associative in H in this sense. This would also give direct correlation with the variational principle. For the known preferred extremals this variant is successful partially because the modified gamma matrices need not span the entire tangent space. The space spanned by the modified gammas is not necessarily tangent space. For instance for CP_2 type vacuum extremals the modified gamma matrices are CP_2 gamma matrices plus an additional light-like component from M^4 gamma matrices.

   If the space spanned by modified gammas has dimension D smaller than 3 co-associativity is automatic. If the dimension of this space is $D = 3$ it can happen that the triplet of gammas spans by multiplication entire octonionic algebra. For $D = 4$ the situation is of course non-trivial.

2. For modified gamma matrices the notion of co-associativity can produce problems since modified gamma matrices do not in general span the tangent space. What does co-associativity mean now? Should one replace normal space with orthogonal complement of the space spanned by modified gamma matrices? Co-associativity option must be considered for $D = 4$ only. CP_2 type vacuum extremals provide a good example. In this case the modified gamma matrices reduce to sums of ordinary CP_2 gamma matrices and ligt-like M^4 contribution. The orthogonal complement for the modified gamma matrices consists of dual light-like gamma matrix and two gammas orthogonal to it: this space is subspace of M^4 and trivially associative.

3.4.1 Basic idea behind $M^8 - M^4 \times CP_2$ duality

If four-surfaces $X^4 \subset M^8$ under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly, the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. This correspondence could be called number theoretical compactification or $M^8 - H$ duality.

The hard mathematical facts behind the notion of number theoretical compactification are following.

1. One must assume that M^8 has unique decomposition $M^8 = M^4 \times E^4$. This decomposition generalizes also to the case of M^8_c. This would be most naturally due to Kähler structure in E^4 defined by a self-dual Kähler form defining parallel constant electric and magnetic fields in Euclidian sense. Besides Kähler form there is vector field coupling to sigma matrix

representing the analog of strong isospin: the corresponding octonionic sigma matrix however is imaginary unit times gamma matrix - say ie_1 in M^4 - defining a preferred plane M^2 in M^4. Here it is essential that the gamma matrices of E^4 defined in terms of octonion units commute to gamma matrices in M^4. What is involved becomes clear from the Fano triangle illustrating octonionic multiplication table.

2. The space of hyper-complex structures of the hyper-octonion space - they correspond to the choices of plane $M^2 \subset M^8$ - is parameterized by 6-sphere $S^6 = G^2/SU(3)$. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Fixed complex structure therefore corresponds to a point of S^6.

3. Quaternionic sub-algebras of M^8 (and M^8_c) are parametrized by $G_2/U(2)$. The quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space defined by real and preferred imaginary unit and parametrized by a point of S^6) are parameterized by $SU(3)/U(2) = CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. Thus the space of quaternionic structures can be parametrized by the 10-dimensional space $G_2/U(2)$ decomposing as $S^6 \times CP_2$ locally.

4. The basic result behind number theoretic compactification and $M^8 - H$ duality is that associative sub-spaces $M^4 \subset M^8$ containing a fixed commutative sub-space $M^2 \subset M^8$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

Consider now the formulation of $M^8 - H$ duality.

1. The idea of the standard formulation is that associative manifold $X^4 \subset M^8$ has at its each point associative tangent plane. That is X^4 corresponds to an integrable distribution of $M^2(x) \subset M^8$ parametrized 4-D coordinate x that is map $x \rightarrow S^6$ such that the 4-D tangent plane is hyper-quaternionic for each x.

2. Since the Kähler structure of M^8 implies unique decomposition $M^8 = M^4 \times E^4$, this surface in turn defines a surface in $M^4 \times CP_2$ obtained by assigning to the point of 4-surface point $(m, s) \in H = M^4 \times CP_2$: $m \in M^4$ is obtained as *projection* $M^8 \rightarrow M^4$ (this is modification to the earlier definition) and $s \in CP_2$ parametrizes the quaternionic tangent plane as point of CP_2. Here the local decomposition $G_2/U(2) = S^6 \times CP_2$ is essential for achieving uniqueness.

3. One could also map the associative surface in M^8 to surface in 10-dimensional $S^6 \times CP_2$. In this case the metric of the image surface cannot have Minkowskian signature and one cannot assume that the induced metrics are identical. It is not known whether S^6 allows genuine complex structure and Kähler structure which is essential for TGD formulation.

4. Does duality imply the analog of associativity for $X^4 \subset H$? The tangent space of H can be seen as a sub-space of doubly complexified quaternions. Could one think that quaternionic sub-space is replaced with sub-space analogous to that spanned by real parts of complexified quaternions? The attempts to define this notion do not however look promising. One can however define associativity and co-associativity for the tangent space M^8 of H using octonionization and can formulate it also terms of induced gamma matrices.

5. The associativity defined in terms of induced gamma matrices in both in M^8 and H has the interesting feature that one can assign to the associative surface in H a new associative surface in H by assigning to each point of the space-time surface its M^4 projection and point of CP_2 characterizing its associative tangent space or co-associative normal space. It seems that one continue this series ad infinitum and generate new solutions of field equations! This

brings in mind iteration which is standard manner to generate fractals as limiting sets. This certainly makes the heart of mathematician beat.

6. Kähler structure in $E^4 \subset M^8$ guarantees natural $M^4 \times E^4$ decomposition. Does associativity imply preferred extremal property or vice versa, or are the two notions equivalent or only consistent with each other for preferred extremals?

A couple of comments are in order.

1. This definition generalizes to the case of M^8_c: all that matters is that tangent space-is is complexified quaternionic and there is a unique identification $M^4 \subset M^8_c$: this allows to assign the point of 4-surfaces a point of $M^4 \times CP_2$. The generalization is needed if one wants to formulate the hypothesis about O_c real-analyticity as a manner to build quaternionic space-time surfaces properly.

2. This definition differs from the first proposal for years ago stating that each point of X^4 contains a *fixed* $M^2 \subset M^4$ rather than $M_2(x) \subset M^8$ and also from the proposal assuming integrable distribution of $M^2(x) \subset M^4$. The older proposals are not consistent with the properties of massless extremals and string like objects for which the counterpart of M^2 depends on space-time point and is not restricted to M^4. The earlier definition $M^2(x) \subset M^4$ was problematic in the co-associative case since for the Euclidian signature is is not clear what the counterpart of $M^2(x)$ could be.

3. The new definition is consistent with the existence of Hamilton-Jacobi structure meaning slicing of space-time surface by string world sheets and partonic 2-surfaces with points of partonic 2-surfaces labeling the string world sheets [K10]. This structure has been proposed to characterize preferred extremals in Minkowskian space-time regions at least.

4. Co-associative Euclidian 4-surfaces, say CP_2 type vacuum extremal do not contain integrable distribution of $M^2(x)$. It is normal space which contains $M^2(x)$. Does this have some physical meaning? Or does the surface defined by $M^2(x)$ have Euclidian analog?

   A possible identification of the analog would be as string world sheet at which W boson field is pure gauge so that the modes of the modified Dirac operator [K26] restricted to the string world sheet have well-defined em charge. This condition appears in the construction of solutions of modified Dirac operator.

   For octonionic spinor structure the W coupling is however absent so that the condition does not make sense in M^8. The number theoretic condition would be as commutative or co-commutative surface for which imaginary units in tangent space transform to real and imaginary unit by a multiplication with a fixed imaginary unit! One can also formulate co-associativity as a condition that tangent space becomes associative by a multiplication with a fixed imaginary unit.

   There is also another justification for the distribution of Euclidian tangent planes. The idea about associativity as a fundamental dynamical principle can be strengthened to the statement that space-time surface allows slicing by hyper-complex or complex 2-surfaces, which are commutative or co-commutative inside space-time surface. The physical interpretation would be as Minkowskian or Euclidian string world sheets carrying spinor modes. This would give a connection with string model and also with the conjecture about the general structure of preferred extremals.

5. Minimalist could argue that the minimal definition requires octonionic structure and associativity *only* in M^8. There is no need to introduce the counterpart of Kähler action in M^8 since the dynamics would be based on associativity or co-associativity alone. The objection is that one must assumes the decomposition $M^8 = M^4 \times E^4$ without any justification.

   The map of space-time surfaces to those of $H = M^4 \times CP_2$ implies that the space-time surfaces in H are in well-defined sense quaternionic. As a matter of fact, the standard spinor structure of H can be regarded as quaternionic in the sense that gamma matrices are essentially tensor products of quaternionic gamma matrices and reduce in matrix representation for quaternions to ordinary gamma matrices. Therefore the idea that one should introduce octonionic gamma matrices in H is questionable. If all goes as in dreams, the mere associativity or co-associativity would code for the preferred extremal property of Kähler action in H. One could at least hope that associativity/co-associativity in H is consistent with the preferred extremal property.

6. One can also consider a variant of associativity based on modified gamma matrices - but only in H. This notion does not make sense in M^8 since the very existence of quaternionic tangent plane makes it possible to define $M^8 - H$ duality map. The associativity for modified gamma matrices is however consistent with what is known about extremals of Kähler action. The associativity based on induced gamma matrices would correspond to the use of the space-time volume as action. Note however that gamma matrices are *not* necessary in the definition.

3.4.2 Hyper-octonionic Pauli "matrices" and the definition of associativity

Octonionic Pauli matrices suggest an interesting possibility to define precisely what associativity means at the level of M^8 using gamma matrices (for background see [K79]).

1. According to the standard definition space-time surface $X^4 \subset M^8$ is associative if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is associative. The definition can be given also in terms of octonionic gamma matrices whose definition is completely straightforward.

2. Could/should one define the analog of associativity at the level of H? One can identify the tangent space of H as M^8 and can define octonionic structure in the tangent space and this allows to define associativity locally. One can replace gamma matrices with their octonionic variants and formulate associativity in terms of them locally and this should be enough.

   Skeptic however reminds M^4 allows hyper-quaternionic structure and CP_2 quaternionic structure so that complexified quaternionic structure would look more natural for H. The tangent space would decompose as $M^8 = HQ + ijQ$, weher j is commuting imaginary unit and HQ is spanned by real unit and by units iI_k, where i second commutating imaginary unit and I_k denotes quaternionic imaginary units. There is no need to make anything associative.

   There is however far from obvious that octonionic spinor structure can be (or need to be!) defined globally. The lift of the CP_2 spinor connection to its octonionic variant has questionable features: in particular vanishing of the charged part and reduction of neutral part to photon. Therefore is is unclear whether associativity condition makes sense for $X^4 \subset M^4 \times CP_2$. What makes it so fascinating is that it would allow to iterate duality as a sequences $M^8 \rightarrow H \rightarrow H$.... This brings in mind the functional composition of octonion real-analytic functions suggested to produced associative or co-associative surfaces.

I have not been able to settle the situation. What seems the working option is associativity in both M^8 and H and modified gamma matrices defined by appropriate Kähler action and correlation between associativity and preferred extremal property.

3.4.3 Are Kähler and spinor structures necessary in M^8?

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action defining Kähler function. As found, this leads to the conclusion that the $M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

Are also the 4-surfaces in M^8 preferred extremals of Kähler action?

It would be a mathematical miracle if associative and co-associative surfaces in M^8 would be in 1-1 correspondence with preferred extremals of Kähler action. This motivates the question whether Kähler action make sense also in M^8. This does not exclude the possibility that associativity implies or is equivalent with the preferred extremal property.

One expects a close correspondence between preferred extremals: also now vacuum degeneracy is obtained, one obtains massless extremals, string like objects, and counterparts of CP_2 type vacuum extremals. All known extremals would be associative or co-associative if modified gamma matrices define the notion (possible only in the case of H).

The strongest form of duality would be that the space-time surfaces in M^8 and H have same induced metric same induced Kähler form. The basic difference would be that the spinor connection for surfaces in M^8 would be however neutral and have no left handed components and only em gauge potential. A possible interpretation is that M^8 picture defines a theory in the phase in which electroweak symmetry breaking has happened and only photon belongs to the spectrum.

The question is whether one can define WCW also for M^8. Certainly it should be equivalent with WCW for H: otherwise an inflation of poorly defined notions follows. Certainly the general formulation of the WCW geometry generalizes from H to M^8. Since the matrix elements of symplectic super-Hamiltonians defining WCW gamma matrices are well defined as matrix elements involve spinor modes with Gaussian harmonic oscillator behavior, the non-compactness of E^4 does not pose any technical problems.

Spinor connection of M^8

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units. This is possible but perhaps a more natural option is the introduction of just single Kähler form as in the case of CP_2.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The naive replacement of sigma matrices appearing in the coupling of electroweak gauge fields takes the left handed parts of these fields to zero so that only neutral part remains. Further, gauge fields correspond to curvature of CP_2 which vanishes for E^4 so that only Kähler form form remains. Kähler form couples to 3L and q so that the basic asymmetry between leptons and quarks remains. The resulting field could be seen as analog of photon.

4. The absence of weak parts of classical electro-weak gauge fields would conform with the standard thinking that classical weak fields are not important in long scales. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

Dirac equation for leptons and quarks in M^8

Kähler gauge potential would also couple to octonionic spinors and explain the distinction between quarks and leptons.

1. The complexified octonions representing H spinors decompose to $1 + 1 + 3 + \overline{3}$ under SU(3) representing color automorphisms but the interpretation in terms of QCD color does not make sense. Rather, the triplet and single combine to two weak isospin doublets and quarks and leptons corresponds to to "spin" states of octonion valued 2-spinor. The conservation of quark and lepton numbers follows from the absence of coupling between these states.

2. One could modify the coupling so that coupling is on electric charge by coupling it to electromagnetic charge which as a combination of unit matrix and sigma matrix is proportional to $1 + kI_1$, where I_1 is octonionic imaginary unit in $M^2 \subset M^4$. The complexified octonionic units can be chosen to be eigenstates of Q_{em} so that Laplace equation reduces to ordinary scalar Laplacian with coupling to self-dual em field.

3. One expects harmonic oscillator like behavior for the modes of the Dirac operator of M^8 since the gauge potential is linear in E^4 coordinates. One possibility is Cartesian coordinates is $A(A_x, A_y, A_z, A_t) = k(-y, x, t, -z)$. Thhe coupling would make E^4 effectively a compact space.

4. The square of Dirac operator gives potential term proportional to $r^2 = x^2 + y^2 + z^2 + t^2$ so that the spectrum of 4-D harmonic oscillator operator and $SO(4)$ harmonics localized near origin are expected. For harmonic oscillator the symmetry enhances to $SU(4)$.

   If one replaces Kähler coupling with em charge symmetry breaking of $SO(4)$ to vectorial $SO(3)$ is expected since the coupling is proportional to $1 + ike_1$ defining electromagnetic charge. Since the basis of complexified quaternions can be chosen to be eigenstates of e_1 under multiplication, octonionic spinors are eigenstates of em charge and one obtains two color singles $1 \pm e_1$ and color triplet and antitriplet. The color triplets cannot be however interpreted in terms of quark color.

   Harmonic oscillator potential is expected to enhance $SO(3)$ to SU(3). This suggests the reduction of the symmetry to $SU(3) \times U(1)$ corresponding to color symmetry and em charge so that one would have same basic quantum numbers as tof CP_2 harmonics. An interesting question is how the spectrum and mass squared eigenvalues of harmonics differ from those for CP_2.

5. In the square of Dirac equation $J^{kl}\Sigma_{kl}$ term distinguishes between different em charges ($\Sigma_k l$ reduces by self duality and by special properties of octonionic sigma matrices to a term proportional to iI_1 and complexified octonionic units can be chosen to be its eigenstates with eigen value ± 1. The vacuum mass squared analogous to the vacuum energy of harmonic oscillator is also present and this contribution are expected to cancel themselves for neutrinos so that they are massless whereas charged leptons and quarks are massive. It remains to be checked that quarks and leptons can be classified to triality $T = \pm 1$ and $t = 0$ representations of dynamical $SU(3)$ respectively.

What about the analog of Kähler Dirac equation

Only the octonionic structure in $T(M^8)$ is needed to formulate quaternionicity of space-time surfaces: the reduction to O_c-real-analyticity would be extremely nice but not necessary (O_c denotes complexified octonions needed to cope with Minkowskian signature). Most importantly, there might be no need to introduce Kähler action (and Kähler form) in M^8. Even the octonionic representation of gamma matrices is un-necessary. Neither there is any absolute need to define octonionic Dirac equation and octonionic Kähler Dirac equation nor octonionic analog of its solutions nor the octonionic variants of imbedding space harmonics.

It would be of course nice if the general formulas for solutions of the Kähler Dirac equation in H could have counterparts for octonionic spinors satisfying quaternionicity condition. One can indeed wonder whether the restriction of the modes of induced spinor field to string world sheets defined by integrable distributions of hyper-complex spaces $M^2(x)$ could be interpretated in terms of commutativity of fermionic physics in M^8. $M^8 - H$ correspondence could map the octonionic spinor fields at string world sheets to their quaternionic counterparts in H. The fact that only holomorphy is involved with the definition of modes could make this map possible.

3.4.4 How could one solve associativity/co-associativity conditions?

The natural question is whether and how one could solve the associativity/-co-associativity conditions explicitly. One can imagine two approaches besides $M^8 \to H \to H...$ iteration generating new solutions from existing ones.

Could octonion-real analyticity be equivalent with associativity/co-associativity?

Analytic functions provide solutions to 2-D Laplace equations and one might hope that also the field equations could be solved in terms of octonion-real-analyticity at the level of M^8 perhaps also at the level of H. Signature however causes problems - at least technical. Also the compactness of CP_2 causes technical difficulties but they need not be insurmountable.

For E^8 the tangent space would be genuinely octonionic and one can define the notion octonion-real analytic map as a generalization of real-analytic function of complex variables (the coefficients of Laurent series are real to guarantee associativity of the series). The argument is complexified octonion in $O \oplus iO$ forming an algebra but not a field. The norm square is Minkowskian as difference of two Euclidian octonionic norms: $N(o_1 + io_2) = N(o_1) - N(o_2)$ and vanishes at 15-D light cone boundary. Obviously, differential calculus is possible outside the light-cone boundary. Rational analytic functions have however poles at the light-cone boundary. One can wonder whether the poles at M^4 light-cone boundary, which is subset of 15-D light-cone boundary could have physical significance and relevant for the role of causal diamonds in ZEO.

The candidates for associative surfaces defined by O_c-real-analytic functions (I use O_c for complexified octonions) have Minkowskian signature of metric and are 4-surfaces at which the projection of $f(o_1 + io_2)$ to $Im(O_1)$, $iIm(O_2)$, and $iRe(Q_2) \oplus Im(Q_1)$ vanish so that only the projection to hyper-quaternionic Minkowskian sub-space $M^4 = Re(Q_1) + iIm(Q_2)$ with signature $(1, -1, -, 1-, 1)$ is non-vanishing. The inverse image need not belong to M^8 and in general it belongs to M^8_c but this is not a problem: all that is needed that the tangent space of inverse image is complexified quaternionic. If this is the case then $M^8 - H$ duality maps the tangent space of the inverse image to CP_2 point and image itself defines the point of M^4 so that a point of H is obtained. Co-associative surfaces would be surfaces for which the projections of image to $Re(O_1)$, $iRe(O_2)$, and to $Im(O_1)$ vanish so that only the projection to $iIm(O_2)$ with signature $(-1, -1, -1, -1)$ is non-vanishing.

The inverse images as 4-D sub-manifolds of M^8_c (not M^8!) are excellent candidates for associative and co-associative 4-surfaces since $M^8 - H$ duality assignes to them a 4-surface in $M^4 \times CP_2$ if the tangent space at given point is complexified quaternionic. This is true if one believes on the analytic continuation of the intuition from complex analysis (the image of real axes under the map defined by O_c-real-analytic function is real axes in the new coordinates defined by the map: the intuition results by replacing "real" by "complexified quaternionic"). The possibility to solve field equations in this manner would be of enormous significance since besides basic arithmetic operations also the functional decomposition of O_c-real-analytic functions produces similar functions. One could speak of the algebra of space-time surfaces.

What is remarkable that the complexified octonion real analytic functions are obtained by analytic continuation from single real valued function of real argument. The real functions form naturally a hierarchy of polynomials (maybe also rational functions) and number theoretic vision suggests that there coefficients are rationals or algebraic numbers. Already for rational coefficients hierarchy of algebraic extensions of rationals results as one solves the vanishing conditions. There is a temptation to regard this hierarchy coding for space-time sheets as an analog of DNA.

Note that in the recent formulation there is no need to pose separately the condition about integrable distribution of $M^2(x) \subset M^4$.

Quaternionicity condition for space-time surfaces

Quaternionicity actually has a surprisingly simple formulation at the level of space-time surfaces. The following discussion applies to both M^8 and H with minor modifications if one accepts that also H can allow octonionic tangent space structure, which does not require gamma matrices.

1. Quaternionicity is equivalent with associativity guaranteed by the vanishing of the associator $A(a, b, c) = a(bc) - (ab)c$ for any triplet of imaginary tangent vectors in the tangent space of

the space-time surface. The condition must hold true for purely imaginary combinations of tangent vectors.

2. If one is able to choose the coordinates in such a manner that one of the tangent vectors corresponds to real unit (in the imbedding map imbedding space M^4 coordinate depends only on the time coordinate of space-time surface), the condition reduces to the vanishing of the octonionic product of remaining three induced gamma matrices interpreted as octonionic gamma matrices. This condition looks very simple - perhaps too simple!- since it involves only first derivatives of the imbedding space vectors.

   One can of course whether quaternionicity conditions replace field equations or only select preferred extremals. In the latter case, one should be able to prove that quaternionicity conditions are consistent with the field equations.

3. Field equations would reduce to tri-linear equations in in the gradients of imbedding space coordinates (rather than involving imbedding space coordinates quadratically). Sum of analogs of 3×3 determinants deriving from $a \times (b \times b)$ for different octonion units is involved.

4. Written explicitly field equations give in terms of vielbein projections e^A_α, vielbein vectors e^A_k, coordinate gradients $\partial_\alpha h^k$ and octonionic structure constants f_{ABC} the following conditions stating that the projections of the octonionic associator tensor to the space-time surface vanishes:

$$\begin{aligned} e^A_\alpha e^B_\beta e^C_\gamma A^E_{ABC} &= 0 \ , \\ A^E_{ABC} &= f_{AD}{}^E f_{BC}{}^D - f_{AB}{}^D f_{DC}{}^E \ , \\ e^A_\alpha &= \partial_\alpha h^k e^A_k \ , \\ \Gamma_k &= e^A_k \gamma_A \ . \end{aligned} \tag{3.4.1}$$

   The very naive idea would be that the field equations are indeed integrable in the sense that they reduce to these tri-linear equations. Tri-linearity in derivatives is highly non-trivial outcome simplifying the situation further. These equations can be formulated as the as purely algebraic equations written above plus integrability conditions

$$F^A_{\alpha\beta} = D_\alpha e^A_\beta - D_\beta e^A_\alpha = 0 \ . \tag{3.4.2}$$

   One could say that vielbein projections define an analog of a trivial gauge potential. Note however that the covariant derivative is defined by spinor connection rather than this effective gauge potential which reduces to that in SU(2). Similar formulation holds true for field equations and one should be able to see whether the field equations formulated in terms of derivatives of vielbein projections commute with the associatitivity conditions.

5. The quaternionicity conditions can be formulated as vanishing of generalization of Cayley's hyperdeterminant for "hypermatrix" a_{ijk} with 2-valued indiced (see http://en.wikipedia.org/wiki/Hyperdeterminant). Now one has 8 hyper-matrices with 3 8-valued indices associated with the vanishing $A^E_{BCD} x^B y^C z^D = 0$ of trilinear forms defined by the associators. The conditions say somethig only about the octonioni structure constants and since octonionic space allow quaternionic sub-spaces these conditions must be satisfied.

The inspection of the Fano triangle [A117] expressing the multiplication table for octonionic imaginary units reveals that give any two imaginary octonion units e_1 and e_2 their product $e_1 e_2$ (or equivalently commutator) is imaginary octonion unit (2 times octonion unit) and the three units span together with real unit quaternionic sub-algebra. There it seems that one can generate local quaternionic sub-space from two imaginary units plus real unit. This generalizes to the vielbein components of tangent vectors of space-time surface and one can build the solutions to the quaternionicity conditions from vielbein projections e_1, e_2, their product $e_3 = k(x)e_1e_2$ and real fourth "time-like" vielbein component which must be expressible as a combination of real unit and imaginary units:

$$e_0 = a \times 1 + b^i e_i$$

For static solutions this condition is trivial. Here summation over i is understood in the latter term. Besides these conditions one has integrability conditions and field equations for Kähler action. This formulation suggests that quaternionicity is additional - perhaps defining - property of preferred extremals.

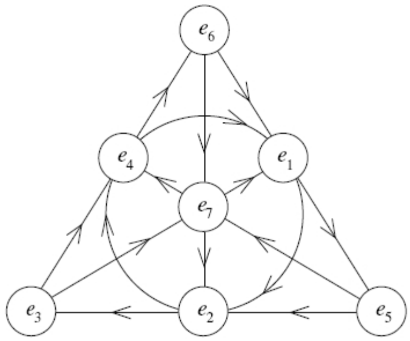

Figure 3.2: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

3.4.5 Quaternionicity at the level of imbedding space quantum numbers

From the multiplication table of octonions as illustrated by Fano triangle [A117] one finds that all edges of the triangle, the middle circle and the three the lines connecting vertices to the midpoints of opposite side define triplets of quaternionic units. This means that by taking real unit and any imaginary unit in quaternionic M^4 algebra spanning $M^2 \subset M^4$ and two imaginary units in the complement representing CP_2 tangent space one obtains quaternionic algebra. This suggests an explanation for the preferred M^2 contained in tangent space of space-time surface (the M^2:s could form an integrable distribution). Four-momentum restricted to M^2 and I_3 and Y interpreted as tangent vectors in CP_2 tangent space defined quaterionic sub-algebra. This could give content for the idea that quantum numbers are quaternionic.

I have indeed proposed that the four-momentum belongs to M^2. If $M^2(x)$ form a distribution as the proposal for the preferred extremals suggests this could reflect momentum exchanges between different points of the space-time surface such that total momentum is conserved or momentum exchange between two sheets connected by wormhole contacts.

3.4.6 Questions

In following some questions related to $M^8 - H$ duality are represented.

Could associativity condition be formulated using modified gamma matrices?

Skeptic can criticize the minimal form of $M^8 - H$ duality involving no Kähler action in M^8 is unrealistic. Why just Kähler action? What makes it so special? The only defense that I can imagine is that Kähler action is in many respects unique choice.

An alternative approach would replace induced gamma matrices with the modified ones to get the correlation In the case of M^8 this option cannot work. One cannot exclude it for H.

1. For Kähler action the modified gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k \gamma_A$, assign to a given point of X^4 a 4-D space which need not be tangent space anymore or even its sub-space.

   The reason is that canonical momentum current contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. In the case of M^8 the duality map to H is therefore lost.

2. The space spanned by the modified gamma matrices need not be 4-dimensional. For vacuum extremals with at most 2-D CP_2 projection modified gamma matrices vanish identically. For massless extremals they span 1- D light-like subspace. For CP_2 vacuum extremals the modified gamma matrices reduces to ordinary gamma matrices for CP_2 and the situation reduces to the quaternionicity of CP_2. Also for string like objects the conditions are satisfied since the gamma matrices define associative sub-space as tangent space of $M^2 \times S^2 \subset M^4 \times CP_2$. It seems that associativity is satisfied by all known extremals. Hence modified gamma matrices are flexible enough to realize associativity in H.

3. Modified gamma matrices in Dirac equation are required by super conformal symmetry for the extremals of action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of associativity in H does not affect in any manner $M^8 - H$ duality necessarily based on induced gamma matrices in M^8 allowing purely number theoretic interpretation of standard model symmetries. One can however argue that the most natural definition of associativity is in terms of induced gamma matrices in both M^8 and H.

Remark: A side comment not strictly related to associativity is in order. The anti-commutators of the modified gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

Now skeptic can ask why should one demand $M^8 - H$ correspondence if one in any case is forced to introduced Kähler also at the level of M^8? Does $M^8 - H$ correspondence help to construct preferred extremals or does it only bring in a long list of conjectures? I can repeat the questions of the skeptic.

Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative?

The 8-dimensionality of M^8 allows to consider both associativity of the tangent space and associativity of the normal space- let us call this co-associativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

Can $M^8 - H$ duality be useful?

Skeptic could of course argue that $M^8 - H$ duality generates only an inflation of unproven conjectures. This might be the case. In the following I will however try to defend the conjecture. One can however find good motivations for $M^8 - H$ duality: both theoretical and physical.

1. If $M^8 - H$ duality makes sense for induced gamma matrices also in H, one obtains infinite sequence if dualities allowing to construct preferred extremals iteratively. This might relate to octonionic real-analyticity and composition of octonion-real-analytic functions.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action as hyper-quaternionic surfaces. Unfortunately, it is not clear whether one should introduce the counterpart of Kähler action in M^8 and the coupling of M^8 spinors to Kähler form. Note that the Kähler form in E^4 would be self dual and have constant components: essentially parallel electric and magnetic field of same constant magnitude.

3. $M^8 - H$ duality provides insights to low energy physics, in particular low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

   This argument does not seem to be consistent with $SU(3) \times U(1) \subset SU(4)$ symmetry for Mx Dirac equation. One can however argue that $SU(4)$ symmetry combines $SO(4)$ multiplets together. Furthermore, $SO(4)$ represents the isometries leaving Kähler form invariant.

$M^8 - H$ duality in low energy physics and low energy hadron physics

$M^8 - H$ can be applied to gain a view about color confinement. The basic idea would be that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quarks using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies.

A possible interpretation is that the space-time surfaces vary so slowly in CP_2 degrees of freedom that can approximate CP_2 with a small region of its tangent space E^4. One could also say that color interactions mask completely electroweak interactions so that the spinor connection of CP_2 can be neglected and one has effectively E^4. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since WCW degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [K49] .

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

3.4.7 Summary

The overall conclusion is that the most convincing scenario relies on the associativity/co-associativity of space-time surfaces define by induced gamma matrices and applying both for M^8 and H. The fact that the duality can be continued to an iterated sequence of duality maps $M^8 \rightarrow H \rightarrow H...$ is what makes the proposal so fascinating and suggests connection with fractality.

The introduction of Kähler action and coupling of spinors to Kähler gauge potentials is highly natural. One can also consider the idea that the space-time surfaces in M^8 and H have same induced metric and Kähler form: for iterated duality map this would mean that the steps in the map produce space-time surfaces which identical metric and Kähler form so that the sequence might stop. M^8_H duality might provide two descriptions of same underlying dynamics: M^8 description would apply in long length scales and H description in short length scales.

3.5 Octo-twistors and twistor space

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in an elegant manner. One can also consider generalization of the notion of spinor and twistor. I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as one of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and $M^4 \times CP_2$ (recall $M^8 - H$ duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the modified gamma matrices both in M^8 and H.

The basic challenge is to achieve twistorial description of four-momenta or even $M^4 \times CP_2$ quantum numbers: this applies both to the momenta of fundamental fermions at the lines of generalized Feynman diagrams and to the massive incoming and outcoming states identified as their composites.

1. A rather attractive way to overcome the problem at the level of fermions propagating along the braid strands at the light-like orbits of partonic 2-surfaces relies on the assumption that generalized Feynman diagrammatics effectively reduces to a form in which all fermions in the propagator lines are massless although they can have non-physical helicity [K63]. One can use ordinary M^4 twistors. This is consistent with the idea that space-time surfaces are quaternionic sub-manifolds of octonionic imbedding space.

2. Incoming and outgoing states are composites of massless fermions and not massless. They are however massless in 8-D sense. This suggests that they could be described using generalization of twistor formalism from M^4 to M^8 and even betterm to $M^4 \times CP_2$.

In the following two possible twistorializations are considered.

3.5.1 Two manners to twistorialize imbedding space

In the following the generalization of twistor formalism for M^8 or $M^4 \times CP_2$ will be considered in more detail. There are two options to consider.

1. For the first option one assigns to $M^4 \times CP_2$ twistor space as a product of corresponding twistor spaces $T(M_4) = CP_3$ and the flag-manifold $T(CP_2) = SU(3)/U(1) \times U(1)$ parameterizing the choices of quantization axes for $SU(3)$: $T_H = T(M^4) \times T(CP_2)$. Quite remarkably, M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. The twistor space is 12-dimensional. The choice of quantization axis is certainly a physically well-define operation so that $T(CP_2)$ has physical interpretation. If all observable physical states are color singlets situation becomes more complex. If one assumes QCC for color quantum numbers Y and I_3, then also the choice of color quantization axis is fixed at the level of Kähler action from the condition that Y and I_3 have classically their quantal values.

2. For the second option one generalizes the usual construction for M^8 regarded as tangent space of $M^4 \times CP_2$ (unless one takes $M^8 - H$ duality seriously).

The tangent space option looks like follows.

1. One can map the points of M^8 to octonions. One can consider 2-component spinors with octonionic components and map points of M^8 light-cone to linear combinations of 2×2 Pauli sigma matrices but with octonionic components. By the same arguments as in the deduction of ordinary twistor space one finds that 7-D light-cone boundary is mapped to 7+8 D space since the octonionic 2-spinor/its conjugate can be multiplied/divided by arbitrary octonion without changing the light-like point. By standard argument this space extends to 8+8-D space. The points of M^8 can be identified as 8-D octonionic planes (analogs of complex sphere CP_1 in this space. An attractive identification is as octonionic projective space OP_2. Remarkably, octonions do not allow higher dimensional projective spaces.

2. If one assumes that the spinors are quaternionic the twistor space should have dimension 7+4+1=12. This dimension is same as for $M^4 \times CP_2$. Does this mean that quaternionicity assumption reduces $T(M^8) = OP_2$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$? Or does it

yield 12-D space $G_2/U(1) \times U(1)$, which is also natural since G_2 has 2-D Cartan algebra? Number theoretical compactification would transform $T(M^8) = G_2/U(1) \times U(1)$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$. This would not be surprising since in $M^8 - H$-duality CP_2 parametrizes (hyper)quaternionic planes containing preferred plane M^2.

Quaternionicity is certainly very natural in TGD framework. Quaternionicity for 8-momenta does not in general imply that they reduce to the observed M^4-momenta unless one identifies M^4 as one particular subspace of M^8. In $M^8 - H$ duality one in principle allows all choices of M^4: it is of course unclear whether this makes any physical difference. Color confinement could be interpreted as a reduction of M^8 momenta to M^4 momenta and would also allow the interpretational problems caused by the fact that CP_2 momenta are not possible.

3. Since octonions can be regarded as complexified quaternions with non-commuting imaginary unit, one can say that quaternionic spinors in M^8 are "real" and thus analogous to Majorana spinors. Similar interpretation applies at the level of H. Could one can interpret the quaternionicity condition for space-time surfaces and imbedding space spinors as TGD analog of Majorana condition crucial in super string models? This would also be crucial for understanding supersymmetry in TGD sense.

3.5.2 Octotwistorialization of M^8

Consider first the twistorialization in 4-D case. In M^4 one can map light-like momoment to spinors satisfying massless Dirac equation. General point m of M^4 can be mapped to a pair of massless spinors related by incidence relation defining the point m. The essential element of this association is that mass squared can be defined as determinant of the 2×2 matrix resulting in the assignment. Light-likeness is coded to the vanishing of the determinant implying that the spinors defining its rows are linearly independent. The reduction of M^4 inner product to determinant occurs because the 2×2 matrix can be regarded as a matrix representation of complexified quaternion. Massless means that the norm of a complexified quaternion defined as the product of q and its conjugate vanishes. Incidence relation $s_1 = xs_2$ relating point of M^4 and pair of spinors defining the corresponding twistor, can be interpreted in terms of product for complexified quaternions.

The generalization to the 8-D situation is straightforward: replace quaternions with octonions.

1. The transition to M^8 means the replacement of quaternions with octonions. Masslessness corresponds to the vanishing norm for complexified octonion (hyper-octonion).

2. One should assign to a massless 8-momentum an 8-dimensional spinor identifiable as octonion - or more precisely as hyper-octonion obtained by multiplying the imaginary part of ordinary octonion with commuting imaginary unit j and defining conjugation as a change of sign of j or that of octonionic imaginar units.

3. This leads to a generalization of the notion of twistor consisting of pair of massless octonion valued spinors (octonions) related by the incidence relation fixing the point of M^8. The incidence relation for Euclidian octonions says $s_1 = xs_2$ and can be interpreted in terms of triality for $SO(8)$ relating conjugate spinor octet to the product of vector octed and spinor octet. For Minkowskian subspace of complexified octonions light-like vectors and s_1 and s_2 can be taken light-like as octonions. Light like x can annihilate s_2.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the modified gamma matrices both in M^8 and H.

3.5.3 Octonionicity, $SO(1,7)$, G_2, and non-associative Malcev group

The symmetries assignable with octonions are rather intricate. First of all, octonions (their hyper-variants defining M^8) have SO(8) (SO(1,7)) as isometries. $G_2 \subset SO(7)$ acts as automorphisms of octonions and $SO(1,7) \to G_2$ clearly means breaking of Lorentz invariance.

John Baez has described in a lucid manner G_2 geometrically (http://math.ucr.edu/home/baez/octonions/node14.html). The basic observation is that that quaternionic sub-space is generated by two linearly independent imaginary units and by their product. By adding a fourth linearly independent imaginary unit, one can generated all octonions. From this and the fact that G_2 represents subgroup of $SO(7)$, one easily deduces that G_2 is 14-dimensional. The Lie algebra of G_2 corresponds to derivations of octonionic algebra as follows infinitesimally from the condition that the image of product is the product of images. The entire algebra $SO(8)$ is direct sum of G_2 and linear transformations generated by right and left multiplication by imaginary octonion: this gives $14 + 14 = 28 = D(SO(8))$. The subgroup $SO(7)$ acting on imaginary octonsions corresponds to the direct sum of derivations and adjoint transformations defined by commutation with imaginary octonions, and has indeed dimension $14 + 7 = 21$.

One can identify also a non-associative group-like structure.

1. In the case of octonionic spinors this group like structure is defined by the analog of phase multiplication of spinor generalizing to a multiplication with octonionic unit expressible as linear combinations of 8 octonionic imaginary units and defining 7-sphere plays appear as analog of automorphisms $o \rightarrow uou^{-1} = uou^*$.

   One can associate with these transformations a non-associative Lie group and Lie algebra like structures by defining the commutators just as in the case of matrices that is as $[a, b] = ab-ba$. One 7-D non-associative Lie group like structure with topology of 7-sphere S^7 whereas G_2 is 14-dimensional exceptional Lie group (having S^6 as coset space $S^6 = G_2/SU(3)$). This group like object might be useful in the treatment of octonionic twistors. In the case of quaternions one has genuine group acting as SO(3) rotations.

2. Octonionic gamma matrices allow to define as their commutators octonionic sigma matrices:

$$\Sigma_{kl} = \frac{i}{2}\left[\gamma_k, \gamma_l\right] \quad . \tag{3.5.1}$$

   This algebra is 14-dimensional thanks to the fact that octonionic gamma matrices are of form $\gamma_0 = \sigma_1 \otimes 1$, $\gamma_i = \sigma_2 \otimes e_i$. Due to the non-associativity of octonions this algebra does not satisfy Jacobi identity - as is easy to verify using Fano triangle - and is therefore not a genuine Lie-algebra. Therefore these sigma matrices do not define a representation of G_2 as I thought first.

   This algebra has decomposition $g = h + t$, $[h, t] \subset t$, $[t, t] \subset h$ characterizing for symmetric spaces. h is the 7-D algebra generated by Σ_{ij} and identical with the non-associative Malcev algebra generated by the commutators of octonionic units. The complement t corresponds to the generators Σ_{0i}. The algebra is clearly an octonionic non-associative analog fo $SO(1, 7)$.

3.5.4 Octonionic spinors in M^8 and real complexified-quaternionic spinors in H?

This above observations about the octonionic sigma matrices raise the problem about the octonionic representation of spinor connection. In $M^8 = M^4 \times E^4$ the spinor connection is trivial but for $M^4 \times CP_2$ not. There are two options.

1. Assume that octonionic spinor structure makes sense for M^8 only and spinor connection is trivial.

2. An alternative option is to identify M^8 as tangent space of $M^4 \times CP_2$ possessing quaternionic structure defined in terms of octonionic variants of gamma matrices. Should one replace sigma matrices appearing in spinor connection with their octonionic analogs to get a sigma matrix algebra which is pseudo Lie algebra. Or should one map the holonomy algebra of CP_2 spinor connection to a sub-algebra of $G_2 \subset SO(7)$ and define the action of the sigma matrices as ordinary matrix multiplication of octonions rather than octonionic multiplication? This seems to be possible formally.

The replacement of sigma matrices with their octonionic counterparts seems to lead to weird looking results. Octonionic multiplication table implies that the electroweak sigma matrices associated with CP_2 tangent space reduce to M^4 sigma matrices so that the spinor connection is quaternionic. Furthermore, left-handed sigma matrices are mapped to zero so that only the neutral part of spinor connection is non-vanishing. This supports the view that only M^8 gamma matrices make sense and that Dirac equation in M^8 is just free massless Dirac equation leading naturally also to the octonionic twistorialization.

One might think that distinction between different H-chiralities is difficult to make but it turns out that quarks and leptons can be identified as different components of 2-component complexified octonionic spinors.

The natural question is what associativization of octonions gives. This amounts to a condition putting the associator $a(bc)-(ab)c$ to zero. It is enough to consider octonionic imaginary units which are different. By using the decomposition of the octonionic algebra to quaternionic sub-algebra and its complement and general structure of structure constants, one finds that quaternionic sub-algebra remains as such but the products of all imaginary units in the complement with different imaginary units vanish. This means that the complement behaves effectively as 4-D flat space-gamma matrix algebra annihilated by the quaternionic sub-algebra whose imaginary part acts like Lie algebra of $SO(3)$.

3.5.5 What the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices could mean?

The basic implication of octonionization is the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices. For M^8 this has no consequences since since spinor connection is trivial.

For $M^4\times CP_2$ situation would be different since CP_2 spinor connection would be replaced with its octonionic variant. This has some rather unexpected consequences and suggests that one should not try to octonionize at the level of $M^4\times CP_2$ but interepret gamma matrices as tensor products of quaternionic gamma matrices, which can be replaced with their matrix representations. There are however some rather intriguing observations which force to keep mind open.

Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1\times\sigma_1 \ \ , \ \ \gamma^i = \gamma^i\otimes\sigma_2 \ \ , \ \ i=1,..,7 \ \ . \tag{3.5.2}$$

7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i=1,...,6 \ \ , \ \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6}\gamma^{6)}_i \ \ . \tag{3.5.3}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1\otimes\sigma_1 \ \ , \ \ \gamma_i = e_i\otimes\sigma_2 \ \ . \tag{3.5.4}$$

where e_i are the octonionic units. $e_i^2=-1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod\gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = je_i \times \sigma_3 \ \ , \ \ \Sigma_{ij} = jf_{ij}\ ^k e_k \otimes 1 \ \ . \tag{3.5.5}$$

   Here j is commuting imaginary unit. These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension $D = 14$ of the non-associative version of sigma matrix algebra algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A66] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [L2]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

Some physical implications of the reduction of $SO(7,1)$ to its octonionic counterpart

The octonization of spinor connection of CP_2 has some weird physical implications forcing to keep mind to the possibility that the octonionic description even at the level of H might have something to do with reality.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonionization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \to sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics.

2. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom and gauge group becomes $SO(2)$ subgroup of rotation group of $E^3 \subset M^4$. This looks like catastrophe. One might say that electroweak interactions are transformed to gravimagnetic interactions.

3. In very optimistic frame of mind one might ask whether this might be a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

4. Interestingly, the condition that electromagnetic charge is well-defined quantum number for the modes of the induced spinor field for $X^4 \subset H$ leads to the proposal that the solutions of the modified Dirac equation are localized to string world sheets in Minkowskian regions of space-time surface at least. For CP_2 type vacuum extremals one has massless Dirac and this allows only covariantly constant right-handed neutrino as solution. One has however only a piece of CP_2 (wormhole contact) so that holomorphic solutions annihilated by two complexified gamma matrices are possible in accordance with the conformal symmetries.

Can one assume non-trivial spinor connection in M^8

1. The simplest option encouraged by the requirement of maximal symmetries is that it is absent. Massless 8-momenta would characterize spinor modes in M^8 and this would give physical justification for the octotwistors.

2. If spinor connection is present at all, it reduces essentially to Kähler connection having different couplings to quarks and leptons identifiable as components of octonionic 2-spinors. It should be $SO(4)$ symmetric and since CP_2 is instant one might argue that now one has also instanton that is self-dual U(1) gauge field in $E^4 \subset M^4 \times E^4$ defining Kähler form. One can loosely say that that one has of constant electric and magnetic fields which are parallel to each other. The rotational symmetry in E^4 would break down to $SO(2)$.

3. Without spinor connection quarks and leptons are in completely symmetric position at the level of M^8: this is somewhat disturbing. The difference between quarks and leptons in H is made possible by the fact that CP_2 does not allow standard spinor structure. Now this problem is absent. I have also consider the possibility that only leptonic spinor chirality is allowed and quarks result via a kind of anyonization process allowing them to have fractional em charges (see `http://www.tgdtheory.fi/public_html/articles/genesis.pdf`).

4. If the solutions of the Kähler Dirac equation in Minkowskian regions are localized to two surfaces identifiable as integrable distributions of planes $M^2(x)$ and characterized by a local light-like direction defining the direction of massless momentum, they are holomorphic (in the sense of hyper-complex numbers) such that the second complexified modified gamma matrix annihilates the solution. Same condition makes sense also at the level of M^8 for solutions restricted to string world sheets and the presence or absence of spinor connection does not affect the situation.

   Does this mean that the difference between quarks and leptons becomes visible only at the imbedding space level where ground states of super-conformal representations correspond to to imbedding space spinor harmonics which in CP_2 cm degrees are different for quarks and leptons?

Octo-spinors and their relation to ordinary imbedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\begin{aligned} \Psi_{L,i} &= e_i \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \quad , \\ \Psi_{q,i} &= e_i \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \quad . \end{aligned} \tag{3.5.6}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2 \pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{3.5.7}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic modified Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

3.6 Abelian class field theory and TGD

The context leading to the discovery of adeles (http://en.wikipedia.org/wiki/Adele_ring) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p\ mod\ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p\ mos\ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (http://en.wikipedia.org/wiki/Class_field_theory) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

3.6.1 Adeles and ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (http://en.wikipedia.org/wiki/Class_field_theory) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [A56, A150, A148]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where imbedding space is replaced with Cartesian product of real imbedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://en.wikipedia.org/wiki/Adele_ring`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod'_p Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (`http://en.wikipedia.org/wiki/Idele_class_group`Idele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

3.6.2 Questions about adeles, ideles and quantum TGD

The intriguing general result of class field theory (`http://en.wikipedia.org/wiki/Class_field_theory`) is that the the maximal Abelian extension for rationals is homomorphic with the multiplicative group of ideles. This correspondence plays a key role in Langlands correspondence.

Does this mean that it is not absolutely necessary to introduce p-adic numbers? This is actually not so. The Galois group of the maximal abelian extension is rather complex objects (absolute Galois group, AGG, defines as the Galois group of algebraic numbers is even more complex!). The ring $\hat{Z}$ of adeles defining the group of ideles as its invertible elements homeomorphic to the Galois group of maximal Abelian extension is profinite group (`http://en.wikipedia.org/wiki/Profinite_group`). This means that it is totally disconnected space as also p-adic integers and numbers are. What is intriguing that p-dic integers are however a continuous structure in the sense that differential calculus is possible. A concrete example is provided by 2-adic units consisting of bit sequences which can have literally infinite non-vanishing bits. This space is formally discrete but one can construct differential calculus since the situation is not democratic. The higher the pinary digit in the expansion is, the less significant it is, and p-adic norm approaching to zero expresses the reduction of the insignificance.

1. Could TGD based physics reduce to a representation theory for the Galois groups of quaternions and octonions?

Number theoretical vision about TGD raises questions about whether adeles and ideles could be helpful in the formulation of TGD. I have already earlier considered the idea that quantum TGD could reduce to a representation theory of appropriate Galois groups. I proceed to make questions.

1. Could real physics and various p-adic physics on one hand, and number theoretic physics based on maximal Abelian extension of rational octonions and quaternions on one hand, define equivalent formulations of physics?

2. Besides various p-adic physics all classical number fields (reals, complex numbers, quaternions, and octonions) are central in the number theoretical vision about TGD. The technical problem is that p-adic quaternions and octonions exist only as a ring unless one poses some additional conditions. Is it possible to pose such conditions so that one could define what might be called quaternionic and octonionic adeles and ideles?

   It will be found that this is the case: p-adic quaternions/octonions would be products of rational quaternions/octonions with a p-adic unit. This definition applies also to algebraic extensions of rationals and makes it possible to define the notion of derivative for corresponding adeles. Furthermore, the rational quaternions define non-commutative automorphisms of quaternions and rational octonions at least formally define a non-associative analog of group of octonionic automorphisms [K93].

3. I have already earlier considered the idea about Galois group as the ultimate symmetry group of physics. The representations of Galois group of maximal Abelian extension (or even that for algebraic numbers) would define the quantum states. The representation space could be group algebra of the Galois group and in Abelian case equivalently the group algebra of ideles or adeles. One would have wave functions in the space of ideles.

   The Galois group of maximal Abelian extension would be the Cartan subgroup of the absolute Galois group of algebraic numbers associated with given extension of rationals and it would be natural to classify the quantum states by the corresponding quantum numbers (number theoretic observables).

   If octonionic and quaternionic (associative) adeles make sense, the associativity condition would reduce the analogs of wave functions to those at 4-dimensional associative sub-manifolds of octonionic adeles identifiable as space-time surfaces so that also space-time physics in various number fields would result as representations of Galois group in the maximal Abelian Galois group of rational octonions/quaternions. TGD would reduce to classical number theory! One can hope that WCW spinor fields assignable to the associative and co-associative space-time surfaces provide the adelic representations for super-conformal algebras replacing symmetries for point like objects.

   This of course involves huge challenges: one should find an adelic formulation for WCWin terms octonionic and quaternionic adeles, similar formulation for WCW spinor fields in terms of adelic induced spinor fields or their octonionic variants is needed. Also zero energy ontology, causal diamonds, light-like 3-surfaces at which the signature of the induced metric changes, space-like 3-surfaces and partonic 2-surfaces at the boundaries of CDs, M^8-H duality, possible representation of space-time surfaces in terms of of O_c-real analytic functions (O_c denotes for complexified octonions), etc. should be generalized to adelic framework.

4. Absolute Galois group is the Galois group of the maximal algebraic extension and as such a poorly defined concept. One can however consider the hierarchy of all finite-dimensional algebraic extensions (including non-Abelian ones) and maximal Abelian extensions associated with these and obtain in this manner a hierarchy of physics defined as representations of these Galois groups homomorphic with the corresponding idele groups.

5. In this approach the symmetries of the theory would have automatically adelic representations and one might hope about connection with Langlands program [K35], [A56, A150, A148].

2. Adelic variant of space-time dynamics and spinorial dynamics?

As an innocent novice I can continue to pose stupid questions. Now about adelic variant of the space-time dynamics based on the generalization of Kähler action discussed already earlier but without mentioning adeles ([K95]).

1. Could one think that adeles or ideles could extend reals in the formulation of the theory: note that reals are included as Cartesian factor to adeles. Could one speak about adelic space-time surfaces endowed with adelic coordinates? Could one formulate variational principle in terms of adeles so that exponent of action would be product of actions exponents associated with various factors with Neper number replaced by p for Z_p. The minimal interpretation would be that in adelic picture one collects under the same umbrella real physics and various p-adic physics.

2. Number theoretic vision suggests that 4:th/8:th Cartesian powers of adeles have interpretation as adelic variants of quaternions/ octonions. If so, one can ask whether adelic quaternions and octonions could have some number theoretical meaning. Adelic quaternions and octonions are not number fields without additional assumptions since the moduli squared for a p-adic analog of quaternion and octonion can vanish so that the inverse fails to exist at the light-cone boundary which is 17-dimensional for complexified octonions and 7-dimensional for complexified quaternions. The reason is that norm squared is difference $N(o_1) - N(o_2)$ for $o_1 \oplus io_2$. This allows to define differential calculus for Taylor series and one can consider even rational functions. Hence the restriction is not fatal.

   If one can pose a condition guaranteeing the existence of inverse for octonionic adel, one could define the multiplicative group of ideles for quaternions. For octonions one would obtain non-associative analog of the multiplicative group. If this kind of structures exist then four-dimensional associative/co-associative sub-manifolds in the space of non-associative ideles define associative/co-associative adeles in which ideles act. It is easy to find that octonionic ideles form 1-dimensional objects so that one must accept octonions with arbitrary real or p-adic components.

3. What about equations for space-time surfaces. Do field equations reduce to separate field equations for each factor? Can one pose as an additional condition the constraint that p-adic surfaces provide in some sense cognitive representations of real space-time surfaces: this idea is formulated more precisely in terms of p-adic manifold concept [K95] (see the appendix of the book). Or is this correspondence an outcome of evolution?

   Physical intuition would suggest that in most p-adic factors space-time surface corresponds to a point, or at least to a vacuum extremal. One can consider also the possibility that same algebraic equation describes the surface in various factors of the adele. Could this hold true in the intersection of real and p-adic worlds for which rationals appear in the polynomials defining the preferred extremals.

4. To define field equations one must have the notion of derivative. Derivative is an operation involving division and can be tricky since adeles are not number field. The above argument suggests this is not actually a problem. Of course, if one can guarantee that the p-adic variants of octonions and quaternions are number fields, there are good hopes about well-defined derivative. Derivative as limiting value $df/dx = \lim(f(x+dx) - f(x))/dx$ for a function decomposing to Cartesian product of real function $f(x)$ and p-adic valued functions $f_p(x_p)$ would require that $f_p(x)$ is non-constant only for a finite number of primes: this is in accordance with the physical picture that only finite number of p-adic primes are active and define "cognitive representations" of real space-time surface. The second condition is that dx is proportional to product $dx \times \prod dx_p$ of differentials dx and dx_p, which are rational numbers. dx goes to xero as a real number but not p-adically for any of the primes involved. dx_p in turn goes to zero p-adically only for Q_p.

5. The idea about rationals as points common to all number fields is central in number theoretical vision. This vision is realized for adeles in the minimal sense that the action of rationals is well-defined in all Cartesian factors of the adeles. Number theoretical vision allows also to talk about common rational points of real and various p-adic space-time surfaces in preferred coordinate choices made possible by symmetries of the imbedding space, and one ends up to the vision about life as something residing in the intersection of real and p-adic number fields. It is not clear whether and how adeles could allow to formulate this idea.

6. For adelic variants of imbedding space spinors Cartesian product of real and p-adc variants of imbedding spaces is mapped to their tensor product. This gives justification for the physical vision that various p-adic physics appear as tensor factors. Does this mean that the generalized induced spinors are infinite tensor products of real and various p-adic spinors and Clifford algebra generated by induced gamma matrices is obtained by tensor product construction? Does the generalization of massless Dirac equation reduce to a sum of d'Alembertians for the factors? Does each of them annihilate the appropriate spinor? If only finite number of Cartesian factors corresponds to a space-time surface which is not vacuum extremal vanishing induced Kähler form, Kähler Dirac equation is non-trivial only in finite number of adelic factors.

3. Objections leading to the identification of octonionic adeles and ideles

The basic idea is that appropriately defined invertible quaternionic/octonionic adeles can be regarded as elements of Galois group assignable to quaternions/octonions. The best manner to proceed is to invent objections against this idea.

1. The first objection is that p-adic quaternions and octonions do not make sense since p-adic variants of quaternions and octonions do not exist in general. The reason is that the p-adic norm squared $\sum x_i^2$ for p-adic variant of quaternion, octonion, or even complex number can vanish so that its inverse does not exist.

2. Second objection is that automorphisms of the ring of quaternions (octonions) in the maximal Abelian extension are products of transformations of the subgroup of $SO(3)$ (G_2) represented by matrices with elements in the extension and in the Galois group of the extension itself. Ideles separate out as 1-dimensional Cartesian factor from this group so that one does not obtain 4-field (8-fold) Cartesian power of this Galois group.

One can define quaternionic/octonionic ideles in terms of rational quaternions/octonions multiplied by p-adic number. For adeles this condition produces non-sensical results.

1. This condition indeed allows to construct the inverse of p-adic quaternion/octonion as a product of inverses for rational quaternion/octonion and p-adic number. The reason is that the solutions to $\sum x_i^2 = 0$ involve always p-adic numbers with an infinite number of pinary digits - at least one and the identification excludes this possibility. The ideles form also a group as required.

2. One can interpret also the quaternionicity/octonionicity in terms of Galois group. The 7-dimensional non-associative counterparts for octonionic automorphisms act as transformations $x \rightarrow gxg^{-1}$. Therefore octonions represent this group like structure and the p-adic octonions would have interpretation as combination of octonionic automorphisms with those of rationals.

3. One cannot assign to ideles 4-D idelic surfaces. The reason is that the non-constant part of all 8-coordinates is proportional to the same p-adic valued function of space-time point so that space-time surface would be a disjoint union of effectively 1-dimensional structures labelled by a subset of rational points of M^8. Induced metric would be 1-dimensional and induced Kähler and spinor curvature would vanish identically.

4. One must allow p-adic octonions to have arbitrary p-adic components. The action of ideles representing Galois group on these surfaces is well-defined. Number field property is lost

but this feature comes in play as poles only when one considers rational functions. Already the Minkowskian signature forces to consider complexified octonions and quaternions leading to the loss of field property. It would not be surprising if p-adic poles would be associated with the light-like orbits of partonic 2-surfaces. Both p-adic and Minkowskian poles might therefore be highly relevant physically and analogous to the poles of ordinary analytic functions. For instance, n-point functions could have poles at the light-like boundaries of causal diamonds and at light-like partonic orbits and explain their special physical role.

The action of ideles in the quaternionic tangent space of space-time surface would be analogous to the action of of adelic linear group $Gl_n(A)$ in n-dimensional space.

5. Adelic variants of octonions would be Cartesian products of ordinary and various p-adic octonions and would define a ring. Quaternionic 4-surfaces would define associative local sub-rings of octonion-adelic ring.

3.7 An attempt to understand preferred extremals of Kähler action

Preferred extremal of Kähler action is one of the basic poorly defined notions of TGD. There are pressing motivations for understanding what "preferred" really means. For instance, the conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [K81]. The problem is however how to assign a complex coordinate with the string world sheet having Minkowskian signature of metric. One can hope that the understanding of preferred extremals could allow to identify two preferred complex coordinates whose existence is also suggested by number theoretical vision giving preferred role for the rational points of partonic 2-surfaces in preferred coordinates. The best one could hope is a general solution of field equations in accordance with the hints that TGD is integrable quantum theory.

3.7.1 What "preferred" could mean?

The first question is what preferred extremal could mean.

1. In positive energy ontology preferred extremal would be a space-time surface assignable to given 3-surface and unique in the ideal situation: since one cannot pose conditions to the normal derivatives of imbedding space coordinates at 3-surface, there is infinity of extremals. Some additional conditions are required and space-time surface would be analogous to Bohr orbit : hence the attribute "preferred". The problem would be to understand what "preferred" could mean. The non-determinism of Kähler action however destroyed this dream in its original form and led to zero energy ontology (ZEO).

2. In ZEO one considers extremals as space-time surfaces connecting two space-like 3-surfaces at the boundaries. One might hope that these 4-surfaces are unique. The non-determinism of Kähler action suggests that this is not the case. At least there is conformal invariance respecting the light-likeness of the 3-D parton orbits at which the signature of the induced metric changes: the conformal transformations would leave the space-like 3-D ends or at least partonic 2-surfaces invariant. This non-determinism would correspond to quantum criticality.

3. Effective 2-dimensionality follows from strong form of general coordinate invariance (GCI) stating that light-like partonic orbits and space-like 3-surfaces at the ends of space-time surface are equivalent physically: partonic 2-surfaces and their 4-D tangent space data would determine everything. One can however worry about how effective 2-dimensionality relates to the the fact that the modes of the induced spinor field are localized at string world sheets and partonic 2-surface. Are the tangent space data equivalent with the data characterizing string world sheets as surfaces carrying vanishing electroweak fields?

There is however a problem: the hierarchy of Planck constants (dark matter) requires that the conformal equivalence classes of light-like surfaces must be counted as physical degrees of freedom so that either space-like or light-like surfaces do not seem to be quite enough.

Should one then include also the light-like partonic orbits to the what one calls 3-surface? The resulting connected 3-surfaces would define analogs of Wilson loops. Could the conformal equivalence class of the preferred extremal be unique without any additional conditions? If so, one could get rid of the attribute "preferred". The fractal character of the many-sheeted space-time however suggests that one can have this kind of uniqueness only in given length scale resolution and that "radiative corrections" due to the non-determinism are always present.

These considerations show that the notion of preferred extremal is still far from being precisely defined and it is not even clear whether the attribute "preferred" is needed. If not then the question is what are the extremals of Kähler action.

3.7.2 What is known about extremals?

A lot is is known about properties of extremals and just by trying to integrate all this understanding, one might gain new visions. The problem is that all these arguments are heuristic and rely heavily on physical intuition. The following considerations relate to the space-time regions having Minkowskian signature of the induced metric. The attempt to generalize the construction also to Euclidian regions could be very rewarding. Only a humble attempt to combine various ideas to a more coherent picture is in question.

The core observations and visions are following.

1. Hamilton-Jacobi coordinates for M^4 (discussed in this chapter) define natural preferred coordinates for Minkowskian space-time sheet and might allow to identify string world sheets for X^4 as those for M^4. Hamilton-Jacobi coordinates consist of light-like coordinate m and its dual defining local 2-plane $M^2 \subset M^4$ and complex transversal complex coordinates $(w, \overline{w})$ for a plane E^2_x orthogonal to M^2_x at each point of M^4. Clearly, hyper-complex analyticity and complex analyticity are in question.

2. Space-time sheets allow a slicing by string world sheets (partonic 2-surfaces) labelled by partonic 2-surfaces (string world sheets).

3. The quaternionic planes of octonion space containing preferred hyper-complex plane are labelled by CP_2, which might be called CP_2^{mod} [K72]. The identification $CP_2 = CP_2^{mod}$ motivates the notion of $M^8 - -M^4 \times CP_2$ duality [K19]. It also inspires a concrete solution ansatz assuming the equivalence of two different identifications of the quaternionic tangent space of the space-time sheet and implying that string world sheets can be regarded as strings in the 6-D coset space $G_2/SU(3)$. The group G_2 of octonion automorphisms has already earlier appeared in TGD framework.

4. The duality between partonic 2-surfaces and string world sheets in turn suggests that the $CP_2 = CP_2^{mod}$ conditions reduce to string model for partonic 2-surfaces in $CP_2 = SU(3)/U(2)$. String model in both cases could mean just hypercomplex/complex analyticity for the coordinates of the coset space as functions of hyper-complex/complex coordinate of string world sheet/partonic 2-surface.

The considerations of this section lead to a revival of an old very ambitious and very romantic number theoretic idea.

1. To begin with express octonions in the form $o = q_1 + Iq_2$, where q_i is quaternion and I is an octonionic imaginary unit in the complement of fixed a quaternionic sub-space of octonions. Map preferred coordinates of $H = M^4 \times CP_2$ to octonionic coordinate, form an arbitrary octonion analytic function having expansion with real Taylor or Laurent coefficients to avoid problems due to non-commutativity and non-associativity. Map the outcome to a point of H to get a map $H \to H$. This procedure is nothing but a generalization of Wick rotation to get an 8-D generalization of analytic map.

2. Identify the preferred extremals of Kähler action as surfaces obtained by requiring the vanishing of the imaginary part of an octonion analytic function. Partonic 2-surfaces and string world sheets would correspond to commutative sub-manifolds of the space-time surface and of imbedding space and would emerge naturally. The ends of braid strands at partonic 2-surface would naturally correspond to the poles of the octonion analytic functions. This would mean a huge generalization of conformal invariance of string models to octonionic conformal invariance and an exact solution of the field equations of TGD and presumably of quantum TGD itself.

3.7.3 Basic ideas about preferred extremals

The slicing of the space-time sheet by partonic 2-surfaces and string world sheets

The basic vision is that space-time sheets are sliced by partonic 2-surfaces and string world sheets. The challenge is to formulate this more precisely at the level of the preferred extremals of Kähler action.

1. Almost topological QFT property means that the Kähler action reduces to Chern-Simons terms assignable to 3-surfaces. This is guaranteed by the vanishing of the Coulomb term in the action density implied automatically if conserved Kähler current is proportional to the instanton current with proportionality coefficient some scalar function.

2. The field equations reduce to the conservation of isometry currents. An attractive ansatz is that the flow lines of these currents define global coordinates. This means that these currents are Beltrami flows [B30] so that corresponding 1-forms J satisfy the condition $J \wedge dJ = 0$. These conditions are satisfied if

$$J = \Phi \nabla \Psi$$

hold true for conserved currents. From this one obtains that Ψ defines global coordinate varying along flow lines of J.

3. A possible interpretation is in terms of local polarization and momentum directions defined by the scalar functions involved and natural additional conditions are that the gradients of Ψ and Φ are orthogonal:

$$\nabla \Phi \cdot \nabla \Psi = 0 \ ,$$

and that the Ψ satisfies massless d'Alembert equation

$$\nabla^2 \Psi = 0$$

as a consequence of current conservation. If Ψ defines a light-like vector field - in other words

$$\nabla \Psi \cdot \nabla \Psi = 0 \ ,$$

the light-like dual of Φ -call it Φ_c- defines a light-like like coordinate and Φ and Φ_c defines a light-like plane at each point of space-time sheet.

If also Φ satisfies d'Alembert equation

$$\nabla^2 \Phi = 0 \ ,$$

also the current

$$K = \Psi \nabla \Phi$$

is conserved and its flow lines define a global coordinate in the polarization plane orthogonal to time-lik plane defined by local light-like momentum direction.

If Φ allows a continuation to an analytic function of the transversal complex coordinate, one obtains a coordinatization of space-time surface by Ψ and its dual (defining hyper-complex coordinate) and $w, \overline{w}$. Complex analyticity and its hyper-complex variant would allow to provide space-time surface with four coordinates very much analogous with Hamilton-Jacobi coordinates of M^4.

This would mean a decomposition of the tangent space of space-time surface to orthogonal planes defined by light-like momentum and plane orthogonal to it. If the flow lines of J defined Beltrami flow it seems that the distribution of momentum planes is integrable.

4. General arguments suggest that the space-time sheets allow a slicing by string world sheets parametrized by partonic 2-surfaces or vice versa. This would mean a intimate connection with the mathematics of string models. The two complex coordinates assignable to the Yangian of affine algebra would naturally relate to string world sheets and partonic 2-surfaces and the highly non-trivial challenge is to identify them appropriately.

Hamilton-Jacobi coordinates for M^4

The earlier attempts to construct preferred extremals [K10] led to the realization that so called Hamilton-Jacobi coordinates (m, w) for M^4 define its slicing by string world sheets parametrized by partonic 2-surfaces. m would be pair of light-like conjugate coordinates associated with an integrable distribution of planes M^2 and w would define a complex coordinate for the integrable distribution of 2-planes E^2 orthogonal to M^2. There is a great temptation to assume that these coordinates define preferred coordinates for M^4.

1. The slicing is very much analogous to that for space-time sheets and the natural question is how these slicings relate. What is of special interest is that the momentum plane M^2 can be defined by massless momentum. The scaling of this vector does not matter so that these planes are labelled by points z of sphere S^2 telling the direction of the line $M^2 \cap E^3$, when one assigns rest frame and therefore S^2 with the preferred time coordinate defined by the line connecting the tips of CD. This direction vector can be mapped to a twistor consisting of a spinor and its conjugate. The complex scalings of the twistor $(u, \overline{u}) \to \lambda u, \overline{u}/\lambda)$ define the same plane. Projective twistor like entities defining CP_1 having only one complex component instead of three are in question. This complex number defines with certain prerequisites a local coordinate for space-time sheet and together with the complex coordinate of E^2 could serve as a pair of complex coordinates (z, w) for space-time sheet. This brings strongly in mind the two complex coordinates appearing in the expansion of the generators of quantum Yangian of quantum affine algebra [K81].

2. The coordinate Ψ appearing in Beltrami flow defines the light-like vector field defining M^2 distribution. Its hyper-complex conjugate would define Ψ_c and conjugate light-like direction. An attractive possibility is that Φ allows analytic continuation to a holomorphic function of w. In this manner one would have four coordinates for M^4 also for space-time sheet.

3. The general vision is that at each point of space-time surface one can decompose the tangent space to $M^2(x) \subset M^4 = M^2_x \times E^2_x$ representing momentum plane and polarization plane $E^2 \subset E^2_x \times T(CP_2)$. The moduli space of planes $E^2 \subset E^6$ is 8-dimensional and parametrized by $SO(6)/SO(2) \times SO(4)$ for a given E^2_x. How can one achieve this selection and what conditions it must satisfy? Certainly the choice must be integrable but this is not the only condition.

Space-time surfaces as associative/co-associative surfaces

The idea that number theory determines classical dynamics in terms of associativity condition means that space-time surfaces are in some sense quaternionic surfaces of an octonionic space-time. It took several trials before the recent form of this hypothesis was achieved.

1. Octonionic structure is defined in terms of the octonionic representaton of gamma matrices of the imbedding space existing only in dimension $D = 8$ since octonion units are in one-one correspondence with tangent vectors of the tangent space. Octonionic real unit corresponds

to a preferred time axes (and rest frame) identified naturally as that connecting the tips of CD. What modified gamma matrices mean depends on variational principle for space-time surface. For volume action one would obtain induced gamma matrices. For Kähler action one obtains something different. In particular, the modified gamma matrices do not define vector basis identical with tangent vector basis of space-time surface.

2. Quaternionicity means that the modified gamma matrices defined as contractions of gamma matrices of H with canonical momentum densities for Kähler action span quaternionic sub-space of the octonionic tangent space [K26]. A further condition is that each quaternionic space defined in this manner contains a preferred hyper-complex subspace of octonions.

3. The sub-space defined by the modified gamma matrices does not co-incide with the tangent space of space-time surface in general so that the interpretation of this condition is far from obvious. The canonical momentum densities need not define four independent vectors at given point. For instance, for massless extremals these densities are proportional to light-like vector so that the situation is degenerate and the space in question reduces to 2-D hyper-complex sub-space since light-like vector defines plane M^2.

The obvious questions are following.

1. Does the analog of tangent space defined by the octonionic modified gammas contain the local tangent space $M^2 \subset M^4$ for preferred extremals? For massless extremals [K10] this condition would be true. The orthogonal decomposition $T(X^4) = M^2 \oplus_\perp E^2$ can be defined at each point if this is true. For massless extremals also the functions Ψ and Φ can be identified.

2. One should answer also the following delicate question. Can M^2 really depend on point x of space-time? CP_2 as a moduli space of quaternionic planes emerges naturally if M^2 is *same* everywhere. It however seems that one should allow an integrable distribution of M^2_x such that M^2_x is same for all points of a given partonic 2-surface.

   How could one speak about fixed CP_2 (the imbedding space) at the entire space-time sheet even when M^2_x varies?

   (a) Note first that G_2 defines the Lie group of octonionic automorphisms and G_2 action is needed to change the preferred hyper-octonionic sub-space. Various $SU(3)$ subgroups of G_2 are related by G_2 automorphism. Clearly, one must assign to each point of a string world sheet in the slicing parameterizing the partonic 2-surfaces an element of G_2. One would have Minkowskian string model with G_2 as a target space. As a matter fact, this string model is defined in the target space $G_2/SU(3)$ having dimension $D = 6$ since $SU(3)$ automorphisms leave given $SU(3)$ invariant.

   (b) This would allow to identify at each point of the string world sheet standard quaternionic basis - say in terms of complexified basis vectors consisting of two hyper-complex units and octonionic unit q_1 with "color isospin" $I_3 = 1/2$ and "color hypercharge" $Y = -1/3$ and its conjugate $\overline{q}_1$ with opposite color isospin and hypercharge.

   (c) The CP_2 point assigned with the quaternionic basis would correspond to the $SU(3)$ rotation needed to rotate the standard basis to this basis and would actually correspond to the first row of $SU(3)$ rotation matrix. Hyper-complex analyticity is the basic property of the solutions of the field equations representing Minkowskian string world sheets. Also now the same assumption is highly natural. In the case of string models in Minkowski space, the reduction of the induced metric to standard form implies Virasoro conditions and similar conditions are expected also now. There is no need to introduce action principle -just the hyper-complex analycitity is enough-since Kähler action already defines it.

3. The WZW model inspired approach to the situation would be following. The parameterization corresponds to a map $g : X^2 \to G_2$ for which g defines a flat G_2 connection at string world sheet. WZW type action would give rise to this kind of situation. The transition $G_2 \to G_2/SU(3)$ would require that one gauges $SU(3)$ degrees of freedom by bringing in

$SU(3)$ connection. Similar procedure for $CP_2 = SU(3)/U(2)$ would bring in $SU(3)$ valued chiral field and $U(2)$ gauge field. Instead of introducing these connections one can simply introduce $G_2/SU(3)$ and $SU(3)/U(2)$ valued chiral fields. What this observation suggests that this ansatz indeed predicts gluons and electroweak gauge bosons assignable to string like objects so that the mathematical picture would be consistent with physical intuition.

The two interpretations of CP_2

An old observation very relevant for what I have called $M^8 - H$ duality [K19] is that the moduli space of quaternionic sub-spaces of octonionic space (identifiable as M^8) containing preferred hyper-complex plane is CP_2. Or equivalently, the space of two planes whose addition extends hyper-complex plane to some quaternionic subspace can be parametrized by CP_2. This CP_2 can be called it CP_2^{mod} to avoid confusion. In the recent case this would mean that the space $E^2(x) \subset E_x^2 \times T(CP_2)$ is represented by a point of CP_2^{mod}. On the other hand, the imbedding of space-time surface to H defines a point of "real" CP_2. This gives two different CP_2s.

1. The highly suggestive idea is that the identification $CP_2^{mod} = CP_2$ (apart from isometry) is crucial for the construction of preferred extremals. Indeed, the projection of the space-time point to CP_2 would fix the local polarization plane completely. This condition for $E^2(x)$ would be purely local and depend on the values of CP_2 coordinates only. Second condition for $E^2(x)$ would involve the gradients of imbedding space coordinates including those of CP_2 coordinates.

2. The conditions that the planes M_x^2 form an integrable distribution at space-like level and that M_x^2 is determined by the modified gamma matrices. The integrability of this distribution for M^4 could imply the integrability for X^2. X^4 would differ from M^4 only by a deformation in degrees of freedom transversal to the string world sheets defined by the distribution of M^2s.

   Does this mean that one can begin from vacuum extremal with constant values of CP_2 coordinates and makes them non-constant but allows to depend only on transversal degrees of freedom? This condition is too strong even for simplest massless extremals for which CP_2 coordinates depend on transversal coordinates defined by $\epsilon \cdot m$ and $\epsilon \cdot k$. One could however allow dependence of CP_2 coordinates on light-like M^4 coordinate since the modification of the induced metric is light-like so that light-like coordinate remains light-like coordinate in this modification of the metric.

   Therefore, if one generalizes directly what is known about massless extremals, the most general dependence of CP_2 points on the light-like coordinates assignable to the distribution of M_x^2 would be dependence on either of the light-like coordinates of Hamilton-Jacobi coordinates but not both.

3.7.4 What could be the construction recipe for the preferred extremals assuming $CP_2 = CP_2^{mod}$ identification?

The crucial condition is that the planes $E^2(x)$ determined by the point of $CP_2 = CP_2^{mod}$ identification and by the tangent space of $E_x^2 \times CP_2$ are same. The challenge is to transform this condition to an explicit form. $CP_2 = CP_2^{mod}$ identification should be general coordinate invariant. This requires that also the representation of E^2 as (e^2, e^3) plane is general coordinate invariant suggesting that the use of preferred CP_2 coordinates - presumably complex Eguchi-Hanson coordinates - could make life easy. Preferred coordinates are also suggested by number theoretical vision. A careful consideration of the situation would be required.

The modified gamma matrices define a quaternionic sub-space analogous to tangent space of X^4 but not in general identical with the tangent space: this would be the case only if the action were 4-volume. I will use the notation $T_x^m(X^4)$ about the modified tangent space and call the vectors of $T_x^m(X^4)$ modified tangent vectors. I hope that this would not cause confusion.

$CP_2 = CP_2^{mod}$ condition

Quaternionic property of the counterpart of $T_x^m(X^4)$ allows an explicit formulation using the tangent vectors of $T_x^m(X^4)$.

1. The unit vector pair (e_2, e_3) should correspond to a unique tangent vector of H defined by the coordinate differentials dh^k in some natural coordinates used. Complex Eguchi-Hanson coordinates [L2] are a natural candidate for CP_2 and require complexified octonionic imaginary units. If octonionic units correspond to the tangent vector basis of H uniquely, this is possible.

2. The pair (e_2, e_3) as also its complexification $(q_1 = e_2 + ie_3, \overline{q}_1 = e_2 - ie_3)$ is expressible as a linear combination of octonionic units $I_2, ...I_7$ should be mapped to a point of $CP_2^{mod} = CP_2$ in canonical manner. This mapping is what should be expressed explicitly. One should express given (e_2, e_3) in terms of $SU(3)$ rotation applied to a standard vector. After that one should define the corresponding CP_2 point by the bundle projection $SU(3) \to CP_2$.

3. The tangent vector pair

$$(\partial_w h^k, \partial_{\overline{w}} h^k)$$

defines second representation of the tangent space of $E^2(x)$. This pair should be equivalent with the pair $(q_1, \overline{q}_1)$. Here one must be however very cautious with the choice of coordinates. If the choice of w is unique apart from constant the gradients should be unique. One can use also real coordinates (x, y) instead of $(w = x + iy, \overline{w} = x - iy)$ and the pair (e_2, e_3). One can project the tangent vector pair to the standard vielbein basis which must correspond to the octonionic basis

$$(\partial_x h^k, \partial_y h^k) \to (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A) e_A) \leftrightarrow (e_2, e_3) \ ,$$

where the e_A denote the octonion units in 1-1 correspondence with vielbein vectors. This expression can be compared to the expression of (e_2, e_3) derived from the knowledge of CP_2 projection.

Formulation of quaternionicity condition in terms of octonionic structure constants

One can consider also a formulation of the quaternionic tangent planes in terms of (e_2, e_3) expressed in terms of octonionic units deducible from the condition that unit vectors obey quaternionic algebra. The expressions for octonionic *resp.* quaternionic structure constants can be found at [A66] *resp.* [A77].

1. The ansatz is

$$\begin{aligned}
\{E_k\} &= \{1, I_1, E_2, E_3\} \ , \\
E_2 &= E_{2k}e^k \equiv \sum_{k=2}^{7} E_{2k}e^k \ , \ E_3 = E_{3k}e^k \equiv \sum_{k=2}^{7} E_{3k}e^k \ , \\
|E_2| &= 1 \ , \ |E_3| = 1 \ .
\end{aligned} \tag{3.7.1}$$

2. The multiplication table for octonionic units expressible in terms of octonionic triangle [A66] gives

$$f^{1kl}E_{2k} = E_{3l} \ , \ f^{1kl}E_{3k} = -E_{2l} \ , \ f^{klr}E_{2k}E_{3l} = \delta_1^r \ . \tag{3.7.2}$$

Here the indices are raised by unit metric so that there is no difference between lower and upper indices. Summation convention is assumed. Also the contribution of the real unit is present in the structure constants of third equation but this contribution must vanish.

3. The conditions are linear and quadratic in the coefficients E_{2k} and E_{3k} and are expected to allow an explicit solution. The first two conditions define homogenous equations which must allow solution. The coefficient matrix acting on (E_2, E_3) is of the form

$$\begin{pmatrix} f_1 & 1 \\ -1 & f_1 \end{pmatrix} ,$$

where 1 denotes unit matrix. The vanishing of the determinant of this matrix should be due to the highly symmetric properties of the structure constants. In fact the equations can be written as eigen conditions

$$f_1 \circ (E_2 \pm iE_3) = \mp i(E_2 \pm iE_3) \ ,$$

and one can say that the structure constants are eigenstates of the hermitian operator defined by I_1 analogous to color hyper charge. Both values of color hyper charged are obtained.

Explicit expression for the $CP_2 = CP_2^{mod}$ conditions

The symmetry under $SU(3)$ allows to construct the solutions of the above equations directly.

1. One can introduce complexified basis of octonion units transforming like $(1,1,3,\overline{3})$ under SU(3). Note the analogy of triplet with color triplet of quarks. One can write complexified basis as $(1, e_1, (q_1, q_2, q_3), (\overline{q}_1 \overline{q}_2, \overline{q}_3))$. The expressions for complexified basis elements are

$$(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(e_2 + ie_3, e_4 + ie_5, e_6 + ie_7) \ .$$

These options can be seen to be possible by studying octonionic triangle in which all lines containing 3 units defined associative triple: any pair of octonion units at this kind of line can be used to form pair of complexified unit and its conjugate. In the tangent space of $M^4 \times CP_2$ the basis vectors q_1, and q_2 are mixtures of E^2_x and CP_2 tangent vectors. q_3 involves only CP_2 tangent vectors and there is a temptation to interpret it as the analog of the quark having no color isospin.

2. The quaternionic basis is real and must transform like $(1, 1, q_1, \overline{q}_1)$, where q_1 is any quark in the triplet and $\overline{q}_1$ its conjugate in antitriplet. Having fixed some basis one can perform SU(3) rotations to get a new basis. The action of the rotation is by 3×3 special unitary matrix. The over all phases of its rows do not matter since they induce only a rotation in (e_2, e_3) plane not affecting the plane itself. The action of $SU(3)$ on q_1 is simply the action of its first row on (q_1, q_2, q_3) triplet:

$$\begin{aligned} q_1 &\rightarrow (Uq)_1 = U_{11}q_1 + U_{12}q_2 + U_{13}q_3 \equiv z_1q_1 + z_2q_2 + z_3q_3 \\ &= z_1(e_2 + ie_3) + z_2(e_4 + ie_5) + z_3(e_6 + ie_7) \ . \end{aligned} \tag{3.7.3}$$

The triplets (z_1, z_2, z_3) defining a complex unit vector and point of S^5. Since overall phase does not matter a point of CP_2 is in question. The new real octonion units are given by the formulas

$$\begin{aligned} e_2 &\rightarrow Re(z_1)e_2 + Re(z_2)e_4 + Re(z_3)e_6 - Im(z_1)e_3 - Im(z_2)e_5 - Im(z_3)e_7 \ , \\ e_3 &\rightarrow Im(z_1)e_2 + Im(z_2)e_4 + Im(z_3)e_6 + Re(z_1)e_3 + Re(z_2)e_5 + Re(z_3)e_7 \ . \end{aligned} \tag{3.7.4}$$

For instance the CP_2 coordinates corresponding to the coordinate patch (z_1, z_2, z_3) with $z_3 \neq 0$ are obtained as $(\xi_1, \xi_2) = (z_1/z_3, z_2/z_3)$.

Using these expressions the equations expressing the conjecture $CP_2 = CP_2^{mod}$ equivalence can be expressed explicitly as first order differential equations. The conditions state the equivalence

$$(e_2, e_3) \quad \leftrightarrow \quad (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A e_A) \ , \tag{3.7.5}$$

where e_A denote octonion units. The comparison of two pairs of vectors requires normalization of the tangent vectors on the right hand side to unit vectors so that one takes unit vector in the direction of the tangent vector. After this the vectors can be equated. This allows to expresses the contractions of the partial derivatives with vielbein vectors with the 6 components of e_2 and e_3. Each condition gives 6+6 first order partial differential equations which are non-linear by the presence of the overal normalization factor for the right hand side. The equations are invariant under scalings of (x, y). The very special form of these equations suggests that some symmetry is involved.

It must be emphasized that these equations make sense only in preferred coordinates: ordinary Minkowski coordinates and Hamilton-Jacobi coordinates for M^4 and Eguchi-Hanson complex coordinates in which $SU(2) \times U(1)$ is represented linearly for CP_2. These coordinates are preferred because they carry deep physical meaning.

Does TGD boil down to two string models?

It is good to look what have we obtained. Besides Hamilton-Jacobi conditions, and $CP_2 = CP_2^{mod}$ conditions one has what one might call string model with 6-dimensional $G_2/SU(3)$ as targent space. The orbit of string in $G_2/SU(3)$ allows to deduce the G_2 rotation identifiable as a point of $G_2/SU(3)$ defining what one means with standard quaternionic plane at given point of string world sheet. The hypothesis is that hyper-complex analyticity solves these equations.

The conjectured electric-magnetic duality implies duality between string world sheet and partonic 2-surfaces central for the proposed mathematical applications of TGD [K34, K35, K70, K82]. This duality suggests that the solutions to the $CP_2 = CP_2^{mod}$ conditions could reduce to holomorphy with respect to the coordinate w for partonic 2-surface plus the analogs of Virasoro conditions. The dependence on light-like coordinate would appear as a parametric dependence.

If this were the case, TGD would reduce at least partially to what might be regarded as dual string models in $G_2/SU(3)$ and $SU(3)/U(2)$ and also to string model in M^4 and X^4! In the previous arguments one ends up to string models in moduli spaces of string world sheets and partonic 2-surfaces. TGD seems to yield an inflation of string models! This not actually surprising since the slicing of space-time sheets by string world sheets and partonic 2-surfaces implies automatically various kinds of maps having interpretation in terms of string orbits.

3.8 In what sense TGD could be an integrable theory?

During years evidence supporting the idea that TGD could be an integrable theory in some sense has accumulated. The challenge is to show that various ideas about what integrability means form pieces of a bigger coherent picture. Of course, some of the ideas are doomed to be only partially correct or simply wrong. Since it is not possible to know beforehand what ideas are wrong and what are right the situation is very much like in experimental physics and it is easy to claim (and has been and will be claimed) that all this argumentation is useless speculation. This is the price that must be paid for real thinking.

Integrable theories allow to solve nonlinear classical dynamics in terms of scattering data for a linear system. In TGD framework this translates to quantum classical correspondence. The solutions of modified Dirac equation define the scattering data. This data should define a real analytic function whose octonionic extension defines the space-time surface as a surface for which its imaginary part in the representation as bi-quaternion vanishes. There are excellent hopes about this thanks to the reduction of the modified Dirac equation to geometric optics.

In the following I will first discuss briefly what integrability means in (quantum) field theories, list some bits of evidence for integrability in TGD framework, discuss once again the question whether the different pieces of evidence are consistent with other and what one really means with various notions. An an outcome I represent what I regard as a more coherent view about integrability of TGD. The notion of octonion analyticity developed in the previous section is essential for the for what follows.

3.8.1 What integrable theories are?

The following is an attempt to get some bird's eye of view about the landscape of integrable theories.

Examples of integrable theories

Integrable theories are typically non-linear 1+1-dimensional (quantum) field theories. Solitons and various other particle like structures are the characteristic phenomenon in these theories. Scattering matrix is trivial in the sense that the particles go through each other in the scattering and suffer only a phase change. In particular, momenta are conserved. Korteveg- de Vries equation [B5]was motivated by the attempt to explain the experimentally discovered shallow water wave preserving its shape and moving with a constant velocity. Sine-Gordon equation [B9] describes geometrically constant curvature surfaces and defines a Lorentz invariant non-linear field theory in 1+1-dimensional space-time, which can be applied to Josephson junctions (in TGD inspired quantum biology it is encountered in the model of nerve pulse [K58]). Non- linear Schrödinger equation [B8] having applications to optics and water waves represents a further example. All these equations have various variants.

From TGD point of view conformal field theories represent an especially interesting example of integrable theories. (Super-)conformal invariance is the basic underlying symmetry and by its infinite-dimensional character implies infinite number of conserved quantities. The construction of the theory reduces to the construction of the representations of (super-)conformal algebra. One can solve 2-point functions exactly and characterize them in terms of (possibly anomalous) scaling dimensions of conformal fields involved and the coefficients appearing in 3-point functions can be solved in terms of fusion rules leading to an associative algebra for conformal fields. The basic applications are to 2-dimensional critical thermodynamical systems whose scaling invariance generalizes to conformal invariance. String models represent second application in which a collection of super-conformal field theories associated with various genera of 2-surface is needed to describe loop corrections to the scattering amplitudes. Also moduli spaces of conformal equivalence classes become important.

Topological quantum field theories are also examples of integrable theories. Because of its independence on the metric Chern-Simons action is in 3-D case the unique action defining a topological quantum field theory. The calculations of knot invariants (for TGD approach see [K34]), topological invariants of 3-manifolds and 4-manifolds, and topological quantum computation (for a model of DNA as topological quantum computer see [K24]) represent applications of this approach. TGD as almost topological QFT means that the Kähler action for preferred extremals reduces to a surface term by the vanishing of Coulomb term in action and by the weak form of electric-magnetic duality reduces to Chern-Simons action. Both Euclidian and Minkowskian regions give this kind of contribution.

$\mathcal{N} = 4$ SYM is the a four-dimensional and very nearly realistic candidate for an integral quantum field theory. The observation that twistor amplitudes allow also a dual of the 4-D conformal symmetry motivates the extension of this symmetry to its infinite-dimensional Yangian variant [A108]. Also the enormous progress in the construction of scattering amplitudes suggests integrability. In TGD framework Yangian symmetry would emerge naturally by extending the symplectic variant of Kac-Moody algebra from light-cone boundary to the interior of causal diamond and the Kac-Moody algebra from light-like 3-surface representing wormhole throats at which the signature of the induced metric changes to the space-time interior [K81].

About mathematical methods

The mathematical methods used in integrable theories are rather refined and have contributed to the development of the modern mathematical physics. Mention only quantum groups, conformal algebras, and Yangian algebras.

The basic element of integrability is the possibility to transform the non-linear classical problem for which the interaction is characterized by a potential function or its analog to a linear scattering problem depending on time. For instance, for the ordinary Schrödinger function one can solve potential once single solution of the equation is known. This does not work in practice. One can however gather information about the asymptotic states in scattering to deduce the potential. One cannot do without information about bound state energies too.

In TGD framework asymptotic states correspond to partonic 2-surfaces at the two light-like boundaries of CD (more precisely: the largest CD involved and defining the IR resolution for momenta). From the scattering data coding information about scattering for various values of energy of the incoming particle one deduced the potential function or its analog.

1. The basic tool is inverse scattering transform known as Gelfand-Marchenko-Levitan (GML) transform described in simple terms in [B11].

   (a) In 1+1 dimensional case the S-matrix characterizing scattering is very simple since the only thing that can take place in scattering is reflection or transmission. Therefore the S-matrix elements describe either of these processes and by unitarity the sum of corresponding probabilities equals to 1. The particle can arrive to the potential either from left or right and is characterized by a momentum. The transmission coefficient can have a pole meaning complex (imaginary in the simplest case) wave vector serving as a signal for the formation of a bound state or resonance. The scattering data are represented by the reflection and transmission coefficients as function of time.

   (b) One can deduce an integral equation for a propagator like function $K(t,x)$ describing how delta pulse moving with light velocity is scattered from the potential and is expressible in terms of time integral over scattering data with contributions from both scattering states and bound states. The derivation of GML transform [B11] uses time reversal and time translational invariance and causality defined in terms of light velocity. After some tricks one obtains the integral equation as well as an expression for the time independent potential as $V(x) = K(x,x)$. The argument can be generalized to more complex problems to deduce the GML transform.

2. The so called Lax pair is one manner to describe integrable systems [B6]. Lax pair consists of two operators L and M. One studies what might be identified as "energy" eigenstates satisfying $L(x,t)\Psi = \lambda\Psi$. λ does not depend on time and one can say that the dynamics is associated with x coordinate whereas as t is time coordinate parametrizing different variants of eigenvalue problem with the same spectrum for L. The operator $M(t)$ does not depend on x at all and the independence of λ on time implies the condition

   $$\partial_t L = [L, M] \ .$$

   This equation is analogous to a quantum mechanical evolution equation for an operator induced by time dependent "Hamiltonian" M and gives the non-linear classical evolution equation when the commutator on the right hand side is a multiplicative operator (so that it does not involve differential operators acting on the coordinate x). Non-linear classical dynamics for the time dependent potential emerges as an integrability condition.

   One could say that $M(t)$ introduces the time evolution of $L(t,x)$ as an automorphism which depends on time and therefore does not affect the spectrum. One has $L(t,x) = U(t)L(0,x)U^{-1}(t)$ with $dU(t)/dt = M(t)U(t)$. The time evolution of the analog of the quantum state is given by a similar equation.

3. A more refined view about Lax pair is based on the observation that the above equation can be generalized so that M depends also on x. The generalization of the basic equation for $M(x,t)$ reads as

   $$\partial_t L - \partial_x M - [L, M] = 0 \ .$$

   The condition has interpretation as a vanishing of the curvature of a gauge potential having components $A_x = L, A_t = M$. This generalization allows a beautiful geometric formulation of the integrability conditions and extends the applicability of the inverse scattering transform. The monodromy of the flat connection becomes important in this approach. Flat connections in moduli spaces are indeed important in topological quantum field theories and in conformal field theories.

4. There is also a connection with the so called Riemann- Hilbert problem [A83]. The monodromies of the flat connection define monodromy group and Riemann-Hilbert problem concerns the existence of linear differential equations having a given monodromy group. Monodromy group emerges in the analytic continuation of an analytic function and the action of the element of the monodromy group tells what happens for the resulting many-valued analytic function as one turns around a singularity once ('mono-'). The linear equations obviously relate to the linear scattering problem. The flat connection (M, L) in turn defines the monodromy group. What is needed is that the functions involved are analytic functions of (t, x) replaced with a complex or hyper-complex variable. Again Wick rotation is involved. Similar approach generalizes also to higher dimensional moduli spaces with complex structures.

   In TGD framework the effective 2-dimensionality raises the hope that this kind of mathematical apparatus could be used. An interesting possibility is that finite measurement resolution could be realized in terms of a gauge group or Kac-Moody type group represented by trivial gauge potential defining a monodromy group for n-point functions. Monodromy invariance would hold for the full n-point functions constructed in terms of analytic n-point functions and their conjugates. The ends of braid strands are natural candidates for the singularities around which monodromies are defined.

3.8.2 Why TGD could be integrable theory in some sense?

There are many indications that TGD could be an integrable theory in some sense. The challenge is to see which ideas are consistent with each other and to build a coherent picture where everything finds its own place.

1. 2-dimensionality or at least effective 2-dimensionality seems to be a prerequisite for integrability. Effective 2-dimensionality is suggested by the strong form of General Coordinate Invariance implying also holography and generalized conformal invariance predicting infinite number of conservation laws. The dual roles of partonic 2-surfaces and string world sheets supports a four-dimensional generalization of conformal invariance. Twistor considerations [K79] indeed suggest that Yangian invariance and Kac-Moody invariances combine to a 4-D analog of conformal invariance induced by 2-dimensional one by algebraic continuation.

2. Octonionic representation of imbedding space Clifford algebra and the identification of the space-time surfaces as quaternionic space-time surfaces would define a number theoretically natural generalization of conformal invariance. The reason for using gamma matrix representation is that vector field representation for octonionic units does not exist. The problem concerns the precise meaning of the octonionic representation of gamma matrices.

   Space-time surfaces could be quaternionic also in the sense that conformal invariance is analytically continued from string curve to 8-D space by octonion real-analyticity. The question is whether the Clifford algebra based notion of tangent space quaternionicity is equivalent with octonionic real-analyticity based notion of quaternionicity.

   The notions of co-associativity and co-quaternionicity make also sense and one must consider seriously the possibility that associativity-co-associativity dichotomy corresponds to Minkowskian-Euclidian dichotomy.

3. Field equations define hydrodynamic Beltrami flows satisfying integrability conditions of form $J \wedge dJ = 0$.

   (a) One can assign local momentum and polarization directions to the preferred extremals and this gives a decomposition of Minkowskian space-time regions to massless quanta analogous to the 1+1-dimensional decomposition to solitons. The linear superposition of modes with 4-momenta with different directions possible for free Maxwell action does not look plausible for the preferred extremals of Kähler action. This rather quantal and solitonic character is in accordance with the quantum classical correspondence giving very concrete connection between quantal and classical particle pictures. For 4-D volume action one does not obtain this kind of decomposition. In 2-D case volume action gives superposition of solutions with different polarization directions so that the situation is nearer to that for free Maxwell action and is not like soliton decomposition.

(b) Beltrami property in strong sense allows to identify 4 preferred coordinates for the space-time surface in terms of corresponding Beltrami flows. This is possible also in Euclidian regions using two complex coordinates instead of hyper-complex coordinate and complex coordinate. The assumption that isometry currents are parallel to the same light-like Beltrami flow implies hydrodynamic character of the field equations in the sense that one can say that each flow line is analogous to particle carrying some quantum numbers. This property is not true for all extremals (say cosmic strings).

(c) The tangent bundle theoretic view about integrability is that one can find a Lie algebra of vector fields in some manifold spanning the tangent space of a lower-dimensional manifolds and is expressed in terms of Frobenius theorem [A34]). The gradients of scalar functions defining Beltrami flows appearing in the ansatz for preferred exremals would define these vector fields and the slicing. Partonic 2-surfaces would correspond to two complex conjugate vector fields (local polarization direction) and string world sheets to light-like vector field and its dual (light-like momentum directions). This slicing generalizes to the Euclidian regions.

4. Infinite number of conservation laws is the signature of integrability. Classical field equations follow from the condition that the vector field defined by modified gamma matrices has vanishing divergence and can be identified an integrability condition for the modified Dirac equation guaranteeing also the conservation of super currents so that one obtains an infinite number of conserved charges.

5. Quantum criticality is a further signal of integrability. 2-D conformal field theories describe critical systems so that the natural guess is that quantum criticality in TGD framework relates to the generalization of conformal invariance and to integrability. Quantum criticality implies that Kähler coupling strength is analogous to critical temperature. This condition does affects classical field equations only via boundary conditions expressed as weak form of electric magnetic duality at the wormhole throats at which the signature of the metric changes.

   For finite-dimensional systems the vanishing of the determinant of the matrix defined by the second derivatives of potential is similar signature and applies in catastrophe theory. Therefore the existence of vanishing second variations of Kähler action should characterize criticality and define a property of preferred extremals. The vanishing of second variations indeed leads to an infinite number of conserved currents [K26, K10] following the conditions that the deformation of modified gamma matrix is also divergenceless and that the modified Dirac equation associated with it is satisfied.

3.8.3 Questions

There are several questions which are not completely settled yet. Even the question what preferred extremals are is still partially open. In the following I try to de-learn what I have possibly learned during these years and start from scratch to see which assumptions might be un-necessarily strong or even wrong.

3.8.4 Could TGD be an integrable theory?

Consider first the abstraction of integrability in TGD framework. Quantum classical correspondence could be seen as a correspondence between linear quantum dynamics and non-linear classical dynamics. Integrability would realize this correspondence. In integrable models such as Sine-Gordon equation particle interactions are described by potential in 1+1 dimensions. This too primitive for the purposes of TGD. The vertices of generalized Feynman diagrams take care of this. At lines one has free particle dynamics so that the situation could be much simpler than in integrable models if one restricts the considerations to the lines or Minkowskian space-time regions surrounding them.

The non-linear dynamics for the space-time sheets representing incoming lines of generalized Feynman diagram should be obtainable from the linear dynamics for the induced spinor fields defined by modified Dirac operator. There are two options.

1. Strong form of the quantum classical correspondence states that each solution for the linear dynamics of spinor fields corresponds to space-time sheet. This is analogous to solving the potential function in terms of a single solution of Schrödinger equation. Coupling of space-time geometry to quantum numbers via measurement interaction term is a proposal for realizing this option. It is however the quantum numbers of positive/negative energy parts of zero energy state which would be visible in the classical dynamics rather than those of induced spinor field modes.

2. Only overall dynamics characterized by scattering data- the counterpart of S-matrix for the modified Dirac operator- is mapped to the geometry of the space-time sheet. This is much more abstract realization of quantum classical correspondence.

3. Can these two approaches be equivalent? This might be the case since quantum numbers of the state are not those of the modes of induced spinor fields.

What the scattering data could be for the induced spinor field satisfying modified Dirac equation?

1. If the solution of field equation has hydrodynamic character, the solutions of the modified Dirac equation can be localized to light-like Beltrami flow lines of hydrodynamic flow. These correspond to basic solutions and the general solution is a superposition of these. There is no dispersion and the dynamics is that of geometric optics at the basic level. This means geometric optics like character of the spinor dynamics.

   Solutions of the modified Dirac equation are completely analogous to the pulse solutions defining the fundamental solution for the wave equation in the argument leading from wave equation with external time independent potential to Marchenko-Gelfand-Levitan equation allowing to identify potential in terms of scattering data. There is however no potential present now since the interactions are described by the vertices of Feynman diagram where the particle lines meet. Note that particle like regions are Euclidian and that this picture applies only to the Minkowskian exteriors of particles.

2. Partonic 2-surfaces at the ends of the line of generalized Feynman diagram are connected by flow lines. Partonic 2-surfaces at which the signature of the induced metric changes are in a special position. Only the imaginary part of the bi-quaternionic value of the octonion valued map is non-vanishing at these surfaces which can be said to be co-complex 2-surfaces. By geometric optics behavior the scattering data correspond to a diffeomorphism mapping initial partonic 2-surface to the final one in some preferred complex coordinates common to both ends of the line.

3. What could be these preferred coordinates? Complex coordinates for S^2 at light-cone boundary define natural complex coordinates for the partonic 2-surface. With these coordinates the diffeomorphism defining scattering data is diffeomorphism of S^2. Suppose that this map is real analytic so that maps "real axis" of S^2 to itself. This map would be same as the map defining the octonionic real analyticity as algebraic extension of the complex real analytic map. By octonionic analyticity one can make large number of alternative choices for the coordinates of partonic 2-surface.

4. There can be non-uniqueness due to the possibility of $G_2/SU(3)$ valued map characterizing the local octonionic units. The proposal is that the choice of octonionic imaginary units can depend on the point of string like orbit: this would give string model in $G_2/SU(3)$. Conformal invariance for this string model would imply analyticity and helps considerably

but would not probably fix the situation completely since the element of the coset space would constant at the partonic 2-surfaces at the ends of CD. One can of course ask whether the $G_2/SU(3)$ element could be constant for each propagator line and would change only at the 2-D vertices?

This would be the inverse scattering problem formulated in the spirit of TGD. There could be also dependence of space-time surface on quantum numbers of quantum states but not on individual solution for the induced spinor field since the scattering data of this solution would be purely geometric.

Acknowledgements

I want to thank Tony Smith and Carlos Castro for useful discussions and references related to quaternions and octonions.

Chapter 4

TGD as a Generalized Number Theory III: Infinite Primes

4.1 Introduction

The third part of the multi-chapter discussing the idea about physics as a generalized number theory is devoted to the possible role of infinite primes in TGD.

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also for their complexifications and one can speak about infinite primes in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors.

4.1.1 The notion of infinite prime

The original motivation for the notion of infinite prime came from the first attempts to construct TGD inspired theory of consciousness (around 1995) [K73] . Suppose very naively that the 4-surfaces in a given sector of the "world of classical worlds" (WCW) are labelled by a fixed p-adic prime. The natural expectation is that evolution by quantum jumps means dispersion in the space of these sectors and leads to the increase of the p-adic prime characterizing the Universe. As one moves backwards in subjective time (sequence of quantum jumps) one ends up to the situation in which the prime characterizing the universe was $p = 2$. Should one assume that there was the first quantum jump when everything began? If not, then it would seem that the p-adic prime characterizing the Universe must be infinite. Second problem is that the p-adic length scales are finite and if the size scale of Universe is given by p-adic length scale the Universe has finite sized: this does not make sense in TGD framework. The only way out of the problems is the assumption that the p-adic prime characterizing the entire Universe is literally infinite and that p-adic primes characterizing space-time sheets are finite.

These argument, which are by no means central for the recent view about p-adic primes, motivated the attempt to construct a theory of infinite primes and to extend quantum TGD accordingly. This turns out to be possible. The recipe for constructing infinite primes is structurally equivalent with a repeated second quantization of an arithmetic super-symmetric quantum field theory. At the lowest level one has fermionic and bosonic states labeled by finite primes and infinite primes correspond to many particle states of this theory. Also infinite primes analogous to bound states are predicted. This hierarchy of quantizations can be continued indefinitely by taking the many particle states of the previous level as elementary particles at the next level. It must be also emphasized that the notion of infinity is relativistic. With respect to the p-adic norm infinite primes have unit norm for all finite and infinite primes so that there is nothing to become scared of!

Construction could make sense also for hyper-quaternionic and hyper-octonionic primes although non-commutativity and non-associativity pose technical challenges. One can also construct infinite number of real units as ratios of infinite integers with a precise number theoretic anatomy. The fascinating finding is that the quantum states labeled by standard model quantum numbers allow a representation as wave functions in the discrete space of these units. Space-time point becomes infinitely richly structured in the sense that one can associate to it a wave function in the space of real (or octonionic) units allowing to represent the WCW spinor fields. One can speak about algebraic holography or number theoretic Brahman=Atman identity and one can also say that the points of imbedding space and space-time surface are subject to a number theoretic evolution. In philosophical mood one can of course also ask whether there exists a hierarchy of imbedding spaces in which the imbedding space at the lower level represents something with infinitesimal size in the sense of real topology and whether this hierarchy is accompanied also by a hierarchy of conscious entities.

This picture suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [A193] providing a rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively. Same generalization could make sense for all classical number fields.

4.1.2 Infinite primes and physics in TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. It turns out that this idea makes sense when one considers discrete wave functions in the space of infinite primes and that one can indeed represent standard model quantum numbers in this manner.

4. The number theoretical supersymmetry suggests also space-time supersymmetry TGD framework. Space-time super-symmetry in its standard form is not possible in TGD Universe and this cheated me to believe that this supersymmetry is completely absent in TGD Universe.

> The progress in the understanding of the properties of the modified Dirac action however led to a generalization of the space-time super-symmetry as a dynamical and broken symmetry of quantum TGD [K27] .

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [K80] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [K19] .

Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

G_2 acts as automorphisms of hyper-octonions and $SU(3)$ as its subgroup respecting the choice of a preferred imaginary unit. The discrete subgroups of $SU(3)$ permuting to each other hyper-octonionic primes are analogous to Galois group and turned out to play a crucial role in the understanding of the correspondence between infinite hyper-octonionic primes and quantum states predicted by quantum TGD.

The notion of finite measurement resolution as the key concept

TGD predicts several hierarchies: the hierarchy of space-time sheets, the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K80] , the dark matter hierarchy characterized by increasing values of $\hbar$ [K25] , the hierarchy of extensions of a given p-adic number field. TGD inspired theory of consciousness predictes the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. These hierarchies should be closely related.

The notion of finite measurement resolution turns out to be the key concept: the p-adic norm of the rational defined by the infinite prime characterizes the angle measurement resolution for given p-adic prime p. It is essential that one has what might be called a state function reduction selecting a fixed p-adic prime which could be also infinite. This gives direct connections with cognition and with the p-adicization program relying also on angle measurement resolution. Also the value the integers characterizing the singular coverings of CD and CP_2 defining as their product Planck constant characterize the measurement resolution for a given p-adic prime in CD and CP_2 degrees of freedom. This conforms with the fact that elementary particles are characterized by two infinite primes. Hence finite measurement resolution ties tightly together the three threads of the number theoretic vision. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory with finite measurement resolution.

Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. This conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

Quantum classical correspondence requires the map of the quantum numbers of configuration space spinor fields to space-time geometry. The modified Dirac equation with measurement interaction term realizes this requirement. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries. The notion of finite measurement resolution allows to deduce much more detailed about this correspondence. In particular, the rational defined by the infinite prime classifies the finite sub-manifold geometry defined by the discretization of the partonic 2-surface implied by the finite measurement resolution. Also a direct correlation between integers defining Planck constant and the "fermionic" part of the infinite prime emerges.

4.1.3 Infinite primes, cognition, and intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. One can define the notion of prime also for the algebraic extensions of rationals. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. The introduction of infinite primes, integers, and rationals leads also to a generalization of classical number fields since an infinite algebra of real (complex, etc...) units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum of infinitesimals (real zeros) is replaced by multiplication of real units meaning that the set of real and also more general units becomes infinitely degenerate.

3. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point -or more generally wave functions in the space of the units associated with the point- might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

4. In zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

5. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower

level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- Infinite primes [L21]

4.2 Infinite primes, integers, and rationals

The definition of the infinite integers and rationals is a straightforward procedure and structurally similar to a repeated second quantization of a super-symmetric quantum field theory but including also the number theoretic counterparts of bound states.

4.2.1 The first level of hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{aligned} P &= 1+X \ , \\ X &= \prod_p p \ . \end{aligned} \tag{4.2.1}$$

If P were divisible by finite prime then $P-X=1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N=P-k$, $k>1$, are certainly not primes since k can be taken as a factor. The number $P'=P-2=-1+X$ could however be prime. P is certainly not divisible by $P-2$. It seems that one cannot express P and $P-2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p\in U} p+q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P-2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{aligned} P(\pm,n) &= \pm 1+nX \ , \\ k(p) &= 0,1,..... \ , \\ n &= \prod_p p^{k(p)} \ , \\ X &= \prod_p p \ , \end{aligned} \tag{4.2.2}$$

where $k(p)\neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer

n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good candidates for infinite primes is the same as above. No finite prime p with $k(p) \neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{array}{l} P(\pm, m, n, s) = n\frac{X}{s} \pm ms \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{4.2.3}$$

Here $x|y$ means 'x divides y'. To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm, m, n, s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+, 1, 1, 1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$\begin{array}{l} P(\pm, m, n, s|r) = nY^r \pm ms \ , \\ Y = \frac{X}{s} \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{4.2.4}$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$\begin{array}{l} N = (n_1Y + m_1s)(n_2Y + m_2s) = \frac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\ Y = \frac{X}{s} \ , \\ n_1m_2 - n_2m_1 = 0 \ . \end{array}$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteeing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n,m,r)(Y) = nY^r + m$ with integer valued coefficients ($n > 0$) defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$\begin{array}{l} P(\pm,m,n,s|r_1,r_2) = nY^{r_1} \pm ms \ , \\ m = P_{r_2}(Y)Y + m_0 \ , \\ Y = \frac{X}{s} \ , \\ m_0 = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{4.2.5}$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteeing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm,m,n,s)$ labeled by rational numbers n/s and integers m plus the primes $P(\pm,m,n,s|r_1,r_2)$ constructed as r_1:th or r_2:th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p \ mod \ 4 = 3$ condition guaranteeing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm,m,n,s)$. $P(\pm,1,1,1) \ mod \ 4$ is either 3 or 1. The value of $P(\pm,m,n,s) \ mod \ 4$ for odd s on n only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. For even s the value of $P(\pm,m,n,s) \ mod \ 4$ depends on m only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+,m,n,s)$ or $P(-,m,n,s)$ but not both are physically interesting infinite primes ($2m \ mod \ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

4.2.2 Infinite primes form a hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm,m,n,s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction recipe. In present case 'vacuum primes' at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X\prod_{P(\pm,m,n,s)} P(\pm,m,n,s) \ , \\ S = s\prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{4.2.6}$$

S is product or ordinary primes p and infinite primes $P_i(\pm,m,n,s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type 'bosons' respectively. The non-negative integer-valued function $K(P)=K(\pm,m,n,s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type 'bosons'. More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\frac{P_1(\pm,m_1,n_1,s_1K_1,S_1}{P_2(\pm,m_2,n_2,s_2,K,S_2)} = \frac{n_1s_2}{n_2s_1}\prod_{\pm,m,n,s}(\frac{n}{s})^{K_1^+(\pm,n,m,s)-K_2^+(\pm,n,m,s)} \ . \tag{4.2.7}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1P_2 \rightarrow P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

4.2.3 Construction of infinite primes as a repeated quantization of a super-symmetric arithmetic quantum field theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symmetric arithmetic quantum field theory in which single particle fermion and boson states are labeled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actually use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labeled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labeling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labeled by primes dividing s.

2. The multiplication of the 'vacuum' X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ 'p-bosons' in mode of type X/s and multiplication of the 'vacuum' s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ 'p-bosons'. in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \tag{4.2.8}$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P \ mod\ 4 = 3$ and $P \ mod\ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labeled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labeled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labeled by natural number whereas now modes are labeled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,...,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the imbedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about ...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

4.2.4 Construction in the case of an arbitrary commutative number field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [K71] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [K71]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [K71]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

4.2.5 Mapping of infinite primes to polynomials and geometric objects

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \to x_{\pm} \pm \frac{m}{sn} \ . \tag{4.2.9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on ....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

4.2.6 How to order infinite primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the 'large' and the 'small' part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

4.2.7 What is the cardinality of infinite primes at given level?

The basic problem is to decide whether Nature allows also integers S , $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the

number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

4.2.8 How to generalize the concepts of infinite integer, rational and real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \tag{4.2.10}$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \tag{4.2.11}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \tag{4.2.12}$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labeled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \tag{4.2.13}$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultra-metricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \quad . \tag{4.2.14}$$

Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n \geq n_0} x_n p^{-n} \quad , \\ x_n &\in \{0, .., p-1\} \quad . \end{aligned} \tag{4.2.15}$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \quad , \\ N &= \sum_i m_i M_i \quad , \end{aligned} \tag{4.2.16}$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part*. Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labeled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N |N> \quad . \tag{4.2.17}$$

The natural inner product is

$$\langle X, Y \rangle = x_0 y_0 + \sum_N x_N y_N \quad . \tag{4.2.18}$$

The inner product is well defined if the number of N:s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X + Y = x_0 + y_0 + \sum_N (x_N + y_N) p^{-N} \ , \tag{4.2.19}$$

The product XY is expressible in the form

$$XY = x_0 y_0 + x_0 Y + X y_0 + \sum_{N_1, N_2} x_{N_1} y_{N_2} p^{-N_1 - N_2} \ , \tag{4.2.20}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums N_1+N_2 by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x = \sum_k x_k p^{-k} \ \rightarrow \ x_p = \sum_k x_k p^k \ ,$$

generalizes to the form

$$x = x_0 + \sum_N x_N p^{-N} \ \rightarrow \ (x_0)_p + \sum_N (x_N)_p p^N \ , \tag{4.2.21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \ \rightarrow \ x_0 + \sum_N x_N p_2^{-N} \ . \tag{4.2.22}$$

   The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the imbedding space provokes the question whether it might be possible to regard the infinite-dimensional WCW, or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

4.2.9 Comparison with the approach of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement 'Set is Many allowing to regard itself as One' really means and to the fact that there is no obvious connection with physics.

The proposed approach is based on the introduction of the concept of prime as a basic concept whereas partial ordering is based on the use of ratios: using these one can recursively define partial ordering and get precise quantitative information based on finite reals. The ordering is only partial and there is infinite number of ratios of infinite integers giving rise to same real unit which in turn leads to the idea about number theoretic anatomy of real point.

The 'Set is Many allowing to regard itself as One' is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as 'One' and its decomposition to a product of primes corresponds to the set as 'Many'. The concept of prime, the ultimate 'One', has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

4.3 Can one generalize the notion of infinite prime to the non-commutative and non-associative context?

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [K72] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [K72] . Also the hierarchy of infinite primes should generalize as also the representation of infinite primes as polynomials although associativity is expected to pose technical problems.

4.3.1 Quaternionic and octonionic primes and their hyper counterparts

The loss of commutativity and associativity implies that the definitions of quaternionic and octonionic primes are not completely straightforward.

Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space if one uses octonionic representation for the gamma matrices [K72] . Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many manners and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labeling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio $3:1$. For other Lie-groups this ratio is either 2:1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

Quaternionic and octonionic primes

Quaternionic primes with $p \ mod \ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p \ mod \ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p \ mod \ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p \ mod \ 4 = 3$ with norm $z\overline{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

Hyper primes

The notion of prime generalizes to hyper-quaternionic and octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic prime has one particualr representative as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$.

Notice that the interpretation of hyper-quaternionic primes (or integers) as four-momenta implies that it is not possible to find rest system for them if one assumes the entire quaternionic prime as four-momentum: only a system where energy is minimum is possible. The introduction of a preferred hyper-complex plane necessary for several reasons- in particular for the possibility to identify standard model quantum numbers in terms of infinite primes- allows to identify the momentum of particle in the preferred plane as the first two components of the hyper prime in fixed coordinate frame. Note that this leads to a universal spectrum for mass squared.

For time like hyper-primes the momentum is always time like for hyper-primes. In this case it is possible to find a rest frame by applying a hyper-primeness preserving G_2 transformation so that the resulting momentum has no component in the preferred frame. As a matter fact, SU(3) rotation is enough for a suitable choice of SU(3). These transformations form a discrete subgroup of SU(3) since hyper-integer property must be preserved. Massless states correspond to a null norm for the corresponding hyper integer unless one allows also tachyonic hyper primes with minimal representatives $(n_3, n_3 - 1, 0, ...)$, $n_3 = (p-1)/2$. Note that Gaussian primes with $p \ mod 4 \ = 1$ are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$ and would correspond to genuine tachyons. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

The notion of "irreducible" (see Appendix of [K71]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

4.3.2 Hyper-octonionic infinite primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance. It is however not possible to interpret them as as 8-momenta with mass squared equal to prime. The proper identification of standard model quantum numbers will be discussed later.

Should infinite primes be commutative and associative?

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes. Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

The original idea was that infinite hyper-octonionic primes could be mapped to polynomials and one could assign to these space-time surfaces in analogy with the identification of surfaces as zero locii of polynomals. Although this idea has been given up, it is good to make clear its problematic aspects.

1. The sums of products of monomials of generating infinite primes define higher level infinite primes and also here non-commutativity and associativity cause potential technical difficulties. The assignment of a monomial to a quaternionic or octonionic infinite prime is not unique since the rational obtained by dividing the finite part mr with the integer n associated with infinite part can be defined either as $(1/n) \times mr$ or $mr \times (1/n)$ and the resulting non-commuting rationals are different.

2. If the polynomial associated with infinite prime has real-rational coefficients, these difficulties do not appear. The problem is that the polynomials as such would not contain information about the number field in question.

3. Commutativity requirement for infinite primes allows real-rationals or possibly algebraic extensions of them as the coefficients of the polynomials formed from hyper-octonionic infinite primes. If only infinite primes with complex rational coefficients are allowed and only the vacuum state $V_{\pm} = X \pm 1$ involving product over all primes of the number field, would reveal the number field. One could thus construct the generating infinite primes using the notion of hyper-octonionic prime for any algebraic extension of rationals.

The idea about mapping of infinite primes to polynomials in turn defining space-time surfaces is non-realistic. The recent view is more abstract and based on the mapping of wave functions in the space of hyper-octonion units assignable to single imbedding space point by its number-theoretic anatomy. The basic idea would be simple: the virtual four-momenta defined in terms of generalized eigenvalues of Chern-Simons Dirac action correspond to hyper-quaternionic primes serving as building bricks of an infinite prime reprsenting the many fermion state. In this approach one cannot assume commutatitivity of primes involved at any level. The problems due to non-commutativity and non-associativity are however circumvented by assuming that permutations and associations of are represented as phase factors and therefore do not change the quantum state. This means the introduction of association statistics besides permutation statistics. Besides Fermi and Bose statistics one can consider braid statistics. Note that Fermi statistics makes sense only when the fermionic finite primes appearing in the state do not commute.

The construction recipe for associative (co-associative) infinite primes

The following argument represents the construction recipe for the first level associative (co-associative) primes without the restriction to rational infinite primes. If the reduction is possible always by a suitable G_2 rotation then the construction of the infinite primes analogous to bound states is obtained in trivial manner from that for rational variants of these primes. The recipe generalizes to the higher levels in trivial manner.

Each hyper-octonionic prime has a number of conjugates obtained by applying transformations of G_2 respecting the property of being hyper-octonionic integer.

1. The number of conjugates of given finite prime depends on the number of non-vanishing components of the the prime with norm p in the minimal representation having minimal energy. Several primes with a given norm p not related by a multiplication with unit or by automorphism are in principle possible. The degeneracy is determined by the number of elements of a subgroup of Galois group acting non-trivially on the prime.

   Galois group contains the permutations of 7 imaginary units and 7 conjugations of units consistent with the octonionic product. X is proportional to $p^{N(p)}$ where $N(p)$ in principle depends on p.

   There could exist also G_2 transformations which change the number of components of the infinite prime. They satisfy tight number theoretical constraints since the quantity $\sum_{i=1}^{7} n_i^2$ must be preserved. For instance, for the transformation from standard form with two components to that with more than two components one has $n_1^2(i) = \sum_k n_k^2(f)$. For the transformation from 2-component prime to 3-component prime one has a condition characterizing Pythagorean triangle. One can however consider also a situation when no such G_2 transformation exist so that one has several G_2 orbits corresponding to the same rational prime.

The construction itself would be relatively straightforward. Consider first the construction of the "vacuum" primes.

1. In the case of ordinary infinite primes there are two different vacuum primes $X \pm 1$. This is the case also now. I turns out that this degeneracy corresponds to the spin and orbital degrees of freedom for the spinor fields of WCW.

2. The product X of all hyper-octonionic irreducibles can be regarded as the counterpart of Dirac vacuum in a rather concrete sense. Moreover, in the hyper-quaternionic and octonionic case the norm of X is analogous to the Dirac determinant of a fermionic field theory with prime valued mass spectrum and integer valued momentum components. The inclusion of only irreducible eliminates from the infinite product defining Dirac determinant product over various Lorentz boosts of $p^k\gamma_k - m$.

3. Infinite prime property requires that X must be defined by taking one representative from each G_2 equivalence class representing irreducible and forming the product of all its G_2 conjugates. The standard representative for the hyper-octonionic primes can be taken to be time-like positive energy prime unless one allows also tachyonic primes in which case a natural representative has a vanishing real component. The conjugates of each irreducible appear in X so for a given norm p the net result is real for each rational prime p.

The construction of non-vacuum primes is equally straighforward.

1. If the conjectured effective 2-dimensionality holds true, it is enough to construct hyper-complex primes first. To the finite hyper-complex primes appearing in these infinite primes one can apply transformations of G_2 mapping hyper-octonionic integers to hyper-octonionic integers. The infinite prime would have degeneracy defined by the product of G_2 orbits of finite primes involved. Every finite prime would be like particle possessing finite number of quantum states. If there are several G_2 orbits corresponding to the same finite prime exist they must be also included and the conjectured effective 2-dimensionality fails.

2. An interesting question is what happens when the finite part of an infinite prime is multiplied by light like integer k. The first guess is that k describes the presence of a massless particle. If the resulting infinite integer is multiplied with conjugates $k_{c,i}$ of k an integer of form $\prod_i k_{c,i} mX/n$ having formally zero norm results. It would thus seem that there is a kind of gauge invariance in the sense that infinite primes for which both finite and infinite part are multiplied with the same light-like primes, are divisors of zero and correspond to gauge degrees of freedom. This conclusion is supported by the interpretation of the projection of infinite prime to the preferred hyper-complex plane as momentum of particle in a preferred M^2 plane assigned by the hierarchy of Planck constants to each CD and also required by the p-adicization.

3. More complex infinite hyper-octonionic primes can be constructed from rational hyper-complex and complex infinite primes using a representation in terms of polynomials and then acting on the finite primes appering in their expression by elements of G_2 preserving integer property. This construction works at all levels of the hierarchy and one might hope that it is all that is needed. If there are several G_2 orbits for given finite prime p one encounters a problem since hyper-octonionic primes with more than 2 components do not allow associative and commutative polynomial representations. The interpretation as bound states is suggestive.

4.4 How to interpret the infinite hierarchy of infinite primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

4.4.1 Infinite primes and hierarchy of super-symmetric arithmetic quantum field theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. It turns also possible to assign to infinite prime space-time surface as a geometric correlate although the original proposal for how to achieve this failed. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-symplectic representations (for instance, ordinary particles correspond to states of this kind).

Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \ .$$

2. Form the vacuum states

$$V_{\pm} = X \pm 1 \ .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \rightarrow X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \rightarrow mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_{\pm}(m, n, s) = \frac{mX}{s} \pm ns \ ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed. As found, the conjectured effective 2-dimensionality for hyper-octonionic primes allows the reduction of polynomial representation of hyper-octonionic primes to that for hyper-complex primes. This would be in accordance with the effective 2-dimensionality of the basic objects of quantum TGD.

The physical counterpart of n:th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

How infinite rationals correspond to quantum states and space-time surfaces?

The most promising answer to the question how infinite rationals correspond to space-time surfaces is discussed in detail in the next section. Here it is enough to give only the basic idea.

1. In zero energy ontology hyper-octonionic units (in real sense) defined by ratios of infinite integers have an nterpretation as representations for pairs of positive and negative energy states. Suppose that the quantum number combinations characterizing positive and negative energy quantum states are representable as superpositions of real units defined by ratios of infinite integers at each point of the space-time surface. If this is true, the quantum classical correspondence coded by the measurement interaction term of the modified Dirac action maps the quantum numbers also to space-time geometry and implies a correspondence between infinite rationals and space-time surfaces.

2. The space-time surface associated with the infinite rational is in general not a union of the space-time surfaces associated with the primes composing the integers defining the rational. There the classical description of interactions emerges automatically. The description of classical states in terms of infinite integers would be analogous to the description of many particle states as finite integers in arithmetic quantum field theory. This mapping could in principle make sense both in real and p-adic sectors of WCW.

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime.

What is the interpretation of the higher level infinite primes?

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

4.4.2 Infinite primes, the structure of many-sheeted space-time, and the notion of finite measurement resolution

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in a rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface. It turns out that the notion of finite measurement resolution is the key concept: infinite prime characterizes angle measurement resolution. This gives a direct connection with the p-adicization program relying also on angle measurement resolution as well as a connection with the hierarchy of Planck constants. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory so that the characterization of the finite measurement resolution, which has been the ugly ducling of theoretical physics transforms to a beautiful swan.

The first intuitions

The concrete prediction of the general vision is that the hierarchy of infinite primes should somehow correspond to the hierarchy of space-time sheets or partonic 2-surfaces if one accepts the effective

2-dimensionality. The challenge is to find space-time counterparts for infinite primes at the lowest level of the hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labeled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.

2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by join along boundaries bonds to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at the lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on... . A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles. The concrete realization of multi-p p-adicity would be in terms of infinite integers coming as power series $\sum x_n N^n$ and having interpretation as p-adic numbers for any prime dividing N.

2. Effective 2-dimensionality would suggest that the individual p-adic topologies could be assigned with the 2-dimensional partonic surfaces. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. There are however reasons to think that even single partonic 2-surface corresponds to a multi-p p-adic topology.

Do infinite primes code for the finite measurement resolution?

The above describe heuristic picture is not yet satisfactory. In order to proceed, it is good to ask what determines the finite prime or set of them associated with a given partonic 2-surface. It is good to recall first the recent view about the p-adicization program relying crucially on the notion of finite measurement resolution.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M and measurement resolution does not depend on on the value of M. Situation is different if one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer. For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$ if modular arithetics is used to define the the measurement resolution. This works at the first level of the hierarchy but not at higher

levels. The alternative manner to assign a finite measurement resolution to M/N for given p is as $\Delta\phi = 2\pi|N/M|_p = 2\pi/p^n$. In this case the small fermionic part of the infinite prime would fix the measurement resolution. The argument below shows that only this option works also at the higher levels of hierarchy and is therefore more plausible.

3. p-Adicization conditions in their strong form require that the notion of integration based on harmonic analysis in symmetric spaces makes sense even at the level of partonic 2-surfaces. These conditions are satisfied if the partonic 2-surfaces in a given measurement resolution can be regarded as algebraic continuations of discrete surfaces whose points belong to the discrete variant of the $\delta M^4_\pm \times CP_2$. This condition is extremely powerful since it effectively allows to code the geometry of partonic 2-surfaces by the geometry of finite sub-manifold geometries for a given measurement resolution. This condition assigns the integer N to a given partonic surface and all primes appearing as factors of N define possible effective p-adic topologies assignable to the partonic 2-surface.

How infinite primes could then code for the finite measurement resolution? Can one identify the measurement resolution for $M/N = M/(Rp^n)$ as $\Delta\phi = ((M/R)\ mod\ p^n) \times 2\pi/p^n$ or as $\Delta\phi = 2\pi/p^n$? The following argument allows only the latter option.

1. Suppose that p-adic topology makes sense also for infinite primes and that state function reduction selects power of infinite prime P from the product of lower level infinite primes defining the integer N in M/N. Suppose that the rational defined by infinite integer defines measurement resolution also at the higher levels of the hierarchy.

2. The infinite primes at the first level of hierarchy representing Fock states are in one-one correspondence with finite rationals M/N for which integers M and N can be chosen to characterize the infinite bosonic part and finite fermionic part of the infinite prime. This correspondence makes sense also at higher levels of the hierarchy but M and N are infinite integers. Also other option obtained by exchanging "bosonic" and "fermionic" but later it will be found that only the first identification makes sense.

3. The first guess is that the rational M/N characterizing the infinite prime characterizes the measurement resolution for angles and therefore partially classifies also the finite sub-manifold geometry assignable to the partonic 2-surface. One should define what $M/N = ((M/R)\ mod\ P^n) \times P^{-n}$ is for infinite primes. This would require expression of M/R in modular arithmetics modulo P^n. This does not make sense.

4. For the second option the measurement resolution defined as $\Delta\phi = 2\pi|N/M|_P = 2\pi/P^n$ makes sense. The Fourier basis obtained in this manner would be infinite but all states $exp(ik/P^n)$ would correspond in real sense to real unity unless one allows k to be infinite P-adic integer smaller than P^n and thus expressible as $k = \sum_{m<n} k_m P^m$, where k_m are infinite integers smaller than P. In real sense one obtains all roots $exp(iq2\pi)$ of unity with $q < 1$ rational. For instance, for $n = 1$ one can have $0 < k/P < 1$ for a suitably chosen infinite prime k. Thus one would have essentially continuum theory at higher levels of the hierarchy. The purely fermionic part N of the infinite prime would code for both the number of Fourier components in discretization for each power of prime involved and the ratio characterize the angle resolution.

The proposed relation between infinite prime and finite measurement resolution implies very strong number theoretic selection rules on the reaction vertices.

1. The point is that the vertices of generalized Feynman diagrams correspond to partonic 2-surfaces at which the ends of light-like 3-surfaces describing the orbits of partonic 2-surfaces join together. Suppose that the partonic 2-surfaces appearing a both ends of the propagator lines correspond to same rational as finite sub-manifold geometries. If so, then for a given p-adic effective topology the integers assignable to all lines entering the vertex must contain this p-adic prime as a factor. Particles would correspond to integers and only the particles having common prime factors could appear in the same vertex.

2. In fact, already the work with modelling dark matter [K25] led to ask whether particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions. It also seemed natural to assume that that only the space-time sheets containing common primes in this collection can interact. This inspired the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The possibility of multi-p p-adicity raises the question about how to fix the p-adic prime characterizing the mass of the particle. The mass scale of the contribution of a given throat to the mass squared is given by $p^{-n/2}$, where $T = 1/n$ corresponds to the p-adic temperature of throat. Hence the dominating contribution to the mass squared corresponds to the smallest prime power p^n associated with the throats of the particle. This works if the integers characterizing other particles than graviton are divisible by the gravitonic p-adic prime or a product of p-adic primes assignable to graviton. If the smallest power p^n assignable to the graviton is large enough, the mass of graviton is consistent with the empirical bounds on it. The same consideration applies in the case of photons. Recall that the number theoretically very natural condition that in zero energy ontology the number of generalized Feynman graphs contributing to a given process is finite is satisfied if all particles have a non-vanishing but arbitrarily small p-adic thermal mass [K26] .

Interpretational problem

The identification of infinite prime as a characterizer of finite measurement resolution looks nice but there is an interpretational problem.

1. The model characterizing the quantum numbers of WCW spinor fields to be discussed in the next section involves a pair of infinite primes P_+ and P_- corresponding to the two vacuum primes $X \pm 1$. Do they correspond to two different measurement resolutions perhaps assignable to CD and CP_2 degrees of freedom?

2. Different measurement resolutions in CD and CP_2 degrees of freedom need not be not a problem as long as one considers only the discrete variants of symmetric spaces involved. What might be a problem is that in the general case the p-adic primes associated with CD and CP_2 degrees of freedom would not be same unless the integers N_+ and N_- are assumed to have have same prime factors (they indeed do if $p^0 = 1$ is formally counted as prime power factors).

3. The idea of assigning different p-adic effective topologies to CD and CP_2 does not look attractive. Both CD and CP_2 and thus also partonic 2-surface could however possess simultaneously both p-adic effective topologies. This kind of option might make sense since the integers representable as infinite powers series of integer N can be regarded as p-adic integers for all prime factors of N. As a matter fact, this kind of multi-p p-adicity could make sense also for the partonic 2-surfaces characterized by a measurement resolution $\Delta\phi = 2\pi M/N$. One would have what might be interpreted as N_+N_--adicity.

4. It will be found that quantum measurement means also the measurement of the p-adic prime selecting same p-adic prime from N_+ and N_-. If $N_\pm$ is divisible only by $p^0 = 1$, the corresponding angle measurement resolution is trivial. From the point of view of consciousness state function reduction selects also the p-adic prime characterizing the cognitive representation which is very natural since quantum superpositions of different p-adic topologies are not natural physically.

4.4.3 How the hierarchy of Planck constants could relate to infinite primes and p-adic hierarchy?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K80] , the dark matter hierarchy characterized by increasing values of $\hbar$ [K23, K21] , the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations give hopes about developing a more quantitative vision about the relationship between these hierarchies, in particular between the hierarchy of infinite primes, p-adic length scale hierarchy, and the hierarchy if Planck constants.

If infinite primes code for the hierarchy of measurement resolutions, the correlations between the p-adic hierarchy and the hierarchy of Planck constants indeed suggest themselves and allow also to select between two interpretations for the fact that two infinite primes N_+ and N_- are needed to characterize elementary particles (see the next section).

Recall that the hierarchy of Planck constants in the most general situation corresponds to a replacement M^4 and CP_2 factors of the imbedding space with singular coverings and factor spaces. The condition that Planck constant is integer valued allows only singular coverings characterized by two integers n_a *resp.* n_b assignable to CD *resp.* CP_2. This condition also guarantees that a given value of Planck constant corresponds to only a finite number of pages of the "Big Book" and therefore looks rather attractive mathematically. This option also forces evolution as a dispersion to the pages of the books characterized by increasing values of Planck constant.

Concerning the correspondence between the hierarchy of Planck constants and p-adic length scale hierarchy there seems to be only single working option. The following assumptions make precise the relationship between finite measurement resolution, infinite primes and hierarchy of Planck constants.

1. Measurement resolution CD *resp.* CP_2 degrees of freedom is assumed to correspond to the rational M_+/N_+ *resp.* M_-/N_-. $N_\pm$ is identified as the integer assigned to the fermionic part of the infinite integer..

2. One must always fix the consideration to a fixed p-adic prime. This process could be regarded as analogous to fixing the quantization axes and p would also characterize the p-adic cognitive space-time sheets involved. The p-adic prime is therefore same for CD and CP_2 degrees of freedom as required by internal consistency.

3. The relationship to the hierarchy of Planck constants is fixed by the identifications $n_a = n_+(p)$ and $n_b = n_-(p)$ so that the number of sheets of the covering equals to the number of bosons in the fermionic mode p of the quantum state defined by infinite prime.

4. A physically attractive hypothesis is that number theoretical bosons *resp.* fermions correspond to WCW orbital *resp.* spin degrees of freedom. The first ones correspond to the symplectic algebra of WCW and the latter one to purely fermionic degrees of freedom.

Consider now the basic consequences of these assumptions from the point of view of physics and cognition.

1. Finite measurement resolution reduces for a given value of p to

$$\Delta\phi = \frac{2\pi}{p^{n_\pm(p)+1}} = \frac{2\pi}{p^{n_{a/b}}} \quad ,$$

   where $n_\pm(p) = n_{a/b} - 1$ is the number of bosons in the mode p in the fermionic part of the state. The number theoretical fermions and bosons and also their probably existing physical counterparts are necessary for a non-trivial angle measurement resolution. The value of Planck constant given by

$$\frac{\hbar}{\hbar_0} = n_a n_b = (n_+(p) + 1) \times (n_-(p) + 1)$$

tells the total number of bosons added to the fermionic mode p assigned to the infinite prime.

2. The presence of $\hbar > \hbar_0$ partonic 2-surfaces is absolutely essential for a Universe able to measure its own state. This is in accordance with the interpretation of hierarchy of Planck constants in TGD inspired theory of consciousness. One can also say that $\hbar = 0$ sector does not allow cognition at all since $N_{\pm} = 1$ holds true. For given p $\hbar = n_a n_b = 0$ means that given fermionic prime corresponds to a fermion in the Dirac sea meaning $n_{\pm}(p) = -1$. Kicking out of fermions from Direac sea makes possible cognition. For purely bosonic vacuum primes one has $\hbar = 0$ meaning trivial measurement resolution so that the physics is purely classical and would correspond to the purely bosonic sector of the quantum TGD.

3. For $\hbar = \hbar_0$ the number of bosons in the fermionic state vanishes and the general expression for the measurement resolution reduces to $\Delta\phi = 2\pi/p$. When one adds $n_{\pm}(p)$ bosons to the fermionic part of the infinite prime, the measurement resolution increases from $\Delta\phi = 2\pi/p$ to $\Delta\phi = 2\pi/p^{n_{\pm}(p)+1}$. Adding a sheet to the covering means addition of a number theoretic boson to the fermionic part of infinite prime. The presence of both number theoretic bosons and fermions with the values of p-adic prime $p_1 \neq p$ does not affect the measurement resolution $\Delta\phi = 2\pi/p^n$ for a given prime p.

4. The resolutions in CD and CP_2 degrees of freedom correspond to the same value of the p-adic prime p so that one has dicretizations based on $\Delta\phi = 2\pi/p^{n_a}$ in CD degrees of freedom and $\Delta\phi = 2\pi/p^{n_b}$ in CP_2 degrees of freedom. The finite sub-manifold geometries make sense in this case and since the effective p-adic topology is same, the continuation to continuous p-adic partonic 2-surface is possible.

p-Adic thermodynamics involves the p-adic temperature $T = 1/n$ as basic parameter and the p-adic mass scale of the particle comes as $p^{-(n+1)/2}$. The natural question is whether one could assume the relation $T_{\pm} = 1/(n_{\pm}(p) + 1)$ between p-adic temperature and infinite prime and thus the relations $T_a = 1/n_a(p)$ and $T_b = 1/n_b(p)$. This identification is not consistent with the recent physical interpretation of the p-adic thermodynamics nor with the view about dark matter hierarchy and must be given up.

1. The minimal non-trivial measurement resolution with $n_i = 1$ and $\hbar = \hbar_0$ corresponds to the p-adic temperature $T_i = 1$. p-Adic mass calculations indeed predict $T = 1$ for fermions for $\hbar = \hbar_0$. In the case of gauge bosons $T \geq 2$ is favored so that gauge bosons would be dark. This would require that gauge bosons propagate along dark pages of the Big Book and become "visible" before entering to the interaction vertex.

2. p-Adic thermodynamics also assumes same p-adic temperature in CD and CP_2 degrees of freedom but the proposed identification allows also different temperatures. In principle the separation of the super-conformal degrees of freedom of CD and CP_2 might allow different p-adic temperatures. This would assign to different p-adic mass scales to the particles and the larger mass scale should give the dominant contribution.

3. For dark particles the p-adic mass scale would be by a factor $1/\sqrt{p}^{n_i(p)-1}$ lower than for ordinary particles. This is in conflict with the assumption that the mass of the particle does not depend on $\hbar$. This prediction would kill completely the recent vision about the dark matter.

4.5 How infinite primes could correspond to quantum states and space-time surfaces?

The hierarchy of infinite primes is in one-one correspondence with a hierarchy of second quantizations of an arithmetic quantum field theory. The additive quantum number in question is energy like quantity for ordinary primes and given by the logarithm of prime whereas p-adic length scale hypothesis suggests that the conserved quantity is proportional to the inverse of prime or its square root. For infinite primes at the first level of hierarchy these quantum numbers label single particles states having interpretation as ordinary elementary particles. For octonionic and hyper-octonionic primes the quantum number is analogous to a momentum with 8 components. The question is whether these number theoretic quantum numbers could have interpretation as genuine quantum numbers. Quantum classical correspondence raises another question. Is it possible to label space-time surfaces by infinite primes? Could this correspondence be even one-to-one?

I have considered these questions already more than decade ago. The discussion at that time was necessarily highly speculative and just a mathematical exercise. After that time however a lot of progress has taken place in quantum TGD and it is highly interaction to see what comes out from the interaction of the notion of infinite prime with the notions of zero energy ontology and generalized imbedding space, and with the recent vision about how Chern-Simons Dirac term in the Kähler-Dirac action allows to code information about quantum numbers to the space-time geometry. The possibility of this coding allows to simplify the discussion dramatically. If one could map infinite hyper-octonionic or hyper-quaternionic primes to quantum numbers of the standard model naturally, then the their map of to the geometry of space-time surfaces would realize the coding of space-time surfaces by infinite primes (and more generally by integers and rationals). Also a detailed realization of number theoretic Brahman=Atman identity would emerge as an outcome.

4.5.1 A brief summary about various moduli spaces and their symmetries

It is good to sum up the number theoretic symmetries before trying to construct an overall view about the situation. Several kinds of number theoretical symmetry groups are involved corresponding to symmetries in the moduli spaces of hyper-octonionic and hyper-quaternionic structures, symmetries mapping hyper-octonionic primes to hyper-octonionic primes, and translations acting in the space of causal diamonds (CDs) and shifting. The moduli space for CDs labeled by pairs of its tips that its pairs of points of $M^4 \times CP_2$ is also in important role.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit. $SO(7,1)$ acts as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for hyper-quaternionic structures.

2. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

3. Color group $SU(3)$ is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. For given hyper-octonionic prime one can identify a subgroup of SU(3) generating a finite set of hyper-octonionic primes for it at sphere S^7. This suggests wave function at the orbit of given hyper-octonionic prime in turn generalizing to wave functions in the space of infinite primes.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of M^8 giving rise to the preferred point of H.

These symmetries deserve a more detailed discussion.

1. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures. $SO(7)$ respects the choice of the real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as substructures of hyper-octonionic structure. The choice of global hyper-octonionic structures involves also a choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of CD. Since the integers representing physical states must be

hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3) \times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in M^8 picture. The choice of both tips of CD reduces $SO(1,3)$ to $SO(3)$.

2. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic M^8 primes 8_v and to fermionic M^8 primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

3. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit meaning a choice of preferred hyper-complex plane $M^4 \subset M^4$. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement in number theoretical and as it turns also in physical sense. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

4.5.2 Associativity and commutativity or only their quantum variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Hyper-quaternionicity formulated in terms of the modified gamma matrices defined by Kähler action fixes classical space-time dynamics and a very beautiful algebra formulation of quantum TGD in terms of hyper-octonionic local Clifford algebra of imbedding space emerges. There is no need for the use of hyper-octonion real analytic maps although one cannot exclude the possibility that they might be involved with the construction of hyper-quaternionic space-time surfaces.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to complex rational infinite primes. Since one can decompose complex rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative and commutative. In case of associativity this means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

4.5.3 The correspondence between infinite primes and standard model quantum numbers

I have considered several candidates for the correspondence between infinite primes and standard model quantum numbers. The confusing aspect has been the dual nature of hyper-octonionic primes. One one hand they could be interpreted as components of 8-D momentum representing perhaps momentum and other quantum numbers. On the other hand, they transform like representations of $SU(3) \subset G_2$ and behave like color singlets and triplets so that the idea about quantum superpositions of infinite primes related by $SU(3)$ action is attractive. The second puzzling feature is that there are two kinds of infinite primes corresponding to two signs for the "small" part of the infinite prime. The following proposal leads to an interpretation for these aspects.

1. The number of components of hyper-octonionic prime is 8 as is the dimension of the Cartan algebra of the product of Poincare group, color group SU(3) and electro-weak gauge group $SU(2)_L \times U(1)$ defining the quantum numbers of particles. One might therefore dream about a number theoretic interpretation of elementary particle quantum numbers by interpreting hyper-octonionic prime as 8-momentum. This form of the big idea fails. The point is that complexified basis for octonions consists of two color singlets and color triplet and its conjugate. For a given hyper-octonionic prime one can construct new primes by using a subgroup G of SU(3) by definition respecting the property that the values of the components of prime as integers and as a consequence also the modulus squared so that the primes are at sphere S^7. This group is analogous to Galois group. Identifying prime as an element of basis of quantum states, one can form wave functions at the discrete orbit of given prime transforming according to irreducible representations of color group. Triality $t \pm 1$ states correspond to color partial waves associated with quarks and antiquarks and triality $t = 0$ states to gluons and leptons and their color excitations. The states can be chosen to be eigenstates of the preferred hyper-octonionic imaginary unit ie_1. Additive four-momentum could be assigned the M^2 part of the hyper-octonion as will be found. Therefore the construction applies in special but natural coordinates assignable to the particle required also by zero energy ontology and hierarchy of Planck constants as well as by p-adicization program.

2. This construction gives only the quantum numbers assignable to color partial waves in WCW degrees of freedom. Also the quantum numbers assignable to imbedding space spinors are wanted. Luckily, there are two kinds of infinite primes, which might be denoted by $P_\pm$ because the sign of the "small" part of the infinite prime can be chosen freely. Super-conformal symmetry suggests that quantum numbers associated with spinorial and WCW degrees freedom can be assigned to the infinite primes of these two types.

   (a) In the case of spinor degrees of freedom one can restrict the multiplets to those generated by SU(2) subgroup of SU(3) identified as rotation group. The interpretation is in terms of automorphism group of quaternions. Discrete subgroups of SU(2) generate the orbit of given hyper-octonionic prime and one obtains finite number of SU(2) multiplets having interpretation in terms of rotational degrees of freedom associated with the light-cone boundary. In the case of fermions (bosons) only half odd integer (integer) spins are allowed.

   (b) Remarkably, four of the hyper-octonionic units remain invariant under SU(2). Also now only the hyper-complex projection in $M^2 \subset M^4$ can be interpreted as four-momentum in the preferred frame and the interpretation as a counterpart of Dirac equation eliminating four complex non-physical helicities of the imbedding spinor of given chirality. The states of same spin associated with the two spin doublets have interpretation as electro-weak doublets. As a representation of SU(3) electro-weak doublets would correspond to quark and antiquark in color isospin doublet. This leaves two additional quantum numbers assignable to the color isospin singlets. The natural interpretation is in terms of electromagnetic charge and weak isospin. An analogous picture emerges also in the description of super-symmetric QFT limit of TGD [K27] replacing massless particles identified as light-like geodesics of M^4 with light like geodesics of $M^4 \times CP_2$ and assigning to them two quantum numbers in the Cartan algebra of $SU(3)$ and identified as electro-weak charges. Also conformal weight expressible in terms of stringy mass formula allows a description in terms of infinite primes. What is not achieved is the number theoretical description of genus of the partonic 2-surface and wave functions in the moduli space of the partonic 2-surfaces.

3. In this picture leptons, gauge bosons, and gluons correspond to an infinite prime of type P_+ or P_- whereas quarks as well as color excitations of leptons correspond to a pair of primes of type P_+ and P_-. One can fix the notations by assigning color quantum numbers to P_+ and and spinorial quantum numbers to P_-. Both P_+ and P_- contribute to four-momentum.

Each pair of infinite primes of this kind defines a finite-dimensional space of quantum states assignable to the subgroups of $SU(3)$ and $SU(2)$ respecting the prime property. Needless to say, this prediction is extremely powerful and fixes the spectrum of the quantum numbers almost completely!

4. An interesting question is whether one can require number theoretical color confinement in the sense that the physical states resulting as tensor products of states assignable to a given infinite prime in P_+ are color singlets. This might be necessary to guarantee associativity. G_2 singletness would be even stronger condition but not possible for massless states. What is interesting is that spin and color in well-defined sense separate from each other. One can wonder whether this relates somehow to the spin puzzle of proton meaning that quarks do not seem to contribute to baryonic spin.

5. The appearance of discrete subgroups of $SU(3)$ and $SU(2)$ strongly suggests a connection with the inclusions of the hyper-finite factors of type II_1 characterized by these subgroups, which are expected to play a fundamental role in quantum TGD. An interesting question is whether also infinite subgroups could be involved. For instance, one can consider the subgroups generated by discrete subgroup and infinite cyclic group and these might be involved with the inclusions for which the index is equal to four. The appearance of these groups suggests also a connection with the hierarchy of Planck constants and one can ask how the singular coverings defining the pages of the book like structure relate to the moduli space of causal diamonds.

The rather unexpected conclusion is that the wave functions in the discrete space defined by infinite primes are able to code for the quantum numbers of WCW spinor fields and thus for WCW spinor fields. A fascinating possibility is that even M-matrix- which is nothing but a characterization of zero energy state- could find an elegant formulation as entanglement coefficients associated with the pair of the integer and inverse integer characterizing the positive and negative energy states.

1. The great vision is that associativity and commutativity conditions fix the number theoretical quantum dynamics completely. Quantum associativity states that the wave functions in the space of infinite primes, integers, and rationals are invariant under associations of finite hyper-octonionic primes ($A(BC)$ and $(AB)C$ are the basic associations), physics requires associativity only apart from a phase factor, in the simplest situation $+1/-1$ but in more general case phase factor. The condition of commutativity poses a more familiar condition implying that permutations induce only a phase factor which is +/- 1 for boson and fermion statistics and a more general phase for quantum group statistics for the anyonic phases, which correspond to nonstandard values of Planck constant in TGD framework. These symmetries induce time-like entanglement for zero energy stats and perhaps non-trivial enough M-matrix.

2. One must also remember that besides the infinite primes defining the counterparts of free Fock states of supersymmetric QFT, also infinite primes analogous to bound states are predicted. The analogy with polynomial primes illustrates what is involved. In the space of polynomials with integer coefficients polynomials of degree one correspond free single particle states and one can form free many particle states as their products. Higher degree polynomials with algebraic roots correspond to bound states being not decomposable to a product of polynomials of first degree in the field of rationals. Could also positive and negative energy parts of zero energy states form a analog of bound state giving rise to highly non-trivial M-matrix?

4.5.4 How space-time geometry could be coded by infinite primes?

Second key question is whether space-time geometry could be characterized in terms of infinite primes (and integers and rationals in the most general case) and how this is achieved. The question is how the quantum states consisting of fundamental fermions serving as building bricks of elementary particles could be coded by infinite quaternionic integeres to which one can assign ordinary finite quaternionic primes.

The basic idea is roughly that at the first level of the hierarchy the finite primes appearing as building blocks of infinite prime correspond to structures formed by pairs or wormhole contacts assigned with elementary particles.

1. The partonic orbits defined by wormhole throats could be characterized by finite primes specifying the preferred p-adic topology assignable to the p-adic "cognitive representation" of the throat.
2. One could assign hyper-quaternionic integer to the real particle as its four-momentum. In this case the mass shell condition would fix the hyper-quaternionic integer to a high extent. All discrete Lorentz boosts of the particle state taking hyper-quaternionic integers to hyper-quaternionic integers would correspond to the same p-adic integer (prime) defined by the length of the Lorentz boosted hyper-quaternionic integer. The p-adic prime characterizing virtual particle would be one of the primes appearing in the factorization of this integer to a product of powers of prime, most naturally the one whose power is largest.

   Note that p-adic length scale hypothesis suggests that the p-adic primes near powers of two are favored for on mass shell particles and perhaps also for the virtual particles.
3. For fundamental fermions associated with boundaries of string world sheets and appearing as building bricks of particles the masses would vanish on mass shell so that the hyper-quaternionic integer would in this case have vanishing norm.

   The virtual four-momentum assigned to a virtual fermion line as a generalized eigenvalue of Chern-Simons Dirac operator would correspond to hyper-quaternionic integer. In this case p-adic prime would be defined as for physical particles and would depend on the mass of the virtual particle. If the integration over virtual momenta by residue calculus effectively leads to an integral over on mass shell massless virtual momenta with non-physical spinor helicities then also virtual fundamental fermions would correspond to zero norm hyper-quaternionic integers.
4. The correlation between particle's four-momentum and the p-adic prime characterizing corresponding cognitive representation would be in accordance with quantum classical correspondence.
5. The hyperquaternionic primes appearing as largest factors in the factorizion of hyper-quaternionic integers assignable with physical particles could be interpreted as building bricks of an infinite hyperquaternionic prime characterizing the many-particle state and at least the boundaries of string world sheets. The idea that p-adic space-time surfaces defined "cognitive representations" as p-adic chart maps of real space-time surfaces and vice versa (as the TGD based definition of p-adic manifolds assumes) suggests that the p-adic primes in question characterize also space-time regions rather than only the boundaries of string world sheets.

A couple of comments about this speculation are in order.

1. Zero energy ontology implies a hierarchy of CDs within CDs and this hierarchy as well as the hierarchy of space-time sheets corresponds naturally to the hierarchy of infinite primes. One can assign standard model quantum numbers to various partonic 2-surfaces with positive and negative energy parts of the quantum state assignable to the light-like boundaries of CD. Also infinite integers and rationals are possible and the inverses of infinite primes would naturally correspond to elementary particles with negative energy. The condition that zero energy state has vanishing net quantum numbers implies that the ratio of infinite integers assignable to zero energy state equals to real unit in real sense and has has vanishing total quantum numbers.
2. Neither quantum numbers nor infinite primes coding them cannot characterize the partonic 2-surface itself completely since they say nothing about the deformation of the space-time surface but only about labels characterizing the WCW spinor field. Also the topology of partonic 2-surface fails to be coded. Quantum classical correspondence however suggests that this correspondence could be possible in a weaker sense. In the Gaussian approximation for functional integral over the world of classical worlds space-time surface and thus the

collection of partonic 2-surfaces is effectively replaced with the one corresponding to the maximum of Kähler function, and in this sense one-one correspondence is possible unless the situation is non-perturbative. In this case the physics implied by the hierarchy of Planck constants could however guarantee uniqueness.

One of the basic ideas behind the identification of the dark matter as phases with non-standard value of Planck constant is that when perturbative description of the system fails, a phase transition increasing the value of Planck constant takes place and makes perturbative description possible. Geometrically this phase transition means a leakage to another sector of the imbedding space realized as a book like structure with pages partially labeled by the values of Planck constant. Anyonic phases and fractionization of quantum numbers is one possible outcome of this phase transition. An interesting question is what the fractionization of the quantum numbers means number theoretically.

4.5.5 How to achieve consistency with p-adic mass formula

The first argument against the proposal that infinite primes could code for four-momentum in preferred coordinates is that the logarithms of finite primes and even less those of hyper-octonionic primes are natural from the point of view of p-adic mass calculations predicting that the mass squared of particle behaves as $1/p$ for $T_p = 1$ (fermions) and $1/p^2$ for $T_p = 1/2$ (gauge bosons). This difficulty might be circumvented.

Ordinary primes

Consider first ordinary primes for which the inverse always exists.

1. One can map finite primes p to phase factors $exp(i2\pi/p)$. The roots of unity play the role of primes in the decomposition of the roots of unity $exp(i2\pi/n)$, $n = \prod_i p_i^{n_i}$. $1/n$ is expressible as a sum of form

$$\begin{aligned} \frac{1}{n} &= \sum_i P_i \ , \\ P_i &= \frac{k_i}{p_i^{n_i}} \ . \end{aligned} \tag{4.5.1}$$

giving

$$exp(\frac{i2\pi}{n}) = exp(i2\pi \textstyle\sum_i P_i) = exp(i2\pi \sum_i \frac{k_i}{p_i^{n_i}}) \ . \tag{4.5.2}$$

Apart from a common normalization factor one can interpret the coefficients P_i as energy like quantities assigned to the single particle states. The power $p_i^{n_i}$ would correspond to various p-adic inverse temperature $1/T_p = 2n_i$ in this expansion.

2. The representation in terms of phase factors is not unique since P_i^k and $P_i^k + np_i^k$ define the same phase. This non-uniqueness is completely analogous to the non-uniqueness of momentum in the presence of a discrete translational symmetry and can be interpreted in terms of lattice momentum. Physically this corresponds to a finite measurement resolution. Also in the formulation of symplectic QFT defining one part of quantum TGD only phases defined by the roots of unity appear and similar non-uniqueness emerges and is due to the discretization serving as a space-time correlate for a finite measurement resolution implying UV cutoff.

3. Mass squared is proportional to $1/p_i^2$ so that only the p-adic temperatures $T_p = 1/2n_i$ are possible for rational primes. For more general primes one can however have also a situation in which the modulus square of prime is ordinary prime. For instance, Gaussian (complex) primes $P = m + in$ satisfy $|P|^2 = p$ for $p\ mod\ 4 = 1$ and $|P|^2 = p^2$ for $p\ mod\ 4 = 3$ (for example, rational prime 5 decomposes as $5 = (2+i)(2-i)$). Therefore it is possible to have states satisfying $M^2 \propto 1/p$, p ordinary prime for hyper-octonionic primes. These primes correspond to the rational primes decomposing to the products of ordinary primes and also also higher roots of p might be possible. The finite prime assignable to the hyper-octonionic prime has a natural interpretation as the p-adic prime assignable to an elementary particle. In zero energy ontology this assignment makes sense also for virtual particles having interpretation as pairs of positive and negative energy on mass shell particles assignable to the light-like throats of wormhole contact.

Hyper-octonionic primes with inverse

Consider next the situation for hyper-octonionic primes when the integers in question have inverse. We are interested only in the longitudinal part of infinite prime in M^2. The phase factor makes sense also in the case of hyper-octonionic primes if the condition $|P| > 0$ holds true so that one has massive particles in 8-D sense possibly resulting via p-adic thermodynamics. If the imaginary unit appearing in the exponent is the imaginary unit i appearing in the complexification of octonions, the exponent has the character of a phase factor for hyper-octonionic primes. The reason is that $1/P = P^*/|P|^2$ is hyper-octonionic number of form $O_0 + iO_1$, where O_1 is a purely imaginary octonion. The exponent in the phase factor is therefore $2\pi(iO_0 - O_1)$ and involves only imaginary units, and one can write $exp(i2\pi(O_0 + iO_1)) = exp(iO_0) \times exp(-O_1)$. Both factors are phase factors. This condition analogous to unitarity is one further good reason for hyper-octonions and Minkowskian signature.

Light-like hyper-octonionic primes

The proposed representation as a phase factor fails for massless particles since light-like hyper-primes do not possess an inverse. One must therefore define the notion of primeness differently to see what might be the physical interpretation of these primes. Since the multiplication of hyper-octonionic integer by light-like prime yields zero norm prime, the natural interpretation would be as a gauge transformation and one might consider gauge transformations obtained by exponentiating Lie algebra with light-like coefficients.

One can consider two options depending on whether one requires that the relevant algebra has unit or not.

1. For the first option hyper-octonionic light-like integers are of form $n(1+e)$ and the product of two light-like integers $n_i(1+e)$ is of form $2n_1n_2(1+e)$. Here e could be arbitrary hyper-octonionic imaginary unit consistent with the prime property. This does not however allow unit light-like integer acting like unit since one has $(1+e)^2 = 2(1+e)$. All odd integers would be primes.

2. The number $E = (1+e)/2$ behaves as a unit. If one requires that unit is included in the algebra integers can be defined as numbers of form nE so that their product is n_1n_2E and equivalent with the ordinary product of integers so that primes correspond to ordinary primes.

One can construct the first level infinite primes from these primes just as in the case of ordinary primes. Now however $X = \prod p_i$ is replaced with $X = \prod_n[(2n+1)(1+e)]$ for the first option and equal to the $X = E\prod p_i$ for the second option.

The multiplicative phase factor could be defined for both options as $exp(i2\pi E/N)$ where N is a light-like hyper-octonionic integer. This definition would eliminate the singular $1/E$ factor and the situation reduces essentially to that for ordinary primes in the case of massless states. If the infinite prime $P_\pm$ is such that one can assign to it non-trivial multiplets in color or rotational degrees of freedom (half odd integer spin for fermions) it must have a part in the complement of M^2. For standard model elementary particles this is always the case. The energy spectrum is of form $1/2(2m+1)$ or $1/p$. For light-like hyper-octonions the projection to M^2 is in general time-like and quantized. If one does not allow the unit E in exponent the phase factor is ill-defined and one must identify the light-like hyper-octonionic primes as gauge degrees of freedom.

M^2 momentum is light-like only for states which are spinless color and electro-weak singlets having no counterpart in standard model counterpart nor in quantum TGD. Therefore light-like hyper-octonionic primes reducing to M^2 could correspond to gauge degrees of freedom. M^2 momentum is of form $P = (1,1)/2(2m+1)$ for the first option and of form $P = (1,1)/p$ for the second option. Even for graviton, photon, gluons, and right handed neutrino either hyper-octonionic prime is space-like if the state is massless. Light-like hyper-octonions can however characterize massive states but the proposed interpretation in terms of gauge degrees of freedom is highly suggestive.

If one interprets hyper-octonionic prime as 8-D momentum, which is of course not necessary in the recent framework, one could worry about conflict with TGD variant of twistor program. In accordance with associativity the role of 8-momentum in fermionic propagator is however taken by its projection to the hyper-quaternionic sub-space defined by the modified gamma matrices at given point of space-time sheet and masslessness holds for this projection so that 8-D tachyons are possible [K79] . This is highly analogous to the identification of the four-momentum as M^2 projection of hyperfinite prime.

The treatment of zero modes

There are also zero modes which are absolutely crucial for quantum measurement theory. They entangle with quantum fluctuating degrees of freedom in quantum measurement situation and thus map quantum numbers to positions of pointers. The interior degrees of freedom of space-time interior must correspond to zero modes and they represent space-time correlates for quantum states realized at light-like partonic 3-surfaces. Quantum measurement theory suggests 1-1 correspondence between zero modes and quantum fluctuating degrees of freedom so that also super-symmetry should have zero mode counterpart. The recent progress in understanding of the modified Dirac action [K26] leads to a concrete identification of the super-conformal algebra of zero modes as related to the deformation of the space-time surface defining vanishing second variations of Kähler action.

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see fig. `http://www.tgdtheory.fi/appfigures/planckhierarchy.jpg`, which is also in the appendix of this book).

4.5.6 Complexification of octonions in zero energy ontology

The complexification of octonions plays a crucial role in the number theoretical vision and could be regarded as its weakest point. It has however a natural physical interpretation in zero energy ontology.

1. CD has two tips, which correspond to the points of M^4. For M^4 the fixing of the quantization axes requires choosing a time-like direction fixing the rest system. This direction is naturally defined by the tips of CD. The moduli space for CDs is $M^4 \times M^4_+$. The realization of the hierarchy of Planck constants forces also a choice of a space-like direction fixing the quantization axes of spin.

2. In the case of CP_2 the choice of the quantization axes requires fixing of a preferred point of CP_2 remaining invariant under $U(2)$ subgroup of $SU(3)$ acting linearly on complex coordinates having origin at this point and containing also the Cartan subgroup. This fixes the quantization axes of color hyper-charge. If the preferred CP_2 points associated with the light-like boundaries of CD are different they fix a unique geodesic circle of CP_2 fixing the quantization axes for color isospin. The moduli space is therefore $(CP_2)^2$.

3. The full moduli space is $M^4 \times M^4_+ \times (CP_2)^2$. In M^8 description the moduli space would naturally correspond to pairs of points of M^4 and E^4 so that the moduli space for the choices CDs and quantization axes would be $M^4 \times M^4_+ \times (E^4)^2$. This space can be regarded locally as the space of complexified octonions.

4. p-Adic length scale hypothesis follows if the time-like distance between the tips of CDs is quantized in powers of two so that a union of 3-D proper-time constant hyperboloids of M^4_+ results. Hierarchy of Planck constants implies rational multiples of these basic distances. Hyperboloids are coset spaces of Lorentz group and this suggests even more general quantization in which one replaces the hyperboloids with spaces obtained by identifying the points related by the action of a discrete subgroup of Lorentz group. This would give the analog of lattice cell obtained and one would obtain a lattice like structure consisting of unit cells labeled by the elements of the sub-group of Lorentz group. The interpretation of the moduli space of CDs as a discrete momentum space dual to WCW is suggestive. In the case of CP_2 similar quantization could correspond to the replacement of CP_2 with equivalence classes of points of CP_2 under action of a discrete subgroup of $SU(3)$.

5. Could this discrete space be identified as the space of hyper-octonionic primes as looks natural? In other words, could the discrete points of the dual space $M^4_+ \times CP_2$ decompose to subsets in one-one corresponds with the orbits of G_+ and G_- appearing in the reductions $SO(7,1) \rightarrow SO(7) \rightarrow G_2 \rightarrow SU(3) \rightarrow G_+$ for primes in P_+ and $SO(7,1) \rightarrow SO(7) \rightarrow G_2 \rightarrow SU(3) \rightarrow SU(2) \rightarrow G_-$ in P_-? One can also consider the subgroups of G_2 respecting the hyperbolic prime property. This would allow to integrate $G_+ \times G_-$ multiplets to larger multiplets and get an over all view about multiplet structure. An interesting question is whether $SO(7,1)$ could contain non-compact discrete subgroups with infinite number of elements and respecting the property of being hyper-octonionic prime. If this idea is correct, the dual space $M^4_+ \times CP_2$ would play a role of heavenly sphere providing a representation for the quantum numbers labeling WCW spinor fields.

4.5.7 The relation to number theoretic Brahman=Atman identity

Number theoretic Brahman=Atman identity -one might also use the term algebraic holography - states the number theoretic anatomy of single space-time point is enough to code for both WCW and and WCW spinors fields- the quantum states of entire Universe or at least the sub-Universe defined by CD. The entire quantum TGD could be represented in terms of 8-D imbedding space with the notion of number generalized to allow real units defined as ration of infinite integers and having number theoretical anatomy.

Before continuing it is perhaps good to represent the most obvious objection against the idea. The correspondence between WCW and WCW spinors with infinite rationals and their discreteness means that also WCW (world of classical worlds) and space of WCW spinors should be discrete. First this looks non-sensible but is indeed what one obtains if space-time surfaces correspond to light-like 3-surfaces expressible in terms of algebraic equations involving rational functions with rational coefficients.

By the above considerations it is indeed clear that zero energy states correspond to ratios of infinite integers boiling down to a hyper-octonionic unit with vanishing net four-momentum and electro-weak charges. WCW spinor fields can be mapped to wave functions in the space of these units and even the reduced configuration space consisting of the maxima of Kähler function could be coded by these wave functions. The wave functions in the space of hyper-octonion units would be induced by the discrete wave functions associated with the orbits of hyper-octonionic finite primes appearing in the decomposition of the infinite hyper-octonionic primes of type P_+ and P_-. The net color and quantum numbers and spin associated with the wave function in the space of hyper-octonionic units are vanishing. Clearly, a detailed realization of number theoretic Brahman=Atman identity emerges predicting reducing even the spectrum of possible quantum numbers to number theory.

In the original formulation of Brahman-Atman identity the description based on H was used. This leads to the conclusion that that the analog of a complex Schrödinger amplitude in the space of number-theoretic anatomies of a given imbedding space point represented by single point of H and represented as 8-tuples of real units should naturally represent the dependence of WCW spinors understood as ground states of super-conformal representations obtained as an 8-fold tensor power of a fundamental representation or product of representations perhaps differing somehow. The 8-tuples define a number theoretical analog of $U(1)^8$ group in terms of which all number theoretical symmetries are represented. This description should be equivalent with the use of single hyper-octonion unit.

4.6 Infinite primes and mathematical consciousness

The mathematics of infinity relates naturally with the mystery of consciousness and religious and mystic experience. In particular, mathematical cognition might have as a space-time correlate the infinitely structured space-time points implied by the introduction of infinite-dimensional space of real units defined by infinite (hyper-)octonionic rationals having unit norm in the real sense. I hope that the reader takes this section as a noble attempt to get a glimpse about unknown rather than final conclusions.

4.6.1 Algebraic Brahman=Atman identity

The proposed view about cognition and intentionality emerges from the notion of infinite primes, which was actually the first genuinely new mathematical idea inspired by TGD inspired consciousness theorizing. Infinite primes, integers, and rationals have a precise number theoretic anatomy. For instance, the simplest infinite primes correspond to the numbers $P_\pm = X \pm 1$, where $X = \prod_k p_k$ is the product of all finite primes. Indeed, $P_\pm \ mod \ p = 1$ holds true for all finite primes. The construction of infinite primes at the first level of the hierarchy is structurally analogous to the quantization of super-symmetric arithmetic quantum field theory with finite primes playing the role of momenta associated with fermions and bosons. Also the counterparts of bound states emerge. This process can be iterated at the second level the product of infinite primes constructed at the first level replaces X and so on.

The structural similarity with repeatedly second quantized quantum field theory strongly suggests that physics might in some sense reduce to a number theory for infinite rationals M/N and that second quantization could be followed by further quantizations. As a matter fact, the hierarchy of space-time sheets could realize this endless second quantization geometrically and have also a direct connection with the hierarchy of logics labeled by their order. This could have rather breathtaking implications.

1. One is forced to ask whether this hierarchy corresponds to a hierarchy of realities for which level below corresponds in a literal sense infinitesimals and the level next above to infinity.
2. Second implication is that there is an infinite number of infinite rationals behaving like real units ($M/N \equiv 1$ in real sense) so that space-time points could have infinitely rich number theoretical anatomy not detectable at the level of real physics. Infinite integers would correspond to positive energy many particle states and their inverses (infinitesimals with number theoretic structure) to negative energy many particle states and $M/N \equiv 1$ would be a counterpart for zero energy ontology to which oneness and emptiness are assigned in mysticism.
3. Single space-time point, which is usually regarded as the most primitive and completely irreducible structure of mathematics, would take the role of Platonia of mathematical ideas being able to represent in its number theoretical structure even the quantum state of entire Universe. Algebraic Brahman=Atman identity and algebraic holography would be realized in a rather literal sense.

Number theoretic anatomy of space-time point

This number theoretical anatomy should relate to mathematical consciousness in some manner. For instance, one can ask whether it makes sense to speak about quantum jumps changing the number theoretical anatomy of space-time points and whether these quantum jumps give rise to mathematical ideas. In fact, the identifications of Platonia as spinor fields in WCW on one hand and as the set number theoretical anatomies of point of imbedding space force the conclusion that WCW spinor fields (recall also the identification as correlates for logical mind) can be realized in terms of the space for number theoretic anatomies of imbedding space points. Therefore quantum jumps would be correspond to changes in the anatomy of the space-time points. Or more precisely, to the changes of the WCW spinor fields regarded as wave functions in the set of imbedding space points which are equivalent in real sense. Imbedding space would be experiencing genuine number theoretical evolution. The whole physics would reduce to the anatomy of numbers. All mathematical notions which are more than mere human inventions would be imbeddable to the Platonia realized as the number theoretical anatomies of single imbedding space point.

To realize this picture would require that WCW spinor fields and perhaps even WCW allow a mapping to the number theoretic anatomies of space-time point. In finite-dimension Euclidian spaces momentum space labelling plane wavs is dual to the space. One could hope that also now the "orbital" quantum numbers of WCW spinor fields could code for WCW in given measurement resolution. The construction of the previous sections realize the mapping of the quantum states defined by WCW spinors fields assignable to given CD to wave function in the space of hyper-octonionic units. These wave functions can be also regarded as linear combinations of these units if the coefficients are complex numbers formed using the commuting imaginary unit of complexified octonions so that that the Hilbert space like structure in question would have purely number theoretic meaning. The rationals defined by infinite primes characterize also measurement resolution and classify the the finite sub-manifold geometries associated with partonic two-surfaces. At higher levels one has rationals defined by ratios of infinite integers and one can ask whether this interpretation generalizes.

Note that one must distinguish between two kinds of hyper-octonionic units.

1. Already in the case of complex numbers one has rational complex units defined in terms of Pythagorean triangle and their products generate infinite dimensional space. The hyper-octonionic units defined as ratios U of infinite integers and suggested to provide a representation of WCW spinor fields correspond to these. The powers U^m define roots of unity which can be regarded analogous to $exp(i2\pi x)$, where x is not rational but the exponent itself is complex rational.

2. Besides this there are roots of unity which are in general algebraic complex numbers. These roots of unit correspond to phases $exp(i2\pi M/N)$, where M/N is ratio of real infinite integers and i is the commuting hyper-octonionic imaginary unit. These real infinite integers can be assigned to hyper-octonionic integers by replacing everywhere finite hyper-octonionic primes with their norm which is ordinary prime. By the previous considerations only the phases $exp(i2\pi M/P^n)$ make sense p-adically for infinite primes P.

4.6.2 Leaving the world of finite reals and ending up to the ancient Greece

If strong number theoretic vision is accepted, all physical predictions of quantum TGD would be numbers in finite algebraic extensions of rationals at the first level of hierarchy. Just the numbers which ancient Greeks were able to construct by the technical means at use! This seems rather paradoxical but conforms also with the hypothesis that the dicrete algebraic intersections of real and p-adic 2-surfaces provide the fundamental cognitive representations.

The proposed construction for infinite primes gives a precise division of infinite primes to classes: the ratios of primes in given class span a *subset of rational numbers*. These classes give much more refined classification of infinities than infinite ordinals or alephs. They would correspond to separate phases in the evolution of consciousness identified as a sequence of quantum jumps defining sequence of primes $\rightarrow p_1 \rightarrow p_2$...... Infinite primes could mean a transition from space-time level to the level of function spaces. WCW is example of a space which can be parameterized by a space of functions locally.

The minimal assumption is that infinite primes reflect their presence only in the possibility to multiply the coordinates of imbedding space points by real units formed as ratios of infinite integers. The correspondence between polynomials and infinite primes gives hopes of mapping at least the reduced WCW consisting of the the maxima of Kähler function to the anatomy of space-time point. Also WCW spinors and perhaps also the the modes of WCW spinor fields would allow this kind of map.

One can consider also the possibility that infinite integers and rationals give rise to a hierarchy of imbedding spaces such that given level represents infinitesimals from the point of view of higher levels in hierarchy. Even 'simultaneous' time evolutions of conscious experiences at different aleph levels with completely different time scales (to put it mildly) are possible since the time values around which the contents of conscious experience are possibly located, are determined by the quantum jump: also multi-snapshots containing snapshots also from different aleph levels are possible. Un-integrated conscious experiences with all values of p could be contained in given quantum jump: this would give rise to a hierarchy of conscious beings: the habitants above given level could be called Gods with full reason: those above us would probably call us just 'epsilons' if ready to admit that we exist at all except in non-rigorous formulations of elementary calculus!

4.6.3 Infinite primes and mystic world view

The proposed interpretation deserves some additional comments from the point of consciousness theory.

1. An open problem is whether the finite integer S appearing in the infinite prime is product of only finite or possibly even infinite number of lower level primes at a given level of hierarchy. The proposed physical identification of S indeed allows S to be a product of infinitely many primes. One can allow also M and N appearing in the infinite and infinite part to be contain infinite number of factors. In this manner one obtains a hierarchy of infinite primes expressible in the form

$$\begin{aligned}
P &= nY^{r_1} + mS \ \ , \quad r = 1, 2, ... \\
m &= m_0 + P_{r_2}(Y) \ \ , \\
Y &= \frac{X}{S} \ \ , \\
S &= \prod_i P_i \ \ .
\end{aligned}$$

   Note that this ansatz is in principle of the same general form as the original ansatz $P = nY + mS$. These primes correspond in physical analogy to states containing infinite number of particles.

   If one poses no restrictions on S this implies that that the cardinality for the set of infinite primes at first level would be $c = 2^{alef_0}$ ($alef_0$ is the cardinality of natural numbers). This is the cardinality for *all* subsets of natural numbers equal to the cardinality of reals. At the next level one obtains the cardinality 2^c for *all* subsets of reals, etc....

   If S were always a product of *finite number of primes* and $k(p)$ would differ from zero for finite number of primes only, the cardinality of infinite primes would be $alef_0$ at each level. One could pose the condition that mS is infinitesimal as compared to nX/S. This would guarantee that the ratio of two infinite primes at the same level would be well defined and equal to n_1S_2/n_2S_1. On the other hand, the requirement that all rationals are obtained as ratios of infinite primes requires that no restrictions are posed on $k(p)$: in this case the cardinality coming from possible choices of $r = ms$ is the cardinality of reals at first level.

   The possibility of primes for which also S is finite would mean that the algebra determined by the infinite primes must be generalized. For the primes representing states containing infinite number of bosons and/or fermions it would be be possible to tell how P_1P_2 and P_2P_1 differ and these primes would behave like elements of free algebra. As already found, this kind of free algebra would provide single space-time point with enormous algebraic representative power and analog of Brahman=Atman identity would result.

2. There is no physical subsystem-complement decomposition for the infinite primes of form $X \pm 1$ since fermionic degrees of freedom are not excited at all. Mystic could interpret it as a state of consciousness in which all separations vanish and there is no observer-observed distinction anymore. A state of pure awareness would be in question if bosonic and fermionic excitations represent the contents of consciousness! Since fermionic many particle states identifiable as Boolean statements about basic statements are identified as representation for reflective level of consciousness, $S = 1$ means that the reflective level of consciousness is absent: enlightment as the end of thoughts according to mystics.

   The mystic experiences of oneness ($S = 1!$), of emptiness (the subset of primes defined by S is empty!) and of the absence of all separations (there is no subsystem-complement separation and hence no division between observer and observed) could be related to quantum jumps to this kind of sectors of the WCW. In super-symmetric interpretation $S = 1$ means that state contains no fermions.

3. There is entire hierarchy of selves corresponding to the hierarchy of infinite primes and the relationship between selves at different levels of the hierarchy is like the relationship between God and human being. Infinite primes at the lowest level would presumably represent elementary particles. This implies a hierarchy for moments of consciousness and it would be un-natural to exclude the existence of higher level 'beings' (one might call them Angels, Gods, etc...).

4.6.4 Infinite primes and evolution

The original argument leading to the notion of infinite primes was simple. Generalized unitarity implies evolution as a gradual increase of the p-adic prime labeling the WCW sector D_p to which the localization associated with quantum jump occurs. Infinite p-adic primes are forced by the requirement that p-adic prime increases in a statistical sense and that the number of quantum jumps already occurred is infinite (assuming finite number of these quantum jumps and therefore the first quantum jump, one encounters the problem of deciding what was the first WCW spinor field).

Quantum classical correspondence requires that p-adic evolution of the space-time surface with respect to geometric time repeats in some sense the p-adic evolution by quantum jumps implied by the generalized unitarity [K29] . Infinite p-adic primes are in a well defined sense composites of the primes belonging to lower level of infinity and at the bottom of this de-compositional hierarchy are finite primes. This decomposition corresponds to the decomposition of the space-time surface into p-adic regions which in TGD inspired theory of consciousness correspond to selves. Therefore the increase of the composite primes at lower level of infinity induces the increase of the infinite p-adic prime. p-Adic prime can increase in two manners.

1. One can introduce the concept of the p-adic sub-evolution: the evolution of infinite prime P is induced by the sub-evolution of infinite primes belonging to a lower level of infinity being induced by .... being induced by the evolution at the level of finite primes. For instance, the increase of the cell size means increase of the p-adic prime characterizing it: neurons are indeed very large and complicated cells whereas bacteria are small. Sub-evolution occurs both in subjective and geometric sense.

   (a) For a given value of geometric time the p-adic prime of a given space-time sheet gradually increases in the evolution by quantum jumps: our geometric past evolves also!

   (b) The p-adic prime characterizing space-time sheet also increases as the geometric time associated with the space-time sheet increases (say during morphogenesis).

   The notion of sub-evolution is in accordance with the "Ontogeny recapitulates phylogeny" principle: the evolution of organism, now the entire Universe, contains the evolutions of the more primitive organisms as sub-evolutions.

2. Infinite prime increases also when entirely new finite primes emerge in the decomposition of an infinite prime to finite primes. This means that entirely new space-time sheets representing new structures emerge in quantum jumps. The creation of space-time sheets in quantum jumps could correspond to this process. By quantum classical correspondence this process corresponds at the space-time level to phase transitions giving rise to new material space-time sheets with more and more refined effective p-adic effective topology.

4.7 Does the notion of infinite-P p-adicity make sense?

In this section speculations related to infinite-P p-adicity are represented in the form of shy questions in order to not irritate too much the possible reader. The basic open question causing the tension is whether infinite primes relate only to the physics of cognition or whether they might allow to say something non-trivial about the physics of matter too.

The following list of questions is rather natural with the background provided by the p-adic physics.

1. Can one generalize the notion of p-adic norm and p-adic number field to include infinite primes? Could one define the counterpart of p-adic topology for literally infinite values of p? Does the topology R_P for infinite values of P approximate or is it equivalent with real topology as p-adic topology at the limit of infinite p is assumed to do (at least in the sense that p-adic variants of Diophantine equations at this limit correspond to ordinary Diophantine equations)? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale.

2. Canonical identification maps p-adic numbers of unit norm to real numbers in the range $[0, p]$. Does the canonical identification map the p-adic numbers R_P associated with infinite prime to reals? Could the number fields R_P provide alternative formulations/generalizations of the non-standard analysis based on the hyper-real numbers of Robinson [A193] ?

3. The notion of finite measurement resolution for angle variables given naturally as a hierarchy $2\pi/p^n$ of resolutions for a given p-adic prime defining aa hierarchy of algebraic extension of p-adic numbers is central in the attempts to formulate p-adic variants of quantum TGD and fuse them with real number based quantum TGD [K71] . If p is replaced with an infinite prime, the angular resolution becomes ideal and the roots of unity $exp(2\pi m/p^n)$ are replaced with real units unless also the integer m is replaced with an infinite integer M so that the ratio M/P^n is finite rational number. Could this approach be regarded as alternative for real number based notion of phase angle?

The consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p. Using infinite primes one might obtain the real theory in this approximation.

The question discussed in this section is whether the notion of p-adic number field makes sense makes sense for infinite primes and whether it might have some physical relevance. One can formally introduce power series in powers of any infinite prime P and the coefficients can be taken to belong to any ordinary number field. In the representation by polynomials P-adic power series correspond to Laurent series in powers of corresponding polynomial and are completely finite.

For straightforward generalization of the norm all powers of infinite-P prime have vanishing norm. The infinite-p p-adic norm of infinite-p p-adic integer would be given by its finite part so that in this sense positive powers of P would represent infinitesimals. For Laurent series this would mean that the lowest term would give the whole approximation in the real topology. For finite-primes one could however replace the norm as a power of p by a power of some other number. This would allow to have a finite norm also for P-adic primes. Since the simplest P-adic primes at the lowest level of hierarchy define naturally a rational one might consider the possibility of defining the norm of P as the inverse of this rational.

4.7.1 Does infinite-P p-adicity reduce to q-adicity?

Any non-vanishing p-adic number is expressible as a product of power of p multiplied by a p-adic unit which can be infinite as a normal integer and has pinary expansion in powers of p:

$$x = p^n(x_0 + \sum_{k>0} x_k p^k) \ , \ x_k \in \{0,..,p-1\} \ , \ x_0 > 0 \ . \tag{4.7.1}$$

The p-adic norm of x is given by $N_p(x) = p^{-n}$. Each unit has p-adic inverse which for finite integers is always infinite as an ordinary integer.

To define infinite-P p-adic numbers one must generalize the pinary expansion to a infinite-P p-adic expansion of an infinite rational. In particular, one must identify what the statement 'infinite integer modulo P' means when P is infinite prime, and what are the infinite integers N satisfying the condition $N < P$. Also one must be able to construct the p-adic inverse of any infinite prime.

The correspondence of infinite primes with polynomials allows to construct infinite-P p-adics in a straightforward manner.

Consider first the infinite integers at the lowest level.

1. Infinite-P p-adics at the first level of hierarhcy correspond to Laurent series like expansions using an irreducible polynomial P of degree n representing infinite prime. The coefficients of the series are numbers in the coefficient fields. Modulo p operation is replaced with modulo polynomial P operation giving a unique result and one can calculate the coefficients of the expansion in powers of P by the same algorithm as in the case of the ordinary p-adic numbers. In the case of n-variables the coefficients of Taylor series are naturally rational functions of at most $n-1$ variables. For infinite primes this means rationals formed from lower level infinite-primes.

2. Infinite-P p-adic units correspond to expansions of this type having non-vanishing zeroth order term. Polynomials take the role of finite integers. The inverse of a infinite integer in P-adic number field is obtained by developing the polynomial counterpart of $1/N$ in the following manner. Express N in the form $N = N_0(1+x_1P+..)$, where N_0 is polynomial with degree at most equal to $n-1$. The factor $1/(1+x_1P+...)$ can be developed in geometric series so that only the calculation of $1/N_0$ remains. Calculate first the inverse $\hat{N}_0^{-1}$ of N_0 as an element of the 'finite field' defined by the polynomials modulo P: a polynomial having degree at most equal to $n-1$ results. Express $1/N_0$ as

$$\frac{1}{N_0} = \hat{N}_0^{-1}(1+y_1P+...)$$

   and calculate the coefficients in the expansion iteratively using the condition $N \times (1/N) = 1$ by applying polynomial modulo arithmetics. Generalizing this, one can develop any rational function to power series with respect to polynomial prime P. The expansion with respect to a polynomial prime can in turn be translated to an expansion with respect to infinite prime and also mapped to a superposition of Fock states.

3. What about the norm of infinite-P p-adic integers? Ultra-metricity suggest a straightforward generalization of the usual p-adic norm. The direct generalization of the finite-p p-adic norm would mean the identification of infinite-P p-adic norm as P^{-n}, where n corresponds to the lowest order term in the polymomial expansion. Thus the norm would be infinite for $n<0$, equal to one for $n=0$ and vanish for $n>0$. Any polynomial integer N would have vanishing norm with respect to those infinite-P p-adics for which P divides N. Essentially discrete topology would result.

This seems too trivial to be interesting. One can however replace P^{-n} with a^{-n}, where a is any finite number a without losing the multiplicativity and ultra-metricity properties of the norm. The function space associated with the polynomial defined by P serves as a guideline also now. This space is naturally q-adic for some rational number q. At the lowest level the infinite prime defines naturally an ordinary rational number as the zero of the polynomial as is clear from the definition of the polynomial. At higher levels of the hierarchy the rational number is rational function of lower level infinite primes and by continuing the assignments of lower level rational functions to the infinite primes one ends up with an assignment of a unique rational number with a given infinite prime serving as an excellent candidate for a rational defining the q-adicity.

4.7.2 q-Adic topology determined by infinite prime as a local topology of WCW

Since infinite primes correspond to polynomials, infinite-P p-adic topology, which by previous considerations would be actually q-adic topology, is a natural candidate for a topology in function spaces, in particular in the WCW.

This view conforms also with the idea of algebraic holography. The sub-spaces of WCW can be modelled in terms of function spaces of rational functions, their algebraic extensions, and their P-adic completions. The mapping of the elements of these spaces to infinite rationals would make possible the correspondence between WCW and number theoretic anatomy of point of the imbedding space.

The q-adic norm for these function spaces is in turn consistent with the ultra-metricity for the space of maxima of Kähler functions conjectured to be all that is needed to construct S-matrix. Ultra-metricity conforms nicely with the expected four-dimensional spin glass degeneracy due to the enormous vacuum degeneracy meaning that maxima of Kähler function define the analog of spin glass free energy landscape. That only maxima of Kähler function would be needed would mean that radiative corrections to WCW integral would vanish as quantum criticality indeed requires. This TGD can be regarded as an analog of for an integrable quantum theory. Quantum criticality is absolutely essential for guaranteeing that S-matrix and U-matric elements are algebraic numbers which in turn guarantees number theoretic universality of quantum TGD.

4.7.3 The interpretation of the discrete topology determined by infinite prime

Also $p = 1$-adic topology makes formally sense and corresponds to a discrete topology in which all rationals have unit norm. It results also results if one naively generalizes p-adic topology to infinite-p p-adic topology by defining the norm of infinite prime at the lowest level of hierarchy as $|P|_P = 1/P = 0$. In this topology the distance between two points is either 1 or 0 and this topology is the roughest possible topology one can imagine.

It must be however noticed that if one maps infinite-P p-adics to real by the formal generalization of the canonical identification then one obtains real topology naturally if coefficients of powers of P are taken to be reals. This would mean that infinite-P p-adic topology would be equivalent with real topology.

Consider now the possible interpretations.

1. At the level of function spaces infinite-p p-adic topology in the naive sense has a completely natural interpretation and states that the replacement of the Taylor series with its lowest term.

2. The formal possibility of $p = 1$-adic topology at space-time level suggests a possible interpretation for the mysterious infinite degeneracy caused by the presence of the absolute minima of the Kähler function: one can add to any absolute minimum a vacuum extremal, which behaves completely randomly except for the constraints forcing the surface to be a vacuum extremal. This non-determinism is much more general than the non-determinism involving a discrete sequence of bifurcations (I have used the term association sequence about this kind of sequences). This suggests that one must replace the concept of 3-surface with a more general one, allowing also continuous association sequences consisting of a continuous family of space-like 3-surfaces with infinitesimally small time like separations. These continuous association sequences would be analogous to vacuum bubbles of the quantum field theories.

One can even consider the possibility that vacuum extremals are non-differentiable and even discontinuous obeying only effective $p = 1$-adic topology. Also modified Dirac operator vanishes identically in this case. Since vacuum surfaces are in question, $p = 1$ regions cannot correspond to material sheets carrying energy and also the identification as cognitive space-time sheets is questionable. Since $p = 1$, the smallest possible prime in generalized sense, it must represent the lowest possible level of evolution, primordial chaos. Quantum classical correspondence suggests that $p = 1$ level is indeed present at the space-time level and might realized by the mysterious vacuum extremals.

4.8 How infinite primes relate to other views about mathematical infinity?

Infinite primes is a purely TGD inspired notion. The notion of infinity is number theoretical and infinite primes have well defined divisibility properties. One can partially order them by the real norm. p-Adic norms of infinite primes are well defined and finite. The construction of infinite primes is a hierarchical procedure structurally equivalent to a repeated second quantization of a supersymmetric arithmetic quantum field theory. At the lowest level bosons and fermions are labelled by ordinary primes. At the next level one obtains free Fock states plus states having interpretation as bound many particle states. The many particle states of a given level become the single particle states of the next level and one can repeat the construction ad infinitum. The analogy with quantum theory is intriguing and I have proposed that the quantum states in TGD Universe correspond to octonionic generalizations of infinite primes.

It is interesting to compare infinite primes (and integers) to the Cantorian view about infinite ordinals and cardinals. The basic problems of Cantor's approach which relate to the axiom of choice, continuum hypothesis, and Russell's antinomy: all these problems relate to the definition of ordinals as sets. In TGD framework infinite primes, integers, and rationals are defined purely algebraically so that these problems are avoided. It is not surprising that these approaches are not equivalent. For instance, sum and product for Cantorian ordinals are not commutative unlike for infinite integers defined in terms of infinite primes.

Set theory defines the foundations of modern mathematics. Set theory relies strongly on classical physics, and the obvious question is whether one should reconsider the foundations of mathematics in light of quantum physics. Is set theory really the correct approach to axiomatization?

1. Quantum view about consciousness and cognition leads to a proposal that p-adic physics serves as a correlate for cognition. Together with the notion of infinite primes this suggests that number theory should play a key role in the axiomatics.

2. Algebraic geometry allows algebraization of the set theory and this kind of approach suggests itself strongly in physics inspired approach to the foundations of mathematics. This means powerful limitations on the notion of set.

3. Finite measurement resolution and finite resolution of cognition could have implications also for the foundations of mathematics and relate directly to the fact that all numerical approaches reduce to an approximation using rationals with a cutoff on the number of binary digits.

4. The TGD inspired vision about consciousness implies evolution by quantum jumps meaning that also evolution of mathematics so that no fixed system of axioms can ever catch all the mathematical truths for the simple reason that mathematicians themselves evolve with mathematics.

I will discuss possible impact of these observations on the foundations of physical mathematics assuming that one accepts the TGD inspired view about infinity, about the notion of number, and the restrictions on the notion of set suggested by classical TGD.

4.8.1 Cantorian view about infinity

The question which I have but repeatedly under the rug during the last fifteen years concerns the relationship of infinite primes to the notion of infinity as Cantor and his followers have understood it. I must be honest: I have been too lazy to even explain to myself what Cantor really said. Therefore the reading of the New Scientist article "The Ultimate logic: to infinity and beyond" [A139] was a pleasant surprise since it gave a bird's eye of view about how the ideas about infinity have evolved after Cantor as a response to severe difficulties in the set theoretic formulation for the foundations of Mathematics.

Cantor's paradize

I try to summarize Cantor's view about infinity first. Cantor was the pioneer of set theory, in particular the theory of infinite sets. Cantor started his work around 1870. His goal was to formulate all notions of mathematics in terms of sets, in particular natural numbers. Cardinals and ordinals define two kind of infinite numbers in Cantor's approach.

1. Cantor realized that real numbers are "more numerous" than natural numbers and understood the importance of one-to-one correspondence (bijection) in set theory. One can say

that two sets related by bijection have same cardinality. This led to the notion of cardinal number. Cardinals are represented as sets and two cardinals are same if a bijection exists between the corresponding sets. For instance, the infinite cardinals assignable to natural numbers and reals are different since no bijection between them exists.

2. The definition of ordinal relies on successor axiom of natural numbers generalized to allow infinitely large ordinals. Given ordinal can be identified as the union of all ordinals strictly smaller than it. Well ordering is a closely related notion and states that every subset of ordinals has smallest element. One can classify ordinals to three types: 0, elements with predecessor, and elements without predecessor such as ω, which corresponds to the ordinal defined as the union of all natural numbers.

   The number of ordinals much larger than the number of cardinals. This is clear since the notion of ordinal involves additional structure coming from their ordering. A given cardinal corresponds to infinitely many ordinals and one can identify the cardinal as the smallest ordinal of this kind. For instance, ω and $\omega+n$ correspond to same cardinal α_0 (countable infinity) for all finite values of n.

3. Cantor introduced the notion of power set as the set of all subsets of the set and proved that the cardinality of the power set is larger than that of set. Cantor introduced also the continuum hypothesis stating that there are no cardinals between the cardinal $\aleph_0$ *resp.* $\aleph_1$ assignable to natural numbers *resp.* reals. Hilbert represented continuum hypothesis as one of his 23 problems in his talk at the 1900 International Congress of Mathematicians in Paris. Hilbert was also a defender of Cantor and introduced the term Cantor's paradise.

4. Cantor developed the arithmetics of ordinals based on sum, product, and power: each of these operations is expressible in terms of set theoretic concepts. For infinite ordinals multiplication and sum are not commutative anymore. This looks highly counter intuitive and requires detailed definition of the sum and product. Sum means just writing the ordered sequences representing ordinals in succession. To see the non-commutativity of sum it is enough to notice that the number of elements having predecessor is not the same for $\omega+n$ and $n+\omega$.

   To see the non-commutativity of product it is enough to notice that the product is define as cartesian product $S\times T$ of the ordered sets representing the ordinals. This means that every element of T is replaced with S. It is easy to see that $n\times\omega$ and $\omega\times n$ are different.

   One can define also the powers (exponentials) in the arithmetics of ordinals: exponent must reduce to the notion of power set X^Y, which can be realized as the set of maps $Y\rightarrow X$ and has formally $\#X^{\#Y}$ elements.

It is pity that the we physicists have so pragmatic attitude to mathematics that we do not have time to realize the beauty of the idea about reduction of all mathematics to set theory. This is even more regrettable since it might well be that the manner to make progress in physics might require replacing the mathematics with a mathematics which does not rely on set theory alone.

Snakes in Cantor's paradise

Cantor's paradise is extremely beautiful place but there are snakes there. Continuum hypothesis looked to Cantor intuitively obvious but the attempts to prove it failed. Betrand Russel showed in 1901 that the logical basis of Cantor's set theory was flawed. This manifested itself via a simple paradox. Assume that it makes sense to speak about the set of all ordinals. This is by definition ordinal itself since ordinal is a set consisting of all ordinals strictly smaller than it. But this would mean that the set of all ordinals is its own member! The famous barber's paradox is a more concrete manner to express Russel's antinomy. One cannot speak of the set of ordinals and must introduce the notion of class. Russell introduced also the notion of types and type theory.

At 1920 Ernst Zermelo and Abraham Fraenkel devised a series of rules for manipulating sets but these rules did not allow to resolve the status of the continuum hypothesis. The stumbling block was the rule known as "axiom of choice" stating that if you have a collection of sets you can form a new set by picking one element from each of them. At first this sounds rather obvious but in the case when there is no obvious rule telling how to do it, situation becomes non-trivial.

Then Polish mathematicians Stefan Banach and Alfred Tarski managed to show how the axiom would allow the division of a spherical ball to six subsets which can then be arranged to two balls with the same size as the original ball using only rotations and translations. These six sets are non-measurable in terms of Lebesque measure. The non-intuitive outcome must relate to the definition of the volume of the ball that is integration or measure theory: the axioms of measure theory should bring in constraints preventing construction of the six sets.

Around 1931 Kurt Gödel proved the incompleteness theorem that it is not possible to axiomaticize arithmetics using any axiom system. There always remain unprovable propositions, which are true and cannot be proved to be true. This kind of statement is analogous to "I am a statement which cannot be proved to be true". If this statement could be proved to be true it would not be true.

Constructing logical universes

The attempts to expel the snakes from Cantor's paradise led to the idea that by posing some constraints it might be possible to construct logically consistent set theory obeying Zermelo-Fraenkel axioms such that continuum hypothesis and the axiom of choice would hold true and which would be free of paradoxes such as Banach-Tarski paradox.

Around 1938 Gödel introduced what he called " constructible universe" or L world satisfying these constraints. The structure of L world is hierarchical and one can say that the successor idea manifests itself directly in the construction. The levels are labeled by ordinals and one can always add a new level. The introduction of a new level to the hierarchy means that new axioms are introduced to the system bringing in meta level to the mathematical structure. The axiom system can be extended indefinitely. Gödel's theorem holds true at given level of hierarchy but by adding new levels non-probable truths can be made provable.

1963 Paul Cohen however demonstrated that there is infinite number of this kind of L worlds. In some of them continuum hypothesis holds true, in some of them the number of cardinals between $\aleph_0$ andα_1 can be arbitrary large - even infinite. This initiated a boom of constructions brings in mind the inflation of GUTs in particle physics and the endless variety of brane constructions and the landscape misery of M-theory. From the point of view of physicist the non-uniqueness in foundations of mathematics does not seem to matter much since the everyday mathematics would remain the familiar one. One can of course ask what about quantum theory: should quantum physics replace classical physics in the formulation of fundamental fo mathematics.

For instance, von Neuman proposed one particular L world. In von Neumann unverse one starts from natural numbers and constructs its power set and at each step in the construction one consideres power set assigned to the sete obtained at the previous level. It is clear that one imagine several options. One could consider all subsets, only finite subsets, or only subsets which have cardinality smaller than the set itself. Power sets identified as the set of all finite subsets would give minimal option. Power set identified as the set of all subsets would give the maximal option.

The work of Hugh Woodin represented in 2010 International Congress of Mathematicians in Hyderabad, India represents the last twist in the story. Woodin argues that one must step outside the system that is conventional mathematical world to solve the problem. Woodin has introduced so called Woodin cardinals whose existence implies that all "projective" subsets of reals have a measurable size: it is not an accident that the word "measure" appears here when one recalls what Banach-Tarski paradox states. Woodin was motivated by the problems of set theory. He expresses this by saying "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable".

Woodin proposed his own constructive universe which he calls ultimate L. It has all the desired properties: in particular, continuum hypothesis holds true. Physicists reader need not get frustrated if he fails to intuit why this is the case: for a decade ago Wooding himself did not believe in this. Also this L world is infinite tower to which one can add new levels.

4.8.2 The notion of infinity in TGD Universe

The construction of infinite primes, integers, and rationals brings strongly in mind the L worlds of Gödel and followers and this inspires the idea about concrete comparison of these approaches to see the differences.

Rule of thumb

It is good to start with a rule of thumb allowing to make strong conclusions about the cardinalities of infinite primes. If one considers the set formed by all finite subsets of a countable set you get a countable set because these subsets can be expressed as bit sequences with finite number if non-vanishing binary digits telling whether given element of set belongs to the subset or not: this bit sequence corresponds to a unique integer. If *all* subsets (also infinite) are allowed the set is not countably finite. If continuum hypothesis holds true it has at least as many elements as real line.

2-adic integers are good example. Consider first all 2-adic numbers with a *finite* number of non-vanishing bits (finite as real numbers). You get a countably infinite set since you can map these bit sequences to natural numbers in an obvious manner.

Consider next all possible bit sequences: most of them have infinite number bits. These numbers form naturally 2-adic continuum with 2-adic topology and differentiability. 2-adics can be mapped to real continuum in simple manner: canonical identification allows to do this continuously. The cardinality of these bit sequences is same as for reals as the rule of thumb would predict.

The hierarchy of infinite integers is based on number theoretical view about infinity and it would seem that these infinities are between the countable infinity and infinity defining the number of points of real axis. This reflects the fact that number theoretic infinity is much more refined notion than the infinities associated with cardinals and even ordinals. For instance, one can divide these infinities whereas Cantorian arithmetics contains only sum, product and power.

How Cantor's ordinals relate to the construction of infinite primes?

The fascinating question is whether the comparison of the construction of infinite primes, integers and rationals could relate to the work of Cantor and Gödel and his followers could provide new insights about infinite primes themselves.

1. What is intriguing that L-worlds are defined as infinite hierarchies just as the hierarchy of infinite primes and associated hierarchies. The great idea is that these constructions are essentially set theoretic in accordance with the vision that mathematics should reduce to set theory. As already noticed, naive set theory however leads to paradoxes which motivates the work of Gödel and followers. The basic physical philosophy is the identification of physical state as a set: this is essentially a notion belonging to classical physics.

2. TGD approach is algebraic rather than set theoretic. The construction is based on explicit formulas assuming the existence of weird quantities defined as product of all primes at previous level. These quantities can be taken as purely algebraic notions without any attempt to find a set theoretic definition.

   The possibility to interpret the construction as a repeated second quantization of a supersymmetric arithmetic quantum field theory with bosons and fermions labeled by ordinary primes at the loweset level of hierarchy replaces the set theoretic picture. These weird products of all primes represent Dirac sea at a given level of hierarchy and the many particles states of previous level become elementary particles at the new level of hierarchy. This construction is proposed to have a direct physical realization in terms of many-sheeted space-time and generalized to the level of octonionic primes is suggested to allow number theoretic interpretation of standard model quantum numbers.

   Perhaps it is not mere arrogance of quantum physics to argue that the classical set theoretic view about physical state is replaced with quantum view about it. Algebra replaces set theory and real and p-adic topologies are essential: for instance, infinite primes are infinite only in real topology.

One can raise many interesting questions. Although the underlying philosophies are very different, one can ask whether it might be possible to reduce TGD inspired construction to set theory playing key role in the construction of ordinals?

1. Can one assign to a given infinite integer a set in a natural manner? At the lowest level of hierarchy infinite prime can be mapped to a rational. Could one assign to this rational a set in cartesian product $N \times N$? Does this argument generalize to higher levels? Could the construction discussed in [K44] allow to realize the set theoretic representation?

2. The notion of divisibility and explicit formulas for infinite integers obviously imply that the number of infinite numbers is much larger than cardinals of Cantor. This is true also for the ordinals of Cantor. How infinite integers relate to the ordinals of Cantor for which successor axiom is true? Also now it makes sense to form successors and in general they correspond to products of infinite primes which can be mapped to polynomials of several variables. For infinite integers however also the predecessor always exists. For instance $X \pm 1$ are infinite primes, where X represents the product of primes at previous level. Only zero fails to have predecessor for infinite natural numbers.

3. In TGD framework one loses the very essential notion of well-orderedness stating that every ordinal corresponds to a set with smallest element: that is element without predecessor. For instance, the infinite numbers known as limits and by definition are infinite and have no predecessor, the simplest example about limit is ω, which corresponds to the union of all natural numbers. The study of predecessors allowed to conclude that the sum and product are non-commutative for ordinals. Since the notion of well-ordered set does not make sense for infinite integers, one cannot identify infinite integers as ordinals.

   One must however remember that just the well-orderedness hypothesis together with successor axiom allows to express ordinal as a union of strictly smaller ordinals. This in turn leads to the conclusion that ordinals cannot form a set and to Russel's antinomy and are responsible for the many problems of set theory forcing Wooding to sigh "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable". Maybe the well-orderedness axiom is simply too strong for infinite ordinals.

4. Sum, product, and power are the basic operations in the arithmetics of ordinals. All they reduce to set theoretic constructions. One can however define these operations purely algebraically. The algebraic definition of sum and product makes sense since one can map the infinite integers to polynomials of several variables. The possibly existing set theoretic definition of infinite integers using infinite sets cannot be consistent with the commutativity of sum and product defined algebraically. Either algebra or set theory but not both!

5. Also the notion of power makes sense for ordinals and relies on the notion of power set. Could the algebraic definition of exponential make sense? If the exponent N of M^N is finite integer, then the exponent makes sense for infinite M. If N is infinite integer it does not. Hence it seems that the analogs of numbers like ω^ω do not exist in TGD inspired L universe.

6. The failure of set theoretic reductionism brings in mind the motivic approach to integration as purely algebraic approach applied to the symbol defining the integral instead of a number approach based on set theoretic notions. The motivation of the motivic approach in p-adic context is that p-adic numbers are not well-ordered so that one loses the notion of boundary and orientation as topological concepts although they can make sense algebraically.

For the hierarchy infinite integers the notion of infinity relies on real norm, which is essentially length rather than on the cardinality of a set. This infinity is essentially non-Cantorian and it is perhaps useless to try to relate it to that of ordinal or cardinal. There is just an infinite hierarchy of infinities which replaces the hierarchy of ordinals and for which the real norm of ratio makes possible partial ordering. Clearly the notion of infinity is extremely slippery and one must carefully specify what one means with infinite.

Cardinals in TGD Universe

What about cardinals in TGD framework? There seems to be no reason for giving them up and the first guess is that TGD replaces cardinals and ordinals of Cantor with cardinals and the hierarchy of infinite primes, integers, and rationals.

1. The first question is what is the cardinal assignable to infinite primes at the first level of hierarchy. For the set of finite primes the cardinal is $\aleph_0$. For the first level of infinite primes the situation is not so simple. The simple infinite primes correspond to free Fock states constructed from fermions and bosons labelled by primes. They are in one-one correspondence with rationals. There is however infinite number of many particle bound states representable as products of irreducible polynomials of one variable with integer coefficients and having finite number of roots which are algebraic numbers. The set of algebraic numbers is countable. This suggests that the cardinality of set of infinite primes at the first level of hierarchy corresponds to $\aleph_0$. This if course assuming that infinite integers and rationals for a set although they themselves cannot be described as sets.

   If one allows Fock states containing infinite number of particles and having thus infinite energy one obtains formally polynomials of infinite degree identifiable as Taylor expansions. In this case the roots can be transcendental numbers and one expects that a cardinal larger than $\aleph_0$, say $\aleph_1$ emerges. In von Neumann's Universe one indeed allows all subsets and $\aleph_1$ appears already at the first level. The higher cardinals appear at higher levels.

   One cannot exclude the Fock states containing infinite number of quanta if one accepts the idea that infinite prime representing quantum state characterizing entire Universe make sense. Does this mean that $\aleph_1$ has meaning only for entire universe and for states carrying infinite energy (in ZEO the positive energy part of zero energy state would carry the infinite energy)?

2. What happens at the next levels of the hierarchy? One possibility is that infinite primes at each level define a countable set. The point is that in polynomials representation one considers only finite degree polynomials depending on finite number of variables, having rational coefficients. Therefore everything at the level of definitions is countable and finite and the product X of primes of previous level is just an algebraic symbol identifiable as a variable of polynomial.

3. In an alternative construction of infinite integers suggested in [**?**]ne considers the first level of the hierarchy the set of finite subsets of algebraic numbers and the set of finite subsets of this set at the next level and so on. All these sets are countable which suggests that the number of infinite primes at each level of the hierarchy is countable and that only the completion of algebraic number to reals or p-adic can give rise to $\aleph_1$. This would conform with the fact that quantum physics is basically based on counting of quanta and that finite measurement resolution is an essential restriction on what we can know.

What about the axiom of choice?

Axiom of choice has several variants. One variant is axiom of countable choice. Second variant is generalized continuum hypothesis states that the cardinality of an infinite set is between that of infinite set S and its power set: in other words there is no cardinal satisfying $\aleph_\alpha < \lambda < 2^{\aleph_\alpha}$ or equivalently: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$. For a finite collection of sets it can be proved but already when on has a countable collection of nonempty set and in the case that one cannot uniquely specify some preferred element of each set, axiom of choice must be postulated. For instance, each subset of natural numbers has smallest element so that there is no need to postulate axiom of choice separately. Also closed intervals of real axis have smallest element.

What happens to the axiom of choice in TGD Universe. TGD is a physical theory and this means that the laws of classical physics strong considerations on the allowed sets. Classical physics is in TGD framework the dictated by the Kähler action and by a principle selecting its preferred extremals. Although several almost formulations for this principle exist, it is far from being well-understood and it is not clear whether one can give explicit formula for preferred extremals. One formulation is as quaternionic sub-manifolds of 8-D imbedding space allowing octonionic structure in its tangent space and defined by octonionic representation of the gamma matrices defining the Clifford algebra.

1. The world of classical worlds can be regarded as the space of preferred extremals of Kähler action identifiable as certain 4-surfaces in $M^4 \times CP_2$. The mere extremal property implies also smoothness of the partonic 2-surfaces so that very powerful constraints are involved:

therefore situation is very far from the extreme generality of set theory where one does assumes neither continuity nor smoothness. Zero energy ontology means that this space effectively reduces to a collection of spaces assignable to causal diamonds. Strong form of holography reduces this space effectively to the space consisting of collectinons of partonic 2-surfaces at the light-like boundaries of CD plus 4-D tangent space data at them which very probably cannot be chosen freely.

2. In this kind of situation it might well happen that all collections of sets, say are finite or in the case that that they are countable they allow a unique choice of preferred point. Axiom of choice would not be needed. The specification of a preferred point of every 4-surface in the collection does not look a problem for a pragmatic physicist, since one can restrict the consideration to the boundaries of causal diamonds and consider for instance minimum of light-like radial coordinate. In fact, finite measurement resolution leads to the effective replacement of partonic 2-surfaces with the collection of ends of braid strands and the ends of braid strands define the preferred points. One might say, that physics with finite measurement resolution performs the choice automatically. A stronger form of this choice is that the points in question are rational points for a natural choice of the imbedding space coordinates.

Generalization of real numbers inspired by infinite integers

Surreal numbers define a generalization of reals obtained by introducing a hierarchy of infinite reals and infinitesimals as their inverses. Infinite integers and rationals in TGD sense could give rise to a similar generalization so that one would have an infinite hierarchy of 8-D imbedding space such that at given level previous level would represent infinitesimals.

TGD suggests another generalization of reals. One can construct from infinite integers rationals with unit norm. A possible interpretation would be as zero energy states with denominator and numerator representation positive and negative energy parts of the zero energy state and vanishing of total quantum numbers represented by real unit property. These numbers would have arbitrarily complex number theoretical anatomy however.

This structure has enormous representative power and one could dream that the world of classical worlds and spinor fields in this space could allow representation in terms of these real units. Brahman Atman Identity would be realized: the structure of single space-time point invisible to ordinary physics would represent the world of classical worlds! Single space-time point would be the Platonia!

Could one say that the space of all infinite rationals which are real units is countable? If previous arguments are correct this would seem to be true. If this is true, then TGD inspired notion of infinity would be extremely conservative as compared to the view proposed by Cantor and his followers using the Cantorian criteria. Just $\aleph_n$, $n = 0, 1$ and hierarchy of infinite integers which are countable sets. One can of course, ask how many surfaces WCW contains, what $\aleph$ is in question. This depends on the properties of preferred extremals. If partonic 2-surfaces can be chosen freely at the boundaries of CDs the restrictions come only from smoothness of the imbedding of the partonic 2-surfaces and tangent space data. The space of all functions from reals to reals has cardinality $2^{\aleph_1}$ which suggests that the cardinality is not larger than this, perhaps smaller since continuity and smoothness poses strong conditions. The natural guess is that the tangent space of WCW can be modelled as and infinite-dimensional separable Hilbert space which has cardinality $\aleph_1$.

TGD leads also a second generalization of the number concept motivated by number theoretical universality inspiring the attempt to glue different number fields (reals and various p-adics) together among common numbers -rationals in particular- to form a larger structure [K71].

To sum up, the distinctions between Cantorian and TGD inspired approaches are clear. Cantorian approach relies on set theory and TGD on number theory. What is common is the hierarchy of infinities.

4.8.3 What could be the foundations of physical mathematics?

Theoretical physicists do not spend normally their time for questioning the foundations of mathematics. They calculate. There are exceptions: Von Neumann was both a theoretical physicist developing mathematical foundations of quantum theory and mathematician building the mathematics of quantum theory and also proposing his own L world for foundations of mathematics.

A physicist posing the question "What should be done for the foundations of mathematics?" sounds blasphemous and the physicist should add the attribute "physical" to "mathematics" to avoid irritation. In any case, the fact is that the problems plaguing set theory and therefore the foundations of mathematics had been discovered roughly century ago and no commonly accepted solution to these problems have been found. The foundations of mathematics rely on classical physics and quantum view about existence suggests that the foundations of mathematics might need a revision.

Again the work of von Neumann comes readily into mind. The goal of von Neuman was to build a non-commutative measure theory: the outcome was the three algebras bearing his name and defining the mathematical backbone of three kinds of quantum theories. Factors of type I are natural for wave mechanism with finite number of degrees of freedom. In QFT hyperfinite factors of type III appear. In TGD framework hyperfinite factors of type II (and possibly of type III) are natural.

Connes who has studied von Neumann algebras highly relevant to quantum physics proposed the notion of non-commutative geometry in terms of a spectral triplet defined by C^* algebra A, Hilbert space H, and Dirac operator D with some additional properties. As a special case one re-discovers Riemannian manifolds using commutative function algebra, the Hilbert space of continuous functions, and certain kind of Dirac operator.

Physicists are usually mathematical opportunists and do not want to use time to ponder the foundations of mathematics My belief is that physicists should get rid of this attitude and make fool of themselves by posing the childish questions of physicist in the hope that some real mathematician might get interested. In order to not irritate mathematicians too much I will talk about physical mathematics instead of mathematics in the sequel.

The proposal that infinite primes, integers, and rationals should replace Cantor's ordinals and surreal numbers [K44] has been already made. This would allow to get rid of Russell's antinomy, leave the notion of cardinal intact. Also axiom of choice looks too strong from the point of view of physics.

Does it make sense to speak about physical set theory?

For the physicist set theory looks quite too general. In the recent day physical theories sets are typically manifolds, sub-manifolds, or orbifolds. Feynman diagrams represent example of 1-D singular manifolds and in TGD generalized Feynman diagrams of TGD fail to be 3-manifolds only at the vertices represented as 2-D partonic surfaces. In string theories and in twistor approach to gauge theories algebraic geometry is important. Branes are typically algebraic surfaces. The spaces are endowed with various structures: besides metric induced topology one differential structure, differential forms, metric, spinor structure, complex and Kähler structure, etc...

1. In algebraic geometry sets are replaced with varieties and basic set theoretic operations such as intersection and union are algebraized. Physicists should not fail to realize how profound this algebraization of the set theory is. The price that must be paid is that varieties are manifolds only locally. What limitations does this mean for set theory? Is it enough to formulate set theory algebraically? In TGD framework this could be possible in the intersection of real and p-adic worlds (WCWs) since set theoretic operations would have algebraic representation. For instance, $A \subset B$ would be formulated by adding additional functions for which the intersection of zero locus with B defines A.

   The algebraic notion of set as a variety is extremely restrictive: maybe the problems of set theory are partially due to the neglect of the fact that allowed sets must have a physical realization. Every physicists of course has her own pet theory, which he regards as the real physics, and one natural condition on any acceptable physics is that it can emulate sufficiently general spaces - to act as a kind of mathematical Turing machine. At least real and complex manifolds with arbitrary dimension should have some kind of physical representation. One can imagine this kind of representation in terms of unions of partonic 2-surfaces since union can be regarded also as a Cartesian product as long as the surfaces do not intersect.

2. The introduction of topology is the first step in bringing structure to the set theoretic primordial chaos. Metric topology is standard in physics at space-time level. More refined topologies can be certainly found in highly technical mathematical physics articles. In algebraic geometry Zariski topology is important but has its problems realized by Groethendienck in his attempts to build a universal cohomology theory working in all number fields. The closed sets of Zariski topology are varieties. Their complements would be open sets open also in norm based topology. Zariski topology is obviously much rougher than the metric topology. Zariski topology makes sense also for p-adic number fields. This kind of topology might make sense in TGD framework if one restricts the consideration to the intersection of real and p-adic worlds identified at the level of WCW as the space of algebraic surfaces defined using polynomials with rational coefficients and having finite degree.

3. In TGD framework preferred extremals of Kähler action define space-time surfaces and strong form of holography makes the situation effectively 2-dimensional. The conjecture is that preferred extremals correspond to quaternionic surfaces of octonionic 8-space. Octonionic structure is associated with the octonionic representation of the imbedding space gamma matrices (not actually matrices any more!) defining the Clifford algebra. Associativity would be the basic dynamical principle. Does this mean that number theory- in particular classical number fields- should appear in the formulation of the foundations of physical mathematics? This idea is attractive even when one does not assume that TGD Universe is the Universe.

What is beautiful that algebraic geometry brings in also number theory. One might hope that the foundations of physical mathematics could be based on the fusion of set theory, geometry, algebra, and number theory.

Quantum Boolean algebra instead of Boolean algebra?

Mathematical logic relies on the notion of Boolean algebra, which has a well-known representation as the algebra of sets which in turn has in algebraic geometry a representation in terms of algebraic varieties. This is not however attractive at space-time level since the dimension of the algebraic variety is different for the intersection *resp.* union representing AND *resp.* OR so that only only finite number of ANDs can appear in the Boolean function. TGD inspired interpretation of the fermionic sector of the theory in terms of Boolean algebra inspires more concrete ideas about the the realization of Boolean algebra at both quantum level and classical space-time level and also suggests a geometric realization of the basic logical functions respecting the dimension of the representative objects.

1. In TGD framework WCW spinors correspond to fermionic Fock states and an attractive interpretation for the basis of fermionic Fock states is as Boolean algebra. In zero energy ontology one consider pairs of positive and negative energy states and zero energy states could be seen as physical correlates for statements $A \rightarrow B$ or $A \leftrightarrow B$ with individual state pairs in the quantum superposition representing various instances of the rule $A \rightarrow B$ or $A \leftrightarrow B$. The breaking of time reversal invariance means that either the positive or negative energy part of the state (but not both) can correspond to a state with precisely define number of particles with precisely defining quantum numbers such as four-momentum. At the second end one has scattered state which is a superposition of this kind of many-particle states. This would suggest that $A \rightarrow B$ is the correct interpretation.

2. In quantum group theory [A76] the notion of co-algebra [A19] is very natural and the binary algebraic operations of co-algebra are in a well-defined sense time reversals of those of algebra. Hence there is a great temptation to generalize Boolean algebra to include its co -algebra [A185] so that one might speak about quantum Boolean algebra. The vertices of generalized Feynman diagrams represent two topological binary operations for partonic two surfaces and there is a strong temptation to interpret them as representations for the operations of Boolean algebra and its co-algebra.

   (a) The first vertex corresponds to the analog of a stringy trouser diagram in which partonic 2-surface decays to two and the reversal of this representing fusion of partonic 2-surfaces.

In TGD framework this diagram does not represent classically particle decay or fusion but the propagation of particle along two paths after the decay or the reversal of this process. The Boolean analog would be logical OR ($A \vee B$) or set theoretical union $A \cup B$ *resp.* its co-operation. The partonic two surfaces would represent the arguments (*resp.* co-arguments) A and B.

(b) Second one corresponds to the analog of 3-vertex for Feynman diagram: the three 3-D "lines" of generalized Feynman diagram meet at the partonic 2-surface. This vertex (co-vertex) is the analog of Boolean AND ($A \wedge B$) or intersection $A \cap B$ of two sets *resp.* its co-operation.

(c) I have already earlier ended up with the proposal that only three-vertices appear as fundamental vertices in quantum TGD [K17]. The interpretation of generalized Feynman diagrams as a representation of quantum Boolean algebra would give a deeper meaning for this proposal.

These vertices could therefore have interpretation as a space-time representation for operations of Boolean algebra and its co-algebra so that the space-time surfaces could serve as classical correlates for the generalized Boolean functions defined by generalized Feynman diagrams and expressible in terms of basic operations of the quantum Boolean algebra. For this representation the dimension of the variety representing the value of Boolean function at classical level is the same as as the dimension of arguments: that is two. Hence this representation is not equivalent with the representation provided by algebraic geometry for which the dimension of the geometric variety representing $A \wedge B$ and $A \vee B$ in general differs from that for A and B. If one however restricts the algebra to that assignable to braid strands, statements would correspond to points at partonic level, so that one would have discrete sets and the set theoretic representation of quantum Boolean algebra could make sense. Discrete sets are indeed the only possibility since otherwise the dimension of intersection and union are different if algebraic varieties are in question.

3. The breaking of time reversal invariance is accompanied by a generation of entropy and loss of information. The interpretation at the level of quantum Boolean algebra would be following. The Boolean function and and OR assign to two statements a single statement: this means a gain of information and at the level of physics this is indeed the case since entropy is reduced in the process reducing the number of particles. The occurrence of co-operations of AND and OR corresponds to particle decays and uncertainty about the path along which particle travels (dispersion of wave packet) and therefore loss of information.

   (a) The "most logical" interpretation for the situation is in conflict with the identification of the arrow of logic implication with the arrow of time: the direction of Boolean implication arrow and the arrow of geometric time would be opposite so that final state could be said to imply the initial state. The arrow of time would weaken logical equivalence to implication arrow.

   (b) If one naively identifies the arrows of logical implication and geometric time so that initial state can be said to imply the final state, second law implies that logic becomes fuzzy. Second law would weak logical equivalence to statistical implication arrow.

   (c) The natural question is whether just the presence of both algebra and co-algebra operations causing a loss of information in generalized Feynman diagrams could lead to what might be called fuzzy Boolean functions expressing the presence of entropic element appears at the level of Boolean cognition.

4. This picture requires a duality between Boolean algebra and its co-algebra and this duality would naturally correspond to time reversal. Skeptic can argue that there is no guarantee about the existence of the extended algebra analogous to Drinfeld double [A161] that would unify Boolean algebra and its dual. Only the physical intuition suggests its existence.

These observations suggest that generalized Feynman diagrams and their space-time counterparts could have a precise interpretation in quantum Boolean algebra and that one should perhaps consider the extension of the mathematical logic to quantum logic. Alternatively, one could argue that quantum Boolean algebra is more like a model for what mathematical cognition could be in the real world.

The restrictions of mathematical cognition as a guideline?

With the birth of quantum theory physicists ceased to be outsiders since it was impossible to consider quantum measurement as something not affecting the measured system in any way. With the advent of consciousness theory physicists have been forced to give up the idea about unidirectional action with with reality and have become a part of quantum Universe - self. This also requires dramatic modification of the basic ontology forcing to give up the physicalistic dogmas. Consciousness involves free will manifested in ability to select and create something completely new in each quantum jump. Physical Universe is not given but is re-created again and again and evolves.

In standard mathematics mathematician is still a complete outsider, and the possible limitations of mathematical cognition are not considered seriously in the attempts to formulate the foundations of mathematics. Mathematicians still choose effortlessly one element from each set of infinite collection of sets. We know that in numerics one is always bound to introduce cutoff on the number of bits and use finite subset of rational numbers but also this has not been taken into account in the formulation of foundations as far as I know. If one takes consciousness theory seriously one is led to wonder what are the physical restrictions on mathematical cognition and therefore on physical mathematics. What looks obvious that the idea about mathematics based on fixed axiomatics must be given up. The evolution of the physical universe and of consciousness means also the evolution of (at least physical) mathematics. The paradox of self reference plaguing conventional view about consciousness and leading to infinite regress disappears when this regress is replaced with evolution.

Suppose that life resides and cognitive representations are realized in the intersection of real and p-adic worlds reducing to intersections of real and p-adic variants of partonic 2-surfaces at space-time level. At the level of WCW the intersection of real and p-adic worlds could correspond to the space of partonic 2-surfaces defined by rational functions constructed using polynomials of finite degree with rational coefficients.

What kind of restrictions of this picture poses set theory, topology, and logic? The reader can of course imagine restrictions on some other fields of mathematics involved. The question in the case of the set theory and topology has been already touched. In the case of logic the key question seems to concern the operational meaning of $\forall$ and $\exists$, when the finite resolution of measurements and cognitive representation are taken into account. What these universal quantors really mean: what is their domain of definition?

Consider first the domain of definition at space-time level.

1. Should all theorems be formulated using $\forall$ and $\exists$ restricted to the dense subset rationals of 8-D imbedding space. Since continuous function is fixed from its values in a dense subset, this assumption is not so strong unless there are other restrictions.

2. At space-time surface and partonic 2-surfaces the situation is different. The assumption that only the common rational points of real and p-adic surfaces define cognitive representations poses a strong limitation since typically the number of rational points of 2-surface is expected to be finite. Algebraic extensions of p-adic numbers extend the number of common points and one can imagine an evolutionary hierarchy of mathematics realized in terms of geometry of partonic 2-surfaces reflecting itself as the geometry of space-time surfaces by strong form of holography.

3. The orbits of the rational points selected at the ends of partonic 2-surfaces are braids along light-like 3-surfaces. At space-time level one has world sheets or strings which form in general case 2-braids. This picture leads to a what I have used to call almost topological QFT.

What about the domain of definition of existence quantors at the level of WCW? The natural conjecture is that the surfaces in the intersection of real and p-adic worlds form a dense set of full WCW so that everything holding true in the intersection would hold true generally and one coul dhope that systems which are living in the proposed sense are able to discover interesting mathematics.

Suppose that the partonic 2-surfaces decompose into patches such that in each patch the surface is a zero locus of polynomials with rational coefficients. Since polynomials can be seen as Taylor series with cutoff one can hope that they form a dense subset. Since rationals are dense subset of reals, one can hope that also the restriction to rational coefficients preserves the dense subset property and living subsystems are able to represent all that is needed and completion takes care of the rest as it does for rationals. The notion of completion leading from rationals to various algebraic numbers fields and also to reals and complex numbers would become the fundamental principle leading from number theory to metric topology.

Physicist reader has certainly noticed that "rational point" does not represent a general coordinate invariant notion.

1. The coordinates of point are rational in preferred coordinates and the symmetries of the 8-D imbedding space suggest families of preferred coordinates. The moduli space for CDs would be characterized by the choice of these preferred coordinates dictating also the choice of quantization axes so that quantum measurement theory would be realized as a decomposition of WCW to a union corresponding to different choices. State function reduction would involve also a localization determining quantization axes.

2. There are many possible choices of quantization axes/preferred coordinates and this means a restriction of general coordinate invariance from group of all coordinate transformations to a discrete subgroup of isometries which is not unique. Cognition would break the general coordinate invariance. The world in which the mathematician thinks using spherical coordinates differs in some subtle manner from the world in which she thinks using Cartesian coordinates. Mathematician does not remain outside Platonia anymore just as quantum physicists is not outside the physical Universe!

Axiom of choice relates to selection, which can be regarded as a cognitive act. The question whether axiom of choice is needed at all has been already discussed but a couple of clarifying comments are in order.

1. At quantum level selection would be naturally assigned with state function reduction, also the state function reduction selecting quantization axes. The cascade of state function reductions - starting from the scale of CD and proceeding fractally downwards sub-CD by sub-CD and stopping when only negentropic entanglement stabilized by NMP remains - could be how Nature performs the choice. State function reduction would involve also the choice of quantization axes dictating possible subsequence choices. Note that non-deterministic element would be involved with the quantum choice.

2. If life and cognitive representations are at the intersection of real and p-adic worlds, it would seem that rational points are chosen at space-time level and algebraic 2-surfaces at WCW level. As explained, it is easy to imagine the collection of sets from which one selects points is always finite or that there is a natural explicit criterion allowing to select preferred point from each set. Finite measurement resolution implying braids and string world sheets could provide this criterion. If so, the axiom of choice would be un-necessary in physical mathematics. Finite measurement resolution suggests that for partonic 2-surfaces the ends of braid strands define preferred points.

Platonia is a strange place about which many mathematicians claim to visit regularly. I already proposed that the generalization of space-time point by bringing in the infinite number theoretical anatomy of real (and octonionic) units might allow to realize number theoretical Brahman=Atman identity by representing WCW in terms of the number theoretic anatomy of space-time points. This kind of representation would certainly be the most audacious idea that physical mathematician could dare to think of.

Is quantal Boolean reverse engineering possible?

The quantal version of Boolean algebra means that the basic logical functions have quantum inverses. The inverse of $C = A \wedge B$ represents the quantum superposition of all pairs A and B

for which $A \wedge B = C$ hols true. Same is true for $\vee$. How could these additional quantum logical functions with no classical counterparts extend the capacities of logician?

What comes in mind is logical reverse engineering. Consider the standard problem solving situation repeatedly encountered by my hero Hercule Poirot. Someone has been murdered. Who could have done it? Who did it? Actually scientists who want to explain instead of just applying the method to get additional items to the CVC, meet this kind of problem repeatedly. One has something which looks like an experimental anomaly and one has to explain it. Is this anomaly genuine or is it due to a systematic error in the information processing? Could the interpretation of data be somehow wrong? Is the model behind experiments based on existing theory really correct or has something very delicate been neglected? If a genuine anomaly is in question (someone has been really murdered- this is always obvious in the tales about the deeds of Hercule Poirot since the mere presence of Hercule guarantees the murder unless it has been already done), one encounters what might be called Poirot problem in honor of my hero. As a matter fact, from the point of view of Boolean algebra, one has the same reverse Boolean engineering problem irrespective of whether it was a genuine anomaly or not.

This brings in my mind the enormously simplified problem. The logical statement C is found to be true. Which pairs A, B could have implied C as $C = A \wedge B$ (or $A \vee B$). Of course, much more complex situations can be considered where C corresponds to some logical function $C = f(A_1, A_2, ..., A_n)$. Quantum Poirot could use quantum computer able to realize the co-gates for gates AND and OR (essentially time reversals) and write a quantum computer program solving the problem by constructing the Boolean co-function of Boolean function f.

What would happen in TGD Universe obeying zero energy ontology (ZEO) is following.

1. The statement C is represented as as positive energy part of zero energy state (analogous to initial state of physical event) and $A_1, ..A_n$ is represented as one state in the quantum superposition of final states representing various value combinations for $A_1, ..., A_n$. Zero energy states (rather than only their evolution) represents the arrow of time. The M-matrix characterizing time-like entanglement between positive and negative energy states generalizes generalizes S-matrix. S-matrix is such that initial states have well defined particle numbers and other quantum numbers whereas final states do not. They are analogous to the outcomes of quantum measurement in particle physics.

2. Negentropy Maximization Principle [K42] maximizing the information contents of conscious experience (sic!) forces state function reduction to one particular $A_1, ..., A_n$ and one particular value combination consistent with C is found in each state function reduction. At the ensemble level one obtains probabilities for various outcomes and the most probable combination might represent the most plausible candidate for the murderer in quantum Poirot problem. Also in particle physics one can only speak about plausibility of the explanation and this leads to the endless n sigma talk. Note that it is absolutely essential that state function reduction occurs. Ironically, quantum problem solving causes dissipation at the level of ensemble but the ensemble probabilities carry actually information! Second law of thermodynamics tells us that Nature is a pathological problem solver- just like my hero!

3. In TGD framework basic logical binary operations have a representation at the level of Boolean algebra realized in terms of fermionic oscillator operators. They have also space-time correlates realized topologically. $\wedge$ has a representation as the analog of three-vertex of Feynman graph for partonic 2-surfaces: partonic 2-surfaces are glued along the ends to form outgoing partonic 2-surface. $\vee$ has a representation as the analog of stringy trouser vertex in which partonic surfaces fuse together. Here TGD differs from string models in a profound manner.

To conclude, I am a Boolean dilettante and know practically nothing about what quantum computer theorists have done- in particular I do not know whether they have considered quantum inverse gages. My feeling is that only the gates with bits replaced with qubits are considered: very natural when one thinks in terms of Boolean logic. If this is really the case, quantal co-AND and co-OR having no classical counterparts would bring a totally new aspect to quantum computation in solving problems in which one cannot do without (quantum) Poirot and his little gray (quantum) brain cells.

How to understand transcendental numbers in terms of infinite integers?

Santeri Satama made in my blog a very interesting question about transcendental numbers. The reformulation of Santeri's question could be "How can one know that given number defined as a limit of rational number is genuinely algebraic or transcendental?". I answered to the question and since it inspired a long sequence of speculations during my morning walk on sands of Tullinniemi I decided to expand my hasty answer to a blog posting.

The basic outcome was the proposal that by bringing TGD based view about infinity based on infinite primes, integers, and rationals one could regard transcendental numbers as algebraic numbers by allowing genuinely infinite numbers in their definition.

1. In the definition of any transcendental as a limit of algebraic number (root of a polynomial and rational in special case) in which integer n approaches infinity one can replace n with any infinite integer. The transcendental would be an algebraic number in this generalized sense. Among other things this might allow polynomials with degree given by infinite integer if they have finite number of terms. Also mathematics would be generalized number theory, not only physics!

2. Each infinite integer would give a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to real norm they would be identical.

3. This would extend further the generalization of number concept obtained by allowing all infinite rationals which reduce to units in real sense and would further enrich the infinitely rich number theoretic anatomy of real point and also of space-time point. Space-time point would be the Platonia. One could call this number theoretic Brahman=Atman identity or algebraic holography.

1. How can one know that the real number is transcendental?

The difficulty of telling whether given real number defined as a limit of algebraic number boils down to the fact that there is no numerical method for telling whether this kind of number is rational, algebraic, or transcendental. This limitation of numerics would be also a restriction of cognition if p-adic view about it is correct. One can ask several questions. What about infinite-P p-adic numbers: if they make sense could it be possible to cognize also transcendentally? What can we conclude from the very fact that we cognize transcendentals? Transcendentality can be proven for some transcendentals such as π. How this is possible? What distinguishes "knowably transcendentals" like π and e from those, which are able to hide their real number theoretic identity?

1. Certainly for "knowably transcendentals" there must exist some process revealing their transcendental character. How π and e are proven to be transcendental? What in our mathematical cognition makes this possible? First of all one starts from the definitions of these numbers. e can be defined as the limit of the rational number $(1 + 1/n)^n$ and 2π could be defined as the limit for the length of the circumference of a regular n-side polygon and is a limit of an algebraic number since Pythagoras law is involved in calculating the length of the side. The process of proving "knowable transcendentality" would be a demonstration that these numbers cannot be solutions of any polynomial equation.

2. Squaring of circle is not possible because π is transcendental. When I search Wikipedia for squaring of circle I find a link to Weierstrass theorem allowing to prove that π and e are transcendentals. In the formulation of Baker this theorem states the following: If α_1,...,α_n are distinct algebraic numbers then the numbers e^{α_1},...,e^{α_n} are linearly independent over algebraic numbers and therefore transcendentals. One says that the extension $Q(e^{\alpha_1}, ..., e^{\alpha_n})$ of rationals has transcendence degree n over Q. This is something extremely deep and unfortunately I do not know what is the gist of the proof. In any case the proof defines a procedure of demonstrating "knowable transcendentality" for these numbers. The number of these transcendentals is huge but countable and therefore vanishingly small as compared to the uncountable cardinality of all transcendentals.

3. This theorem allows to prove that π and e are transcendentals. Suppose on the contrary that π is algebraic number. Then also $i\pi$ would be algebraic and the previous theorem would imply that $e^{i\pi} = -1$ is transcendental. This is of course a contradiction. Theorem also implies that e is transcendental. But how do we know that $e^{i\pi} = -1$ holds true? Euler deduced this from the connection between exponential and trigonometric functions understood in terms of complex analysis and related number theory. Clearly, rational functions and exponential function and its inverse -logarithm- continued to complex plane are crucial for defining e and π and proving also $e^{i\pi} = -1$. Exponent function and logarithm appear everywhere in mathematics: in group theory for instance. All these considerations suggest that "knowably transcendental" is a very special mathematical property and deserves a careful analysis.

2. Exponentiation and formation of set of subsets as transcendence

What is so special in exponentiation? Why it sends algebraic numbers to "knowably transcendentals". One could try to understand this in terms of exponentiation which for natural numbers has also an interpretation in terms of power set just as product has interpretation in terms of Cartesian product.

1. In Cantor's approach to the notion of infinite ordinals exponentiation is involved besides sum and product. All three binary operations - sum, product, exponent are expressed set theoretically. Product and sum are "algebraic" operations. Exponentiation is "non-algebraic" binary operation defined in terms of power set (set of subsets). For m and n defining the cardinalities of sets X and Y, m^n defines the cardinality of the set Y^X defining the number of functions assigning to each point of Y a point of X. When X is two-element set (bits 0 and 1) the power set is just the set of all subsets of Y which bit 1 assigned to the subset and 0 with its complement. If X has more than two elements one can speak of decompositions of Y to subsets colored with different colors- one color for each point of X.

2. The formation of the power set (or of its analog for the number of colors larger than 2) means going to the next level of abstraction: considering instead of set the set of subsets or studying the set of functions from the set. In the case of Boolean algebras this means formation of statements about statements. This could be regarded as the set theoretic view about transcendence.

3. What is interesting that 2-adic integers would label the elements of the power set of integers (all possible subsets would be allowed, for finite subsets one would obtain just natural numbers) and p-adic numbers the elements in the set formed by coloring integers with p colors. One could thus say that p-adic numbers correspond naturally to the notion of cognition based on power sets and their finite field generalizations.

4. But can one naively transcend the set theoretic exponent function for natural numbers to that defined in complex plane? Could the "knowably transcendental" property of numbers like e and π reduce to the transcendence in this set theoretic sense? It is difficult to tell since this notion of power applies only to integers m, n rather than to a pair of transcendentals e, π. Concretization of $e^{i\pi}$ in terms of sets seems impossible: it is very difficult to imagine what sets with cardinality e and π could be.

3. Infinite primes and transcendence

TGD suggests also a different identification of transcendence not expressible as formation of a power set or its generalizations.

1. The notion of infinite primes replaces the set theoretic notion of infinity with purely number theoretic one.

   (a) The mathematical motivation could be the need to avoid problems like Russell's antinomy. In Cantorian world a given ordinal is identified as the ordered set of all ordinals smaller than it and the set of all ordinals would define an ordinal larger than every ordinal and at the same time member of all ordinals.

(b) The physical motivation for infinite primes is that their construction corresponds to a repeated second quantization of an arithmetic supersymmetric quantum field theory such that the many particle states of the previous level become elementary particles of the new level. At the lowest level finite primes label fermionic and bosonic states. Besides free many-particle states also bound states are obtained and correspond at the first level of the hierarchy to genuinely algebraic roots of irreducible polynomials.

(c) The allowance of infinite rationals which as real numbers reduce to real units implies that the points of real axes have infinitely rich number theoretical anatomy. Space-time point would become the Platonia. One could speak of number theoretic Brahman=Atman identity or algebraic holography. The great vision is that the World of Classical Worlds has a mathematical representation in terms of the number theoretical anatomy of space-time point.

2. Transcendence in purely number theoretic sense could mean a transition to a higher level in the hierarchy of infinite primes. The scale of new infinity defined as the product of all prime at the previous level of hierarchy would be infinitely larger than the previous one. Quantization would correspond to abstraction and transcendence.

This idea inspires some questions.

1. Could infinite integers allow the reduction of transcendentals to algebraic numbers when understood in general enough sense. Could real algebraic numbers be reduced to infinite rationals with finite real values (for complex algebraic numbers this is certainly not the case)? If so, then all real numbers would be rationals identified as ratios of possibly infinite integers and having finite value as real numbers? This turns out to be too strong a statement. The statement that all real numbers can be represented as finite or infinite algebraic numbers looks however sensible and would reduce mathematics to generalized number theory by reducing limiting procedure involved with the transition from rationals to reals to algebraic transcendence. This applies also to p-adic numbers.

2. p-Adic cognition for finite values of prime p does not explain why we have the notions of π and e and more generally, that of transcendental number. Could the replacement of finite-p p-adic number fields with infinite-P p-adic number fields allow us to understand our own mathematical cognition? Could the infinite-P p-adic number fields or at least integers and corresponding space-time sheets make possible mathematical cognition able to deduce analytic formulas in which transcendentals and transcendental functions appear making it possible to leave the extremely restricted realm of numerics and enter the realm of mathematics? Lie group theory would represent a basic example of this transcendental aspect of cognition. Maybe this framework might allow to understand why we can have the notion of transcendental number!

4. Identification of real transcendentals as infinite algebraic numbers with finite value as real numbers

The following observations suggests that it could be possible to reduce transcendentals to generalized algebraic numbers in the framework provided by infinite primes. This would mean that not only physics but also mathematics (or at least "physical mathematics") could be seen as generalized number theory.

1. In the definition of any transcendental as an $n \to \infty$ limit of algebraic number (root of a polynomial and rational in special case), one can replace n with any infinite integer if n appears as an argument of a function having well defined value at this limit. If n appears as the number of summands or factors of product, the replacement does not make sense. For instance, an algebraic number could be defined as a limit of Taylor series by solving the polynomial equation defining it. The replacement of the upper limit of the series with infinite integer does not however make sense. Only transcendentals (and possibly also some algebraic numbers) allowing a representation as $n \to \infty$ limit with n appearing as argument

of expression involving a finite number of terms can have representation as infinite algebraic number. The rule would be simple.

Transcendentals or algebraic numbers allowing an identification as infinite algebraic number must correspond to a term of a sequence with a fixed number of terms rather than sum of series or infinite product.

2. Each infinite integer gives a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to the real norm they would be identical.

3. The heuristic guess is that any genuine algebraic number has an expression as Taylor series obtained by writing the solution of the polynomial equation as Tarylor expansion. If so, algebraic numbers must be introduced in the standard manner and do not allow a representation as infinite rationals. Only transcendentals would allow a representation as infinite rationals or infinite algebraic numbers. The infinite variety of representation in terms of infinite integers would enormously expand the number theoretical anatomy of the real point. Do all transcendentals allow an expression containing a finite number of terms and N appearing as argument? Or is this the defining property of only "knowably transcendentals"?

One can consider some examples to illustrate the situation.

1. The transcendental π could be defined as $\pi_N = -iN(e^{i\pi/N} - 1)$, where $e^{i\pi/N}$ is N:th root of unity for infinite integer N and as a real number real unit. In real sense the limit however gives π. There are of course very many definitions of π as limits of algebraic numbers and each gives rise to infinite variety of number theoretic anatomies of π.

2. One can also consider the roots $exp(i2\pi n/N)$ of the algebraic equation $x^N = 1$ for infinite integer N. One might define the roots as limits of Taylor series for the exponent function but it does not make sense to define the limit when the cutoff for the Taylor series approaches some infinite integer. These roots would have similar multiplicative structure as finite roots of unity with p^n:th roots with p running over primes defining the generating roots. The presence of N^{th} roots of unity f or infinite N would further enrich the infinitely rich number theoretic anatomy of real point and therefore of space-time points.

3. There would be infinite variety of Neper numbers identified as $e_N = (1 + 1/N)^N$, N any infinite integer. Their number theoretic anatomies would be different but as real numbers they would be identical.

To conclude, the talk about infinite primes might sound weird in the ears of a layman but mathematicians do not lose their peace of mind when they here the word "infinity". The notion of infinity is relative. For instance, infinite integers are completely finite in p-adic sense. One can also imagine completely "real-worldish" realizations for infinite integers (say as states of repeatedly second quantized arithmetic quantum field theory and this realization might provide completely new insights about how to understand bound states in ordinary QFT).

4.9 Local zeta functions, Galois groups, and infinite primes

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.

2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.

3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

4.9.1 Zeta function and infinite primes

Fermionic Zeta function is expressible as a product of fermionic partition functions $Z_{F,p} = 1 + p^{-z}$ and could be seen as an image of X under algebraic homomorphism mapping prime p to $Z_{F,p}$ defining an analog of prime in the commutative function algebra of complex numbers. For hyper-octonionic infinite primes the contribution of each p to the norm of X is same finite power of p since only single representative from each Lorentz equivalence class is included, and there is one-one correspondence with ordinary primes so that an appropriate power of ordinary ζ_F might be regarded as a representation of X also in the case of hyper-octonionic primes.

Infinite primes suggest a generalization of the notion of ζ function. Real-rational infinite prime $X \pm 1$ would correspond to $\zeta_F \pm 1$. General infinite prime is mapped to a generalized zeta function by dividing ζ_F with the product of partition functions $Z_{F,p}$ corresponding to fermions kicked out from sea. The same product multiplies '1'. The powers p^n present in either factor correspond to the presence of n bosons in mode p and to a factor $Z^n_{p,B}$ in corresponding summand of the generalized Zeta. In the case of hyper-octonionic infinite primes some power of Z_F multiplied by p-dependent powers $Z_{F,p}^{n(p)}$ of fermionic partition functions with $n(p) \rightarrow 0$ for $p \rightarrow \infty$ should replace the image of X. If effective 2-dimensionality holds true $n(p) = 2$ holds true for $p > 2$.

For zeros of ζ_F which are same as those of Riemann ζ the image of infinite part of infinite prime vanishes and only the finite part is represented faithfully. Whether this might have some physical meaning is an interesting question.

4.9.2 Local zeta functions and Weil conjectures

Riemann Zeta is not the only zeta [A1, A110]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1 - p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [A104] associated with finite fields $G(p, k)$ and thus to single prime. The extensions $G(p, nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s) \times \zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [A104] .

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1 - N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

4.9.3 Galois groups, Jones inclusions, and infinite primes

Langlands program [A56, A150] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p, k)$ and their extensions $G(p, nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n, Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [A217] . In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on WCW-spinor fields. One can also speak about WCW spinor s invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \rightarrow K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the p-adic length scale range 10 nm-5 μm contains as many as four scaled up electron Compton lengths assignable to Gaussian Mersennes $M_k = (1 + i)^k - 1$, $k = 151, 157, 163, 167$, which suggests that the emergence of living matter means an improved cognitive resolution.

Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \rightarrow \sum x_n z^n$ [A61] . Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [L9] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of imbedding space point allows to represent WCW (the world of classical worlds associated with the light-cone of a given point of H) and WCW spinor fields emerges naturally [L9] .

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [L11] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on WCW spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. WCW spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

4.9.4 Prime Hilbert spaces and infinite primes

There is a result of quantum information science providing an additional reason why for p-adic physics. Suppose that one has N-dimensional Hilbert space which allows $N+1$ unbiased basis. This means that the moduli squared for the inner product of any two states belonging to different basis equals to $1/N$. If one knows all transition amplitudes from a given state to all states of all $N+1$ mutually unbiased basis, one can fully reconstruct the state. For $N = p^n$ dimensional $N+1$ unbiased basis can be found and the article of Durt [A134] gives an explicit construction of these basis by applying the properties of finite fields. Thus state spaces with p^n elements - which indeed emerge naturally in p-adic framework - would be optimal for quantum tomography. For instance, the discretization of one-dimensional line with length of p^n units would give rise to p^n-dimensional Hilbert space of wave functions.

The observation motivates the introduction of prime Hilbert space as as a Hilbert space possessing dimension which is prime and it would seem that this kind of number theoretical structure for the category of Hilbert spaces is natural from the point of view of quantum information theory. One might ask whether the tensor product of mutually unbiased bases in the general case could be constructed as a tensor product for the bases for prime power factors. This can be done but since the bases cannot have common elements the number of unbiased basis obtained in this manner is equal to $M+1$, where M is the smallest prime power factor of N. It is not known whether additional unbiased bases exists.

Hierarchy of prime Hilbert spaces characterized by infinite primes

The notion of prime Hilbert space provides also a new interpretation for infinite primes, which are in 1-1 correspondence with the states of a supersymmetric arithmetic QFT. The earlier interpretation was that the hierarchy of infinite primes corresponds to a hierarchy of quantum states. Infinite primes could also label a hierarchy of infinite-D prime Hilbert spaces with product and sum for infinite primes representing unfaithfully tensor product and direct sum.

1. At the lowest level of hierarchy one could interpret infinite primes as homomorphisms of Hilbert spaces to generalized integers (tensor product and direct sum mapped to product and sum) obtained as direct sum of infinite-D Hilbert space and finite-D Hilbert space. (In)finite-D Hilbert space is (in)finite tensor product of prime power factors. The map of N-dimensional

Hilbert space to the set of N-orthogonal states resulting in state function reduction maps it to N-element set and integer N. Hence one can interpret the homomorphism as giving rise to a kind of shadow on the wall of Plato's cave projecting (shadow quite literally!) the Hilbert space to generalized integer representing the shadow. In category theoretical setting one could perhaps see generalize integers as shadows of the hierarchy of Hilbert spaces.

2. The interpretation as a decomposition of the universe to a subsystem plus environment does not seem to work since in this case one would have tensor product. Perhaps the decomposition could be to degrees of freedom to those which are above and below measurement resolution. One could of course consider decomposition to a tensor product of bosonic and fermionic state spaces.

3. The construction of the Hilbert spaces would reduce to that of infinite primes. The analog of the fermionic sea would be infinite-D Hilbert space which is tensor product of all prime Hilbert spaces H_p with given prime factor appearing only once in the tensor product. One can "add n bosons" to this state by replacing of any tensor factor H_p with its n+1:th tensor power. One can "add fermions" to this state by deleting some prime factors H_p from the tensor product and adding their tensor product as a finite-direct summand. One can also "add n bosons" to this factor.

4. At the next level of hierarchy one would form infinite tensor product of all infinite-dimensional prime Hilbert spaces obtained in this manner and repeat the construction. This can be continued ad infinitum and the construction corresponds to abstraction hierarchy or a hierarchy of statements about statements or a hierarchy of n:th order logics. Or a hierarchy of space-time sheets of many-sheeted space-time. Or a hierarchy of particles in which certain many-particle states at the previous level of hierarchy become particles at the new level (bosons and fermions). There are many interpretations.

5. Note that at the lowest level this construction can be applies also to Riemann Zeta function. ζ would represent fermionic vacuum and the addition of fermions would correspond to a removal of a product of corresponding factors ζ_p from ζ and addition of them to the resulting truncated ζ function. The addition of bosons would correspond to multiplication by a power of appropriate ζ_p. The analog of ζ function at the next level of hierarchy would be product of all these modified ζ functions and might well fail to exist since the product might typically converge to either zero or infinity.

Hilbert spaces assignable to infinite integers and rationals make also sense

1. Also infinite integers make sense since one can form tensor products and direct sums of infinite primes and of corresponding Hilbert spaces. Also infinite rationals exist and this raises the question what kind of state spaces inverses of infinite integers mean.

2. Zero energy ontology suggests that infinite integers correspond to positive energy states and their inverses to negative energy states. Zero energy states would be always infinite rationals with real norm which equals to real unit.

3. The existence of these units would give for a given real number an infinite rich number theoretic anatomy so that single space-time point might be able to represent quantum states of the entire universe in its anatomy (number theoretical Brahman=Atman). Also the world of classical worlds (light-like 3-surfaces of the imbedding space) might be imbeddable to this anatomy so that basically one would have just space-time surfaces in 8-D space and WCW would have representation in terms of space-time based on generalized notion of number. Note that infinitesimals around a given number would be replaced with infinite number of number-theoretically non-equivalent real units multiplying it.

Should one generalize the notion of von Neumann algebra?

Especially interesting are the implications of the notion of prime Hilbert space concerning the notion of von Neumann algebra -in particular the notion of hyper-finite factors of type II_1 playing a key role in TGD framework. Does the prime decomposition bring in additional structure? Hyper-finite factors of type II_1 are canonically represented as infinite tensor power of 2×2 matrix algebra having a representation as infinite-dimensional fermionic Fock oscillator algebra and allowing a natural interpretation in terms of spinors for the world of classical worlds having a representation as infinite-dimensional fermionic Fock space.

Infinite primes would correspond to something different: a tensor product of all $p\times p$ matrix algebras from which some factors are deleted and added back as direct summands. Besides this some factors are replaced with their tensor powers. Should one refine the notion of von Neumann algebra so that one can distinguish between these algebras as physically non-equivalent? Is the full algebra tensor product of this kind of generalized hyper-finite factor and hyper-finite factor of type II_1 corresponding to the vibrational degrees of freedom of 3-surface and fermionic degrees of freedom? Could p-adic length scale hypothesis - stating that the physically favored primes are near powers of 2 - relate somehow to the naturality of the inclusions of generalized von Neumann algebras to HFF of type II_1?

4.10 Miscellaneous

This section is devoted to what might be called miscellaneous since it does not relate directly to quantum TGD.

4.10.1 The generalization of the notion of ordinary number field

The notion of infinite rationals leads also to the generalization of the notion of a finite number. The obvious generalization would be based on the allowance of infinitesimals. Much more interesting approach is however based on the observation that one obtains infinite number of real units by taking two infinite primes with a finite rational valued ratio q and by dividing this ratio by ordinary rational number q. As a real number the resulting number differs in no manner from ordinary unit of real numbers but in p-adic sense the points are not equivalent. This construction generalizes also to quaternionic and octonionic case.

Space-time points would become structured since infinite rationals normed to unity define naturally a gigantically infinite-dimensional free algebra generated by the units serving in well-define sense as Mother of All Algebras. The units of the algebra multiplying ordinary rational numbers (and also other elements) of various number fields are invisible at the level of real physics so that the interpretation as the space-time correlate of mathematical cognition realizing the idea of monad is natural. Universe would be an algebraic hologram with single point being able to represent the state of the Universe in its structure. Infinite rationals would allow the realization of the Platonia of all imaginable mathematical constructs at the level of space-time.

The generalized units for quaternions and octonions

In the case of real and complex rationals the group of generalized units generated by primes *resp.* infinite Gaussian primes is commutative. In the case of unit quaternions and hyper-quaternions group becomes non-commutative and in case of unit hyper-octonions the group is replaced by a kind non-associative generalization of group.

1. For infinite primes for which only finite number of bosonic and fermionic modes are excited it is possible to tell how the products AB and BA of two infinite primes explicitly since finite hyper-octonionic primes can be assumed to multiply the infinite integer X from say left.

2. Situation changes if an infinite number of bosonic excitations are present since one would be forced to move finite H- or O-primes past a infinite number of primes in the product AB. Hence one must simply assume that the group G generated by infinite units with infinitely many bosonic excitations is a free group. Free group interpretation means that non-associativity is safely localized inside infinite primes and reduced to the non-associativity of ordinary hyper-octonions. Needless to say free group is the best one can hope of achieving since free group allows maximal number of factor groups.

The free group G can be extended into a free algebra A by simply allowing superpositions of units with coefficients which are real-rationals or possibly complex rationals. Again free algebra fulfils the dreams as system with a maximal representative power. The analogy with quantum states defined as functions in the group is highly intriguing and unit normalization would correspond to the ordinary normalization of Schrödinger amplitudes. Obviously this would mean that single point is able to mimic quantum physics in its structure. Could state function reduction and preparation be represented at the level of space-time surfaces so that initial and final 3-surfaces would represent pure states containing only bound state entanglement or negentropic entanglement represented algebraically, and could the infinite rationals generating the group of quaternionic units (no sums over them) represent pure states?

The free algebra structure of A together with the absolutely gigantic infinite-dimensionality of the endless hierarchy of infinite rational units suggests that the resulting free algebra structure is universal in the sense that any algebra defined with coefficients in the field of rationals can be imbedded to the resulting algebra or represented as a factor algebra obtained by the sequence $A \rightarrow A_1 = A/I_1 \rightarrow A_1/I_2...$ where the ideal I_k is defined by $k : th$ relation in A_{k-1}.

Physically the embedding would mean that some field quantities defined in the algebra are restricted to the subalgebra. The representation of algebra B as an iterated factor algebra would mean that some field quantities defined in the algebra are constant inside the ideals I_k of A defined by the relations. For instance, the induced spinor field at space-time surface could have the same value for all points of A which differ by an element of the ideal. At WCW level, the WCW spinor field would be constant inside an ideal associated with the algebra of A-valued functions at space-time surfaces.

The units can be interpreted as defining an extension of rationals in C, H, or O. Galois group is defined as automorphisms of the extension mapping the original number field to itself and obviously the transformations $x \rightarrow gxg^{-1}$, where g belongs to the extended number field act as automorphisms. One can regard also the extension by real units as the extended number field and in this case the automorphisms contain also the automorphisms induced by the multiplication of each infinite prime Π_i by a real unit U_i: $\Pi_i \rightarrow \hat{\Pi}_i = U_i\Pi_i$.

The free algebra generated by generalized units and mathematical cognition

One of the deepest questions in theory of consciousness concerns about the space-time correlates of mathematical cognition. Mathematician can imagine endlessly different mathematical structures. Platonist would say that in some sense these structures exist. The claim classical physical worlds correspond to certain 4-surfaces in $M^4_+ \times CP_2$ would leave out all these beautiful mathematical structures unless they have some other realization than the physical one.

The free algebra A generated by the generalized multiplicative units of rationals allows to understand how Platonia is realized at the space-time level. A has no correlate at the level of real physics since the generalized units correspond to real numbers equal to one. This holds true also in quaternionic and octonionic cases since one can require that the units have net quaternionic and octonionic phases equal to one. By its gigantic size A and free algebra character might be able represent all possible algebras in the proposed manner. Also non-associative algebras can be represented.

Algebraic equations are the basic structural building blocks of mathematical thinking. Consider as a simple example the equation $AB = C$. The equations are much more than tautologies since they contain the information at the left hand side about the variables of the algebraic operation giving the outcome on the right hand side. For instance, in the case of multiplication $AB = C$ the information about the factors is present although it is completely lost when the product is evaluated. These equations pop up into our consciousness in some mysterious manner and the question is what are the space-time correlates of these experiences suggested to exist by quantum-classical correspondence.

The algebra of units is an excellent candidate for the sought for correlate of mathematical cognition. Leibniz might have been right about his monads! The idealization is in complete accordance with the idea about the Universe as an algebraic hologram taken to its extreme. One might perhaps say that each point represents an equation.

One could also try interpret generalized Feynman diagrams as sequences of mathematical operations. For instance, the scattering $AB \rightarrow CD$ by exchange of particle C could be seen as an arithmetic operation $AB \rightarrow (AE^{-1})(EB) = CD$. If this is really the case, then at least tree diagrams might allow interpretation in terms of arithmetic operations for the complexified octonionic units. In case of loop diagrams it seems that one must allow sums over units.

When two points are cobordant?

Topological quantum field theories have led to a dramatic success in the understanding of 3- and 4-dimensional topologies and cobordisms of these manifolds (two n-manifolds are cobordant if there exists an $n+1$-manifold having them as boundaries). In his thought-provoking and highly inspiring article Pierre Cartier [A127] poses a question which at first sounds absurd. What might be the the counterpart of cobordism for points? The question is indeed absurd unless the points have some structure.

If one takes seriously the idea that each point of space-time sheet corresponds to a unit defined by an infinite rational, the obvious question is under what conditions there is a continuous line connecting these points with continuity being defined in some generalized sense. In real sense the line is continuous always but in p-adic sense only if all p-adic norms of the two units are identical. Since the p-adic norm of the unit of $Y(n/m) = X/\Pi(n/m)$ is that of $q = n/m$, the norm of two infinite rational numbers is same only if they correspond to the same ordinary rational number.

Suppose that one has

$$\begin{aligned} Y_I &= \frac{\prod_i Y(q^I_{1i})}{\prod_i Y(q^I_{2i})} \ , \quad Y_F = \frac{\prod_i Y(q^F_{1i})}{\prod_i Y(q^F_{2i})} \ , \\ q^I_{ki} &= \frac{n^I_{k_i}}{m^I_{k_i}} \ , \qquad q^F_{ki} = \frac{n^F_{k_i}}{m^F_{k_i}} \ , \end{aligned} \tag{4.10.1}$$

Here $m^{\cdot}_{\cdot}$ representing arithmetic many-fermion state is a square free integer and $n^{\cdot}_{\cdot}$ representing arithmetic many-boson state is an integer having no common factors with $m^{\cdot}_{\cdot}$.

The two units have same p-adic norm in all p-adic number fields if the rational numbers associated with Y_I and Y_F are same:

$$\frac{\prod_i q^I_{1i}}{\prod_i q^I_{2i}} = \frac{\prod_i q^F_{1i}}{\prod_i q^F_{2i}} \ . \tag{4.10.2}$$

The logarithm of this condition gives a conservation law of energy encountered in arithmetic quantum field theories, where the energy of state labeled by the prime p is $E_p = log(p)$:

$$\begin{aligned} E^I &= \sum_i log(n^I_{1i}) - \sum_i log(n^I_{2i}) - \sum_i log(m^I_{1i}) + \sum_i log(m^I_{2i}) = \\ &= \sum_i log(n^F_{1i}) - \sum_i log(n^F_{2i}) - \sum_i log(m^F_{1i}) + \sum_i log(m^F_{2i}) = E^F \ . \end{aligned} \tag{4.10.3}$$

There are both positive and negative energy particles present in the system. The possibility of negative energies is indeed one of the basic predictions of quantum TGD distinguishing it from standard physics. As one might have expected, Y^I and Y^F represent the initial and final states of a particle reaction and the line connecting the two points represents time evolution giving rise to the particle reaction. In principle one can even localize various steps of the reaction along the line and different lines give different sequences of reaction steps but same overall reaction. This symmetry is highly analogous to the conformal invariance implying that integral in complex plane depends only on the end points of the curve.

Whether the entire four-surface should correspond to the same value of topological energy or whether E can be discontinuous at elementary particle horizons separating space-time sheets and represented by light-like 3-surfaces around wormhole contacts remains an open question. Discontinuity through elementary particle horizons would make possible the arithmetic analogs of poles and cuts of analytic functions since the limiting values of Y from different sides of the horizon are different. Note that the construction generalizes to the quaternionic and octonionic case.

TGD inspired analog for d-algebras

Maxim Kontsevich has done deep work with quantizations interpreted as a deformation of algebraic structures and there are deep connections with this work and braid group [A182] . In particular, the Grothendienck-Teichmueller algebra believed to act as automorphisms for the deformation structures acts as automorphisms of the braid group at the limit of infinite number of strands. I must admit that my miserable skills in algebra do not allow to go to the horrendous technicalities but occasionally I have the feeling that I have understood some general ideas related to this work. In his article "Operads and Motives in Deformation Quantization" Kontsevich introduces the notions of operad and d-algebras over operad. Without going to technicalities one can very roughly say that d-algebra is essentially d-dimensional algebraic structure, and that the basic conjecture of Deligne generalized and proved by Kontsevich states in its generalized form that $d+1$-algebras have a natural action in all d-algebras.

In the proposed extension of various rationals a notion resembling that of universal d-algebra to some degree but not equivalent with it emerges naturally. The basic idea is simple.

1. Points correspond to the elements of the assumed to be universal algebra A which in this sense deserves the attribute $d = 0$ algebra. By its universality A should be able to represent any algebra and in this sense it cannot correspond $d = 0$-algebra of Kontsevich defined as a complex, that is a direct sum of vector spaces V_n and possessing d operation $V_n \rightarrow V_{n+1}$, satisfying $d^2 = 0$. Each point of a manifold represents one particular element of 0-algebra and one could loosely say that multiplication of points represents algebraic multiplication. This algebra has various subalgebras, in particular those corresponding to reals, complex numbers and quaternions. One can say that sub-algebra is non-associative, non-commutative, etc.. if its real evaluation has this property.

2. Lines correspond to evolutions for the elements of A which are continuous with respect to real (trivially) and all p-adic number fields. The latter condition is nontrivial and allows to interpret evolution as an evolution conserving number theoretical analog of total energy. Universal 1-group would consist of curves along which one has the analog of group valued field (group being the group of generalized units) having values in the universal 0-group G. The action of the 1-group in 0-group would simply map the element of 0-group at the first end of the curve its value at the second end. Curves define a monoid in an obvious manner. The interpretation as a map to A allows pointwise multiplication of these mappings which generalizes to all values of d.

   One could also consider the generalization of local gauge field so that there would be gauge potential defined in the algebra of units having values on A. This potential would define holonomy group acting on 0-algebra and mapping the element at the first end of the curve to its gauge transformed variant at the second end. In this case also closed curves would define non-trivial elements of the holonomy group. In fact, practically everything is possible since probably any algebra can be represented in the algebra generated by units.

3. Two-dimensional structures correspond to dynamical evolutions of one-dimensional structures. The simplest situation corresponds to 2-cubes with the lines corresponding to the initial and final values of the second coordinate representing initial and final states. One can also consider the possibility that the two-surface is topologically non-trivial containing handles and perhaps even holes. One could interpret this cognitive evolution as a 1-dimensional flow so that the initial points travel to final points. Obviously there is symmetry breaking involved since the second coordinate is in the role of time and this defines kind of time orientation for the surface.

4. The generalization to 4- and higher dimensional cases is obvious. One just uses d-manifolds with edges and uses their time evolution to define $d+1$-manifolds with edges. Universal 3-algebra is especially interesting from the point of view of braid groups and in this case the maps between initial and final elements of 2-algebra could be interpreted as braid operations if the paths of the elements along 3-surface are entangled. For instance field lines of Kähler gauge potential or of magnetic field could define this kind of braiding.

5. The d-evolutions define a monoid since one can glue two d-evolutions together if the outcome of the first evolution equals to the initial state of the second evolution. $d+1$-algebra also acts naturally in d-algebra in the sense that the time evolution $f(A \rightarrow B)$ assigns to the d-algebra valued initial state A a d-algebra valued final state and one can define the multiplication as $f(A \rightarrow B)C = B$ for $A = C$, otherwise the action gives zero. If time evolutions correspond to standard cubes one gets more interesting structure in this manner since the cubes differing by time translation can be identified and the product is always non-vanishing.

6. It should be possible to define generalizations of homotopy groups to what might be called "cognitive" homotopy groups. Effectively the target manifold would be replaced by the tensor product of an ordinary manifold and some algebraic structure represented in A. All kinds of "cognitive" homotopy groups would result when the image is cognitively non-contractible. Also homology groups could be defined by generalizing singular complex consisting of cubes with cubes having the hierarchical decomposition into time evolutions of time evolutions of... in some sub-algebraic structure of A. If one restricts time evolutions to sub-algebraic structures one obtains all kinds of homologies. For instance, associativity reduces 3-evolutions to paths in rational $SU(3)$ and since $SU(3)$ just like any Lie group has non-trivial 3-homology, one obtains nontrivial "cognitive" homology for 3-surfaces with non-trivial 3-homology.

The following heuristic arguments are inspired by the proposed vision about algebraic cognition and the conjecture that Grothendienck-Teichmueller group acts as automorphisms of Feynman diagrammatics relating equivalent quantum field theories to each other.

1. The operations of $d+1$-algebra realized as time evolution of d-algebra elements suggests an interpretation as cognitive counterparts for sequences of algebraic manipulations in d-algebra which themselves become elements of $d+1$ algebra. At the level of paths of points the sequences of algebraic operations correspond to transitions in which the number of infinite primes defining an infinite rational can change in discrete steps but is subject to the topological energy conservation guaranteeing the p-adic continuity of the process for all primes. Different paths connecting a and b represent different but equivalent manipulations sequences.

   For instance, at $d = 2$ level one has a pile of these processes and this in principle makes it possible an abstraction to algebraic rules involved with the process by a pile of examples. Higher values of d in turn make possible further abstractions bringing in additional parameters to the system. All kinds of algebraic processes can be represented in this manner. For instance, multiplication table can be represented as paths assigning to an the initial state product of elements a and b represented as infinite rationals and to the final state their product ab represented as single infinite rational. Representation is of course always approximate unless the algebra is finite. All kinds abstract rules such as various commutative diagrams, division of algebra by ideal by choosing one representative from each equivalence class of A/I as end point of the path, etc... can be represented in this manner.

2. There is also second manner to represent algebraic rules. Entanglement is a purely algebraic notion and it is possible to entangle the many-particle states formed as products of infinite rationals representing inputs of an algebraic operation A with the outcomes of A represented in the same manner such that the entanglement is consistent with the rule.

3. There is nice analogy between Feynman diagrams and sequences of algebraic manipulations. Multiplication ab corresponds to a map $A \otimes A \rightarrow A$ is analogous to a fusion of elementary particles since the product indeed conserves the number theoretical energy. Co-algebra operations are time reversals of algebra operations in this evolution. Co-multiplication Δ assigns to $a \in A$ an element in $A \otimes A$ via algebra homomorphism and corresponds to a decay of initial state particle to two final state particles. It defines co-multiplication assign to $a \otimes b \in A \otimes A$ an element of $A \otimes A \rightarrow A \otimes A \otimes A$ and corresponds to a scattering of elementary particles with the emission of a third particle. Hence a sequence of algebraic manipulations is like a Feynman diagram involving both multiplications and co-multiplications and thus containing also loops. When particle creation and annihilation are absent, particle number is conserved and the process represents algebra endomorphism $A \rightarrow A$. Otherwise a more general operation is in question. This analogy inspires the question whether particle reactions could serve as a blood and flesh representation for $d = 4$ algebras.

4. The dimension $d = 4$ is maximal dimension of single space-time evolution representing an algebraic operation (unless one allows the possibility that space-time and imbedding space dimensions are come as multiples of four and 8). Higher dimensions can be effectively achieved only if several space-time sheets are used defining $4n$-dimensional WCW. This could reflect some deep fact about algebras in general and also relate to the fact that 3- and 4-dimensional manifolds are the most interesting ones topologically.

4.10.2 One element field, quantum measurement theory and its q-variant, and the Galois fields associated with infinite primes

John Baez talked in This Weeks Finds (Week 259) [B12] about one-element field - a notion inspired by the $q = exp(i2\pi/n) \rightarrow 1$ limit for quantum groups. This limit suggests that the notion of one-element field for which 0=1 - a kind of mathematical phantom for which multiplication and sum should be identical operations - could make sense. Physicist might not be attracted by this kind of identification.

In the following I want to articulate some comments from the point of view of quantum measurement theory and its generalization to q-measurement theory which I proposed for some years ago and which is represented above.

I also consider and alternative interpretation in terms of Galois fields assignable to infinite primes which form an infinite hierarchy. These Galois fields have infinite number of elements but the map to the real world effectively reduces the number of elements to 2: 0 and 1 remain different.

$q \rightarrow 1$ limit as transition from quantum physics to effectively classical physics?

The $q \rightarrow 1$ limit of quantum groups at q-integers become ordinary integers and n-D vector spaces reduce to n-element sets. For quantum logic the reduction would mean that 2^N-D spinor space becomes 2^N-element set. N qubits are replaced with N bits. This brings in mind what happens in the transition from wave mechanism to classical mechanics. This might relate in interesting manner to quantum measurement theory.

Strictly speaking, $q \rightarrow 1$ limit corresponds to the limit $q = exp(i2\pi/n)$, $n \rightarrow \infty$ since only roots of unity are considered. This also correspond to Jones inclusions at the limit when the discrete group Z_n or or its extension-both subgroups of $SO(3)$- to contain reflection has infinite elements. Therefore this limit where field with one element appears might have concrete physical meaning. Does the system at this limit behave very classically?

In TGD framework this limit can correspond to either infinite or vanishing Planck constant depending on whether one consider orbifolds or coverings. For the vanishing Planck constant one should have classicality: at least naively! In perturbative gauge theory higher order corrections come as powers of $g^2/4\pi\hbar$ so that also these corrections vanish and one has same predictions as given by classical field theory.

Q-measurement theory and $q \rightarrow 1$ limit

Q-measurement theory differs from quantum measurement theory in that the coordinates of the state space, say spinor space, are non-commuting. Consider in the sequel q-spinors for simplicity.

Since the components of quantum spinor do not commute, one cannot perform state function reduction. One can however measure the modulus squared of both spinor components which indeed commute as operators and have interpretation as probabilities for spin up or down. They have a universal spectrum of eigen values. The interpretation would be in terms of fuzzy probabilities and finite measurement resolution but may be in different sense as in case of HFF:s. Probability would become the observable instead of spin for q not equal to 1.

At $q \rightarrow 1$ limit quantum measurement becomes possible in the standard sense of the word and one obtains spin down or up. This in turn means that the projective ray representing quantum states is replaced with one of n possible projective rays defining the points of n-element set. For HFF:s of type II_1 it would be N-rays which would become points, N the included algebra. One might also say that state function reduction is forced by this mapping to *single* object at $q \rightarrow 1$ limit.

On might say that the set of orthogonal coordinate axis replaces the state space in quantum measurement. We do this replacement of space with coordinate axis all the time when at blackboard. Quantum consciousness theorist inside me adds that this means a creation of symbolic representations and that the function of quantum classical correspondences is to build symbolic representations for quantum reality at space-time level.

$q \to 1$ limit should have space-time correlates by quantum classical correspondence. A TGD inspired geometro-topological interpretation for the projection postulate might be that quantum measurement at $q \to 1$ limit corresponds to a leakage of 3-surface to a dark sector of imbedding space with $q \to 1$ (Planck constant near to 0 or ∞ depending on whether one has $n \to \infty$ covering or division of M^4 or CP_2 by a subgroup of $SU(2)$ becoming infinite cyclic - very roughly!) and Hilbert space is indeed effectively replaced with n rays. For $q \neq 1$ one would have only probabilities for different outcomes since things would be fuzzy.

In this picture classical physics and classical logic would be the physical counterpart for the shadow world of mathematics and would result only as an asymptotic notion.

Could 1-element fields actually correspond to Galois fields associated with infinite primes?

Finite field G_p corresponds to integers modulo p and product and sum are taken only modulo p. An alternative representation is in terms of phases $exp(ik2\pi/p)$, $k = 0, ..., p-1$ with sum and product performed in the exponent. The question is whether could one define these fields also for infinite primes by identifying the elements of this field as phases $exp(ik2\pi/\Pi)$ with k taken to be finite integer and Π an infinite prime (recall that they form infinite hierarchy). Formally this makes sense. 1-element field would be replaced with infinite hierarchy of Galois fields with infinite number of elements!

The probabilities defined by components of quantum spinor make sense only as real numbers and one can indeed map them to real numbers by interpreting q as an ordinary complex number. This would give same results as $q \to 1$ limit and one would have effectively 1-element field but actually a Galois field with infinite number of elements.

If one allows k to be also infinite integer but not larger than than Π in the real sense, the phases $exp(ik2\pi/\Pi)$ would be well defined as real numbers and could differ from 1. All real numbers in the range $[-1, 1]$ would be obtained as values of $cos(k2\pi/\Pi)$ so that this limit would effectively give real numbers.

This relates also interestingly to the question whether the notion of p-adic field makes sense for infinite primes. The p-adic norm of any infinite-p p-adic number would be power of π either infinite, zero, or 1. Excluding infinite normed numbers one would have effectively only p-adic integers in the range $1, ...\Pi - 1$ and thus only the Galois field $G < sub > \Pi < /sub >$ representable also as quantum phases.

I conclude with a nice string of text from John'z page:

What's a mathematical phantom? According to Wraith, it's an object that doesn't exist within a given mathematical framework, but nonetheless "obtrudes its effects so convincingly that one is forced to concede a broader notion of existence".

and unashamedly propose that perhaps Galois fields associated with infinite primes might provide this broader notion of existence! In equally unashamed tone I ask whether there exists also hierarchy of conscious entities at $q = 1$ levels in real sense and whether we might identify ourselves as this kind of entities? Note that if cognition corresponds to p-adic space-time sheets, our cognitive bodies have literally infinite geometric size in real sense.

One-element field realized in terms of real units with number theoretic anatomy

One-element field looks rather self-contradictory notion since 1 and 0 should be represented by same element. The real units expressible as ratios of infinite rationals could however provide a well-defined realization of this notion.

1. The condition that same element represents the neutral element of both sum and product gives strong constraint on one-element field. Consider an algebra formed by reals with sum and product defined in the following manner. Sum, call it $\oplus$, corresponds to the ordinary product $x \times y$ for reals whereas product, call it $\otimes$, is identified as the non-commutative

product $x \otimes y = x^y$. $x = 1$ represents both the neutral element (0) of $\oplus$ and the unit of $\otimes$. The sub-algebras generated by 1 and multiple powers $P_n(x) = P_{n-1}(x) \otimes x = x \otimes ... \otimes x$ form commutative sub-algebras of this algebra. When one restricts the consideration to $x = 1$ one obtains one-element field as sub-field which is however trivial since $\oplus$ and $\otimes$ are identical operations in this subset.

2. One can get over this difficulty by keeping the operations $\oplus$ and $\otimes$, by assuming one-element property only with respect to the real and various p-adic norms, and by replacing ordinary real unit 1 with the algebra of real units formed from infinite primes by requiring that the real and various p-adic norms of the resulting numbers are equal to one. As far as real and various p-adic norms are considered, one has commutative one-element field. When number theoretic anatomy is taken into account, the algebra contains infinite number of elements and is non-commutative with respect to the product since the number theoretic anatomies of x^y and y^x are different.

4.10.3 A little crazy speculation about knots and infinite primes

D-dimensional knots correspond to the isotopy equivalence classes of the imbeddings of spheres S^d to S^{d+2}. One can consider also the isotopy equivalence classes of more general manifolds $M^d \subset M^{d+2}$. Knots [A53] are very algebraic objects. The product (or sum, I prefer to talk about product) of knots is defined in terms of connected sum. Connected sum quite generally defines a commutative and associative product, and one can decompose any knot into prime knots.

Knots can be mapped to Jones polynomials $J(K)$ (for instance - there are many other polynomials and there are very general mathematical results about them [A53]) and the product of knots is mapped to a product of corresponding polynomials. The polynomials assignable to prime knots should be prime in a well-defined sense, and one can indeed define the notion of primeness for polynomials $J(K)$: prime polynomial does not factor to a product of polynomials of lower degree in the extension of rationals considered.

This raises the idea that one could define the notion of zeta function for knots. It would be simply the product of factors $1/(1-J(K)^{-s})$ where K runs over prime knots. The new (to me) but very natural element in the definition would be that ordinary prime is replaced with a polynomial prime. This observation led to the idea that the hierarchy of infinite primes could correspond to the hierarchy of knots in various dimensions and this in turn stimulated quite fascinating speculations.

Do knots correspond to the hierarchy of infinite primes?

A very natural question is whether one could define the counterpart of zeta function for infinite primes. The idea of replacing primes with prime polynomials would resolve the problem since infinite primes can be mapped to polynomials. For some reason this idea however had not occurred to me earlier.

The correspondence of both knots and infinite primes with polynomials inspires the question whether d = 1-dimensional prime knots might be in correspondence (not necessarily 1-1) with infinite primes. Rational or Gaussian rational infinite primes would be naturally selected these are also selected by physical considerations as representatives of physical states although quaternionic and octonionic variants of infinite primes can be considered.

If so, knots could correspond to the subset of states of a super-symmetric arithmetic quantum field theory with bosonic single particle states and fermionic states labeled by quaternionic primes.

1. The free Fock states of this QFT are mapped to first order polynomials and irreducible polynomials of higher degree have interpretation as bound states so that the non-decomposability to a product in a given extension of rationals would correspond physically to the non-decomposability into many-particle state. What is fascinating that apparently free arithmetic QFT allows huge number of bound states.

2. Infinite primes form an infinite hierarchy, which corresponds to an infinite hierarchy of second quantizations for infinite primes meaning that n-particle states of the previous level define single particle states of the next level. At space-time level this hierarchy corresponds to a hierarchy defined by space-time sheets of the topological condensate: space-time sheet containing a galaxy can behave like an elementary particle at the next level of hierarchy.

3. Could this hierarchy have some counterpart for knots?In one realization as polynomials, the polynomials corresponding to infinite prime hierarchy have increasing number of variables. Hence the first thing that comes into my uneducated mind is as the hierarchy defined by the increasing dimension d of knot. All knots of dimension d would in some sense serve as building bricks for prime knots of dimension $d+1$ or possibly $d+2$ (the latter option turns out to be the more plausible one). A canonical construction recipe for knots of higher dimensions should exist.

4. One could also wonder whether the replacement of spherical topologies for d-dimensional knot and $d+2$-dimensional imbedding space with more general topologies could correspond to algebraic extensions at various levels of the hierarchy bringing into the game more general infinite primes. The units of these extensions would correspond to knots which involve in an essential manner the global topology (say knotted non-contractible circles in 3-torus). Since the knots defining the product would in general have topology different from spherical topology the product of knots should be replaced with its category theoretical generalization making higher-dimensional knots a groupoid in which spherical knots would act diagonally leaving the topology of knot invariant. The assignment of d-knots with the notion of n-category, n-groupoid, etc.. by putting d=n is a highly suggestive idea. This is indeed natural since are an outcome of a repeated abstraction process: statements about statements about .....

5. The lowest ($d = 1, D = 3$) level would be the fundamental one and the rest would be (somewhat boring!) repeated second quantization. This is why the dimension $D = 3$ (number theoretic braids at light-like 3-surfaces!) would be fundamental for physics.

Further speculations

Some further speculations about the proposed structure of all structures are natural.

1. The possibility that algebraic extensions of infinite primes could allow to describe the refinements related to the varying topologies of knot and imbedding space would mean a deep connection between number theory, manifold topology, sub-manifold topology, and n-category theory.

2. Category theory appears already now in fundamental role in the construction of the generalization of M-matrix unifying the notions of density matrix and S-matrix. Generalization of category to n-category theory and various n-structures would have very direct correspondence with the physics of TGD Universe if one assumes that repeated second quantization makes sense and corresponds to the hierarchical structure of many-sheeted space-time where even galaxy corresponds to elementary fermion or boson at some level of hierarchy.

   This however requires that the unions of light-like 3-surfaces and of their sub-manifolds at different levels of topological condensate are able to represent higher-dimensional manifolds physically albeit not in the standard geometric sense since imbedding space dimension is just 8. This might be possible.

3. As far as physics is considered, the disjoint union of sub-manifolds of dimensions d_1 and d_2 behaves like a $d_1 + d_2$-dimensional Cartesian product of the corresponding manifolds. This is of course used in standard manner in wave mechanics (the WCW of n-particle system is identified as E^{3n}/S_n with division coming from statistics).

4. If the surfaces have intersection points, one has a union of Cartesian product with punctures (intersection points) and of lower-dimensional manifold corresponding to the intersection points.

5. Note also that by posing symmetries on classical fields one can effectively obtain from a given n-manifold manifolds (and orbifolds) with quotient topologies.

The megalomanic conjecture is that this kind of physical representation of d-knots and their imbedding spaces is possible using many-sheeted space-time. Perhaps even the entire magnificent mathematics of n-manifolds and their sub-manifolds might have a physical representation in terms of sub-manifolds of 8-D $M^4 \times CP_2$ with dimension not higher than space-time dimension $d = 4$.

The idea survives the most obvious killer test

All this looks nice and the question is how to give a death blow to all this reckless speculation. Torus knots are an excellent candidate for performing this unpleasant task but the hypothesis survives!

1. Torus knots [A97] are labeled by a pair integers (m, n), which are relatively prime. These are prime knots. Torus knots for which one has $m/n = r/s$ are isotopic so that any torus knot is isotopic with a knot for which m and n have no common prime power factors.

2. The simplest infinite primes correspond to free Fock states of the supersymmetric arithmetic QFT and are labeled by pairs (m, n) of integers such that m and n do not have any common prime factors. Thus torus knots would correspond to free Fock states! Note that the prime power $p^{k(p)}$ appearing in m corresponds to $k(p)$-boson state with boson "momentum" p and the corresponding power in n corresponds to fermion state plus $k(p) - 1$ bosons.

3. A further property of torus knots is that (m, n) and (n, m) are isotopic: this would correspond at the level of infinite primes to the symmetry $mX + n \rightarrow nX + m$, X product of all finite primes. Thus infinite primes are in $2 \rightarrow 1$ correspondence with torus knots and the hypothesis survives also this murder attempt. Probably the assignment of orientation to the knot makes the correspondence 1-1 correspondence.

How to realize the representation of the braid hierarchy in many-sheeted space-time?

One can consider a concrete construction of higher-dimensional knots and braids in terms of the many-sheeted space-time concept.

1. The basic observation is that ordinary knots can be constructed as closed braids so that everything reduces to the construction of braids. In particular, any torus knot labeled by (m,n) can be made from a braid with m strands: the braid word in question is $(\sigma_1....\sigma_{m-1})^n$ or by $(m, n) = (n, m)$ equivalence from n strands. The construction of infinite primes suggests that also the notion of d-braid makes sense as a collection of d-braids in $d + 2$-space, which move and and define $d + 1$-braid in $d + 3$ space (the additional dimension being defined by time coordinate).

2. The notion of topological condensate should allow a concrete construction of the pairs of d- and $d + 2$-dimensional manifolds. The 2-D character of the fundamental objects (partons) might indeed make this possible. Also the notion of length scale cutoff fundamental for the notion of topological condensate is a crucial element of the proposed construction.

3. Infinite primes have also interpretation as physical states and the representation in terms of knots would mean a realization of quantum classical correspondence.

The concrete construction would proceed as follows.

1. Consider first the lowest non-trivial level in the hierarchy. One has a collection of 3-D light-like 3-surfaces X^3_i representing ordinary braids. The challenge is to assign to them a 5-D imbedding space in a natural manner. Where do the additional two dimensions come from? The obvious answer is that the new dimensions correspond to the partonic 2-surface X^2 assignable to the $3-D$ light-like surface X^3 at which these surfaces have suffered topological condensation. The geometric picture is that X^3_i grow like plants from ground defined by X^2 at 7-dimensional $\delta M^4_+ \times CP_2$.

2. The degrees of freedom of X^2 should be combined with the degrees of freedom of X^3_i to form a 5-dimensional space X^5. The natural idea is that one first forms the Cartesian products $X^5_i = X^3_i \times X^2$ and then the desired 5-manifold X^5 as their union by posing suitable additional conditions. Braiding means a translational motion of X^3_i inside X^2 defining braid as the orbit in X^5. It can happen that X^3_i and X^3_j intersect in this process. At these points of the union one must obviously pose some additional conditions. Same applies to intersection of more than two X^3_i.

Finite (p-adic) length scale resolution suggests that all points of the union at which an intersection between two or more light-like 3-surfaces occurs must be regarded as identical. In general the intersections would occur in a 2-d region of X^2 so that the gluing would take place along 5-D regions of X_i^5 and there are therefore good hopes that the resulting 5-D space is indeed a manifold. The imbedding of the surfaces X_i^3 to X^5 would define the braiding.

3. At the next level one would consider the 5-d structures obtained in this manner and allow them to topologically condense at larger 2-D partonic surfaces in the similar manner. The outcome would be a hierarchy consisting of $2n+1$-knots in $2n+3$ spaces. A similar construction applied to partonic surfaces gives a hierarchy of $2n$-knots in $2n+2$-spaces.

4. The notion of length scale cutoff is an essential element of the many-sheeted space-time concept. In the recent context it suggests that d-knots represented as space-time sheets topologically condensed at the larger space-time sheet representing $d+2$-dimensional imbedding space could be also regarded effectively point-like objects (0-knots) and that their d-knottiness and internal topology could be characterized in terms of additional quantum numbers. If so then d-knots could be also regarded as ordinary colored braids and the construction at higher levels would indeed be very much analogous to that for infinite primes.

Part II

TGD AND P-ADIC NUMBERS

Chapter 5

p-Adic Numbers and Generalization of Number Concept

5.1 Introduction

In this chapter basic facts about p-adic numbers and the question about their relation to real numbers are discussed. Also the basic technicalities related to the notion of p-adic physics are discussed.

5.1.1 Problems

It is far from obvious what the p-adic counterpart of real physics could mean and how one could fuse together real and p-adic physics. Therefore it is good to list the basic problems and proposals for their solution.

The first problem concerns the correspondence between real and p-adic numbers.

1. The success of p-adic mass calculations involves the notions of p-adic probability, thermodynamics, and the mapping of p-adic probababilities to the real ones by a continuous correspondence $x = \sum x_n p^n \rightarrow Id(x) = \sum x_n p^{-n}$ that I have christened canonical identification. The problem is that I n does not respect symmetries defined by isometries and also general coordinate invariance is possible only if one can identify preferred imbedding space coordinates. The reason is that I does not commute with the basic arithmetic operations. I allows several variants and it is possible to have correspondence which respects symmetries in arbitrary accuracy in preferred coordinates. Thus I can play a role at space-time level only if one defines symmetries modulo measurement resolution. I would make sense only in the interval defining the measurement resolution for a given coordinate variable and the p-adic effective topology would make sense just because the finite measurement resolution does not allow to well-order the points.

2. The identification of real and p-adic numbers via rationals common to all number fields - or more generally along algebraic extension of rationals- respects symmetries and algebra but is not continuous. At the imbedding space level preferred coordinates are required also now. The maximal symmetries of the imbedding space allow identification of this kind of coordinates. They are not unique. For instance, M^4 linear coordinates look very natural but for CP_2 trigonometric functions of angle like coordinates look more suitablel and Fourier analysis suggests strongly the introduction of algebraic extensions involving roots of unity. Partly the non-uniqueness has an interpretation as an imbedding space correlate for the selection of the quantization axes. The symmetric space property of WCW gives hopes that general coordinate invariance in quantal sense can be realized. The existence of p-adic harmonic analysis suggests a discretization of the p-adic variant of imbedding space and WCW based on roots of unity.

3. One can consider a compromise between the two correspondences. Discretization via common algebraic points can be completed to a p-adic continuum by assigning to each real discretization interval (say angle increment $2\pi/N$) p-adic numbers with norm smaller than one.

Second problem relates to integration and Fourier analysis. Both these procedures are fundamental for physics -be it classical or quantum. The p-adic variant of definite integral does not exist in the sense required by the action principles of physics although classical partial differential equations assigned to a particular variational principle make perfect sense. Fourier analysis is also possible only if one allows algebraic extension of p-adic numbers allowing a sufficient number of roots of unity correlating with the measurement resolution of angle. The finite number of them has interpretation in terms of finite angle resolution. Fourier analysis provides also an algebraic realization of definite integral when one integrates over the entire manifold as one indeed does in the case of WCW. If the space in question allows maximal symmetries as WCW and imbedding space do, there are excellent hopes of having p-adic variants of both integration and harmonic analysis and the above described procedure allows a precise completion of the discretized variant of real manifold to its continuous p-adic variant.

The third problem relates to the definitions of the p-adic variants of Riemannian, symplectic, and Kähler geometries. It is possible to generalize formally the notion of Riemann metric although non-local quantities like areas and total curvatures do not make sense if defined in terms of integrals. If all relevant quantities assignable to the geometry (family of Hamiltonians defining isometries, Killing vector fields, components of metric and Kähler form, Kähler function, etc...) are expressible in terms of rational functions involving only rational numbers as coefficients of polynomials, they allow an algebraic continuation to the p-adic context and the p-adic variant of the geometry makes sense.

The fourth problem relates to the question what one means with p-adic quantum mechanics. In TGD framework p-adic quantum theory utilizes p-adic Hilbert space. The motivation is that the notions of p-adic probability and unitarity are well defined. From the beginnning it was clear that the straightforward generalization of Schrödinger equation is not very interesting physically and gradually the conviction has developed that the most realistic approach must rely on the attempt to find the p-adic variant of the TGD inspired quantum physics in all its complexity. The recent approach starts from a rather concrete view about generalized Feynman diagrams defining the points of WCW and leads to a rather detailed view about what the p-adic variants of QM could be and how they could be fused with real QM to a larger structure. Even more, just the requirement that this p-adicization exists, gives very powerful constraints on the real variant of the quantum TGD.

The fifth problem relates to the notion of information in p-adic context. p-Adic thermodynamics leads naturally to the question what p-adic entropy might mean and this in turn leads to the realization that for rational or even algebraic probabilities p-adic variant of Shannon entropy can be negative and has minimum for a unique prime. One can say that the entanglement in the intersection of real and p-adic worlds is negentropic. This leads to rather fascinating vision about how negentropic entanglement makes it possible for living systems to overcome the second law of thermodynamics. The formulation of quantum theory in the intersection of real and living worlds becomes the basic challenge.

The proposed solutions to the technical problems could be rephrased in terms of the notion of algebraic universality. Various p-adic physics are obtained as algebraic continuation of real physics through the common algebraic points of real and p-adic worlds and by performing completion in the sense that the interval corresponding to finite measurement resolution are replaced with their p-adic counterpart via canonical identification. This allows to have exact symmetries as their discrete variants and also a continuous correspondence if desired. Particular p-adicization is characterized by a choice fo preferred imbedding space coordinates, which has interpretation in terms of a particular cognitive representation. Hence one is forced to refine the view about general coordinate invariance. Different coordinate choices correspond to different cognitive representations having delicate effects on physics if it is assumed to include also the effects of cognition.

5.1.2 Program

These ideas lead to a reasonably well defined p-adicization program. Try to define precisely the concepts of the p-adic space-time and configuration space (WCW), formulate the finite-p p-adic versions of quantum TGD. Try to fuse together real and various p-adic quantum TGDs are to form a full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, and the notions of probability and unitarity to the p-adic context. Also new physical thinking and philosophy is needed. The notions of zero energy ontology, hierarchy of Planck constants and the generalization of the notion of imbedding space required by it are essential but not discussed in detail in this chapter.

5.1.3 Topics of the chapter

The topics of the chapter are the following:

1. p-Adic numbers, their extensions (also those involving transcendentals) are described. The existence of a square root of an ordinary p-adic number is necessary in many applications of the p-adic numbers (p-adic group theory, p-adic unitarity, Riemannian geometry) and its existence implies a unique algebraic extension, which is 4-dimensional for $p > 2$ and 8-dimensional for $p = 2$. Contrary to the first expectations, all possible algebraic extensions are possible and one cannot interpret the algebraic dimension of the algebraic extension as a physical dimension.

2. The concepts of the p-adic differentiability and analyticity are discussed. The notion of p-adic fractal is introduced the properties of the fractals defined by p-adically differentiable functions are discussed.

3. Various approaches to the problem of defining p-adic valued definite integral are discussed. The only reasonable generalizations rely on algebraic continuation and correspondence via common rationals. p-Adic field equations do not necessitate p-adic definite integral but algebraic continuation allows to assign to a given real space-time sheets a p-adic space-time sheets if the definition of space-time sheet involves algebraic relations between imbedding space coordinates. There are also hopes that one can algebraically continue the value of Kähler action to p-adic context if finite-dimensional extensions are allowed.

4. Symmetries are discussed from p-adic point of view starting from the identification via common rationals. Also possible p-adic generalizations of Fourier analysis are considered. Besides a number theoretical approach, group theoretical approach providing a direct generalization of the ordinary Fourier analysis based on the utilization of exponent functions existing in algebraic extensions containing some root of e and its powers up to e^{p-1} is discussed. Also the generalization of Fourier analysis based on the Pythagorean phases is considered.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- p-Adic number fields [L26]
- p-Adic physics [L27]
- p-Adic number fields and cognition and intention [L25]
- p-Adic length scale hypothesis [L28]
- p-Adic manifold [L23]
- p-Adic mass calculations [L24]

5.2 Summary of the basic physical ideas

In the following various manners to end up with p-adic physics and with the idea about p-adic topology as an effective topology of space-time surface are described.

5.2.1 p-Adic mass calculations briefly

p-Adic mass calculations based on p-adic thermodynamics with energy replaced with the generator $L_0 = zd/dz$ of infinitesimal scaling are described in the first part of [K46] .

1. p-Adic thermodynamics is justified by the randomness of the motion of partonic 2-surfaces restricted only by the light-likeness of the orbit.

2. It is essential that the conformal symmetries associated with the light-like coordinates of parton and light-cone boundary are not gauge symmetries but dynamical symmetries. The point is that there are two kinds of conformal symmetries: the super-symplectic conformal symmetries assignable to the light-like boundaries of $CD \times CP_2$ and super Kac-Moody symmetries assignable to light-like 3-surfaces defining fundamental dynamical objects. In so called coset construction the differences of super-conformal generators of these algebras annihilate the physical states. This leads to a generalization of equivalence principle since one can assign four-momentum to the generators of both algebras identifiable as inertial *resp.* gravitational four-momentum. A second important consequence is that the generators of either algebra do not act like gauge transformations so that it makes sense to construct p-adic thermodynamics for them.

3. In p-adic thermodynamics scaling generator L_0 having conformal weights as its eigen values replaces energy and Boltzmann weight $exp(H/T)$ is replaced by p^{L_0/T_p}. The quantization $T_p = 1/n$ of conformal temperature and thus quantization of mass squared scale is implied by number theoretical existence of Boltzmann weights. p-Adic length scale hypothesis states that primes $p \simeq 2^k$, k integer. A stronger hypothesis is that k is prime (in particular Mersenne prime or Gaussian Mersenne) makes the model very predictive and fine tuning is not possible.

The basic mystery number of elementary particle physics defined by the ratio of Planck mass and proton mass follows thus from number theory once CP_2 radius is fixed to about 10^4 Planck lengths. Mass scale becomes additional discrete variable of particle physics so that there is not more need to force top quark and neutrinos with mass scales differing by 12 orders of magnitude to the same multiplet of gauge group. Electron, muon, and τ correspond to Mersenne prime $k = 127$ (the largest non-super-astrophysical Mersenne), and Mersenne primes $k = 113, 107$. Intermediate gauge bosons and photon correspond to Mersenne M_{89}, and graviton to M_{127}.

Mersenne primes are very special also number theoretically because bit as the unit of information unit corresponds to $log(2)$ and can be said to exists for M_n-adic topology. The reason is that $log(1+p)$ existing always p-adically corresponds for $M_n = 2^n - 1$ to $log(2^n) \equiv nlog(2)$ so that one has $log(2 \equiv log(1+M_n)/n$. Since the powers of 2 modulo p give all integers $n \in \{1, p-1\}$ by Fermat's theorem, one can say that the logarithms of all integers modulo M_n exist in this sense and therefore the logarithms of all p-adic integers not divisible by p exist. For other primes one must introduce a transcendental extension containing $log(a)$ where are is so called primitive root. One could criticize the identification since $log(1+M_n)$ corresponding in the real sense to n bits corresponds in p-adic sense to to a very small information content since the p-adic norm of the p-adic bit is $1/M_n$.

The value of k for quark can depend on hadronic environment [K49] and this would produce precise mass formulas for low energy hadrons. This kind of dependence conforms also with the indications that neutrino mass scale depends on environment [C1]. Amazingly, the biologically most relevant length scale range between 10 nm and 4 μm contains four Gaussian Mersennes $(1+i)^n - 1$, $n = 151, 157, 163, 167$ and scaled copies of standard model physics in cell length scale could be an essential aspect of macroscopic quantum coherence prevailing in cell length scale.

p-Adic mass thermodynamics is not quite enough: also Higgs boson is needed and wormhole contact carrying fermion and anti-fermion quantum numbers at the light-like wormhole throats is excellent candidate for Higgs [K39] . The coupling of Higgs to fermions can be small and induce only a small shift of fermion mass: this could explain why Higgs has not been observed. Also the Higgs contribution to mass squared can be understood thermodynamically if identified as absolute value for the thermal expectation value of the eigenvalues of the modified Dirac operator having interpretation as complex square root of conformal weight.

The original belief was that only Higgs corresponds to wormhole contact. The assumption that fermion fields are free in the conformal field theory applying at parton level forces to identify all gauge bosons as wormhole contacts connecting positive and negative energy space-time sheets [K39] . Fermions correspond to topologically condensed CP_2 type extremals with single light-like wormhole throat. Gravitons are identified as string like structures involving pair of fermions or gauge bosons connected by a flux tube. Partonic 2-surfaces are characterized by genus which explains family replication phenomenon and an explanation for why their number is three emerges [K17] . Gauge bosons are labeled by pairs (g_1, g_2) of handle numbers and can be arranged to octet and singlet representations of the resulting dynamical SU(3) symmetry. Ordinary gauge bosons are SU(3) singlets and the heaviness of octet bosons explains why higher boson families are effectively absent. The different character of bosons could also explain why the p-adic temperature for bosons is $T_p = 1/n < 1$ so that Higgs contribution to the mass dominates.

5.2.2 p-Adic length scale hypothesis, zero energy ontology, and hierarchy of Planck constants

Zero energy ontology and the hierarchy of Planck constants realized in terms of the generalization of the imbedding space lead to a deeper understanding of the origin of the p-adic length scale hypothesis.

Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the light-like boundaries of CD. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. "Any physical state is creatable from vacuum" becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe?, Is theory building completely useless if only single solution of field equations is realized?). At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events.

Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [K18] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra spanned by the gamma matrices of the "world of classical worlds" represents a von Neumann algebra [A132] known as hyperfinite factor of type II_1 (HFF) [K18, K80, K25] . HFF [A128, A171] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [A9] . The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [A212] , anyons [D3] , quantum groups and conformal field theories [A172] , and knots and topological quantum field theories [A201, A217] .

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

One can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with M-matrix.

The temporal distance between the tips of CD corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where R is CP_2 size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$ since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is .1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [K18] . The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics. Connes tensor product [A128] provides a mathematical description of the finite measurement resolution but does not fix the M-matrix as was the original hope. The remaining challenge is the calculation of M-matrix and the progress induced by zero energy ontology during last years has led to rather concrete proposal for the construction of M-matrix.

How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

Zero energy ontology in which zero energy states have as imbedding space correlates CDs for which the distance between the tips of future and past directed light-cones are power of 2 multiples of fundamental time scale ($T_n = 2^nT_0$) implies in a natural manner coupling constant evolution. A weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^nT_0$ (or $T_p = pT_0$) induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section $X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^nT_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^kT_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around .1 eV would correspond to $L(169) \simeq 5$ μm (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, p would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of WCW.

Mersenne primes and Gaussian Mersennes

The generalization of the imbedding space required by the postulated hierarchy of Planck constants [K25] means a book like structure for which the pages are products of singular coverings or factor spaces of CD (causal diamond defined as intersection of future and past directed light-cones) and of CP_2 [K25] . This predicts that Planck constants are rationals and that a given value of Planck constant corresponds to an infinite number of different pages of the Big Book, which might be seen as a drawback. If only singular covering spaces are allowed the values of Planck constant are products of integers and given value of Planck constant corresponds to a finite number of pages given by the number of decompositions of the integer to two different integers. The definition of the book like structure assigns to a given CD preferred quantization axes and so that quantum measurement has direct correlate at the level of moduli space of CDs.

TGD inspired quantum biology and number theoretical considerations suggest preferred values for $r = \hbar/\hbar_0$. For the most general option the values of $\hbar$ are products and ratios of two integers n_a and n_b. Ruler and compass integers defined by the products of distinct Fermat primes and power of two are number theoretically favored values for these integers because the phases $exp(i2\pi/n_i)$, $i \in \{a, b\}$, in this case are number theoretically very simple and should have emerged first in the number theoretical evolution via algebraic extensions of p-adics and of rationals. p-Adic length scale hypothesis favors powers of two as values of r.

One can however ask whether a more precise characterization of preferred Mersennes could exist and whether there could exists a stronger correlation between hierarchies of p-adic length scales and Planck constants. Mersenne primes $M_k = 2^k - 1$, $k \in \{89, 107, 127\}$, and Gaussian Mersennes $M_{G,k} = (1 + i)k - 1$, $k \in \{113, 151, 157, 163, 167, 239, 241..\}$ are expected to be physically highly interesting and up to $k = 127$ indeed correspond to elementary particles. The number theoretical miracle is that all the four p-adic length scales with $k \in \{151, 157, 163, 167\}$ are in the biologically highly interesting range 10 nm-2.5 μm). The question has been whether these define scaled up copies of electro-weak and QCD type physics with ordinary value of $\hbar$. The proposal that this is the case and that these physics are in a well-defined sense induced by the dark scaled up variants of corresponding lower level physics leads to a prediction for the preferred values of $r = 2^{k_d}$, $k_d = k_i - k_j$.

What induction means is that dark variant of exotic nuclear physics induces exotic physics with ordinary value of Planck constant in the new scale in a resonant manner: dark gauge bosons transform to their ordinary variants with the same Compton length. This transformation is natural since in length scales below the Compton length the gauge bosons behave as massless and free particles. As a consequence, lighter variants of weak bosons emerge and QCD confinement scale becomes longer.

This proposal will be referred to as Mersenne hypothesis. It leads to strong predictions about EEG [K22] since it predicts a spectrum of preferred Josephson frequencies for a given value of membrane potential and also assigns to a given value of $\hbar$ a fixed size scale having interpretation as the size scale of the body part or magnetic body. Also a vision about evolution of life emerges. Mersenne hypothesis is especially interesting as far as new physics in condensed matter length scales is considered: this includes exotic scaled up variants of the ordinary nuclear physics and their dark variants. Even dark nucleons are possible and this gives justification for the model of dark nucleons predicting the counterparts of DNA,RNA, tRNa, and amino-acids as well as realization of vertebrate genetic code [K75] .

These exotic nuclear physics with ordinary value of Planck constant could correspond to ground states that are almost vacuum extremals corresponding to homologically trivial geodesic sphere of CP_2 near criticality to a phase transition changing Planck constant. Ordinary nuclear physics would correspond to homologically non-trivial geodesic sphere and far from vacuum extremal property. For vacuum extremals of this kind classical Z^0 field proportional to electromagnetic field is present and this modifies dramatically the view about cell membrane as Josephson junction. The model for cell membrane as almost vacuum extremal indeed led to a quantitative breakthrough in TGD inspired model of EEG and is therefore something to be taken seriously. The safest option concerning empirical facts is that the copies of electro-weak and color physics with ordinary value of Planck constant are possible only for almost vacuum extremals - that is at criticality against phase transition changing Planck constant.

5.2.3 p-Adic physics and the notion of finite measurement resolution

Canonical identification mapping p-adic numbers to reals in a continuous manner plays a key role in some applications of TGD and together with the discretization necessary to define the p-adic variants of integration and harmonic analysis suggests that p-adic topology identified as an effective topology could provide an elegant manner to characterize finite measurement resolution.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong in the modelling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with a pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

This interpretation would suggest that there is an infinite hierarchy of measurement resolutions characterized by the value of the p-adic prime. This would mean quite interesting refinement of the notion of finite measurement resolution. At the level of quantum theory it could be interpreted as a maximization of p-adic entanglement negentropy as a function of the p-adic prime. Perhaps one might say that there is a unique p-adic effective topology allowing to maximize the information content of the theory relying on finite measurement resolution.

5.2.4 p-Adic numbers and the analogy of TGD with spin-glass

The vacuum degeneracy of the Kähler action leads to a precise spin glass analogy at the level of the WCW geometry and the generalization of the energy landscape concept to TGD context leads to the hypothesis about how p-adicity could be realized at the level of WCW. Also the concept of p-adic space-time surface emerges rather naturally.

Spin glass briefly

The basic characteristic of the spin glass phase [B14] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants. Free energy as a function of the coupling constants defines 'energy landscape' and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [A199] .

Vacuum degeneracy of Kähler action

The Kähler action defining WCW geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the CP_2 projection of the space-time surface is Lagrange manifold of CP_2: these manifolds are at most two-dimensional and any canonical transformation of CP_2 creates a new Lagrange manifold. An explicit representation for Lagrange manifolds is obtained using some canonical coordinates P_i, Q_i for CP_2: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ , \ i = 1,2 \ ,$$

where f arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of CP_2 for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of P_i and Q_i can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary (p_i, q_i) phase space.

Since vacuum degeneracy is removed only by the classical gravitational interaction there are good reasons to expect large ground state degeneracy, when the system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass but is 4-dimensional.

Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with CP_2 projection, which is a Lagrangian sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a given preferred extremal of Kähler action one obtains a new one by adding arbitrary space-time surfaces which is vacuum extremal and deforming them.

The symplectic invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of CH. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function become possible. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe [?] . In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multi-furcation. This characterization works when non-determinism has discrete nature. For CP_2 type extremals which are bosonic vacua, basic objects are essentially four-dimensional since M^4_+ projection of CP_2 type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with CP_2 type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate preferred extremals. This non-determinism is expected to make the preferred extremals of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{5.2.1}$$

which does not depend on the pinary digits x_n, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the preferred extremals are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible 'engineering' at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible 'engineering' at the level of the real world.

5.2.5 Life as islands of rational/algebraic numbers in the seas of real and p-adic continua?

The possibility to define entropy differently for rational/algebraic entanglement and the fact that number theoretic entanglement entropy can be negative raises the question about which kind of systems can possess this kind of entanglement. I have considered several identifications but the most elegant interpretation is based on the idea that living matter resides in the intersection of real and p-adic worlds, somewhat like rational numbers live in the intersection of real and p-adic number fields.

The observation that Shannon entropy allows an infinite number of number theoretic variants for which the entropy can be negative in the case that probabilities are algebraic numbers leads to the idea that living matter in a well-defined sense corresponds to the intersection of real and p-adic worlds. This would mean that the mathematical expressions for the space-time surfaces (or at least 3-surfaces or partonic 2-surfaces and their 4-D tangent planes) make sense in both real and p-adic sense for some primes p. Same would apply to the expressions defining quantum states. In particular, entanglement probabilities would be rationals or algebraic numbers so that entanglement can be negentropic and the formation of bound states in the intersection of real and p-adic worlds generates information and is thus favored by NMP.

This picture has also a direct connection with consciousness.

1. Algebraic entanglement is a prerequisite for the realization of intentions as transformations of p-adic space-time sheets to real space-time sheets representing actions. Essentially a leakage between p-adic and real worlds is in question and makes sense only in zero energy ontology.

since various quantum numbers in real and p-adic sectors are not in general comparable in positive energy ontology so that conservation laws would be broken. Algebraic entanglement could be also called cognitive. The transformation can occur if the partonic 2-surfaces and their 4-D tangent space-distributions are representable using rational functions with rational coefficients in preferred coordinates for the imbedding space dictated by symmetry considerations. Intentional systems must live in the intersection of real and p-adic worlds. For the minimal option life would be also effectively 2-dimensional phenomenon and essentially a boundary phenomenon as also number theoretical criticality suggests.

2. The generation of non-rational (non-algebraic) bound state entanglement between the system and external world means that the system loses consciousness during the state function reduction process following the U-process generating the entanglement. What happens that the Universe corresponding to given CD decomposes to two un-entangled subsystems, which in turn decompose, and the process continues until all subsystems have only entropic bound state entanglement or negentropic algebraic entanglement with the external world.

3. If the sub-system generates entropic bound state entanglement in the the process, it loses consciousness. Note that the entanglement entropy of the sub-system is a sum over entanglement entropies over all subsystems involved. This hierarchy of subsystems corresponds to the hierarchy if sub-CDs so that the survival without a loss of consciousness depends on what happens at all levels below the highest level for a given self. In more concrete terms, ability to stay conscious depends on what happens at cellular level too. The stable evolution of systems having algebraic entanglement is expected to be a process proceeding from short to long length scales as the evolution of life indeed is.

4. U-process generates a superposition of states in which any sub-system can have both real and algebraic entanglement with the external world. This would suggest that the choice of the type of entanglement is a volitional selection. A possible interpretation is as a choice between good and evil. The hedonistic complete freedom resulting as the entanglement entropy is reduced to zero on one hand, and the algebraic bound state entanglement implying correlations with the external world and meaning giving up the maximal freedom on the other hand. The hedonistic option is risky since it can lead to non-algebraic bound state entanglement implying a loss of consciousness. The second option means expansion of consciousness - a fusion to the ocean of consciousness as described by spiritual practices.

5. This formulation means a sharpening of the earlier statement "Everything is conscious and consciousness can be only lost" with the additional statement "This happens when non-algebraic bound state entanglement is generated and the system does not remain in the intersection of real and p-adic worlds anymore". Clearly, the quantum criticality of TGD Universe seems has very many aspects and life as a critical phenomenon in the number theoretical sense is only one of them besides the criticality of the space-time dynamics and the criticality with respect to phase transitions changing the value of Planck constant and other more familiar criticalities. How closely these criticalities relate remains an open question.

A good guess is that algebraic entanglement is essential for quantum computation, which therefore might correspond to a conscious process. Hence cognition could be seen as a quantum computation like process, a more appropriate term being quantum problem solving. Living-dead dichotomy could correspond to rational-irrational or to algebraic-transcendental dichotomy: this at least when life is interpreted as intelligent life. Life would in a well defined sense correspond to islands of rationality/algebraicity in the seas of real and p-adic continua.

The view about the crucial role of rational and algebraic numbers as far as intelligent life is considered, could have been guessed on very general grounds from the analogy with the orbits of a dynamical system. Rational numbers allow a predictable periodic decimal/pinary expansion and are analogous to one-dimensional periodic orbits. Algebraic numbers are related to rationals by a finite number of algebraic operations and are intermediate between periodic and chaotic orbits allowing an interpretation as an element in an algebraic extension of any p-adic number field. The projections of the orbit to various coordinate directions of the algebraic extension represent now periodic orbits. The decimal/pinary expansions of transcendentals are un-predictable being analogous to chaotic orbits. The special role of rational and algebraic numbers was realized already by Pythagoras, and the fact that the ratios for the frequencies of the musical scale are rationals supports the special nature of rational and algebraic numbers. The special nature of the Golden Mean, which involves $\sqrt{5}$, conforms the view that algebraic numbers rather than only rationals are essential for life.

5.2.6 p-Adic physics as physics of cognition and intention

The vision about p-adic physics as physics of cognition has gradually established itself as one of the key idea of TGD inspired theory of consciousness. There are several motivations for this idea.

The strongest motivation is the vision about living matter as something residing in the intersection of real and p-adic worlds. One of the earliest motivations was p-adic non-determinism identified tentatively as a space-time correlate for the non-determinism of imagination. p-Adic non-determinism follows from the fact that functions with vanishing derivatives are piecewise constant functions in the p-adic context. More precisely, p-adic pseudo constants depend on the pinary cutoff of their arguments and replace integration constants in p-adic differential equations. In the case of field equations this means roughly that the initial data are replaced with initial data given for a discrete set of time values chosen in such a manner that unique solution of field equations results. Solution can be fixed also in a discrete subset of rational points of the imbedding space. Presumably the uniqueness requirement implies some unique pinary cutoff. Thus the space-time surfaces representing solutions of p-adic field equations are analogous to space-time surfaces consisting of pieces of solutions of the real field equations. p-Adic reality is much like the dream reality consisting of rational fragments glued together in illogical manner or pieces of child's drawing of body containing body parts in more or less chaotic order.

The obvious looking interpretation for the solutions of the p-adic field equations is as a geometric correlate of imagination. Plans, intentions, expectations, dreams, and cognition in general are expected to have p-adic space-time sheets as their geometric correlates. This in the sense that p-adic space-time sheets somehow initiate the real neural processes providing symbolic counterparts for the cognitive representations provided by p-adic space-time sheets and p-adic fermions. A deep principle seems to be involved: incompleteness is characteristic feature of p-adic physics but the flexibility made possible by this incompleteness is absolutely essential for imagination and cognitive consciousness in general.

p-Adic space-time regions can suffer topological phase transitions to real topology and vice versa in quantum jumps replacing space-time surface with a new one. This process has interpretation as a topological correlate for the mind-matter interaction in the sense of transformation of intention to action and symbolic representation to cognitive representation. p-Adic cognitive representations could provide the physical correlates for the notions of memes [J1] and morphic fields [I1] . p-Adic real entanglement makes possible makes possible cognitive measurements and cognitive quantum computation like processes, and provides correlates for the experiences of understanding and confusion.

At the level of brain the fundamental sensory-motor loop could be seen as a loop in which real-to-p-adic phase transition occurs at the sensory step and its reverse at the motor step. Nerve pulse patterns would correspond to temporal sequences of quark like sub-CDs of duration 1 millisecond inside electronic sub-CD of duration .1 s with the states of sub-CDs allowing interpretation as a bit (this would give rise to memetic code). The real space-time sheets assignable to these sub-CDs are transformed to p-adic ones as sensory input transforms to thought. Intention in transforms to action in the reverse process in motor action. One can speak about creation of matter from vacuum in these time scales.

Although p-adic space-time sheets as such are not conscious, p-adic physics would provide beautiful mathematical realization for the intuitions of Descartes. The formidable challenge is to develop experimental tests for p-adic physics. The basic problem is that we can perceive p-adic reality only as 'thoughts' unlike the 'real' reality which represents itself to us as sensory experiences. Thus it would seem that we should be able generalize the physics of sensory experiences to physics of cognitive experiences.

5.3 p-Adic numbers

5.3.1 Basic properties of p-adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A121] . p-Adic numbers are representable as power expansion of the prime number p of form:

$$x = \sum_{k \geq k_0} x(k)p^k , \; x(k) = 0, ..., p-1 \; . \tag{5.3.1}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \; . \tag{5.3.2}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \; , \tag{5.3.3}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \leq max\{d(x, y), d(y, z)\} \; . \tag{5.3.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x, y) \leq D \; . \tag{5.3.5}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

2. Distances of points x and y inside single class are smaller than distances between different classes.

3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B32] . The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

5.3.2 Algebraic extensions of p-adic numbers

Algebraic democracy suggests that all possible real algebraic extensions of the p-adic numbers are possible. This conclusion is also suggested by various physical requirements, say the fact that the eigenvalues of a Hamiltonian representable as a rational or p-adic $N \times N$-matrix, being roots of N:th order polynomial equation, in general belong to an algebraic extension of rationals or p-adics. The dimension of the algebraic extension cannot be interpreted as physical dimension. Algebraic extensions are characteristic for cognitive physics and provide a new manner to code information. A possible interpretation for the algebraic dimension is as a dimension for a cognitive representation of space and would explain how it is possible to mathematically imagine spaces with all possible dimensions although physical space-time dimension is four (TGD as a number theory vision suggest that also space-time dimensions which are multiples of four are possible). The idea of algebraic hologram and other ideas related to the physical interpretation of the algebraic extensions of p-adics are discussed in the chapter "TGD as a generalized number theory".

It seems however that algebraic democracy must be extended to include also transcendentals in the sense that finite-dimensional extensions involving also transcendental numbers are possible: for instance, Neper number e defines a p-dimensional extension. It has become clear that these extensions fundamental for understanding how p-adic physics as physics of cognition is able to mimic real physics. The evolution of mathematical cognition can be seen as a process in which p-adic space-time sheets involving increasing value of p-adic prime p and increasing dimension of algebraic extension appear in quantum jumps.

Recipe for constructing algebraic extensions

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible
[A121] . The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in R_p assumed to have rational coefficients: irreducibility means that the polynomial does not possess roots in R_p so that one cannot decompose it into a product of lower order R_p valued polynomials. This condition is equivalent with the condition with irreducibility in the finite field $G(p,1)$, that is modulo p in R_p.

Denoting one of the roots of $P_n(t)$ by θ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k \theta^k \ . \tag{5.3.6}$$

Since θ is root of the polynomial in R_p it follows that θ^n is expressible as a sum of lower powers of θ so that these numbers indeed form an n-dimensional linear space with respect to the p-adic topology.

Especially simple odd-dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$\begin{aligned} P_n(t) &= t^n + \epsilon d \ , \\ \epsilon &= \pm 1 \ . \end{aligned} \tag{5.3.7}$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if d does not possess n:th root in R_p. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case θ is one of the roots of the equation

$$t^n = \pm d \ , \tag{5.3.8}$$

where d is a p-adic integer with a finite number of pinary digits. It is possible although not necessary to identify the roots as complex numbers. There exists n complex roots of d and θ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$\begin{aligned} \theta &= d^{1/n}exp(i\phi(m)),\ m = 0, 1, ..., n-1 \ , \\ \phi(m) &= \frac{m2\pi}{n} \text{ or } \frac{m\pi}{n} \ . \end{aligned} \tag{5.3.9}$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to an algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of the algebraically extended p-adic numbers with the complex numbers plays a fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

p-Adic valued norm for numbers in algebraic extension

The p-adic valued norm of an algebraically extended p-adic number x can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \rightarrow^e R_p^n$ defined by the multiplication with x: $y \rightarrow xy$

$$N(x) = det(x)^{\alpha} \ , \ \alpha > 0 \ . \tag{5.3.10}$$

Real valued norm can be defined as the p-adic norm of $N(x)$. The requirement that the norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of R^n) implies the condition $\alpha = 1/n$, where n is the dimension of the algebraic extension.

The canonical correspondence between the points of R_+ and R_p generalizes in obvious manner: the point $\sum_k x_k\theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, ..,)$ of R^n using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces a linear structure in R_+^n and p-adic multiplication induces a multiplication for the vectors of R_+^n.

Algebraic extension allowing square root of ordinary p-adic numbers

The existence of a square root of an ordinary p-adic number is a common theme in various applications of the p-adic numbers and for long time I erratically believed that only this extension is involved with p-adic physics. Despite this square root allowing extension is of central importance and deserves a more detailed discussion.

1. The p-adic generalization of the representation theory of the ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of the p-adic numbers allowing square roots of the 'real' p-adic numbers is used. The reason is that the matrix elements of the raising and lowering operators in Lie-algebras as well as of oscillator operators typically involve square roots. The existence of square root might play a key role in various p-adic considerations.

2. The existence of a square root of a real p-adic number is also a necessary ingredient in the definition of the p-adic unitarity and probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is ordinary p-adic number leads to expressions involving square roots.

3. p-Adic length scales hypothesis states that the p-adic length scale is proportional to the square root of p-adic prime.

4. Simple metric geometry of polygons involves square roots basically via the theorem of Pythagoras. p-Adic Riemannian geometry necessitates the existence of square root since the definition of the infinitesimal length ds involves square root. Note however that p-adic Riemannian geometry can be formulated as a mere differential geometry without any reference to global concepts like lengths, areas, or volumes.

The original belief that square root allowing extensions of p-adic numbers are exceptional seems to be wrong in light of TGD as a generalized number theory vision. All algebraic extensions of p-adic numbers a possible and the interpretation of algebraic dimension of the extension as a physical dimension is not the correct thing to do. Rather, the possibility of arbitrarily high algebraic dimension reflects the ability of mathematical cognition to imagine higher-dimensional spaces. Square root allowing extension of the p-adic numbers is the simplest one imaginable, and it is fascinating that it indeed is the dimension of space-time surface for $p > 2$ and dimension of imbedding space for $p = 2$. Thus the square root allowing extensions deserve to be discussed.

The results can be summarized as follows.

1. In $p > 2$ case the general form of extension is

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ , \tag{5.3.11}$$

   where the condition $\theta^2 = x$ for some p-adic number x not allowing square root as a p-adic number. For $p \ mod \ 4 = 3$ θ can be taken to be imaginary unit. This extension is natural for p-adication of space-time surface so that space-time can be regarded as a number field locally. Imbedding space can be regarded as a cartesian product of two 4-dimensional extensions locally.

2. In $p = 2$ case 8-dimensional extension is needed to define square roots. The extension is defined by adding $\theta_1 = \sqrt{-1} \equiv i$, $\theta_2 = \sqrt{2}$, $\theta_3 = \sqrt{3}$ and the products of these so that the extension can be written in the form

$$Z = x_0 + \sum_k x_k\theta_k + \sum_{k<l} x_{kl}\theta_{kl} + x_{123}\theta_1\theta_2\theta_3 \ . \tag{5.3.12}$$

   Clearly, $p = 2$ case is exceptional as far as the construction of the conformal field theory limit is considered since the structure of the representations of Virasoro algebra and groups in general changes drastically in $p = 2$ case. The result suggest that in $p = 2$ limit space-time surface and H are in same relation as real numbers and complex numbers: space-time surfaces defined as the absolute minima of 2-adiced Kähler action are perhaps identifiable as surfaces for which the imaginary part of 2-adically analytic function in H vanishes.

The physically interesting feature of p-adic group representations is that if one doesn't use $\sqrt{p}$ in the extension the number of allowed spins for representations of $SU(2)$ is finite: only spins $j < p$ are allowed. In $p = 3$ case just the spins $j \leq 2$ are possible. If 4-dimensional extension is used for $p = 2$ rather than 8-dimensional then one gets the same restriction for allowed spins.

5.3.3 Is e an exceptional transcendental?

One can consider also the possibility of transcendental extensions of p-adic numbers and an open problem is whether the infinite-dimensional extensions involving powers of π and logarithms of primes make sense and whether they should be allowed. For instance, it is not clear whether the allowance of powers of π is consistent with the extensions based on roots of unity. This question is not academic since Feynman amplitudes in real context involve powers of π and algebraic universality forces the consider that also they p-adic variants might involve powers of π.

Neper number obviously defines the simplest transcendental extension since only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$ so that the extension is finite-dimensional. In the case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

An interesting question is whether e is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

1. Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.
2. The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).
3. The differential equation allows to develop $f(x)$ in power series, say in origin

$$f(x) = \sum f_n \frac{x^n}{n!}$$

   such that f_{n+m} is expressible as a rational function of the m lower derivatives and is therefore a rational number.

   The series converges when the p-adic norm of x satisfies $|x|_p \leq p^k$ for some k. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of e are possible. Any polynomial has n roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of e and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

5.4 What is the correspondence between p-adic and real numbers?

There must be some kind of correspondence between reals and p-adic numbers. This correspondence can depend on context. In p-adic mass calculations one must map p-adic mass squared values to real numbers in a continuous manner and canonical identification is a natural guess. Presumably also p-adic probabilities should be mapped to their real counterparts. One can wonder whether p-adic valued S-matrix has any physical meaning or whether one should assume that the elements of S-matrix are algebraic numbers allowing interpretation as real or p-adic numbers: this would pose extremely strong constraints on S-matrix. If one wants to introduce p-adic physics at space-time level one must be able to relate p-adic and real space-time regions to each other and the identification along common rational points of real and various p-adic variants of the imbedding space suggests itself here.

5.4.1 Generalization of the number concept

The recent view about the unification of real and p-adic physics is based on the generalization of number concept obtained by fusing together real and p-adic number fields along common rationals (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also).

Rational numbers as numbers common to all number fields

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers appearing in the extension of p-adic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional algebraic and perhaps even transcendental extensions of p-adic numbers adds additional pages to this "Big Book".

This leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common points. The outcome of experimentation is that this generalization makes sense under very high symmetries and that it is safest to lean strongly on the physical picture provided by quantum TGD.

1. The most natural guess is that the coordinates of common points are rational or in some algebraic extension of rational numbers. General coordinate invariance and preservation of symmetries require preferred coordinates existing when the manifold has maximal number of isometries. This approach is especially natural in the case of linear spaces- in particular Minkowski space M^4. The natural coordinates are in this case linear Minkowski coordinates. The choice of coordinates is not completely unique and has interpretation as a geometric correlate for the choice of quantization axes for a given CD.

2. As will be found, the need to have a well-defined integration based on Fourier analysis (or its generalization to harmonic analysis in symmetric spaces) poses very strong constraints and allows p-adicization only if the space has maximal symmetries. Fourier analysis requires the introduction of an algebraic extension of p-adic numbers containing sufficiently many roots of unity.

   (a) This approach is especially natural in the case of compact symmetric spaces such as CP_2.

   (b) Also symmetric spaces such the 3-D proper time $a = constant$ hyperboloid of M^4- call it $H(a)$ -allowing Lorentz group as isometries allows a p-adic variant utilizing the hyperbolic counterparts for the roots of unity. $M^4 \times H(a = 2^n a_0)$ appears as a part of the moduli space of CDs.

   (c) For light-cone boundaries associated with CDs $SO(3)$ invariant radial coordinate r_M defining the radius of sphere S^2 defines the hyperbolic coordinate and angle coordinates of S^2 would correspond to phase angles and $M^4_\pm$ projections for the common points of real and p-adic variants of partonic 2-surfaces would be this kind of points. Same applies to CP_2 projections. In the "intersection of real and p-adic worlds" real and p-adic partonic 2-surfaces would obey same algebraic equations and would be obtained by an algebraic continuation from the corresponding equations making sense in the discrete variant of $M^4_\pm \times CP_2$. This connection with discrete sub-manifolds geometries means very powerful constraints on the partonic 2-surfaces in the intersection.

3. The common algebraic points of real and p-adic variant of the manifold form a discrete space but one could identify the p-adic counterpart of the real discretization intervals $(0, 2\pi/N)$ for angle like variables as p-adic numbers of norm smaller than 1 using canonical identification or some variant of it. Same applies to the the hyperbolic counterpart of this interval. The non-uniqueness of this map could be interpreted in terms of a finite measurement resolution. In particular, the condition that WCW allows Kähler geometry requires a decomposition to a union of symmetric spaces so that there are good hopes that p-adic counterpart is analogous to that assigned to CP_2.

The idea about astrophysical size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves astrophysical length scales.

Generalizing complex analysis by replacing complex numbers by generalized numbers

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions for which polynomials have rational coefficients are obviously functions satisfying this constraint. Algebraic functions for which polynomials have rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed.

For instance, one can ask whether residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the back of the book like structure (in very metaphorical sense) having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense. Contrary to the first expectations the algebraically continued residue calculus does not seem to have obvious applications in quantum TGD.

5.4.2 Canonical identification

Canonical There exists a natural continuous map $Id : R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0,1,..,p-1\} \ . \end{aligned} \tag{5.4.1}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{5.4.2}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^k \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \tag{5.4.3}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite number of pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite number of pinary digits. The finite number of pinary digits expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

Canonical identification is a continuous map of non-negative reals to p-adics

The topology induced by the inverse of the canonical identification map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. 5.4.2) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. This allows two alternative interpretations. Either p-adic image of a physical systems provides a good representation of the system above some pinary cutoff or the physical system can be genuinely p-adic below certain length scale L_p and become in good approximation real, when a length scale resolution L_p is used in its description. The first interpretation is correct if canonical identification is interpreted as a cognitive map. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

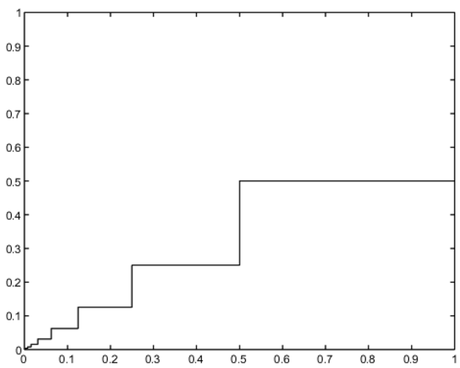

Figure 5.1: The real norm induced by canonical identification from 2-adic norm.

If one considers seriously the application of canonical identification to basic quantum TGD one cannot avoid the question about the p-adic counterparts of the negative real numbers. There is no satisfactory manner to circumvent the fact that canonical images of p-adic numbers are naturally non-negative. This is not a problem if canonical identification applies only to the coordinate interval $(0, 2\pi/N)$ or its hyperbolic variant defining the finite measurement resolution. That p-adicization program works only for highly symmetric spaces is not a problem from the point of view of TGD.

The interpretation of canonical identification in terms of finite measurement resolution

The question what the canonical identification really means could be a key to the understanding of the special aspects of this map. The notion of finite measurement resolution is a good candidate for the needed principle.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong

in the modelling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

The notion of p-adic linearity

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

Does canonical identification define a generalized norm?

Canonical correspondence is quite essential in TGD applications. Canonical identification makes it possible to define a p-adic valued definite integral. Canonical identification is in a key role in the successful predictions of the elementary particle masses. Canonical identification makes also possible to understand the connection between p-adic and real probabilities. These and many other successful applications suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{5.4.4}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{5.4.5}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{5.4.6}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some nonlinear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

5.4.3 The interpretation of canonical identification

During the development of p-adic TGD two seemingly mutually inconsistent competing identifications of reals and p-adics have caused a lot of painful tension. Canonical identification provides one possible identification map respecting continuity whereas the identification of rationals as points common to p-adics and reals respects algebra of rationals. The resolution of the tension came from the realization that canonical identification naturally maps the predictions of p-adic probability theory and thermodynamics to real numbers. Canonical identification also maps p-adic cognitive representations to symbolic ones in the real world world or vice versa. The identification by common rationals is in turn the correspondence implied by the generalized notion of number and natural in the construction of quantum TGD proper.

Canonical identification maps the predictions of the p-adic probability calculus and statistical physics to real numbers

p-Adic mass calculations based on p-adic thermodynamics were the first and rather successful application of the p-adic physics (see the four chapters in [K46] . The essential element of the approach was the replacement of the Boltzmann weight $e^{-E/T}$ with its p-adic generalization p^{L_0/T_p}, where L_0 is the Virasoro generator corresponding to scaling and representing essentially mass squared operator instead of energy. T_p is inverse integer valued p-adic temperature. The predicted mass squared averages were mapped to real numbers by canonical identification.

One could also construct a real variant of this approach by considering instead of the ordinary Boltzmann weights the weights p^{-L_0/T_p}. The quantization of temperature to $T_p = log(p)/n$ would be a completely ad hoc assumption. In the case of real thermodynamics all particles are predicted to be light whereas in case of p-adic thermodynamics particle is light only if the ratio for the degeneracy of the lowest massive state to the degeneracy of the ground state is integer. Immense number of particles disappear from the spectrum of light particles by this criterion. For light particles the predictions are same as of p-adic thermodynamics in the lowest non-trivial order but in the next order deviations are possible.

Also p-adic probabilities and the p-adic entropy can be mapped to real numbers by canonical identification. The general idea is that a faithful enough cognitive representation of the real physics can by the number theoretical constraints involved make predictions, which would be extremely difficult to deduce from real physics.

The variant of canonical identification commuting with division of integers

The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity.

A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p\rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [K43] . In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

Generalized numbers would be regarded in this picture as a generalized manifold obtained by gluing different number fields together along rationals. Instead of a direct identification of real and p-adic rationals, the p-adic rationals in R_p are mapped to real rationals (or vice versa) using a variant of the canonical identification $I_{R\rightarrow R_p}$ in which the expansion of rational number $q = r/s = \sum r_n p^n / \sum s_n p^n$ is replaced with the rational number $q_1 = r_1/s_1 = \sum r_n p^{-n} / \sum s_n p^{-n}$ interpreted as a p-adic number:

$$q = \frac{r}{s} = \frac{\sum_n r_n p^n}{\sum_m s_n p^n} \rightarrow q_1 = \frac{\sum_n r_n p^{-n}}{\sum_m s_n p^{-n}} \quad . \qquad (5.4.7)$$

R_{p1} and R_{p_2} are glued together along common rationals by an the composite map $I_{R \rightarrow R_{p_2}} I_{R_{p_1} \rightarrow R}$.

This variant of canonical identification seems to be excellent candidate for mapping the predictions of p-adic mass calculations to real numbers and also for relating p-adic and real scattering amplitudes to each other [K43] .

p-Adic fractality, canonical identification, and symmetries

The original motivation for the canonical identification and its variants- in particular the variant mapping real rationals with the defining integers below a pinary cutoff to p-adic rationals- was that it defines a continuous map from p-adics to reals and produces beautiful p-adic fractals as a map from reals to p-adics by canonical identification followed by a p-adically smooth map in turn followed by the inverse of the canonical identification.

The first drawback was that the map does not commute with symmetries. Second drawback was that the standard canonical identification from reals to p-adics with finite pinary cutoff is two-valued for finite integers. The canonical real images of these transcendentals are also transcendentals. These are however countable whereas p-adic algebraics and transcendentals having by definition a non-periodic pinary expansion are uncountable. Therefore the map from reals to p-adics is single valued for almost all p-adic numbers.

On the other hand, p-adic rationals form a dense set of p-adic numbers and define "almost all" for the purposes of numerics! Which argument is heavier? The direct identification of reals and p-adics via common rationals commutes with symmetries in an approximation defined by the pinary cutoff an is used in the canonical identification with pinary cutoff mapping rationals to rationals.

Symmetries are of extreme importance in physics. Is it possible to imagine the action of say Poincare transformations commuting with the canonical identification in the sets of p-adic and real transcendentals? This might be the case.

1. Wick rotation is routinely used in quantum field theory to define Minkowskian momentum integrals. One Wick rotates Minkowski space to Euclidian space, performs the integrals, and returns to Minkowskian regime by using the inverse of Wick rotation. The generalization to the p-adic context is highly suggestive. One could map the real Minkowski space to its p-adic counterpart, perform Poincare transformation there, and return back to the real Minkowski space using the inverse of the rational canonical identification.

2. For p-adic transcendentals one would a formal automorph of Poincare group as IPI^{-1} and these Poincare group would be the fractal counterpart of the ordinary Poincare group. Mathematician would regard I as the analog of intertwining operator, which is linear map between Hilbert spaces. This variant of Poincare symmetry would be exact in the transcendental realm since canonical identification is continuous. For rationals this symmetry would fail.

3. For rationals which are constructed as ratios of small enough integers, the rational Poincare symmetry with group elements involving rationals constructed from small enough integers would be an exact symmetry. For both options the use of preferred coordinates, most naturally linear Minkowski coordinates would be essential since canonical identification does not commute with general coordinate transformations.

4. Which of these Poincare symmetries corresponds to the physical Poincare symmetry? The above argument does not make it easy to answer the question. One can however circumvent it. Maybe one could distinguish between rational and transcendental regime in the sense that Poincare group and other symmetries would be realized in different manner in these regimes?

Note that the analog of Wick rotation could be used also to define p-adic integrals by mapping the p-adic integration region to real one by some variant of canonical identification continuously, performing the integral in the real context, and mapping the outcome of the integral to p-adic number by canonical identification. Again preferred coordinates are essential and in TGD framework such coordinates are provided by symmetries. This would allow a numerical treatment of the p-adic integral but the map of the resulting rational to p-adic number would be two valued. The difference between the images would be determined by the numerical accuracy when p-adic expansions are used. This method would be a numerical analog of the analytic definition of p-adic integrals by analytic continuation from the intersection of real and p-adic worlds defined by rational values of parameters appearing in the expressions of integrals.

5.5 p-Adic differential and integral calculus

p-Adic differential calculus differs from its real counterpart in that piecewise constant functions depending on a finite number of pinary digits have vanishing derivative. This property implies p-adic nondeterminism, which has natural interpretation as making possible imagination if one identifies p-adic regions of space-time as cognitive regions of space-time.

One of the stumbling blocks in the attempts to construct p-adic physics have been the difficulties involved with the definition of the p-adic version of a definite integral. There are several alternative options as how to define p-adic definite integral and it is quite possible that there is simply not a single correct version since p-adic physics itself is a cognitive model.

1. The first definition of the p-adic integration is based on three ideas. The ordering for the limits of integration is defined using canonical correspondence. $x < y$ holds true if $x_R < y_R$ holds true. The integral functions can be defined for Taylor series expansion by defining indefinite integral as the inverse of the differentiation. If p-adic pseudo constants are present in the integrand one must divide the integration range into pieces such that p-adic integration constant changes its value in the points where new piece begins.

2. Second definition is based on p-adic Fourier analysis based on the use of p-adic plane waves constructed in terms of Pythagorean phases. This definition is especially attractive in the definition of p-adic QFT limit and is discussed in detail later in the section 'p-Adic Fourier analysis'. In this case the integral is defined in the set of rationals and the ordering of the limits of integral is therefore not a problem.

3. For p-adic functions which are direct canonical images of real functions, p-adic integral can be defined also as a limit of Riemann sum and this in principle makes the numerical evaluation of p-adic integrals possible. As found in the chapter 'Mathematical Ideas', Riemann sum representation leads to an educated guess for an *exact formula for the definite integral* holding true for functions which are p-adic counterparts of real-continuous functions and for p-adically analytic functions. The formula provides a calculational recipe of p-adic integrals, which converges extremely rapidly in powers of p. Ultrametricity guarantees the absence of divergences in arbitrary dimensions provided that integrand is a bounded function. It however seems that this definition of integral cannot hold true for the p-adically differentiable function whose real images are not continuous.

5.5.1 p-Adic differential calculus

The rules of the p-adic differential calculus are formally identical to those of the ordinary differential calculus and generalize in a trivial manner for the algebraic extensions.

The class of the functions having vanishing p-adic derivatives is larger than in the real case: any function depending on a finite number of positive pinary digits of p-adic number and of arbitrary number of negative pinary digits has a vanishing p-adic derivative. This becomes obvious, when one notices that the p-adic derivative must be calculated by comparing the values of the function at nearby points having the same p-adic norm (here is the crucial difference with respect to real case!). Hence, when the increment of the p-adic coordinate becomes sufficiently small, p-adic constant doesn't detect the variation of x since it depends on finite number of positive p-adic pinary digits only. p-Adic constants correspond to real functions, which are constant below some length scale $\Delta x = 2^{-n}$. As a consequence p-adic differential equations are non-deterministic: integration constants are arbitrary functions depending on a finite number of the positive p-adic pinary digits. This feature is central as far applications are considered and leads to the interpretation of p-adic physics as physics of cognition which involves imagination in essential manner. The classical non-determinism of the Kähler action, which is the key feature of quantum TGD, corresponds in a natural manner to the non-determinism of volition in macroscopic length scales.

p-analytic maps $g : R_p \to R_p$ satisfy the usual criterion of differentiability and are representable as power series

$$g(x) = \sum_k g_k x^k \ . \tag{5.5.1}$$

Also negative powers are in principle allowed.

5.5.2 p-Adic fractals

p-Adically analytic functions induce maps $R_+ \rightarrow R_+$ via the canonical identification map. The simplest manner to get some grasp on their properties is to plot graphs of some simple functions (see Fig. 5.5.2 for the graph of p-adic x^2 and Fig. 5.5.2 for the graph of p-adic $1/x$). These functions have quite characteristic features resulting from the special properties of the p-adic topology. These features should be universal characteristics of cognitive representations and should allow to deduce the value of the p-adic prime p associated with a given cognitive system.

1. p-Analytic functions are continuous and differentiable from right: this peculiar asymmetry is a completely general signature of the p-adicity. As far as time dependence is considered, the interpretation of this property as a mathematical counterpart of the irreversibility looks attractive. This suggests that the transition from the reversible microscopic dynamics to irreversible macroscopic dynamics could correspond to the transition from the ordinary topology to an effective p-adic topology.

2. There are large discontinuities associated with the points $x = p^n$. This implies characteristic threshold phenomena. Consider a system whose output $f(n)$ is a function of input, which is integer n. For $n < p$ nothing peculiar happens but for $n = p$ the real counterpart of the output becomes very small for large values of p. In the bio-systems threshold phenomena are typical and p-adicity might be the key to their understanding. The discontinuities associated with the powers of $p = 2$ are indeed encountered in many physical situations. Auditory experience has the property that a given frequency ω_0 and its multiples $2^k\omega_0$, octaves, are experienced as the same frequency, this suggests that the auditory response function for a given frequency ω_0 is a 2-adicallly analytic function. Titius-Bode law states that the mutual distances of planets come in powers of 2, when suitable unit of distance is used. In turbulent systems period doubling spectrum has peaks at frequencies $\omega = 2^k\omega_0$.

3. A second signature of the p-adicity is "p-plicity" appearing in the graph of simple p-analytic functions. As an example, consider the graph of the p-adic x^2 demonstrating clearly the decomposition into p steps at each interval $[p^k, p^{k+1})$.

4. The graphs of the p-analytic functions are in general ordered fractals as the examples demonstrate. For example, power functions x^n are self-similar (the values of the function at some any interval (p^k, p^{k+1}) determines the function completely) and in general p-adic x^n with non-negative (negative) n is smaller (larger) than real x^n expect at points $x = p^n$ as the graphs of p-adic x^2 and $1/x$ show (see Fig. 5.5.2 and 5.5.2) These properties are easily understood from the properties of the p-adic multiplication. Therefore the first guess for the behavior of a p-adically analytic function is obtained by replacing x and the coefficients g_k with their p-adic norms: at points $x = p^n$ this approximation is exact if the coefficients of the power series are powers of p. This step function approximation is rather reasonable for simple functions such as x^n as the figures demonstrate. Since p-adically analytic function can be approximated with $f(x) \sim f(x_0) + b(x - x_0)^n$ or as $a(x - x_0)^n$ (allowing non-analyticity at x_0) around any point the fractal associated with p-adically analytic function has universal geometrical form in sufficiently small length scales.

p-Adic analyticity is well defined for the algebraic extensions of R_p, too. The figures 5.5.2 and 5.5.2 visualize the behavior of the real and imaginary parts of the 2-adic z^2 function as a function of the real x and y coordinates in the parallelpiped I^2,$I = [1 + 2^{-7}, 2 - 2^{-7}]$. An interesting possibility is that the order parameters describing various phases of some physical systems are p-adically differentiable functions. The p-analyticity would therefore provide a means for coding the information about ordered fractal structures.

The order parameter could be one coordinate component of a p-adically analytic map $R^n \to R^n$, $n = 3, 4$. This is analogous to the possibility to regard the solution of the Laplace equation in two dimensions as a real or imaginary part of an analytic function. A given region V of the order parameter space corresponds to a given phase and the volume of the ordinary space occupied by this phase corresponds to the inverse image $g^{-1}(V)$ of V. Very beautiful images are obtained if the order parameter is the the real or imaginary part of a p-analytic function $f(z)$. A good example is p-adic z^2 function in the parallelpiped $[a, b] \times [a, b]$, $a = 1 + 2^{-9}$, $b = 2 - 2^9$ of C-plane. The value range of the order parameter can be divided into, say, 16 intervals of the same length so that each interval corresponds to a unique color. The resulting fractals possess features, which probably generalize to higher-dimensional extensions.

1. The inverse image is an ordered fractal and possesses lattice/cell like structure, with the sizes of cells appearing in powers of p. Cells are however not identical in analogy with the differentiation of the biological cells.

2. p-Analyticity implies the existence of a local vector valued order parameter given by the p-analytic derivative of $g(z)$: the geometric structure of the phase portrait indeed exhibits the local orientation clearly.

A second representation of the fractals is obtained by dividing the value range of z into a finite number of intervals and associating different color to each interval. In a given resolution this representation makes obvious the presence of 0, 1- and 2-dimensional structures not obvious from the graph representation used in the figures of this book.

These observations suggests that p-analyticity might provide a means to code the information about ordered fractal structures in the spatial behavior of order parameters (such as enzyme concentrations in bio-systems). An elegant manner to achieve this is to use purely real algebraic extension for 3-space coordinates and for the order parameter: the image of the order parameter $\Phi = \phi_1 + \phi_2\theta + \phi_3\theta^2$ under the canonical identification is real and positive number automatically and might be regarded as concentration type quantity.

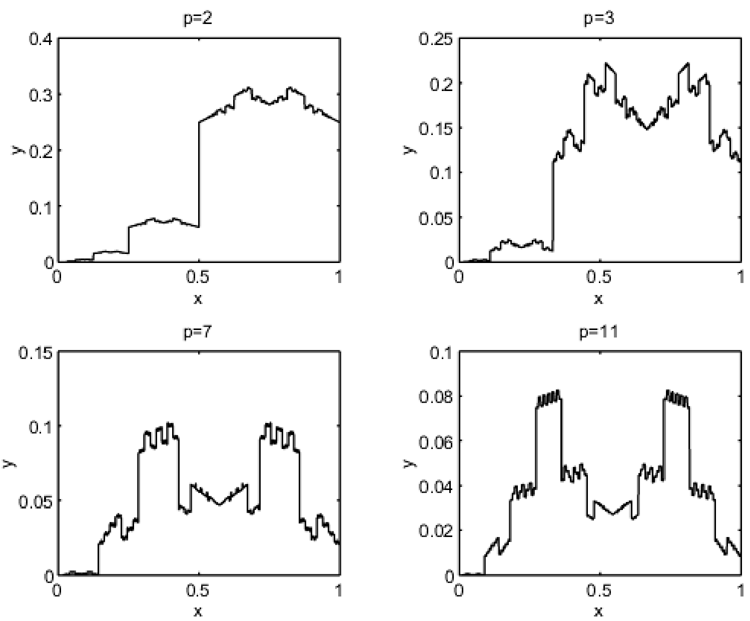

Figure 5.2: p-Adic x^2 function for some values of p

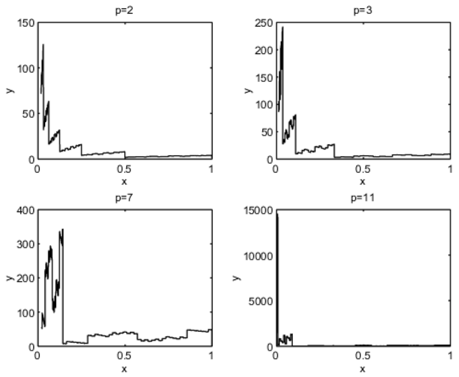

Figure 5.3: p-Adic $1/x$ function for some values of p

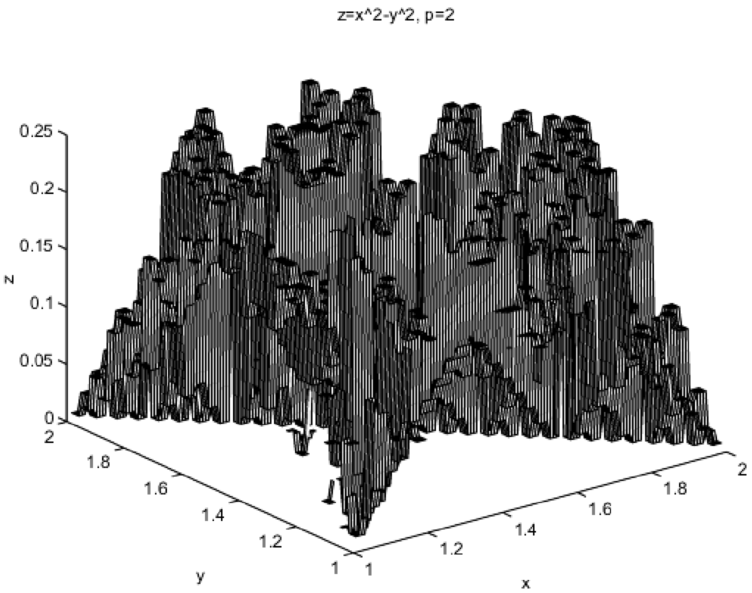

Figure 5.4: The graph of the real part of 2-adically analytic $z^2 =$ function.

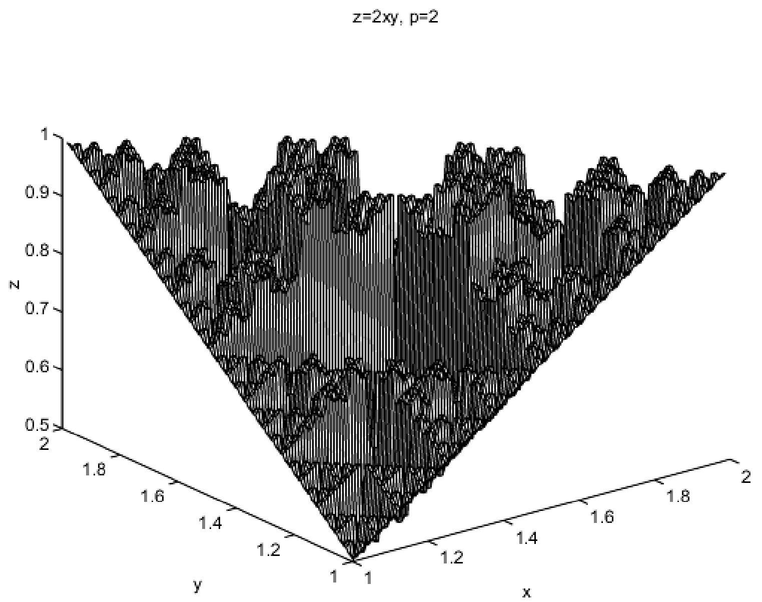

Figure 5.5: The graph of 2-adically analytic $Im(z^2) = 2xy$ function.

5.5.3 p-Adic integral calculus

The basic problems of the integration with p-adic values of integral are caused by the facts that p-adic numbers are not well-ordered and by the properties of p-adic norm. The general idea that p-adic physics can mimic real physics only at the algebraic level, leads to the idea that p-adic integration could be algebraized whereas numerical approaches analogous to Riemann sum are not possible. In the following three examples are discussed.

1. Definite integral can be defined using integral function and by defining integration limits via canonical identification: the drawback is the loss of general coordinate invariance. A more elegant general coordinate invariant approach is based on the identification of rationals as common to both reals and p-adics. This works for rational valued integration limits.

2. residue calculus allows to realize integrals of analytic functions over closed curves of complex plane. The generalization of the residue calculus makes possible to realize conformal invariance at elementary particle horizons which are metrically 2-dimensional and allow conformal invariance and has also p-adic counterpart.

3. The perturbative series using Gaussian integration is the only to perform in practice infinite-dimensional functional integrals and being purely algebraic procedure, allows a straightforward p-adic generalization. This is the only option for p-adicizing configuration space integral.

Definition of the definite integral using integral function concept and canonical identification or identification by common rationals

The concept of the p-adic definite integral can be defined for functions $R_p \to C$ [A123] using translationally invariant Haar measure for R_p. In present context one is however interested in defining a p-adic valued definite integral for functions $f : R_p \to R_p$: target and source spaces could of course be also some some algebraic extensions of the p-adic numbers.

What makes the definition nontrivial is that the ordinary definition as the limit of a Riemann sum doesn't seem to work: it seems that Riemann sum approaches to zero in the p-adic topology since, by ultra-metricity, the p-adic norm of a sum is never larger than the maximum p-adic norm for the summands. The second difficulty is related to the absence of a well-ordering for the p-adic numbers. The problems might be avoided by defining the integration essentially as the inverse of the differentiation and using the canonical correspondence to define ordering for the p-adic numbers. More generally, the concepts of the form, cohomology and homology are crucially based on the concept of the boundary. The concept of boundary reduces to the concept of an ordered interval and canonical identification makes it indeed possible to define this concept.

The definition of the p-adic integral functions defining integration as inverse of the differentiation is straightforward and one obtains just the generalization of the standard calculus. For instance, one has $\int z^n = \frac{z^{n+1}}{(n+1)} + C$ and integral of the Taylor series is obtained by generalizing this. One must however notice that the concept of integration constant generalizes: any function $R_p \to R_p$ depending on a finite number of pinary digits only, has a vanishing derivative.

Consider next the definite integral. The absence of the well ordering implies that the concept of the integration range (a, b) is not well defined as a purely p-adic concept. As already mentioned there are two solutions of the problem.

1. The identification of rational numbers as common to both reals and p-adics allows to order the integration limits when the end points of the integral are rational numbers. This is perhaps the most elegant solution of the problem since it is consistent with the restricted general coordinate invariance allowing rational function based coordinate changes. This approach works for rational functions with rational coefficients and more general functions if algebraic extension or extension containing transcendentals like e and logarithms of primes are allowed. The extension containing e, π, and $log(p)$ is finite-dimensional if e/π and $\pi/log(p)$ are rational numbers for all primes p. Essentially algebraic continuation of real integral to p-adic context is in question.

2. An alternative resolution of the problem is based on the canonical identification. Consider p-adic numbers a and b. It is natural to define a to be smaller than b if the canonical images of a and b satisfy $a_R < b_R$. One must notice that $a_R = b_R$ does not imply $a = b$, since the inverse of the canonical identification map is two-valued for the real numbers having a finite number of pinary digits. For two p-adic numbers a, b with $a < b$, one can define the integration range (a, b) as the set of the p-adic numbers x satisfying $a \leq x \leq b$ or equivalently $a_R \leq x_R \leq b_R$. For a given value of x_R with a finite number of pinary digits, one has two values of x and x can be made unique by requiring it to have a finite number of pinary digits.

One can define definite integral $\int_a^b f(x)dx$ formally as

$$\int_a^b f(x)dx = F(b) - F(a) \ , \tag{5.5.2}$$

where $F(x)$ is integral function obtained by allowing only ordinary integration constants and $b_R > a_R$ holds true. One encounters however a problem, when $a_R = b_R$ and a and b are different. Problem is avoided if the integration limits are assumed to correspond to p-adic numbers with a finite number of pinary digits.

One could perhaps relate the possibility of the p-adic integration constants depending on finite number of pinary digits to the possibility to decompose integration range $[a_R, b_R]$ as $a = x_0 < x_1 < ....x_n = b$ and to select in each subrange $[x_k, x_{k+1}]$ the inverse images of $x_k \leq x \leq x_{k+1}$, with x having finite number of pinary digits in two different manners. These different choices correspond to different integration paths and the value of the integral for different paths could correspond to the different choices of the p-adic integration constant in integral function. The difference between a given integration path and 'standard' path is simply the sum of differences $F(x_k) - F(y_k)$, $(x_k)_R = (y_k)_R$.

This definition has several nice features.

1. The definition generalizes in an obvious manner to the higher dimensional case. The standard connection between integral function and definite integral holds true and in the higher-dimensional case the integral of a total divergence reduces to integral over the boundaries of the integration volume. This property guarantees that p-adic action principle leads to same field equations as its real counterpart. It this in fact this property, which drops other alternatives from the consideration.

2. The basic results of the real integral calculus generalize as such to the p-adic case. For instance, integral is a linear operation and additive as a set function.

The ugly feature is the loss of the general coordinate invariance due to the fact that canonical identification does not commute with coordinate changes (except scalings by powers of p) and it seems that one cannot use canonical identification at the fundamental level to define definite integrals.

Definite integrals in p-adic complex plane using residue calculus

residue calculus allows to calculate the integrals $\oint_C f(z)dz$ around complex curves as sums over poles of the function inside the curve:

$$\oint f(z)dz = i2\pi \sum_k Res(f(z_k)) \ , \tag{5.5.3}$$

where $Res(f(z_k))$ at pole $z = z_k$ is defined as $Res(f(z_k)) = \lim_{z \rightarrow z_k}(z - z_k)f(z)$. This definition applies in case of 2-dimensional $\sqrt{-1}$-containing algebraic extension of p-adic numbers ($p \ mod \ 4 = 3$) but its seems that this is not relevant for quantum TGD.

Quaternion conformal invariance corresponds to the conformal invariance associated with topologically 3-dimensional elementary particle horizons surrounding wormhole contacts which have

Euclidian signature of induced metric. The induced metric is degenerate at the elementary particle horizon so that these surfaces are metrically two-dimensional. This implies a generalization of conformal invariance analogous to that at light cone cone boundary. In particular, a subfield of quaternions isomorphic with complex numbers is selected. One expects that residue calculus generalizes.

Elementary particle horizons are defined by a purely algebraic condition stating that the determinant of the induced metric vanishes, and thus the notion makes sense for p-adic space-time sheets too. Also residue calculus should make sense for all algebraic extensions of p-adic numbers and the algebra of quaternion conformal invariance would generalize to the p-adic context too. Note however that the notion of p-adic quaternions does not make sense: the reason is that p-adic Euclidian length squared for a non-vanishing p-adic quaternion can vanish so that the inverse of quaternion is not well defined always. In the set of rational numbers this failure does not however occur and this might be enough for p-adicization to work.

Definite integrals using Gaussian perturbation theory

In quantum field theories functional integrals are defined by Gaussian perturbation theory. For real infinite-dimensional Gaussians the procedure has a rigorous mathematical basis deriving from measure theory. For the imaginary infinite-dimensional Gaussians defining the Feynman path integrals of quantum field theory the rigorous mathematical justification is lacking.

In TGD framework the integral over WCW of three surface can be reduced to a real Gaussian perturbation theory around the maxima of Kähler function. The integration is over quantum fluctuating degrees of freedom defining infinite-dimensional symmetric space for given values of zero modes. According to the more detailed arguments about how to construct p-adic counterpart of real WCW physics described in the chapter "Construction of Quantum Theory", the following conjectures are trued.

1. The symmetric space property implies that there is only one maximum of Kähler function for given values of zero modes.

2. The generalization of Duistermaat-Heecke theorem holding true in finite-dimensional case suggests that by symmetric space property the integral of the exponent of Kähler gives just the exponent of Kähler function at the maximum and Gaussian determinant and metric determinant cancel each other.

3. The fact that free Gaussian field theory corresponds to a flat symmetric space inspires the hypothesis that S-matrix elements involving WCW spinor fields in the representations of the isometry group reduce to those given by free field theory with propagator defined by the inverse of WCW covariant Kähler metric evaluated in the tangent space basis defined by the isometry currents at the maximum of Kähler function. This implies that there is no perturbation series which would spoil any hopes about proving the rationality. The reduction to a free field theory does not make quantum TGD non-interacting since interactions are described as topologically (as decays and fusions of 3-surfaces) rather than algbraically as non-linearities of local action.

4. If the exponent function is a rational function with rational coefficients in the sense that for the points of WCW having finite number of rational valued coordinates (also zero modes), then the exponent $e^{K_{max}}$ is a rational number for rational values of zero modes. From the rationality of the exponent of the Kähler function follows the rational valuedness of the matrix elements of the metric. The undeniably very optimistic conclusion is that for rational values of the zero modes the S-matrix elements would be rational valued or have values if finite extension of rationals, so that they could be continued to the p-adic sectors of WCW. The S-matrix would have the same form in all number fields.

5. One could also interpret the outcome as an algebraic continuation of the rational quantum physics to real and p-adic physics. WCW-integrals can be thought of as being performed in the rational WCW. Of course, one can define also ordinary integrals over R^n numerically using Riemann sums by considering the division of the integration region to very small n-cubes for which the sides have rational-number valued lengths and such that the value of the function is taken at rational valued point inside each cube.

The finite-dimensional real one-dimensional Gaussian $exp(-ax^2/2)$ provides a natural testing ground for this rather speculative picture. The integral of the Gaussian is $(2\pi)^{1/2}/\sqrt{a}$: in n-dimensional case where a is replaced by a quadratic form defined by a matrix A one obtains $(2\pi)^{n/2}/\sqrt{det(A)}$ in n-dimensional case. The integral of a function $exp(-ax^2+kx^n)x^k$ reduces to a perturbation series as sum of graphs containing single vertex containing k lines and arbitrary number of vertices containing n lines and endowed with a factor k, and assigning with the lines the propagator factor $1/a$. For n-dimensional case the propagator factor would be inverse of the matrix A.

The result makes sense in the p-adic context if a and k are rational numbers. In the n-dimensional case matrix A and the coefficients defining the polynomial defining the interaction term must be rational numbers. The only problematic factor is the power of 2π, which seems to require algebraic extension containing π. Of course, one could define the normalization of the functional integral by dividing it by $(2\pi)^{n/2}$ to get rid of this fact. In the definition of S-matrix elements this normalization factor always disappears so that this problem has no physical significance.

In the case of free scalar quantum field theory n-point functions the perturbation theory are simply products of 2-point functions defined by the inverse of the infinite-dimensional Gaussian matrix. For plane wave basis for scalar field labelled by 4-momentum k the inverse of the Gaussian matrix reduces to the propagator ($i/(k^2+i\epsilon)$ for scalar field), which is rational function of the square of 4-momentum vector. In case of interacting quantum field the infinite summation over graphs spoils the hopes of obtaining end result which could be proven to be rational valued for rational values of incoming and outgoing four-momenta. The loop integrals are source of divergence problems and also number-theoretically problematic.

5.6 p-Adic symmetries and Fourier analysis

5.6.1 p-Adic symmetries and generalization of the notion of group

The most basic questions physicist can ask about the p-adic numbers are related to symmetries. It seems obvious that the concept of a Lie-group generalizes: nothing prevents from replacing the real or complex representation spaces associated with the definitions of the classical Lie-groups with the linear space associated with some algebraic extension of the p-adic numbers: the defining algebraic conditions, such as unitarity or orthogonality properties, make sense for the algebraically extended p-adic numbers, too.

For orthogonal groups one must replace the ordinary real inner product with the inner product $\sum_k X_k^2$ with a Cartesian power of a purely real extension of p-adic numbers. In the unitary case one must consider the complexification of a Cartesian power of a purely real extension with the inner product $\sum \bar{Z}_k Z_k$. Here $p\ mod\ 4=3$ is required. It should be emphasized however that the p-adic inner product differs from the ordinary one so that the action of, say, p-adic counterpart of a rotation group in R_p^3 induces in R^3 an action, which need not have much to do with ordinary rotations so that the generalization is physically highly nontrivial. Extensions of p-adic numbers also mean extreme richness of structure.

The exponentiation $t \rightarrow exp(tJ)$ of the Lie-algebra element J is a central element of Lie group theory and allows to coordinatize that elements of Lie group by mapping tangent space points the points representing group elements. Without algebraic extensions involving e or its roots one can exponentiate only the group parameters t satisfying $|t|_p<1$. Thus the values of the exponentiation parameter which are too small/large in real/p-adic sense are not possible and one can say that the standard p-adic Lie algebra is a ball with radius $|t|_p=1/p$.

The study of ordinary one-dimensional translations gives an idea about what it is involved. For finite values of the p-adic integer t the exponentiated group element corresponds in the case of translation group to a power of e so that the points reached by exponentiation cannot correspond to rational points. Since logarithm function exist as an inverse of p-adic exponent and since rationals correspond to infinite but periodic pinary expansions, rational points having the same p-adic norm can be reached by p-adic exponentials using t which is infinite as ordinary integer. This result is expected to generalize to the case of groups represented using rational-valued matrices.

One can define a hierarchy of p-adic Lie-groups by allowing extensions allowing e and even its roots such that the algebras have p-adic radii p^k. Hence the fact that the powers $e, ..., e^{p-1}$ define a finite-dimensional extensions of p-adic numbers seems to have a deep group theoretical meaning. One can define a hierarchy of increasingly refined extensions by taking the generator of extension to be $e^{1/n}$. For instance, in the case of translation group this makes possible p-adic variant of Fourier analysis by using discrete plane wave basis.

One can generalize also the notion of group by using the generalized notion of number. This means that one starts from the restriction of the group in question to a group acting in say rational and complex rational linear space and requires that real and p-adic groups have rational group transformations as common. By performing various completions one obtains a generalized group having the characteristic book like structure. In this kind of situation the relationship between various groups is clear and also the role of extensions of p-adic numbers can be understood. The notion of Lie-algebra generalizes also to form a book like structure. Coefficients of the pages of the Lie-algebra belong to various number fields and rational valued coefficients correspond to a part partially (because of the restriction $|t|_p < p^k$) common to all Lie-algebras.

$SO(2)$ as example

A simple example is provided by the generalization of the rotation group $SO(2)$. The rows of a rotation matrix are in general n orthonormalized vectors with the property that the components of these vectors have p-adic norm not larger than one. In case of $SO(2)$ this means the the matrix elements $a_{11} = a_{22} = a, a_{12} = -a_{21} = b$ satisfy the conditions

$$\begin{aligned} a^2 + b^2 &= 1 \ , \\ |a|_p &\leq 1 \ , \\ |b|_p &\leq 1 \ . \end{aligned} \tag{5.6.1}$$

One can formally solve a as $a = \sqrt{1-b^2}$ but the solution doesn't exists always. There are various possibilities to define the orthogonal group.

1. One possibility is to allow only those values of a for which square root exists as p-adic number. In case of orthogonal group this requires that both $b = sin(\Phi)$ and $a = cos(\Phi)$ exist as p-adic numbers. If one requires further that a and b make sense also as ordinary rational numbers, they define a Pythagorean triangle (orthogonal triangle with integer sides) and the group becomes discrete and cannot be regarded as a Lie-group. Pythagorean triangles emerge for rational counterpart of any Lie-group.

2. Other possibility is to allow an extension of the p-adic numbers allowing a square root of any ordinary p-adic number. The minimal extensions has dimension 4 (8) for $p > 2$ ($p = 2$). Therefore space-time dimension and imbedding space dimension emerge naturally as minimal dimensions for spaces, where p-adic $SO(2)$ acts 'stably'. The requirement that a and b are real is necessary unless one wants the complexification of $so(2)$ and gives constraints on the values of the group parameters and again Lie-group property is expected to be lost.

3. The Lie-group property is guaranteed if the allowed group elements are expressible as exponents of a Lie-algebra generator Q. $g(t) = exp(iQt)$. This exponents exists only provided the p-adic norm of t is smaller than one. If one uses square root allowing extension, one can require that t satisfies $|t| \leq p^{-n/2}$, $n > 0$ and one obtains a decreasing hierarchy of groups $G_1, G_2, ...$ For the physically interesting values of p (typically of order $p = 2^{127} - 1$) the real counterparts of the transformations of these groups are extremely near to the unit element of the group. These conclusions hold true for any group. An especially interesting example physically is the group of 'small' Lorentz transformations with $t = O(\sqrt{p})$. If the rest energy of the particle is of order $O(\sqrt{p})$: $E_0 = m = m_0\sqrt{p}$ (as it turns out) then the Lorentz boost with velocity $\beta = \beta_0\sqrt{p}$ gives particle with energy $E = m/\sqrt{1-\beta_0^2 p} = m(1 + \frac{\beta_0^2 p}{2} + ..)$ so that $O(p^{1/2})$ term in energy is Lorentz invariant. This suggests that non-relativistic regime corresponds to small Lorentz transformations whereas in genuinely relativistic regime one must include also the discrete group of 'large' Lorentz transformations with rational transformations matrices.

4. One can extend the group to contain products G_1G_2, such that G_1 is a rational matrix belonging to the restriction of the Lie-group to rational matrices not obtainable from a unit matrix p-adically by exponentiation, and G_2 is a group element obtainable from unit element by exponentiation. For instance, rational CP_2 is obtained from the group of rational 3×3 unitary matrices as by dividing it by the $U(2)$ subgroup of rational unitary matrices.

Even the construction of the representations of the translation group raises nontrivial issues since the construction of p-adic Fourier analysis is by no means a nontrivial task. One can however define the concept of p-adic plane wave group theoretically and p-adic plane waves are orthogonal with respect to the inner product defined by the proposed p-adic integral.

The representations of 3-dimensional rotation group $SO(3)$ can be constructed as homogenous functions of Cartesian coordinates of E^3 and in this case the phase factors $exp(im\phi)$ typically appearing in the expressions of spherical harmonics do not pose any problems. The construction of p-adic spherical harmonics is possible if one assumes that allowed spherical angles (θ,ϕ) correspond to Pythagorean triangles.

A similar situation is encountered also in the case of CP_2 spherical harmonics in in fact, quite generally. This number theoretic quantization of angles could be perhaps interpreted as a kind of cognitive quantum effect consistent with the fact that only rationals can be visualized concretely and relate directly to the sensory experience. More generally, the possibility to realize only rationals numerically might reflect the facts that only rationals are common to reals and p-adics and that cognition is basically p-adic.

Fractal structure of the p-adic Poincare group

p-Adic Poincare group, just as any other p-adic Lie group, contains entire fractal hierarchy of subgroups with the same Lie-algebra. For instance, translations $m^k \rightarrow m^k + p^N a^k$, where a^k has p-adic norm not larger than one form subgroup for all values of N. The larger the value of N is, the smaller this subgroup is. Quite generally this implies orbits within orbits and representations within representations like structure so that p-adic symmetry concept contains hologram like aspect. This property of the p-adic symmetries conforms nicely with the interpretation of p-adic symmetries as cognitive representations of real symmetries since the symmetries can be realized in a p-adically finite spatiotemporal volume of the cognitive space-time sheet. Even more, this volume can be p-adically arbitrarily small. If one identifies both p-adics and reals as a completion of rationals, the corresponding real volumes are however strictly speaking infinite in absence of a pinary cutoff.

The hierarchy of subgroups implies that M^4_+ decomposes in a natural manner to 4-cubes with side $L_0 = N_p(L)L_p$, where $N_p(L) = p^{-N}$ denotes the p-adic norm of L such that these 4-cubes are invariant under the group of sufficiently small Poincare transformations. In real context these cubes define a hierarchy of exteriors of cubes with decreasing sizes. One can have full p-adic Poincare invariance in p-adically arbitrarily small volume. Only those Poincare transformations, which leave the minimal p-adic cube invariant are symmetries. Also this picture suggest that the p-adic space-time sheets providing cognitive representations about finite space-time regions by canonical identification can have very large size.

The construction of the p-adic Fourier analysis is a nontrivial problem. The usual exponent functions $f_P(x) = exp(iPx)$, providing a representation of the p-adic translations do not make sense as a Fourier basis: f_P is not a periodic function; f_P does not converge if the norm of Px is not smaller than one and the natural orthogonalization of the different momentum eigenstates does not seem to be possible using the proposed definition of the definite integral.

This state of affairs suggests that p-adic Fourier analysis involves number theory. It turns out that one can construct what might be called number theoretical plane waves and that p-adic momentum space has a natural fractal structure in this case. The basic idea is to reduce p-adic Fourier analysis to a Fourier analysis in a finite field $G(p,1)$ plus fractality in the sense that all p^m-scaled versions of the $G(p,1)$ plane waves are used. This means that p-adic plane waves in a given interval $[n,n+1)p^m$ are piecewise constant plane waves in a finite field $G(p,1)$. Number theoretical p-adic plane waves are pseudo constants so that the construction does not work for p-adically differentiable functions. The pseudo-constancy however turns out to be a highly desirable feature in the construction of the p-adic QFT limit of TGD based on the mapping of the real H-quantum fields to p-adic quantum fields using the canonical identification.

The unsatisfactory feature of this approach is that number theoretic p-adic plane waves do not behave in the desired manner under translations. It would be nice to have a p-adic generalization of the plane wave concept allowing a generalization of the standard Fourier analysis and a direct connection with the theory of the representations of the translation group. A natural idea is to to define exponential function as a solution of a p-adic differential equation representing the action of a translation generator and to introduce multiplicative pseudo constant making possible to define exponential function for all values of its argument. One can develop an argument suggesting that the plane waves obtained in this manner are indeed orthogonal.

Infinitesimal form of translational symmetry might be argued to be too strong requirement since p-adically infinitesimal translations typically correspond to real translations which are arbitrarily large: this is not consistent with the idea that cognitive representations with a finite spatial resolution are in question. This motivates a third approach to the p-adic Fourier analysis. The basic requirement is that discrete subgroup of translations commutes with the map of the real plane waves to their p-adic counterparts. This means that the products of the real phase factors are mapped to the products of the corresponding p-adic phase factors. This is possible if the phase factor is a rational complex number so that the phase angle corresponds to a Pythagorean triangle. The p-adic images of the real plane waves are defined for the momenta $k = nk_G$, $k_G = \phi_G/\Delta x$, where $\phi_G \in [0, 2\pi]$ is a Pythagorean phase angle and where the points $x_n = n\Delta x$ define a discretization of x-space, Δx being a rational number. These plane waves form a complete and orthogonalized set.

5.6.2 p-Adic Fourier analysis: number theoretical approach

Contrary to the original expectations, number theoretical Fourier analysis is probably not basic mathematical tools of p-adic QFT since it fails to provide irreducible representation for the translational symmetries. Despite this it deserves documentation.

Fourier analysis in a finite field $G(p,1)$

The p-adic numbers of unit norm modulo p reduce to a finite field $G(p,1)$ consisting of the integers $0,1,...,p-1$ with arithmetic operations defined by those of the ordinary integers taken modulo p. Since the elements $1,...,p-1$ form a multiplicative group there must exists an element a of $G(p,1)$ (actually several) such that $a^{p-1}=1$ holds true in $G(p,1)$. This kind of element is called primitive root. If n is a factor of $p-1$: $(p-1)=nm$, then also $a^m=1$ holds true. This reflects the fact that Z_{p-1} decomposes into a product $Z_{m_1}^{n_1}Z_{m_2}^{n_2}...Z_{m_s}^{n_s}$ of commuting factors Z_{m_i}, such that $m_i^{n_i}$ divides $p-1$.

A Fourier basis in $G(p,1)$ can be defined using p functions $f_k(n)$, $k=0,..,p-1$. For $k=0,1,...,p-2$ these functions are defined as

$$f_k(n) \quad = \quad a^{nk} \ , \ n=0 \ ,...,p-1 \ , \tag{5.6.2}$$

and satisfy the periodicity property

$$f_k(0)=f_k(p-1) \ .$$

The problem is to identify the lacking p:th function. Since $f_k(n)$ transforms irreducibly under translations $n\rightarrow n+m$ it is natural to require that also the p:th function transforms in a similar manner and satisfies the periodicity property. This is achieved by defining

$$f_{p-1}(n) \quad = \quad (-1)^n \ . \tag{5.6.3}$$

The counterpart of the complex conjugation for f_k for $k\neq p-1$ is defined as $f_k\rightarrow f_{p-1-k}$. f_{p-1} is invariant under the conjugation. The inner product is defined as

$$\langle f_k, f_l\rangle = \sum_{n=0}^{p-2} f_{p-1-k}(n)f_l(n) = \delta(k,l)(p-1) \ . \tag{5.6.4}$$

The dual basis $\hat{f}_k$ clearly differs only by the normalization factor $1/(p-1)$ from the basis f_{p-k}. The counterpart of Fourier expansion for any real function in $G(p,1)$ can be obviously constructed using this function basis and Fourier components are obtained as the inner products of the dual Fourier basis with the function in question.

A natural interpretation for the integer k is as a p-adic momentum since in the translations $n \rightarrow n+m$ the plane wave with $k \neq p-1$ changes by a phase factor a^{km}. For $k = p-1$ it transforms by $(-1)^m$ so that also now an eigen state of finite field translations is in question.

p-Adic Fourier analysis based on p-adic plane waves

The basic idea is to reduce p-adic Fourier analysis to the Fourier analysis in $G(p,1)$ by using fractality.

1. Let the function $f(x)$ be such that the maximum p-adic norm of $f(x)$ is p^{-m}. One can uniquely decompose $f(x)$ to a sum of functions $f_n(x)$ such that $|f_n(x)|_p = p^n$ or vanishes in the entire range of definition for f:

$$\begin{aligned} f(x) &= \sum_{n\geq m} f_n(x) \ , \\ f_n(x) &= g_n(x)p^n \ , \\ |g_n(x)| &= 1 \ \text{for} \ g(x) \neq 0 \ . \end{aligned} \tag{5.6.3}$$

   The higher the value of n, the smaller the contribution of f_n. The expansion converges extremely rapidly for the physically interesting large values of p.

2. Assume that $f(x)$ is such that for each value of n one can find some resolution $p^{m(n)}$ below which $g_n(x)$ is constant in the sense that for all intervals $[r, r+1)p^{m(n)}$ (defined in terms of the canonical identification) the function $f_n(x)$ is constant. For p-adically differentiable functions this cannot be the case since they would be pseudo constants if this were true. In the physical situation CP_2 size provides a natural p-adic cutoff so that only a finite number of f_n:s are needed and the resolution in question corresponds to CP_2 length scale. Hence ordinary plane waves (possibly with a natural UV cutoff) should have an expansion in terms of the p-adic plane waves.

3. The assumption implies that in each interval $(r, r+1)p^{m(n)-1}$, g_n can be regarded as a function in $G(p,1)$ identified as the set $x = (r+sp)p^{m(n)-1}$, $s = 0, 1, ..., p-1$. Hence one can Fourier expand $f_n(x)$ using $G(p,1)$ plane waves f^{ks}. In this manner one obtains a rapidly converging expansion using p-adic plane waves.

Periodicity properties of the number theoretic p-adic plane waves

The periodicity properties of the p-adic plane waves make it possible to associate a definite wavelength with a given p-adic plane wave. For the p-adic momenta k not dividing $p-1$, the wavelength corresponds to the entire range $(n, n+1)p^m$ and its real counterpart is

$$\lambda = p^{-m-1/2}l \ ,$$

where $l \sim 10^4\sqrt{\hbar G}$ is the fundamental p-adic length scale. If k divides $p-1 = \prod_i m_i^{n_i}$, the period is m_i and the real wavelength is

$$\lambda(m_i) = m_i p^{-m-1-1/2}l \ .$$

One might wonder whether this selection of preferred wavelengths has some physical consequences. The first thing to notice is that p-adic plane waves do *not* replace ordinary plane waves in the construction of the p-adic QFT limit of TGD. Rather, ordinary plane waves are expanded using the p-adic plane waves so that the selection of the preferred wavelengths, if it occurs at all, must be a dynamical process. The average value of the prime divisors, and hence the number of different wavelengths for a given value of p, counted with the degeneracy of the divisor is given by [A204]

$$\Omega(n) = ln(ln(n)) + 1.0346 \ ,$$

and is surprisingly small, or order 6 for numbers of order M_{127}! If one can apply probabilistic arguments or [A204] to the numbers of form $p-1$, too then one must conclude that very few wavelengths are possible for general prime p! This in turn means that to each p there are associated only very few characteristic length scales, which are predictable. Furthermore, all the p^k-multiples of these scales are also possible if p-adic fractality holds true in macroscopic length scales.

Mersenne primes M_n can be considered as an illustrative example of the phenomenon. From [A141] one finds that $M_{127}-1$ has 11 distinct prime factors and 3 and 7 occurs three and 2 times respectively. The number of distinct length scales is $3\cdot 2^{11}-1 \sim 2^{12}$. $M_{107}-1$ and $M_{89}-1$ have 7 and 11 singly occurring factors so that the numbers of length scales are $2^7-1 = 127 = M_7$ and $2^{11}-1$. Note that for hadrons (M_{107}) the number of possible wavelengths is especially small: does this have something to do with the collective behavior of color confined quarks and gluons? An interesting possibility is that this length scale generation mechanism works even macroscopically (for p-adic length scale hypothesis at macroscopic length scales see the third part of the book). One cannot exclude the possibility that long wavelength photons, gravitons and neutrinos might therefore provide a completely new mechanism for generating periodic structures with preferred sizes of period.

5.6.3 p-Adic Fourier analysis: group theoretical approach

The problem with the straightforward generalization of the Fourier analysis is that the standard Taylor expansion of the plane wave $exp(ikx)$ converges only provided x has p-adic norm smaller than one and that the p-adic exponential function does not have the periodicity properties of the ordinary exponential function guaranteeing orthogonality of the functions of the Fourier basis. Besides this one must assume $p\ mod\ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as ordinary p-adic number.

The approach based on algebraic extensions allowing trigonometry

In an attempt to construct Fourier analysis the safest approach is to start from the ordinary Fourier analysis at circle or that for a particle in a one-dimensional box. The function basis uses as the basic building blocks the functions $e^{in\phi}$ in the case of circle and functions $e^{in\pi x/L}$ in the case of a particle in a box of side L.

The view about rationals as common to both reals and p-adics, and the possibility of finite-dimensional extensions of p-adics generated by the roots $e^{i2\pi/p^k}$ suggest how to realize this idea.

1. Consider first the case of the circle. Fix some value of N and select a set of points $\phi_n = in2\pi/p^k$ at which the phases are defined meaning p^{k+1}-dimensional algebraic extension. That powers of p appear is consistent with p-adic fractality. If so spin 1/2 *resp.* spin 1 particles would be inherently 2-adic *resp.* 3-adic. The plane wave basis corresponds $exp(ik\phi_n)$, $k = 0, ..., N-1$. In the case of particle in the one-dimensional box such that L corresponds to a rational number, the box is decomposed into N intervals of length L/N.

2. One can assign to the phases a well defined angular momentum as integer $n = 0, ..., N-1$ whereas the momentum spectrum for a particle in a box are given by $n\pi/L$. It is possible to continue the phase factor to the neighborhood of each point by requiring that the differential equation

$$\frac{d}{dx}exp(ikx) = ikexp(ikx)$$

defining the exponential function is satisfied.

3. The inner product of the plane waves $f_{k_1})$ and f_{k_2} can be defined as the sum

$$\langle k_1 \rangle \equiv \sum_n \overline{f}_{k_1}(x_n) f_{k_2}(x_n) \ , \tag{5.6.4}$$

and orthogonality and completeness differ by no means from those of ordinary Fourier analysis.

p-Adic Fourier analysis, Pythagorean phases, and Gaussian primes

An alternative approach is based on Pythagorean phases and discretization in x-space, which might be a natural thing to do if p-adic field theory is taken as a cognitive model rather than 'real' physics. This is also natural because rational Minkowski space is in the algebraic approach the fundamental object and reals and p-adics emerge as its completions.

Rational phase factors are common to the complexified p-adics ($p\ mod\ 4 = 3$) and reals and this suggests that one should define p-adic plane waves so that their values are in the set of the Pythagorean phases. Pythagorean phases are in one-one correspondence with the phases of the squares of Gaussian integers N_G and thus generated as products of squares of Gaussian primes π_G, which are complex integers with modulus squared equal to prime $p\ mod\ 4 = 1$. Thus the set of phases $\phi(\pi_G)$ for the phases for π_G^2 form an algebraically infinite-dimensional linear space in the sense that the phases representable as superpositions

$$2\phi_G = \sum_{\pi_G} n_{\pi_G} 2\phi(\pi_G)$$

of these phases with integer coefficients belong to the set.

Consider now the definition of the plane wave basis based on Pythagorean phases and the identification of the p-adics and reals via common rationals.

1. Let $x_0 = q = m/n$ denote a value of x-coordinate and let k denote some value of momentum. If $exp(ikx_0)$ is a Pythagorean phase then also the multiples nk correspond to Pythagorean phases. k itself cannot be a rational number so that k is not defined as an ordinary p-adic number: this could be seen as a defect of the approach since one cannot speak of a well-defined momentum. Neither can k be a rational multiple of π so that Pythagorean phases have nothing to do with the phases defined by algebraic extensions containing the phase $exp(i\pi/n)$ already discussed.

   For a given value of $x_0 = q$ the momenta k for which $exp(ikq)$ is a Pythagorean phase are in one-one correspondence with Pythagorean phases. Moreover, Pythagorean phases result in the lattice defined by the multiples of the x_0. Thus a natural definition of the p-adic plane waves emerges predicting a maximal momentum spectrum with one-one correspondence with Pythagorean phases, and selecting a preferred lattice of points at the real axis. This definition is also in accordance with the idea that p-adic plane waves are related with a cognitive representation for real physics.

2. Pythagorean phases are in one-one correspondence with the phase factors associated with the squares of the Gaussian integers and generating phases correspond to the phases $\phi(\pi_G)$ associated with the squares of Gaussian primes π_G. The moduli squared for the Gaussian primes correspond to squares of rational primes $p\ mod\ 4 = 1$. Thus set of allowed momenta k_G for given spatial resolution m/n is the set

$$\{k_G(q)\} = \{\tfrac{2\phi_G/}{q} + \tfrac{2\pi n}{q} | n \in Z\} \ ,$$

$$\{\phi_G\} = \{\textstyle\sum_{\pi_G} n_{\pi_G} \phi(\pi_G)\} \ .$$

   When the spatial resolution $x_0 = q$ is replaced with $q_1 = r/s$, the spectrum is scaled by a rational factor q/q_1. The set of momenta is a dense subset of the real axis. There is no observable difference between the real momenta differing by a multiple of $2\pi/q$ and one must drop them from consideration. This conclusion is forced also by the fact that p-adically the momenta $k = nk_0$ do not exist, it is only the phase factors which exist.

3. It is easy to see that the p-adic plane waves with different momenta are orthogonal to each other as complex rational numbers:

$$\sum_n exp\left[in(k_G(1) - k_G(2))\right] = 0 \ \ .$$

4. Also completeness relations are satisfied in the sense that the condition

$$\sum_{k_G} exp\left[i(n_1 - n_2)k_G\right] = 0$$

is satisfied for $n_1 \neq n_2$. This is due to the fact that all integer multiples of k_G define Pythagorean phases. This means that the Fourier series of a function with respect to Pythagorean phases makes sense and one can expand p-adic-valued functions of space-time coordinates as Fourier series using Pythagorean phases. In particle expansion of the the imbedding space coordinates as functions of p-adic space-time coordinates might be carried out in this manner.

5. One can criticize this approach for the fact that there is no unique continuation of the phase factors from the set of the rationals $x_n = nx_0$ to p-adic numbers neighborhoods of these points. Although eigen states of finite translations are in question one cannot regard the states as eigen states of infinitesimal translations since the momenta are not well defined as p-adic numbers. One could of course arbitrarily assign momentum eigenstate $e^{in\pi(x-x_k)}$ the point x_k to the eigenstate characterized by the dimensionless momentum n but the momentum spectrum associated with different Pythagorean phases would be same.

5.6.4 How to define integration and p-adic Fourier analysis, integral calculus, and p-adic counterparts of geometric objects?

p-Adic differential calculus exists and obeys essentially the same rules as ordinary differential calculus. The only difference from real context is the existence of p-adic pseudo-constants: any function which depends on finite number of pinary digits has vanishing p-adic derivative. This implies non-determinism of p-adic differential equations. One can defined p-adic integral functions using the fact that indefinite integral is the inverse of differentiation. The basis problem with the definite integrals is that p-adic numbers are not well-ordered so that the crucial ordering of the points of real axis in definite integral is not unique. Also p-adic Fourier analysis is problematic since direct counterparts of ep(ix) and trigonometric functions are not periodic. Also exp(-x) fails to converse exponentially since it has p-adic norm equal to 1. Note also that these functions exists only when the p-adic norm of x is smaller than 1.

The following considerations support the view that the p-adic variant of a geometric objects, integration and p-adic Fourier analysis exists but only when one considers highly symmetric geometric objects such as symmetric spaces. This is welcome news from the point of view of physics. At the level of space-time surfaces this is problematic. The field equations associated with Kähler action and modified Dirac equation make sense. Kähler action defined as integral over p-adic space-time surface fails to exist. If however the Kähler function identified as Kähler for a preferred extremal of Kähler action is rational or algebraic function of preferred complex coordinates of WCW with rational coefficients, its p-adic continuation is expected to exist.

Circle with rotational symmetries and its hyperbolic counterparts

Consider first circle with emphasis on symmetries and Fourier analysis.

1. In this case angle coordinate ϕ is the natural coordinate. It however does not make sense as such p-adically and one must consider either trigonometric functions or the phase $exp(i\phi)$

instead. If one wants to do Fourier analysis on circle one must introduce roots $U_{n,N} = exp(in2\pi/N)$ of unity. This means discretization of the circle. Introducing all roots $U_{n,p} = exp(i2\pi n/p)$, such that p divides N, one can represent all $U_{k,n}$ up to $n = N$. Integration is naturally replaced with sum by using discrete Fourier analysis on circle. Note that the roots of unity can be expressed as products of powers of roots of unity $exp(in2\pi/p^k)$, where p^k divides N.

2. There is a number theoretical delicacy involved. By Fermat's theorem $a^{p-1}\ mod\ p = 1$ for $a = 1, ...p-1$ for a given p-adic prime so that for any integer M divisible by a factor of $p-1$ the M:th roots of unity exist as ordinary p-adic numbers. The problem disappears if these values of M are excluded from the discretization for a given value of the p-adic prime. The manner to achieve this is to assume that N contains no divisors of $p-1$ and is consistent with the notion of finite measurement resolution. For instance, $N = p^n$ is an especially natural choice guaranteeing this.

3. The p-adic integral defined as a Fourier sum does not reduce to a mere discretization of the real integral. In the real case the Fourier coefficients must approach to zero as the wave vector $k = n2\pi/N$ increases. In the p-adic case the condition consistent with the notion of finite measurement resolution for angles is that the p-adic valued Fourier coefficients approach to zero as n increases. This guarantees the p-adic convergence of the discrete approximation of the integral for large values of N as n increases. The map of p-adic Fourier coefficients to real ones by canonical identification could be used to relate p-adic and real variants of the function to each other.

This finding would suggests that p-adic geometries -in particular the p-adic counterpart of CP_2, are discrete. Variables which have the character of a radial coordinate are in natural manner p-adically continuous whereas phase angles are naturally discrete and described in terms of algebraic extensions. The conclusion is disappointing since one can quite well argue that the discrete structures can be regarded as real. Is there any manner to escape this conclusion?

1. Exponential function $exp(ix)$ exists p-adically for $|x|_p \leq 1/p$ but is not periodic. It provides representation of p-adic variant of circle as group $U(1)$. One obtains actually a hierarchy of groups $U(1)_{p,n}$ corresponding to $|x|_p \leq 1/p^n$. One could consider a generalization of phases as products $Exp_p(N, n2\pi/N + x) = exp(in2\pi n/N)exp(ix)$ of roots of unity and exponent functions with an imaginary exponent. This would assign to each root of unity p-adic continuum interpreted as the analog of the interval between two subsequent roots of unity at circle. The hierarchies of measurement resolutions coming as $2\pi/p^n$ would be naturally accompanied by increasingly smaller p-adic groups $U(1)_{p,n}$.

2. p-Adic integration would involve summation plus possibly also an integration over each p-adic variant of discretization interval. The summation over the roots of unity implies that the integral of $\int exp(inx)dx$ would appear for $n = 0$. Whatever the value of this integral is, it is compensated by a normalization factor guaranteeing orthonormality.

3. If one interprets the p-adic coordinate as p-adic integer without the identification of points differing by a multiple of n as different points the question whether one should require p-adic continuity arises. Continuity is obtained if $U_n(x+mp^m) = U_n(x)$ for large values of m. This is obtained if one has $n = p^k$. In the spherical geometry this condition is not needed and would mean quantization of angular momentum as $L = p^k$, which does not look natural. If representations of translation group are considered the condition is natural and conforms with the spirit of the p-adic length scale hypothesis.

The hyperbolic counterpart of circle corresponds to the orbit of point under Lorentz group in two 2-D Minkowski space. Plane waves are replaced with exponentially decaying functions of the coordinate η replacing phase angle. Ordinary exponent function $exp(x)$ has unit p-adic norm when it exists so that it is not a suitable choice. The powers p^n existing for p-adic integers however approach to zero for large values of $x = n$. This forces discretization of η or rather the hyperbolic phase as powers of p^x, $x = n$. Also now one could introduce products of $Exp_p(nlog(p) + z) = p^n exp(x)$ to achieve a p-adic continuum. Also now the integral over the discretization interval is compensated by orthonormalization and can be forgotten. The integral of exponential function would reduce to a sum $\int Exp_p dx = \sum_k p^k = 1/(1-p)$. One can also introduce finite-dimensional but non-algebraic extensions of p-adic numbers allowing e and its roots $e^{1/n}$ since e^p exists p-adically.

Plane with translational and rotational symmetries

Consider first the situation by taking translational symmetries as a starting point. In this case Cartesian coordinates are natural and Fourier analysis based on plane waves is what one wants to define. As in the previous case, this can be done using roots of unity and one can also introduce p-adic continuum by using the p-adic variant of the exponent function. This would effectively reduce the plane to a box. As already noticed, in this case the quantization of wave vectors as multiples of $1/p^k$ is required by continuity.

One can take also rotational symmetries as a starting point. In this case cylindrical coordinates (ρ, ϕ) are natural.

1. Radial coordinate can have arbitrary values. If one wants to keep the connection $\rho = \sqrt{x^2+y^2}$ with the Cartesian picture square root allowing extension is natural. Also the values of radial coordinate proportional to odd power of p are problematic since one should introduce $\sqrt{p}$: is this extension internally consistent? Does this mean that the points $\rho \propto p^{2n+1}$ are excluded so that the plane decomposes to annuli?

2. As already found, angular momentum eigen states can be described in terms of roots of unity and one could obtain continuum by allowing also phases defined by p-adic exponent functions.

3. In radial direction one should define the p-adic variants for the integrals of Bessel functions and they indeed might make sense by algebraic continuation if one consistently defines all functions as Fourier expansions. Delta-function renormalization causes technical problems for a continuum of radial wave vectors. One could avoid the problem by using exponentially decaying variants of Bessel function in the regions far from origin, and here the already proposed description of the hyperbolic counterparts of plane waves is suggestive.

4. One could try to understand the situation also using Cartesian coordinates. In the case of sphere this is achieved by introducing two coordinate patches with Cartesian coordinates. Pythagorean phases are rational phases (orthogonal triangles for which all sides are integer valued) and form a dense set on circle. Complex rationals (orthogonal triangles with integer valued short sides) define a more general dense subset of circle. In both cases it is difficult to imagine a discretized version of integration over angles since discretization with constant angle increment is not possible.

The case of sphere and more general symmetric space

In the case of sphere spherical coordinates are favored by symmetry considerations. For spherical coordinates $sin(\theta)$ is analogous to the radial coordinate of plane. Legedre polynomials expressible as polynomials of $sin(\theta)$ and $cos(\theta)$ are expressible in terms of phases and the integration measure $sin^2(\theta)d\theta d\phi$ reduces the integral of S^2 to summation. As before one can introduce also p-adic continuum. Algebraic cutoffs in both angular momentum l and m appear naturally. Similar cutoffs appear in the representations of quantum groups and there are good reasons to expect that these phenomena are correlated.

Exponent of Kähler function appears in the integration over WCW. From the expression of Kähler gauge potential given by $A_\alpha = J_\alpha{}^\theta \partial_\theta K$ one obtains using $A_\alpha = cos(\theta)\delta_{\alpha,\phi}$ and $J_{\theta\phi} = sin(\theta)$ the expression $exp(K) = sin(\theta)$. Hence the exponent of Kähler function is expressible in terms of spherical harmonics.

The completion of the discretized sphere to a p-adic continuum- and in fact any symmetric space- could be performed purely group theoretically.

1. Exponential map maps the elements of the Lie-algebra to elements of Lie-group. This recipe generalizes to arbitrary symmetric space G/H by using the Cartan decomposition $g = t + h$,

$[h,h] \subset h$,$[h,t] \subset t$,$[t,t] \subset h$. The exponentiation of t maps t to G/H in this case. The exponential map has a p-adic generalization obtained by considering Lie algebra with coefficients with p-adic norm smaller than one so that the p-adic exponent function exists. As a matter fact, one obtains a hierarchy of Lie-algebras corresponding to the upper bounds of the p-adic norm coming as p^{-k} and this hierarchy naturally corresponds to the hierarchy of angle resolutions coming as $2\pi/p^k$. By introducing finite-dimensional transcendental extensions containing roots of e one obtains also a hierarchy of p-adic Lie-algebras associated with transcendental extensions.

2. In particular, one can exponentiate the complement of the $SO(2)$ sub-algebra of $SO(3)$ Lie-algebra in p-adic sense to obtain a p-adic completion of the discrete sphere. Each point of the discretized sphere would correspond to a p-adic continuous variant of sphere as a symmetric space. Similar construction applies in the case of CP_2. Quite generally, a kind of fractal or holographic symmetric space is obtained from a discrete variant of the symmetric space by replacing its points with the p-adic symmetric space.

3. In the N-fold discretization of the coordinates of M-dimensional space t one $(N-1)^M$ discretization volumes which is the number of points with non-vanishing t-coordinates. It would be nice if one could map the p-adic discretization volumes with non-vanishing t-coordinates to their positive valued real counterparts by applying canonical identification. By group invariance it is enough to show that this works for a discretization volume assignable to the origin. Since the p-adic numbers with norm smaller than one are mapped to the real unit interval, the p-adic Lie algebra is mapped to the unit cell of the discretization lattice of the real variant of t. Hence by a proper normalization this mapping is possible.

The above considerations suggest that the hierarchies of measurement resolutions coming as $\Delta\phi = 2\pi/p^n$ are in a preferred role. One must be however cautious in order to avoid too strong assumptions. The following arguments however support this identification.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M unless one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. Otherwise the measurement resolution is just $\Delta\phi = 2\pi/p^n$. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer and assume that state function reduction selects one particular prime (no superposition of quantum states with different p-adic topologies). For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$. One could assign to a given measurement resolution all the p-adic primes appearing as factors in N so that the notion of multi-p p-adicity would make sense. One can also consider the identification of the measurement resolution as $\Delta\phi = |N/M|_p = 2\pi/p^k$. This interpretation is supported by the approach based on infinite primes [K70] .

What about integrals over partonic 2-surfaces and space-time surface?

One can of course ask whether also the integrals over partonic 2-surfaces and space-time surface could be p-adicized by using the proposed method of discretization. Consider first the p-adic counterparts of the integrals over the partonic 2-surface X^2.

1. WCW Hamiltonians and Kähler form are expressible using flux Hamiltonians defined in terms of X^2 integrals of JH_A, where H_A is $\delta CD \times CP_2$ Hamiltonian, which is a rational function of the preferred coordinates defined by the exponentials of the coordinates of the sub-space

t in the appropriate Cartan algebra decomposition. The flux factor $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ is scalar and does not actually depend on the induced metric.

2. The notion of finite measurement resolution would suggest that the discretization of X^2 is somehow induced by the discretization of $\delta CD \times CP_2$. The coordinates of X^2 could be taken to be the coordinates of the projection of X^2 to the sphere S^2 associated with $\delta M^4_\pm$ or to the homologically non-trivial geodesic sphere of CP_2 so that the discretization of the integral would reduce to that for S^2 and to a sum over points of S^2.

3. To obtain an algebraic number as an outcome of the summation, one must pose additional conditions guaranteeing that both H_A and J are algebraic numbers at the points of discretization (recall that roots of unity are involved). Assume for definiteness that S^2 is $r_M = constant$ sphere. If the remaining preferred coordinates are functions of the preferred S^2 coordinates mapping phases to phases at discretization points, one obtains the desired outcome. These conditions are rather strong and mean that the various angles defining CP_2 coordinates -at least the two cyclic angle coordinates- are integer multiples of those assignable to S^2 at the points of discretization. This would be achieved if the preferred complex coordinates of CP_2 are powers of the preferred complex coordinate of S^2 at these points. One could say that X^2 is algebraically continued from a rational surface in the discretized variant of $\delta CD \times CP_2$. Furthermore, if the measurement resolutions come as $2\pi/p^n$ as p-adic continuity actually requires and if they correspond to the p-adic group $G_{p,n}$ for which group parameters satisfy $|t|_p \leq p^{-n}$, one can precisely characterize how a p-adic prime characterizes the real partonic 2-surface. This would be a fulfilment of one of the oldest dreams related to the p-adic vision.

A even more ambitious dream would be that even the integral of the Kähler action for preferred extremals could be defined using a similar procedure. The conjectured slicing of Minkowskian space-time sheets by string world sheets and partonic 2-surfaces encourages these hopes.

1. One could introduce local coordinates of H at both ends of CD by introducing a continuous slicing of $M^4 \times CP_2$ by the translates of $\delta M^4_\pm \times CP_2$ in the direction of the time-like vector connecting the tips of CD. As space-time coordinates one could select four of the eight coordinates defining this slicing. For instance, for the regions of the space-time sheet representable as maps $M^4 \rightarrow CP_2$ one could use the preferred M^4 time coordinate, the radial coordinate of δM^4_+, and the angle coordinates of $r_M = constant$ sphere.

2. Kähler action density should have algebraic values and this would require the strengthening of the proposed conditions for X^2 to apply to the entire slicing meaning that the discretized space-time surface is a rational surface in the discretized $CD \times CP_2$. If this condition applies to the entire space-time surface it would effectively mean the discretization of the classical physics to the level of finite geometries. This seems quite strong implication but is consistent with the preferred extremal property implying the generalized Bohr rules. The reduction of Kähler action to 3-dimensional boundary terms is implied by rather general arguments. In this case only the effective algebraization of the 3-surfaces at the ends of CD and of wormhole throats is needed [K33] . By effective 2-dimensionality these surfaces cannot be chosen freely.

3. If Kähler function and WCW Hamiltonians are rational functions, this kind of additional conditions are not necessary. It could be that the integrals of defining Kähler action flux Hamiltonians make sense only in the intersection of real and p-adic worlds assumed to be relevant for the physics of living systems.

Tentative conclusions

These findings suggest following conclusions.

1. Exponent functions play a key role in the proposed p-adicization. This is not an accident since exponent functions play a fundamental role in group theory and p-adic variants of real geometries exist only under symmetries- possibly maximal possible symmetries- since otherwise the notion of Fourier analysis making possible integration does not exist. The inner product defined in terms of integration reduce for functions representable in Fourier

basis to sums and can be carried out by using orthogonality conditions. Convolution involving integration reduces to a product for Fourier components. In the case of imbedding space and WCW these conditions are satisfied but for space-time surfaces this is not possible.

2. There are several manners to choose the Cartan algebra already in the case of sphere. In the case of plane one can consider either translations or rotations and this leads to different p-adic variants of plane. Also the realization of the hierarchy of Planck constants leads to the conclusion that the extended imbedding space and therefore also WCW contains sectors corresponding to different choices of quantization axes meaning that quantum measurement has a direct geometric correlate.

3. The above described 2-D examples represent symplectic geometries for which one has natural decomposition of coordinates to canonical pairs of cyclic coordinate (phase angle) and corresponding canonical conjugate coordinate. p-Adicization depends on whether the conjugate corresponds to an angle or non-compact coordinate. In both cases it is however possible to define integration. For instance, in the case of CP_2 one would have two canonically conjugate pairs and one can define the p-adic counterparts of CP_2 partial waves by generalizing the procedure applied to spherical harmonics. Products of functions expressible using partial waves can be decomposed by tensor product decomposition to spherical harmonics and can be integrated. In particular inner products can be defined as integrals. The Hamiltonians generating isometries are rational functions of phases: this inspires the hope that also WCW Hamiltonians also rational functions of preferred WCW coordinates and thus allow p-adic variants.

4. Discretization by introducing algebraic extensions is unavoidable in the p-adicization of geometrical objects but one can have p-adic continuum as the analog of the discretization interval and in the function basis expressible in terms of phase factors and p-adic counterparts of exponent functions. This would give a precise meaning for the p-adic counterparts of the imbedding space and WCW if the latter is a symmetric space allowing coordinatization in terms of phase angles and conjugate coordinates.

5. The intersection of p-adic and real worlds would be unique and correspond to the points defining the discretization.

5.7 Generalization of Riemann geometry

Geometrization of physics program requires Riemann geometry and its variants such as Kähler geometry in the p-adic context. The notion of the p-adic space-time surface and its relationship to its real counterpart should be also understood. In this section the basic problems and ideas related to these challenges are discussed.

5.7.1 p-Adic Riemannian geometry depends on cognitive representation

p-Adic Riemann geometry is a direct formal generalization of the ordinary Riemann geometry. In the minimal purely algebraic generalization one does not try to define concepts like arch length and volume involving definite integrals but simply defines the p-adic geometry via the metric identified as a quadratic form in the tangent space of the p-adic manifold. Canonical identification would make it possible to define p-adic variant of Riemann integral formally allowing to calculate arc lengths and similar quantities but looks like a trick. The realization that the p-adic variant of harmonic analysis makes it possible to define definite integrals in the case of symmetric space became possible only after a detailed vision about what quantum TGD is [K26] had emerged.

Symmetry considerations dictate the p-adic counterpart of the Riemann geometry for $M^4_+ \times CP_2$ to a high degree but not uniquely. This non-uniqueness might relate to the distinction between different cognitive representations. For instance, in the case of Euclidian plane one can introduce linear or cylindrical coordinates and the manifest symmetries dictating the preferred coordinates correspond to translational and rotational symmetries in these two cases and give rise to different p-adic variants of the plane. Both linear and cylindrical coordinates are fixed only modulo the action of group consisting of translations and rotations and the degeneracy of choices can be interpreted in terms of a choice of quantization axes of angular momentum and momenta.

The most natural looking manner to define the p-adic counterpart of M^4 is by using a p-adic completion for a subset of rational points in coordinates which are preferred on physical basis. In case of M^4 linear Minkowski coordinates are an obvious choice but also the counterparts of Robertson-Walker coordinates for M^4_+ defined as $[t,(z,x,y)] = a\times[cosh(\eta), sinh(\eta)(cos(\theta), sin(\theta)cos(\phi), sin(\theta)sin($ expressible in terms of phases and their hyperbolic counterparts and transforming nicely under the Cartan algebra of Lorentz group are possible. p-Adic variant is obtained by introducing finite measurement resolution for angle and replacing angle range by finite number of roots of unity. Same applies to hyperbolic angles.

Rational CP_2 could be defined as a coset space $SU(3,Q)/U(2,Q)$ associated with complex rational unitary 3×3-matrices. CP_2 could be defined as coset space of complex rational matrices by choosing one point in each coset $SU(3,Q)/U(2,Q)$ as a complex rational 3×3-matrix representable in terms of Pythagorean phases [A72] and performing a completion for the elements of this matrix by multiplying the elements with the p-adic exponentials $exp(iu)$, $|u|_p < 1$ such that one obtains p-adically unitary matrix.

This option is not very natural as far as integration is considered. CP_2 however allows the analog of spherical coordinates for S^2 expressible in terms of angle variables alone and this suggests the introduction of the variant of CP_2 for which the coordinate values correspond to roots of unity. Completion would be performed in the same manner as for rational CP_2. This non-uniqueness need not be a drawback but could reflect the fact that the p-adic cognitive representation of real geometry are geometrically non-equivalent. This means a refinement of the principle of General Coordinate Invariance taking into account the fact that the cognitive representation of the real world affects the world with cognition included in a delicate manner.

5.7.2 p-Adic imbedding space

The construction of both quantum TGD and p-adic QFT limit requires p-adicization of the imbedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the imbedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost. The essential element is the notion of finite measurement resolution leading to discretization in large and to p-adicization below the resolution scale. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification mapping p-adic mass squared to its real counterpart.

The identification of the fundamental p-adic length scale

The fundamental p-adic length scale corresponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the 'radius' R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the 'cosmological constant' Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 = 2\Lambda . \tag{5.7.1}$$

The relationship between R and l is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} . \qquad (5.7.2)$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π:s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

   (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p\ mod\ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

   (b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of $1/3$ rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

$$l \equiv \pi R \ ,$$

   rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counterpart of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p\ mod\ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal plane waves. p-Adic plane waves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naively expected p, in the expression of the p-adic length scale can be understood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{5.7.2}$$

where m_{kl} is the standard M^4 metric $(1,-1,-1,-1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k:th coordinate axis the expression

$$s = R\sqrt{p}m^k \ . \tag{5.7.3}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{5.7.4}$$

The p-adic distance along the k:th coordinate axis from the origin to the point $m^k = (p-1)(1+p+p^2+...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p}l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R\frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{5.7.4}$$

What is remarkable is that the shortest distance in the range $m^k = 1,..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

The two variants of CP_2

As noticed, CP_2 allows two variants based on rational discretization and on the discretiation based on roots of unity. The root of unity option corresponds to the phases associated with $1/(1+r^2) = tan^2(u/2) = (1-cos(u))/(1+cos(u))$ and implies that integrals of spherical harmonics can be reduced to summations when angular resolution $\Delta u = 2\pi/N$ is introduced. In the p-adic context, one can replace distances with trigonometric functions of distances along zig zag curves connecting the points of the discretization. Physically this notion of distance is quite reasonable since distances are often measured using interferometer.

In the case of rational variant of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/columns are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their non-vanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_k X_k) \ ; \ |t_k|_p < 1\} = \{x = 1 + iy \ ; \ |y|_p < 1\} \ . \tag{5.7.5}$$

The diagonal elements of the matrices in this group are of the form $1 + O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1 + \phi_2))\} \ . \tag{5.7.6}$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1 + O(p)$. For $p \ mod \ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1 z_2 z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the 'physical' $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \ mod \ 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic sub-manifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \tag{5.7.6}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \ .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \quad . \tag{5.7.7}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1+r^2)^2} \quad . \tag{5.7.7}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

5.7.3 Topological condensate as a generalized manifold

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlates for cognition and intentionality. The vision about quantum TGD as a generalized number theory provides possible solutions to the basic problems associated with the precise definition of topological condensate.

Generalization of number concept and fusion of real and p-adic physics

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also). The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as e^p? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

This notion of manifold implies a generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in a discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

It took some time to end up with this vision. The first picture was based on the notion of real and p-adic space-time sheets glued together by using canonical identification or some of its variants but led to insurmountable difficulties since p-adic topology is so different from real topology. One can of course ask whether one can speak about p-adic counterparts of notions like boundary of 3-surface or genus of 2-surface crucial for TGD based model of family replication phenomenon. It seems that these notions generalize as purely algebraically defined concepts which supports the view that p-adicization of real physics must be a purely algebraic procedure.

How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x+p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \to \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n+y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about infinite size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves cosmic length scales.

What determines the p-adic primes assignable to a given real space-time sheet?

The p-adic realization of the Slaving Principle suggests that various levels of the topological condensate correspond to real matter like regions and p-adic mind like regions labelled by p-adic primes p. The larger the length scale, the larger the value of p and the course the induced real topology. If the most interesting values of p indeed correspond Mersenne primes, the number of most interesting levels is finite: at most 12 levels below electron length scale: actually also primes near prime powers of two seem to be physically important.

The intuitive expectation is that the p-adic prime associated with a given real space-time sheet characterizes its effective p-adic topology. As a matter fact, several p-adic effective topologies can be considered and the attractive hypothesis is that elementary particles are characterized by integers defined by the product of these p-adic primes and the integers for particles which can have direct interactions possess common prime factors.

The intuitive view is that those primes are favored for with the p-adic space-time sheet obtained by an algebraic continuation has as many rational or algebraic space-time points as possible in common with the real space-time sheet. The rationale is that if the real space-time sheet is generated in a quantum jump in which p-adic space-time sheet is transformed to a real one, it must have a large number of points in common with the real space-time sheet if the probability amplitude for this process involves a sum over the values of an n-point function of a conformal field theory over all common n-tuples and vanishes when the number of common points is smaller than n.

5.8 Appendix: p-Adic square root function and square root allowing extension of p-adic numbers

The following arguments demonstrate that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$.

5.8.1 $p > 2$ resp. $p = 2$ corresponds to $D = 4$ resp. $D = 8$ dimensional extension

What is important is that only the square root of ordinary p-adic numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find a finite-dimensional extension allowing square root for all ordinary p-adic numbers numbers. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.

1. For $p > 2$ a p-adic number y in the range $(0, p-1)$ allows square root only provided there exists a p-adic number $x \in \{0, p-1\}$ satisfying the condition $y = x^2 \ mod \ p$. Let x_0 be the smallest integer, which does not possess a p-adic square root and add the square root θ of x_0 to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p-1\}$ as is easy to see. p-adic numbers $mod \ p$ form a finite field $G(p, 1)$ [A121] so that any p-adic number y, which does not possess square root can be written in the form $y = x_0 u$, where u possesses square root. Since θ is by definition the square root of x_0 then also y possesses square root. The extension does not depend on the choice of x_0.

   The square root of -1 does not exist for $p \ mod \ 4 = 3$ [A113] and $p = 2$ but the addition of θ guarantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by θ. $\sqrt{(-1+p)-p}$ can be developed in power in powers of p and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p-1$ does not possess a square root, one can take θ to be equal to $\sqrt{-1}$.

2. The next step is to add the square root of p so that the extension becomes 4-dimensional and an arbitrary number in the extension can be written as

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ . \tag{5.8.1}$$

In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since n:th Taylor coefficient is proportional to 2^{-n} and possesses 2-adic norm 2^n. If x_0 possesses norm 1 then x_1 must possess norm smaller than 1/8 for the series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one obtains 8-dimensional extension.

The emergence of the dimensions $D = 4$ and $D = 8$ for the algebraic extensions allowing the square root of an ordinary p-adic number stimulates an obvious question: could one regard space-time as this kind of an algebraic extension for $p > 2$ and the imbedding space $H = M^4_+ \times CP_2$ as a similar 8-dimensional extension of the 2-adic numbers? Contrary to the first expectations, it seems that algebraic dimension cannot be regarded as a physical dimension, and that quaternions and octonions provide the correct framework for understanding space-time and imbedding space dimensions. One could perhaps say that algebraic dimensions are additional dimensions of the world of cognitive physics rather than those of the real physics and there presence could perhaps explain why we can imagine all possible dimensions mathematically.

By construction, any ordinary p-adic number in the extension allows square root. The square root for an arbitrary number sufficiently near to p-adic axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ at a point of p-adic axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to R^4 and the points of the extension allowing a square root consist of disjoint converge cubes forming a structure resembling future light cone in certain respects.

5.8.2 p-Adic square root function for $p > 2$

The study of the properties of the series representation of a square root function shows that the definition of the square root function is possible in certain region around the real p-adic axis. What is nice that this region can be regarded as the p-adic analog (not the only one) of the future light cone defined by the condition

$$N_p(Im(Z)) \quad < \quad N_p(t = Re(Z)) = p^k \ , \tag{5.8.2}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as a time coordinate and the imaginary part of the p-adic coordinate is identified as a spatial coordinate. The p-adic norm for the four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to a parallelpiped rather than ball.

The result can be understood heuristically as follows.

1. For the four-dimensional extension allowing square root ($p > 2$) one can construct square root at each point $x(k,s) = sp^k$ represented by ordinary p-adic number, $s = 1, ..., p-1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.

2. Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

$$N_p(Z - sp^k) \quad \leq \quad R(k) \ , \tag{5.8.3}$$

   and having radius $R(k) = p^d, d \in Z$ around the expansion point.

3. A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultra-metricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function only in the union of p-adic convergence spheres associated with the points $x(k,s) = sp^k$ which correspond to ordinary p-adic numbers.

4. By the scaling properties of the square root function the convergence radius $R(x(k,s)) \equiv R(k)$ is related to $R(x(0,s)) \equiv R(0)$ by the scaling factor p^{-k}:

$$R(k) \quad = \quad p^{-k}R(0) \ , \tag{5.8.4}$$

   so that the convergence sphere expands as a function of the p-adic time coordinate. The study of the convergence reduces to the study of the series at points $x = s = 1, ..., k-1$ with a unit p-adic norm.

5. Two neighboring points $x = s$ and $x = s+1$ cannot belong to the same convergence sphere: this would lead to a contradiction with the basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

$$R(0) \quad < \quad 1 \ . \tag{5.8.5}$$

   The requirement that the convergence is achieved at all points of the real axis implies

$$R(0) = \frac{1}{p} ,$$
$$R(p^k s) = \frac{1}{p^{k+1}} . \quad (5.8.5)$$

If the convergence radius is indeed this, then the region, where the square root is defined, corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is the p-adic analog of the M^4 light-cone. If the convergence radius is smaller, the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$\begin{aligned} N_p(Z) &= (N_p(det(Z)))^{\frac{1}{4}} , \\ &= (N_p((x^2 + y^2)^2 + 2p^2((xv - yu)^2 + (xu - yv)^2) + p^4(u^2 + v^2)^2))^{\frac{1}{4}} , \end{aligned} \quad (5.8.4)$$

where $det(Z)$ is the determinant of the linear map defined by a multiplication with Z. The definition of the convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2 + v^2) + p^4(u^2 + v^2)^2)) < 1 . \quad (5.8.5)$$

For physically interesting case $p \ mod \ 4 = 3$ the points (y, u, v) satisfying the conditions

$$\begin{aligned} N_p(y) &\leq \frac{1}{p} , \\ N_p(u) &\leq 1 , \\ N_p(v) &\leq 1 , \end{aligned} \quad (5.8.4)$$

belong to the sphere of convergence: it is essential that for all u and v satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers, the real counterpart of the sphere $r = t$ is now the parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with an average velocity of light in discrete steps at times $t = p^k$.

5.8.3 Convergence radius for square root function

In the following it will be shown that the convergence radius of $\sqrt{t + Z}$ is indeed non-vanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t + Z}$ reads as

$$\begin{aligned} \sqrt{t + Z} &= = \sqrt{x} \sum_n a_n , \\ a_n &= (-1)^n \frac{(2n - 3)!!}{2^n n!} x^n , \\ x &= \frac{Z}{t} . \end{aligned} \quad (5.8.3)$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \rightarrow \infty$. The p-adic norm of the n:th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n - 3)!!}{n!})N_p(x^n) < N_p(x^n)N_p(\frac{1}{n!}) . \quad (5.8.4)$$

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \ for \ \ N_p(x) < 1 \ , \tag{5.8.5}$$

holds true. The strategy is as follows:
a) The norm of x^n can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for U is derived at the limit of large n.

p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \leq n(i) < p$ be the p-adic expansion of n. Then $N_p(n!)$ can be expressed in the form

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \ , \\ N(1) &= \frac{1}{p} \ , \\ N(i+1) &= N(i)^{p-1}p^{-i} \ . \end{aligned} \tag{5.8.4}$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \ . \tag{5.8.5}$$

Proof: $n!$ can be written as a product

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} X(i, n(i)) \ , \\ X(k, n(k)) &= N_p((n(k)p^k)!) \ , \\ X(k-1, n(k-1)) &= N_p(\prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k + i)) = N_p((n(k-1)p^{k-1})!) \ , \\ X(k-2, n(k-2)) &= N_p(\prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k + n(k-1)p^{k-1} + i) \ ,) \\ &= N_p((n(k-2)p^{k-2})!) \ , \\ X(k-i, n(k-i)) &= N_p((n(k-i)p^{k-i})!) \ . \end{aligned} \tag{5.8.1}$$

The factors $X(k, n(k))$ reduce in turn to the form

$$\begin{aligned} X(k, n(k)) &= \prod_{i=1}^{n(k)} Y(i, k) \ , \\ Y(i, k) &= \prod_{m=1}^{p^k} N_p(ip^k + m) \ . \end{aligned} \tag{5.8.1}$$

The factors $Y(i, k)$ in turn are indentical and one has

$$\begin{aligned} X(k, n(k)) &= X(k)^{n(k)} \ , \\ X(k) &= N_p(p^k!) \ . \end{aligned} \tag{5.8.1}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of k:
1) $X(1) = N_p(p!) = p^{-1}$.
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ($p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to p^2.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$\begin{aligned} N_p(n!) &= p^{-\sum_{i=0}^{k} n(i)A(i)} \ , \\ A(i) &= \sum_{m=1}^{i} m(p-1)^{i-m} \ . \end{aligned} \quad (5.8.1)$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$\begin{aligned} A(i) &= (\frac{p-1}{p-2})^2(p-1)^{i-1}(1+\frac{i}{(p-1)^{i+1}}-\frac{(i+1)}{(p-1)^i}) \ , \\ &\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \end{aligned} \quad (5.8.1)$$

Upper bound for $N_p(\frac{x^n}{n!})$ for $p > 2$

By using the expressions $n = \sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_p n!$ as well as the upper bound

$$A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \quad (5.8.2)$$

For $A(i)$ one obtains the upper bound

$$N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^{k} n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \ . \quad (5.8.2)$$

It is clear that for $N_p(x) < 1$ that is $K \geq 1$ the upper bound goes to zero. For $p > 3$ exponents are negative for all values of i: for $p = 3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as a majorant.

5.8.4 $p = 2$ case

In $p = 2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^{k} n(i)\frac{i(i+1)}{2^{i+1}}} \ . \quad (5.8.3)$$

At the limit $n \to \infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K > 1$, so that one has

$$N_p(Z) \equiv (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \ . \quad (5.8.4)$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$\sqrt{s+\frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \ , \tag{5.8.5}$$

so that one can develop square as a series in all half odd integer points of the p-adic axis (points which are ordinary p-adic numbers). As a consequence, the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Space-time has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces).

Chapter 6

p-Adic Physics: Physical Ideas

6.1 Introduction

p-Adic topologies form an infinite hierarchy and p-adic physics leads to a vision about many-sheeted space-time as a hierarchical structure consisting of p-adic and real space-time sheets of increasing size and increasing value of prime p. These surfaces are glued together using topological sum or join along boundaries bonds. Contrary to the original expectations, p-adic space-time regions represent 'mind-stuff' rather than 'matter' which is also present and represented by real and infinite-p p-adic regions. Thus p-adic provide 'cognitive representations' for matter like regions and this is why their physics provides a manner to understand real physics. If p-adic-to-real phase transitions are possible, one can understand why it is possible to assign p-adic prime even to real regions. In fact, the hypothesis that p-adic regions provide a cognitive model for real physics, poses very strong constraints on real physics.

There is a "holy trinity" of non-determinisms in TGD in the sense that there is the non-determinism associated with the quantum jumps, the classical non-determinism of the Kähler action and p-adic non-determinism. The non-determinism of quantum jumps can involve also a selection between various multi-furcations for various absolute minima of the Kähler action in which case it represents a genuine volitional act. p-Adic non-determinism in turn corresponds to the non-determinism of pure imagination with no material consequences. Also real space-time sheets with finite time duration are also possible and they might represent what might be called 'sensory space-time sheets' as opposed to cognitive space-time sheets. Cognitive space-time sheets can be transformed to real ones in quantum jumps inducing change of control parameters of the polynomial defining space-time surface: if the change is such that the p-adic root is replaced with a real root, one can say that thought is transformed into action. The reverse of this process is the transformation of sensory input into cognition.

"Holy trinity" implies that it should be possible to determine the p-adic prime characterizing a given space-time region (or space-time sheet) by observing a large number of quantum time developments of this system. The characteristic p-adic fractality, that is the presence of time scales $T(p,k) = p^k T_p$, should become manifest in the statistical properties of the cognitive time developments which in should turn reflect the properties of the real physics since cognitive representations are in question. For instance, quantum jumps with especially large amplitude would tend to occur at time scales $T(p,k) = p^k T_p$. $T(p,k)$ could also provide series of characteristic correlation times. Needless to say, this prediction means definite departure from the non-determinism of ordinary quantum mechanics and only at the limit of infinite p the predictions should be identical. An interesting possibility is that $1/f$ noise [D1] is a direct manifestation of the classical non-determinism: if this is the case, it should be possible to associate a definite value of p to $1/f$ noise. Also transformations of the p-adic cognitive space-time sheets to real space-time sheets of a finite time duration and vice versa might be involved with the $1/f$ noise so that $1/f$ noise would be a direct signature of cognitive consciousness.

The 'physical' building blocks of p-adic TGD, as opposed to the philosophical mathematical ones briefly summarized above, and in more detail in previous chapters, are spin glass analogy leading to the general picture about how finite-p p-adicity emerges from quantum TGD, the identification of elementary particles as CP_2 type extremals, and elementary particle black hole analogy. These building blocks have been present as stable pieces of theory from beginning whereas philosophical ideas and interpretations have undergone rather wild fluctuations during an almost last decade of p-adic TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- p-Adic physics [L27]
- p-Adic length scale hypothesis [L28]
- p-Adic manifold [L23]
- p-Adic mass calculations [L24]

6.2 p-Adic numbers and spin glass analogy

Spin glass phase decomposes into regions in which the direction of the magnetization varies randomly with respect to spatial coordinates but remains constant in time. What makes spin glass special is that the boundary regions between regions of different magnetization do not give rise to large surface energies. Spin glass structure emerges in two manners in TGD framework.

1. Spin glass behavior at the level of real physics is encountered in TGD framework because of the classical non-determinism of the Kähler action. The classical non-determinism of CP_2 type extremals represents the manifestation of the spin glass analogy at the level of elementary particle physics. In macroscopic length scales real physics spin glass analogy makes possible 'real world engineering'.

2. Spin glass behavior at the level of cognition is encountered because of the p-adic non-determinism and makes possible what might be called imagination or 'cognitive engineering'. The point is that any piecewise constant function has a vanishing p-adic derivative. Therefore any function of the spatial coordinates depending on a finite number of the pinary digits is a pseudo constant. The discontinuities of this kind in the field variables do not lead to infinite surface energies in the p-adic context as they would in the real context.

Spin glass energy landscape is characterized by an ultra-metric distance function. The reduced WCW CH_{red} consisting of the maxima of the Kähler function with respect to quantum fluctuating degrees of freedom and zero modes defines the TGD counterpart of the spin glass energy landscape. This notion makes sense only in real context since p-adic space-time regions do not contribute to the Kähler function and all p-adic configurations are equally probable. The original vision was that if the ultra-metric distance function in CH_{red} is induced from a p-adic norm, a connection between p-adic physics and real physics also at the level of space-time might emerge somehow. It seems however that the ultra-metricity of CH_{red} need not directly relate to the p-adicity at the space-time level which can be understood if p-adic space-time regions give rise to cognitive representations of the real regions.

Of course, it *might* be that the p-adic prime characterizing cognitive representation of a real region characterizes also the reduced WCW associated with the region in question (one must of course assume that the reduced WCW approximately decomposes into a Cartesian product of the reduced WCWs associated with real regions).

6.2.1 General view about how p-adicity emerges

In TGD classical theory is exact part of the quantum theory and in a well defined sense appears already at the level of the configuration space geometry: the definition of WCW Kähler metric

[K33] associates a unique space-time surface to a given 3-surface. The vacuum functional of the theory (exponent of the Kähler function) is analogous to the exponent $exp(H/T_c)$ appearing in the definition of the partition function of a critical system so that the Universe described by TGD is quantum critical system. Critical system is characterized by the presence of two phases, which can be present in arbitrary large volumes. The TGD counterpart of this seems to be the presence of two kinds of 3-surfaces for which either Kähler electric or Kähler magnetic field energy dominates. These 3-surfaces have outer boundaries for purely topological reasons and these boundaries can be of a macroscopic size. Therefore it seems that 3-space should be regarded as what could be called topological condensate with a hierarchical, fractal like structure: there are 3-surfaces (with boundaries) condensed on 3-surfaces condensed on...... .

This leads to a radically new manner to see the world around us. The outer surfaces of the macroscopic bodies correspond to the boundaries of 3-surfaces in the condensate so that one can see the 3-topology in all its complexity just by opening one's eyes! A rather compelling evidence for the basic ideas of TGD if one is willing to give up the nebulous concept of "material object in topologically trivial 3-space" and to allow nontrivial 3-topology in macroscopic length scales. A second rather radical departure from the conventional picture of the 3-space is that 3-space is not connected in TGD Universe but contains arbitrary many disjoint components. In fact the actual Universe should consist of infinitely many 3-surfaces condensed on each other.

In two-dimensional critical systems conformal transformations act as symmetries and conformal invariance implies the Universality of critical systems. This suggests that one should try to find the generalization of the conformal invariance to higher dimensional, in particular, 4-dimensional case. If finally turned out that quaternion-conformal invariance realizes quantum criticality four 4-surfaces imbedded to 8-dimensional space. As a by product an explanation for space-time and imbedding space dimensions results.

In this approach the p-adic regions of the space-time surface result dynamically. Space-time surface is defined by the vanishing condition of a polynomial of two quaternion-valued variables q and p. This condition gives p as a function of q. It can however occur that some components of p become complex numbers. They must be however real so that the solution fails to exist in the real sense. It might be however possible to perform the completion of the rational space-time surface to a p-adic space-time surface and for some values of the p-adic prime the series defining the power series representing $p = f(q)$ might converge to a number in some algebraic extension of the ordinary p-adic numbers. Even more general rational-adic topologies in which norm is power of a rational number are possible. p-Adic numbers would thus be very closely related with quaternion-conformal invariance and criticality.

p-Adic topologies form an infinite hierarchy and p-adic physics leads to a vision about many-sheeted space-time as a hierarchical structure consisting of p-adic 4-surfaces of increasing size and increasing value of prime p. These surfaces are glued together using topological sum operation. Contrary to the original expectations, this hierarchy is the hierarchy for the regions of space-time representing 'mind-stuff' rather than 'matter' which is also present and represented by real and infinite-p p-adic regions. p-Adic provide 'cognitive representations' for matter-like regions and this is why their physics provides a manner to understand real physics.

6.2.2 p-Adic numbers and the analogy of TGD with spin-glass

The vacuum degeneracy of the Kähler action leads to precise spin glass analogy at the level of the WCW geometry and the generalization of the energy landscape concept to TGD context leads to the hypothesis about how p-adicity is realized at the level of the WCW. Also the concept of p-adic space-time surface emerges rather naturally.

Spin glass briefly

The basic characteristic of the spin glass phase [B14] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants. Free energy as a function of the coupling constants defines 'energy landscape' and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [A199] .

Vacuum degeneracy of Kähler action

The Kähler action defining WCW geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the CP_2 projection of the space-time surface is Lagrange manifold of CP_2: these manifolds are at most two-dimensional and any canonical transformation of CP_2 creates a new Lagrange manifold. An explicit representation for Lagrange manifolds is obtained using some canonical coordinates P_i, Q_i for CP_2: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ ,$$

where f arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of CP_2 for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of P_i and Q_i can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary (p_i, q_i) phase space.

Since vacuum degeneracy is removed only by classical gravitational interaction there are good reasons to expect large ground state degeneracy, when system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass.

Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with CP_2 projection, which is a Legendre sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a given absolute minimum or more general preferred extremal of Kähler action one obtains a new one by adding arbitrary space-time surfaces which is vacuum extremal. Uniqueness of the absolute minima in the sense that real regions of space-time $X^4(X^3)$ are unique could be achieved by requiring that vacuum regions are p-adic and represent thus cognitive regions whereas real regions carry non-vanishing induced Kähler field.

The symplectic invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of CH. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function for given values of the zero modes, become possible. Thus locally, the space maxima of Kähler function should look like a union of copies of the space of zero modes. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe. In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multi-furcation. This characterization works when non-determinism has discrete nature. For CP_2 type extremals which are bosonic vacua, basic objects are essentially four-dimensional since M^4_+ projection of CP_2 type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with CP_2 type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate absolute minima. This non-determinism is expected to make the absolute minima of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

The real effective action is expected to be Einstein-Yang-Mills action for the induced gauge fields. This action does not possess any vacuum degeneracy. The space-time surfaces are certainly absolute minima of the Kähler action and EYM-action could take a dynamical role only in the sense that extremality with respect to classical part of EYM action selects one of the degenerate absolute minima of the Kähler action. On the other hand, the construction of S-matrix suggests that the choice of particular parameter values characterizing zero modes affects only the coupling constants and propagators of the effective Einstein-Yang-Mills theory, and that one must perform averaging over the predictions of these theories. Thus EYM action could at most fix a gauge.

p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{6.2.0}$$

which does not depend on the pinary digits x_n, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the absolute minima (defined by the correspondence with infinite primes) are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible 'engineering' at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible 'engineering' at the level of the real world.

Localization in zero modes

The Kähler function defining WCW metric possesses infinite number of zero modes which represent non-quantum-fluctuating degrees of freedom. The requirement that physics is local at the level of zero modes implies that each quantum jump involves a localization in zero modes. This localization could be complete or in a region whose size is determined by the p-adic length scale hypothesis.

Localization would mean an enormous calculational simplification: functional integral reduces into ordinary functional integral over the quantum-fluctuating degrees of freedom and there is no need to integrate over the zero modes. The complete or partial localization in zero modes would explain why the world of conscious experience looks classical. Perhaps the complete localization is however too much to wish for: it could however be that one must use wave functionals in the zero modes only in the case that one is interested in a comparison of the transition rates associated with different values of zero modes rather than in transition rates with the condition that a localization has occurred to definite values of zero modes.

The functional integral over the fiber degrees of freedom can be approximated by a Gaussian integrals around maxima. Classical non-determinism would suggest the possibility of several maxima in fiber degrees of freedom but the symmetric space property of the fiber suggests that there is only single maximum of Kähler function. The existence of single maximum gives good hopes that the configuration space integration reduces effectively to Gaussian integration of free field theory.

6.2.3 The notion of the reduced WCW

Quantum jumps occur with highest probability to those values of zero modes which correspond to the maxima of the Kähler function and a simplified description of the situation is obtained by considering the reduced WCW CH_{red} consisting of the maxima of Kähler function with respect to both zero modes and and quantum fluctuating degrees of freedom.

The hypothesis that the space CH_{red} is an enumerable set is a natural first guess. In macroscopic length scales, one might indeed hope that the generation of Kähler electric fields reducing the vacuum degeneracy could imply a discrete degeneracy for the maxima of the Kähler action.

In elementary particle length scales this hypothesis fails and it is good to analyze the situation in more detail since it gives some about how complex the situation can be. For the so called CP_2 type extremals the classical non-determinism gives rise to a functional continuum of degenerate maxima of the Kähler function. The degenerate maxima correspond to random zitterbewegung orbits for which the 'time parameter' u is an arbitrary function of CP_2 coordinates. In this case however zero modes characterizing light like random curve representing the zitterbewegung orbit behave exactly like conformal gauge degrees of freedom. The choice of the 'time parameter' u however affects S-matrix elements: dependence is very weak and only through the volumes of the propagator lines determined by the selection of u (Kähler action for CP_2 type extremal is proportional to its volume) occurring in quantum jump. Effectively the functional continuum is replaced with the real continuum of the volume of the propagator line varying from zero to the volume of CP_2.

A localization for the positions of the vertices of the Feynman diagrams defined by CP_2 type extremals cannot however be assumed. Neither can one assume that only single Feynman diagram is selected if one wants that a generalization of ordinary Feynman diagrammatics results. There are several alternative identifications.

1. The degrees represented by Feynman diagrams with varying positions of vertices represent fiber degrees of freedom so that there would be slight dependence of the Kähler function on the positions of the vertices. Certainly the Feynman diagrams with different topologies have different value of Kähler action and must correspond to fiber degrees of freedom. The reason is that vertex regions of the Feynman diagrams must involve deformations of CP_2 extremals since otherwise Feynman diagrams are singular as 4-manifolds. Note that the idea about localization in fiber degrees of freedom is not favored by this example.

2. The positions for the vertices of the Feynman diagram are excellent candidates for zero modes and localization is not possible now. The fact that these degrees of freedom correspond to center of mass degrees of freedom related to the isometries of the theory might distinguish between them and other zero modes. One can consider also a refinement for localization in

the zero modes hypothesis: localization occurs only in length scale resolution defined by the p-adic length scale. In fact, the assumption that CP_2 type extremals have suffered topological condensation on space-time sheets with size of order p-adic length scale characterizing the elementary particle implies this.

Whether the notion of CH_{red} makes sense for the p-adic space-time regions is not at all obvious. For the proposed construction of the WCW metric p-adic regions do not contribute to the Kähler function which is real-valued. Only in case that the p-adic contribution is rational number, it could be interpreted as a real valued contribution to the Kähler function. In case of CP_2 type extremals this is not the case although the exponent of the Kähler function for a full CP_2 type extremal is a rational number if the proposed model for the p-adic evolution of Kähler coupling strength is correct. If it does not make sense to distinguish between the maxima of the Kähler function in the p-adic context, one cannot define CH_{red} on basis of this criterion. From the point of view of cognition this means maximal freedom of imagination.

An interesting question is whether one must count the cognitive degeneracy as a degeneracy of physical states. If localization occurs in each quantum jump with respect to both real and p-adic zero mode degeneracy, and if all cognitive options are equally probable, then the only conclusion seems to be that space-time surfaces for which the cognitive degeneracy is highest, represent the most probable final states. This would mean that the systems with the highest cognitive resources would be winners in the struggle for survival. An alternative manner to see the same thing is that systems with a high cognitive degeneracy are able to undergo a rich repertoire of p-adic-to-real phase transitions and thus to adapt with the environment.

Explicit definition of the ultra-metric distance function for energy landscape

The points of CH_{red} are completely analogous to the minima of the free energy and the precise analogy with spin glass suggests that CH_{red} must possess naturally an ultra-metric topology. One can quite generally construct an explicit ultra-metric distance function for the set of energy minima in a given energy landscape describing energy as a function of the coordinates of some WCW using existing recipes [B31] . The concept is useful when the energy landscape has fractal like structure. An attractive metaphor is to regard energy as a height function for a landscape with mountains.

The distance function between two energy minima should describe the difficulty of getting from a given minimum to another one. A concrete measure for this difficulty is obtained by considering all possible paths from x to y. The height for the highest point on this path, absolute maximum $h_{max}(\gamma)$ of the height function on this path gives the measure for the difficulty for reaching y along the path γ. There exists some easiest path from x to y. The difficulty to reach y from x can be defined as the height of the highest point associated with the easiest path and hence the minimum of $h_{max}(\gamma)$ in the set of all possible paths from x to y:

$$d(x,y) = Min(h_{max}(\gamma(x,y)) \ .$$

It is easy check that this distance function is ultra-metric:

$$d(x,z) \leq Max\{d(x,y), d(y,z)\} \ .$$

All what is needed is to notice that for any path $x \to z$ going through y highest point of the path is either the highest point associated with the path from $x \to y$ or $y \to z$: from this the inequality follows trivially since one can in principle find also easier paths.

Identification of the height function in the case of the reduced WCW?

Obviously the negative for the maximum of Kähler function as function of zero modes is the counterpart of free energy. This function could well be many valued but this is an unessential complication. It is not clear whether K is negative definite (there are strong reasons to believe that this is the case). One can however consider any positive definite function of K as a height function defining an ultra-metric norm in the manner suggested. The requirement that p-adic norm results should fix the definition uniquely.

The exponential $exp(-K_{max})$ of the maximum of Kähler function as function of the zero modes, which is the inverse for the vacuum functional of the theory, is the first guess for the height function defining the ultra-metric norm (the wandering from 3-surface X^3 to Y^3 corresponds to quantum tunnelling physically.). The justification for this identification is that the integration over the fiber degrees of freedom gives Gaussian determinant cancelling the metric determinant and leaves on the exponent of Kähler function to the functional integral over zero modes. The intuitive expectation is that ultra-metric norm is p-adic for some p and that the space of zero modes decomposes into regions D_p In order to get a power of p as required by p-adicity, one can expand h as powers of p and identify p-adic norm as p^n for the highest pinary digit n with non-vanishing coefficient.

The height function can have a normalization factor and this factor could be chosen so that the ultra-metric norm is a power of p for CP_2 type extremals, which are certainly very important building blocks of absolute minimum space-time surfaces. The argument relating the gravitational coupling constant to the Kähler coupling strength and fixing the dependence of the Kähler coupling strength on the prime p, suggests that one must define the height function as

$$h_p = \frac{exp(-K(p))}{exp(-K(p=1))} \ ,$$

where the Kähler function at $p = 1$ is formally obtained by regarding the value of the Kähler coupling strength as a function in the set of all natural numbers.

Does the proposed height function h_p define p-adic topology?

The great question is whether one can obtain p-adic ultra-metricity in this manner. There is some evidence for this.

1. Criticality and spin glass analogy suggests that $exp(K)$ as a function of zero modes is fractal. If it is p-adic fractal then p-adic topology is expected to be a natural consequence: in this case the map of CH_{red} to its p-adic counterpart could make it possible to replaced CH_{red} with a smooth function.

2. CP_2 type extremals, the counterparts of black holes and a model of elementary particle in TGD, have finite negative Kähler action. One can glue CP_2 type extremals to any space-time surface to lower the Kähler action. 3-surfaces Z^3 on path from X^3 to Y^3 containing CP_2 extremals on $X^4(Z^3)$ are excellent candidates for 'mountains' in the landscape metaphor. The height of Z^3 is roughly described by the number of CP_2 type extremals glued on $X^4(Z^3)$.

3. The argument leading to a correct prediction of gravitational constant in terms of assuming that Kähler coupling strength α_K depends on zero modes only through the p-adic prime assumed to characterize a given region D_p of WCW for which the set of maxima of Kähler function as function of zero modes should obey has p-adic topology. The crucial input is the relationship

$$exp(K_p(CP_2))\frac{R^2}{G} = \frac{1}{p} \ ,$$

   which is equivalent with $G = exp(K_p(CP_2)L_p^2$, where $L_p \simeq \sqrt{p} \times R$ is the p-adic length scale and $R \simeq 10^4\sqrt{G}$ is CP_2 size and the fundamental p-adic length scale. This formula is a dimensional estimate for gravitational coupling strength in terms of the p-adic length scale squared and the exponential of Kähler function for CP_2 type extremal describing graviton. The exponent gives the probability for the appearance of one virtual graviton in a given quantum state. The probability is very small since the exponent is negative for CP_2type extremal and gravitation is consequently a very weak interaction.

4. If one makes the identification

$$\frac{R^2}{G}(\sim 10^8) = exp(-K_{p=1}),$$

   then the function

$$h_p = \frac{exp(-K_p)}{exp(-K_{p=1})}/$$

is the n:th power of p for a vacuum extremal to which n CP_2 type extremals are glued. This is just the p-adic norm p^n! If h_p were p^n-valued in the general case it would be a p-adic pseudo constant and rather tame as a fractal. Very probably, this is not true in the general case and the p-adic norm of the p-adic counterpart of h_p in the canonical identification

$$\begin{array}{l} N_p \equiv |Id(h_p)|_p \ , \\ Id(\sum x_n p^n) = \sum_n x_n p^{-n} \ . \end{array}$$

depending on the most significant pinary digit of h_p only, is a good candidate for a p-adically ultra-metric height function having also a correct normalization. In any case, it seems that the number of virtual CP_2 type extremals (gravitons!) glued to an absolute minimum space-time surface $X^4(X^3)$ could define the height function. p-Adicity would emerge naturally and would have a direct physical meaning. Of course, this identification works for $n \geq 0$ only: the physical interpretation of the p-adic norm in $n < 0$ case is open.

A possible interpretation in terms of virtual graviton emission suggests the interpretation of the factor $\frac{R^2}{G} = exp(-K_{p=1})$ as a Gaussian determinant $\sqrt{det_G}$ associated with the integration over the zero modes around the maximum. The definition of Gaussian determinant in the real context is problematic and p-adicization plus adelic decomposition of the functional integral might provide a precise definition of $\sqrt{det_G}$. The divergence of the Gaussian determinant in the real context would lead to the vanishing of the gravitational constant. This picture is in accordance with the assumption that gravitational constant does not appear in quantum TGD as a fundamental constant and that the curvature scalar term in the low energy effective action essentially results from radiative corrections and hence derives from the logarithm of det_G.

6.3 p-Adic numbers and quantum criticality

TGD Universe is quantum critical in the sense that the value of Kähler coupling constant is completely analogous to critical temperature. Therefore the obvious question is how p-adicity might relate to quantum criticality.

6.3.1 Connection with quantum criticality

p-Adicization of the reduced WCW relates in an interesting manner to quantum criticality. At quantum criticality the number of the absolute minima of Kähler action for a surface Y^3 belonging to light cone boundary measures the cognitive resources of this surface and of its diffeomorphs. N_d is assumed to behave as $N_d \sim exp(-K_{cr})$, where Kähler function is evaluated for the critical value α_{cr} of the Kähler coupling strength. α_{cr} is like Hagedorn temperature appearing in the thermodynamics of strings. Above α_{cr} the theory might not be mathematically well defined since (at least real) the sum over the WCW integrals associated with the maxima of Kähler function would diverge exponentially at the limit when the value of Kähler function increases. In string thermodynamics this corresponds to the growth of number $g(E)$ of the states of given energy more rapidly than the inverse of the Boltzmann factor $exp(-E/T_H)$. Below α_{cr} the theory is certainly well defined but in TGD framework the cognitive resources of the Universe would not be maximal since vacuum functional would differ significantly from zero for very few space-time surfaces only. At quantum criticality the situation is optimal but it is not clear whether the real theory makes sense at quantum criticality: at least in string thermodynamics the partition function diverges also at Hagedorn temperature.

The cognitive resources of p-adic space-time sheet are measured by the entropy type quantity $log(N_d)/log(2)$ having lower bound $log(p)/log(2)$ bits for the 3-surfaces allowed by the vacuum functional. For instance, the maximal cognitive resources of electronic space-time sheet ($M_{127} = 2^{127} - 1$) would be 127 bits. In TGD one must allow even infinite primes and for these cognitive resources can be literally infinite.

6.3.2 Geometric description of the critical phenomena?

The idea that critical systems might have a geometric description is not new. There is a lot of evidence that simple, purely geometric lattice models based on the bond concept reproduce same critical exponents as the thermal models [B35] . The probability for a bond to exist corresponds to temperature in these models. For example, in a bond percolation model it is possible to relate the critical exponents to various fractal dimensions. This provides a nice manner to reduce the problem of predicting critical temperature to that of predicting the critical probability for the bond. This problem is local and once the temperature dependence of the bond probability and critical bond probability are known one can calculate the critical temperature.

What is new that in TGD approach the concept of bond ceases to be a phenomenological concept related to the simple modelling of the critical systems. TGD predicts that the boundaries of 3-surfaces can have arbitrarily large sizes. Furthermore, the formation of the join along boundaries bonds connecting the boundaries of two disjoint 3-surfaces seems to provide the basic mechanism for the formation of macroscopic quantum systems with long range correlations. This means that phase transitions should basically correspond to changes in the connectedness of the boundary of the 3-space. The description of the super fluidity, super conductivity and Quantum Hall effect based on the join along boundaries bond concept is suggested in [K36, K78] and also other phase transitions might be describable in the same manner. In hadronic length scale join along boundaries bonds correspond to color flux tubes connecting valence quarks. In nuclear length scale the short range part of the nuclear force corresponds to the formation of join along boundaries bonds between nucleons.

p-Adic approach suggests a concrete description for the phase transition changing the connectedness of the 3-surface. Disjoint 3-surfaces are labelled by p-adic numbers, whose p-adic expansion does not contain powers p^n with $n > N$, where N is some finite integer: the larger the value of N the larger the degree of disjointness. This means that phase transitions (say evaporation or condensation) changing the connectedness of the 3-surface should correspond to transitions changing the value of N. In evaporation process N increases and in condensation process N decreases. Also catastrophic processes like the breaking of a solid object to pieces might correspond to increase in N. Typical self organization processes such as biological growth and healing might correspond to a gradual decrease of N.

Fractal like configurations with a discrete scale invariance are known to play important role in the description of the critical phenomena: they are the most probable configurations at the critical point. The idea that fractal corresponds to a fixed point of a discrete scaling transformation, is in accordance with the definition of the fractals as fixed points for a set of affine transformations acting on subsets of some metric space [A118] . A natural candidate for the discrete scaling transformation is the transformation of the 4-surface induced by the multiplication of the p-adic argument Z of H-coordinate $h(Z)$ by a power of p: $Z \rightarrow p^n Z$. A tempting idea is that most probable 3-spaces indeed are invariant under these scalings. This even suggests that something, which might be called "Mandelbrot cosmology", might provide a description of the Universe in all length scales as a 4-dimensional analog of Mandelbrot set. The breaking of the discrete scaling invariance is bound to occur, when one considers finite subsystem instead of the whole Universe. p-Adic cutoff might provide an elegant description for the breaking of the exact scaling invariance: 3-surface in question depends on finite number of the pinary digits of Z only.

6.3.3 Initial value sensitivity and p-adic differentiability

Initial value sensitivity is one of the basic properties of the critical systems and implies unpredictability in practice. p-Adic differentiability seems to be related to this property in a very general manner. Consider a configuration of an initial value sensitive system, which can possess very high dimension. For definiteness, assume that the dynamics is described by some differential equations, which can be reduced to equations of first order for WCW coordinates X (we do not bother to write indices):

$$\frac{dX}{dt} = J(X) \ . \tag{6.3.1}$$

Space-time coordinate is a p-adic number one can assume that time coordinate is a p-adic number, too.

The purely p-adic feature of this differential equation follows from the fact that any function depending on a finite number of pinary digits of a p-adic number possesses a vanishing p-adic derivative! This implies that the integration constants are not just ordinary constants but functions of the p-adic number t depending on finite number of pinary digits of t! Obviously this implies classical non-determinism in long time scales! One can construct solutions of the differential equation in the form $X(t) = X_0(t) + X_1(t)$, where $X_0(t)$ depends on a finite number of pinary digits of the p-adic time t and equations reduce to

$$\frac{dX_1}{dt} = J(X_0 + X_1) \ . \tag{6.3.2}$$

Of course, one must be careful in defining what "finite number of pinary digits" means, when p-adic cutoff is actually present. The simplest integration constants depend on the p-adic norm of t (or on the lowest pinary digit of t) only.

The result is in accordance with the so called Slaving Principle [B25] . One can think that the dynamics in long time scales (low pinary digits of p-adic number t) is given by the integration constants having arbitrary dependence on these pinary digits and the dynamics in short length scales is determined by the differential equations in the "background" given by these time dependent integration constants.

Initial value sensitivity implies effectively non-deterministic behavior and p-adic numbers perhaps provide a possibility to describe it properly. The properties of the Kähler function suggests that the classical non-determinism might be in fact actual. The point is that the classical space time surface associated with a given 3-surface need not be unique. This surface is determined as an absolute minimum of the so called Kähler action and Kähler action possesses enormous vacuum degeneracy [K10] : the most general vacuum extremal has 2- dimensional CP_2 projection, which is so called Lagrange manifold possessing a vanishing induced Kähler form. Symplectic transformations and $Diff(M^4)$ act as exact dynamical symmetries of the vacuum extremals and $Diff(M^4)$ contains p-adically analytic transformations of M^4 as subgroup. It might well happen that those absolute minima, which are obtainable as small deformations of the vacuum extremals inherit the characteristic degeneracy of the vacuum extremals.

The classical macroscopic non-determinism might be essential to the possibility of the quantum measurements. In TGD the state function reduction is described as 'jump between histories' that is two deterministic time developments [K41] . In quantum measurement microscopic and macroscopic system are strongly correlated and microscopic transition induces a phase transition like phenomenon in a macroscopic critical system. The general belief is that quantum effects become unimportant in macroscopic systems. The situation need not be this if macroscopic system is critical, or even non-deterministic.

In the TGD inspired theory of 'thinking systems', conscious thoughts correspond to quantum jumps selecting one of the possible time developments in the quantum superposition of several quantum average effective space-time times allowed by the non-determinism. p-Adic pseudo constants could provide a mathematical description for this non-determinism. These 'cognitive' quantum jumps are certainly involved with a realistic description of a quantum measurement modelling also the presence of the observer quantum mechanically.

In turns out that quantum non-determinism, classical non-determinism of Kähler action and p-adic non-determinism are very closely related in quantum TGD: one could even speak of a holy trinity of non-determinisms. Quantum non-determinism corresponds closely to the classical non-determinism of Kähler action: quantum jumps select between various branches of the branches of multi-furcations of classical space-time surface. The p-adic counterparts of these branches are in turn obtained by varying pseudo constants in the solution of the p-adic Euler-Lagrange equations for the Kähler action: this requirement in fact makes it possible to assign unique p-adic prime to a given, sufficiently small space-time region.

6.3.4 There are very many p-adic critical orbits

An interesting connection between the p-adicity and initial value sensitive systems is related to the possibility to replace also the WCW (possibly infinite dimensional) with an algebraic extension of the p-adic numbers. The underlying motivation is the need to get a proper mathematical description of the finite accuracy for the observables and p-adic cutoff provides this description.

This in turn suggests Universality in some aspects of the dynamical behavior. The dynamical equations $dX/dt = J(X)$ define a flow that is a diffeomorphism $X \rightarrow F(X,t)$ of WCW. This flow contains as integration constants arbitrary functions of the p-adic time coordinate t depending on a finite number of pinary digits of t so that classical non-determinism is present. By p-adic conformal invariance this diffeomorphism ought to be p-adically analytic map that is representable as a power series of the algebraically extended p-adic numbers x and t.

The p-adic analyticity of the dynamic diffeomorphism gives strong constraints on the properties of the dynamic map. A particularly interesting map is in this respect Poincare map. One can ask several interesting questions. How does the Universal behavior of one- dimensional and 2-dimensional analytic iterated maps generalize to the p-adic case? What do attractors look like? What are the counterparts of Julia set and Mandelbrot set? What about routes to chaos? Could p-adic hypothesis provide deeper explanation for the fact that period doubling seems to be a rather general mechanism for the transition to turbulence. It might be possible to answer these questions since p-adic analyticity is very strong constraint on the behavior of the maps.

Already the study of the simplest p-adic complex maps reveal some surprises. The simplest map to study is the map $Z \rightarrow Z^n$ for any extension of p-adic numbers (dimension is arbitrary!). The repeller consists of the points p-adic norm equal to one. Due to the roughness of the p-adic topology, the real counterpart of the repeller is of same dimension as WCW itself so that the critical orbits form a set with a non-vanishing measure! For example, in the 2-dimensional case and for the 2-adic extension, the set of the critical orbits corresponds in the real plane to a square $(1/2, 1] \times (1/2, 1]$.

How do the small deformations of $Z \rightarrow Z^n$ of form $Z \rightarrow Z^n + \epsilon Z^m$ affect the set of the critical orbits? If the norm of the parameter ϵ is sufficiently small, the previous repeller belongs to the repeller also now. Also new points can appear in repeller. These considerations suggest that the repellers/attractors of the p-adically analytic maps have rather simple structure as compared to their real and complex counterparts. An interesting possibility is that in general case these sets are fractal like objects resembling the fractals associated with p-adic order parameters.

The fact that set of critical orbits is n-dimensional rather than $(n-1)$ or lower-dimensional in the p-adic case suggests an interesting physical interpretation in accordance with the general idea that p-adic topology corresponds to criticality. In ordinary situation these orbits are not very interesting because a small deformation spoils their criticality. In p-adic case the situation is different since the critical orbits are meta-stable and their are very many of them. In TGD one can even identify good candidates for the set of of these meta-stable critical orbits as small deformations of the vacuum extremals of the Kähler action. Needless to emphasize, this vacuum degeneracy is a phenomenon not encountered in the standard field theories.

6.4 p-Adic Slaving Principle and elementary particle mass scales

The understanding of the elementary particle mass scales is a fundamental problem in the unified field theories. The attempts to understand the generation of the mass scales dynamically have not been successful. The basic problem is the fine tuning difficulty: the predicted mass scale hierarchy is not stable under the small changes of the model parameters. A possible explanation for the failure is that the fundamental mass scales are really fundamental and therefore cannot depend on the details of the dynamical model.

Criticality is known to imply Universality and criticality indeed is the fundamental property of Kähler action. Therefore the derivation of the elementary particle length scale(s) should be based on a proper formulation of the criticality concept. p-Adic numbers indeed provide a promising tool in this respect and the following arguments show that it is possible not only to understand some general elementary particle length scale but leptonic, hadronic and intermediate gauge boson length scales plus a small number of shorter length scales in terms of primes near prime powers of two. The most important length scales correspond to Mersenne primes: there are only sixteen Mersenne primes below electron length scale and the remaining Mersenne primes correspond to super astronomical length scales.

What is nice that the p-adic hypothesis makes possible to express these length scales as square roots of Mersenne primes and possibly Fermat primes, that is prime numbers of type $p = 2^m \pm 1$. What is amusing is that Mersenne primes are closely related to the so called Perfect Numbers $n = 2^{m-1}(2^m - 1)$ representable not only as a product of their prime factors but also as a sum of their proper divisors. The ancient number mystics believed that this property makes these numbers very exceptional in the World Order!

6.4.1 p-Adic length scale hypothesis

p-Adic length scale hypothesis has served as a basic hypothesis of p-adic TGD for several years. This hypothesis states that the scales $L_p = \sqrt{p}l$, $l = 1.376 \cdot 10^4\sqrt{G}$ are fundamental length scale at p-adic condensate level p. The original interpretation of the hypothesis was following:

1. Above the length scale L_p p-adicity sets on and effective course grained space-time topology is p-adic rather than ordinary real topology.

2. The length scale L_p serves as a p-adic length scale cutoff for the field theory description of particles. This means that space-time begins to look like Minkowski space so that quantum field theory $M^4 \to CP_2$ becomes a realistic approximation. Below this length scale string like objects and other particle like 3-surfaces dominate.

3. It is un-natural to assume that just single p-adic field would be chosen from the infinite number of possibilities. Rather, there is an infinite number of cutoff length scales. To each prime p there corresponds a cutoff length scale L_p above which p-adic quantum field theory $M^4 \to CP_2$ makes sense and one has a hierarchy of p-adic quantum field theories. These different p-adic field theories correspond to different hierarchically levels possibly present in the topological condensate. Hierarchical ordering $< p_1 < p_2 < ...$ means that only the surface $p_1 < p_2$ can condense on the surface p_2. The condensed surface can in practice be regarded as a point like particle at level p_2 described by the p-adic conformal field theory below length scale L_{p_2}.

The work with p-adic QFT has however demonstrated that the hypothesis a) and b) are probably wrong and the following interpretation is closer to the truth.

1. The length scale $L_p = \sqrt{p}l$ defines an *infrared* cutoff rather than ultraviolet cutoff for a p-adic quantum field theory formulated in terms of quarks and leptons and gauge bosons. For instance, for hadrons this length scale is of order hadron size and L_p defines UV cutoff for possibly existing field theory describing hadrons as basic objects. Above L_p real topology effectively replaces the p-adic one (real continuity implies p-adic continuity) and if length scale resolution L_p is used real physics is excellent approximation.

2. p-Adic QFT is free of UV divergences with any UV cutoff and there is no need to assume that p-adicity fails below some length scale. Rather, p-adicity is completely general property of the effective quantum average space-time defined by the Quantum TGD, which is based on the real number field. The concept of the effective space-time, or topological condensate, is in turn necessary for the formulation of field theory limit of TGD. The analogy of Quantum TGD with spin glass phase gives strong support for the p-adic topological condensate consisting of p-adic regions with different p glued together along their boundaries.

p-Adic topologies form a hierarchy of increasingly coarser topologies. The p-adic norm $N(x_p)$ defines a function of a real argument via the canonical identification of the non-negative real numbers and p-adic numbers. The p-adic norm is same as ordinary real norm for $x = p^k$ and is constant at each interval $[p^k, p^{k+1})$. This means that

1. p-adic topologies are coarser than real topologies so that the functions, which are continuous in the p-adic topology need not be continuous in the real topology.

2. p-adic topologies are ordered: the larger the value of p, the coarser the topology in the long length scales. In short length scales the situation is just the opposite.

6.4.2 Slaving Principle and p-adic length scale hypothesis

Slaving Principle states that there exists a hierarchy of dynamics with increasing characteristic length (time) scales and the dynamical variables of a given length scale obey dynamics, where the dynamical variables of the longer length (time) scale serve as "masters" that is effectively as external parameters or integration constants. The dynamics of the "slave" corresponds to a rapid adaptation to the conditions posed by the "master".

p-Adic length scale hierarchy suggests a quantitive realization of this philosophy.

1. By the previous considerations there is an infinite hierarchy of length scales L_p such that the space-time surfaces below the length scale L_p look like Minkowski space and p-adic quantum field theory $M^4 \to CP_2$ makes sense below the length scale L_p. These length scales are associated with the different condensation levels present in the topological condensate and define the typical size of the p-adic surface in absence of the collective quantum effects, which should correspond to the formation of the join along boundaries bonds between objects with size of order L_p. The reason why the typical size is just this is that the imbedding of the p-adic coordinate space into space H has strongest discontinuities in the real topology, when coordinate values correspond to powers of p so that a typical imbedding decomposes into separate pieces with size of order L_p. Of course, this kind of discontinuity is possible for all powers of p but is not observable in shorter length scales for the physically most interesting values of p due to the extreme smallness of the corresponding length scales.

2. The lowest level of the hierarchy corresponds to 2-adic dynamics and this field theory makes sense below the cutoff length scale $L_2 = \sqrt{2}l$ defining the typical size for a 2-adic surface. Solutions of the 2-adic field equations are non-deterministic due to the possibility of the integration constants depending on finite number of binary digits. The dependence on a finite number of positive bits of the real coordinates only means that they are genuine constants below some length scale $L_2(lower) < L_2$, which in principle depends on the state of the system.

3. 2-adic pseudo-constants are analogous to external parameters and should be determined by the dynamics associated with the longer length and time scales. The properties of the p-adic numbers suggest that these constants in turn are p-adically differentiable functions of their argument with some value of $p_1 > 2$ determined by the p_1-adic dynamics describing the interaction between $p = 2$ surface condensed on $p = p_1$ level and $p = p_1$ background surface. The p_1-adic integration constants associated with these functions are actual constants above the length scale $L_{p_1}(lower) \geq L_2(lower)$ but also these in principle depend on a finite number of pinary digits and their values are determined by the interaction of p_1 level with the next level in the condensation hierarchy.

4. At the next level p_1 one encounters p_1-adic dynamics and new p-adic integration constants. The net effect is that one obtains a hierarchy of p-adic numbers $2 < p_1 < p_2 < ...$ in correspondence with the length and time scales $L_2 < L_{p_1} < L_{p_2} < ...$: the higher the boss the larger the p. In TGD it is very tempting to interpret the various levels of the slaving hierarchy as the levels of the topological condensate so that the surfaces at level p are condensed on the surfaces of level $p_1 > p$ (see Fig. 6.4.2). Not all values of p need be present in the hierarchy and it might well happen that certain values of p are in an exceptional position physically.

6.4.3 Primes near powers of two and Slaving Hierarchy: Mersenne primes

All values of p are in principle present in the Slaving Hierarchy but the assumption that all values of p are equally important physically is not realistic. The point is that the number $N(n)$ of primes smaller than n behaves as $N(n) \sim n/ln(n)$ and there are just too many prime numbers. For example, for $n = 10^{38}$ there are about one prime number per 87 natural numbers!

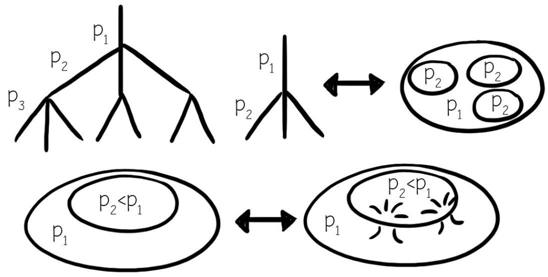

Figure 6.1: Two-dimensional visualization of topological condensate concept

A natural looking assumption is that a new physically important length scale emerges, when a fixed number of powers of 2 combine to form a new length scale. The reason is that a given interval $[2^k, 2^{k+1})$ forms an independent fractal unit (for the simplest fractals these intervals are related by a similarity, see figures in [K47] and it is therefore unnatural to cut this unit into pieces as would happen if p were far from a power of two. This breaking would indeed happen since p-adically differentiable functions have sharp gradients at points p^k. This non-breaking or "synergy" is reached provided the allowed primes are as close as possible to powers of 2: $p \simeq 2^m$. It should be noticed that this condition also guarantees that the frequency peaks associated with various powers of p in good approximation correspond to period doubling frequencies characteristic to fractal and chaotic systems.

The best approximation achievable corresponds to Fermat and Mersenne primes

$$p = 2^m \pm 1 \ . \tag{6.4.1}$$

It can be shown that for Fermat primes (+) the condition $m = 2^k$ must be satisfied and for Mersenne primes (-) m must be itself prime.

How abundant are the prime numbers of type $p = 2^m \pm 1$? The great surprise was that there are very few numbers of this kind!

1. The primes of type $2^m + 1$, Fermat primes, are very rare: only 5 numbers in the range $1 < n < 2^{2^{21}} \simeq 10^{10^6}$(!) [A113] and there are good arguments suggesting that the number of the Fermat primes is finite! The known Fermat primes correspond to $m = 2^k$, with $k = 0, 1, 2, 3, 4$. The corresponding primes are $p = 3, 5, 17, 257, 65537$. Note that the lowest Fermat prime 3 is also a Mersenne prime. It will be later found that p-adic conformal invariance is in TGD possible for primes p satisfying the condition $p \ mod \ 4 = 3$ and this condition is not satisfied by Fermat primes $F > 3$.

2. The primes of form $2^m - 1$, Mersenne primes, are also there as follows from the requirement that m is prime. The list of allowed exponents of m consists of the following numbers:

$$2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, ...$$

.

One can make two observations about these numbers:

1. $m = 127$ corresponds to the number 10^{38} fundamental to Physics. The square root of this number gives the ratio of the proton length scale to Planck length scale. This suggests the possibility that fundamental physical length scales are given by square roots of Mersenne and possibly Fermat primes using some length scale of order Planck scale as a unit.

2. $m = 61$ corresponds to the number of order 10^{19}: this in turn allows the possibility that fundamental physical length scales are linearly related to Fermat and Mersenne primes. This alternative however turns out to be not the correct one.

These observations lead to following scenario for the fundamental length scales:

1. The p-adic length scale L_p, below which p-adic quantum field theory approximation makes sense, is proportional to the square root of p and these length scales are p-adically the most interesting length scales:

$$\begin{aligned} L_p &= \sqrt{p}l \ , \\ l &\sim k\cdot 10^4\sqrt{G} \ , \\ k &\simeq 1.376 \ . \end{aligned} \tag{6.4.0}$$

   Only quite recently the physical interpretation of the length scale l was found. Contrary to the original expectations, CP_2 is not of order Planck length but of order l. At this length scale Euclidian regions of space-time, in particular CP_2 type extremals representing elementary particles, become important. Above this length scale a field theory in Minkowski space is expected to be a good approximation to quantum physics.

2. Physically the most interesting length scales correspond to the p-adic cutoff length scales L_p associated with the Mersenne primes M_n.

3. The fact that l is of the same order of magnitude as the length scale at which the coupling constants of the standard model become approximately equal, is not probably an accident. Below l it is not anymore sensible to speak about the topological condensation of CP_2 type extremals since CP_2 type extremals themselves have size of order l. Hence the symmetry breaking effects caused by the topological condensation cannot be present in the string model type description applying below l.

The predictions are as follows:

1. $m = 127$ corresponds to electron Compton length.

2. $m = 107$ corresponds to proton Compton length L_P.

3. $m = 89$ corresponds to length scale of order 1/256 times proton Compton length and is identifiable approximately as $L_W/2\sqrt{2}$, where L_W is intermediate boson length scale of about $L_P/100$.

4. $m = 61$ corresponds to length scale of the order of $10^{-6}L_P$ is not reachable by the present day accelerators.

5. $m = 521$ corresponds to a completely super-astronomical length scale of order 10^{27} light years!

It seems that the proposed scenario might have caught something essential in the problem of the elementary particle mass scales: it predicts correctly 3 fundamental length scales associated with leptons, hadrons and intermediate gauge bosons from number theory; there is extremely large gap in the length scale hierarchy after electron Compton length and new shorter length scales exist but unfortunately they are outside the reach of the present day experiments. The calculations of the third part of the book show that not only the mass scales can be understood but also particle masses can be predicted with errors below one per cent using the length scale hypothesis combined with the p-adic Super Virasoro invariance and p-adic thermodynamics.

6.4.4 Length scales defined by prime powers of two and Finite Fields

Above M_{127} there is an extremely large gap for Mersenne primes and this suggests that there must be also other physically important primes. Certainly all primes near powers of two define physically interesting length scales by 2-adic fractality but there are two many of them. The first thing, which comes into mind is to consider the set of primes near prime powers of two containing as special case Mersenne primes. The following argument is one of the many arguments in favor of these length scales developed during last years.

TGD Universe is critical at quantum level and criticality is related closely to the scaling invariance. This suggests that unitary irreducible representations of p-adic scalings $x \to p^m x$, $m \in Z$ should play central role in quantum theory. Unitarity requires that scalings are represented by a multiplication with phase factor and the reduction to a representation of a finite cyclic group Z_m requires that scalings $x \to p^m x$, m some integer, act trivially. In ordinary complex case the representations in question correspond to the phase factors $\Psi_k(x) = |x|^{(\frac{ik2\pi}{ln(p)})} = exp(iln(|x|)\frac{k2\pi}{ln(p)})$, $k \in Z$ and the reduction to a representation of Z_m is also possible but there is no good reason for restricting the consideration to discrete scalings.

1. The Schrödinger amplitudes in question are p-adic counterparts of the ordinary complex functions $\Psi_k(x) = exp(iln(|x|)k\frac{ik2\pi}{ln(p)})$, $k \in Z$. They have a unit p-adic norm, they are analogous to plane waves, they depend on p-adic norm only and satisfy the scaling invariance condition

$$\begin{aligned} \Psi_k(p^m x|p \to p_1) &= \Psi_k(x|p \to p_1) \ , \\ \Psi_k(x|p \to p_1) &= \Psi_k(|x|_p|p \to p_1) \ , \\ |\Psi_k(x|p \to p_1)|_p &= 1 \ , \end{aligned} \tag{6.4.-1}$$

which guarantees that these functions are effectively functions on the set of the p-adic numbers with cutoff performed in m:th power.

2. The solution to the conditions is suggested by the analogy with the real case:

$$\begin{aligned} \Psi_k(x|p \to p_1) &= exp(i\frac{kn(x)2\pi}{m}) \ , \\ n(x) &= ln_p(N(x)) \in N \ , \end{aligned} \tag{6.4.-1}$$

where $n(x)$ is integer (the exponent of the lowest power of the p-adic number) and $k = 0, 1, ..., m-1$ is integer. The existence of the functions is however not obvious. It will be shortly found that the functions in question exist in $p > 2$-adic for all m relatively prime with respect to p but exist for all odd m and $m = 2$ in the 2-adic case.

3. If m is prime (!) the functions $K = \Psi_k$ form a finite field $G(m,1) = Z_m$ with respect to the p-adic sum defined as the p-adic product of the Schrödinger amplitudes

$$K + L = \Psi_{k+l} = \Psi_k \Psi_l \ , \tag{6.4.0}$$

and multiplication defined as

$$KL = \Psi_{kl} \ . \tag{6.4.1}$$

Hence, if the proposed Schödinger amplitudes possessing definite scaling invariance properties are physically important, then the length scales defined by the prime powers of two must be physically special since Schrödinger amplitudes or equivalently, the p-adic scaling momenta k labeling them, have a natural finite field structure. By the Slaving Hierarchy Hypothesis, also the p-adic length scales near prime powers of two (and perhaps of prime $p > 2$, too) are therefore physically interesting. p-Adic scalings correspond to p-adic translations if p-adic coordinates correspond to exponentials of the ordinary linear coordinates so that translations are represented by scalings.

The generalized plane waves exist p-adically if nontrivial $N = p$:th root of the quantity $exp(i2\pi) = 1$ exists.

1. $N = 2$:th roots of 1 exist trivially for all values of p.

2. In 2-adic case the roots exist always for odd values of N and especially so for prime values of N: the trick is to write $1^{1/N} = -(-1)^{1/N} = -(1-2)^{1/N}$ and use the Taylor series

$$\begin{aligned} (1+x)^{1/N} &= \sum_n \frac{A_n}{n!}x^n \ , \\ A_n &= \prod_{k=0}^{n-1}(\frac{1}{N}-k)(-1)^n \ , \\ x &= -2 \ . \end{aligned} \tag{6.4.0}$$

to show the existence of one root different from the trivial root. In 2-adic case the powers of $x = 2$ converge to zero rapidly and compensate the powers of 2 coming from $n!$ in the denominator. The coefficients A_n possess 2-adic norm not larger than 1.

3. For $p > 2$ nontrivial $N = p$:th roots do not allow representation as plane waves for the simple reason that only the trivial p:th root of 1 exists p-adically. Roots of unity must have p-adic norm equal to one and by writing the condition modulo p one obtains a condition $a^N \ mod \ p = 1$ in $G(p,1)$. The roots of unity in $G(p,1)$ satisfy always $a^{p-1} = 1$ and the possible orders N are factors of $p-1$. In particular, prime roots with $p_1 > p-1$ are not possible. The number of prime factors is typically quite small. For instance, for primes of order $p = 2^{127}$ the number of prime roots is of order 6.

The conclusion is that for $p > 2$ only those finite fields $G(p_1,1)$ for which p_1 is factor of $p-1$ are realizable as representation of phase factors whereas for $p = 2$ all fields $G(p_1,1)$ allow this kind of representation. Therefore $p = 2$-adic numbers are clearly exceptional. In the p-adic case the functions $\Psi_p(x,|p \to p_1)$ give irreducible representations for the group of p-adic scalings $x \to p^m x$, $m \in Z$ and the integers k can be regarded as scaling momenta. This suggests that these functions should play the role of the ordinary momentum eigenstates in the quantum theory of fractal structures. The result motivates the hypothesis that prime powers of two and also of p define physically especially interesting p-adic length scales: this hypothesis will be of utmost importance in future applications of TGD.

The ordinary (number theoretic) p-adic plane waves associated with the translations can be constructed as functions $f_k(x) = a^{kx}$, $k = 0,...,n$, $a^n = 1$. For $p > 2$ these plane waves are periodic with period n, which is factor of $p-1$ so that wavelengths correspond to factors of $p-1$ and generate a finite number of physically favored length scales. The p-adic plane waves with the momenta $k = 0,...,p-2$ form finite field $G(p,1)$, when p-adic arithmetics is replaced with the modulo p arithmetics, that is to accuracy $O(p)$ (note that the definition of the arithmetic operations is *not* the same as in the previous case). The square roots of the p-adic plane waves are also well defined

The important property of the p-adic plane waves is that they are pseudo constants: this property played profound role in the earlier formulations of the p-adic QFT limit. It took a considerable time to discover that the counterparts of the ordinary real plane waves providing representations for translation group exists and satisfy the appropriate orthogonality relations. Therefore number theoretic plane waves do not play so essential role in p-adic QFT as was originally believed.

6.5 CP_2 type extremals

CP_2 type extremals are perhaps the most important vacuum extremals of the Kähler action. The reason is that they are vacuum extremals with a negative and finite Kähler action and hence favored both by the absolute minimization of the Kähler action and criticality (randomness of light-like projection to M^4 implies criticality). It seems that also other identification of preferred extremals allow CP_2 type vacuum extremals and actually all known extremals. On the other hand, maximization of Kähler function does not favor CP_2 type extremals because the virtual CP_2 type extremals are exponentially suppressed. CP_2 type extremals seem to play the same role as black holes possess in General Relativity. p-Adic thermodynamics, leading to excellent predictions for the masses of the elementary particles, predicts that elementary particles should possess p-adic entropy and Hawking-Bekenstein law for the entropy generalizes.

In GRT based cosmology black holes populate the most probable Universe, which is of course a problem: in TGD black holes are replaced by elementary particles. The second law of thermodynamics requires that the very early Universe should have a low entropy and hence that black holes should populate the recent day Universe: in TGD the very early cosmology is dominated by cosmic strings, which is a low entropy state. The absolute minimization of the Kähler action would imply that most cosmic strings would decay to elementary particles and produce p-adic entropy. It is not clear whether also criticality implies this. To get a grasp of the orders of magnitude, it is good to notice that electron, which corresponds to $p = M_{127} = 2^{127} - 1$, has entropy equal to 127 bits.

The basic observation is that the M^4_+ projection of the CP_2 type extremal corresponds to a light like random curve and the quantization of this motion leads to Virasoro algebra and Kac Moody algebra characterizing quantized transversal motion superposed with the cm motion. CP_2 type extremals allow covariantly constant right handed neutrino spinors as solutions of the Dirac equation for the induced spinors in the interior and this leads to $N = 1$ super symmetry and a generalization of the Virasoro invariance to Super Virasoro invariance.

The previous p-adic mass calculations were based on this picture but it turned out that the Super Virasoro invariance and related Kac Moody symmetries generalize to the level of WCW geometry and in an extended form provide the basic symmetries of the quantum TGD. Although the quantization of the zitterbewegung motion of the CP_2 type extremals is a phenomenological procedure only, and is not needed in the fundamental theory, it deserves to be described because of its key role in the development of quantum TGD. There were however some strange features involved: for instance, $N = 1$ super-symmetry generated by right-handed neutrino was exact only for minimal surfaces.

The realization that super-symmetry requires modified Dirac action led to the final breakthrough. CP_2 type extremals allow quaternion-conformal symmetries and the super-generators associated with quark and lepton numbers are non-vanishing despite the fact that vacuum extremals are in question. Even Super-Kac-Moody generators are non-vanishing. Even more, CP_2 type extremals cease to be vacua for Dirac action. Especially beautiful feature of CP_2 type extremals is that they can describe also massive states and zitterbewegung is the geometric correlate of massivation.

6.5.1 Zitterbewegung motion classically

The M^4_+ projection of a CP_2 type extremal is a random light like curve. Also Dirac equation, which gives also classically rise to a motion with light velocity and this motivates the term 'zitterbewebung'. Zitterbewegung occurs at the light of velocity and any given 3-velocity gives rise to the solution of light likeness condition if one fixes the time component of velocity to be

$$\frac{dm^0}{d\tau} = \sqrt{m_{ij}\frac{dm^i}{d\tau}\frac{dm^j}{d\tau}} \ . \tag{6.5.0}$$

The vanishing of CP_2 part of the second fundamental form requires that velocity and acceleration are orthogonal:

$$m_{kl}\frac{dm^k}{d\tau}\frac{d^2m^l}{d\tau^2} = 0 \ . \tag{6.5.1}$$

This condition is identically satisfied.

A very general solution to the conditions is provided by the equations

$$\frac{d^2m^k}{d\tau^2} = F^{kl}\frac{dm^l}{d\tau} \ , \tag{6.5.2}$$

describing the motion the of massless charged particle in external Maxwell field.

6.5.2 Basic properties of CP_2 type extremals

CP_2 type extremal has the following explicit representation

$$m^k = f^k(u(s^k)) \ , \quad m_{kl}\frac{df^k}{du}\frac{df^l}{du} = 0 \ . \tag{6.5.3}$$

The function $u(s^k)$ is an arbitrary function of CP_2 coordinates and serves effectively as a time parameter in CP_2 defining a slicing of CP_2 to time=constant sections. The functions f^k are arbitrary apart from the restriction coming from the light likeness. When one expands the functions f^k to Fourier series with respect to the parameter u, light likeness conditions reduce to classical Virasoro conditions $L_n = 0$.

It is possible to write the expression for m^k in a physically more transparent form by separating the center of mass motion and by introducing p-adic length scale L_p as a normalization factor.

$$\frac{m^k}{L_p} = m_0^k + p_0^k u + \sum_n \frac{1}{\sqrt{n}}a_n^k exp(i2\pi nu) + c.c. \ . \tag{6.5.4}$$

The first term corresponds to the center of mass term responsible for rectilinear motion along geodesic line and second term corresponds to the zitterbewegung motion. p^k serves as an effective classical momentum which can be normalized as $p_kp^k = \epsilon$, $\epsilon = \pm 1$ or $\epsilon = 0$. What has significance is whether p^k is time like, light like, or space like. Conformal invariance corresponds to the freedom to replace u with a new 'time parameter' $f(u)$.

The physically most natural representation of u is as a function $f(U)$ of the fractional volume U for a 4-dimensional sub-manifold of CP_2 spanned by the 3-surfaces $X^3(U = 0)$ and $X^3(U)$:

$$u = f(U) \ , \quad U = \frac{V(s^k)}{V(CP_2)} = \frac{S_K(u)}{S_K(CP_2)} \ . \tag{6.5.5}$$

The range of the values for U is bounded from above: $U \leq V_{max}/V(CP_2)$ and the value $U = 1$ is possible only if CP_2 type extremal begins and ends as a point. U represents also Kähler action using the value of the Kähler action for CP_2 as a unit.

The requirement that CP_2 type extremal extends over an infinite time and spatial scale implies the requirement

$$f(U_{max}) = \infty \ . \tag{6.5.6}$$

For $f(U_{max}) < \infty$ CP_2 type extremal can exist only in a finite temporal and spatial interval for finite values of 'momentum ' components p^k. This suggest a precise geometric distinction between real and virtual particles: virtual particles correspond to the functions $f(U_{max}) < \infty$ in contrast to the incoming and outgoing particles for which one has $f(U_{max}) = \infty$. This hypothesis, although it looks like an ad hoc assumption, is at least worth of studying.

The mere requirement that virtual CP_2 type extremal extends over a temporal or spatial distance of order $L > L_p$ implies that for $L < L_p$ the value of U is smaller than one. Kähler action, which is given by

$$S_K(X^4) = U \times S_K(CP_2) \ , \tag{6.5.7}$$

remains small for distances much smaller than L. For $f(U_{max}) = \infty$ this is even more true. This has an important implication: below a certain length scale the exponential of the Kähler action associated with the internal line of a Feynman diagram does not give rise to a suppression factor whereas above some characteristic length L and time scale there is an exponential suppression of the propagator by the factor $exp(-S_K(CP_2))$ practically hindering the propagation over distances larger than this length scale.

The presence of the exponential obviously introduces an effective infrared cutoff: this cutoff is prediction of the fundamental theory rather than ad hoc input as in quantum field theories. Of course, infrared cutoff results also from the condition $f(U_{max}) < \infty$. Physically the infrared cutoff results from the topological condensation of the CP_2 type extremals to larger space-time sheets. These could correspond to massless extremals (MEs). p-Adic length scale L_p is an excellent candidate for the cutoff length scale in the directions transversal to ME.

The suppression factor coming from the exponent of the Kähler action implies a distance dependent renormalization of the propagators. In the long length scale limit the suppression factor approaches to a constant value

$$exp\left[-\frac{V_{max}}{V(CP_2)}S_K(CP_2)\right] \ ,$$

and can be absorbed to the coupling constant so that the dependence on the maximal length of the internal lines can be interpreted as an effective coupling constant evolution. For instance, the smallness of the gravitational constant could be understood as follows. Since gravitons propagate over macroscopic distances, the virtual CP_2 type extremals develops a full Kähler action and there is huge suppression factor reducing the value of the gravitational coupling to its observed value: at short length scales the values of the gravitational coupling approaches to $G_{short} = L_p^2$ which means strong gravitation for momentum transfers $Q^2 > 1/L_p^2$. The values of V_{max} and thus those of the suppression factor can vary: only at the limit when CP_2 extremal has point like contact with the lines it joins together, one has $V_{max} = V(CP_2)$. If the boundary component characterizing elementary particle family belongs to CP_2 type extremal (it could be associated with a larger space-time sheet), CP_2 type extremal contains a hole: also this reduces the maximal volume of the CP_2 extremal.

6.5.3 Quantized zitterbewegung and Super Virasoro algebra

Calculating various Fourier components of right left hand side of the light likeness condition $m_{kl}p^kp^l = 0$ for $p^k = dm^k/du$ explicitly using the general expansion for m^k separating center of mass motion from zitterbewegung, one obtains classical Virasoro conditions

$$\begin{aligned} p_0^2 &= L_0 \ , \\ L_n|phys\rangle &= 0 \ , \ . \end{aligned} \tag{6.5.7}$$

where L_n are defined by by their classical expressions as bi-linears of the Fourier coefficients. Therefore interior degrees of freedom give Virasoro algebra and zitterbewegung is more or less equivalent with the classical string dynamics.

It is not however not obvious whether a quantization of this dynamics is needed. If quantization is needed (perhaps to formulate the unitarity conditions in zero modes properly), it corresponds to the construction of the bosonic wave functionals in zero modes defined by the zitterbewegung degrees of freedom. Quantization could be carried out in the same manner as in string models.

The simplest assumption motivated by the Euclidian metric of CP_2 type extremal is that the commutator of p^k and m^k is proportional to a delta function as in ordinary quantization. One can Fourier expand m^k and p_k in the form

$$
\begin{aligned}
m^k &= m_0^k + p_0^k s + \frac{1}{K}\sum \frac{1}{n} a_n^{k,\dagger} exp(inKs) + \sum \frac{1}{n} a_n^k exp(-inKs) \ , \\
p^k &= p_0^k + i\sum a_n^{k,\dagger} exp(inKs) - i\sum a_n^k exp(-inKs) \ . \qquad (6.5.7)
\end{aligned}
$$

Here cm motion has been extracted and the formula is identical with the formula expressing the motion for a fixed point of string. The parameter K is Kac Moody central charge. Note that the exponents $exp(iKns)$ exist provided that Ks is p-adically of order $O(p)$ or, if algebraic extension by introducing $\sqrt{p}$ is allowed, of order $O(\sqrt{p})$.

The commutator of p_i and m^j is of the standard form if the oscillator operators obey Kac-Moody algebra

$$
\begin{aligned}
\left[p_{i,0}, m_0^j\right] &= m_i^{\ j} \ , \\
Comm(a_{i,m}^{\dagger}, a_n^j) &= Km\delta(m,n)m_i^{\ j} \ . \qquad (6.5.7)
\end{aligned}
$$

Here K appears Kac-Moody central charge, which must be integer in the real context at least.

Expressing the light likeness condition as quantum condition, one obtains an infinite series of conditions, which give the quantum counterparts of the Virasoro conditions

$$
\begin{aligned}
p_0^2 &= kL_0 \ , \\
L_n|phys\rangle &= 0 \ , \ n<0 \ . \qquad (6.5.7)
\end{aligned}
$$

k is some proportionality constant. One can solve these conditions by going to the transverse gauge in which physical states are created by oscillator operators orthogonal to an arbitrarily chosen light like vector. What quantization means physically is that zitterbewegung amplitudes are constrained by a Gaussian vacuum functional. A good guess motivated by the p-adic considerations is that the width of the ground state Gaussian is given by a p-adic length scale L_p: this is achieved if m^k is replaced with m^k/L_p in the general expression for $m^k(u)$. The experience with string models would suggests that vacuum functionals might be crucial for the understanding of graviton emission.

6.5.4 Zitterbewegung at the level of the modified Dirac action

At the level of the modified Dirac action zitterbewegung motion implies that the conserved momentum associated with CP_2 type extremal, besides being conserved and non-vanishing, is also time like. This means that zitterbewegung creates massive particles besides massless particles as well as off-mass-shell versions of both and Super Virasoro conditions imply the quantization of the mass squared spectrum.

This means that in quantum TGD Feynman diagrammatics is topologized in the sense that the lines of Feynman diagram correspond to CP_2 type extremals which in general performing zitterbewegung. The non-determinism of the CP_2 type extremals means that one obtains a sum over over all possible diagrams with vertices at arbitrary space-time locations just as in quantum field theory approach. What is so nice that the time-development operator associated with an individual line of the diagram is the exponent of the Hamiltonian operator identified as the Poincare energy associated with the modified Dirac action. This operator is that associated with a free theory and contains no nonlinear terms. Interactions result from criticality property of the extremals of Kähler action. In particular, one gets rid of the divergences of the interacting quantum field theories by the topologization of the Feynman diagrammatics.

6.6 Black-hole-elementary particle analogy

String models have provided considerable insights into black hole thermodynamics by reducing it to ordinary thermodynamics for stringy black holes [B23] although one still does not understand, which is the mechanism of the thermalization. In TGD context elementary particles are regarded as thermodynamical systems in p-adic sense. This is something new since the standard theories of particle physics describe elementary particles as pure quantum states. The resulting thermal description of the the particle massivation is extremely successful. The fact that one can associate a well defined entropy to an elementary particle, suggests an analogy between black holes and elementary particles and this analogy indeed exists in a quite precise form as will be found. It also leads to a partial explanation for the p-adic length scale hypothesis serving as the corner stone of the p-adic mass calculations. The identification of the CP_2 type extremal as a cognitive representation of elementary particle suggests that p-adic entropy characterizes information associated with a cognitive representation provided by CP_2 type extremal.

6.6.1 Generalization of the Hawking-Bekenstein law briefly

In TGD elementary particles are modelled as so called CP_2 type extremals, which are surfaces with a size of order Planck length having metric with Euclidian signature. These vacuum surfaces are isometric with CP_2 itself and have a one-dimensional, random light like curve as the M^4_+ projection. A natural candidate for the TGD counterpart of the black hole horizon is the surface at which the Euclidian signature of the metric associated with the CP_2 type extremal is changed to the Minkowskian signature of the background space-time. The radius r of this surface is the crucial length scale for the topological condensation and the simplest guess is that it is of the order of the size of the CP_2 radius and hence of the fundamental p-adic length scale. The hope is that the generalization of the black hole thermodynamics, with r replacing the radius of the black hole horizon, could give this information.

p-Adic mass calculations indeed give the p-adic counterpart of the Hawking-Bekenstein formula $S \propto GM^2$ as an identity at p-adic level:

$$S_p = -\frac{1}{T_p}(M_p^2/m_0^2) \ ,$$

where $1/T_p = n$ is the the integer valued inverse of the p-adic temperature and the mass scale $m_0^2/3$ corresponds to unit p-adic number in the unit used. The peculiar looking sign of S_p does not have in the p-adic context the same significance as in real context since the real counterpart of S_p is positive. Although p-adic entropy and mass squared are linearly related, the real counterparts are not in such a simple relation. In case of massive particles the real counterpart of the entropy is in excellent approximation equal to $S = log(p)$ whereas the mass is of order $1/p$ (p is of order 10^{38} for electron!). For massless (or nearly massless) particles one has $S \leq log(p)/p$. The large difference between fermionic and photonic entropies does not favor pair annihilation and this suggests that matter antimatter asymmetry is generated thermodynamically. For instance, via the topological condensation of fermions and anti-fermions on different space-time sheets during the early cosmology.

The generalization of the Hawking-Bekenstein formula in the form of the area law $S = A/4G$ reads as

$$S = \frac{xA}{4l^2} \ ,$$

where the fundamental p-adic length scale $l \simeq 1.376 \cdot 10^4\sqrt{G}$ replaces Planck length $\sqrt{G}$ and x is a numerical constant near unity. The radius of the elementary particle horizon is in an excellent approximation given by $r(p) = \sqrt{\frac{log(p)}{\pi x}}l$. Particles are thus surrounded by an Euclidian region of the space-time with radius r. Thus the fundamental p-adic length scale l of order CP_2 size has a direct geometric meaning. For instance, in the energy scales below $1/l$ the induced metric of the space-time becomes Euclidian and it might be possible to describe particle physics using Euclidian field theory: essentially QFT in a small deformation of CP_2 would be in question. It is encouraging, that l is also the length scale at which the standard model couplings become identical and super symmetry is expected to become manifest.

The p-adic length scale hypothesis stating that the primes p near prime powers of two are the physically most interesting p-adic primes, is the cornerstone of p-adic mass calculations but there is no really convincing argument for why should it be so. The proportionality of r to $\sqrt{log(p)}$ suggests an explanation for the p-adic length scale hypothesis. The point is that for $p \simeq 2^k$, k prime, one has $r \propto L(k)$ and if the numerical constant x is chosen to be $x = \frac{log(2)}{\pi}$, the radius of elementary particle horizon is in excellent approximation $r(p \simeq 2^k) = L(k)$. Note also that the area of the elementary particle horizon becomes quantized in multiples of prime. This suggests that the precise value of $p \simeq 2^k$ is such that this condition is satisfied optimally and that physics is k-adic below r and $p \simeq 2^k$-adic above r.

$M^4_+ \times CP_2$ allows the imbedding of Schwartshild metric in the region below Schwartchild radius but the imbedding fails for too small values of the radial variable [K74] . An interesting possibility is that black hole entropy is just the sum of the elementary particle entropies topologically condensed below the horizon. This would give $S_{TGD} \propto \sum m_i^2 < S_{GRT} \propto (\sum m_i)^2$. An interesting problem is related to the detailed definition of p-adic entropy: are the entropies of particles with same value of p additive as p-adic numbers or does the additivity hold true for the real counterparts of the p-adic entropies. A related question is whether it might be that also in case of black holes additivity holds true, not for the mass as it is usually assumed, but for the p-adic mass squared for a given p (in TGD inspired model of hadron this is true for quark masses). This could be understood as a result of strong gravitational interactions. The additivity with respect to mass squared would give an upper bound of order $10^{-4}/\sqrt{G}$ for the contribution of a given p-adic prime to the total mass. For instance, the total contribution of electrons to the mass would be always below this mass irrespective of the number of electrons!

6.6.2 In what sense CP_2 type extremals behave like black holes?

CP_2 type extremals are in some respects classically black hole like objects since their metric is Euclidian. When this kind of surface is glued to Minkowskian background there must exist a two-dimensional surface, where the signature of the induced metric changes from the Minkowskian $(1,-1,-1,-1)$ to the Euclidian $(-1,-1,-1,-1)$. On this surface, which could be called elementary particle horizon, the metric is degenerate and has the signature $(0,-1,-1,-1)$. Physically elementary particle horizon can be visualized as the throat of the wormhole feeding the elementary particle gauge fluxes to the background space-time. Of course, one cannot exclude the presence of several wormholes for a given space-time sheet.

This surface indeed behaves in certain respects like horizon. Time like geodesic lines cannot go through this surface. The reason is that the square of the four velocity associated with the geodesic is conserved:

$$v_\mu v^\mu = 1 \ , \ 0 \text{ or } \ -1 \ ,$$

depending on whether the geodesic is time like, light like or space like. Clearly, a time like geodesic cannot enter from the external world to the interior of the CP_2 type extremal. If a space like geodesic starts from the interior of the CP_2 type extremal it can in principle continue as a space like geodesic into the exterior. These analogies should not be taken too seriously: it does not make sense to identify particles orbits as geodesics in these length scales shorter than the actual sizes of particle.

These analogies suggest that Hawking-Bekenstein formula $S = A/4G$ relating black hole entropy to the area of the black hole horizon, might have a generalization to the elementary particle context with the radius of the elementary particle horizon replacing the black hole horizon. The unit of the area need not be determined by Planck length $\sqrt{G}$, it could be replaced by the fundamental p-adic length scale $l \sim 10^4\sqrt{G}$: this length scale indeed replaces Planck length as a fundamental length scale in TGD.

6.6.3 Elementary particles as p-adically thermal objects?

In the p-adic mass calculations elementary particles were assumed to be thermal objects in the p-adic sense. What is new that energy is replaced with mass squared and the thermalization is believed to result from the interactions of a topologically condensed CP_2 type extremal with the background space-time surface of a much larger size. The thermalization mixes massless states with Planck mass states and gives rise to particle massivation. Super Virasoro invariance $-$ abstracted from the Virasoro invariance of the CP_2 type extremals $-$ together with the general symmetry considerations based on the symmetries of $M^4_+ \times CP_2$, leads to the realization of the mass squared operator essentially as the Virasoro generator L_0 in certain representations of the Super Virasoro algebra constructed using the representations of various Kac Moody algebras associated with Lorentz group, electro-weak group and color group.

$-L_0$ takes thus the role of a Hamiltonian in the partition function:

$$exp(-H/T) \rightarrow p^{L_0/T_p} \ ,$$

where T_p is the p-adic temperature, which by number theoretic reasons is quantized to $1/T_p = n$, n a positive integer. Mass squared is essentially the thermal expectation of L_0. The real mass squared is the real counterpart of the p-adic mass squared in the canonical identification $x = \sum x_n p^n \rightarrow \sum x_n p^{-n} \equiv x_R$ mapping p-adics to reals. Assuming that elementary particles correspond to p-adic primes near prime powers of two, one obtains excellent predictions, not only for the mass scales of elementary particles but also for the particle mass ratios. For instance, electron corresponds to the Mersenne prime $M_{127} = 2^{127} - 1$.

It should be noticed that the real counterpart of the p-adic inverse temperature $1/T_p$ is naturally defined as

$$(\frac{1}{T_p})_r = (\frac{1}{T_p})_R log(p) \ ,$$

where $log(p)$ factor results from the definition of Boltzmann weights as powers of p rather than power of e. The real counterpart T_r of T_p can be identified as

$$T_r = \frac{1}{nlog(p)} \ . \qquad (6.6.1)$$

One might wonder about whether the sign of T_p should be taken as negative since positive exponent of L^0 appears in the Boltzmann weights. The sign is correct; for the opposite sign T_r would be in good approximation equal to $\frac{1}{(p-n)log(p)}$, which is not consistent with the fact that physically temperature decreases when n increases.

As already explained, the new vision about p-adics and cognition forces to modify this early vision by interpreting CP_2 type extremals as cognitive representations of elementary particles rather than genuine elementary particles.

p-Adic mass squared

The thermal expectation of the p-adic mass squared operator is proportional to the thermal expectation of the Virasoro generator L_0:

$$\begin{aligned} M_p^2 &= k\langle L_0\rangle \ , \\ k &= 1 \ . \end{aligned} \qquad (6.6.1)$$

The correct choice for the value of the rational number k is $k = 1$ as became clear in the recent reconstruction of the quantum TGD [K39] .

The real mass squared M^2 is identified as

$$\begin{aligned} M^2 &= \frac{M_R^2 \pi^2}{l^2} \ , \\ l &\simeq 1.376 \cdot 10^4 \sqrt{G} \ , \end{aligned} \qquad (6.6.1)$$

where l is the fundamental p-adic length scale and M_R^2 is the real counterpart of M_p^2 in the canonical identification. $\sqrt{G}$ is Planck length scale.

p-Adic entropy is proportional to p-adic mass squared

The definition of the p-adic entropy involves some number theory. The general definition

$$S = -p_n log(p_n) \ ,$$

in terms of the probabilities p_n of various states does not work as such since the e-based logarithm $log(p_n)$ does not exist p-adically. Since p-adic Boltzmann weights are integer powers of p it is natural to modify somehow the p-based logarithm $log_p(x)$ so that the resulting logarithm $Log_p(x)$ exists for any p-adic number and has the basic property

$$Log_p(xy) = Log_p(x) + Log_p(y) \ ,$$

guaranteeing the additivity of the p-adic entropy for non-interacting systems. The definition satisfying these constraints is

$$Log_p(x = \sum_{n \geq n_0} x_n p^n) \equiv n_0 \ . \tag{6.6.2}$$

The lowest power in the expansion of x in powers of p fixes the value of the logarithm in the same way as it determines also the norm of the p-adic number. This leads to the definition of p-adic entropy as

$$S_p = -\sum_p p_n Log_p(p_n) \ . \tag{6.6.3}$$

In p-adic thermodynamics the p-adic probabilities have the general form

$$p_n = \frac{p^{L_0(n)/T_p}}{Z} \ .$$

Here $L_0(n)$ denotes the eigenvalue of the Virasoro generator L_0, which is integer. The partition function $Z = trace(p^{L_0/T_p})$ has unit p-adic norm if the ground state is massless, so that its p-adic logarithm vanishes in this case: $Log_p(Z) = 0$. This implies $Log_p(p_n) = Log_p(p^{L_0(n)/T_p}) = L_0(n)/T_p$ so that the p-adic entropy reduces to

$$S_p = \frac{1}{T_p}\langle L_0 \rangle \ , \tag{6.6.4}$$

ane hence that the p-adic mass squared and p-adic entropy are proportional to each other

$$S_p = -\frac{1}{kT_p}M_p^2 \ . \tag{6.6.5}$$

By noticing that the entropy for Schwartschild black hole is given by

$$S = 4\pi G M^2 \ , \tag{6.6.6}$$

one finds that in the p-adic context the analog of the Hawking-Bekenstein formula indeed holds as an identity.

The proposed identification of the entropy is in accordance with the formula $dE = TdS$. In the p-adic context E should clearly be replaced by $\langle -L_0 \rangle$ and T by T_p. The differentials do not however make sense since the thermodynamical quantities are now discrete. Since only $\langle -L_0 \rangle$ and T_p appear as variables one could define

$$\langle -L_0 \rangle = T_p S_p \ .$$

This definition gives $S_p = -\frac{1}{kT_p}M_p^2$ and is in accordance with the standard definition of the Shannon entropy. The definition for the real counterpart of the p-adic entropy is

$$S = log(p) S_R \ .$$

The inclusion of $log(p)$-factor maximizes the resemblance with the usual Shannon entropy defined in terms of the e-based logarithm and makes it possible to compare the real counterpart of entropy with other kind of entropies.

The real counterparts of entropy and mass squared are not linearly related

Due to the delicacies related to the canonical identification, the real counterparts of entropy and mass squared differ drastically from each other and there is no simple relationship between the two quantities. The reason is that the vacuum expectation of $-L_0$ is of order $-np$ for particles having $T_p = 1$ and, essentially due to the presence of minus sign, one has $S_R(p) = 1$ in an excellent approximation, whereas the real counterpart of M_p^2 is of order n/p. For photon and other (nearly) massless bosons the entropy vanishes or is very small.

The fundamental difference in the thermal properties of fermions and massless bosons should have observable consequences. For instance, the annihilation of fermion-anti-fermion pair to massless particles means a considerable reduction of the p-adic entropy and would not be a favorable process thermodynamically. Thus the second law of thermodynamics would favor the presence of net fermion and anti-fermion number densities. For instance, fermions and anti-fermions could suffer a topological condensation on different space-time sheets to avoid annihilation during early cosmology or anti-fermions could even suffer topological evaporation as suggested in [K30, K30] . This in turn would lead to the generation of matter-antimatter asymmetry. It should be noticed that lare entropies are in accordance with the second law of thermodynamics.

Hawking-Bekenstein area formula in elementary particle context

Hawking-Bekenstein formula in the p-adic form $S_p \propto M_p^2$ holds true on basis of the previous considerations although ther are no hopes of deriving the area law from the first principles at this stage. Hawking-Bekenstein formula can be also written in the form

$$S = \frac{A}{4G} \quad ,$$

relating black hole entropy to the area of the black hole horizon. One might hope that in the real context a generalization of the area law to the form

$$S = x\frac{A}{4L^2} \quad ,$$

where L is some fundamental length scale analogous to the gravitational constant G and x is some numerical constant near unity, would hold true. Since the size of CP_2 defines the fundamental p-adic length scale and replaces $\sqrt{G}$ as a fundamental length scale in TGD, it is conceivable that L is of the order of the CP_2 size $l \sim 10^4\sqrt{G}$. The area in question would be most naturally the area of the elementary particle horizon, where the signature of the induced metric for the topologically condensed CP_2 type extremal changes from Euclidian to Minkowskian. It is well known that l is also the length scale at which the couplings of the standard model become identical and supersymmetry is expected to become manifest. This is what is expected since above cm energy $1/l$ one would have an Euclidian quantum field theory in CP_2.

The radius r of the elementary particle horizon is of order

$$r \simeq \sqrt{log(p)}L \quad . \tag{6.6.7}$$

This means that the # contacts connecting the CP_2 type extremal to the background space-time are surrounded by an Euclidian region with a size of order L.

It is interesting to look for the detailed form of the Hawking-Bekenstein law for elementary particles. One obtains the following general relationship

$$\begin{aligned} S &\equiv log(p)S_R = log(p)(\langle\frac{-L_0}{T_p}\rangle)_R == Xlog(p)M_R^2 = X \times log(p)\frac{l^2}{\pi^2}M^2 \quad , \\ X &\equiv \frac{M_R^2}{S_R} \quad . \end{aligned} \tag{6.6.7}$$

For massive particles $X \sim p$ holds true. Hence the entropy is related by a factor $p \cdot 10^8$ to the corresponding black hole entropy:

$$\begin{aligned} S &= a^2 S_{BH} \ , \\ S_{BH} &= 4\pi G M^2 \\ a &= \sqrt{\frac{log(p)X}{4\pi^3}}\frac{l}{\sqrt{G}} \sim 10^4 \ , \\ l &\simeq 1.376 \cdot 10^4 \sqrt{G} \ . \end{aligned} \tag{6.6.5}$$

6.6.4 p-Adic length scale hypothesis and p-adic thermodynamics

The basic assumption of p-adic mass calculations is that physically interesting p-adic primes correspond to prime powers of two:

$$p \simeq 2^k \ , \ k \text{ prime} \ .$$

There are several arguments in favor of this hypothesis but no really convincing argument. The area law however leads to a very attractive, if not even convincing, explanation of the p-adic length scale hypothesis.

The proportionality of the elementary particle horizon radius to $\sqrt{log(p)}$ suggests quite attractive partial explanation for the p-adic length scale hypothesis. The point is that for $p \simeq 2^k$, k prime one has $r \propto L(k)$. Thus, if the numerical constant x is chosen suitably, it is possible to obtain very precisely

$$r(p \simeq 2^k) = L(k) \ .$$

The reason is that the p-adic entropy is in thermal equilibrium very near to its maximum value. The required value of the coefficient x is

$$x = \frac{log(2)}{\pi} \ . \tag{6.6.6}$$

The requirement that r_F (r_B) is as near as possible to the appropriate p-adic length scale $L(k)$ ($L(k)\sqrt{p}$) fixes also the precise value of the p-adic prime $p \simeq 2^k$.

This hypothesis means that the area of the elementary particle horizon is quantized in the multiples of prime k:

$$A = kA_1 \ . \tag{6.6.7}$$

The quantization law for the area has been proposed also in the context of the non-perturbative quantum gravity. A suggestive possibility is that physics is k-adic below the elementary particle horizon and $p \simeq 2^k$-adic above it. The appearance of an additional k-adic length scale suggests that for $p \simeq 2^k$ the degeneracy of the effective space-time surfaces is especially large due to the additional k-adic degeneracy and that the p-adic scattering amplitudes are be especially large for this reason. Hence the favored p-adic primes would emerge purely dynamically.

It must be noticed that k-adic fractality allows also more general primes of type $p \simeq 2^{k^n}$, where k is prime and n is integer. For these primes the radius of the elementary particle horizon is $\sqrt{k^{n-1}}L(k)$ and hence also a natural k-adic length scale. There are very few physically interesting length scales of this type. As the p-adic mass calculations show, the best fit to the neutrino mass squared differences is obtained for $p_\nu \simeq 2^{13^2=169}$ rather than $p \simeq 2^{167}$. The length scale $L(p_\nu)$ is also the natural length scale associated with the double cell layers appearing very frequently in bio-systems ($k = 167$ corresponds to the typical size of a cell)!

6.6.5 Black hole entropy as elementary particle entropy?

In TGD Schwartshild metric does not allow a global imbedding as a surface in $M^4_+ \times CP_2$. One can however find imbeddings, which extend also below the Schwartshild radius. This suggests that particles in the interior of the black hole are topologically condensed below the radius r_s. The problem is whether the single particle entropies are additive as real numbers or as p-adic numbers.

Additivity of real entropies?

Consider first the additivity as real numbers. With this assumption the sum for the real counterparts of the p-adic entropies of various particles gives a lower bound for the black hole entropy:

$$S = \sum_i S(i) = \sum_i km_i^2 \ .$$

This entropy is by a factor is $10^8 \cdot p$ larger than the corresponding black hole entropy so that black hole-elementary particle analogy does not work at quantitative level. For sufficiently large particle numbers elementary particle entropy becomes smaller than the black hole entropy, which behaves as $(\sum m_i)^2$. In case of protons $p = M_{107} = 2^{107} - 1$ the critical value of N would be roughly $N \sim 10^{32}$, which would mean black hole with a mass of order 100 kilograms.

Additivity of the p-adic entropies?

One can consider also a different definition of the black hole entropy. In p-adic thermodynamics the natural additive quantity for many particle systems is the Virasoro generator L_0 (mass squared essentially) rather than energy. The additivity works quite nicely for the TGD based model of a hadron as a bound state of quarks. Therefore one could consider the possibility that also for black holes the mass squared of elementary particles with same value of p-adic prime p is p-adically additive

$$(m_p^2)_R = (\sum_i m_p^2(i))_R \ \text{rather than} \ m = \sum m_i \ .$$

Therefore for a black hole containing only particles with single value of the p-adic prime p, the Hawking-Bekenstein formula in the form

$$S_p \propto M_p^2$$

would hold true. For the real counterparts this proportionality does not hold.

When the particle number N exceeds p/n, the mass squared of the system reduces from its upper bound $10^{-4}/\sqrt{G}$ by a factor of order $1/\sqrt{p}$. Thus the mass of, say, the electrons inside black hole, is always below this upper bound irrespective of the number of the electrons!

If particles with several p-adic primes are present inside the black hole then the formula for the black hole entropy reads as

$$S = \sum_p S(p) = \sum_p k(p)M^2(p) \ ,$$

so that the proportionality to the total mass squared does not hold true except approximately (in the case that the mass is in good approximation given by the total mass of a particular particle species).

6.6.6 Why primes near prime powers of two?

The great challenge of TGD is to predict the p-adic prime associated with a given elementary particle. The problem decomposes into the following subproblems.

1. One must understand why there is a definite value of the p-adic prime associated with a given real region of space-time surface (in particular, the space-time time surface describing elementary particle) and how this prime is determined. The new view about p-adicity allows to understand the possibility to label elementary particles by p-adic primes if p-adic–real phase transitions occur already at elementary particle level or if real elementary particle regions are accompanied by p-adic space-time sheets possible providing some kind of a cognitive model of particle. The great question mark is the correlation of the p-adic prime characterizing the particle with the quantum numbers of the particle: is this correlation due to the intrinsic properties of the particle or perhaps a result of some kind of adaptation at elementary particle length scales. In the latter case sub-cosmologies with quite different elementary particle mass spectra are possible. On the other and, quantum self-organization does not allow too

many final state patterns, so that elementary particle mass spectrum could be more or less a constant of Nature.

2. One must understand why quantum evolution by quantum jumps has led to a situation in which elementary particle like surfaces correspond to some preferred primes. It indeed seems that an evolution at elementary particle level is in question (how p-adic evolution follows from simple number theoretic consistency conditions is discussed in the [K29] . It seems that the degeneracy due to the p-adic space-time regions associated with the system must be counted as giving rise to different final states in a quantum jump between quantum histories. If the number $N_d(X^3)$ of the physically equivalent cognitive variants of the space-time surface is especially high, this particular physical state dominates over the other final states of the quantum jump. Highly cognitive systems are winners in the fight for survival. Thus in TGD framework evolution is also, and perhaps basically, evolution of cognition.

3. One should also understand why the primes $p \simeq 2^k$ near prime powers of two are favored physically and to predict the value of k for an elementary particle with given quantum numbers. The analogy between elementary particles and black holes suggests only a partial explanation for the prime powers of 2 and the real explanation should probably involve enhanced cognitive resources for these primes.

In order to formulate the argument supporting p-adic length scale hypothesis one must first describe the general conceptual background.

1. WCW of the 3-surfaces decomposes into regions D_P labelled by infinite p-adic primes. In each quantum jump localization of CH spinor field to single sector D_P must occur if localization in zero modes occurs. Quantum time development corresponds to a sequence of quantum jumps between quantum histories and the value of the infinite-p p-adic prime P characterizing the 3-surface associated with the entire universe increases in a statistical sense. This has natural interpretation as evolution. In a well defined sense the infinite prime characterizing infinitely large universe is a composite of finite p-adic primes characterizing various real regions (space-time sheets) of the space-time. The effective infinite-p p-adic topology associated with this infinite prime is very much like real topology since canonical identification mapping infinite number to its real counterpart just drops the infinitesimals of infinite-p p-adic number. Therefore real physics is an excellent approximation at this level. If the S-matrix is complex rational, the approximation is in fact exact. Note that real topology is quite possible also at the level of WCW and WCW might consist of both real and infinite-P p-adic regions.

2. The requirement that quantum jumps correspond to quantum measurements in the sense of QFT, implies that also localization in zero modes occurs in each quantum jump: localization could occur also in the length scale resolution defined by the p-adic length scale L_p. The strongest hypothesis suggested by the properties of thermodynamical spin glasses is that quantum jump occurs to a state localized around single maximum of the Kähler function.

3. This picture suggests that evolution has occurred already at the elementary particle level and selected preferred p-adic primes characterizing the space-time regions associated with the elementary particles. A crucial question is whether this evolution could have occurred for isolated elementary particles or whether the interaction of the elementary like space-time regions with the surrounding space-time has served as a selective pressure. It might well be that the latter option is the correct one. If this is the case, one can say that the winners in the fight for survival correspond to infinite primes, which are composites of preferred finite primes, perhaps the finite primes given by the p-adic length scale hypothesis.

4. In TGD framework evolution is also evolution of cognition and the most plausible guess is that p-adic non-determinism is what makes cognition possible. Of course, also the classical non-determinism of Kähler action is also present and also important. Perhaps one should call the space-time sheets of finite time duration made possible by this non-determinism as 'sensory space-time sheets' as opposed to p-adic space-time sheets. Certainly this non-determinism should be responsible for volition. In any case, the degenerate space-time sheets

are not physically equivalent in this case as they are in case of the p-adic non-determinism. The number $N_d(X^3)$ of the p-adically degenerate and physically equivalent absolute minima $X^4(X^3)$ of Kähler action is the measure for the cognitive resources of the 3-surface. The basic idea is simple: if $N_d(X^3)$ is very large then quantum jumps lead with high probability to some degenerate physically equivalent maximum of the Kähler function associated with given value of p. One can see this also from the point of view of an elementary particle: the high cognitive degeneracy plus the possibility of p-adic–real phase transitions mean that the particle can adapt to the environment: the surviving elementary particles would be the most intelligent ones! What one should be able to show is that cognitive degeneracy is especially large for some preferred primes so that evolution selects these primes as the most intelligent ones.

In this conceptual framework one can develop more precise variants for arguments supporting the p-adic length scales hypothesis.

1. The simplest possibility is that single maximum of Kähler function is selected in the quantum jump. In this case the relative rate for quantum jumps to a given physical final state with fixed physical configuration is proportional to the p-adic cognitive degeneracy $N_d(N)$, where N denotes the infinite primes characterizing the interacting space-time surface associated with the final state. N decomposes into a product of infinite primes p and $N_d(N)$ decomposes decomposes into a product $N = \prod_P N_d(P)$ $N_d(N)$ is maximized if $N_d(P)$ is maximzes. The elementary systems for which $N_d(P)$ is especially large are winners.

2. The situation reduces to the level of finite p-adic primes if takes seriously the argument allowing to estimate the value of the gravitational constant. The argument was based on the assumption that P decomposes in a well defined sense into passive primes p_i and active prime p characterizing elementary particle: thus there would be the correspondence $P \leftrightarrow p$. This suggests that it is possible to understand the finite p-adic prime p associated with the elementary particle by restricting the consideration to the 3-surfaces describing topologically condensed elementary particles: that is, CP_2 type extremals glued to a space-time sheet with size of order Compton length. p-Adic cognitive degeneracy $N_d(p)$ should be especially high for p-adic primes predicted by the p-adic length scale hypothesis.

3. The interpretation of p-adic regions as cognitive regions suggests a more concrete explanation for the p-adic length scale hypothesis. The degeneracy due to p-adic non-determinism for the p-adic CP_2 type extremals presumably depends on the value of the p-adic prime characterizing the cognitive version of elementary particle. If p-adic–real phase transitions representing transformation of thought-to-action and viceversa are possible for CP_2 type extremals, one could understand the origin of the p-adic length scale hypothesis. p-Adic primes near prime powers of two are winners because the the degeneracy due to p-adic non-determinism is especially larger for them. The observed elementary particles would thus dominate in the Universe simply because the thoughts about them are winners in the fight for survival.

4. The black hole-elementary particle analogy suggests that the primes $p \simeq 2^k$, k prime, are especially interesting since the radius of the elementary particle horizon is the p-adic length scale $L(k)$. This could be understood since k-adicity provides an additional cognitive degeneracy for the absolute minima of Kähler function coming from the region of size $L(k)$ surrounding a topologically condensed elementary particle and any # contact. This enhances the value of $N_d(p)$ further by a multiplicative factor $N_d(k)$ so that $N_d(P)$ becomes especially large.

5. These arguments do not yet tell how to deduce the prime k associated with a given elementary particle. Cognitive resources are measured by a negative on an negentropy type quantity proportional to $N_c = log(N_d(p))$. A natural guess is that N_c is dominated by a term proportional to $log(p)$: $N_c = A(p) + log(p)$. For $p \simeq 2^k$ one has an additional source of cognitive degeneracy which gives $N_c = log(k) + log(p)$ instead of $N_c = log(p)$ and these primes thus correspond to the local maxima of cognitive resources as a function of p. Quite generally, the larger the p, the more probable is its appearance as elementary particle prime (neglecting the constraints coming from, say, the cosmic temperature). Hence it seems that the p-adic

evolution of a given elementary particle is frozen to some local maximum of $N_d(p(k))$, with $p(k)$ given by the p-adic length scale hypothesis.

6. Freezing can be understood if the transition probabilities $P(k \to k_1)$ are so small that further evolution by quantum jumps is impossible. A possible interpretation of the transition $k_i \to k_j$ is a p-adic phase transition changing the elementary particle horizon from radius L_{k_i} to L_{k_j} so that $P(k_i \to k_j)$ would describe the probability of this phase transition. For neutrinos the transition probabilities $P(k_i \to k_j)$ between different sectors allowed by the p-adic length scale hypothesis seem to be largest whereas for higher quark generations they seem to be smallest. Furthermore, k is smaller for higher generations. In particular, $P(k_i \to k_j)$ seems to be largest for spherical boundary topology. This suggests that the (phase) transition probabilities $P(k_i \to k_j)$ decrease as a function of the strength of the dominating particle interaction and of the genus of the particle (reflecting itself via the modular contribution to the particle mass increasing as a function of genus).

To sum up, the proposed formula would dictate the evolution of α_s from the evolution of the electro-weak parameters without any need for perturbative computations. Although the formula of proposed kind is encouraged by the strong constraints between classical gauge fields in TGD framework, it should be deduced in a rigorous manner from the basic assumptions of TGD before it can be taken seriously.

6.7 General vision about coupling constant evolution

Zero energy ontology, the construction of M-matrix as time like entanglement coefficients defining Connes tensor product characterizing finite measurement resolution in terms of inclusion of hyper-finite factors of type II_1, the realization that symplectic invariance of N-point functions provides a detailed mechanism eliminating UV divergences, and the understanding of the relationship between super-symplectic and super Kac-Moody symmetries: these are the pieces of the puzzle whose combination making possible a rather concrete vision about coupling constant evolution in TGD Universe and even a rudimentary form of generalized Feynman rules.

p-adic coupling constant evolution is discrete by p-adic length scale hypothesis justified by zero energy ontology. Discreteness means that continuous mass scale is replaced by mass scales coming as half octaves of CP_2 mass. One key question has been whether it is Kähler coupling strength α_K or gravitational coupling constant, which remains invariant under p-adic coupling constant evolution. Second problem relates to the value of α_K.

The realization that modified Dirac action could be the fundamental variational principle initiated the process, which led to an answer to these and many other questions. The idea that some kind of Dirac determinant gives the vacuum functional identifiable as exponent of Kähler function in turn identifiable as Kähler action S_K for a preferred extremal came first. The basic challenges are to understand the conditions fixing the preferred extremal of Kähler action and how to define the Dirac determinant. After experimentation with several alternatives it became clear that the modified Dirac action contains besides the term defined by Kähler action also a measurement interaction term guaranteeing quantum classical correspondence. An alternative idea inspired by TGD as almost topological QFT vision and quantum holography was that 3-D Chern-Simons action for light-like 3-surfaces at which the induced metric of the space-time surface changes its signature could be enough. This turned out be to not the case.

The most important outcome is a formula for Kähler coupling strength in terms of a calculable and manifestly finite Dirac determinant without any need for zeta function regularization. The formula fixes completely the number theoretic anatomy of Kähler coupling strength and of other gauge coupling strengths. When the formula for the gravitational constant involving Kähler coupling strength and the exponent of Kähler action for CP_2 type vacuum extremal - which remains still a conjecture - is combined with the number theoretical results and with the constraints from the predictions of p-adic mass calculations, one ends up to an identification of Kähler coupling strength as fine structure constant at electron length scale characterized by p-adic prime M_{127}. Also the number theoretic anatomy of the ratio $R^2/\hbar G$, where R is CP_2 size, can be understood to high degree and a relationship between the p-adic evolutions of electromagnetic and color coupling strengths emerges.

6.7.1 General ideas about coupling constant evolution

Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the boundaries of future and past direct light-cones forming a causal diamond. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. "Any physical state is creatable from vacuum" becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe, Is theory building completely useless if only single solution of field equations is realized?).

At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events. The proposed realization of Equivelence Principle at quantum level is based on the identification of classical Noether charges in Cartan algebra with the eigenvalues of their quantum counterparts assignable to Kähler-Dirac action. At classical level EP follows at GRT limit obtained by lumping many-sheeted space-time to M^4 with effective metric satisfying Einstein's equations as a reflection of the underlying Poincare invariance.

Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [K18] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra spanned by the gamma matrices of the "world of classical worlds" represents a von Neumann algebra [A132] known as hyperfinite factor of type II_1 (HFF) [K18, K80, K25] . HFF [A128, A171] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [A9] . The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [A212] , anyons [D3] , quantum groups and conformal field theories [A172] , and knots and topological quantum field theories [A201, A217] .

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy states are associated with causal diamond formed by a pair of future and past directed light-cones having positive and negative energy parts of state at their boundaries. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

On can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with M-matrix.

The temporal distance between the tips of light-cones corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where R is CP_2 size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$ since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is .1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [K18] . The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics.

How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

Zero energy ontology in which zero energy states have as imbedding space correlates causal diamonds for which the distance between the tips of future and past directed light-cones are power of 2 multiples of fundamental time scale ($T_n = 2^n T_0$) implies in a natural manner coupling constant evolution. A weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section $X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around .1 eV would correspond to $L(169) \simeq 5$ μm (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, p would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of WCW.

6.7.2 The bosonic action defining Kähler function as the effective action associated with the induced spinor fields

One could *define* the classical action defining Kähler function as the bosonic action giving rise to the divergences of the isometry currents. In this manner bosonic action, especially the value of the Kähler coupling strength, would come out as prediction of the theory containing no free parameters.

Thus the Kähler action S_B of preferred extremal of Käction defining Kähler function could be *defined* by the functional integral over the Grassmann variables for the exponent of the massless Dirac action. Formally the functional integral is defined as

$$\begin{aligned} exp(S_B(X^4)) &= \int exp(S_F) D\Psi D\bar{\Psi} \ , \\ S_F &= \bar{\Psi}\left[\hat{\Gamma}^\alpha D^{\rightarrow}_\alpha - D^{\leftarrow}_\alpha \hat{\Gamma}^\alpha\right]\Psi\sqrt{g} \ . \end{aligned} \tag{6.7.-1}$$

Formally the bosonic effective action is expressible as a logarithm of the fermionic functional determinant resulting from the functional integral over the Grassmann variables

$$\begin{aligned} S_B(X^4) &= log(det(D)) \ , \\ D &= \hat{\Gamma}^\alpha D^{\rightarrow}_\alpha \ . \end{aligned} \tag{6.7.-1}$$

Formula for the Kähler coupling strength

The identification of exponent of Kähler function as Dirac determinant leads to a formula relating Kähler action for the preferred extremal to the Dirac determinant. The eigenvalues are proportional to $1/\alpha_K$ since the matrices $\hat{\Gamma}^\alpha$ have this proportionality. This gives the formula

$$exp(\frac{S_K(X^4(X^3))}{8\pi\alpha_K}) = \prod_i \lambda_i = \frac{\prod_i \lambda_{0,i}}{\alpha_K^N} \ . \tag{6.7.0}$$

Here $\lambda_{0,i}$ corresponds to $\alpha_K = 1$. $S_K = \int J^* J$ is the reduced Kähler action.

For $S_K = 0$, which might correspond to so called massless extremals [K10] one obtains the formula

$$\alpha_K = (\prod_i \lambda_{0,i})^{1/N} \ . \tag{6.7.1}$$

Thus for $S_K = 0$ extremals one has an explicit formula for α_K having interpretation as the geometric mean of the eigenvalues $\lambda_{0,i}$. Several values of α_K are in principle possible.

p-Adicization suggests that $\lambda_{0,i}$ are rational or at most algebraic numbers. This would mean that α_K is N:th root of this kind of number. S_K in turn would be

$$S_K = 8\pi\alpha_K log(\frac{\prod_i \lambda_{0,i}}{\alpha_K^N}) \ . \tag{6.7.2}$$

so that S_K would be expressible as a product of the transcendental π, N:th root of rational, and logarithm of rational. This result would provide a general answer to the question about number theoretical anatomy of Kähler coupling strength and S_K. Note that S_K makes sense p-adically only if one adds π and its all powers to the extension of p-adic numbers. The exponent of Kähler function however makes sense also p-adically.

6.7.3 A revised view about coupling constant evolution

The development of the ideas related to number theoretic aspects has been rather tortuous and based on guess work since basic theory has been lacking.

1. The original hypothesis was that Kähler coupling strength is invariant under p-adic coupling constant evolution. Later I gave up this hypothesis and replaced it with the invariance of gravitational coupling since otherwise the prediction would have been that gravitational coupling strength is proportional to p-adic length scale squared. Second first guess was that Kähler coupling strength equals to the value of fine structure constant at electron length scale corresponding to Mersenne prime M_{127}. Later I replaced fine structure constant with electro-weak U(1) coupling strength at this length scale. The recent discussion returns back to the roots in both aspects.

2. The recent discussion relies on the progress made in the understanding of quantum TGD at partonic level [K15] . What comes out is an explicit formula for Kähler couplings strength in terms of Dirac determinant involving only a finite number of eigenvalues of the modified Dirac operator. This formula dictates the number theoretical anatomy of g_K^2 and also of other coupling constants: the most general option is that α_K is a root of rational. The requirement that the rationals involved are simple combined with simple experimental inputs leads to very powerful predictions for the coupling parameters.

3. A further simplification is due to the discreteness of p-adic coupling constant evolution allowing to consider only length scales coming as powers of $\sqrt{2}$. This kind of discretization is necessary also number theoretically since logarithms can be replaced with 2-adic logarithms for powers of 2 giving integers. This raises the question whether $p \simeq 2^k$ should be replaced with 2^k in all formulas as the recent view about quantum TGD suggests.

4. The prediction is that Kähler coupling strength α_K is invariant under p-adic coupling constant evolution and from the constraint coming from electron and top quark masses very near to fine structure constant so that the identification as fine structure constant is natural. Gravitational constant is predicted to be proportional to p-adic length scale squared and corresponds to the largest Mersenne prime (M_{127}), which does not correspond to a completely super-astronomical p-adic length scale. For the parameter R^2/G p-adicization program allows to consider two options: either this constant is of form e^q or 2^q: in both cases q is rational number. $R^2/G = exp(q)$ allows only M_{127} gravitons if number theory is taken completely seriously. $R^2/G = 2^q$ allows all p-adic length scales for gravitons and thus both strong and weak variants of ordinary gravitation.

5. A relationship between electromagnetic and color coupling constant evolutions based on the formula $1/\alpha_{em} + 1/\alpha_s = 1/\alpha_K$ is suggested by the induced gauge field concept, and would mean that the otherwise hard-to-calculate evolution of color coupling strength is fixed completely. The predicted value of α_s at intermediate boson length scale is correct.

It seems fair to conclude that the attempts to understand the implications of p-adicization for coupling constant evolution have begun to bear fruits.

Identifications of Kähler coupling strength and gravitational coupling strength

To construct an expression for gravitational constant one can use the following ingredients.

1. The exponent $exp(2S_K(CP_2))$ defining the value of Kähler function in terms of the Kähler action $S_K(CP_2)$ of CP_2 type extremal representing elementary particle expressible as

$$S_K(CP_2) = \frac{S_{K,R}(CP_2)}{8\pi\alpha_K} = \frac{\pi}{8\alpha_K} \ . \tag{6.7.3}$$

Since CP_2 type extremals suffer topological condensation, one expects that the action is modified:

$$S_K(CP_2) \rightarrow a \times S_K(CP_2) \ . \tag{6.7.4}$$

$a < 1$ conforms with the idea that a piece of CP_2 type extremal defining a wormhole contact is in question. One must however keep mind open in this respect.

2. The p-adic length scale L_p assignable to the space-time sheet along which gravitational interactions are mediated. Since Mersenne primes seem to characterized elementary bosons and since the Mersenne prime $M_{127} = 2^{127}-1$ defining electron length scale is the largest non-super-astronomical length scale it is natural to guess that M_{127} characterizes these space-time sheets.

1. The formula for the gravitational constant

A long standing basic conjecture has been that gravitational constant satisfies the following formula

$$\begin{aligned} \hbar G &\equiv r\hbar_0 G = L_p^2 \times exp(-2aS_K(CP_2)) \ , \\ L_p &= \sqrt{p}R \ . \end{aligned} \tag{6.7.4}$$

Here R is CP_2 radius defined by the length $2\pi R$ of the geodesic circle. What was noticed before is that this relationship allows even constant value of G if a has appropriate dependence on p.

This formula seems to be correct but the argument leading to it was based on two erratic assumptions compensating each other.

1. I assumed that modulus squared for vacuum functional is in question: hence the factor $2a$ in the exponent. The interpretation of zero energy state as a generalized Feynman diagram requires the use of vacuum functional so that the replacement $2a \rightarrow a$ is necessary.

2. Second wrong assumption was that graviton corresponds to CP_2 type vacuum extremal- that is wormhole contact in the recent picture. This does allow graviton to have spin 2. Rather, two wormhole contacts represented by CP_2 vacuum extremals and connected by fluxes associated with various charges at their throats are needed so that graviton is string like object. This saves the factor $2a$ in the exponent.

The highly non-trivial implication to be discussed later is that ordinary coupling constant strengths should be proportional to $exp(-aS_K(CP_2))$.

The basic constraint to the coupling constant evolution comes for the invariance of g_K^2 in p-adic coupling constant evolution:

$$\begin{aligned} g_K^2 &= \frac{a(p,r)\pi^2}{log(pK)} \ , \\ K &= \frac{R^2}{\hbar G(p)} = \frac{1}{r}\frac{R^2}{\hbar_0 G(p)} \equiv \frac{K_0(p)}{r} \ . \end{aligned} \tag{6.7.4}$$

2. How to guarantee that g_K^2 is RG invariant and N:th root of rational?

Suppose that g_K^2 is N:th root of rational number and invariant under p-adic coupling constant evolution.

1. The most general manner to guarantee the expressibility of g_K^2 as N:th root of rational is guaranteed for both options by the condition

$$a(p,r) = \frac{g_K^2}{\pi^2} log(\frac{pK_0}{r}) \ . \tag{6.7.5}$$

That a would depend logarithmically on p and $r = \hbar/\hbar_0$ looks rather natural. Even the invariance of G under p-adic coupling constant evolution can be considered.

2. The condition

$$\frac{r}{p} < K_0(p) \ . \tag{6.7.6}$$

must hold true to guarantee the condition $a > 0$. Since the value of gravitational Planck constant is very large, also the value of corresponding p-adic prime must very large to guarantee this condition. The condition $a < 1$ is guaranteed by the condition

$$\frac{r}{p} > exp(-\frac{\pi^2}{g_K^2}) \times K_0(p) \ . \tag{6.7.7}$$

The condition implies that for very large values of p the value of Planck constant must be larger than $\hbar_0$.

3. The two conditions are summarized by the formula

$$K_0(p) \times exp(-\frac{\pi^2}{g_K^2}) < \frac{r}{p} < K_0(p) \tag{6.7.8}$$

characterizing the allowed interval for r/p. If G does not depend on p, the minimum value for r/p is constant. The factor $exp(-\frac{\pi^2}{g_K^2})$ equals to 1.8×10^{-47} for $\alpha_K = \alpha_{em}$ so that $r > 1$ is required for $p \geq 4.2 \times 10^{-40}$. $M_{127} \sim 10^{38}$ is near the upper bound for p allowing $r = 1$. The constraint on r would be roughly $r \geq 2^{k-131}$ and $p \simeq 2^{131}$ is the first p-adic prime for which $\hbar > 1$ is necessarily. The corresponding p-adic length scale is .1 Angstroms.

This conclusion need not apply to elementary particles such as neutrinos but only to the space-time sheets mediating gravitational interaction so that in the minimal scenario it would be gravitons which must become dark above this scale. This would bring a new aspect to vision about the role of gravitation in quantum biology and consciousness.

The upper bound for r behaves roughly as $r < 2.3 \times 10^7 p$. This condition becomes relevant for gravitational Planck constant GM_1M_2/v_0 having gigantic values. For Earth-Sun system and for $v_0 = 2^{-11}$ the condition gives the rough estimate $p > 6 \times 10^{63}$. The corresponding p-adic length scale would be of around $L(215) \sim 40$ meters.

4. p-Adic mass calculations predict the mass of electron as $m_e^2 = (5 + Y_e)2^{-127}/R^2$ where $Y_e \in [0, 1)$ parameterizes the not completely known second order contribution. Top quark mass favors a small value of Y_e (the original experimental estimates for m_t were above the range allowed by TGD but the recent estimates are consistent with small value Y_e [K49]). The range $[0, 1)$ for Y_e restricts $K_0 = R^2/\hbar_0 G$ to the range $[2.3683, 2.5262] \times 10^7$.

5. The best value for the inverse of the fine structure constant is $1/\alpha_{em} = 137.035999070(98)$ and would correspond to $1/g_K^2 = 10.9050$ and to the range $(0.9757, 0.9763)$ for a for $\hbar = \hbar_0$ and $p = M_{127}$. Hence one can seriously consider the possibility that $\alpha_K = \alpha_{em}(M_{127}$ holds true. As a matter fact, this was the original hypothesis but was replaced later with the hypothesis that α_K corresponds to electro-weak $U(1)$ coupling strength in this length scale. The fact that M_{127} defines the largest Mersenne prime, which does not correspond to super-astrophysical length scale might relate to this co-incidence.

To sum up, the recent view about coupling constant evolution differs strongly from previous much more speculative scenarios. It implies that g_K^2 is root of rational number, possibly even rational, and can be assumed to be equal to e^2. Also $R^2/\hbar G$ could be rational. The new element is that G need not be proportional to p and can be even invariant under coupling constant evolution since the the parameter a can depend on both p and r. An unexpected constraint relating p and r for space-time sheets mediating gravitation emerges.

Are the color and electromagnetic coupling constant evolutions related?

Classical theory should be also able to say something non-trivial about color coupling strength α_s too at the general level. The basic observations are following.

1. Both classical color YM action and electro-weak U(1) action reduce to Kähler action.

2. Classical color holonomy is Abelian which is consistent also with the fact that the only signature of color that induced spinor fields carry is anomalous color hyper charge identifiable as an electro-weak hyper charge.

Suppose that α_K is a strict RG invariant. One can consider two options.

1. The original idea was that the sum of classical color action and electro-weak U(1) action is RG invariant and thus equals to its asymptotic value obtained for $\alpha_{U(1)} = \alpha_s = 2\alpha_K$. Asymptotically the couplings would approach to a fixed point defined by $2\alpha_K$ rather than to zero as in asymptotically free gauge theories.

   Thus one would have

$$\frac{1}{\alpha_{U(1)}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K} \ . \tag{6.7.9}$$

   The relationship between $U(1)$ and em coupling strengths is

$$\begin{aligned} \alpha_{U(1)} &= \frac{\alpha_{em}}{cos^2(\theta_W)} \simeq \frac{1}{104.1867} \ , \\ sin^2(\theta_W)_{|10\ MeV} &\simeq 0.2397(13) \ , \\ \alpha_{em}(M_{127}) &= 0.00729735253327 \ . \end{aligned} \tag{6.7.8}$$

   Here Weinberg angle corresponds to 10 MeV energy is reasonably near to the value at electron mass scale. The value $sin^2(\theta_W) = 0.2397(13)$ corresponding to 10 MeV mass scale [E4] is used. Note however that the previous argument implying $\alpha_K = \alpha_{em}(M_{127})$ excludes $\alpha = \alpha_{U(1)}(M_{127})$ option.

2. Second option is obtained by replacing $U(1)$ with electromagnetic gauge $U(1)_{em}$.

$$\frac{1}{\alpha_{em}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K}. \tag{6.7.9}$$

   Possible justifications for this assumption are following. The notion of induced gauge field makes it possible to characterize the dynamics of classical electro-weak gauge fields using only the Kähler part of electro-weak action, and the induced Kähler form appears only in the electromagnetic part of the induced classical gauge field. A further justification is that em and color interactions correspond to unbroken gauge symmetries.

The following arguments are consistent with this conclusion.

1. In TGD framework coupling constant is discrete and comes as powers of $\sqrt{2}$ corresponding to p-adic primes $p \simeq 2^k$. Number theoretic considerations suggest that coupling constants g_i^2 are algebraic or perhaps even rational numbers, and that the logarithm of mass scale appearing as argument of the renormalized coupling constant is replaced with 2-based logarithm of the p-adic length scale so that one would have $g_i^2 = g_i^2(k)$. g_K^2 is predicted to be N:th root of rational but could also reduce to a rational. This would allow rational values for other coupling strengths too. This is possible if $sin(\theta_W)$ and $cos(\theta_W)$ are rational numbers which would mean that Weinberg angle corresponds to a Pythagorean triangle as proposed already earlier. This would mean the formulas $sin(\theta_W) = (r^2 - s^2)/(r^2 + s^2)$ and $cos(\theta_W) = 2rs(r^2 + s^2)$.

2. A very strong prediction is that the beta functions for color and $U(1)$ degrees of freedom are apart from sign identical and the increase of $U(1)$ coupling compensates the decrease of the color coupling. This allows to predict the hard-to-calculate evolution of QCD coupling constant strength completely.

3. $\alpha(M_{127}) = \alpha_K$ implies that M_{127} defines the confinement length scale in which the sign of α_s becomes negative. TGD predicts that also M_{127} copy of QCD should exist and that M_{127} quarks should play a key role in nuclear physics [K68, L5] , [L5] . Hence one can argue that color coupling strength indeed diverges at M_{127} (the largest not completely super-astrophysical Mersenne prime) so that one would have $\alpha_K = \alpha(M_{127})$. Therefore the precise knowledge of $\alpha(M_{127})$ in principle fixes the value of parameter $K = R^2/G$ and thus also the second order contribution to the mass of electron.

4. $\alpha_s(M_{89})$ is predicted to be $1/\alpha_s(M_{89}) = 1/\alpha_K - 1/\alpha(M_{89})$. $sin^2(\theta_W) = .23120$, $\alpha_{em}(M_{89}) \simeq 1/127$, and $\alpha_{U(1)} = \alpha_{em}/cos^2(\theta_W)$ give $1/\alpha_{U(1)}(M_{89}) = 97.6374$. $\alpha = \alpha_{em}$ option gives $1/\alpha_s(M_{89}) \simeq 10$, which is consistent with experimental facts. $\alpha = \alpha_{U(1)}$ option gives $\alpha_s(M_{89}) = 0.1572$, which is larger than QCD value. Hence $\alpha = \alpha_{em}$ option is favored.

Can one deduce formulae for gauge couplings?

The improved physical picture behind gravitational constant allows also to consider a general formula for gauge couplings.

1. The natural guess for the general formula would be as

$$g^2(p,r) \quad = \quad kg_K^2 \times exp[-a_g(p,r) \times S_K(CP_2)] \ . \tag{6.7.10}$$

   here k is a numerical constant.

2. The condition
   $g_K^2 = e^2(M_{127})$ fixes the value of k if it's value does not depend on the character of gauge interaction:

$$k \quad = \quad exp[a_{gr}(M_{127}, r=1) \times S_K(CP_2)] \ . \tag{6.7.11}$$

   Hence the general formula reads as

$$g^2(p,r) \quad = \quad g_K^2 \times exp[(-a_g(p,r) + a_{gr}(M_{127}), r=1)) \times S_K(CP_2)] \ . \tag{6.7.11}$$

   The value of $a(M_{127}, r=1)$ is near to its maximum value so that the exponential factor tends to increase the value of g^2 from e^2. The formula can reproduce α_s and various electro-weak couplings although it is quite possibile that Weinberg angle corresponds to a group theoretic factor not representable in terms of $a_g(p,r)$. The volume of the CP_2 type vacuum extremal would characterize gauge bosons. Analogous formula should apply also in the case of Higgs.

3. α_{em} in very long length scales would correspond to

$$e^2(p \to \infty, r=1) \quad = \quad e^2 \times exp[(-1 + a(M_{127}), r=1)) \times S_K(CP_2)] = e^2 x \ , \tag{6.7.11}$$

   where x is in the range $[0.6549, 0.6609]$.

Chapter 7

Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

7.1 Introduction

The notion of p-adicization has for a long time been a somewhat obscure attempt to provide a theoretical justification for the successes of the p-adic mass calculations. The reduction of quantum TGD to a generalized number theory and the developments in TGD inspired theory of consciousness have however led to a better understanding what the p-adicization possibly means.

7.1.1 What p-adic physics means?

Contrary to the original expectations finite-p p-adic physics means the physics of the p-adic cognitive representations about real physics rather than 'real physics'. This forces to update the prejudices about what p-adicization means. The original hypothesis was that p-adicization is a strict one-to-one map from real to p-adic physics and this led to technical problems with symmetries.

The new vision about quantum TGD the notion of the p-adic space-time emerges dynamically and p-adic space-time regions are absolutely 'real' and certainly not 'p-adicized' in any sense. Furthermore, the new view also encourages the hypothesis that p-adic regions provide cognitive models for the real matter like regions becoming more and more refined in the evolutionary self-organization process by quantum jumps. p-Adic region can serve as a cognitive model for particle itself or for the external world. The model is defined by some cognitive map of real region to its p-adic counterpart. This cognitive map need not be unique. At the level of TGD inspired theory of consciousness the p-adicization becomes modelling of how cognition works.

In this conceptual framework the successes of the p-adic mass calculations can be understood only if p-adic mass calculations provide a model a 'cognitive model' of an elementary particle. The successes of the p-adic mass calculations, and also the fact that they rely on the fundamental symmetries of quantum TGD, encourages the idea that one could try to mimic Nature. Thus p-adic physics could be seen as an abstract mimicry for what Nature already does by constructing explicitly p-adic cognitive representations. This new view about p-adic physics allows much more flexibility since p-adicization can be interpreted as a cognitive map mapping real world physics to p-adic physics. In this view p-adicization need not and cannot be a unique procedure.

7.1.2 Number theoretic vision briefly

The number theoretic vision [K71, K72, K70] about the classical dynamics of space-time surfaces is now relatively detailed although it involves unproven conjectures inspired by physical intuition.

1. *Hyper-quaternions and octonions*

The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional number fields of quaternions and octonions.

The difficulties caused by the Euclidian metric signature of the number theoretical norm have however forced to give up the original idea as such, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannian geometry performed already in Special Relativity.

The problem is that $H = M^4 \times CP_2$ cannot be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

2. Space-time-surface as a hyper-quaternionic sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic sub-manifold of the hyper-octonionic space in the sense that the tangent space of the space-time surface defines a hyper-quaternionic sub-algebra of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Second possibility is that the tangent space-algebra of the space-time surface is either associative or co-associative at each point. One can also consider possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface is algebraically closed in the sense that tangent space at each point has this property. Also the possibility that the property in question is associated with the normal space at each point of X^4 can be considered. Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If normal part of the product is projected out the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH--M^4 \times CP_2$ duality allows to construct a foliation of HO by hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteeing that the distribution of hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter family co-hyper-quaternionic space-time surfaces.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity.

Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that the absolute minima of Kähler action correspond to the hyper-quaternion analytic surfaces. This conjecture has several variants. It could be that only asymptotic behavior corresponds to hyper-quaternion analytic function but that that hyper-quaternionicity is general property of absolute minima. It could also be that maxima of Kähler function correspond to this kind of 4-surfaces. The encouraging hint is the fact that Hamilton-Jacobi coordinates appear naturally also in the construction of general solutions of field equations.

3. The representation of infinite hyper-octonionic primes as 4-surfaces

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers). Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : OH \rightarrow OH$ and hence also a foliation of OH and $H = M^4 \times CP_2$ by hyper-quaternionic 4-surfaces and notion of Kähler calibration. Therefore space-time surface could be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the preferred extremals of Kähler action.

7.1.3 p-Adic space-time sheets as solutions of real field equations continued algebraically to p-adic number field

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlations for cognition and intentionality. This requires a generalization of the notion of number by gluing reals and various p-adic number fields together along common rationals. This in turn implies generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

7.1.4 The notion of pinary cutoff

The notion of pinary cutoff is central for p-adic TGD and it should have some natural definition and interpretation in the new approach. The presence of p-adic pseudo constants implies that there is large number of cognitive representations with varying degrees of faithfulness. Pinary cutoff must serve as a measure for how faithful the p-adic cognitive representation is. Since the cognitive maps are not unique, one cannot even require any universal criterion for the faithfulness of the cognitive map. One can indeed imagine two basic criteria corresponding to self-representations and representations for external world.

1. The subset of rationals common to the real and p-adic space-time surface could define the resolution. In this case, the average distance between common rational points of these two surfaces would serve as a measure for the resolution. Pinary cutoff could be defined as the smallest number of pinary digits in expansions of functions involved above which the resolution does not improve. Physically the optimal resolution would mean that p-adic space-time surface, 'cognitive space-time sheet', has a maximal number of intersections with the real space-time surface for which it provides a self-representation. This purely algebraic notion of faithfulness does not respect continuity: two rational points very near in real sense could be arbitrary far from each other with respect to the p-adic norm.

2. One could base the notion of faithfulness on the idea that p-adic space-time sheet provides almost continuous map of the real space-time sheet belonging to the external world by the basic properties of the canonical identification. The real canonical image of the p-adic space-time sheet and real space-time sheet could be compared and some geometric measure for the nearness of these surfaces could define the resolution of the cognitive map and pinary cutoff could be defined in the same manner as above.

7.1.5 Program

These ideas lead to a rather well defined p-adicization program. Define precisely the concepts of the p-adic space-time and reduced configuration space, formulate the finite-p p-adic versions of quantum TGD and construct the p-adic variants of TGD. Of course, the aim is not to just construct p-adic version of the real quantum TGD but to understand how real and p-adic quantum TGD:s fuse together to form the full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, probability and unitary concepts to p-adic context. Also new physical thinking and philosophy is needed and this long chapter is devoted to the description of the new elements. Before going to the detailed exposition it is appropriate to give a brief overall view of the basic mathematical tools.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- Physics as generalized number theory [L29]
- Quantum physics as generalized number theory [L30]

- p-Adic number fields [L26]
- p-Adic physics [L27]
- p-Adic length scale hypothesis [L28]
- p-Adic manifold [L23]

7.2 p-Adic numbers and consciousness

The idea that p-adic physics provides the physics of cognition and intentionality has become more and more attractive during the 12 years or so that I have spent with p-adic numbers and I feel that it is good to add a summary about these ideas here.

7.2.1 p-Adic physics as physics of cognition

p-Adic physics began from p-adic mass calculations. The next step in the progress was the idea that p-adic physics serves as a correlate for cognition and this thread gradually led to the recent view requiring the generalization of the number concept.

Decomposition of space-time surface into p-adic and real regions as representation for matter-mind duality

Space-time surfaces contain genuinely p-adic and possibly even rational-adic regions so that no p-adicization is performed by Nature itself at at this level and it is enough to mimic the Nature. One manner to end up with the idea about p-adic space-time sheets is following.

Number theoretic vision leads to the idea that space-time surfaces can be associated with a hierarchy of polynomials to which infinite primes are mapped. It can happen that the components of quaternion are not always in algebraic extension of rationals but become complex. In this case the equations might however allow smooth solutions in some algebraic extension of p-adics for some values of prime p. It could also happen that real and p-adic roots exist simultaneously. In both cases the interpretation would be that the p-adic space-time sheets resulting as roots of the rational function provide self-representations for the real space-time sheets represented by real roots. This p-adicization would occur in the regions where some roots of the rational polynomial is complex or real roots exist also in the p-adic sense.

The dynamically generated p-adic space-time sheets could have a common boundary with the real surface in the following sense. At this surface a real root is transformed to a p-adic root and this surface corresponds to a boundary of catastrophe region in catastrophe theory. This boundary provides information about external real world very much in accordance with how nervous system receives information about the external world and makes possible cognitive representations about external world. Since the conditions defining the space-time surface expresses the vanishing of a derivative, the solution involves p-adic pseudo constants so that the cognitive representations are not unique and system can have more or less faithful cognitive representations about itself and about external world.

Rational points of the imbedding space and thus also of space-time surfaces are common to p-adics and reals and p-adic and real space-time surfaces differ only in that completion is different. This fixes the geometric interpretation of the cognitive maps involved with the p-adicization.

Different kinds of cognitive representations

At the level of the space-time surfaces and imbedding space p-adicization boils down to the task of finding a map mapping real space-time region to a p-adic space-time region. These regions correspond to definite regions of the rational imbedding space so that the map has a clear geometric interpretation at the level of rational physics.

The basic constraint on the map is that both real and p-adic space-time regions satisfy field equations: p-adic field equations make sense even if the integral defining the Kähler action does not exist p-adically. p-Adic nondeterminism makes possible this map when one allows finite pinary cutoff characterizing the resolution of the cognitive representation.

There are three basic types of cognitive representations which might be called self-representations and representations of the external world and the the map mediating p-adicization is different for these two maps.

1. The correspondence induced by the common rational points respects algebraic structures and defines self-representation. Real and p-adic space-time surfaces have a subset of rational points (defined by the resolution of the cognitive map) as common. The quality of the representation is defined by the resolution of the map and pinary cutoff for the rationals in pinary expansion is a natural measure for the resolution just as decimal cutoff is a natural measure for the resolution of a numerical model.

2. Canonical identification maps rationals to rationals since the periodic pinary expansion of a rational is mapped to a periodic expansion in the canonical identification. The rationals $q = m/n$ for which n is not divisible by p are mapped to rationals with p-adic norm not larger than unity. Canonical identification respects continuity. Real numbers with real norm larger than p are mapped to real numbers with norm smaller than one in canonical identification whereas reals with real norm in the interval $[1, p)$ are mapped to p-adics with p-adic norm equal to one. Obviously the generalization of the canonical identification can map the world external to a given space-time region into the interior of this region and provides an example of an abstract cognitive representation of the external world. Also now pinary cutoff serves as a natural measure for the quality of the cognitive map.

3. The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity. A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \to R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries.

   This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

   It seems that this option, the discovery of which took almost a decade, must be used to relate p-adic transition amplitudes to real ones and vice versa [K43] . In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

A fascinating possibility is that cognitive self-maps and maps of the external world at the level of human brain are basically realized by using these two basic types of mappings. Obviously canonical identification performed separately for all coordinates is the only possibility if this map is required to be maximally continuous.

p-Adic physics as a mimicry of p-adic cognitive representations

The success of the p-adic mass calculations suggests that one could apply the idea of p-adic cognitive representation even at the level of quantum TGD to build models which have maximal simplicity and calculational effectiveness. p-Adic mass calculations represent this kind of model: now canonical identification is performed for the p-adic mass squared values and can be interpreted as a map from cognitive representation back to real world.

The basic task is the construction of the cognitive self-map or a cognitive map of external world: the laws of p-adic physics define the cognitive model itself automatically. For the cognitive representations of external world involving some variant of canonical identification mapping the exterior of the imbedding space region inside this region. For self-representations situation is much more simpler. In practice, the direct modelling of p-adic physics without explicit construction of the cognitive map could give valuable information about real physics.

In the earlier approach based on phase preserving canonical identification to the mapping of real space-time surface to its p-adic counterpart led to the requirement about existence of unique (almost) imbedding space coordinates. In present case the selection of the quaternionic coordinates for the imbedding space is unique only apart from quaternion-analytic change of coordinates. This does not seem however pose any problems now. One must also remember that only cognitive representations are in question. These representations are not unique and selection of quaternionic coordinates might be even differentiate between different cognitive representations.

Since infinite primes serve as a bridge between classical and quantum, this map also assigns to a real Fock state associated with infinite prime its p-adic version identifiable as the ground state of a superconformal representation. Thus the map respects quantum symmetries automatically. If the construction of the states of the representation is a completely algebraic process, there are hopes of constructing the p-adic counterpart of S-matrix. If S-matrix is complex rational it can be mapped to its real counterpart. If the localization in zero modes occurs in each quantum jump the predictions of the theory could reduce to the integration in fiber degrees of freedom of CH reducible in turn to purely algebraic expressions making sense also p-adically.

7.2.2 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts what might be called zero energy ontology [K19, K18] .

Zero energy ontology classically

In TGD inspired cosmology [K65] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [K74] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

Zero energy ontology at quantum level

Also the construction of S-matrix [K18] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state define a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

Also the transitions between zero energy states are possible but general arguments lead to the conclusion that the corresponding S-matrix is almost trivial. This finding, which actually forced the new view about S-matrix, is highly desirable since it explains why positive energy ontology works so well if one forgets effects related to intentional action.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by an appropriate p-adic time scale and the integer characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also.

Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [K80] . This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by WCW gamma matrices (WCW understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [K80] . $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly, see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [K18] . The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

7.3 An overall view about p-adicization of TGD

In this section the basic problems and ideas related to the p-adicization of quantum TGD are discussed. One should define the notions of Riemann geometry and its variants such as Kähler geometry in the p-adic context. The notion of the p-adic space-time surface and its relationship to its real counterpart should be understood. Also the construction of Kähler geometry of "world of classical worlds" (WCW) in p-adic context should be carried out and the notion of WCW spinor fields should be defined in the p-adic context. The crucial technical problems relate to the notion of integral and Fourier analysis, which are the central elements of any physical theory. The basic challenge is to overcome the fact that although the field equations assignable to a given variational principle make sense p-adically, the action defined as an integral over arbitrary space-time surface has no natural p-adic counterpart as such in the generic case. What raises hopes that these challenges could be overcome is the symmetric space property of WCW and the idea of algebraic continuation. If WCW geometry is expressible in terms of rational functions with rational coefficients it allows a generalization to the p-adic context. Also integration can be reduced to Fourier analysis in the case of symmetric spaces.

7.3.1 p-Adic imbedding space

The construction of both quantum TGD and p-adic QFT limit requires p-adicization of the imbedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the imbedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost. The essential element is the notion of finite measurement resolution leading to discretization in large and to p-adicization below the resolution scale. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification mapping p-adic mass squared to its real counterpart.

The identification of the fundamental p-adic length scale

The fundamental p-adic length scale corresponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the 'radius' R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the 'cosmological constant' Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 = 2\Lambda \ . \tag{7.3.1}$$

The relationship between R and l is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} \ . \tag{7.3.2}$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π:s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

    (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p \ mod \ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

(b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of 1/3 rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

$$l \equiv \pi R \ ,$$

rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counterpart of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal plane waves. p-Adic plane waves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naively expected p, in the expression of the p-adic length scale can be understood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{7.3.2}$$

where m_{kl} is the standard M^4 metric $(1, -1, -1, -1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k:th coordinate axis the expression

$$s = R\sqrt{p} m^k \ . \tag{7.3.3}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{7.3.4}$$

The p-adic distance along the k:th coordinate axis from the origin to the point $m^k = (p-1)(1+p+p^2+...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p}l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R\frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{7.3.4}$$

What is remarkable is that the shortest distance in the range $m^k = 1,..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

The two variants of CP_2

As noticed, CP_2 allows two variants based on rational discretization and on the discretiation based on roots of unity. The root of unity option corresponds to the phases associated with $1/(1+r^2) = tan^2(u/2) = (1-cos(u))/(1+cos(u))$ and implies that integrals of spherical harmonics can be reduced to summations when angular resolution $\Delta u = 2\pi/N$ is introduced. In the p-adic context, one can replace distances with trigonometric functions of distances along zig zag curves connecting the points of the discretization. Physically this notion of distance is quite reasonable since distances are often measured using interferometer.

In the case of rational variant of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/columns are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their non-vanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_kX_k) \ ; \ |t_k|_p < 1\} = \{x = 1+iy \ ; \ |y|_p < 1\} \ . \tag{7.3.5}$$

The diagonal elements of the matrices in this group are of the form $1+O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1+\phi_2)\} \ . \tag{7.3.6}$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1+O(p)$. For $p \ mod \ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1 z_2 z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the 'physical' $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \ mod \ 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic sub-manifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \tag{7.3.6}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \ .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \ . \tag{7.3.7}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1 + r^2)^2} \ . \tag{7.3.7}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

7.3.2 Infinite primes, cognition and intentionality

Somehow it is obvious that infinite primes must have some very deep role to play in quantum TGD and TGD inspired theory of consciousness. What this role precisely is has remained an enigma although I have considered several detailed interpretations, one of them above.

In the following an interpretation allowing to unify the views about fermionic Fock states as a representation of Boolean cognition and p-adic space-time sheets as correlates of cognition is discussed. Very briefly, real and p-adic partonic 3-surfaces serve as space-time correlates for the bosonic super algebra generators, and pairs of real partonic 3-surfaces and their algebraically continued p-adic variants as space-time correlates for the fermionic super generators. Intentions/actions are represented by p-adic/real bosonic partons and cognitions by pairs of real partons and their p-adic variants and the geometric form of Fermi statistics guarantees the stability of cognitions against intentional action. It must be emphasized that this interpretation is not identical with the one discussed above since it introduces different identification of the space-time correlates of infinite primes.

Infinite primes very briefly

Infinite primes have a decomposition to infinite and finite parts allowing an interpretation as a many-particle state of a super-symmetric arithmetic quantum field theory for which fermions and bosons are labelled by primes. There is actually an infinite hierarchy for which infinite primes of a given level define the building blocks of the infinite primes of the next level. One can map infinite primes to polynomials and these polynomials in turn could define space-time surfaces or at least light-like partonic 3-surfaces appearing as solutions of Chern-Simons action so that the classical dynamics would not pose too strong constraints.

The simplest infinite primes at the lowest level are of form $m_B X/s_F + n_B s_F$, $X = \prod_i p_i$ (product of all finite primes). The simplest interpretation is that X represents Dirac sea with all states filled and $X/s_F + s_F$ represents a state obtained by creating holes in the Dirac sea. m_B, n_B, and s_F are defined as $m_B = \prod_i p_i^{m_i}$, $n_B = \prod_i q_i^{n_i}$, and $s_F = \prod_i q_i$, m_B and n_B have no common prime factors. The integers m_B and n_B characterize the occupation numbers of bosons in modes labelled by p_i and q_i and $s_F = \prod_i q_i$ characterizes the non-vanishing occupation numbers of fermions.

The simplest infinite primes at all levels of the hierarchy have this form. The notion of infinite prime generalizes to hyper-quaternionic and even hyper-octonionic context and one can consider the possibility that the quaternionic components represent some quantum numbers at least in the sense that one can map these quantum numbers to the quaternionic primes.

The obvious question is whether WCW degrees of freedom and WCW spinor (Fock state) of the quantum state could somehow correspond to the bosonic and fermionic parts of the hyper-quaternionic generalization of the infinite prime. That hyper-quaternionic (or possibly hyper-octonionic) primes would define as such the quantum numbers of fermionic super generators does not make sense. It is however possible to have a map from the quantum numbers labelling super-generators to the finite primes. One must also remember that the infinite primes considered are only the simplest ones at the given level of the hierarchy and that the number of levels is infinite.

Precise space-time correlates of cognition and intention

The best manner to end up with the proposal about how p-adic cognitive representations relate bosonic representations of intentions and actions and to fermionic cognitive representations is through the following arguments.

1. In TGD inspired theory of consciousness Boolean cognition is assigned with fermionic states. Cognition is also assigned with p-adic space-time sheets. Hence quantum classical correspondence suggets that the decomposition of the space-time into p-adic and real space-time sheets should relate to the decomposition of the infinite prime to bosonic and fermionic parts in turn relating to the above mention decomposition of physical states to bosonic and fermionic parts.

   If infinite prime defines an association of real and p-adic space-time sheets this association could serve as a space-time correlate for the Fock state defined by WCW spinor for given

3-surface. Also spinor field as a map from real partonic 3-surface would have as a space-time correlate a cognitive representation mapping real partonic 3-surfaces to p-adic 3-surfaces obtained by algebraic continuation.

2. Consider first the concrete interpretation of integers m_B and n_B. The most natural guess is that the primes dividing $m_B = \prod_i p^{m_i}$ characterize the effective p-adicities possible for the real 3-surface. m_i could define the numbers of disjoint partonic 3-surfaces with effective p_i-adic topology and associated with with the same real space-time sheet. These boundary conditions would force the corresponding real 4-surface to have all these effective p-adicities implying multi-p-adic fractality so that particle and wave pictures about multi-p-adic fractality would be mutually consistent. It seems natural to assume that also the integer n_i appearing in $m_B = \prod_i q_i^{n_i}$ code for the number of real partonic 3-surfaces with effective q_i-adic topology.

3. Fermionic statistics allows only single genuinely q_i-adic 3-surface possibly forming a pair with its real counterpart from which it is obtained by algebraic continuation. Pairing would conform with the fact that n_F appears both in the finite and infinite parts of the infinite prime (something absolutely essential concerning the consistency of interpretation!).

   The interpretation could be as follows.

   (a) Cognitive representations must be stable against intentional action and fermionic statistics guarantees this. At space-time level this means that fermionic generators correspond to pairs of real effectively q_i-adic 3-surface and its algebraically continued q_i-adic counterpart. The quantum jump in which q_i-adic 3-surface is transformed to a real 3-surface is impossible since one would obtain two identical real 3-surfaces lying on top of each other, something very singular and not allowed by geometric exclusion principle for surfaces. The pairs of boson and fermion surfaces would thus form cognitive representations stable against intentional action.

   (b) Physical states are created by products of super algebra generators Bosonic generators can have both real or p-adic partonic 3-surfaces as space-time correlates depending on whether they correspond to intention or action. More precisely, m_B and n_B code for collections of real and p-adic partonic 3-surfaces. What remains to be interpreted is why m_B and n_B cannot have common prime factors (this is possible if one allows also infinite integers obtained as products of finite integer and infinite primes).

   (c) Fermionic generators to the pairs of a real partonic 3-surface and its p-adic counterpart obtained by algebraic continuation and the pictorial interpretation is as fermion hole pair.

   (d) This picture makes sense if the partonic 3-surfaces containing a state created by a product of super algebra generators are unstable against decay to this kind of 3-surfaces so that one could regard partonic 3-surfaces as a space-time representations for a configuration space spinor field.

4. Are alternative interpretations possible? For instance, could $q = m_B/m_B$ code for the effective q-adic topology assignable to the space-time sheet. That q-adic numbers form a ring but not a number field casts however doubts on this interpretation as does also the general physical picture.

Number theoretical universality of S-matrix

The discreteness of the intersection of the real space-time sheet and its p-adic variant obtained by algebraic continuation would be a completely universal phenomenon associated with all fermionic states. This suggests that also real-to-real S-matrix elements involve instead of an integral a sum with the arguments of an n-point function running over all possible combinations of the points in the intersection. S-matrix elements would have a universal form which does not depend on the number field at all and the algebraic continuation of the real S-matrix to its p-adic counterpart would trivialize. Note that also fermionic statistics favors strongly discretization unless one allows Dirac delta functions.

7.3.3 p-Adicization of second quantized induced spinor fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at the 3-dimensional light-like partonic 3-surfaces and the solutions of the modified Dirac equation can be written explicitly [K19, K18] as simple algebraic functions involving powers of the preferred coordinate variables very much like various operators in conformal field theory can be expressed as Laurent series in powers of a complex variable z with operator valued coefficients. This means that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure. The second quantization of these induced spinor fields as free fields is needed to construct WCW geometry and anti-commutation relation between spinor fields are fixed from the requirement that WCW gamma matrices correspond to super-symplectic generators.

The idea about rational physics as the intersection of the physics associated with various number fields inspires the hypothesis that induced spinor fields have only modes labelled by rational valued quantum numbers. Quaternion conformal invariance indeed implies that zero modes are characterized by integers. This means that same oscillator operators can define oscillator operators are universal. Powers of the quaternionic coordinate are indeed well-define in any number field provided the components of quaternion are rational numbers since p-adic quaternions have in this case always inverse.

7.3.4 Should one p-adicize at the level of WCW?

If Duistermaat-Heckman theorem [A133] holds true in TGD context, one could express WCW functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function defining what might be called reduced WCW CH_{red}. The huge super-conformal symmetries raise the hope that the rest of S-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of S-matrix.

The optimal situation would be that S-matrix elements reduce to algebraic numbers for rational valued incoming momenta and that p-adicization trivializes in the sense that it corresponds only to different interpretations for the imbedding space coordinates (interpretation as real or p-adic numbers) appearing in the equations defining the 4-surfaces. For instance, space-time coordinates would correspond to preferred imbedding space coordinates and the remaining imbedding space coordinates could be rational functions of the latter with algebraic coefficients. Algebraic points in a given extension of rationals would thus be common to real and p-adic surfaces. It could also happen that there are no or very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$.

This picture is probably too simple. The intuitive expectation is that ordinary S-matrix elements are proportional to a factor which in the real case involves an integration over the arguments of an n-point function of a conformal field theory defined at a partonic 2-surface. For p-adic-real transitions the integration should reduce to a sum over the common rational or algebraic points of the p-adic and real surface. Same applies to $p_1 \rightarrow p_2$ type transitions.

If this picture is correct, the p-adicization of WCW would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. If CH_{red} is a discrete subset of CH ultra-metric topology induced from finite-p p-adic norm is indeed natural for it. 'Discrete set in CH' need not mean a discrete set in the usual sense and the reduced WCW could be even finite-dimensional continuum. Finite-p p-adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

A basic technical problem is, whether the integral defining the Kähler action appearing in the exponent of Kähler function exists p-adically. Here the hypothesis that the exponent of the Kähler function is identifiable as a Dirac determinant of the modified Dirac operator defined at the light-like partonic 3-surfaces [K15] suggests a solution to the problem. By restricting the generalized eigen values of the modified Dirac operator to an appropriate algebraic extension of rationals one could obtain an algebraic number existing both in the real and p-adic sense if the number of the contributing eigenvalues is finite. The resulting hierarchy of algebraic extensions of R_p would have interpretation as a cognitive hierarchy. If the maxima of Kähler function assignable to the functional integral are such that the number of eigenvalues in a given algebraic extension is finite this hypothesis works.

If Duistermaat-Heckman theorem generalizes, the p-adicization of the entire WCW would be un-necessary and it certainly does not look a good idea in the light of preceding considerations.

1. For a generic 3-surface the number of the eigenvalues in a given algebraic extension of rationals need not be finite so that their product can fail to be an algebraic number.

2. The algebraic continuation of the exponent of the Kähler function from CH_{red} to the entire CH would be analogous to a continuation of a rational valued function from a discrete set to a real or p-adic valued function in a continuous set. It is difficult to see how the continuation could be unique in the p-adic case.

7.4 p-Adic probabilities

p-Adic Super Virasoro representations necessitate p-adic QM based on the p-adic unitarity and p-adic probability concepts. The concept of a p-adic probability indeed makes sense as shown by [A180] . p-Adic probabilities can be defined as relative frequencies N_i/N in a long series consisting of total number N of observations and N_i outcomes of type i. Probability conservation corresponds to

$$\sum_i N_i \quad = \quad N \ , \tag{7.4.1}$$

and the only difference as compared to the usual probability is that the frequencies are interpreted as p-adic numbers.

The interpretation as p-adic numbers means that the relative frequencies converge to probabilities in a p-adic rather than real sense in the limit of a large number N of observations. If one requires that probabilities are limiting values of the frequency ratios in p-adic sense one must pose restrictions on the possible numbers of the observations N if N is larger than p. For N smaller than p, the situation is similar to the real case. This means that for $p = M_{127} \simeq 10^{38}$, appropriate for the particle physics experiments, p-adic probability differs in no observable manner from the ordinary probability.

If the number of observations is larger than p, the situation changes. If N_1 and N_2 are two numbers of observations they are near to each other in the p-adic sense if they differ by a large power of p. A possible interpretation of this restriction is that the observer at the p:th level of the condensate cannot choose the number of the observations freely. The restrictions to this freedom come from the requirement that the sensible statistical questions in a p-adically conformally invariant world must respect p-adic conformal invariance.

7.4.1 p-Adic probabilities and p-adic fractals

p-Adic probabilities are natural in the statistical description of the fractal structures, which can contain same structural detail with all possible sizes.

1. The concept of a structural detail in a fractal seems to be reasonably well defined concept. The structural detail is clearly fixed by its topology and p-adic conformal invariants associated with it. Clearly, a finite resolution defined by some power of p of the p-adic cutoff scale must be present in the definition. For example, p-adic angles are conformal invariants in the p-adic case, too. The overall size of the detail doesn't matter. Let us therefore assume that it is possible to make a list, possibly infinite, of the structural details appearing in the p-adic fractal.

2. What kind of questions related to the structural details of the p-adic fractal one can ask? The first thing one can ask is how many times i:th structural detail appears in a finite region of the fractal structure: although this number is infinite as a real number it might possess (and probably does so!) finite norm as a p-adic number and provides a useful p-adic invariant of the fractal. If a complete list about the structural details of the fractal is at use one can calculate also the total number of structural details defined as $N = \sum_i N_i$. This means that one can also define p-adic probability for the appearance of i:th structural detail as a relative frequency $p_i = N_i/N$.

3. One can consider conditional probabilities, too. It is natural to ask what is the probability for the occurrence of the structural detail subject to the condition that part of the structural detail is fixed (apart from the p-adic conformal transformations). In order to evaluate these probabilities as relative frequencies one needs to look only for those structural details containing the substructure in question.

4. The evaluation of the p-adic probabilities of occurrence can be done by evaluating the required numbers N_i and N in a given resolution. A better estimate is obtained by increasing the resolution and counting the numbers of the hitherto unobserved structural details. The increase in the resolution greatly increases the number of the observations in case of p-adic fractal and the fluctuations in the values of N_i and N increase with the resolution so that N_i/N has no well defined limit as a real number although one can define the probabilities of occurrence as a resolution dependent concept. In the p-adic sense the increase in the values of N_i and fluctuations are small and the procedure should converge rapidly so that reliable estimates should result with quite a reasonable resolution. Notice that the increase of the fluctuations in the real sense, when resolution is increased is in accordance with the criticality of the system.

5. p-Adic frequencies and probabilities define via the canonical correspondence real valued invariants of the fractal structure.

It must be emphasized that this picture can have practical applications only for small values of p, which could also be important in the macroscopic length scales. In elementary particle physics L_p is of the order of the Compton length associated with the particle and already in the first step CP_2 length scale is achieved and it is questionable whether it makes sense to continue the procedure below the length scale l. In particle physics context the renormalization is related to the the change of the reduction of the p-adic length scale L_p in the length scale hierarchy rather than p-adic fractality for a fixed value of p.

The most important application of the p-adic probability in this book is the description of the particle massivation based on p-adic thermodynamics. Instead of energy, Virasoro generator l is thermalized and in the low temperature phase temperature is quantized in the sense that the counterpart of the Boltzmann weight $exp(H/T)$ is $p^{L_0/T}$, where $T = 1/n$ from the requirement that Boltzmann weight exists (L_0 has integer spectrum). The surprising success of the mass calculations shows that p-adic probability theory is much more than a formal possibility.

7.4.2 Relationship between p-adic and real probabilities

There are uniqueness problems related to the mapping of p-adic probabilities to real ones. These problems find a nice resolution from the requirement that the map respects probability conservation. The implied modification of the original mapping does not change measurably the predictions for the masses of light particles.

How unique the map of p-adic probabilities and mass squared values are mapped to real numbers is?

The mapping of p-adic thermodynamical probabilities and mass squared values to real numbers is not completely unique.

1. Symplectic identification $I : \sum x_n p^n \rightarrow \sum x_n p^{-n}$ takes care of this mapping but does not respect the sum of probabilities so that the real images $I(p_n)$ of the probabilities must be normalized. This is a somewhat alarming feature.

2. The modification of the canonical identification mapping rationals by the formula $I(r/s) = I(r)/I(s)$ has appeared naturally in various applications, in particular because it respects unitarity of unitary matrices with rational elements with $r < p, s < p$. In the case of p-adic thermodynamic the formula $I(g(n)p^n/Z) \rightarrow I(g(n)p^n)/I(Z)$ would be very natural although Z need not be rational anymore. For $g(n) < p$ the real counterparts of the p-adic probabilities would sum up to one automatically for this option. One cannot deny that this option is more convincing than the original one. The generalization of this formula to map p-adic mass squared to a real one is obvious.

3. Options 1) and 2) differ dramatically when the $n = 0$ massless ground state has ground state degeneracy $D > 1$. For option 1) the real mass is predicted to be of order CP_2 mass whereas for option 2) it would be by a factor $1/D$ smaller than the minimum mass predicted by the option 1). Thus option 2) would predict a large number of additional exotic states. For those states which are light for option 1), the two options make identical predictions as far as the significant two lowest order terms are considered. Hence this interpretation would not change the predictions of the p-adic mass calculations in this respect. Option 2) is definitely more in accord with the real physics based intuitions and the main role of p-adic thermodynamics would be to guarantee the quantization of the temperature and fix practically uniquely the spectrum of the "Hamiltonian".

Under what conditions the mapping of p-adic ensemble probabilities to real probabilities respects probability conservation?

One can consider also a more general situation. Assume that one has an ensemble consisting of independent elementary events such that the number of events of type i is N_i. The probabilities are given by $p_i = N_i/N$ and $N = \sum N_i$ is the total number of elementary events. Even in the case that N is infinite as a real number it is natural to map the p-adic probabilities to their real counterparts using the rational canonical identification $I(p_i) = I(N_i)/I(N)$. Of course, N_i and N exist as well defined p-adic numbers under very stringent conditions only.

The question is under what conditions this map respects probability conservation. The answer becomes obvious by looking at the pinary expansions of N_i and N. If the integers N_i (possibly infinite as real integers) have pinary expansions having no common pinary digits, the sum of probabilities is conserved in the map. Note that this condition can assign also to a finite ensemble with finite number of a unique value of p.

This means that the selection of a basis for independent events corresponds to a decomposition of the set of integers labelling pinary digits to disjoint sets and brings in mind the selection of orthonormalized basis of quantum states in quantum theory. What is physically highly non-trivial that this "orthogonalization" alone puts strong constraints on probabilities of the allowed elementary events. One can say that the probabilities define distributions of pinary digits analogous to non-negative probability amplitudes in the space of integers labelling pinary digits, and the probabilities of independent events must be orthogonal with respect to the inner product defined by point-wise multiplication in the space of pinary digits.

p-Adic thermodynamics for which Boltzmann weights $g(E)exp(-E/T)$ are replaced by $g(E)p^{E/T}$ such that one has $g(E) < p$ and E/T is integer valued, satisfies this constraint. The quantization of E/T to integer values implies quantization of both T and "energy" spectrum and forces so called super conformal invariance in TGD applications, which is indeed a basic symmetry of the theory.

There are infinitely many ways to choose the elementary events and each choice corresponds to a decomposition of the infinite set of integers n labelling the powers of p to disjoint subsets. These subsets can be also infinite. One can assign to this kind of decomposition a resolution which is the poorer the larger the subsets involved are. p-Adic thermodynamics would represent the situation in which the resolution is maximal since each set contains only single pinary digit. Note the analogy with the basis of completely localized wave functions in a lattice.

How to map p-adic transition probabilities to real ones?

p-Adic variants of TGD, if they exist, give rise to S-matrices and transition probabilities P_{ij}, which are p-adic numbers.

1. The p-adic probabilities defined by rows of S-matrix mapped to real numbers using canonical identification respecting the $q = r/s$ decomposition of rational number or its appropriate generalization should define real probabilities.

2. The simplest example would simple renormalization for the real counterparts of the p-adic probabilities $(P_{ij})_R$ obtained by canonical identification (or more probably its appropriate modification).

$$\begin{aligned} P_{ij} &= \sum_{k\geq 0} P_{ij}^k p^k \ , \\ P_{ij} &\rightarrow \sum_{k\geq 0} P_{ij}^k p^{-k} \equiv (P_{ij})_R \ , \\ (P_{ij})_R &\rightarrow \frac{(P_{ij})_R}{\sum_j (P_{ij})_R} \equiv P_{ij}^R \ . \end{aligned} \tag{7.4.-1}$$

   The procedure converges rapidly in powers of p and resembles renormalization procedure of quantum field theories. The procedure automatically divides away one four-momentum delta function from the square of S-matrix element containing the square of delta function with no well defined mathematical meaning. Usually one gets rid of the delta function interpreting it as the inverse of the four-dimensional measurement volume so that transition rate instead of transition probability is obtained. Of course, also now same procedure should work either as a discrete or a continuous version.

3. Probability interpretation would suggest that the real counterparts of p-adic probabilities sum up to unity. This condition is rather strong since it would hold separately for each row and column of the S-matrix.

4. A further condition would be that the real counterparts of the p-adic probabilities for a given prime p are identical with the transition probabilities defined by the real S-matrix for real space-time sheets with effective p-adic topology characterized by p. This condition might allow to deduce all relevant phase information about real and corresponding p-adic S-matrices using as an input only the observable transition probabilities.

What it means that p-adically independent events are not independent in real sense?

A further condition would be that p-adic quantum transitions represent also in the real sense independent elementary events so that the real counterpart for a sum of the p-adic probabilities for a finite number of transitions equals to the sum of corresponding real probabilities. This condition is definitely too strong since only a single transition could correspond to a given p-adic norm of transition probability P_{ij} with i fixed.

The crucial question concerns the physical difference between the real counterpart for the sum of the p-adic transition probabilities and for the sum of the real counterparts of these probabilities, which are in general different:

$$(\sum_j P_{ij})_R \neq \sum_j (P_{ij})_R \ . \tag{7.4.0}$$

The suggestion is that p-adic sum of the transition probabilities corresponds to the experimental situation, when one does not monitor individual transitions but using some common experimental signature only looks whether the transition leads to this set of the final states or not. When one looks each transition separately or effectively performs different experiment by considering only one transition channel in each experiment one must use the sum of the real probabilities. More precisely, the choice of the experimental signatures divides the set U of the final states to a disjoint union $U = \cup_i U_i$ and one must define the real counterparts for the transition probabilities P_{iU_k} as

$$\begin{aligned}
P_{iU_k} &= \sum_{j \in U_k} P_{ij} \ , \\
P_{iU_k} &\rightarrow (P_{iU_k})_R \ , \\
(P_{iU_k})_R &\rightarrow \frac{(P_{iU_k})_R}{\sum_l (P_{iU_l})_R} \equiv P^R_{iU_k} \ .
\end{aligned} \tag{7.4.-2}$$

The assumption means deep a departure from the ordinary probability theory. If p-adic physics is the physics of cognitive systems, there need not be anything mysterious in the dependence of the behavior of system on how it is monitored. At least half-jokingly one might argue that the behavior of an intelligent system indeed depends strongly on whether the boss is nearby or not. The precise definition for the monitoring could be based on the decomposition of the density matrix representing the entangled subsystem into a direct sum over the subspaces associated with the degenerate eigenvalues of the density matrix. This decomposition provides a natural definition for the notions of the monitoring and resolution.

The renormalization procedure is in fact familiar from standard physics. Assume that the labels j correspond to momenta. The division of momentum space to cells of a given size so that the individual momenta inside cells are not monitored separately means that momentum resolution is finite. Therefore one must perform p-adic summation over the cells and define the real probabilities in the proposed manner. p-Adic effects resulting from the difference between p-adic and real summations could be the counterpart of the renormalization effects in QFT. It should be added that similar resolution can be defined also for the initial states by decomposing them into a union of disjoint subsets.

An alternative interpretation for the degenerate eigenvalues has emerged years after writing this. The sub-spaces corresponding to given eigenvalue of density matrix represent entangled states resulting in state function reduction interpreted as measurement of density matrix. This entanglement would be negentropic and represent a rule/concept, whose instances the superposed state pairs are. The information measure would Shannon entropy based on the replacement of the probability appearing as argument of logarithm with its p-adic norm. This entropy would be negative and therefore measure the information associated with the entanglement. This number theoretic entropy characterizes two particle state rather than single particle state and has nothing to do with the ordinary Shannon entropy.

Maybe one could say that finite measurement resolution implies automatically conceptualization and rule building. Abstractions are indeed obtained by dropping out the details.

7.4.3 p-Adic thermodynamics

The p-adic field theory limit as such is not expected to give a realistic theory at elementary particle physics level. The point is that particles are expected to be either massless or possess mass of order 10^{-4} Planck mass. The p-adic description of particle massivation described in the third part of the book shows that p-adic thermodynamics provides the proper formulation of the problem. What is thermalized is Virasoro generator L_0 (mass squared contribution is not included to L_0 so that states do not have fixed conformal weight). Temperature is quantized purely number theoretically in low temperature limit ($exp(H/kT) \rightarrow p^{L_0/T}$, $T = 1/n$): in fact, partition function does not even exist in high temperature phase. The extremely small mixing of massless states with Planck mass states implies massivation and predictions of the p-adic thermodynamics for the fermionic masses are in excellent agreement with experimental masses. Thermodynamic approach also explains the emergence of the length scale L_p for a given p-adic condensation level and one can develop arguments explaining why primes near prime powers of two are favored.

It should be noticed that rational p-adic temperatures $1/T = k/n$ are possible, if one poses the restriction that thermal probabilities are non-vanishing only for some subalgebra of the Super Virasoro algebra isomorphic to the Super Virasoro algebra itself. The generators L_{kn},G_{kn}, where k is a positive integer, indeed span this kind of a subalgebra by the fractality of the Super Virasoro algebra and $p^{L_0/T}$ is integer valued with this restriction.

One might apply thermodynamics approach should also in the calculation of S-matrix. What is is needed is thermodynamical expectation value for the transition amplitudes squared over incoming and outgoing states. In this expectation value 3-momenta are fixed and only mass squared varies.

7.4.4 Generalization of the notion of information

TGD inspired theory of consciousness, in particular the formulation of Negentropy Maximization Principle (NMP) in p-adic context, has forced to rethink the notion of the information concept. In TGD state preparation process is realized as a sequence of self measurements. Each self measurement means a decomposition of the sub-system involved to two unentangled parts. The decomposition is fixed highly uniquely from the requirement that the reduction of the entanglement entropy is maximal.

The additional assumption is that bound state entanglement is stable against self measurement. This assumption is somewhat ad hoc and it would be nice to get rid of it. The only manner to achieve this seems to be a generalized definition of entanglement entropy allowing to assign a negative value of entanglement entropy to the bound state entanglement, so that bound state entanglement would actually carry information, in fact conscious information (experience of understanding). This would be very natural since macro-temporal quantum coherence corresponds to a generation of bound state entanglement, and is indeed crucial for ability to have long lasting non-entropic mental images.

The generalization of the notion of number concept leads immediately to the basic problem. How to generalize the notion of entanglement entropy that it makes sense for a genuinely p-adic entanglement? What about the number-theoretically universal entanglement with entanglement probabilities, which correspond to finite extension of rational numbers? One can also ask whether the generalized notion of information could make sense at the level of the space-time as suggested by quantum-classical correspondence.

In the real context Shannon entropy is defined for an ensemble with probabilities p_n as

$$S = -\sum_n p_n log(p_n) \ . \qquad (7.4.\text{-}1)$$

As far as theory of consciousness is considered, the basic problem is that Shannon entropy is always non-negative so that as such it does not define a genuine information measure. One could define information as a change of Shannon entropy and this definition is indeed attractive in the sense that quantum jump is the basic element of conscious experience and involves a change. One can however argue that the mere ability to transfer entropy to environment (say by aggressive behavior) is not all that is involved with conscious information, and even less so with the experience of understanding or moment of heureka. One should somehow generalize the Shannon entropy without losing the fundamental additivity property.

p-Adic entropies

The key observation is that in the p-adic context the logarithm function $log(x)$ appearing in the Shannon entropy is not defined if the argument of logarithm has p-adic norm different from 1. Situation changes if one uses an extension of p-adic numbers containing $log(p)$: the conjecture is that this extension is finite-dimensional. One might however argue that Shannon entropy should be well defined even without the extension.

p-Adic thermodynamics inspires a manner to achieve this. One can replace $log(x)$ with the logarithm $log_p(|x|_p)$ of the p-adic norm of x, where log_p denotes p-based logarithm. This logarithm is integer valued ($log_p(p^n) = n$), and is interpreted as a p-adic integer. The resulting p-adic entropy

$$\begin{aligned} S_p &= \sum_n p_n k(p_n) \ , \\ k(p_n) &= -log_p(|p_n|) \ . \end{aligned} \qquad (7.4.\text{-}1)$$

is additive: that is the entropy for two non-interacting systems is the sum of the entropies of composites. Note that this definition differs from Shannon's entropy by the factor $log(p)$. This entropy vanishes identically in the case that the p-adic norms of the probabilities are equal to one. This means that it is possible to have non-entropic entanglement for this entropy.

One can consider a modification of S_p using p-adic logarithm if the extension of the p-adic numbers contains $log(p)$. In this case the entropy is formally identical with the Shannon entropy:

$$S_p = -\sum_n p_n log(p_n) = -\sum_n p_n \left[-k(p_n)log(p) + p^{k_n} log(p_n/p^{k_n}\right] \ . \tag{7.4.0}$$

It seems that this entropy cannot vanish.

One must map the p-adic value entropy to a real number and here canonical identification can be used:

$$\begin{aligned} S_{p,R} &= (S_p)_R \times log(p)) \ , \\ (\sum_n x_n p^n)_R &= \sum_n x_n p^{-n} \ . \end{aligned} \tag{7.4.0}$$

The real counterpart of the p-adic entropy is non-negative.

Number theoretic entropies and bound states

In the case that the probabilities are rational or belong to a finite-dimensional extension of rationals, it is possible to regard them as real numbers or p-adic numbers in some extension of p-adic numbers for any p. The visions that rationals and their finite extensions correspond to islands of order in the seas of chaos of real and p-adic transcendentals suggests that states having entanglement coefficients in finite-dimensional extensions of rational numbers are somehow very special. This is indeed the case. The p-adic entropy entropy $S_p = -\sum_n p_n log_p(|p_n|)log(p)$ can be interpreted in this case as an ordinary rational number in an extension containing $log(p)$.

What makes this entropy so interesting is that it can have also negative values in which case the interpretation as an information measure is natural. In the real context one can fix the value of the value of the prime p by requiring that S_p is maximally negative, so that the information content of the ensemble could be defined as

$$I \equiv Max\{-S_p, \ p \text{ prime}\} \ . \tag{7.4.1}$$

This information measure is positive when the entanglement probabilities belong to a finite-dimensional extension of rational numbers. Thus kind of entanglement is stable against NMP, and has a natural interpretation as bound state entanglement. The prediction would be that the bound states of real systems form a number theoretical hierarchy according to the prime p and and dimension of algebraic extension characterizing the entanglement.

Number theoretically state function reduction and state preparation could be seen as information generating processes projecting the physical states from either real or p-adic sectors of the state space to their intersection. Later an argument that these processes have a purely number theoretical interpretation will be developed based on the generalized notion of unitarity allowing the U-matrix to have matrix elements between the sectors of the state space corresponding to different number fields.

Number theoretic information measures at the space-time level

Quantum classical correspondence suggests that the notion of entropy should have also space-time counterpart. Entropy requires ensemble and both the p-adic non-determinism and the non-determinism of Kähler action allow to define the required ensemble as the ensemble of strictly deterministic regions of the space-time sheet. One can measure various observables at these space-time regions, and the frequencies for the outcomes are rational numbers of form $p_k = n(k)/N$, where N is the number of strictly deterministic regions of the space-time sheet. The number theoretic entropies are well defined and negative if p divides the integer N. Maximum is expected to result for the largest prime power factor of N. This would mean the possibility to assign a unique prime to a given real space-time sheet.

The classical non-determinism resembles p-adic non-determinism in the sense that the space-time sheet obeys effective p-adic topology in some length and time scale range is consistent with this idea since p-adic fractality suggests that N is power of p.

7.5 About p-adic Quantum Mechanics

An interesting question is whether p-adic quantum mechanics might exist in some sense. The purely formal generalization of the ordinary QM need not be very interesting physically and the following considerations describe p-adic QM as a limiting case of the p-adic field theory limit of TGD to be constructed later. This particular p-adic QM is based on the p-adic Hilbert-space, p-adic unitarity and p-adic probability concepts whereas the physical interpretation is based on the correspondence between the p-adic and real probabilities given by the canonical correspondence. p-Adic QM is expected to apply -if it applies at all- below the p-adic length scale $L_p = \sqrt{p}l$ and above L_p ordinary QM should work, when length scale resolution L_p is used.

Although one can define p-adic Schrödinger equation formally without any difficulty it is not at all obvious whether it emerges from the p-adic QFT limit of TGD. Therefore the following considerations - my first reaction to the question what p-adic quantum theory look like- should be taken as mere warming up exercises perhaps helping to get some familiarity with new concepts. In the next chapter "Negentropy Maximization Principle" a more serious approach starting directly from the condition that real and p-adic approaches must allow fusion to larger coherence whole will be discussed.

7.5.1 p-Adic modifications of ordinary Quantum Mechanics

One can consider several modifications of the ordinary quantum mechanics depending on what kind of p-adicizations one is willing to make.

p-Adicization in dynamical degrees of freedom

The minimal alternative is to replace time- and spatial coordinates with their p-adic counterparts so that the space time is a Cartesian power of R_p. A more radical possibility is to replace the 3-space with a 3-dimensional algebraic extension of the p-adic numbers. This means that space time is replaced with a Cartesian product of R_p and its 3-dimensional extension. The most radical possibility, suggested by the relativistic considerations, is a four-dimensional algebraic extension treating space and time degrees of freedom in an equal position: this alternative is encountered in the formulation of the p-adic field theory limit of TGD.

In practice the formulation of the quantum theory involves an action principle defining the so called classical theory and this is defined by using the integral of the the action density. These integrals certainly exists as real quantities and are defined by the Haar measure for the p-adic numbers. Algebraic continuation of real integrals seems to be the only reasonable manner to defined these integrals.

p-Adicization at Hilbert space level

One can imagine essentially two different manners to p-adicize Hilbert space.

1. The first approach, followed in [H1] , is to keep Schrö-dinger amplitudes complex. In this case it is better to consider a Cartesian power of R_p instead of an algebraic extension as a coordinate space. The canonical identification allows to replace the expressions of the coordinate and momentum operators via their p-adic counterparts. For example, $x \times \Psi$ is replaced with $x \times_p \Psi$, where p-adic multiplication rule is used. Derivative corresponds to a p-adic derivative. It was the lack of the canonical identification replacement, which forced to give up the straightforward generalization of standard QM in the approach followed in [A123] , [H1] . What this approach effects, is the replacement of the ordinary continuity and differentiability and concepts with the p-adic differentiability and the approach looks

rather reasonable manner to construct a fractal quantum mechanics. This approach however is not applicable in the present context.

2. A more radical approach uses Schrödinger amplitude with values in some complex extension, say a square root allowing extension of the p-adic numbers. p-Adic inner product implies that the ordinary unitarity and probability concepts are replaced with there p-adic counterparts. This approach looks natural for various reasons. The representation theory for the Lie-groups generalizes to p-adic case and the replacement implies certain mathematical elegance since p-analyticity and the realization of the p-adic conformal invariance becomes possible. It will be found that p-adic valued inner product is the natural inner product for the quantized harmonic oscillator and for Super Virasoro representations. The concept of the p-adic probability makes sense as first shown by [A180] . The physical interpretation of the theory is however always in terms of the real numbers and the canonical identification provides the needed tool to map the predictions of the theory to real numbers. That physical observables are always real numbers is suggested by the success of the p-adic mass calculations. p-Adic probabilities can be mapped to real probabilities and in the last chapter of the third part of the book it is shown that this correspondence predicts genuinely novel physical effects.

The p-adic representations of the Super Virasoro algebra to be used are defined in the p-adic Hilbert space and everything is well defined at algebraic level if 4- ($p > 2$) or 8- ($p = 2$) dimensional algebraic extension allowing square roots is used. Unitarity concept generalizes in a straightforward manner to the p-adic context and the elements of the S-matrix should have values in the same extension of the p-adic numbers. The requirement that the squares of S-matrix elements are p-adically real numbers gives strong constraints on the S-matrix elements since the quantities $S(m,n)\bar{S}(m,n)$ in general belong to the 4- (2-) dimensional real subspace $x+\theta y+\sqrt{p}z+\sqrt{p}\theta u$ of the 8- (4-) dimensional extension and p-adic reality implies the conditions:$y = z = .. = u = 0$. Reality conditions can be solved always since the solution involves only square roots of rational functions. What is exciting is that space time and imbedding space dimensions for the extension allowing square roots are forced by the quantum mechanical probability concept, by p-adic group theory and by the p-adic Riemannian Geometry.

The existence of the p-adic valued definite integral is crucial concerning the practical construction of the p-adic Quantum Mechanics.

1. In the ordinary wave mechanics the inner product involves an integration over WCW degrees of freedom. This inner product can be generalized to the p-adic integral of $\bar{\Psi}_1\Phi_2$ over the 3-space using p-adic valued integration defined in the first chapter, which works for all analytic functions and also for p-adic counterparts of the plane waves (non-analytic functions).

2. The perturbative formulation QM in terms of the time development operator

$$U(t) = P(exp(i\int exp(\int dt\ V)) \ , \tag{7.5.1}$$

generalizes to the p-adic context. In particular, the concept of the time ordered product $P(...)$ appearing in the definition of the time development operator generalizes since the canonical identification induces ordering for the values of the p-adic time coordinate: $t_1 < t_2$ if $(t_1)_R < (t_2)_R$ holds true. Non-trivialities are related to the p-adic existence of the time development operator: for sufficiently larger values of the time coordinate, the exponent appearing in the time development operator does not exist p-adically and this implies infrared cutoff time and length scale in the p-adic QM.

One can define the action of the time development operator for longer time intervals only if one makes some restrictions on the physical states appearing in the matrix elements. This could explain color confinement number theoretically. For sufficiently long time intervals the color interaction part of the interaction Hamiltonian is so large for colored states that p-adic time development operator fails to exist number theoretically and one must restrict the physical states to be color singlets.

The generalization of the p-adic formula for Riemann integral [K47] suggests an exact formula for the time ordered product. The first guess is that one simply forms the product

$$\begin{aligned} Pexp(i\int_0^t Hdt) &\equiv P\prod_n exp\left[iV(t(n))\Delta t(n)\right] \ , \\ \Delta t(n) &= t_+(n)-t_-(n)=(1+p)p^{m(n)} \ , \end{aligned} \tag{7.5.1}$$

to obtain the value of the time ordered product for time values t having finite number of pinary digits. The product is over all points $t(n)$ having finite number of pinary digits and $m(n)$ is the highest pinary digit in the expansion of $t(n)$ and $t_\pm(n)$ denote the two p-adic images of the real coordinate $t(n)_R$ under canonical identification. $\Delta t(n)$ corresponds to the difference of the p-adic time coordinates, which are mapped to the same value of the real time coordinate in canonical identification so that one can regard the time ordered product as a limiting case in which real time coordinate differences are exactly zero in the time ordered product.

The time ordering of the product is induced by canonical identification from real time ordering. This time development operator is defined for time values with finite number of pinary digits only and defines p-adic pseudo constant. The hope is that the inherent non-determinism of the p-adic differential equations, implied by the existence of the p-adic pseudo constants, makes it possible to continue this function to a p-adically differentiable function of the p-adic time coordinate satisfying the counterpart of the Schrödinger equation for the time development operator.

Not surprisingly, number theoretical problems are encountered also now: the exponential $exp\left[iV(t(n))\Delta t(n)\right]$ need not exist p-adically. The possibility of p-adic pseudo constants suggests that one could simply drop off the troublesome exponentials: this has far reaching physical consequences [K43] .

7.5.2 p-Adic inner product and Hilbert spaces

Concerning the physical applications of algebraically extended p-adic numbers the problem is that p-adic norm is not in general bilinear in its arguments and therefore it does not define inner product and angle. One can however consider a generalization of the ordinary complex inner product $\bar{z}z$ to a p-adic valued inner product. It turns out that p-adic quantum mechanics in the sense as it is used in p-adic TGD can be based on this inner product.

The algebraic generalization of the ordinary Hilbert space inner product is bilinear and symmetric, defines p-adic valued norm. The norm can however for non-vanishing states. This inner product leads to p-adic generalization of unitarity and probability concept. The solution of the unitarity condition $\sum_k S_{mk}\bar{S}_{nk}=\delta(m,n)$ involves square root operations and therefore the minimal extension for the Hilbert space is 4-dimensional in $p>2$ case and 8-dimensional in $p=2$ case. Of course, extensions of arbitrary dimension are allowed.

The inner product associated with a minimal extension allowing square root near real axis provides a natural generalization of the real and complex Hilbert spaces respectively. Instead of real or complex numbers, a square root allowing algebraic extension extension appears as the multiplier field of the Hilbert space and one can understand the points of Hilbert space as infinite sequences $(Z_1, Z_2, ..., Z_n, ...)$, where Z_i belongs to the extension. The inner product $\sum_k \langle Z_k^1, Z_2^k\rangle$ is completely analogous to the ordinary Hilbert space inner product.

The generalization of the the Hilbert space of square integrable functions to a p-adic context is far from trivial since definite integral in in general ill defined procedure. Second problem is posed by the fact that p-adic counterparts of say oscillator operator wave functions do not exist in the entire p-adic variant of the configuration space. Algebraic definition of the inner product by using the rules of Gaussian integration provides a possible solution to the problem.

For Fock space generated by anti-commuting fermionic and commuting bosonic oscillator operators the p-adic counterpart exists naturally and it seems that Fock spaces can be seen as universal Hilbert spaces with rational coefficients identifiable as subspaces of both real Fock space and of all p-adic Fock spaces.

7.5.3 p-Adic unitarity and p-adic cohomology

p-Adic unitarity and probability concepts lead to highly nontrivial conclusions concerning the general structure of the p-adic S-matrix. The most general S-matrix is a product of a complex rational (extended rationals are also possible) unitary S-matrix S_Q and a genuinely p-adic S-matrix S_p which deviates only slightly from unity

$$\begin{aligned} S &= 1+i\sqrt{p}T \ , \\ T &= O(p^0) \ . \end{aligned} \tag{7.5.1}$$

for $p \, mod \, 4=3$ allowing imaginary unit in its four-dimensional algebraic extension. In perturbative context one expects that the p-adic S-matrix differs only slightly from unity. Using the form $S=1+iT$, $T=O(p^0)$ one would obtain in general transition rates of order inverse of Planck mass and theory would have nothing to do with reality. Unitarity requirement implies iterative expansion of T in powers of p and the few lowest powers of p give excellent approximation for the physically most interesting values of p.

The unitarity condition implies that the moduli squared of the matrix T in $S=1+iT$ are of order $O(p^{1/2})$ if one assumes a four-dimensional p-adic extension allowing square root for the ordinary p-adic numbers and one can write

$$\begin{aligned} S &= 1+i\sqrt{p}T \ , \\ i(T-T^{\dagger}) &+ \sqrt{p}TT^{\dagger}=0 \ . \end{aligned} \tag{7.5.1}$$

This expression is completely analogous to the ordinary one since $i\sqrt{p}$ is one of the units of the four-dimensional algebraic extension. Unitarity condition in turn implies a recursive solution of the unitary condition in powers of p:

$$\begin{aligned} T &= \sum_{n\geq 0} T_n p^{n/2} \ , \\ T_n - T_n^{\dagger} &= \frac{1}{i}\sum_{k=0,..,n-1} T_{n-1-k}T_k^{\dagger} \ . \end{aligned} \tag{7.5.1}$$

If algebraic extension is not allowed then the expansion is in powers of p instead of $\sqrt{p}$. Note that the real counterpart of the series converges extremely rapidly for physically interesting primes (such as $M_{127}=2^{127}-1$).

In the p-adic context S-matrix $S=1+T$ satisfies the unitarity conditions

$$T+T^{\dagger} = -TT^{\dagger} \tag{7.5.2}$$

if the conditions

$$\begin{aligned} T &= T^{\dagger} \ , \\ T^2 &= 0 \ . \end{aligned} \tag{7.5.2}$$

defining what might be called p-adic cohomology, are satisfied [K2] . In the real context these conditions are not possible to satisfy as is clear from the fact that the total scattering rate from a given state, which is proportional to T^2_{mm} vanishes.

p-Adic cohomology defines a symmetry analogous to BRST symmetry: if T satisfies unitarity conditions and T_0 satisfies the conditions

$$\begin{aligned} &T_0 = T_0^{\dagger} \ , \qquad\qquad T_0^2 = 0 \ , \\ &\{T_0,T\} = T_0T+TT_0 = 0 \ , \end{aligned} \tag{7.5.3}$$

unitary conditions are satisfies also by the matrix $T_1 = T+T_0$. The total scattering rates are same for T and T_1.

7.5.4 The concept of monitoring

The relationship between p-adic and real probabilities involves the hypothesis that real transition probabilities depend on the cognitive resolution. Cognitive resolution is defined by the decomposition of the state space H into direct sum $H = \oplus H_i$ so that the experimental situation cannot differentiate between different states inside H_i. Each resolution defines different real transition probabilities unlike in ordinary quantum mechanics. Physically this means that the arrangement, where each state in H_i is monitored separately differs from the situation, when one only looks whether the state belongs to H_i. One can say that monitoring affects the behavior of a p-adic subsystem. Of course, these exotic effects relate to the physics of cognition rather than real physics.

Standard probability theory, which also lies at the root of the standard quantum theory, predicts that the probability for a certain outcome of experiment does not depend on how the system is monitored. For instance, if system has N outcomes $o_1, o_2, ...o_N$ with probabilities $p_1, ..., p_N$ then the probability that o_1 or o_2 occurs does not depend on whether common signature is used for o_1 and o_2 or whether observer also detects which of these outcomes occurs. The crucial signature of p-adic probability theory is that monitoring affects the behavior of the system.

Physically monitoring is represented by quantum entanglement [K42] , and differentiates between two eigen states of the density matrix only provided the eigenvalues of the density matrix are different. If there are several degenerate eigenvalues, quantum jump occurs to any state in the eigen space and one can predict only the total probability for the quantum jump into this eigen space: the real probabilities for jumps into individual states are obtained by dividing total real probability by the degeneracy factor. Hence the p-adic probability for a quantum jump to a given eigenspace of density matrix is p-adic sum of probabilities over the eigen states belonging to this eigenspace:

$$P_i = \frac{(n(i)P(i))_R}{\sum_j (n(j)P(j))_R} \quad .$$

Here n_i are dimensions of various eigenspaces.

If the degeneracy of the eigenvalues is removed by an arbitrary small perturbation, the total probability for the transition to the same subspace of states becomes the sum for the real counterparts of probabilities and one has in good approximation:

$$P^R = \frac{n(i)P(i)_R}{[\sum_{j\neq i}\sum_j (n(j)P(j))_R + n(i)P(i)_R]} \quad .$$

Rather dramatic effects could occur. Suppose that that the entanglement probability $P(i)$ is of form $P(i) = np$, $n \in \{0, p-1\}$ and that n is large so that $(np)_R = n/p$ is a considerable fraction of unity. Suppose that this state becomes degenerate with a degeneracy m and $mn > p$ as integer. In this kind of situation modular arithmetics comes into play and $(mnp)_R$ appearing in the real probability $P(1\ or\ 2)$ can become very small. The simplest example is $n = (p+1)/2$: if two states i and j have *very nearly equal but not identical* entanglement probabilities $P(i) = (p+1)p/2 + \epsilon$, $P(j) = (p+1)p/2 - \epsilon$, monitoring distinguishes between them for arbitrary small values of ϵ and the total probability for the quantum jump to this subspace is in a good approximation given by

$$\begin{aligned} P(1\ or\ 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= 2\left[(p+1)p/2\right]_R \quad . \end{aligned} \tag{7.5.3}$$

and is rather large. For instance, for Mersenne primes $x \simeq 1/2$ holds true. If the two states become degenerate then one has for the total probability

$$\begin{aligned} P(1\ or\ 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= \frac{1}{p} \quad . \end{aligned} \tag{7.5.3}$$

The order of magnitude for $P(1\ or\ 2)$ is reduced by a factor of order $1/p$!

Since p-adicity is essential for the exotic effects related to monitoring, the exotic phenomena of monitoring should be related to the quantum physics of cognition rather than real quantum physics. A test for quantum TGD would be provided by the study of the dependence of the transition rates of quantum systems on the resolution of monitoring defined by the dimensions of the degenerate eigenspaces of the subsystem density matrix. One could even consider the possibility of measuring the value of the p-adic prime in this manner. The behavior of living systems is known to be sensitive to monitoring and an exciting possibility is that this sensitivity, if it really can be shown to have statistical nature, could be regarded as a direct evidence for TGD inspired theory of consciousness. Note that the mapping of the physical quantities to entanglement probabilities could provide an ideal manner to compare physical quantities with huge accuracy! Perhaps bio-systems have invented this possibility before physicists and this could explain the miraculous accuracy of biochemistry in realizing genetic code. The measurement of the monitoring effect could provide a manner to determine the value of p_i for each p-adic region of space-time.

An alternative interpretation for the degenerate eigenvalues has emerged years after writing this. The sub-spaces corresponding to given eigenvalue of density matrix represent entangled states resulting in state function reduction interpreted as measurement of density matrix. This entanglement would be negentropic and represent a rule/concept, whose instances the superposed state pairs are. The information measure would Shannon entropy based on the replacement of the probability appearing as argument of logarithm with its p-adic norm. This entropy would be negative and therefore measure the information associated with the entanglement. This number theoretic entropy characterizes two particle state rather than single particle state and has nothing to do with the ordinary Shannon entropy.

Maybe one could say that finite measurement resolution implies automatically conceptualization and rule building. Abstractions are indeed obtained by dropping out the details.

7.5.5 p-Adic Schrödinger equation

The emergence of the p-adic infrared cutoff

The experience with the construction of the p-adic counterpart of the standard model shows that p-adic quantum theory involves in practice infrared cutoff length scale in both time and spatial directions. The cutoff length scale comes out purely number theoretically. In the time like direction the cutoff length scale comes out from the exponent of the time ordered integral: p-adic exponent function $exp(x)$ does not exist unless the p-adic norm of the argument is smaller than one and this in turn means that $P(exp(i\int_0^t Vdt))$ does not exist for too larger values of time argument. A more concrete manner to see this is to consider time dependence for the eigenstates of Hamiltonian: the exponent $exp(iEt)$ exists only for $|Et|_p < 1$. The necessity of the spatial cutoff length scale is seen by considering concrete examples. For instance, the p-adic counterparts of the harmonic oscillator Gaussian wave functions are defined only in a finite range of the argument. As far as the definition of exponent function is considered one must keep in mind that the formal exponent function does not have the usual periodicity properties. The definition as a p-adic plane wave gives the needed periodicity properties but also in this case the infrared cutoff is necessary.

One should be able to construct also global solutions of the p-adic Schrödinger equation. The concept of p-adic integration constant might make this possible: by multiplying the solution of thhe Schrödinger equation with a constant depending on a finite number of the pinary digits, one can extend the solution to an arbitrary large region of the space time. What one cannot however avoid is the decomposition of the space time into disjoint quantization volumes.

One of the original motivation to introduce p-adic numbers was to introduce ultraviolet cutoff as a p-adic cutoff but, as the considerations of the second part of the book show, UV divergences are absent in the p-adic case and short distance contributions to the loops are negligibly small so that the mere p-adicization eliminates automatically UV divergences. Rather, it seems that the length scale L_p serves as an infrared cutoff and, if a length scale resolution rougher than L_p is used, ordinary real theory should work. Only in the length scales $L \leq L_p$ should the p-adic field theory and Quantum Mechanics be useful. The applicability of the real QM for length scale resolution $L \geq L_p$ is in accordance with the fact that the real continuity implies p-adic continuity.

Formal p-adicization of the Schrödinger equation

The formal p-adic generalization of the Schrödinger equation is of the following general form

$$\theta\frac{d\Psi}{dt} = H\Psi \ , \tag{7.5.4}$$

where H is in some sense Hermitian operator. If Schödinger amplitudes are complex values θ can be taken to be imaginary unit i. The same identification is possible if Ψ possesses values in the extension of p-adic allowing square root and the condition $p \ mod \ 4 = 3$ or $p = 2$ guaranteeing that $\sqrt{-1}$ does not exist as an ordinary p-adic number, is satisfied. For $p \ mod \ 4 = 1$ the situation is more complicated since imaginary unit i does not in general belong to the generators of the minimal extension allowing a square root. An open problem is whether one could replace θ appearing in the quadratic extension and define complex conjugation as the operation $\theta \rightarrow -\theta$. The analogy with the ordinary quantum mechanics suggests the form

$$H = -\frac{\nabla^2}{2m} + V \ , \tag{7.5.4}$$

for the Hamiltonian in $p \ mod \ 4 = 3$ case. In the complex case ∇^2 is obtained by replacing the ordinary derivatives with the p-adic derivatives and V is a p-adically differentiable function of the coordinates typically obtained from a p-analytic function via the canonical identification.

Although the formal p-adicization is possible, it is not at all obvious whether one can get anything physically interesting from the straightforward p-adicization of the Schrödinger equation. The study of the the p-adic hydrogen atom shows that formal p-adicization need not have anything to do with physics. For instance, Coulomb potential contains a factor $1/4\pi$ not existing p-adically, the energy eigenvalues depend on π and the straightforward p-adic counterparts of the exponentially decreasing wave functions are not exponentially decreasing functions p-adically and do not even exist for sufficiently large values of the argument r. It seems that a more realistic manner to define the p-adic Schrödinger equation is as limiting case of the p-adic field theory. Of course, it might also be that p-adic Schrödinger equation does not make sense. A more radical solution of the problems is the allowance of finite-dimensional extensions of p-adic numbers allowing also transcendental numbers.

p-Adic harmonic oscillator

The formal treatment of the p-adic oscillator using oscillator operator formalism is completely analogous to that of the ordinary harmonic oscillator. The only natural inner product is the p-adic valued one. That the treatment is correct is suggested by the fact that it is purely algebraic involving only the p-adic counterpart of the oscillator algebra. The matrix elements of the oscillator operators $a^\dagger$ and a involve square roots and they exist provided the minimal extension allowing square roots appears as a coefficient ring of the Hilbert space. If two-dimensional quadratic extension not containing $\sqrt{p}$ is used occupation number must be restricted to the range $[0, p-1]$. If the Hilbert space inner product based on non-degenerate p-adic inner product $Z_cZ + \hat{Z}_c\hat{Z}$ the extension implies a characteristic degeneracy of states with complex amplitudes related to the conjugation $\sqrt{p} \rightarrow -\sqrt{p}$. 2-adic and p-adic cases differ in radical manner since the dimensions of the extension are 4 for $p > 2$ and 8 for $p = 2$. Since the representations of the Kac Moody and Super Virasoro algebras are based on oscillator operators this means that there is deep difference between $p = 2$ and $p > 2$ p-adic conformal field theories.

The p-adic energy eigen values are $E_n = (n + 1/2)\omega_0$ and their real counterparts form a quasi-continuous spectrum in the interval $(2, 4)$ for $p = 2$ and $(1, p)$ for $p > 2$! If p is very large (of order 10^{38} in TGD applications) the small quantum number limit $n < p$ gives the quantum number spectrum of the ordinary quantum mechanics. The occupation numbers $n > p$ have no counterpart in the conventional quantum theory and it seems that the classical theory with a quasi-continuous spectrum but with energy cutoff $p\omega_0$ is obtained at the limit of the arbitrarily large occupation numbers. The limit $p \to \infty$ gives essentially the classical theory with no upper bound for the energy.

The results suggests the idea that p-adic QM might be somewhere halfway between ordinary QM and classical mechanics. This need not however be the case as the study of the p-adic thermodynamics suggests. p-Adic thermodynamics allows a low temperature phase $exp(E_n/T) \equiv p^{n/T_k}$, $T_k = 1/k$, with quantized value of temperature. In this phase the probabilities for the energy eigenstates E_n, $n = \sum_k n_k p^k$ are extremely small except for the smallest values of n so that low temperature thermodynamics does not allow the effective energy continuum. One might argue that situation changes in the high temperature phase. The problem is that p-adic thermodynamics for the harmonic oscillator allows only formally high temperature phase $T = t_0\omega_0/p^k$, $k = 1, 2, ...$, $|t_0| = 1$. The reason is that Boltzmann weights $exp(-E_n/T) = exp(np^k/t_0)$ have p-adic norm equal to 1 so that the sum of probabilities giving free energy converges only formally. If one accepts the formal definition of the free energy as $exp(F) \equiv 1/(1 - exp(-E_0/T))$ then the real counterpart of the energy spectrum indeed becomes continuum also in the thermodynamic sense.

Consider next what a more concrete treatment using Schrödinger equation gives. The p-adic counterpart of the Schrödinger equation is formally the same as the ordinary Schrödinger equation. Ψ is assumed to have values in a minimal extension of p-adic numbers allowing square root and possessing imaginary unit so that the condition $p \ mod \ 4 = 3$ or $p = 2, 3$ must hold true. For the energy momentum eigenstates the equation reduces to

$$(-\frac{d^2}{dy^2} + y^2)\Psi = 2e\Psi \ , \tag{7.5.5}$$

where the dimensionless variables $y = \sqrt{\omega}x$ and $e = \frac{E}{\omega}$ have been introduced. This transformation makes sense provided ω possesses p-adic square root.

The solution ansatz to this equation can be written in the general form $\Psi = exp(-y^2/2)H_{e-1/2}(y)$, where H is the p-adic counterpart of a Hermite polynomial. The first thing to notice is that vacuum wave function does not converge in a p-adic sense for all values of y. A typical term in series is of the form $X_n = \frac{y^{2n}}{2^n n!}$. In ordinary situation the factors, in particular $n!$, in the numerator imply convergence but in present case the situation is exactly the opposite.

In 2-adic case both the factor 2^n and the factor $n!$ in the denominator cause troubles whereas for $p > 2$ the p-adic norm of 2^n is equal to one. $n!$ gives at worst the power 2^{n-1} to the 2-adic norm. Therefore the 2-adic norm of X_n behaves as $N(X_n) \simeq |y_2|^{2n}2^n2^{n-1}$. The convergence is therefore achieved for $|y|_2 \leq 1/4$ only. For $p > 2$ the convergence is achieved for $|y|_p \leq 1/p$. One can continue the oscillator Gaussian to a globally defined function of y by observing that the scaling $y \to y/\sqrt{2}$ corresponds to taking a square root of the oscillator Gaussian and this square root exists if minimal quadratic extension allowing square root is used. In the usual situation the function $H_e(y)$ must be polynomial since otherwise it behaves as $exp(y^2)$ and does not converge: this implies the quantization of energy also now.

The inner product, which should orthogonalize the states is the p-adic valued inner product based on the p-adic generalization of the definite integral. The generalizations of the analytic formulas encountered in the real case should hold true also now. The guess motivated by the formal treatment is that p-adic energies are quantized according to the usual formula and classical energies form a continuum below the upper bound $e_R \leq 4$ in 2-adic case and $e_R =\leq p$ in p-adic case. In fact, the mere requirement $|e|_p \leq 1$ implies that energy is quantized according to the formula $e = n + 1/2$ in p-adic case.

p-Adic fractality in the temporal domain

The assumption that p-adic physics gives faithful cognitive representation of the real physics leads to highly nontrivial predictions, the most important prediction being p-adic fractality with long range temporal correlations and microtemporal chaos.

In p-adic context the diagonalization of the Hamiltonian for N-dimensional state space in general requires N-dimensional algebraic extension of p-adic numbers even when the matrix elements of the Hamiltonian are complex rational numbers. TGD as a generalized number theory vision allows all algebraic extensions of p-adic numbers so that this is not a problem. The necessity to decompose p-adic Hamiltonian to a complex rational free part and p-adically small interaction part could provide the fundamental reason for why Hamiltonians have the characteristic decomposition into free and interaction parts. Of course, it might be that Hamiltonian formalism does not make sense in the p-adic context and should be replaced with the approach based on Lagrangian formalism: at least in case of p-adic QFT limit of TGD this approach seems to be more promising. One could also argue that the very fact that p-adic physics provides a cognitive representations of TGD based physics gives a valuable guide to the real physics itself, and that one should try to identify the constraints on real physics from the requirement that its p-adic counterpart exists. The following discussion is motivated by this kind of attitude.

The emergence of various dynamical time scales is a very general phenomenon. For instance, it seems that strong and weak interactions correspond to different time scales in well defined sense and that it is a good approximation to neglect strong interaction in weak time scales and vice versa. p-Adic framework gives hopes of finding a more precise formulation for this heuristics using number theoretical ideas. The basic observation is that the time ordered exponential of a given interaction Hamiltonian exists only over a finite time interval of length $T_p(n) = p^n L_p$. This suggests that one should distinguish between the time developments associated with various p-adic time scales $T_n = p^n L_p/c$: obviously temporal fractality would be in question.

More concretely, the p-adic exponential $exp(iH\Delta t)$ of the free Hamiltonian exists p-adically only if one assumes that Δt is a small rational proportional to a positive power of p: $\Delta t \propto p^n$. Of course, this restriction to the allowed values of Δt might be interpreted as a failure of the cognitive representation rather than a real physical effect. Alternatively, one might argue that the emergence of the p-adic time scales is a real physical effect and that one must define a separate S-matrix for each p-adic time scale $\Delta t \propto p^n$. Thus p-adic S-matrices for time intervals that differ from each other by arbitrarily long real time interval could be essentially identical. This would mean extremely precise fractal long range correlations and chaos in short time scales also at the level of real physics. This is certainly a testable and rather dramatic prediction in sharp contrast with standard physics views. $1/f$ noise could be seen as one manifestation of these long range correlations.

What would distinguish between different times scales would be different decomposition of the Hamiltonian to free and interaction parts to achieve interaction part which is p-adically small in the time scale involved. For instance, it could be possible to understand color confinement in this manner: in quark gluon plasma phase below the length scale L_p many quark states without any constraints on color are the natural state basis whereas above the length scale L_p physical states must be color singlets since otherwise time evolution operator does not exist.

In case of the cognitive representations of the external world canonical identification maps long external time and length scales to short internal time and length scales and vice versa. Thus p-adic fractality of the cognitive dynamics induces at the level of cognitive representation order in short length and time scales and chaos in long length and time scales: this is of course natural since sensory information comes mainly from the nearby spatiotemporal regions of the system. For self-representations there is chaos in short time scales and fractal long range correlations (so that our temptation to see our life as a coherent temporal pattern would not be self deception!). This kind of fractality is of course absolutely essential in order to understand bio-systems as intentional systems able to plan their future behavior. This prediction is about behavioral patterns of cognitive systems and also testable.

One can get a more quantitative grasp on this idea by studying the time development operator associated with a diagonalizable Hamiltonian. If the eigenvalues E_n of the diagonalized Hamiltonian have p-adic norms $|E_n|_p \leq p^{-m}$, the time evolution determined by this Hamiltonian is defined at most over a time interval of length norm $T_p(m) = p^{m-1}L_p$ since for time intervals longer than this the eigenvalues $exp(iE_nt)$ of $exp(iHt)$ do not exists as a p-adic numbers for all energy eigenstates. Thus one must restrict the time evolution to time scale $t \leq p^{m-1}L_p$: this is consistent with a p-adic hierarchy of interaction time scales.

An alternative approach is based on the requirement that the complex phase factors $exp(iET)$ for the eigenstates of the diagonal part of the Hamiltonian are complex rational phases forming a multiplicative group. This means that one can map the phase factors $exp(iET)$ directly to their p-adic counterparts as complex rational numbers. With suitable constraints on the energy spectrum this makes sense if the interaction time T is quantized so that it is proportional to a power of p. The decomposition of the Hamiltonian to free and interacting parts could be done in such a manner that the exponential of Hamiltonian decomposes to a product of diagonal part representable as complex rational phases and interaction part which is of higher order in p so that ordinary exponential exists for sufficiently small values of interaction time. This decomposition depends on the p-adic time scale.

How to define time ordered products?

In perturbation theory one must deal with the p-adic counterpart of the time ordered exponential $\prod_n Pexp\left[\int_0^t H_{int}(n)dt\right]$ appearing in the definition of the time development operator. In the case of a non-diagonal, time dependent interaction Hamiltonian the very definition of the p-adic counterpart of the time ordered integral is far from obvious since p-adic numbers do not allow natural ordering. Perhaps the simplest possibility is based on Fourier analysis based on the use of Pythagorean phases. This automatically involves the introduction of a time resolution $\Delta t = q = m/n$ and discretization of the time coordinate. Depending on the p-adic norm of Δt one obtains a hierarchy of S-matrices corresponding to different p-adic fractalities. Time ordering would be naturally induced from the ordering of ordinary integers since only the integer multiples of Δt are involved in the discretized version of integral defined by the inner product for the Pythagorean plane waves. The requirement that all time values have same p-adic norm implies $T = n\Delta t$, $n = 0, ..., p-1$. If one assumes that long range fractal temporal order is present one can also allow time intervals $T = n\delta t + mp^k$ which correspond to arbitrarily long real time intervals.

p-Adic particle stability is not equivalent with real stability

It is natural to require that single hadron states are eigenstates for that part of the total Hamiltonian, which consists of the kinetic part of the Hamiltonian. If this the case, one can require that the effect of $exp(iH_0t)$ is just a multiplication by the factor $exp(iEt)$. The fact that particles are not stable against decay to many-particle states suggests that E must be complex. Generalizing the construction of the p-adic plane waves one could define this prefactor for all values of time even in this case. One can however criticize this approach: the introduction of the decay width as imaginary part of E is is category error since decay width characterizes the statistical aspects of the dynamics associated with quantum jumps rather than the dynamics of the Schrödinger equation.

p-Adic unitarity concept suggests a more elegant description. The truncated S-matrix describing the transitions $H_p \rightarrow H_p$ is unitary despite the fact that the transitions between different sectors are possible. This makes sense because the total p-adic transition probability from H_p to H_q, $q \neq p$, vanishes by generalized unitarity conditions. Generalizing, the p-adic representations of elementary particles and even hadrons would p-adically stable in the sense that the total p-adic decay probability would vanishes for them. One could also say that in absence of monitoring p-adic cognitive representation of particle would be stable. This picture is consistent with the notion of p-adic cohomology reducing unitarity conditions for S-matrix $S = 1 + iT$ to the conditions $T = T^\dagger$ and $T^2 = 0$. Of course, it would apply only at the level of cognitive physics.

7.6 Generalization of the notion of configuration space

The number theoretic variants of Shannon entropy make sense for rational and even algebraic entanglement probabilities in finite-dimensional algebraic extensions of rationals and can have negative values so that negentropic entanglement (see fig. `http://www.tgdtheory.fi/appfigures/cat.jpg` or fig. 21 in the appendix of this book) becomes possible. This leads to the vision that life resides in the intersection of real and p-adic worlds for which partonic 2-surfaces- the basic geometric objects- allow a mathematical definition making sense both in real and p-adic sense in preferred coordinates dictated to a high degree by imbedding space symmetries. Rational functions with rational or algebraic coefficients provide a basic example of this kind of functions as also algebraic functions. This vision together with Negentropy Maximization Principle leads to an overall view about how the standard physics picture must be modified in TGD framework (see the next chapter [K42]).

The identification of life as something in the intersection means that there should be also physics outside it. In the real context this poses no problems of principle. But should one allow the continuation of the coefficients of rational functions to p-adic integers infinite as real integers? This seems to raise formidable looking challenges.

1. One should be able to formulate the geometry of the world of classical worlds (WCW) in p-adic sense and also construct p-adic counterparts for the integration over WCW. Since no physically acceptable p-adic variant of definite integral does exists, algebraic continuation seems to be the only possible manner to meet this challenge.

2. One must construct the p-adic counterparts of Kähler function or of its exponent (or both), Kähler metric and Kähler form at the level of WCW.

3. Kähler function identified as Kähler action for preferred extremal and defined as integral does not make sense as such in p-adic context and the only manner to define the p-adic variant of Kähler function is by algebraic continuation from the real sector through the intersection of real and p-adic worlds.

7.6.1 Is algebraic continuation between real and p-adic worlds possible?

It seems that algebraic continuation is the only reasonable manner to tackle these challenges. The following considerations suggests that there are some hopes.

1. Recall that the basic geometric objects can be identified either as light-like 3-surfaces connecting the boundaries of causal diamond (intersection of future and past directed light-cones) or as space-like 3-surfaces at the boundaries of CD. The condition that the identifications are equivalent implies effective 2-dimensionality: the partonic 2-surfaces at the boundaries of causal diamonds (CDs) together with the distribution of four-dimensional tangent planes of space-time surface at the points of the partonic surface, are the basic geometric objects. The tangent space distribution codes for various quantum numbers such as four-momentum so that also these must be rational valued in the common sector. In the following I will just speak about partonic 2-surfaces. It is this space-time 2-surfaces for a given CD, which should be geometrized. 2-dimensionality obviously suggests a connection with algebraic geometry.

2. Number theoretic vision [K72] leads to the conclusion that the space-time sheets are quaternionic in the sense that the modified gamma matrices assignable to the Kähler action in their octonionic representations span quaternionic (co-quaternionic) and thus associative (co-associative) subspace of complexified octonions at each point of the space-time surface. Quaternionicity would be realized in Minkowskian regions and co-quaternionicity in the space-like regions defining geometrization of Feynman diagrams. This notion is independent of the number field so that the notion of p-adic space-time sheet seems to make sense. Note that also the field equations and criticality condition for the preferred extremals [K26] make sense p-adically as purely algebraic conditions.

3. The representability of WCW as a union of symmetric spaces means an enormous simplification since everything reduces to a single point, most naturally the maximum of Kähler function for given values of zero modes. If this maximum is always an algebraic surface and if the Kähler function or its exponent for it is algebraic number (there is infinity of tunings of zero modes guaranteeing this), the maxima make sense also in suitable algebraic extensions of p-adic numbers. The maxima would obviously define the intersection of real and p-adic worlds.

   One might in fact argue that this is as it must be. What is cognitively representable is in the intersection of realities and p-adicities and mathematician can cognitively represent only these maxima and do perturbation theory around them and hope for a complete integrability.

4. What comes naturally in mind is that only p-adically small deformations of the partonic 2-surfaces in the intersections of the p-adic and real worlds are allowed at the p-adic side. If the exponent of Kähler function exists in some algebraic extension at the common point, its small perturbations can be expanded in powers of p as a functional of the coefficients of rational functions extended to p-adic numbers. Symmetric space structure of WCW raises the hope that TGD is a completely integrable theory in the sense that the functional integral

reduces to the exponent of Kähler action due to the cancellation of metric and Gaussian determinants the n-point functions. One would have effectively free field theory. If this is the case the functional integral would make sense also in p-adic context as algebraic continuation.

Consider now in more detail what the algebraic continuation could mean.

1. Kähler function is not uniquely defined since one can add to it a real part of a holomorphic function of WCW complex coordinates (associated with quantum fluctuating degrees of freedom) without affecting Kähler metric. By a suitable choice of this function algebraicity could be guaranteed for any partonic 2-surface. This symmetry is however much like gauge invariance, which suggests that functional integral expressions for n-point functions involving also normalization factors do not depend on the exponent of Kähler function at maximum. In the perturbative approach to quantum field theory the exponents indeed cancel from n-point functions. This would suggest that the algebraicity of Kähler function is only needed. One should be however be very cautious. The Kähler action for CP_2 type vacuum extremals has a deep meaning in TGD and would have interpretation in terms of a non-perturbative effect. If one allows the introduction of a finite-dimensional non-algebraic extension involving powers of some root of e (e^p exists p-adically) both the exponent of Kähler function and Kähler function exist p-adically if Kähler function is a rational number.

2. WCW Kähler metric can be defined in terms of second partial derivatives of the exponent of Kähler function and is algebraic if Kähler function or its exponent are algebraic functions of the preferred WCW coordinates defined by WCW symmetries. The tangent space distribution at X^2 codes information about quantum numbers - in particular four-momenta- which define a measurement interaction terms in Kähler action and by supersymmetry also in Kähler-Dirac action [K26]: by analogy with thermodynamics these terms are simply Lagrangian multiplier terms equating classical conserved charges of space-time surfaces in the quantum superposition with the quantal counterparts. By holography Kähler function or its exponent is expressible in terms of the data associated with X^2 and its tangent space and should be algebraic function of these data.

3. If Kähler function or its exponent is rational function of the parameters charactering partonic 2-surfaces, the continuation to the p-adic sectors at rational points is in principle possible. If Kähler function is proportional to a positive power of p its exponent exists automatically in p-adic context. For Kähler function this would mean that given partonic 2-surface would correspond to a finite number of primes only. The continuation of the exponent of Kähler function is not however very useful since WCW integral cannot be defined except by algebraic continuation. Exponent function behaves also completely differently in p-adic context than in real context (its p-adic norm equals always to one for instance). p-Adic thermodynamics would in turn suggest that the exponent function should be replaced by a power of p since it has desired convergence properties so that Kähler function divided by $log(p)$ should be rational (allowing roots of p in the algebraic extension).

4. The perturbative approach relies on n-point functions involving WCW Hamiltonians and their super-counterparts at the intersection. One would obtain algebraic expressions for the n-point functions involving also contravariant metric of WCW of as a propagator. If one always works in effectively finite-dimensional space (coefficients of polynomials with finite degree in the definition of partonic 2-surfaces involved and rational valued momenta) one has finite-dimensional space of partonic 2-surraces, and the propagator is an algebraic object as the inverse of the Kähler metric defined by the second derivatives of the Kähler function if K or is exponent is algebraic function. p-Adicization also means the continuation of the momenta to the p-adic sector.

5. WCW Hamiltonians and their super-counterparts are defined as integrals over partonic 2-surface and it is not at all obvious that the result is algebraic number even if these quantities themselves are rational functions even in the partonic 2-surfaces themselves are rational surfaces. The condition for being in the intersection should therefore include also the condition about the algebraic character of these objects.

6. One could of course wonder whether coupling constant renormalization involving logarithmic functions of mass scales and powers of π in QFT context could spoil this nice picture and force to introduce infinite-dimensional transcendental extensions of p-adic numbers. There is indeed the danger that symmetric space property is not enough to avoid infinite perturbation series coming from the expansions of WCW Hamiltonians and their super counterparts. This kind of series would obviously spoil the algebraic character. There are however hopes. First of all, tinite measurement resolution is one of the key aspects of quantum TGD and could boild down to a cutoff for the perturbation series. Secondly, the key idea of quantum criticality is that for the maxima of Kähler function the perturbative corrections sum up to zero since they are coded to the Kähler action itself since the scale of induced metric is proportional to the square of $\hbar$.

If this optimistic picture is correct, the algebraic continuation to p-adic sector would reduce to an algebraic continuation of the expressions for n-point functions and the U-matrix in real sector to the p-adic sector, and would be almost trivial since only continuation in momenta and WCW coordinates parametrizing partonic 2-surfaces representing maxima of Kähler function would be in question. Everything could be computed in the real sector. A practically oriented theoretician might of course have suggested this from the beginning. It must be added that this vision is the latest one and need not completely consistent with all what is represented in the sequel.

7.6.2 p-Adic counterparts of WCW Hamiltonians

One must continue the δM^4_+ local CP_2 Hamiltonians appearing in the integrals defining WCW Hamiltonians to various p-adic sectors. CP_2 harmonics are homogeneous polynomials with rational coefficients and do not therefore produce any trouble since normalization factors involve only square roots. The p-adicization of δM^4_+ function basis defining representations of Lorentz group involves more interesting aspects.

p-Adicization of representations of Lorentz group

In the light cone geometry Poincare invariance is strictly speaking broken to Lorentz invariance with respect to the dip of the light cone and at least cosmologically a more natural basis is characterized by the eigenvalues of angular momentum and boost operator in a given direction. The eigenvalue spectrum of the boost operator is continuous without further conditions. One can study these conditions in the realization of the unitary representations of Lorenz group as left translations in the Lorentz group itself by utilizing homogenous functions of four complex variables z^1, z^2, z^3, z^4 satisfying the constraint $z_1z_4 - z_2z_3 = 1$ expressing the fact that they correspond to the homogenous coordinates of the Lorentz group defined by that matrix elements of the SL(2,C) matrix

$$\left(\begin{array}{cc} z_1 & z_3 \\ z_2 & z_4 \end{array} \right) \ .$$

The function basis consists of

$$\begin{array}{lll} f^{a_1,a_2,a_3,a_4}(z_1,z_2,z_3,z_4) & = & z_1^{a_1}z_2^{a_2}z_3^{a_3}z_4^{a_4} \ , \\ & & \\ a_1 = m_1 + i\alpha, & & a_2 = m_2 - i\alpha \ , \\ a_3 = m_3 - i\alpha, & & a_4 = m_4 + i\alpha \ , \\ m_1 + m_2 = M \ , & & m_3 + m_4 = M \ . \end{array}$$

The action of Lorentz transformation is given by

$$\left(\begin{array}{cc} z_1 & z_3 \\ z_2 & z_4 \end{array} \right) \rightarrow \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{cc} z_1 & z_3 \\ z_2 & z_4 \end{array} \right) \ . \tag{7.6.1}$$

and unimodular ($ad - bc = 1$). Lorentz transformation preserves the imaginary parts $i\alpha$ of the complex degrees $d_i = m \pm i\alpha$ of $z_k^{\pm i\alpha + m_k}$ (as can be seen by using binomial series representations for the transformed coordinates). Also the sums $m_1 + m_2 = M$ and $m_3 + m_4 = M$ are Lorentz invariants. Hence the representation is characterized by the the pair (α, M). M corresponds to the minimum angular momentum for the $SU(2)$ decomposition of the representation.

The imaginary parts $i\alpha$ of the complex degrees correspond to the eigen values of Lorentz boost in the direction of the quantization axis of angular momentum. The eigen functions are proportional to the factor

$$\begin{aligned} &\rho_1^{i2\alpha}\rho_2^{-i2\alpha}\rho_3^{-i2\alpha}\rho_4^{i2\alpha}\ per, \\ &\rho_i = \sqrt{z_i\overline{z}_i}\ . \end{aligned}$$

Since one can write $\rho^{i2\alpha} = e^{i2log(\rho)\alpha}$, these are nothing but the logarithmic plane waves. The value set of $\alpha \geq 0$ is continuous in the real context.

The requirement that the logarithmic plane waves are continuable to p-adic number fields and exist p-adically for rational values of $\rho_i = m/n$, quantizes the values of α. This condition is satisfied if the quantities $p^{i2\alpha_i} = e^{i2log(p)\alpha_i}$ exist p-adically for any prime. As shown in [K61] , there seems to be no number theoretical obstructions for the simplest hypothesis $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$, with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes. The existence of p^{iy} in a finite-dimensional extension would require that α_i is proportional to π by a coefficient which for a given prime p_1 has sufficiently small p-adic norm so that the exponent can be expanded in powers series.

Obviously p-adicization gives strong quantization conditions. There is also a second possibility. As discussed in the same chapter, the allowance of infinite primes changes the situation. Let $X = \prod p_i$ be the product of all finite primes. $1 + X$ is the simplest infinite prime and the ratio $Y = X/(1+X)$ equals to 1 in real sense and has p-adic norm $1/p$ for all finite primes. If one allows α to be proportional to a power Y, then the p-adic norm of α can be so small for all primes that the expansion converges without further conditions. Infinite primes will be discussed later in more detail.

Exactly similar exponents (p^{iy}) appear in the partition function decomposition of the Riemann Zeta, and the requirement that these quantities exist in a finite algebraic extension of p-adic numbers for the zeros $z = 1/2 + iy$ of ζ requires that $e^{ilog(p)y}$ is in a finite-dimensional extension involving algebraic numbers and e. One could argue that for the extensions of p-adics the zeros of Zeta define a universal spectrum of the eigen values of the Lorentz boost generator. This might have implications in hadron physics, where the so called rapidity distribution correspond to the distributions of the particles with respect to the variable characterizing finite Lorentz boosts.

Although the realization of the using the functions in Lorentz group differs from the discussed one, the conclusion is same also for them, in particular for the representation realized at the boundary of the light cone which is one of the homogenous spaces associated with Lorentz group.

Function basis of δM^4_+

One can consider two function basis for δM^4_+ and both function basis allow continuation to p-adic values under similar conditions.

1. Spherical harmonic basis

The first basis consists of functions $Y^l_m \times (r_M/r_0)^{n/2+i\rho}$, $n = -2, -1, 0, ...$ For $n = -2$ these functions define a unitary representation of Lorentz group. The spherical harmonics Y^l_m require a finite-dimensional algebraic extension of p-adic numbers. Radial part defines a logarithmic wave $exp[i\rho log(r_M/r_0)]$ and the existence of this for finite-dimensional extension of p-adic numbers for rational values ρ and r_M is guaranteed by $log(p) = q_1exp(q_2)/\pi$ ansatz under the conditions already discussed.

2. Basis consisting of eigen functions of angular momentum and boost

Another function basis of δM^4_+ defining a non-unitary representation of Lorentz group and of conformal algebra consists of eigen states of rotation generator and Lorentz boost and is given by

$$f_{m,n,k} = e^{im\phi}\frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k\ . \qquad (7.6.2)$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of S^2. The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{1+\rho^2)^2} d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2 - 1)/(\rho^2 + 1), 2\rho/(\rho^2 + 1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta) - 1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n - k)$. This suggests that the condition

$$|m| \leq 2(n - k) \tag{7.6.3}$$

is satisfied quite generally.

The emergence of the three quantum numbers (m, n, k) can be understood. Light cone boundary can be regarded as a coset space $SO(3, 1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m, n, k) have interpretation as quantum numbers associated with this Cartan algebra. The representations of the Lorentz group are characterized by half-integer valued parameter $l_0 = m/2$ and complex parameter l_1. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

It is interesting to compare the representations in question to the unitary representations of Lorentz group discussed in [A168] .

1. The unitary representations discussed in [A168] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labelled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labelled by three different quantum numbers.

2. The representations of [A168] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labelling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteeing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity in this sense is not possible.

There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, WCW spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

Logarithmic waves and possible connections with number theory and fundamental physics

Logarithmic plane waves labelled by eigenvalues of the scaling momenta appear also in the definition of the Riemann Zeta defined as $\zeta(z) = \sum_n n^{-z}$, n positive integer [K61] . Riemann Zeta is expressible as a product of partition function factors $1/(1+p^{-x-iy})$, p prime and the powers n^{-x-iy} appear as summands in Riemann Zeta. Riemann hypothesis states that the non-trivial zeros of Zeta reside at the line $x = 1/2$. There are indeed intriguing connections. $Log(p)$ corresponds now to the $log(r_M/r_{min})$ and -x-iy corresponds to the scaling momentum $k_1 + ik_2$ so that the special physical role of the conformal weights $k_1 = 1/2 + iy$ corresponds to Riemann hypothesis. The appearance of powers of p in the definition of the Riemann Zeta corresponds to p-adic length scale hypothesis, ($r_M/r_0 = p$ in ζ and corresponds to a secondary p-adic length scale).

The assumption that the logarithmic plane waves are algebraically continuable from the rational points $r_M/r_{min} = m/n$ to p-adic plane waves using a finite-dimensional extension of p-adic numbers leads to the $log(p) = q_1 exp(q_2)/\pi$ ansatz. Similar hypothesis is inspired by the hypothesis that Riemann Zeta is a universal function existing simultaneously in all number fields. This inspires several interesting observations.

1. p-adic length scale hypothesis stating that $r_{max}/r_{min} = p^n$ is consistent with the number theoretical universality of the logarithmic waves. The universality of Riemann Zeta inspires the hypothesis that the zeros of Riemann Zeta correspond to rational numbers and to preferred values $k_1 + ik_2$ of the scaling momenta appearing in the logarithmic plane waves. In the recent context the most general hypothesis would be that the allowed momenta k_2 correspond to the linear combinations of the zeros of Riemann Zeta with integer coefficients.

2. Hardmuth Mueller [B3] claims on basis of his observations that gravitational interaction involves logarithmic radial waves for which the nodes come as $r/r_{min} = e^n$. This is true if the the scaling momenta k_2 satisfy the condition $k_2/\pi \in Z$. Perhaps Mueller's logarithmic waves really could be seen as a direct signature of the fundamental symmetries of WCW. In particular, this would require $r_{max}/r_{min} = e^m$.

3. The special role of Golden Mean $\Phi = (1 + \sqrt{5})/2$ in Nature could be understood if also $log(\Phi) = q_1 exp(q_2)/\pi$ or more general ansatz holds true. This would imply that the nodes of logarithmic waves can correspond also to the powers of Φ.

   One could of course argue that the number theory at the moment of Big Bang cannot have strong effects on what is observed in laboratory. This might be the case. On the other hand, the non-determinism of the Kähler action however strongly suggests that the construction of the WCW geometry involves all possible light like 3-surfaces of the future light cone so that logarithmic waves would appear in all length scales. Be as it may, it would be amazing if such an abstract mathematical structure as WCW geometry would have direct implications to cosmology and to the physics of living systems.

7.6.3 WCW integration

Assuming that U-matrix exists simultaneously in all number fields (allowing finite-dimensional extensions of p-adics), the immediate question is whether also the construction procedure of the real S-matrix could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. Not only the configuration space but also Kähler function and its exponent, Kähler metric, and WCW functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

Does symmetric space structure allow algebraization of WCW integration?

The basic structure is the rational WCW whose points have rational valued coordinates. This space is common to both real and p-adic variants of WCW. Therefore the construction of the generalized WCW as such is not a problem.

The assumption that WCW decomposes into a union of symmetric spaces labeled by zero modes means that the left invariant metric for each space in the union is dictated by isometries. It should be possible to interpret the matrix elements of WCW metric in the basis of properly normalized isometry currents as p-adic numbers in some finite extension of p-adic numbers allowing perhaps also some transcendentals. Note that the Kähler function is proportional to the inverse of Kähler coupling strength α_K which depends on p-adic prime p, and does seem to be a rational number if one takes seriously various arguments leading to the hypothesis $1/\alpha_K = klog(K^2)$, $K^2 = p \times 2 \times 3 \times 5.. \times 23$, and $k = \pi/4$ or $k = 137/107$ for the two alternative options discussed in [K61] . If so then the most general transcendentals required and allowed in the extensions used correspond to roots of polynomials with coefficients in an extension of rationals by e and algebraic numbers. As already discussed, infinite primes might provide the ultimate solution to the problem of continuation.

The continuation of the exponent of Kähler function and of WCW spinor fields to p-adic sectors would require some selection of a subset of points of the rational WCW. On the other hand, the minimum requirement is that it is possible to define WCW integration in the p-adic context. The only manner to achieve this is by defining WCW integration purely algebraically by perturbative expansion. For free field theory Gaussian integrals are in question and one can calculate them trivially. The Gaussian can be regarded as a Kähler function of a flat Kähler manifold having maximal translational and rotational symmetries. Physically infinite number of harmonic oscillators are in question. The origin of the symmetric space is preferred point as far as Kähler function is considered: metric itself is invariant under isometries.

Algebraization of WCW functional integral

WCW is a union of infinite-dimensional symmetric spaces labelled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem [A133] generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each WCW spinor field would define a vertex from which lines representing the propagators defined by the contravariant WCW metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic WCW integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the WCW integrals would exist also in the p-adic sense.

Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in WCW inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the WCW would be possible in the entire WCW. Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also WCW spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X_l^3 to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the WCW. This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub-WCWs induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [K15] . The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_{C-S} are algebraic numbers for 3-surfaces X_l^3 for which the coefficients characterizing the rational functions defining X_l^3 are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics and physics of cognition and intention support also this conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad .$$

Coupling constant evolution and number theory

The coupling constant evolution associated with the Kähler action might be at least partially understood number-theoretically.

A given space-time sheet is connected by wormhole contacts to the larger space-time sheets. The induced metric within the wormhole contact has an Euclidian signature so that the wormhole contact is surrounded by elementary particle horizons at which the metric is degenerate so that the horizons are metrically effectively 2-dimensional giving rise to quaternion conformal invariance. Because of the causal horizon it would seem that Kähler coupling strength can depend on the space-time sheet via the p-adic prime characterizing it. If so the exponent of the Kähler function would be simply the product of the exponents for the space-time sheets and one would have finite-dimensional extension as required.

If the exponent of the Kähler function is rational function, also the components of the contravariant Kähler metric are rational functions. This would suggest that one function of the coupling constant evolution is to keep the exponent rational.

From the point of view of p-adicization the ideal situation results if Kähler coupling strength is invariant under the p-adic coupling constant evolution as I believed originally. For a long time it however seemed that this option cannot be realized since the prediction $G = L_p^2 exp(-2S_K(CP_2))$ for the gravitational coupling constant following from dimensional considerations alone implies that

G increases without limit as a function of p-adic length scale if α_K is RG invariant. If one however assumes that bosonic space-time sheets correspond to Mersenne primes, situation changes since M_{127} defining electron length scale is the largest Mersenne prime for which p-adic length scale is not super-astronomical and thus excellent candidate for characterizing gravitonic space-time sheets. There is much stronger motivation for this hypothesis coming from the fact that a nice picture about evolution of electro-weak and color coupling strengths emerges just from the physical interpretation of the fact that classical color action and electro-weak $U(1)$ action are proportional to Kähler action [K80] .

The recent progress in the understanding of the definition of the exponent of Kähler function as Dirac determinant [K15] leads to rather detailed picture about the number theoretic anatomy of α_K and other coupling constant strengths as well as the number theoretic anatomy of $R^2/\hbar G$ [K4] . By combining these results with the constraints coming from p-adic mass calculations one ends up to rather strong predictions for α_K and $R^2/\hbar G$.

Consistency check in the case of CP_2

It is interesting to look whether this vision works or fails in a simple finite-dimensional case. For CP_2 the Kähler function is given by $K = -log(1 + r^2)$. This function exists if an extension containing the logarithms of primes is used. $log(1 + x)$, $x = O(p)$ exists as an ordinary p-adic number and a logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p-1$ besides $log(p)$ should be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. Also logarithms of roots of integers and their products would exist. The problem is however that the powers of $log(m)$ and $log(p)$ would generate an infinite-dimensional extension since finite-dimensional extension leads to a contradiction as shown in [K61] .

The exponent of Kähler function as well as Kähler metric and Kähler form have rational-valued elements for rational values of the standard complex coordinates for CP_2. The exponent of the Kähler function is $1/(1+r^2)$ and exists as a rational number at 3-spheres of rational valued radius. The negative of the Kähler function has a single maximum at $r = 0$ and vanishes at the coordinate singularity $r \rightarrow \infty$, which corresponds to the geodesic sphere S^2.

If one wants to cognize about geodesic length, areas of geodesic spheres, and about volume of CP_2, π must be introduced to the extension of p-adics and means infinite-dimensional extension by the arguments of [K61] . The introduction of π is not however necessary for introducing of spherical coordinates if one expresses everything in terms of trigonometric functions. For ordinary spherical coordinates this means effectively replacing θ and ϕ by $u = \theta/\pi$ and $v = \phi/2\pi$ as coordinates. By allowing u and v to have a finite number of rational values requires only the introduction of a finite-dimensional algebraic extension in order to define cosines and sines of the angle variables at these values. What seems clear is that the evolution of cognition as the emergence of higher-dimensional extensions corresponds quite concretely to the emergence of finer discretizations.

7.6.4 Number theoretical Quantum Mechanics

The vision about life as something in the intersection of the p-adic and real worlds requires a generalization of quantum theory to describe the U-process properly. One must answer several questions. What it means mathematically to be in this intersection? What the leakage between different sectors does mean? Is it really possible to formally extend quantum theory so that direct sums of Hilbert spaces in different number fields make sense? Or should one consider the possibility of using only complex, algebraic, or rational Hilbert spaces also in p-adic sectors so that p-adicization would take place only at the level of geometry?

What it means to be in the intersection of real and p-adic worlds?

The first question is what one really means when one speaks about a partonic 2-surface in the intersection of real and p-adic worlds or in the intersection of two p-adic worlds.

1. Many algebraic numbers can be regarded also as ordinary p-adic numbers: square roots of roughly one half of integers provide a simple example about this. Should one assume that all algebraic numbers representable as ordinary p-adic numbers belong to the intersection of the

real and p-adic variants of partonic 2-surface (or to the intersection of two different p-adic number fields)? Is there any hope that the listing of the points in the intersection is possible without a complete knowledge of the number theoretic anatomy of p-adic number fields in this kind of situation? And is the set of common algebraic points for real and p-adic variants of the partonic 2-surface X^2 quite too large- say a dense sub-set of X^2?

This hopeless looking complexity is simplified considerably if one reduces the considerations to algebraic extensions of rationals since these induce the algebraic extensions of p-adic numbers. For instance, if the p-adic number field contains some n:th roots of integers in the range $(1, p-1)$ as ordinary p-adic numbers they are identified with their real counterparts. In principle one should be able to characterize the -probably infinite-dimensional- algebraic extension of rationals which is representable by a given p-adic number field as p-adic numbers of unit norm. This does not look very practical.

2. At the level WCW one must direct the attention to the function spaces used to define partonic 2-surfaces. That is the spaces of rational functions or even algebraic functions with coefficients of polynomials in algebraic extensions of rational numbers making sense with arguments in all number fields so that algebraic extensions of rationals provide a neat hierarchy defining also the points of partonic 2-surfaces to be considered. If one considers only the algebraic points of X^2 belonging to the extension appearing in the definition the function space as common to various number fields one has good hopes that the number of common points is finite.

3. Already the ratios of polynomials with rational coefficients lead to algebraic extensions of rationals via their roots. One can replace the coefficients of polynomials with numbers in algebraic extensions of rationals. Also algebraic functions involving roots of rational functions can be considered and force to introduce the algebraic extensions of p-adic numbers. For instance, an n:th root of a polynomial with rational coefficients is well defined if n:th roots of p-adic integers in the range $(1, p-1)$ are well well-defined. One clearly obtains an infinite hierarchy of function spaces. This would give rise to a natural hierarchy in which one introduces n:th roots for a minimum number of p-adic integers in the range $(1, p-1)$ in the range $1 \leq n \leq N$. Note that also the roots of unity would be introduced in a natural manner.

The situation is made more complex because the partonic 2-surface is in general defined by the vanishing of six rational functions so that algebraic extensions are needed. An exception occurs when six preferred imbedding space coordinates are expressible as rational functions of the remaining two preferred coordinates. In this case the number of common rational points consists of all rational points associated with the remaining two coordinates. This situation is clearly non-generic. Usually the number of common points is much smaller (the set of rational points satisfying $x^n + y^n = z^n$ for $n > 2$ is a good example). This however suggests that these surfaces are of special importance since the naive expectation is that the amplitude for transformation of intention to action or its reversal is especially large in this case. This might also explain why these surfaces are easy to understand mathematically.

4. These considerations suggest that the numbers common to reals and p-adics must be defined as rationals and algebraic numbers appearing explicitly in the algebraic extension or rationals associated with the function spaces used to define partonic 2-surfaces. This would make the deduction of the common points of partonic 2-surface a task possible at least in principle. Algebraic extensions of rationals rather than those of p-adic numbers would be in the fundamental role and induce the extensions of p-adic numbers.

Let us next try to summarize the geometrical picture at the level of WCW and WCW spinor fields.

1. WCW decomposes into WCWs associated with CDs and there unions. For the unions one has Cartesian product of WCWs associated with CDs. At the level of WCW spinor fields one has tensor product.

2. The WCW for a given CD decomposes into a union of sectors corresponding to various number fields and their algebraic extensions. The sub-WCW corresponding to the intersection consists of partonic 2-surfaces X^2 (plus distribution of 4-D tangent spaces $T(X^4)$ at X^2 - a complication which will not be considered in the sequel), whose mathematical representation makes sense in real number field and in some algebraic extensions of p-adic number fields. The extension of p-adic number fields needed for algebraic extension of rationals depends on p and is in general sub-extension of the extension of rationals. This sub-WCW is a submanifold of WCW itself. It has also a filtering by sub-manifolds of QCW. For instance, partonic 2-surfaces representable using ratios of polynomials with degree below fixed number N defines an inclusion hierarchy with levels labelled by N.

3. The spaces of WCW spinors associated with these sectors are dictated by the second quantization of induced spinor fields with dynamics dictated by the modified Dirac action in more or less one-one correspondence. The dimension for the modes of induced spinor field (solutions of the modified Dirac equation at the space-time surface holographically assigned with X^2 plus the 4-D tangent space-space distribution) in general depends on the partonic 2-surface and the classical criticality of space-time surface suggests an inclusion hierarchy of super-conformal algebras corresponding to a hierarchy of criticalities. For instance, the partonic 2-surfaces X^2 having polynomial representations in referred coordinates could correspond to simplest possible surfaces nearest to the vacuum extremals and having in a well define sense smallest (but possibly infinite) dimension for the space of spinor modes.

4. For each CD one can decompose the Hilbert space to a formal direct sum of orthogonal state spaces associated with various number fields

$$H = \oplus_F H_F \ . \tag{7.6.4}$$

Here F serves as a label for number fields. For the sake of simplicity and to get idea about what is involved, all complications due to algebraic extensions are neglected in the sequel so that only rational surfaces are regarded as being common to various sectors of WCW.

5. The states in the direct sum make sense only formally since the formal inner product of these states would be a sum of numbers in different number fields unless one assigns complex Hilbert space with each sector or restricts the coefficients to be rational which is of course also possible. This problem is avoided if the state function reduction process induces inside each CD a choice of the number field. One could say that state function is a number theoretical necessity at least in this sense.

   (a) Should the state function reduction in this sense involve a reduction of entanglement between distinct CDs is not clear. One could indeed consider the possibility of a purely number theoretical reduction not induced by NMP and taking place in the absence of entanglement with reduction probabilities determined by the probabilities assignable to various number fields which should be rational or at most algebraic. Hard experience however suggests that one should not make exceptions from principles.

   (b) The alternative is to allow the Hilbert spaces in question to have rational or at most algebraic coefficients in the intersection of real and various p-adic worlds. This means that the entanglement is algebraic and NMP need not lead to a pure state: the superposition of pairs of entangled states is however mathematically well defined since inner products give algebraic numbers. Cognitive entanglement stable under NMP would become possible. The experience of understanding could be a correlate for it. The pairs in the sum defining the entangled state defined the instances of a concept as a mapping of real world state to its symbol structurally analogous to a Boolean rule. The entangled states between different p-adic number fields would define maps between symbolic representations.

6. Assume that each H_F allows a decomposition to a direct sum of two orthogonal parts corresponding to WCW spinor fields localized to the intersection of number fields and to the complements of the intersection:

$$\begin{aligned} H &= H_{nm} \oplus H_m \ , \\ H_{nm} &= \oplus_F H_{nm,F} \ , \ H_m = \oplus_F H_{m,F} \ . \end{aligned} \tag{7.6.4}$$

   Here nm stands for 'no mixing' (no mixing between different number fields and localization to the complement of the intersection) and m for 'mixing' (mixing between different number fields in the intersection). F labels the number fields. Orthogonal direct sum might be mathematically rather singular and un-necessarily strong assumption but the notion of number theoretical criticality favors it.

The general structure of U-matrix neglecting the complexities due to algebraic extensions

M-matrix is diagonal with respect to the number field for obvious reasons. U-matrix can however induce a leakage between different number fields as well as entanglement between different number fields when unions of CDs are considered. The simplest assumption is that this entanglement is induced by the leakage between different number fields for single CD but not directly. For instance, the members of entangled pair of real states associated with two CDs leak to various p-adic sectors and induce in this manner entanglement beween different number fields. One must however notice that the part of U-matrix acting in the tensor product of Hilbert spaces assignable to separate CDs must be considered separately: it seems that the entanglement inducing part of U is diagonal with respect to number field except in the intersection.

To simplify the rather complex situation consider first the U matrix for a given CD by neglecting the possibility of algebraic extensions of the p-adic number fields. Restrict also the consideration to single CD.

1. The unitarity conditions do not make sense in a completely general sense since one cannot add numbers belonging to different number fields. The problem can be circumvented if the U-matrix decomposes into a product of U-matrices, which both are such that unitarity conditions make sense for them. Here an essential assumption is that unit matrix and projection operators are number theoretically universal. In this spirit assume that for a given CD U decomposes to a product of two U-matrices U_{nm} inducing no mixing between different number fields and U_m inducing the mixing in the intersection:

$$U = U_{nm} U_m \ . \tag{7.6.5}$$

   Here the subscript 'nm' (no mixing) having nothing to do with the induces of U as a matrix means that the action is restricted to a dispersion in a sector of WCW characterized by particular number field. The subscript 'm' (mixing) in turn means that the action corresponds to a leakage between different number fields possible in the intersection of worlds corresponding to different number fields and that U_m acts non-trivially in this intersection.

2. Assume that U_{nm} decomposes into a formal direct sum of U-matrices associated with various number fields F:

$$U_{nm} = \oplus_F U_{nm,F} \ . \tag{7.6.6}$$

   $U_{nm,F}$ acts inside H_F in both WCW and spin degrees of freedom, does not mix states belonging to different number fields, and creates a state which is always mathematically completely well defined in particular number field although the direct sum over number fields is only formally defined. Unitarity condition gives a direct sum of projection operators to Hilbert spaces associated with various number fields. One can assume that this object is number theoretically universal.

3. U_m acts in the intersection of the real and p-adic worlds identified in the simplified picture in terms of surfaces representable using ratios of polynomials with rational coefficients. The resulting superposition of WCW spinor fields in different number fields is as such not mathematical sensible although the expression of U_m is mathematically well-defined. If the leakage takes place with same probability amplitude irrespective of the quantum state, U_m is a unitary operator, not affecting at all the spinor indices of WCW spinor fields characterizing quantum numbers of the state and whose action is analogous to unitary mixing of the identical copies of the state in various number fields.

   The probability with which the intention is realized as action would not therefore depend at all on the quantum number fields, but only on the data at points common to the variants of the partonic 2-surface in various number fields. Intention would reduce completely to the algebraic geometry of partonic 2-surfaces. This assumption allows to write U in the form

$$U = U_{nm}U_m \ , \tag{7.6.7}$$

   where U_m acts as an identity operator in H_{nm}.

The general structure of U-matrix when algebraic extensions of rationals are allowed

Consider now the generalization of the previous argument allowing also algebraic extensions.

1. For each algebraic extension of rationals one can express WCW as a union of two parts. The first one corresponds to to 2-surfaces, which belong to the intersection of real and p-adic worlds. The second one corresponds to 2-surfaces in the algebraic extension of genuine p-adic numbers and having necessarily infinite size in real sense. Therefore the decomposition of U to a product $U = U_{nm}U_m$ makes sense also now.

2. It is natural to assume that U_m decomposes to a product of two operators: $U_m = U_H U_Q$. The strictly horizontal operator U_H connects only same algebraic extensions of rationals assigned to different number fields. Here one must think that p-adic number fields represent a large number of algebraic extensions of rationals without need for an algebraic extension in the p-adic sense. The second unitary operator U_Q describes the leakage between different algebraic extensions of rationals. Number theoretical universality encourages the assumption that this unitary operator reduces to an operator U_Q acting on algebraic extensions of rationals regarded effectively as quantum states so that it would be same for all number fields. One can even consider the possibility that U_Q depends on the extensions of rationals only and not at all on partonic 2-surfaces. One cannot assume that U_Q corresponds just to an inclusion to a larger state space since this would give an infinite number of identical copies of same state and imply a non-normalizable state. Physically U_Q would define dispersion in the space of algebraic extension of rationals defining the rational function space giving rise to the sub-WCW. The simplest possibility is that U_Q between different algebraic extensions is just the projection operator to their intersection multiplied by a numerical constant determined number theoretical in terms of ratios of dimensions of the algebraic extensions so that the diffusion between extensions products unit norm states.

One must take into account the consistency conditions from the web of inclusions for the algebraic extensions of rationals inducing extensions of p-adic numbers.

1. There is an infinite inverted pyramide-like web of natural inclusions of WCWs associated with algebraic extensions of rational numbers and one can assign a copy of this web to all number fields if a given p-adic number field is characterized by a web defined by algebraic

extensions of rationals numbers, which it is able to represent without explicit introduction of the algebraic extension, so that the pyramide is same for all number fields. For instance, the WCW corresponding to p-adic numbers proper is included to the WCWs associated with any of its genuine algebraic extensions and defines the lower tip of the inverted pyramide. From this tip an arrow emerges connecting it to every algebraic extension defining a node of this web. Besides these arrows there are arrows from a given extension to all extensions containing it.

2. These geometric inclusions induce inclusions of the corresponding Hilbert spaces defined by rational functions and possibly by algebraic functions in which case sub-web must be considered (all n:th roots of integers in the range $(1, p-1)$ must be introduced simultaneously). Leakage can occur between different extensions only through WCW spinor fields located in the common intersection of these spaces containing always the rational surfaces. The intersections of WCWs associated with various extensions of p-adic number fields correspond to WCWs assignable to rational functions with coefficients in various algebraic extensions of rationals using preferred coordinates of CD and CP_2.

Together with unitarity conditions this web poses strong constraints on the unitary matrices U_m and U_Q expressible conveniently in terms of commuting diagrams. There are two kinds of webs. The vertical webs are defined by the algebraic extensions of rationals. These form a larger web in which lines connect the nodes of identical webs associated with various p-adic number fields and represent algebraic extensions of rationals.

1. One has the general product decomposition $U = U_{nm}U_QU_m$, where U_{nm} does not induce mixing between number fields, and U_m does it purely horizontally but without affecting quantum states in WCW spin degrees of freedom, and $P(H_{nm})$ projects to the complement of the intersection of number fields holds true also now.

2. Each algebraic extension of rationals gives unitary conditions for the corresponding $U_{nm,F}$ for each p-adic number field with extensions included. These conditions are relatively simple and no commuting diagrams are needed.

3. In the horizontal web U_m mixes the states in the intersections of two number fields but connects only same algebraic extensions so that the lines are strictly horizontal. U_Q acts strictly vertically in the web formed by algebraic extension of rationals and its action is unitary. One has infinite number of commuting diagrams involving U_m and U_Q since the actions along all routes connecting given points between p_1 and p_2 must be identical.

4. If algebraic universality holds in the sense that U_m is expressible using only the data about the common points of 2-surfaces in the intersection defined by particular extensions using some universal functions, and U_Q is purely number theoretical unitary matrix having no dependence on partonic 2-surfaces, one can hope that the constraints due to commuting diagrams in the web of horizontal inclusions can be satisfied automatically and only the unitarity constraints remain. This web of inclusions brings strongly in mind the web of inclusions of hyper-finite factors.

7.7 How to define generalized Feynman diagrams?

S-matrix codes to a high degree the predictions of quantum theories. The longstanding challenge of TGD has been to construct or at least demonstrate the mathematical existence of S-matrix- or actually M-matrix which generalizes this notion in zero energy ontology (ZEO) [K60] . This work has led to the notion of generalized Feynman diagram and the challenge is to give a precise mathematical meaning for this object. The attempt to understand the counterpart of twistors in TGD framework [K79] has inspired several key ideas in this respect but it turned out that twistors themselves need not be absolutely necessary in TGD framework.

1. The notion of generalized Feyman diagram defined by replacing lines of ordinary Feynman diagram with light-like 3-surfaces (elementary particle sized wormhole contacts with throats

carrying quantum numbers) and vertices identified as their 2-D ends - I call them partonic 2-surfaces is central. Speaking somewhat loosely, generalized Feynman diagrams (plus background space-time sheets) define the "world of classical worlds" (WCW). These diagrams involve the analogs of stringy diagrams but the interpretation is different: the analogs of stringy loop diagrams have interpretation in terms of particle propagating via two different routes simultaneously (as in the classical double slit experiment) rather than as a decay of particle to two particles. For stringy diagrams the counterparts of vertices are singular as manifolds whereas the entire diagrams are smooth. For generalized Feynman diagrams vertices are smooth but entire diagrams represent singular manifolds just like ordinary Feynman diagrams do. String like objects however emerge in TGD and even ordinary elementary particles are predicted to be magnetic flux tubes of length of order weak gauge boson Compton length with monopoles at their ends as shown in accompanying article. This stringy character should become visible at LHC energies.

2. Zero energy ontology (ZEO) and causal diamonds (intersections of future and past directed light-cones) define second key ingredient. The crucial observation is that in ZEO it is possible to identify off mass shell particles as pairs of on mass shell fermions at throats of wormhole contact since both positive and negative signs of energy are possible and one obtains also space-like total momenta for wormhole contact behaving as a boson. The localization of fermions to string world sheets and the fact that super-conformal generator G carries fermion number combined with twistorial consideration support the view that the propagators at fermionic lines are of form $(1/G)ip^k\gamma_k(1/G^\dagger + h.c.$ and thus hermitian. In strong models $1/G$ would serve as a propagator and this requires Majorana condition fixing the dimension of the target space to 10 or 11.

3. A powerful constraint is number theoretic universality requiring the existence of Feynman amplitudes in all number fields when one allows suitable algebraic extensions: roots of unity are certainly required in order to realize p-adic counterparts of plane waves. Also imbedding space, partonic 2-surfaces and WCW must exist in all number fields and their extensions. These constraints are enormously powerful and the attempts to realize this vision have dominated quantum TGD for last two decades.

4. Representation of 8-D gamma matrices in terms of octonionic units and 2-D sigma matrices is a further important element as far as twistors are considered [K79] . Modified gamma matrices at space-time surfaces are quaternionic/associative and allow a genuine matrix representation. As a matter fact, TGD and WCW could be formulated as study of associative local sub-algebras of the local Clifford algebra of 8-D imbedding space parameterized by quaternionic space-time surfaces.

5. A central conjecture has been that associative (co-associative) 4-surfaces correspond to preferred extremals of Kähler action [K15]. It took long time to realize that in zero energy ontology the notion of preferred extremal might be un-necessary! The reason is that 3-surfaces are now pairs of 3-surfaces at boundaries of causal diamonds and for deterministic dynamics the space-time surface connecting them is expected to be more or less unique. Now the action principle is non-deterministic but the non-determinism would give rise to additional discrete dynamical degrees of freedom naturally assignable to the hierarchy of Planck constants $h_{eff} = n \times h$, n the number of space-time surface with same fixed ends at boundaries of CD and with same values of Kähler action and of conserved quantities. One must be however cautions: this leaves the possibility that there is a gauge symmetry present so that the n sheets correspond to gauge equivalence classes of sheets. Conformal invariance is associated with criticality and is expected to be present also now.

   One can of course also ask whether one can assume that the pairs of 3-surfaces at the ends of CD are totally un-correlated. If this assumption is not made then preferred extremal property would make sense also in ZEO and imply additional correlation between the members of these pairs. This kind of correlations would correspond to the Bohr orbit property, which is very attractive space-time correlate for quantum states. This kind of correlates are also expected as space-time counterpart for the correlations between initial and final state in quantum dynamics.

6. A further conjecture has been that preferred extremals are in some sense critical (second variation of Kähler action could vanish for infinite number of deformations defining a super-conformal algebra). The non-determinism of Kähler action implies this property for $n > 0$ in $h_{eff} = nh$. If the criticality is present, it could correspond to conformal gauge invariance defined by sub-algebras of conformal algebra with conformal weights coming as multiples of n and isomorphic to the conformal algebra itself.

7. As far as twistors are considered, the first key element is the reduction of the octonionic twistor structure to quaternionic one at space-time surfaces and giving effectively 4-D spinor and twistor structure for quaternionic surfaces.

Quite recently quite a dramatic progress took place in this approach [K26, K79] .

1. The progress was stimulated by the simple observation that on mass shell property puts enormously strong kinematic restrictions on the loop integrations. With mild restrictions on the number of parallel fermion lines appearing in vertices (there can be several since fermionic oscillator operator algebra defining SUSY algebra generates the parton states)- all loops are manifestly finite and if particles has always mass -say small p-adic thermal mass also in case of massless particles and due to IR cutoff due to the presence largest CD- the number of diagrams is finite. Unitarity reduces to Cutkosky rules [B15] automatically satisfied as in the case of ordinary Feynman diagrams.

2. Ironically, twistors which stimulated all these development do not seem to be absolutely necessary in this approach although they are of course possible. Situation changes if one does not assume small p-adically thermal mass due to the presence of massless particles and one must sum infinite number of diagrams. Here a potential problem is whether the infinite sum respects the algebraic extension in question.

This is about fermionic and momentum space aspects of Feynman diagrams but not yet about the functional (not path-) integral over small deformations of the partonic 2-surfaces. The basic challenges are following.

1. One should perform the functional integral over WCW degrees of freedom for fixed values of on mass shell momenta appearing in the internal lines. After this one must perform integral or summation over loop momenta. Note that the order is important since the space-time surface assigned to the line carries information about the quantum numbers associated with the line by quantum classical correspondence realized in terms of modified Dirac operator.

2. One must define the functional integral also in the p-adic context. p-Adic Fourier analysis relying on algebraic continuation raises hopes in this respect. p-Adicity suggests strongly that the loop momenta are discretized and ZEO predicts this kind of discretization naturally.

It indeed seems that the functional integrals over WCW could be carried out at general level both in real and p-adic context. This is due to the symmetric space property (maximal number of isometries) of WCW required by the mere mathematical existence of Kähler geometry [K33] in infinite-dimensional context already in the case of much simpler loop spaces [A145] .

1. The p-adic generalization of Fourier analysis allows to algebraize integration- the horrible looking technical challenge of p-adic physics- for symmetric spaces for functions allowing the analog of discrete Fourier decomposition. Symmetric space property is indeed essential also for the existence of Kähler geometry for infinite-D spaces as was learned already from the case of loop spaces. Plane waves and exponential functions expressible as roots of unity and powers of p multiplied by the direct analogs of corresponding exponent functions are the basic building bricks and key functions in harmonic analysis in symmetric spaces. The physically unavoidable finite measurement resolution corresponds to algebraically unavoidable finite algebraic dimension of algebraic extension of p-adics (at least some roots of unity are needed). The cutoff in roots of unity is very reminiscent to that occurring for the representations of quantum groups and is certainly very closely related to these as also to the inclusions of hyper-finite factors of type II¡sub¿1¡/sub¿ defining the finite measurement resolution.

2. WCW geometrization reduces to that for a single line of the generalized Feynman diagram defining the basic building brick for WCW. Kähler function decomposes to a sum of "kinetic" terms associated with its ends and interaction term associated with the line itself. p-Adicization boils down to the condition that Kähler function, matrix elements of Kähler form, WCW Hamiltonians and their super counterparts, are rational functions of complex WCW coordinates just as they are for those symmetric spaces that I know of. This would allow a continuation to p-adic context.

In the following this vision about generalized Feynman diagrams is discussed in more detail.

7.7.1 Questions

The goal is a proposal for how to perform the integral over WCW for generalized Feynman digrams and the best manner to proceed to to this goal is by making questions.

What does finite measurement resolution mean?

The first question is what finite measurement resolution means.

1. One expects that the algebraic continuation makes sense only for a finite measurement resolution in which case one obtains only finite sums of what one might hope to be algebraic functions. The finiteness of the algebraic extension would be in fact equivalent with the finite measurement resolution.

2. Finite measurement resolution means a discretization in terms of number theoretic braids. p-Adicization condition suggests that that one must allow only the number theoretic braids. For these the ends of braid at boundary of CD are algebraic points of the imbedding space. This would be true at least in the intersection of real and p-adic worlds.

3. The question is whether one can localize the points of the braid. The necessity to use momentum eigenstates to achieve quantum classical correspondence in the modified Dirac action [K15] suggests however a de-localization of braid points, that is wave function in space of braid points. In real context one could allow all possible choices for braid points but in p-adic context only algebraic points are possible if one wants to replace integrals with sums. This implies finite measurement resolution analogous to that in lattice. This is also the only possibility in the intersection of real and p-adic worlds.

   A non-trivial prediction giving a strong correlation between the geometry of the partonic 2-surface and quantum numbers is that the total number $n_F + n_{\overline{F}}$ of fermions and anti-fermions is bounded above by the number n_{alg} of algebraic points for a given partonic 2-surface: $n_F + n_{\overline{F}} \leq n_{alg}$. Outside the intersection of real and p-adic worlds the problematic aspect of this definition is that small deformations of the partonic 2-surface can radically change the number of algebraic points unless one assumes that the finite measurement resolution means restriction of WCW to a sub-space of algebraic partonic surfaces.

4. Braids defining propagator lines for fundamental fermions (to be distinguished from observer particles) emerges naturally. Braid strands correspond to the boundaries of string world sheets at which the modes of induced spinor fields are localized from the condition that em charge is well-defined: induced W field and above weak scale also Z^0 field vanish at them.

   In order to obtain non-trivial fermion propagator one must add to Kähler-Dirac action Chern-Simons Dirac term located at partonic orbits at which the signature of the induced metric changes. The modes of induced spinor field can be required to be generalized eigenmodes of C-S-D operator with generalized eigenvalue $p^k\gamma_k$ with p^k identified as virtual momentum so that massless Dirac propagator is obtained. p^k is discretized by periodic boundary conditions at opposite boundaries of CD and has IR and UV cutoffs due to the finite size of CD and finite lower limit for the size of sub-CDs.

   One has also discretization of the relative position of the second tip of CD at the hyperboloid isometric with mass shell. Only the number of braid points and their momenta would matter, not their positions.

By super-symmetry one must add to Kähler action Chern-Simons term located at partonic orbits and this term must cancel the Chern-Simons term coming from Kähler action by weak form of electric-magnetic duality so that Kähler action reduces to the terms associated with space-like ends of the space-time surface. These terms reduce to Chern-Simons terms if one poses weak form of electric magnetic duality also here. The boundary condition for Kähler-Dirac equations states $\Gamma^n\Psi = 0$ so that incoming fundamental fermions are massless and there is a strong temptation to pose the additional condition $\Gamma^n\Psi = p^k\gamma_k\Psi = 0$

The quantum numbers characterizing positive and negative energy parts of zero energy states couple directly to space-time geometry via the measurement interaction terms in Kähler action expressing the equality of classical conserved charges in Cartan algebra with their quantal counterparts for space-time surfaces in quantum superposition. This makes sense if classical charges parametrize zero modes. The localization in zero modes in state function reduction would be the WCW counterpart of state function collapse.

How to define integration in WCW degrees of freedom?

The basic question is how to define the integration over WCW degrees of freedom.

1. What comes mind first is Gaussian perturbation theory around the maxima of Kähler function. Gaussian and metric determinants cancel each other and only algebraic expressions remain. Finiteness is not a problem since the Kähler function is non-local functional of 3-surface so that no local interaction vertices are present. One should however assume the vanishing of loops required also by algebraic universality and this assumption look unrealistic when one considers more general functional integrals than that of vacuum functional since free field theory is not in question. The construction of the inverse of the WCW metric defining the propagator is also a very difficult challenge. Duistermaat-Hecke theorem states that something like this known as localization might be possible and one can also argue that something analogous to localization results from a generalization of mean value theorem.

2. Symmetric space property is more promising since it might reduce the integrations to group theory using the generalization of Fourier analysis for group representations so that there would be no need for perturbation theory in the proposed sense. In finite measurement resolution the symmetric spaces involved would be finite-dimensional. Symmetric space structure of WCW could also allow to define p-adic integration in terms of p-adic Fourier analysis for symmetric spaces. Essentially algebraic continuation of the integration from the real case would be in question with additional constraints coming from the fact that only phase factors corresponding to finite algebraic extensions of rationals are used. Cutoff would emerge automatically from the cutoff for the dimension of the algebraic extension.

How to define generalized Feynman diagrams?

Integration in symmetric spaces could serve as a model at the level of WCW and allow both the understanding of WCW integration and p-adicization as algebraic continuation. In order to get a more realistic view about the problem one must define more precisely what the calculation of the generalized Feynman diagrams means.

1. WCW integration must be carried out separately for all values of the momenta associated with the internal lines. The reason is that the spectrum of eigenvalues λ_i of the modified Dirac operator D depends on the momentum of line and momentum conservation in vertices translates to a correlation of the spectra of D at internal lines.

2. For tree diagrams algebraic continuation to the p-adic context if the expression involves only the replacement of the generalized eigenvalues of D as functions of momenta with their p-adic counterparts besides vertices. If these functions are algebraically universal and expressible in terms of harmonics of symmetric space , there should be no problems.

3. If loops are involved, one must integrate/sum over loop momenta. In p-adic context difficulties are encountered if the spectrum of the momenta is continuous. The integration over on mass shell loop momenta is analogous to the integration over sub-CDs, which suggests that

internal line corresponds to a $sub - CD$ in which it is at rest. There are excellent reasons to believe that the moduli space for the positions of the upper tip is a discrete subset of hyperboloid of future light-cone. If this is the case, the loop integration indeed reduces to a sum over discrete positions of the tip. p-Adizication would thus give a further good reason why for zero energy ontology.

4. Propagator is expressible in terms of the inverse of generalized eigenvalue and there is a sum over these for each propagator line. At vertices one has products of WCW harmonics assignable to the incoming lines. The product must have vanishing quantum numbers associated with the phase angle variables of WCW. Non-trivial quantum numbers of the WCW harmonic correspond to WCW quantum numbers assignable to excitations of ordinary elementary particles. WCW harmonics are products of functions depending on the "radial" coordinates and phase factors and the integral over the angles leaves the product of the first ones analogous to Legendre polynomials $P_{l,m}$, These functions are expected to be rational functions or at least algebraic functions involving only square roots.

5. In ordinary QFT incoming and outgoing lines correspond to propagator poles. In the recent case this would mean that incoming stringy lines at the ends of CD correspond to fermions satisfying the stringy mass formula serving as a generalization of masslessness condition.

7.7.2 Generalized Feynman diagrams at fermionic and momentum space level

Negative energy ontology has already led to the idea of interpreting the virtual particles as pairs of positive and negative energy wormhole throats. Hitherto I have taken it as granted that ordinary Feynman diagrammatics generalizes more or less as such. It is however far from clear what really happens in the verties of the generalized Feynman diagrams. The safest approach relies on the requirement that unitarity realized in terms of Cutkosky rules in ordinary Feynman diagrammatics allows a generalization. This requires loop diagrams. In particular, photon-photon scattering can take place only via a fermionic square loop so that it seems that loops must be present at least in the topological sense.

One must be however ready for the possibility that something unexpectedly simple might emerge. For instance, the vision about algebraic physics allows naturally only finite sums for diagrams and does not favor infinite perturbative expansions. Hence the true believer on algebraic physics might dream about finite number of diagrams for a given reaction type. For simplicity generalized Feynman diagrams without the complications brought by the magnetic confinement since by the previous arguments the generalization need not bring in anything essentially new.

The basic idea of duality in early hadronic models was that the lines of the dual diagram representing particles are only re-arranged in the vertices. This however does not allow to get rid of off mass shell momenta. Zero energy ontology encourages to consider a stronger form of this principle in the sense that the virtual momenta of particles could correspond to pairs of on mass shell momenta of particles. If also interacting fermions are pairs of positive and negative energy throats in the interaction region the idea about reducing the construction of Feynman diagrams to some kind of lego rules might work.

Virtual particles as pairs of on mass shell particles in ZEO

The first thing is to try to define more precisely what generalized Feynman diagrams are. The direct generalization of Feynman diagrams implies that both wormhole throats and wormhole contacts join at vertices.

1. A simple intuitive picture about what happens is provided by diagrams obtained by replacing the points of Feynman diagrams (wormhole contacts) with short lines and imagining that the throats correspond to the ends of the line. At vertices where the lines meet the incoming on mass shell quantum numbers would sum up to zero. This approach leads to a straightforward generalization of Feynman diagrams with virtual particles replaced with pairs of on mass shell throat states of type $++$, $--$, and $+-$. Incoming lines correspond to $++$ type lines and outgoing ones to $--$ type lines. The first two line pairs allow only time like net momenta

whereas $+-$ line pairs allow also space-like virtual momenta. The sign assigned to a given throat is dictated by the the sign of the on mass shell momentum on the line. The condition that Cutkosky rules generalize as such requires $++$ and $--$ type virtual lines since the cut of the diagram in Cutkosky rules corresponds to on mass shell outgoing or incoming states and must therefore correspond to $++$ or $--$ type lines.

2. The basic difference as compared to the ordinary Feynman diagrammatics is that loop integrals are integrals over mass shell momenta and that all throats carry on mass shell momenta. In each vertex of the loop mass incoming on mass shell momenta must sum up to on mass shell momentum. These constraints improve the behavior of loop integrals dramatically and give excellent hopes about finiteness. It does not however seem that only a finite number of diagrams contribute to the scattering amplitude besides tree diagrams. The point is that if a the reactions $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$,, where N_i denote particle numbers, are possible in a common kinematical region for N_2-particle states then also the diagrams $N_1 \rightarrow N_2 \rightarrow N_2 \rightarrow N_3$ are possible. The virtual states N_2 include all all states in the intersection of kinematically allow regions for $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$. Hence the dream about finite number possible diagrams is not fulfilled if one allows massless particles. If all particles are massive then the particle number N_2 for given N_1 is limited from above and the dream is realized.

3. For instance, loops are not possible in the massless case or are highly singular (bringing in mind twistor diagrams) since the conservation laws at vertices imply that the momenta are parallel. In the massive case and allowing mass spectrum the situation is not so simple. As a first example one can consider a loop with three vertices and thus three internal lines. Three on mass shell conditions are present so that the four-momentum can vary in 1-D subspace only. For a loop involving four vertices there are four internal lines and four mass shell conditions so that loop integrals would reduce to discrete sums. Loops involving more than four vertices are expected to be impossible.

4. The proposed replacement of the elementary fermions with bound states of elementary fermions and monopoles $X_\pm$ brings in the analog of stringy diagrammatics. The 2-particle wave functions in the momentum degrees of freedom of fermion and $X_\pm$ might allow more flexibility and allow more loops. Note however that there are excellent hopes about the finiteness of the theory also in this case.

Loop integrals are manifestly finite

One can make also more detailed observations about loops.

1. The simplest situation is obtained if only 3-vertices are allowed. In this case conservation of momentum however allows only collinear momenta although the signs of energy need not be the same. Particle creation and annihilation is possible and momentum exchange is possible but is always light-like in the massless case. The scattering matrices of supersymmetric YM theories would suggest something less trivial and this raises the question whether something is missing. Magnetic monopoles are an essential element of also these theories as also massivation and symmetry breaking and this encourages to think that the formation of massive states as fermion $X_\pm$ pairs is needed. Of course, in TGD framework one has also high mass excitations of the massless states making the scattering matrix non-trivial.

2. In YM theories on mass shell lines would be singular. In TGD framework this is not the case since the propagator is defined as the inverse of the 3-D dimensional reduction of the modified Dirac operator D containing also coupling to four-momentum (this is required by quantum classical correspondence and guarantees stringy propagators),

$$\begin{aligned} D &= i\hat{\Gamma}^\alpha p_\alpha + \hat{\Gamma}^\alpha D_\alpha \ , \\ p_\alpha &= p_k \partial_\alpha h^k \ . \end{aligned} \tag{7.7.0}$$

The propagator does not diverge for on mass shell massless momenta and the propagator lines are well-defined. This is of course of essential importance also in general case. Only for the incoming lines one can consider the possibility that 3-D Dirac operator annihilates the induced spinor fields. All lines correspond to generalized eigenstates of the propagator in the sense that one has $D_3\Psi = \lambda\gamma\Psi$, where γ is modified gamma matrix in the direction of the stringy coordinate emanating from light-like surface and D_3 is the 3-dimensional dimensional reduction of the 4-D modified Dirac operator. The eigenvalue λ is analogous to energy. Note that the eigenvalue spectrum depends on 4-momentum as a parameter.

3. Massless incoming momenta can decay to massless momenta with both signs of energy. The integration measure $d^2k/2E$ reduces to dx/x where $x \geq 0$ is the scaling factor of massless momentum. Only light-like momentum exchanges are however possible and scattering matrix is essentially trivial. The loop integrals are finite apart from the possible delicacies related to poles since the loop integrands for given massless wormhole contact are proportional to dx/x^3 for large values of x.

4. Irrespective of whether the particles are massless or not, the divergences are obtained only if one allows too high vertices as self energy loops for which the number of momentum degrees of freedom is $3N-4$ for N-vertex. The construction of SUSY limit of TGD in [K27] led to the conclusion that the parallelly propagating N fermions for given wormhole throat correspond to a product of N fermion propagators with same four-momentum so that for fermions and ordinary bosons one has the standard behavior but for $N > 2$ non-standard so that these excitations are not seen as ordinary particles. Higher vertices are finite only if the total number N_F of fermions propagating in the loop satisfies $N_F > 3N - 4$. For instance, a 4-vertex from which $N = 2$ states emanate is finite.

Taking into account magnetic confinement

What has been said above is not quite enough. The weak form of electric-magnetic duality [B7] leads to the picture about elementary particles as pairs of magnetic monopoles inspiring the notions of weak confinement based on magnetic monopole force. Also color confinement would have magnetic counterpart. This means that elementary particles would behave like string like objects in weak boson length scale. Therefore one must also consider the stringy case with wormhole throats replaced with fermion-$X_\pm$ pairs ($X_\pm$ is electromagnetically neutral and $\pm$ refers to the sign of the weak isospin opposite to that of fermion) and their super partners.

1. The simplest assumption in the stringy case is that fermion-$X_\pm$ pairs behave as coherent objects, that is scatter elastically. In more general case only their higher excitations identifiable in terms of stringy degrees of freedom would be created in vertices. The massivation of these states makes possible non-collinear vertices. An open question is how the massivation fermion-$X_\pm$ pairs relates to the existing TGD based description of massivation in terms of Higgs mechanism and modified Dirac operator.

2. Mass renormalization could come from self energy loops with negative energy lines as also vertex normalization. By very general arguments supersymmetry implies the cancellation of the self energy loops but would allow non-trivial vertex renormalization [K27] .

3. If only 3-vertices are allowed, the loops containing only positive energy lines are possible if on mass shell fermion-$X_\pm$ pair (or its superpartner) can decay to a pair of positive energy pair particles of same kind. Whether this is possible depends on the masses involved. For ordinary particles these decays are not kinematically possible below intermediate boson mass scale (the decays $F_1 \rightarrow F_2 + \gamma$ are forbidden kinematically or by the absence of flavor changing neutral currents whereas intermediate gauge bosons can decay to on mass shell fermion-anti-fermion pair).

4. The introduction of IR cutoff for 3-momentum in the rest system associated with the largest CD (causal diamond) looks natural as scale parameter of coupling constant evolution and p-adic length scale hypothesis favors the inverse of the size scale of CD coming in powers of two. This parameter would define the momentum resolution as a discrete parameter of the p-adic coupling constant evolution. This scale does not have any counterpart in standard physics. For electron, d quark, and u quark the proper time distance between the tips of

CD corresponds to frequency of 10 Hz, 1280 Hz, and 160 Hz: all these frequencies define fundamental bio-rhythms [K22] .

These considerations have left completely untouched one important aspect of generalized Feynman diagrams: the necessity to perform a functional integral over the deformations of the partonic 2-surfaces at the ends of the lines- that is integration over WCW. Number theoretical universality requires that WCW and these integrals make sense also p-adically and in the following these aspects of generalized Feynman diagrams are discussed.

7.7.3 Harmonic analysis in WCW as a manner to calculate WCW functional integrals

Previous examples suggest that symmetric space property, Kähler and symplectic structure and the use of symplectic coordinates consisting of canonically conjugate pairs of phase angles and corresponding "radial" coordinates are essential for WCW integration and p-adicization. Kähler function, the components of the metric, and therefore also metric determinant and Kähler function depend on the "radial" coordinates only and the possible generalization involves the identification the counterparts of the "radial" coordinates in the case of WCW.

Conditions guaranteeing the reduction to harmonic analysis

The basic idea is that harmonic analysis in symmetric space allows to calculate the functional integral over WCW.

1. Each propagator line corresponds to a symmetric space defined as a coset space G/H of the symplectic group and Kac-Moody group and one might hope that the proposed p-adicization works for it- at least when one considers the hierarchy of measurement resolutions forced by the finiteness of algebraic extensions. This coset space is as a manifold Cartesian product $(G/H)\times(G/H)$ of symmetric spaces G/H associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. Effective 2-dimensionality and ZEO allow to treat the ends of the propagator line independently. This means an enormous simplification. Each line contributes besides propagator a piece to the exponent of Kähler action identifiable as interaction term in action and depending on the propagator momentum. This contribution should be expressible in terms of generalized spherical harmonics. Essentially a sum over the products of pairs of harmonics associated with the ends of the line multiplied by coefficients analogous to $1/(p^2-m^2)$ in the case of the ordinary propagator would be in question. The optimal situation is that the pairs are harmonics and their conjugates appear so that one has invariance under G analogous to momentum conservation for the lines of ordinary Feynman diagrams.

3. Momentum conservation correlates the eigenvalue spectra of the modified Dirac operator D at propagator lines [K15] . G-invariance at vertex dictates the vertex as the singlet part of the product of WCW harmonics associated with the vertex and one sums over the harmonics for each internal line. p-Adicization means only the algebraic continuation to real formulas to p-adic context.

4. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)}+c.c\ \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)}+c.c\ \ ,i=1,2\ \ . \end{aligned} \quad (7.7.0)$$

Here $K_{kin,i}$ define "kinetic" terms and K_{int} defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories.

Symmetric space property -that is isometry invariance- suggests that one has

$$f_{i,n} = f_{2,n} \equiv f_n \ , \ g_{1,n} = g_{2,n} \equiv g_n \tag{7.7.1}$$

such that the products are invariant under the group H appearing in G/H and therefore have opposite H quantum numbers. The exponent of Kähler function does not factorize although the terms in its Taylor expansion factorize to products whose factors are products of holomorphic and antiholomorphic functions.

5. If one assumes that the exponent of Kähler function reduces to a product of eigenvalues of the modified Dirac operator eigenvalues must have the decomposition

$$\lambda_k = \prod_{i=1,2} exp\left[\sum_n c_{k,n} g_n(Z_i)\overline{g_n(Z_i)} + c.c\right] \times exp\left[\sum_n d_{k,n} g_n(Z_1)\overline{g_n(Z_2)} + c.c\right] \tag{7.7.2}$$

Hence also the eigenvalues coming from the Dirac propagators have also expansion in terms of G/H harmonics so that in principle WCW integration would reduce to Fourier analysis in symmetric space.

Generalization of WCW Hamiltonians

This picture requires a generalization of the view about configuration space Hamiltonians since also the interaction term between the ends of the line is present not taken into account in the previous approach.

1. The proposed representation of WCW Hamiltonians as flux Hamiltonians [K16, K15]

$$\begin{aligned} Q(H_A) &= \int H_A(1+K)Jd^2x \ , \\ J &= \epsilon^{\alpha\beta}J_{\alpha\beta} \ , \ J^{03}\sqrt{g_4} = KJ_{12} \ . \end{aligned} \tag{7.7.2}$$

works for the kinetic terms only since J cannot be the same at the ends of the line. The formula defining K assumes weak form of self-duality (^{03} refers to the coordinates in the complement of X^2 tangent plane in the 4-D tangent plane). K is assumed to be symplectic invariant and constant for given X^2. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge g_K gives the condition $K = g_K^2/\hbar$, where g_K is Kähler coupling constant. Within experimental uncertainties one has $\alpha_K = g_K^/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where α_{em} is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

The assumption that Poisson bracket of WCW Hamiltonians reduces to the level of imbedding space - in other words $\{Q(H_A), Q(H_B\} = Q(\{H_A, H_B\})$ - can be justified. One starts from the representation in terms of say flux Hamiltonians $Q(H_A)$ and *defines* $J_{A,B}$ as $J_{A,B} \equiv Q(\{H_A, H_B\})$. One has $\partial H_A/\partial t_B = \{H_B, H_A\}$, where t_B is the parameter associated with the exponentiation of H_B. The inverse J^{AB} of $J_{A,B} = \partial H_B/\partial t_A$ is expressible as $J^{A,B} = \partial t_A/\partial H_B$. From these formulas one can deduce by using chain rule that the bracket $\{Q(H_A), Q(H_B\} = \partial t_C Q(H_A) J^{CD} \partial t_D Q(H_B)$ of flux Hamiltonians equals to the flux Hamiltonian $Q(\{H_A, H_B\})$.

2. One should be able to assign to WCW Hamiltonians also a part corresponding to the interaction term. The symplectic conjugation associated with the interaction term permutes the WCW coordinates assignable to the ends of the line. One should reduce this apparently non-local symplectic conjugation (if one thinks the ends of line as separate objects) to a non-local symplectic conjugation for $\delta CD \times CP_2$ by identifying the points of lower and upper end of CD related by time reflection and assuming that conjugation corresponds to time reflection. Formally this gives a well defined generalization of the local Poisson brackets between time reflected points at the boundaries of CD. The connection of Hermitian conjugation and time reflection in quantum field theories is is in accordance with this picture.

3. The only manner to proceed is to assign to the flux Hamiltonian also a part obtained by the replacement of the flux integral over X^2 with an integral over the projection of X^2 to a sphere S^2 assignable to the light-cone boundary or to a geodesic sphere of CP_2, which come as two varieties corresponding to homologically trivial and non-trivial spheres. The projection is defined as by the geodesic line orthogonal to S^2 and going through the point of X^2. The hierarchy of Planck constants assigns to CD a preferred geodesic sphere of CP_2 as well as a unique sphere S^2 as a sphere for which the radial coordinate r_M or the light-cone boundary defined uniquely is constant: this radial coordinate corresponds to spherical coordinate in the rest system defined by the time-like vector connecting the tips of CD. Either spheres or possibly both of them could be relevant.

   Recall that also the construction of number theoretic braids and symplectic QFT [K18] led to the proposal that braid diagrams and symplectic triangulations could be defined in terms of projections of braid strands to one of these spheres. One could also consider a weakening for the condition that the points of the number theoretic braid are algebraic by requiring only that the S^2 coordinates of the projection are algebraic and that these coordinates correspond to the discretization of S^2 in terms of the phase angles associated with θ and ϕ.

   This gives for the corresponding contribution of the WCW Hamiltonian the expression

$$Q(H_A)_{int} = \int_{S^2_{\pm}} H_A X \delta^2(s_+,s_-) d^2s_{\pm} = \int_{P(X^2_+)\cap P(X^2_-)} \frac{\partial(s^1,s^2)}{\partial(x^1_{\pm},x^2_{\pm})} d^2x_{\pm} \quad . \qquad (7.7.3)$$

   Here the Poisson brackets between ends of the line using the rules involve delta function $\delta^2(s_+,s_-)$ at S^2 and the resulting Hamiltonians can be expressed as a similar integral of $H_{[A,B]}$ over the upper or lower end since the integral is over the intersection of S^2 projections.

   The expression must vanish when the induced Kähler form vanishes for either end. This is achieved by identifying the scalar X in the following manner:

$$\begin{aligned} X &= J_+^{kl} J^-_{kl} \ , \\ J_{\pm}^{kl} &= (1+K_{\pm})\partial_\alpha s^k \partial_\beta s^l J_{\pm}^{\alpha\beta} \quad . \end{aligned} \qquad (7.7.3)$$

   The tensors are lifts of the induced Kähler form of $X^2_{\pm}$ to S^2 (not CP_2).

4. One could of course ask why these Hamiltonians could not contribute also to the kinetic terms and why the brackets with flux Hamiltonians should vanish. This relate to how one *defines* the Kähler form. It was shown above that in case of flux Hamiltonians the definition of Kähler form as brackets gives the basic formula $\{Q(H_A),Q(H_B)\} = Q(\{H_A,H_B\}$ and same should hold true now. In the recent case $J_{A,B}$ would contain an interaction term defined in terms of flux Hamiltonians and the previous argument should go through also now by identifying Hamiltonians as sums of two contributions and by introducing the doubling of the coordinates t_A.

5. The quantization of the modified Dirac operator must be reconsidered. It would seem that one must add to the super-Hamiltonian completely analogous term obtained by replacing $(1+K)J$ with $X\partial(s^1,s^2)/\partial(x^1_\pm,x^2_\pm)$. Besides the anti-commutation relations defining correct anti-commutators to flux Hamiltonians, one should pose anti-commutation relations consistent with the anti-commutation relations of super Hamiltonians. In these anti-commutation relations $(1+K)J\delta^2(x,y)$ would be replaced with $X\delta^2(s^+,s^-)$. This would guarantee that the oscillator operators at the ends of the line are not independent and that the resulting Hamiltonian reduces to integral over either end for $H_{[A,B]}$.

6. In the case of CP_2 the Hamiltonians generating isometries are rational functions. This should hold true also now so that p-adic variants of Hamiltonians as functions in WCW would make sense. This in turn would imply that the components of the WCW Kähler form are rational functions. Also the exponentiation of Hamiltonians make sense p-adically if one allows the exponents of group parameters to be functions $Exp_p(t)$.

Does the expansion in terms of partial harmonics converge?

The individual terms in the partial wave expansion seem to be finite but it is not at all clear whether the expansion in powers of K actually converges.

1. In the proposed scenario one performs the expansion of the vacuum functional $exp(K)$ in powers of K and therefore in negative powers of α_K. In principle an infinite number of terms can be present. This is analogous to the perturbative expansion based on using magnetic monopoles as basic objects whereas the expansion using the contravariant Kähler metric as a propagator would be in positive powers of α_K and analogous to the expansion in terms of magnetically bound states of wormhole throats with vanishing net value of magnetic charge. At this moment one can only suggest various approaches to how one could understand the situation.

2. Weak form of self-duality and magnetic confinement could change the situation. Performing the perturbation around magnetic flux tubes together with the assumed slicing of the space-time sheet by stringy world sheets and partonic 2-surfaces could mean that the perturbation corresponds to the action assignable to the electric part of Kähler form proportional to α_K by the weak self-duality. Hence by $K = 4\pi\alpha_K$ relating Kähler electric field to Kähler magnetic field the expansion would come in powers of a term containing sum of terms proportional to α_K^0 and α_K. This would leave to the scattering amplitudes the exponents of Kähler function at the maximum of Kähler function so that the non-analytic dependence on α_K would not disappear.

A further reason to be worried about is that the expansion containing infinite number of terms proportional to α_K^0 could fail to converge.

1. This could be also seen as a reason for why magnetic singlets are unavoidable except perhaps for $\hbar < \hbar_0$. By the holomorphic factorization the powers of the interaction part of Kähler action in powers of $1/\alpha_K$ would naturally correspond to increasing and opposite net values of the quantum numbers assignable to the WCW phase coordinates at the ends of the propagator line. The magnetic bound states could have similar expansion in powers of α_K as pairs of states with arbitrarily high but opposite values of quantum numbers. In the functional integral these quantum numbers would compensate each other. The functional integral would leave only an expansion containing powers of α_K starting from some finite possibly negative (unless one assumes the weak form of self-duality) power. Various gauge coupling strengths are expected to be proportional to α_K and these expansions should reduce to those in powers of α_K.

2. Since the number of terms in the fermionic propagator expansion is finite, one might hope on basis of super-symmetry that the same is true in the case of the functional integral expansion. By the holomorphic factorization the expansion in powers of K means the appearance of terms with increasingly higher quantum numbers. Quantum number conservation at vertices would leave only a finite number of terms to tree diagrams. In the case of loop diagrams pairs of

particles with opposite and arbitrarily high values of quantum numbers could be generated at the vertex and magnetic confinement might be necessary to guarantee the convergence. Also super-symmetry could imply cancellations in loops.

Could one do without flux Hamiltonians?

The fact that the Kähler functions associated with the propagator lines can be regarded as interaction terms inspires the question whether the Kähler function could contain only the interaction terms so that Kähler form and Kähler metric would have components only between the ends of the lines.

1. The basic objection is that flux Hamiltonians too beautiful objects to be left without any role in the theory. One could also argue that the WCW metric would not be positive definite if only the non-diagonal interaction term is present. The simplest example is Hermitian 2×2-matrix with vanishing diagonal for which eigenvalues are real but of opposite sign.

2. One could of course argue that the expansions of $exp(K)$ and λ_k give in the general powers $(f_n\overline{f_n})^m$ analogous to diverging tadpole diagrams of quantum field theories due to local interaction vertices. These terms do not produce divergences now but the possibility that the exponential series of this kind of terms could diverge cannot be excluded. The absence of the kinetic terms would allow to get rid of these terms and might be argued to be the symmetric space counterpart for the vanishing of loops in WCW integral.

3. In zero energy ontology this idea does not look completely non-sensical since physical states are pairs of positive and negative energy states. Note also that in quantum theory only creation operators are used to create positive energy states. The manifest non-locality of the interaction terms and absence of the counterparts of kinetic terms would provide a trivial manner to get rid of infinities due to the presence of local interactions. The safest option is however to keep both terms.

Summary

The discussion suggests that one must treat the entire Feynman graph as single geometric object with Kähler geometry in which the symmetric space is defined as product of what could be regarded as analogs of symmetric spaces with interaction terms of the metric coming from the propagator lines. The exponent of Kähler function would be the product of exponents associated with all lines and contributions to lines depend on quantum numbers (momentum and color quantum numbers) propagating in line via the coupling to the modified Dirac operator. The conformal factorization would allow the reduction of integrations to Fourier analysis in symmetric space. What is of decisive importance is that the entire Feynman diagrammatics at WCW level would reduce to the construction of WCW geometry for a single propagator line as a function of quantum numbers propagating on the line.

7.8 How to realize the notion of finite measurement resolution mathematically?

One of the basic challenges of quantum TGD is to find an elegant realization for the notion of finite measurement resolution. The notion of resolution involves observer in an essential manner and this suggests that cognition is involved. If p-adic physics is indeed physics of cognition, the natural guess is that p-adic physics should provide the primary realization of this notion.

The simplest realization of finite measurement resolution would be just what one would expect it to be except that this realization is most natural in the p-adic context. One can however define this notion also in real context by using canonical identification to map p-adic geometric objets to real ones.

7.8.1 Does discretization define an analog of homology theory?

Discretization in dimension D in terms of pinary cutoff means division of the manifold to cube-like objects. What suggests itself is homology theory defined by the measurement resolution and by the fluxes assigned to the induced Kähler form.

1. One can introduce the decomposition of n-D sub-manifold of the imbedding space to n-cubes by $n-1$-planes for which one of the coordinates equals to its pinary cutoff. The construction works in both real and p-adic context. The hyperplanes in turn can be decomposed to $n-1$-cubes by $n-2$-planes assuming that an additional coordinate equals to its pinary cutoff. One can continue this decomposition until one obtains only points as those points for which all coordinates are their own pinary cutoffs. In the case of partonic 2-surfaces these points define in a natural manner the ends of braid strands. Braid strands themselves could correspond to the curves for which two coordinates of a light-like 3-surface are their own pinary cutoffs.

2. The analogy of homology theory defined by the decomposition of the space-time surface to cells of various dimensions is suggestive. In the p-adic context the identification of the boundaries of the regions corresponding to given pinary digits is not possible in purely topological sense since p-adic numbers do not allow well-ordering. One could however identify the boundaries sub-manifolds for which some number of coordinates are equal to their pinary cutoffs or as inverse images of real boundaries. This might allow to formulate homology theory to the p-adic context.

3. The construction is especially interesting for the partonic 2-surfaces. There is hierarchy in the sense that a square like region with given first values of pinary digits decompose to p square like regions labelled by the value $0, ..., p-1$ of the next pinary digit. The lines defining the boundaries of the 2-D square like regions with fixed pinary digits in a given resolution correspond to the situation in which either coordinate equals to its pinary cutoff. These lines define naturally edges of a graph having as its nodes the points for which pinary cutoff for both coordinates equals to the actual point.

4. I have proposed earlier [K14] what I have called symplectic QFT involving a triangulation of the partonic 2-surface. The fluxes of the induced Kähler form over the triangles of the triangulation and the areas of these triangles define symplectic invariants, which are zero modes in the sense that they do not contribute to the line element of WCW although the WCW metric depends on these zero modes as parameters. The physical interpretation is as non-quantum fluctuating classical variables. The triangulation generalizes in an obvious manner to quadrangulation defined by the pinary digits. This quadrangulation is fixed once internal coordinates and measurement accuracy are fixed. If one can identify physically preferred coordinates - say by requiring that coordinates transform in simple manner under isometries - the quadrangulation is highly unique.

5. For 3-surfaces one obtains a decomposition to cube like regions bounded by regions consisting of square like regions and Kähler magnetic fluxes over the squares define symplectic invariants. Also Kähler Chern-Simons invariant for the 3-cube defines an interesting almost symplectic invariant. 4-surface decomposes in a similar manner to 4-cube like regions and now instanton density for the 4-cube reducing to Chern-Simons term at the boundaries of the 4-cube defines symplectic invariant. For 4-surfaces symplectic invariants reduce to Chern-Simons terms over 3-cubes so that in this sense one would have holography. The resulting structure brings in mind lattice gauge theory and effective 2-dimensionality suggests that partonic 2-surfaces are enough.

The simplest realization of this homology theory in p-adic context could be induced by canonical identification from real homology. The homology of p-adic object would the homology of its canonical image.

1. Ordering of the points is essential in homology theory. In p-adic context canonical identification $x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ map to reals induces this ordering and also boundary

operation for p-adic homology can be induced. The points of p-adic space would be represented by n-tuples of sequences of pinary digits for n coordinates. p-Adic numbers decompose to disconnected sets characterized by the norm p^{-n} of points in given set. Canonical identification allows to glue these sets together by inducing real topology. The points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ having p-adic norms p^{-n} and p^{-n-1} are mapped to the same real point p^{-n} under canonical identification and therefore the points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ can be said to define the endpoints of a continuous interval in the induced topology although they have different p-adic norms. Canonical identification induces real homology to the p-adic realm. This suggests that one should include canonical identification to the boundary operation so that boundary operation would be map from p-adicity to reality.

2. Interior points of p-adic simplices would be p-adic points not equal to their pinary cutoffs defined by the dropping of the pinary digits corresponding p^n, $n > N$. At the boundaries of simplices at least one coordinate would have vanishing pinary digits for p^n, $n > N$. The analogs of $n-1$ simplices would be the p-adic points sets for which one of the coordinates would have vanishing pinary digits for p^n, $n > N$. $n-k$-simplices would correspond to points sets for which k coordinates satisfy this condition. The formal sums and differences of these sets are assumed to make sense and there is natural grading.

3. Could one identify the end points of braid strands in some natural manner in this cohomology? Points with $n \leq N$ pinary digits are closed elements of the cohomology and homologically equivalent with each other if the canonical image of the p-adic geometric object is connected so that there is no manner to identify the ends of braid strands as some special points unless the zeroth homology is non-trivial. In [K84] it was proposed that strand ends correspond to singular points for a covering of sphere or more general Riemann surface. At the singular point the branches of the covering would co-incide.

   The obvious guess is that the singular points are associated with the covering characterized by the value of Planck constant. As a matter fact, the original assumption was that *all* points of the partonic 2-surface are singular in this sense. It would be however enough to make this assumption for the ends of braid strands only. The orbits of braid strands and string world sheet having braid strands as its boundaries would be the singular loci of the covering.

7.8.2 Does the notion of manifold in finite measurement resolution make sense?

A modification of the notion of manifold taking into account finite measurement resolution might be useful for the purposes of TGD.

1. The chart pages of the manifold would be characterized by a finite measurement resolution and effectively reduce to discrete point sets. Discretization using a finite pinary cutoff would be the basic notion. Notions like topology, differential structure, complex structure, and metric should be defined only modulo finite measurement resolution. The precise realization of this notion is not quite obvious.

2. Should one assume metric and introduce geodesic coordinates as preferred local coordinates in order to achieve general coordinate invariance? Pinary cutoff would be posed for the geodesic coordinates. Or could one use a subset of geodesic coordinates for $\delta CD \times CP_2$ as preferred coordinates for partonic 2-surfaces? Should one require that isometries leave distances invariant only in the resolution used?

3. A rather natural approach to the notion of manifold is suggested by the p-adic variants of symplectic spaces based on the discretization of angle variables by phases in an algebraic extension of p-adic numbers containing n^{th} root of unity and its powers. One can also assign p-adic continuum to each root of unity [K26]. This approach is natural for compact symmetric Kähler manifolds such as S^2 and CP_2. For instance, CP_2 allows a coordinatization in terms of two pairs (P^k, Q^k) of Darboux coordinates or using two pairs $(\xi^k, \overline{\xi}^k)$, $k = 1, 2$, of complex coordinates. The magnitudes of complex coordinates would be treated in the

manner already described and their phases would be described as roots of unity. In the natural quadrangulation defined by the pinary cutoff for $|\xi^k|$ and by roots of unity assigned with their phases, Kähler fluxes would be well-defined within measurement resolution. For light-cone boundary metrically equivalent with S^2 similar coordinatization using complex coordinates $(z, \overline{z})$ is possible. Light-like radial coordinate r would appear only as a parameter in the induced metric and pinary cutoff would apply to it.

7.8.3 Hierachy of finite measurement resolutions and hierarchy of p-adic normal Lie groups

The formulation of quantum TGD is almost completely in terms of various symmetry group and it would be highly desirable to formulate the notion of finite measurement resolution in terms of symmetries.

1. In p-adic context any Lie-algebra gG with p-adic integers as coefficients has a natural grading based on the p-adic norm of the coefficient just like p-adic numbers have grading in terms of their norm. The sub-algebra g_N with the norm of coefficients not larger than p^{-N} is an ideal of the algebra since one has $[g_M, g_N] \subset g_{M+N}$: this has of course direct counterpart at the level of p-adic integers. g_N is a normal sub-algebra in the sense that one has $[g, g_N] \subset g_N$. The standard expansion of the adjoint action gg_Ng^{-1} in terms of exponentials and commutators gives that the p-adic Lie group $G_N = exp(tpg_N)$, where t is p-adic integer, is a normal subgroup of $G = exp(tpg)$. If indeed so then also G/G_N is group, and could perhaps be interpreted as a Lie group of symmetries in finite measurement resolution. G_N in turn would represent the degrees of freedom not visible in the measurement resolution used and would have the role of a gauge group.

2. The notion of finite measurement resolution would have rather elegant and universal representation in terms of various symmetries such as isometries of imbedding space, Kac-Moody symmetries assignable to light-like wormhole throats, symplectic symmetries of $\delta CD \times CP_2$, the non-local Yangian symmetry, and also general coordinate transformations. This representation would have a counterpart in real context via canonical identification I in the sense that $A \rightarrow B$ for p-adic geometric objects would correspond to $I(A) \rightarrow I(B)$ for their images under canonical identification. It is rather remarkable that in purely real context this kind of hierarchy of symmetries modulo finite measurement resolution does not exist. The interpretation would be that finite measurement resolution relates to cognition and therefore to p-adic physics.

3. Matrix group G contains only elements of form $g = 1 + O(p^m)$, $m \geq 1$ and does not therefore involve matrices with elements expressible in terms roots of unity. These can be included by writing the elements of the p-adic Lie-group as products of elements of above mentioned G with the elements of a discrete group for which the elements are expressible in terms of roots of unity in an algebraic extension of p-adic numbers. For p-adic prime p p:th roots of unity are natural and suggested strongly by quantum arithmetics [K83].

Chapter 8

What p-adic icosahedron could mean? And what about p-adic manifold?

8.1 Introduction

This chapter was originally meant to be a summary of what I understoond about the article "The p-Adic Icosahedron" in Notices of AMS [A129]. The original purpose was to summarize the basic ideas and discuss my own view about more technical aspects - in particular the generalization of Riemann sphere to p-adic context which is rather technical and leads to the notion of Bruhat Tits tree and Berkovich space.

About Bruhat-Tits tree there is a nice web article titled p-Adic numbers and Bruhat-Tits tree [A68] describing also basics of p-adic numbers in a very concise form. The Wikipedia article about Berkovich space is written with a jargon giving no idea about what is involved. There are video lectures [A101] about Berkovich spaces. The web article about Berkovich spaces by Temkin [A211] seems too technical for a non-specialist. The slides [A216] however give a concise bird's eye of view about the basic idea behind Berkovich spaces.

The notion of p-adic icosahedron leads to the challenge of constructing p-adic sphere, and more generally p-adic manifolds and this extended the intended scope of the chapter and led to consider the fundamental questions related to the construction of TGD.

Quite generally, there are two approaches to the construction of manifolds based on algebra *resp.* topology.

1. In algebraic geometry manifolds - or rather, algebraic varieties - correspond to solutions of algebraic equations. Algebraic approach allows even a generalization of notions of real topology such as the notion of genus.

2. Second approach relies on topology and works nicely in the real context. The basic building brick is n-ball. More complex manifolds are obtained by gluing n-balls together. Here inequalities enter the game. Since p-adic numbers are not well-ordered they do not make sense in purely p-adic context unless expressed using p-adic norm and thus for real numbers. The notion of boundary is also one of the problematic notions since in purely p-adic context there are no boundaries.

8.1.1 The attempt to construct p-adic manifolds by mimicking topological construction of real manifolds meets difficulties

The basic problem in the application of topological method to manifold construction is that p-adic disks are either disjoint or nested so that the standard construction of real manifolds using partially overlapping n-balls does not generalize to the p-adic context. The notions of Bruhat-Tits tree [A68], building, and Berkovich disk [A216] and Berkovich space [A211] represent attempts to overcome this problem. Berkovich disk is a generalization of the p-adic disk obtained by adding additional points so that the p-adic disk is a dense subset of it. Berkovich disk allows path connected topology which is path connected. The generalization of this construction is used to construct p-adic manifolds using the modification of the topological construction in the real case. This construction provides also insights about p-adic integration.

The construction is highly technical and complex and pragmatic physicist could argue that it contains several un-natural features due to the forcing of the real picture to p-adic context. In particular, one must give up the p-adic topology whose ultra-metricity has a nice interpretation in the applications to both p-adic mass calculations and to consciousness theory.

I do not know whether the construction of Bruhat-Tits tree, which works for projective spaces but not for Q_p^n (!) is a special feature of projective spaces, whether Bruhat-Tits tree is enough so that no completion would be needed, and whether Bruhat-Tits tree can be deduced from Berkovich approach. What is however remarkable that for $M^4 \times CP_2$ p-adic S^2 and CP_2 are projective spaces and allow Bruhat-Tits tree. This not true for the spheres associated with the light-cone boundary of $D \neq 4$-dimensional Minkowski spaces.

8.1.2 Two basic philosophies concerning the construction of p-adic manifolds

There exists two basic philosophies concerning the construction of p-adic manifolds: algebraic and topological approach. Also in TGD these approaches have been competing: algebraic approach relates real and p-adic space-time points by identifying common rationals. Finite pinary cutoff is however required to avoid totally wild behavior and has interpretation in terms of finite measurement resolution. Canonical identification maps p-adics to reals and vice versa in a continuous manner but is not consistent with field equations without pinary cutoff.

1. One can try to generalize the theory of real manifolds to p-adic context. Since p-adic balls are either disjoint or nested, the usual construction by gluing partially overlapping balls fails. This leads to the notion of Berkovich disk obtained as a completion of p-adic disk having path connected topology (non-ultra-metric) and containing p-adic disk as a dense subset. This plus the complexity of the construction is heavy price to be paid for path-connectedness. A related notion is Bruhat-Tits tree defining kind of skeleton making p-adic manifold defining its boundary path connected. The notion makes sense for the p-adic counterparts of projective spaces, which suggests that p-adic projective spaces (S^2 and CP_2 in TGD framework) are physically very special.

2. Second approach is algebraic and restricts the consideration to algebraic varieties for which also topological invariants have algebraic counterparts. This approach is very natural in TGD framework, where preferred extremals of Kähler action can be characterized purely algebraically - even in a manner independent of the action principle - so that they make sense also p-adically.

   At the level of WCW algebraic approach combined with symmetries works: the mere existence of Kähler geometry implies infinite-D group of isometries and fixes the geometry uniquely. One can say that infinite-D geometries are the final victory of Erlangen program. At space-time level it however seems that one must have correspondence between real and p-adic worlds since real topology is the "lab topology".

8.1.3 Number theoretical universality and the construction of p-adic manifolds

Construction of p-adic counterparts of manifolds is also one of the basic challenges of TGD. Here the basic vision is that one must take a wider perspective. One must unify real and various p-adic physics to single coherent whole and to relate them. At the level of mathematics this requires fusion of real and p-adic number fields along common rationals and the notion of algebraic continuation between number fields becomes a basic tool.

The number theoretic approach is essentially algebraic and based on the gluing of reals and various p-adic number fields to a larger structure along rationals and also along common algebraic numbers. A strong motivation for the algebraic approach comes from the fact that preferred extremals [K10, K89] are characterized by a generalization of the complex structure to 4-D case both in Euclidian and Minkowskian signature. This generalization is independent of the action principle. This allows a straightforward identification of the p-adic counterparts of preferred extremals. The algebraic extensions of p-adic numbers play a key role and make it possible to realize the symmetries in the same manner as they are realized in the construction of p-adic icosahedron.

The lack of well-ordering of p-adic numbers poses strong constraints on the formulation of number theoretical universality.

1. The notion of set theoretic boundary does not make sense in purely p-adic context. Quite generally, everything involving inequalities can lead to problems in p-adic context unless one is able to define effective Archimedean topology in some natural manner. Canonical identification inducing real topology to p-adic context would allow to achieve this.

2. The question arises about whether real topological invariants such as genus of partonic 2-surface make sense in the p-adic sector: for algebraic varieties this is the case. One would however like to have a more general definition and again Archimedean effective topology is suggestive.

3. Integration poses problems in p-adic context and algebraic continuation from reals to p-adic number fields seems to be the only possible option making sense. The continuation is however not possible for all p-adic number fields for given surface. This has however a beautiful interpretation explaining why real space-time sheets (and elementary particles) are characterized by some p-adic prime or primes. The p-adic prime determining the mass scale of the elementary particle could be fixed number theoretically rather than by some dynamical principle formulated in real context (number theoretic anatomy of rational number does not depend smoothly on its real magnitude!). A more direct approach to integration could rely on canonical integration as a chart map allowing to define integral on the real side.

4. Only those discrete subgroups of real symmetries, which correspond matrices with elements in algebraic extension of p-adic numbers can be realized so that a symmetry breaking to discrete subgroup consistent with the notion of finite measurement resolution and quantum measurement theory takes place. p-Adic symmetry groups can be identified as unions of elements of discrete subgroup of the symmetry group (making sense also in real context) multiplied by a p-adic variant of the continuous Lie group. These genuinely p-adic Lie groups are labelled by powers of p telling the maximum norm of the Lie-algebra parameter. Remarkably, effective values of Planck constant come as powers of p. Whether this interpretation for the hierarchy of effective Planck constants is consistent with the interpretation in terms of n-furcations of space-time sheet remains an open question.

8.1.4 How to achieve path connectedness?

The basic problem in the construction of p-adic manifolds is the total disconnectedness of the p-adic topology implied by ultra-metricity. This leads also to problems with the notion of p-adic integration. Physically it seems clear that the notion of path connectedness should have some physical counterpart.

The notion of open set makes possible path connectedness possible in the real context. In p-adic context Bruhat-Tits tree [A68] and completion of p-adic disk to Berkovich disk [A216] are introduced to achieve the same goal. One can ask whether Berkovich space could allow to achieve a more rigorous formulation for the p-adic counterparts of CP_2, of partonic 2-surfaces, their light-like orbits, preferred extremals of Kähler action, and even the "world of classical worlds" (WCW) [K33, K16]. To me this construction does not look promising in TGD framework but I could of course be wrong.

TGD suggests two alternative approaches to the problem of path connectedness. They should be equivalent.

p-Adic manifold concept based on canonical identification

The TGD inspired solution to the construction of path connected p-adic topology relies on the notion of canonical identification mapping reals to p-adics and vice versa in a continuous manner.

1. Canonical identification is used to map the values of p-adic mass squared predicted by p-adic mass calculations to their real counterparts [K39]. It makes also sense to map p-adic probabilities to their real counterparts by canonical identification. In TGD inspired theory of consciousness canonical identification is a good candidate for defining cognitive representations as representations mapping real preferred extremals to p-adic preferred extremals as also for the realization of intentional action as a quantum jump replacing p-adic preferred extremal representing intention with a real preferred extremal representing action. Could these cognitive representations and their inverses actually define real coordinate charts for the p-adic "mind stuff" and vice versa?

2. The trivial but striking observation was that it satisfies triangle inequality and thus defines an Archimedean norm allowing to induce real topology to p-adic context. Canonical identification with finite measurement resolution defines chart maps from p-adics to reals (rather than p-adics!) and vice versa and preferred extremal property allows to complete the discrete image to a space-time surface unique apart from finite measurement resolution so that topological and algebraic approach are combined. Without preferred extremal property one can complete to smooth real manifold (say) but the completion is much less unique.

3. Also the notion of integration can be defined. If the integral for - say- real curve at the map leaf exists, its value on the p-adic side for its pre-image can be defined by algebraic continuation in the case that it exists. Therefore one can speak about lengths, volumes, action integrals, and similar things in p-adic context. One can also generalize the notion of differential form and its holomomorphic variant and their integrals to the p-adic context. These generalizations allow a generalization of integral calculus required by TGD and also provide a justification for some basic assumptions of p-adic mass calculations.

Could path connectedness have a quantal description?

The physical content of path connectedness might also allow a formulation as a quantum physical rather than primarily topological notion, and could boil down to the non-triviality of correlation functions for second quantized induced spinor fields essential for the formulation of WCW spinor structure. Fermion fields and their n-point functions could become part of a number theoretically universal definition of manifold in accordance with the TGD inspired vision that WCW geometry - and perhaps even space-time geometry - allow a formulation in terms of fermions.

The natural question of physicist is whether quantum theory could provide a fresh number theoretically universal approach to the problem. The basic underlying vision in TGD framework is that second quantized fermion fields might allow to formulate the geometry of "world of classical worlds" (WCW) (for instance, Kähler action for preferred extremals and thus Kähler geometry of WCW would reduce to Dirac determinant [K26]). Maybe even the geometry of space-time surfaces could be expressed in terms of fermionic correlation functions.

This inspires the idea that second quantized fermionic fields replace the K-valued (K is algebraic extension of p-adic numbers) functions defined on p-adic disk in the construction of Berkovich. The ultra-metric norm for the functions defined in p-adic disk would be replaced by the fermionic correlation functions and different Berkovich norms correspond to different measurement resolutions so that one obtains also a connection with hyper-finite factors of type II_1. The existence of non-trivial fermionic correlation functions would be the counterpart for the path connectedness at space-time level. The 3-surfaces defining boundaries of a connected preferred extremal are also in a natural manner "path connected" with "path" being defined by the 4-surface. At the level of WCW and in zero energy ontology (ZEO) [K85] WCW spinor fields are analogous to correlation functions having collections of these disjoint 3-surfaces as arguments. There would be no need to complete p-adic topology to a path connected topology in this approach.

This approach is much more speculative that the first option and should be consistent with it.

8.1.5 Topics of the chapter

The chapter was originally meant to discuss p-adic icosahedron. Although the focus was re-directed to the notion of p-adic manifold - especially in TGD framework - I decided to keep the original starting point since it provides a concrete manner to end up with the deep problems of p-adic manifold theory and illustrates the group theoretical ideas.

- In the first section icosahedron is described in the real context. In the second section the ideas related to its generalization to the p-adic context are introduced. After that I discuss how to define sphere in p-adic context.
- In the section about algebraic universality I consider the problems related to the challenge of defining p-adic manifolds TGD point of view, which is algebraic and involves the fusion of various number fields and number theoretical universality as additional elements.
- The key section of the chapter describes the construction of p-adic space-time topology relying on chart maps of p-adic preferred extremals defined by canonical identification in finite measurement resolution and on the completion of discrete chart maps to real preferred extremals of Kähler action. The needed path-connected topology is the topology induced by canonical identification defining real chart maps for p-adic space-time surface. Canonical identification allows also the definition of p-adic valued integrals and definition of p-adic differential forms crucial in quantum TGD.
- Last section discusses in rather speculative spirit the possibility of defining space-time surfaces in terms of correlation functions of induced fermion fields.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- p-Adic manifold [L23]

8.2 Real icosahedron and its generalization to p-adic context

I summarize first the description of icosahedron in real context allowing a generalization to the p-adic context and consider the the problems related to the precise definition of p-adic icosahedron.

8.2.1 What does one mean with icosahedron in real context?

The notion of icosahedron [A48] is a geometric concept involving the notion of distance. In p-adic context this notion does not make sense since one cannot calculated distances, between points using standard formulas. Same applies to areas and volumes. The reason is that Riemann integral does not generalize and this is due to the fact that p-adic numbers are not well-ordered: one cannot say whether for two p-adic numbers of same norm $a < b$ or $b < a$ holds true.

Platonic solids [A69] are however characterized by their isometry groups and group theory makes sense also in p-adic context. The idea is therefore to characterize the icosahedron or any Platonic solid solely by its isometry group.

In practice this means following. Platonic solid is described as a collection of points. Vertices, midpoints of edges, and barycenters of faces. These points are fixed points for discrete subgroups of the Platonic solid. In the case of icosahedron the isometry group is A_5 the group of even permutations of 5 letters. There are are 6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points are the 12 vertices, 20 barycenters, and 30 midpoints of edges. Thus icosahedron becomes a collection of points with a label telling which is the cyclic subgroup associated with the point. This is something which might be able to generalize to p-adic context since there would be no need to talk about distances. One should however describe also the "solid" aspect of icosahedron.

8.2.2 What does one mean with ordinary 2-sphere?

In order to construct p-adic analog of icosahedron one must construct a space in which the isometry group A_5 of icosahedron acts and is imbedded to a group defining the analog of rotation group.

One could consider two options. The first option would be 3-D Euclidian space $E^3 \equiv R^3$ replaced with its p-adic counterpart Q_p^3. The action of SO(3) however leaves the distance from origin invariant and one can restrict the consideration to 2-sphere. The challenge is to define the counterpart of 2-sphere p-adically.

Before one can say anything about p-adic 2-sphere, one must understand what means with the ordinary 2-sphere identified now as sphere in metric sphere.

1. Riemann sphere is compactification of complex plane and can be regarded as complex projective space $CP_1 = P^1(C)$ is taken as starting point. This space is obtained from C^2 by identified points (z_1, z_2) which differ by a complex scaling: $(z_1, z_2) = \lambda(z_1, z_2)$. One can say that points of $P^1(C)$ are complex lines, which are nothing but Riemann spheres. This manifold requires two coordinate patches corresponding to patch containing North *resp.* South pole but not South *resp.* North pole. The coordinates in a patch containing Northern hemisphere can be taken to be $(u = z_1/z_2, 1)$ by projective equivalence allowing to select point $(z_1/z_2, 1)$ from the projective line with $z_2 \neq 0$. In the region containing Southern hemisphere one can take $v = z_2/z_1)$. In the overlap region around equator the coordinates are related by $v = 1/u$. One can think also $P^1(C)$ as plane with single point ∞ (south pole) added.

2. The group $PGL(2, C)$ and also the Lorentz group $SL(2, C)$ acts at Riemann sphere as Möbius transformations. The complex matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is represented as a Möbius transformation

$$u \rightarrow \frac{au + b}{cu + d} \quad .$$

Note that the matrix elements are complex: what this means in p-adic context is not at all clear!

One can regard the coordinates z_1 and z_2 as spinor components and the action of SO(3) is lifted to the action of covering group SU(2) for which 2π rotation is represented by -1. The group A_5 can be lifted to its covering group have twice as many elements as the original one but the action of SU(2) *resp.* overing of A_5 reduces to that of SO(3) *resp.* A_5 since one considers the action on the ratio z_1/z_2 of the spinor components.

3. $S^2 = P^1(C)$ is a good structure to generalize to p-adic context since one can define it purely algebraically, and one realize the action of isometries in it.

8.2.3 Icosahedron in p-adic context

What does one mean with p-Adic numbers?

The article about p-icosahedron [A129] gives also a concise summary of p-adic numbers. p-Adic number fields define a hierarchy of number fields Q_p labeled by prime $p = 2, 3, 5, ...$ They are completions of rationals so that rationals can be said to be common to reals and p-adics. Each Q_p allows an infinite number of algebraic extensions whereas reals allow only one - complex numbers.

Local topology of p-adic numbers is what distinguishes them from reals. Two points of Q_p are near to each other if they differ by a very large positive power of p. As real numbers these numbers would differ very much. Most p-adic numbers have infinite number pinary digits in the pinary expansion and are infinite as real numbers.

The p-adic norm defining the p-adic topology is defined by p-adic number fixed completely by the lowest pinary digit in the expansion and is therefore very rough and obtains only values p^n for Q_p. The resulting topology is very rough. Indeed all p-adic points define open sets: one says that p-adic topology is totally disconnected. p-Adic norm is non-Archimedean. It satisfies $|x-y| \leq Max\{x,y\}$ whereas real norm satisfies $|x|-|y| \leq |x-y| \leq |x|+|y|$. This property of p-adic topology is known as ultra-metricity.

p-Adic differential calculus exists and differentiation rules are same as for the real calculus. It is however not at all clear whether given real Taylor series with rational coefficients generalizes to its p-adic counterpart since the series need not converge p-adically. Exponential and trigonometric functions have p-adic counterparts but they do not have the properties of their real counterparts: for instance, p-adic trigonometric functions are not periodic. This is a problem when one tries to generalize Fourier analysis.

p-Adic integral calculus is problematic. The reason is that p-adic numbers are not well-ordered. As a consequence, the ordering crucial for Riemann integral does not exist. In fact, formal definition of Riemann integral gives as a limit vanishing integral. The generalization of Fourier analysis based on the integration of plane wave factors $exp(ikx)$ as roots of unity appearing in algebraic extension of p-adic numbers seems to be the only manner to overcome the problem. Algebraic continuation of integrals depending on parameters (such as integration limits) from real to p-adic context is in a central role in TGD framework but requires the fusion of reals and various p-adic number fields to bigger structure along common rationals: each number field would be like one page in a big book (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also).

What does one mean with p-adic complex projective space?

The question is what one should do for the projective space $P^1(C)$ to get its p-adic counterpart? The basic condition is that A_5 acts transitively in the p-adic analog of $P^1(C)$.

1. The first guess would be the replacement of $P^1(C)$ with $P^1(Q_p)$. This is however the p-adic analog of real projective line, not complex projective line and one cannot imbed the complex matrices representing the action of the covering group of A_5 of $PGL(2, Q_p)$.

2. What one should do? The basic observation is that complex numbers C define the only possible algebraic extension of real numbers. Generalizing this, one should consider algebraic extension of Q_p. There is infinite number of these extensions and one must choose that of minimal algebraic dimensions. This means that the phases $exp(i\pi/5)$ (10:th root of unity), $exp(i\pi/3)$ (6:th root of unity), and $exp(i\pi/2) = i$ (4:th root of unity) must be contained by the extension. The reason why one must have $exp(i\pi/5)$ rather than $exp(2\pi/5)$ representing rotation of $2\pi/5$ generating the cyclic group Z_5 is due the fact that one has two fold covering. Same applies to other roots of unity. The solutions of equation $x^{60} = 1$ give the needed roots of unity since $60 = 6 \times 10 = 4 \times 3 \times 5$ contains all the needed roots of unity needed in the representation matrices.

   The extension of Q_p containing those roots of unity which do not reduce to -1 (existing p-adically) would define the extension used. One can calculate the algebraic dimension of this extension but certainly it is much larger than 2 as in the case of complex numbers. The extension - call it K - is not unique but is minimal. There is infinite number of extensions containing this extension.

   To define things precisely one must replace the notions of p-adic integer, prime, and rational p applying in K but this is a technicality. This means that p - the only prime in Q_p - is replaced with π, the only prime in K.

I will leave the detailed construction of the projective space $P^1(Q_p)$ later because it is rather technical procedure. Some comments are however in order:

1. For $p \ mod \ 4 = 1$ (say $p = 5$ or 17) $i \equiv \sqrt{-1}$ belongs to the p-adic number field. Therefore the dimension of algebraic extension is considerably smaller than for $p \ mod \ 4 = 3$ (say $p = 3$ or 7) .

2. The naive question is whether for $p \ mod \ 4 = 3$ a considerably simpler approach could make sense. Use 2-D algebraic extension of p-adic numbers consisting of numbers $x + iy$: call this space C_p. Naive non-specialist might think that in this case the rather intricate complex construction of the projective space $P^1(Q_p)$ based on Bruhat-Tits tree might not be needed. This simpler construction however fails for $p \ mod \ 4 = 1$. It fails also more generally. The reason is that the $exp(i\pi/n)$, $n = 3, 5$ are algebraic numbers and do not belong to C_p. Therefore one must extend C_p to included also the phase factors and it seems that one ends up to the same situation as in general case.

3. Side track to TGD.

   (a) In TGD one encounters the problem "What could be the p-adic counterpart of S^2 and $CP_2 = P^2(C)$?". The above general recipe applies to this problem: replace C with an algebraic extension K of Q_p allowing the imbedding of some discrete subgroup of $SU(2)$ *resp.* $SU(3)$ represented as matrices in $PGL(2, K)$ *resp.* $PGL(3, K)$. The interpretation would be that due to finite measurement resolution the Lie group $SU(2)$ *resp.* $SU(3)$ is replaced with its discrete counterpart.

   This has a direct connection to the inclusions of hyperfinite factors of type II_1 (HFF) [K80], where all discrete subgroups of $SU(2)$ appear also those of SU(3), whose interpretation is in terms of finite measurement resolution with included HFF creating states which cannot be distinguished from the original state in the resolution used. General inclusions correspond to discrete subgroups of rotation group and by McKay correspondence [A192] to Lie groups of ADE type. The isometry groups of Platonic solids are the only simple groups in this hierarchy and correspond to exceptional Lie groups E_6, E_7, E_8.

   (b) One could criticize the approach since the algebraic extension K containing the isometry group is not unique. In TGD framework one however interprets the algebraic extensions in terms of finite measurement resolution. One cannot measure all possible angles p-adically- actually one cannot measure angles at all but only discrete set of phase factors coming as roots $exp(ik2\pi/n)$ of unity. The large the value of n, the better the measurement resolution.

What does one mean with p-adic icosahedron?

Once the projective space $P^1(K)$ generalizing $P^1(C) = S^2$ is constructed such that it allows the action of A_5 (it does not allow the action of entire rotation group!) one can identify the points which remain fixed by the action of various subgroups of A_5 (6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points are the 12 vertices, 20 barycenters, and 30 midpoints of edges). This is a purely algebraic procedure and there is no need to define what edges and faces are.

To obtain a more concrete picture about the situation one must define precisely what $P^1(Q)$ means and here the notion of Bruhat-Tits tree [A68] seems to be unavoidable.

8.3 Trying to explain what $P^1(Q_p)$ could mean technically

The naive approach to the construction of $P^1(Q_p)$ would be following. Do the same things as in the case of $P^1(C)$ or $P^1(R)$. The point pairs (q_1, q_2) in Q_p^2 are identified with pairs $\lambda \times (q_1, q_2)$ where $\lambda \neq 0$ is p-adic number. For some reason this simple approach is not adopted in the article [A129]. The reason is that one cannot introduce the notion of Bruhat-Tits tree [A68] in this approach. Bruhat-Trits tree is needed to obtain path-connectedness - that is connect the fixed points of icosahedron to form a "solid" and to give a more geometric meaning to the notion of icosahedron. One can regard $P^1(Q_p)$ as boundary of Bruhat-Tits tree somewhat like sphere is a boundary of ball in real context.

I am not not sure whether this approach on $P^1(Q_p)$ is equivalent with that of Berkovich [A216] based on the idea of adding some points to $P^1(Q_p)$ to make it path connected space containing $P^1(Q_p)$ as a dense subset. The outcome has rather frightening complexity.

The alternative approach would be purely algebraic. I will discuss later the problem of introducing the counterpart of path connectedness without giving up p-adic topology and by introducing induced real topology as effective topology having the desired path-connectedness.

8.3.1 Generalization of $P^1(C)$ making possible to introduce Bruhat-Tits tree

The following construction looks somewhat artificial but its purpose is to make possible the introduction of Bruhat-Tits tree allowing to realize path-connectedness.

1. The point pairs (q_1, q_2) Q_p^2 are replaced with Z_p lattices in Q_p^2. For given lattices the points are of form (n_1u, n_2v), where u and v are linearly independent (in Q_p) vectors of Q_p^2. Note that the p-adic integers $n_i = \sum_{k\geq 0} n_{i,k}p^k$ can be and typically are infinite as real integers. This is how the lattice differs from the real lattice. Also the p-adic distances between lattices points for which n_i differ by a large power of p are very small.

   Note: Q_p^2 is the p-adic analog of space of 2-spinors. The pairs (u, v) are indeed in 1-1 correspondence with pairs (q_1, q_2).

2. Projective equivalence is realized as for point pairs (q_1, q_2). This means that lattices for which base vectors (u, v) differ by a p-adic scaling are equivalent $(u, v) \equiv (\lambda u, \lambda v)$. Only the ratio u/v defining the "direction" of point of Q_p^2 matters.

   Note: In the complex case one would have two complex vectors and their ratio defines the conformal equivalence class of the plane compactified to torus by identifying the opposite edges of the polygon defined by u/v.

   Note: In the article one speaks about homothety classes: homothety means scaling which in p-adic context need not change p-adic norm.

This is not quite enough yet. Real icosahedron is in a well defined sense a connected coherent structure. Not just a collection of points. p-Adic topological is however totally disconnected. This suggests that one must introduce additional structure making possible to speak about icosahedron as "solid". Bruhat-Tits tree is one possible manner to achieve this. Also TGD inspired view about p-adic manifolds makes this possible.

8.3.2 Why Bruhat-Tits tree?

One introduces Bruhat-Tits tree [A68] as an additional structure having $P^1(Q_p)$ as its boundary in a well-defined sense (one needs its counterpart also in $P^1(K)$). In [A129] it is stated that this relates to a proper *global* definition of p-adic analytic structure in terms of Berkovich disks. As already explained, the basic problem for introducing analytic manifold structure is the total disconnectedness of p-adic topology. In p-adic topology each point is open set and all p-adic open sets are also compact. Moreover, two p-adic balls are either disjoint or nested. Therefore one cannot have partially overlapping p-adic spheres and the basic construction recipe for real manifolds fails. One can overcome this problem for algebraic varieties defined by algebraic equations but they are much less general objects than manifolds in real context.

1. There are no problems in defining p-adic differential calculus (a *local* aspect of the analytic structure) and field equations associated with action principles make sense although the definition of action as integral is problematic. p-Adic differential equations are non-deterministic: integration constants are replaced by piecewise constant functions depending on finite number of pinary digits. This has a nice interpretation in TGD inspired consciousness, where this nondeterminism would be correlate for non-determinism of imagination - one aspect of cognition. Therefore I am not at all sure whether the reinforcement of real number based notions to p-adic context is a good idea.

2. p-Adic integration (a *global* aspect of the analytic structure) is the problem in p-adic calculus and the total disconnectedness relates to the absence of well-ordering. An obvious guess is

that Bruhat-Tits tree could help in the definition of p-adic integral by defining the allowed integration paths.

Note: TGD approach on integration relies on algebraic continuation from real context and is based on what might ge regarded fusion of reals and p-adics along common rationals (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also).

3. Intuitively the Bruhat-Tits tree builds up a "skeleton" connecting points by edges and thus curing the total disconnectedness. This requires some non-locality and the replacement of point pairs (q_1, q_2) with integer lattices spanned by q_1 and q_2 would introduce this non-locality.

4. In any case, what one obtains is a graph with vertices and edges. Vertices are identified as homothety classes $[M]$ of the lattices and are just the points of $P^1(Q_p)$. Two vertices $[M]$ and $[N]$ are connected by an edge iff one can find representatives M and N such that $pM \subset N \subset M$. The representative N is in some sense between pM and M. Note that one has $pM \equiv M$ by homothety so that the use of representatives in the definition is necessary.

   The resulting graph is also a regular $p+1$-valent tree, the number of F_p-rational points of $P^1(F_p)$,which is projective space associated with finite field. One can check this in case of $p = 2$. The points (f_1, f_2) are $(1, 0), (1, 1), (0, 1), (1, 1)$ and by projective equivalence one has just $p = 1+2 = 3$ points in corresponding projective space. The transitive action of $Gl(2, K)$ means that all vertices are $p + 1$-valent and this fixes the structure of the graph completely. I will consider this point in more detail later on basis of the web article [A68].

Bruhat-Tits tree can be seen as a skeleton of the "full" $P^1(K)$ containing also the additional points making it a path connected Berkovich space. The "naive" $P^1(K)$ can be regarded as boundary of the Bruhat-Tits tree.

Bruhat-Tits tree looks very nice notion but there is objection against its construction in the proposed manner. Ordinary p-adic numbers- the simplest possible situation - are not in 1-1 correspondence with the Z_p lattices as will be demonstrated later but with powers of p. Same applies to Q_p^2 where the lattices correspond to $Sl(2, Z_p)$ equivalence classes of elements of Q_p^2. One can of course ask whether projective spaces are p-adically and maybe also physically very special for this reason.

8.3.3 Berkovich disk

Bruhat-Tits tree is not enough for p-adicizing real topologist. Also Berkovic disk is required as the analog of open ball in real context. The slides of Emmy Noether Lecture by Annette Werner [A216] give a concise representation of the basic idea behind Berkovich disk serving as a basic building brick of p-adic manifolds just like real n-disk does in the case of real n-manifolds and also also explains its construction. I must admit that I do not understand well enough the connection between Berkovich disk and Bruhat-Tits tree.

One can motivate the construction with the completion of rationals to reals. By adding all irrationals (algebraic numbers and transcendentals) one obtains reals and these additional numbers glue the rationals to form a continuum so that one can defined calculus and many other nice things. The idea is to mimic this construction.

1. In the example one restricts to the unit disk for an non-archimedean field assumed for simplicity be algebraically closed, which means algebraic completion containing all algebraic numbers considered also by Khrennikov. This notion is very formal and unpractical. The idea is to form a completion of the unit disk for a non-archimedean field K (algebraic extension of Q_p) containing thus K as a dense subset with the property that the resulting topology is path connected and not anymore ultra-metric (somewhat artificial!).

   For this purpose one constructs what is called the space of bounded multiplicative non-Archimedean norms for formal K-valued power series defined in the unit disk reducing to the norm of K for constant functions. It is possible to characterize rather explicitly this space and with topology defined by a pointwise convergence (point is now the K-valued function) of the norm one obtains uniquely path connected topology. The additional points can be said to glue the points of the K-disk to a continuum as its dense subset just as the addition of irrationals glues rationals to form a continuum.

2. The construction generalizes to the construction of the counterparts of p-adic projective spaces and symmetric spaces. Berkovich has also proposed an approach to p-adic integration and harmonic analysis relying on the notion of Berkovic space.

   Note: In TGD framework integration is defined by algebraic continuation in the structure defined by the fusion of real and various p-adic numbers fields and their extensions to form a book like structure. One could perhaps say that this fusion defines a kind of "super-completion": all possible completions of rationals are fused to single book like structure and rationals indeed defined a dense subset of this structure.

The construction is rather technical. From unit disk to a function space defined in it to the space of multiplicative seminorms defined in this function space! For the simple brain of physicist desperately crying for some concreteness this looks hopelessly complicated. Physicists would be happy in finding some concrete physical interpretation for all this.

8.3.4 Bruhat-Tits tree allows to "connect" the points of p-adic icosahedron as a point set of $P^1(K)$

The notion of p-adic icosahedron can be defined also in terms of Bruhat-Tits tree since the $PLG(2,K)$ acts transitively on the homothety class so that one obtains all homothety classes from the one associated with $(u,v) = (1,1)$ and one can speak about orbit of this basic homothety class. This means that one can connect the vertices, mid-points of edges, and barycenters of faces to common origin by edge paths in Bruhat-Tits tree and therefore to each other. This is what path-connectedness means.

How Bruhat-Tits tree allows to build from a set of totally disconnected fixed points a "solid"? One answer is that the addition points of completion make this possible.

1. Bruhat-Tits tree allows to define what is called an end of the Bruhat-Tits tree as an equivalence class of infinite half line with two half lines identified if they differ by a finite number of edges. These ends are in one-one corresponds with the K-rational points of $P^1(K)$ (these are not the only points of $P^1(K)$). One can say that $P^1(K)$ represents the boundary of Bruhat-Tits tree as a p-adic manifold.

   Note: Could this finite number of different edges corresponds to a finite number of pinary digits appearing in p-adic integration "constants")? The identification could mean that all choices of pseudo constants in p-adic differential equations are regarded as equivalent. Physicist might speak about the analog of gauge invariance: the values of pseudo constants do not matter.

2. For a finite set of points of totally disconnected $P^1(K)$ there exists a unique minimal subtree of the entire Bruhat-Tits tree containing the points of this set as its ends [A129]. This subtree is what connects the points of this point set to a coherent structure in the set that one can construct paths connecting the points to single point. There are of course several manners to achieve this but one can define even the analog of the geodesic line as a path with a minimal number of edges so that it becomes possible to speak also about the edges of icosahedron. The length of the geodesic could be simply the number of edges for this minimal edge path.

3. The p-adic counterpart of Platonic solid must be also "solid". This is achieved if the fixed points for the subgroups of the isometry group of Platonic solid (in particular for those of the A_5) defining the Platonic are identified as ends of a unique minimal subtree of Bruhat-Tits tree.

For higher-dimensional projective spaces $P^n(K)$ Bruhat-Tits tree generalizes from 1-D discrete homogenous space $PGl(2,K)/Gl2,Z_K)$ to n-dimensional discrete homogenous space. The reason is that the edges of tree develop higher-dimensional cycles having interpretation as simplexes. One can also define homology groups for this structure. Also now $P^n(K)$ can be regarded as a boundary of the resulting structure.

8.4 Algebraic universality in TGD framework

In TGD framework the algebraic approach looks very promising one - at the first glance perhaps even the only possible one - since under some assumptions the field equations for preferred extremals [K10, K89] reduce to purely algebraic ones and do not even refer to action principle explicitly. The point is that the preferred extremal property means a generalization of complex structure to 4-D situation and is a notion independent of action and the preferred extremals are solutions to field equations of very many general coordinate invariant variational principles (Einstein-Maxwell equations with cosmological term and minimal surface equations hold true). p-Adic variants of these conditions are purely algebraic and make sense so that one can hope that even space-time surfaces might have p-adic counterparts. The underlying assumptons can be questioned and I have indeed done this [K94] by bringing in consideration the possiblity the possibility that TGD could allow cosmological constant like parameters depending on position.

As already noticed, one can consider a compromise between topological and algebraic approach to the definition of p-adic manifolds by using a variant of canonical identification to map rational points of the p-adic preferred extremal to rational points of its real counterpart and completing this skeleton to a preferred extremal in the real context [K95]. This mapping need not be one-to-one. In the intersection of real and p-adic worlds the expression for real preferred extremal makes sense also in p-adic number field, and a direct identification makes sense and is unique.

In the real sector the preferred extremal property would boil down to to the existence of complex structure in Euclidian regions and what I call Hamilton-Jacobi structure in Minkowskian regions. Also the conjecture that preferred extremals are quaternionic surfaces in certain sense [K72] implies independence on action principle. The challenge is to prove that these two algebraic characterizations of preferred extremals are equivalent. These two purely algebraic conditions might make sense also in p-adic context with complex and hypercomplex numbers replaced with appropriate algebraic extensions of p-adic numbers.

The p-adicization program based on the notion of algebraic continuation involves many open questions to be discussed first.

8.4.1 Should one p-adicize entire space-time surfaces or restrict the p-adicization to partonic 2-surfaces and boundaries of string world sheets?

One of the many open questions concerns the objects for which one should be able to find p-adic counterparts. The arguments based on canonical identification and universality of the preferred extremal property support the view that p-adicization can be carried out at 4-D level for space-time surfaces and also at the level of WCW. Later a detailed proposal for how p-adic preferred extremals can be mapped to real preferred extremals with the uniqueness of this correspondence restricted by the finite measurement resolution realized as pinary cutoff will be described.

One can however consider also an alternative approach in which one restricts the p-adicization to 3- or even 2-dimensional objects of some special classes of these objects and this possibility is discussed below.

1. Should one p-adicize only boundaries?

A grave objection against p-adicizing only partonic 2-surfaces and braid strands is that one loses the very powerful constraints provided by the preferred extremal property and coordinate maps defined by the canonical identification in preferred coordinates. Therefore the algebraic continuation of the partonic 2-surface can become highly non-unique ($x^n + y^n = z^n$, $n > 2$, is the basic counter example: in higher dimensions one expects that this kind of situations are very rare!). Furthermore, the restriction to partonic 2-surfaces and braid strands is artificial since imbedding space must be p-adicized in any case. The replacement of the p-adicization of the partonic surface plus 4-D tangent space data with that of the preferred extremal containing it increases the number of constraints dramatically so that holography might even make the p-adicization unique.

Despite this objection one can try to invent arguments for restricting the p-adicization to some subset of objects since this would simplify the situation enormously.

1. The basic underlying idea of homology theory is that the boundary of a boundary is empty.

p-Adic manifolds in turn have no boundaries because of the properties of p-adic topology. Should p-adicization in TGD framework be carried only for boundaries? Light-like 3-surfaces define boundaries between Minkowskian and Euclidian regions of space-time surface. The space-like 3-surfaces defining the ends of space-time surfaces at the boundaries of CD are boundaries. Also 2-D partonic surfaces and boundaries of string world sheets can be considered. One must consider also the boundaries of string world sheets as this kind of objects.

2. Strong form of General Coordinate Invariance implies strong form of holography. Either the data at light-like 3-surfaces (at which the signature of induced metric changes) or space-like 3-surfaces at the ends of CD codes for physics, which implies that partonic 2-surfaces and 4-D tangent space data at them code for physics.

   What 2-D tangent space data could include? The tangent space data are dictated partially by the weak form of electric magnetic duality [K19] stating that the electric component of the induced Kähler field component is proportional to its magnetic component at light-like 3-surfaces. Also the boundaries of string world sheets contribute to 4-D tangent space data and at the end of braid strands at partonic 2-surfaces both light-like and space-like direction are involved.

If space-time interior is not p-adicized (somewhat un-natural option), the p-adicization reduces to the algebraic continuation of Kähler function and Morse function to p-adic sectors of WCW. Both functions reduce to 3-D Chern-Simons terms for selected 3-surfaces. p-Adicization should reduce to algebraic continuation of various geometric parameters appearing as arguments of Kähler action.

In the minimal situation only partonic 2-surfaces and the boundaries of string world sheets - briefly braid strands - need to be p-adicized and the existing results - such as the results of Mumford derived from the existence of p-adic uniformization - could give powerful contraints. One can also ask whether the p-adic string world sheet in some sense is equivalent with the generalization of Bruhat-Tits tree allowing also loops.

Besides the string world sheet boundary and partonic 2-surface also for "4-D tangent space data" fixed at least partially by weak form of electric magnetic duality and string world sheets is needed. There are several open questions.

1. Does weak form of electric-magnetic duality have any meaning if one cannot speak about space-time interior in p-adic sense? This condition would apply only at partonic 2-surfaces. Same question applies in the case of braid strands. Can one effectively reduce space-time interior and string world sheet to their tangent spaces at partonic 2-surface/braid strands.

2. It is not even clear whether the dynamics of light-like 3-surfaces and space-like 3-surfaces is deterministic. Strong form of holography requires either determinism or non-determinism realized as gauge invariance, which could correspond to Kac-Moody type symmetries. Kac-Moody symmetry would favor the idea that p-adicization takes place only for partonic 2-surfaces and for the braid strands. Gauge symmetry would also give hopes that the integral of Chern-Simons term depends only on the data at the end points of braid strands at partonic 2-surfaces and maybe on data at braid strands: this would however require p-adic integration not possible in purely p-adic context. These data should remain invariant under Kac-Moody symmetries.

3. Should one p-adicize the weak form of electric magnetic duality? The duality involves the dual of Kähler form of the partonic surface with respect to the induced four-metric: the normal component of Kähler electric field at partonic surface and/or at string world sheet boundary equals to Kähler magnetic form at the partonic surface at particular point of its orbit (most naturally light-like curve). The induced 4-metric becomes degenerate at the light-like 4-surface and the component of electric field is finite only if weak form of electric-magnetic duality can be satisfied. Should the duality hold true for entire 3-surfaces, for partonic 2-surfaces, or perhaps only for for the braid strands? The purpose of the condition is to guarantee that Kähler electric charge as eletric flux is proportional to Kähler magnetic charge: therefore it should hold along entire 3-surfaces and if these are regarded as real surfaces there are no problems with the p-adicization of the condition.

2. What kind of algebraic 2-surfaces can have p-adic counterparts?

There is no need for a generic algebraic surface to have direct algebric p-adic counterpart for all p-adic primes. If one uses as preferred coordinates a subset of preferred coordinates of the imbedding space and accepts only imbedding space isometries as general coordinate transformations, the algebraic surfaces in the intersection of real and p-adic worlds must satisfy very strong conditions. For instance, a representation in terms of polynomials cannot involve real transcendentals. Even rational coefficients can force algebraic extension of Q_p, when the remaining imbedding space coordinates are expressed in terms of the coordinates of the partonic two-surface.

Mumford is one of the pioneers of p-adicization of the algebraic geometry and has demonstrated that only a restricted set of p-adic algebraic surfaces allow interpretation as p-adic Riemann surfaces if one requires that a generalization of so called uniformization theorem holds true for them [A99]. This theorem says that Riemann surfaces are constructible as factor spaces of either sphere, complex plane, or complex upper plane (hyperbolic space H^2 with the subgroup Γ identified as the finitely generated free subgroup of the isometries of the space in question. The construction does not work for all algebraic surfaces but only for the surfaces satisfying certain additional conditions. This is not a problem in TGD framework in the intersection of real and p-adic worlds since the p-adicization is not expected to be possible always but only in the intersection of real and p-adic worlds.

According to the article Multiloop Calculations in p-Adic String Theory and Bruhat-Tits Trees by Chekhov et al [A143] the construction of higher genus Riemann surfaces as so called Mumford surfaces takes place by starting from Bruhat-Tits tree representing $g = 0$ surface and by taking subgraphs having interpretation as representations for an orbit of so called Schottky group characterizing the higher genus Riemann surface and gluing these graphs together by transversal connections. This indeed represents the genus homologically as a loop of the resulting tree.

Note: The article of Chekhov et al describes a proposal for the construction of complex scattering amplitudes for p-adic strings in real imbedding space so that the situation is not relevant for TGD as such. The amplitudes are constructed in terms of p-adic characteristics and this means that the amplitudes can be interpreted also as numbers in p-adic number fields extended by roots of unity. The characteristics $q = exp(i2pi\tau)$ exist only for the values of q which are of form $q = p^n exp(x)exp(i2\pi/m)$, $|x| < 1$ so that discretization of the p-adic norm and phase of τ is necessary.

3. Should one really restrict the p-adicization to algebraic surfaces?

One could also consider the possibility of restricting p-adicization to algebraic surfaces (they could be also 4-D). Practicing physicist would argue that the restriction of p-adicization to algebraic surfaces is quite too heavy an idealization. In the real world spheres are topological rather than algebraic.

Luckily, if the construction recipe for p-adic manifolds to be discussed later really works, canonical identification with pinary cutoff allows to generalize p-adic algebraic surfaces to p-adic manifolds, and to achieve very close correspondence with the real manifold theory. Given real preferred extremal can correspond to not necessarily unique p-adic preferred extremal for some values of p. Also two p-adic preferred extremals with different values of p-adic prime which correspond to the same real preferred extremal correspond to each other. This provides an elegant solution to all problems discussed hitherto and there is not need to restrict the p-adicization in any manner.

Finite measurement resolution would be a prerequisite for algebraic continuation in the sense that subset of rational and algebraic points defined by pinary cutoff and algebraic extension would be common to the real and p-adic preferred extremals. Therefore finite measurement resolution would make it possible to realize both number theoretical universality and p-adic manifold topology.

8.4.2 Should one p-adicize at the level of WCW?

One can of course challenge the idea about p-adicization at the level of WCW and WCW spinor fields and ask what this procedure gives. One motivation for the p-adicization would be p-adic thermodynamics. p-Adic thermodynamics should emerge at the level of M-matrix which indeed can be regarded as a "complex square root" of hermitian density matrix in zero energy ontology and therefore expressible as a product of hermitian square root of density matrix and unitary S-matrix.

Hence it would seem that the p-adicization at the level of WCW is natural and the representability as a union of symmetric spaces constructible as factor groups of symplectic group of $\delta M^4_\pm \times CP_2$ gives hopes that algebraic approach works also in infinite-dimensional case. Finite measurement resolution and the properties of hyper-finite factors of type II_1 are expecetd to reduce the situation to finite-dimensional case effectively.

8.4.3 Possible problems of p-adicization

The best manner to clarify one's thoughts is to invent all possible objections and in the following I do my best in this respect. The basic point is following. If one accepts the purely algebraic approach without no reference to canonical identification, one must check that everything in TGD - as I recently understand it - can be expressed without inequalities! Boundaries are defined by inequalities and one must check that they can be avoided. If this is not the case, the notion of p-adic manifold relying on the notion of canonical identification seems to remain the only manner to avoid problems.

Wormhole throats are causal rather than topological boundaries

The notion of boundary does not have any counterpart in purely p-adic context since its definition involves inequalities. The original vision was that space-time sheets possess boundaries and the boundaries carry quantum numbers - in particular family replication phenomenon for fermions would have explanation in terms of the genus of 2-dimensional boundary component of 3-surface [K17]. It however turned out that boundary conditions require that the space-time sheet approaches vacuum extremals at boundary and this does not seem to make sense. This led to the view that one must allow only closed space-time "sheets" which can be thought of as being obtained by gluing real space-time sheets together along boundaries.

Also the notion of elementary particle involves preferred extremals - massless extremals in the simplified model [?] connected by wormhole contact structure defining the elementary particle. These preferred extremals must combine to form a closed space-time surface and this is quite possible: the minimal situation corresponds to two space-time sheets glued together as in the model of elementary particles.

Genuine boundaries are replaced by the light-like 3-surfaces -orbits of wormhole throats - at which the signature of the induced metric changes from Minkowskian to Euclidian and four-metric degenerates effectively to 3-D metric locally. These can be defined by purely algebraic conditions and there is no need for inequalities.

Partonic 2-surfaces are identified as intersections of the space-like 3-surfaces at the ends of CD: the ends of CD are defined by purely algebraic equation $t^2 - r^2 = 0$ and $(t-T)^2 - r^2 = 0$ and once the equations of space-time surface are known one can solve the equations for space-like 3-surfaces. The equations defining what light-like 3-surfaces at which the induced four-metric is degenerate are algebraic and express just the degeneracy of the induced four-metric. The condition that algebraic equations for light-like 3-surfaces and space-like 3-surfaces hold true simultaneously define partonic 2-surfaces. Hence it seems that the surfaces can be expressed algebraically.

This approach might look a little bit artificial. Also the idea that only boundaries should be p-adicized should be be p-adicized looks artificial. The best looking option is the use of canonical identification to define p-adic manifolds since it allows to transfer real topological notions to the p-adic context. In particular, the well-ordering of reals induces that of p-adics so that inequalities cease to be a problem and boundaries can be defined.

What about the notion of causal diamond and Minkowski causality?

A possible problem for purely p-adic approach allowing no in-equalities is caused by the notion of causal diamond (CD) defined as intersection of future and past directed light cones (as a matter fact, CP_2 is included to CD as Cartesian factor but I do not bother to mention it again and again). CD has light-like boundaries.

It is not quite clear whether space-time surface must be always localized inside CD. The notion of generalized Feynman diagram indeed suggests that the space-time surfaces can continue also outside the CDs and that CD could be seen as an imbedding space correlate for what might be called spot-light of consciousness. If this were the case quite generally, the p-adicization of space-time sheets would not produce problems even if one does not use canonical identification.

In purely p-adic context, one should however give some meaning for the statement that space-time surface is contained inside CD and this seems to require the notion of boundary for CD. Does this notion of CD make sense in the p-adic context or is the fusion of real and p-adic number fields along common rationals required? The resolution of the problem seems to require the fusion. In the case of algebraic extensions also common algebraics are present.

The first questions concern the notion of Minkowski causality, which relies on light-cone and its complement expressed in terms of inequalities.

1. The first reason of worry is that in purely p-adic context also the equation $t^2 + r^2 = 0$ has a lot of solutions! The reason is that the notion of positive and negative do not make sense for p-adic numbers without some constraints. If one restricts the p-adic numbers to those having finite number of pinary digits - this happens always when one has finite pinary resolution - all p-adic numbers included rationals reduces to finite positive integers as real numbers. Therefore in finite pinary resolution the problems disappear. The condition that rationals points of Minkowski space are common with its p-adic variant, makes finite pinary resolution natural, and one could say that all p-adic numbers - including negatives of finite integers - can be said to be infinitely large positive integers in real sense. Here one must of course be very cautious.

2. The condition $s = t^2 - r^2 < 0$ for the complement of future light-cone has no meaning in the p-adic context for general p-adic numbers. If rational values of Minkowski coordinates correspond to same point in real and p-adic sense, finite pinary resolution means that all pinary cutoffs have $s \geq 0$ and $t \geq 0$ in real sense. This is also true for $a = \sqrt{t^2 - r^2}$ so that one remains inside future light-cone unavoidably. Anything outside future light-cone is unexpressible in finite measurement resolution p-adically.

   Finite temporal and spatial resolution suggest integer quantization of t and r in suitable units and one could say whether s has finite of infinite number of pinary digits - that is are positive or negative. Finite real integer values of t and r have finite number of pinary digits. Their negatives have infinite number of pinary digits and one could argue they correspond to infinite future if they are interpreted as real numbers. The values of s in future light-cone have finite number of pinary digits and correspond to finite real values. Outsider the future light cone the values of s are negative in real sense and have infinite number of pinary digits and thus interpreted as real numbers are in future infinity.

   One can consider also rational values of t and s if one keeps also p-adically track that rational is in question. Rationality means that pinary expansion is periodic after some pinary digit. Therefore it would seem to be possible to distinguish between $s \geq 0$ and $s \leq 0$ also p-adically for finite measurement resolution purely algebraically.

3. Causal diamond is defined as the intersection of future and past directed light cones. The lower light-cone in the intersection decomposes to pieces of hyperplanes $t \geq 0$ with $r \leq t$ and upper light-cone to pieces $T - t \geq 0$, $r \leq T - t$. If these variables are quantized as integer multiples of suitable unit and if these integer multiples can be interpreted in both real and p-adic sense, there is no need for inequalities in p-adic context. Also now rational values can be allowed.

If only boundaries are p-adicized, p-adicization would apply only to the light-like boundaries of CDs, and one would avoid possible problems related the sign of $s = t^2 - r^2$. This would conform with the strong form of holography and allow p-adicization of WCW.

Again one might argue that the number theoretical game above is artificial. The safest alternative seems to be canonical identification with pinary cutoff used to map real preferred extremal to its p-adic counterpart.

Definition of integrals as the basic technical problem

Physicist wants to perform integrals, and the problems related to the notion of integral is what any novice of p-adic physics is doomed to encounter sooner or later. As will be described the definition of p-adic manifold based on canonical identification solves these problems by inducing real integration to the p-adic realm by algebraic continuation.

Before continuing about integration it is however good to summarize the general TGD based view about the relationship between real and p-adic worlds.

1. Intersection of real and p-adic worlds as key concept

In TGD framework the basic notion is the intersection of real and p-adic worlds generalizing the idea that rationals are common to reals and p-adics. Algebraic continuation between real and p-adic worlds takes place through this intersection, in which real formulas allow interpretation as p-adic ones. The notions of intersection and algebraic continuation apply both at space-time level and WCW level.

1. At the space-time level rational (and even some algebraic) points of real surfaces are contained by p-adic surfaces. One can identify these rationals and say that real and p-adic surfaces intersect at these points and define discrete cognitive representation. Among other things this would explain why numerics is necessarily discrete and possible only using rationals with cutoff.

2. One can abstract this idea to the level of WCW. Instead of number fields one considers surfaces (partonic 2-surfaces, 3-surfaces, or space-time surfaces) in various number fields. If the representation of the surface (say in terms of rational functions) makes sense both for reals and p-adic number field in question, one can identify the real and p-adic variants of surfaces. These surfaces can be said to belong to the intersection of real and p-adic worlds (worlds of classical worlds, to be more precise). In TGD inspired theory of consciousness one would say that they belong to the intersection of material/sensory world and the world of cognition. In TGD inspired quantum biology life is identified as something residing in the intersection of realities and p-adicities.

2. Algebraic continuation as a basic tool

With this philosophical background one an consider the algebraic continuation of real integrals from the intersection of real and p-adic worlds defined by surfaces, whose representations in preferred coordinates make sense in real number field and in the p-adic number field to which one wants to continue.

1. Harmonic analysis in coset spaces with discretization defined by the algebraic extension of Q_p might make possible to avoid the problems by reducing the integrals to sums over the discrete points of the coset space. Algebraic continuation is of course central element in the program.

2. The recent progress in the calculation of planar scattering amplitudes in $\mathcal{N}=4$ SYMs gives hopes that M-matrix could be defined in number theoretically universal manner. The reason is that in TGD framework the fermions defining building bricks of elementary particles are massless - a basic prerequisite for the twistor approach - also when they appear as virtual particles. This gives enormously powerful kinematical constraints reducing the number of diagrams dramatically, and allows to express amplitude in terms of on-mass shell amplitudes just as one does in the twistor Grassmannian approach.

   For $\mathcal{N}=4$ SYM (and also more general theories) planar Feynman diagrams boil down to integrals over Grassmannians, which are coset spaces associated with $Gl(n,C)/Gl(n-m,C)\times Gl(m,C)$ allowing the already described generalization to p-adic context. The integrals reduce to multiple residue integrals, which could make sense also in the p-adic context because of the very weak dependence on integration region. The algebraic continuation of the resulting amplitudes to p-adic context replacing C with an appropriate extension of p-adic numbers might well make sense.

3. Two problems as solutions of each other

Unfortunately, the algebraic continuation of integrals is not free of technical problems. Even in the case of rational functions the algebraic continuation of the real integrals is susceptible to p-adic existence problems.

1. The basic problem with definition of ordinary 1-D integrals of rational functions is that the integral function of $1/x$ is $log(x)$ rather than rational function as for other powers. Unless the limits are very special (of form $x = 1+O(p)$), the algebraic continuation requires infinite-dimensional extension of p-adic numbers containing all powers of $log(x)$ for some $1 \leq x < p$. Can one allow infinite-D extensions, which are not algebraic?

2. The appearance of 2π in residue integral formulas which could otherwise make sense in p-adic context provides a second reason for worries: should one also transcendental extension containing powers of 2π?

Often two quite unrelated looking problems turn out to have a common solution. Now the second problem is purely physical: why a given particle should correspond to a particular p-adic prime? At this moment one must be satisfied with the p-adic length scale hypothesis stating that these primes are near powers of 2 and Mersenne primes are favored. I have not been able to identify any convincing dynamical principle explaining why primes near powers of two seem to be favored. It deserves however to be mentioned that the preferred p-adic length scale as a fixed point of p-adic coupling constant evolution (discrete) is one possible explanation meaning vanishing of beta functions, something very natural taking into account the quantum criticality of TGD Universe.

Could this problem define the solution of the first problem and vice versa! Maybe one must just accept that algebraic continuation to given p-adic number field is not always possible!

1. This criterion could strongly constrain the p-adic primes assignable to a given elementary particle. Consider as an example Kähler function defined as Kähler action for Euclidian portion of space-time (generalized Feynman graph) and Morse function defined as Kähler action for Minkowskian portion of space-time. The existence of the p-adic variant of Kähler function (or its real exponent) and Morse function (or its imaginary exponent) would allow to assign to a given space-time surface a highly restricted set of p-adic primes, and the allowed quantum superpositions of space-time surfaces could contain only those for which at least one of the allowed primes is same.

2. For massless particles Kähler action would vanish and algebraic continuation of Kähler action would be possible to all p-adic primes in accordance with the scale invariance of massless particles. Also the breaking of scale invariance and conformal invariance meaning selection of a particular p-adic length scale could be basically a number theoretical phenomenon. This would provide a totally new approach to the mystery of mass scales which in standard model framework requires fine tuning of Higgs mass with a totally unrealistic accuracy (one must avoid both the Landau pole meaning infinite self-coupling of Higgs and vacuum instability preventing massivation by Higgs vacuum expectation).

3. For instance, a function of form $log(m/n)$ can be algebraically continued only to those p-adic number fields for which m and n have form $m = k + O(p)$, and $n = k + O(p)$, $0 < k < p$ so that one has $m/n = 1 + O(p)$. The exponent of Kähler function in turn can be continued to Q_p if it is proportional to power of corresponding prime p. The exponential decay of Kähler function would have p-adic counterpart as decay of p-adic norm (just like Boltzmann weight $exp(-E/T)$ corresponds to p^n in thermodynamics). This could partially answer the question why the space-time surfaces assignable to electron seem correspond to Mersenne prime $M_{127} = 2^{127} - 1$ as suggested by p-adic mass calculations.

4. Number theoretic criterion might also mean that the p-adic prime characterizing particle state is extremely sensitive to the details of the particle state in real sense. The point is that a small modification of rational number in real sense changes its prime decomposition dramatically! Number theoretic anatomy is not continuous in real sense! An extremely small symmetry breaking in real sense modifying the value of Kähler function as function of quantum numbers might modify the value of the p-adic prime dramatically by affecting profoundly the number theoretic anatomy of some rational parameter appearing in the formula for Kähler function. For instance, in the standard framework it is very difficult to imagine any breaking for the SUSY assignable to right-handed neutrinos since they interact only gravitationally. The addition of right handed neutrino transforming particle to sparticle might however modify the p-adic prime (and thus mass scale) assigned to the particle dramatically.

4. What should one achieve?

It is a long way from this heuristic number theoretic vision to the calculation of p-adic valued integrals at space-time level, say to a formula for the p-adic action integral defined by Kähler action density (if needed at all).

1. The reduction to integral of Abelian Chern-Simons form over preferred 3-surfaces would be the first step and the definition of p-adic integral of Chern-Simons form second step. The special properties of preferred extremals give hopes about the reduction of the value of the Kähler action to local data given at discrete points at partonic 2-surfaces. The braid picture for many-fermion states forced by the modified Dirac equation [K89] and motivated by the notion of finite measurement resolution having discretization as a space-time correlate, suggests a reduction of real action integral to a sum of contributions from the ends of braid strands defining the boundaries of string world sheets. The optimistic hope would be that this data allows a continuation to the p-adic realm.

   Note: This kind of reduction might be quite too strong a conditon. All that is required in the approach based on canonical identification is that the values of Kähler function and Morse function exist in the given p-adic number field or its algebraic extension.

2. p-Adic valued functional integral is unavoidable at the level of WCW.

   (a) Algebraic continuation in the framework provided by the fusion of reals and various p-adic number fields looks the only reasonable approach to the p-adic functional integral.

   (b) Second element is Fourier/harmonic analysis in symmetric spaces: WCW is indeed a union of infinite-dimensional symmetric spaces over zero modes which do not contribute to WCW metric. One can hope that one can define the symmetric spaces algebraically in terms of their maximal symmetries since the metric reduces to that in single point of the symmetric space.

   (c) Canonical identification is the third element: p-adic functional integral for given p should be real functional integral restricted to preferred extremals allowing canonical identification map to the p-adic preferred extremal for that value of p. This would mean that real functional integral decomposes into a sum of contributions labelled by p-adic number fields and their algebraic extensions. This decomposition would be analogous to the formula obtained as a logarithm of the adelic formula for the rational as the inverse of the product of its p-adic norms.

Do the topological invariants of real topology make sense in the p-adic context?

In p-adic context the direct construction of topological invariants is not possible. For instance, the homology theory formulated in terms of simplexes fails since the very notion of simplex requires inequalities and well-ordering of the number system to define orientation for the simplex.

Also the notion of boundary is lacking since p-adic sets do not possess boundaries in topological sense. There however exists refined theories of p-adic homology allowing to circumvent this difficulty and the problem is that there are too many theories of this kind. A single universal theory would be needed and this was the dream of Grothendieck.

p-Adic mass calculations assume that the genus of the partonic 2-surface makes sense also in the p-adic context. For algebraic varieties the genus can be defined algebraically. There should be no problems if the partonic 2-surfaces are defined by algebraic equations which make sense for both reals and p-adic numbers. This is true for polynomial equations with rational coefficients and for algebraic extensions with coefficients in algebraic extension. By continuity algebraic continuation should allow to extend the notion of genus to surfaces for which rational coefficients are replaced with general p-adic numbers.

One expects that also more refined topological invariants making sense in the real context make sense also p-adically for algebraic varieties. A possible objection is that in the case of 3-manifolds allowing hyperbolic geometry (constant sectional curvatures) the volume of 3-manifold serves as a topological invariant. Volume is defined as an integral but in purely p-adic context volume integral is ill-defined. Is this a reason for worries? Hyperbolic n-manifolds have purely group theoretic formulation as coset spaces H^n/Γ, where Γ is discrete subgroup of the isometry group $SO(1,n)$ of n-dimensional hyperboloid H^n of $n+1$-D Minkowski space satisfying some additional conditions. Maybe this could allow to overcome the problem.

If canonical identification is used to map real preferred extremals to p-adic ones, boundaries and real topological invariants are mapped to p-adic ones both by algebraic continuation and in topological sense within finite measurement resolution. This even in the case that the real surface is not algebraic surfaces. This applies also to conformal moduli of the partonic 2-surfaces, whose p-adic variants play a key role in p-adic mass calculations.

What about p-adic symmetries?

A further objection relates to symmetries. It has become already clear that discrete subgroups of Lie-groups of symmetries cannot be realized p-adically without introducing algebraic extensions of p-adics making it possible to represent the p-adic counterparts of real group elements. Therefore symmetry breaking is unavoidable in p-adic context: one can speak only about realization of discrete sub-groups for the direct generalizations of real symmetry groups. The interpretation for the symmetry breaking is in terms of discretization serving as a correlate for finite measurement resolution reflecting itself also at the level of symmetries.

1. Definition of p-adic Lie groups

The above observation has led to TGD inspired proposal for the realization of the p-adic counterparts symmetric spaces resembling the construction of $P^1(K)$ in many respects but also differing from it.

1. For TGD option one considers a discrete subgroup G_0 of the isometry group G making sense both in real context and for extension of p-adic numbers. One combines G_0 with a p-adic counterpart of Lie group G_p obtained by exponentiating the Lie algebra by using p-adic parameters t_i in the exponentiation $exp(t_iT_i)$.

2. One obtains actually an inclusion hierarchy of p-adic Lie groups. The levels of the hierarchy are labelled by the maximum p-adic norms $|t_i|_p = p^{-n_i}$, $n_i \geq 1$ and in the special case $n_i = n$ - strongly suggested by group invariance - one can write $G_{p,1} \supset G_{p,2} \subset ...G_{p,n}....$ $G_{p,i}$ defines the p-adic counterpart of the continuous group which gets the smaller the larger the value of n is. The discrete group cannot be obtained as a p-adic exponential (although it can be obtained as real exponential), and one can say that group decomposes to a union of disconnected parts corresponding to the products of discrete group elements with $G_{p,n}$.

   This decomposition to totally uncorrelated disjoint parts is of course worrying from the point of view of algebraic continuation. The construction of p-adic manifolds by using canonical identification to define coordinate charts as real ones allows a correspondence between p-adic and real groups and also allows to glue together the images of the disjoint regions at real side: this induces gluing at p-adic side. The procedure will be discussed later in more detail.

3. A little technicality is needed. The usual Lie-algebra exponential in the matrix representation contains an imaginary unit. For $p \ mod \ 4 = 3$ this imaginary unit can be introduced as a unit in the algebraic extension. For $p \ mod \ 4 = 1$ it can be realized as an algebraic number. It however seems that imaginary unit or its p-adic analog should belong to an algebraic extension of p-adic numbers. The group parameters for algebraic extension of p-adic numbers belong to the algebraic extension. If the algebraic extension contains non-trivial roots of unity $U_{m,n} = exp(i2\pi m/n)$, the differences $U_{m,n} - U^*_{m,n}$ are proportional to imaginary unit as real numbers and one can replace imaginary unit in the exponential with $U_{m,n} - U^*_{m,n}$. In real context this means only a rescaling of the Lie algebra generator and Planck constant by a factor $(2sin(2\pi m/n))^{-1}$. A natural imaginary unit is defined in terms of U_{1,p^n}.

4. This construction is expected to generalize to the case of coset spaces and give rise to a coset space G/H identified as the union of discrete coset spaces associated with the elements of the coset G_0/H_0 making sense also in the real context. These are obtained by multiplying the element of $G_/H_0$ by the p-adic factor space $G_{p,n}/H_{p,n}$.

One has two hierarchies corresponding to the hierarchy of discrete subgroups of G_0 requiring each some minimal algebraic extension of p-adic numbers and to the hierarchy of G_p:s defined by the powers of p. These two hierarchies can be assigned to angles (actually phases coming as roots of unity) and p-adic length scales in the space of group parameters.

2. Does the hierarchy of Planck constants emerge p-adically?

The Lie algebra of the rotation group spanned by the generators L_x, L_y, L_z provides a good example of the situation and leads to the question whether the hierarchy of Planck constants [K25] could be understood p-adically.

1. Ordinary commutation relations are $[L_x, L_y] = i\hbar L_z$. For the hierarchy of Lie groups it is convenient to extend the algebra by introducing the generators $L_i^{n)} = p^n L_i$ and one obtains $[L_x^{m)}, L_y^{n)}] = i\hbar L_z^{m+n)}$. This resembles the commutation relations of Kac-Moody algebra structurally. Since p-adic integers one the replacement of $\hbar = p^k \to np^k$, $n \ mod \ p =\neq 0$ produces same Lie-algebra.

2. For the generators of Lie-algebra generated by $L_i^{m)}$ one has $[L_x^{m)}, L_y^{m)}] = ip^m \hbar L_z^{m)}$. One can say that Planck constant is scaled from $\hbar$ to $p^m \hbar$. It is important to realize that $\hbar_{eff} = mp^k \hbar$ for $m \ mod \ p \ \neq 0$ (p-adic unit property) is equivalent with $\hbar_{eff} = p^k \hbar$ in the sense that p-adically the resulting Lie-algebras are same.

3. The earlier proposal assigns the origin of the effective hierarchy of Planck constants $\hbar_{eff} = n\hbar$ to n-furcations of space-time sheets. Recall that n-furcations are assigned with the non-determinism of Kähler action. In n-furcation the solution becomes n-valued meaning the presence of n alternative branches in the usual interpretation. The proposal is that a space-time counterpart of second quantization occurs. Single branch is in the role of single particle state and "classically" the only possible one. "Quantally" also m-branch states, $1 \leq m \leq n$, are allowed. This makes sense in zero energy ontology if the branching occurs either at the space-like ends of the space-time surface inside CD or at light-like wormhole throats. Otherwise one has problem with conservation laws allowing only single branch. The Kähler action for m-branch state would be roughly m times that for single branch states as a sum of the Kähler actions for branches so that one would have $\hbar_{eff} = m\hbar$. This prediction is inconsistent with p-adic Lie-algebra prediction unless $m = p^k$ holds true.

Can these two views about the effective hierarchy of Planck constants be consistent with each other? The connection between p-adic length scale hierarchy and hierarchy of Planck constants has been conjectured already earlier but the recent form of the conjecture is the most quantitive one found hitherto.

1. It a connection exists, it could be due to a relationship between the inherent non-determinism of Kähler action and the generic p-adic non-determinism of differential equations. Skeptic could of course counter-argue that in p-adic context both non-determinisms are present. One can however argue that by the condition that p-adic space-time sheets are maps of real ones and vice versa, these non-determinisms must be equivalent for preferred extremals.

2. Also p-adic non-determinism induces multi-furcations of preferred extremals. These two kinds of multi-furcations should be consistent with each other. Also in p-adic context one can consider "second quantization" allowing simultaneously several branches of multi-furcation. Suppose that the p-adic non-determinism is characterized by integration pseudo-constants (functions with vanishing derivatives), and that the first p^k digits for these functions can be chosen freely. For each integration pseudo-constant involved one would have p^k branches so that for m independent variables there would be p^{mk} branches altogether.

    (a) The argument based on the sum of Kähler actions for n-branch states would suggests $\hbar_{eff} = n\hbar$, $1 \leq n \leq p^{km}$ not consistent with $\hbar_{eff} = p^{mk}\hbar$. Consistency between the two pictures is achieved if all p^{mk} branches are realized simultaneously so that the state is analogous to a full Fermi sphere. This option looks admittedly artificial.

(b) An alternative possibility is following. Suppose that the p-adic Planck constant is $p^r\hbar$,$r \leq km$, and thus equivalent with $kp^r\hbar$ for all $k \ mod \ p \neq 0$, and that the allowed numbers for branches satisfy $n = n_1p^r \leq p^{mk}$, $n_1 \ mod \ p \neq 0$ so that Planck constant in p-adic sense is equivalent with $p^r\hbar$. This would realize a correspondence between the number of branches of multofurcation and the Planck constant associated with p-adic Lie algebras.

3. Note that also n-adic and even $q = m/n$-adic topology is possible with norms given by powers of integer or rational. Number field is however obtained only for primes. This suggests that if also integer - and perhaps even rational valued scales are allowed for causal diamonds, they correspond to effective n-adic or q-adic topologies and that powers of p are favored.

3. Integration as the problem again

The difficult questions concern again integration. The integrals reduce to sums over the discrete subgroup of G multiplied with an integral over the p-adic variant $G_{p,n}$ of the continuous Lie group. The first integral - that is summation - is number theoretically universal. The latter integral is the problematic one.

1. The easy way to solve the problem is to interpret the hierarchy of continuous p-adic Lie groups $G_{p,n}$ as analogs of gauge groups. But if the wave functions are invariant under $G_{p,n}$, what is the situation with respect to $G_{p,m}$ for $m < n$? Infinitesimally one obtains that the commutator algebras $[G_{p,k}, G_{p,l}] \subset G_p, k+l$ must annihilate the functions for $k + l \geq n$. Does also $G_{p,m}$, $m < n$ annihilate the functions for as a direct calculation demonstrates in the real case. If this is the case also p-adically the hierarchy of groups $G_{p,n}$ would have no physical implications. This would be disappointing.

2. One must however be very cautious here. Lie algebra consists of first order differential operators and in p-adic context the functions annihilated by these operators are pseudo-constants. It could be that the wave functions annihilated by $G_{p,n}$ are pseudo-constants depending on finite number of pinary digits only so that one can imagine of defining an integral as a sum. In the recent case the digits would naturally correspond to powers p^m, $m < n$. The presence of these functions could be purely p-adic phenomenon having no real counterpart and emerge when one leaves the intersections of real and p-adic worlds. This would be just the non-determinism of imagination assigned to p-adic physics in TGD inspired theory of consciousness.

Is there any hope that one could define harmonic analysis in $G_{p,n}$ in a number theoretically universal manner? Could one think of identifying discrete subgroups of $G_{p,n}$ allowing also an interpretation as real groups?

1. Exponentiation implies that in matrix representation the elements of $G_{p,n}$ are of form $g = Id + p^n g_1$: here Id represents real unit matrix. For compact groups like $SU(2)$ or CP_2 the group elements in real context are bounded above by unity so that this kind of sub-groups do not exist as real groups. For non-compact groups like $SL(2,C)$ and T^4 this kind of subgroups make sense also in real context.

2. Zero energy ontology suggests that discrete but infinite sub-groups Γ of $SL(2,C)$ satisfying certain additional conditions define hyperbolic spaces as factor spaces H^3/Γ (H^3 is hyperboloid of M^4 light-cone). These spaces have constant sectional curvature and very many 3-manifolds allow a hyperbolic metric with hyperbolic volume defining a topological invariant. The moduli space of CDs contains the groups Γ defining lattices of H^3 replacing it in finite measurement resolution. One could imagine hierarchies of wave functions restricted to these subgroups or H^3 lattices associated with them. These wave functions would have the same form in both real and p-adic context so that number theoretical universality would make sense and one could perhaps define the inner products in terms of "integrals" reducing to sums.

3. The inclusion hierarchy $G_{p,n} \supset G_{p,n+1}$ would in the case of $SL(2,C)$ have interpretation in terms of finite measurement resolution for four-momentum. If $G_{p,n}$ annihilate the physical

states or creates zero norm states, this inclusion hierarchy corresponds to increasing IR cutoff (note that short length scale in p-adic sense corresponds to long scale in real sense!). The hierarchy of groups $G_{p,n}$ makes sense also in the case of translation group T^4 and also now the interpretation in terms of increasing IR cutoff makes sense. This picture would provide a group theoretic realization for with the vision that p-adic length scale hierarchies correspond to hierarchies of length scale measurement resolutions in M^4 degrees of freedom.

What about general coordinate invariance?

In purely algebraic approach one must introduce some preferred coordinate system in which the action of various symmetry transformations is simple: typically induced from linear transformations as in the case of projective spaces. This requires physically preferred coordinate system if one hopes to avoid problems with general coordinate invariance. This approach applies also to more general space-time surfaces. A more general approach would assume general coordinate invariance only modulo finite measurement resolution.

For $H = M^4 \times CP_2$ preferred coordinate systems indeed exist but are determined only apart from the isometries of H. For M^4 the preferred coordinates correspond most naturally to linear Minkowsksi coordinates having simple behavior under isometries. Spherical coordinates are not favored since angles cannot be represented p-adically without infinite-dimensional algebraic extension. For CP_2 complex coordinates in which $U(2) \subset SU(3)$ is represented linearly are preferred. The great virtue of sub-manifold gravity is that preferred space-time coordinates can be chosen as a suitable subset of these coordinates depending on the region of the space-time surface. This reduces the general coordinate transformations to the isometries of the imbedding space but does yet not mean breaking of general coordinate invariance.

Suppose that one accepts the notion of preferred coordinates and assumes that partonic two-surfaces (at least) can be expressed in terms of rational equations (for algebraic extensions rationals are generalized rationals). General coordinate transformations must preserve this state of affairs. GCI must therefore preserve the property of being a ratio of polynomials with rational coefficients. Only those isometries of H are allowed, which respect the algebraic extensions of p-adic numbers used. This means that only a discrete subgroup of isometries can induce general coordinate transformations in p-adic context.

There is however a continuum of choices of preferred coordinates induced by isometries of H so that one obtains a continuum of choices not equivalent under allowed general coordinate transformations. It would seem that general coordinate invariance is broken. The world containing a conscious observer who has chosen coordinate system M_1 differs from the world in which this coordinate system is M_2!

TGD inspired quantum measurement theory leads to this kind of symmetry breaking also in real sector induced by a selection of quantization axis. In TGD framework this choice has a correlate at the level of moduli space of CDs. For instance, the choice of a preferred rest frame forced also by number theoretical vision and construction of preferred extremals would reflect itself in the properties of the interior of the space-time surface even if it need not affect partonic 2-surfaces.

One can argue that it must be possible to realize general coordinate invariance in more general manner than defining physics using preferred coordinates and simple cubic lattice structures for the imbedding space. Maybe also general coordinate invariance should be defined in finite measurement resolution. The lattice structures defining the discretization for imbedding space with non-preferred coordinates would look deformed lattice structures in the preferred coordinates but difference would be vanishing in the pinary resolution used.

8.5 How to define p-adic manifolds?

What p-adic manifolds are? This is the basic question also in TGD. What p-adic CP_2 could mean, and can one speak about p-adic space-time sheets and about solutions of p-adic field equations in p-adic $M^4 \times CP_2$? Does WCW have p-adic counterpart?

The TGD inspired vision about p-adic space-time sheets as correlates for cognition suggests an approach based on the identification of cognitive representations mapping real preferred extremal to its p-adic counterpart and vice versa in finite pinary resolution so that one would map discrete set of rational points to rational points (rational in algebraic extension of p-adic numbers). One would have real chart leafs for p-adic preferred extremals instead of p-adic ones.

8.5.1 Algebraic and topological approaches to the notion of manifold

There are two approaches to the notion of manifold and they correspond to the division of mathematics to algebra and topology: some-one has talked about the devil of algebra and angel of topology. In the case of infinite-D WCW geometry and p-adic manifolds the roles of devil and angle seem to however change.

1. In the algebraic approach manifolds are regarded as purely algebraic objects - algebraic varieties - and thus number theoretically universal: only algebraic equations are allowed. Inequalities are not accepted. This notion of manifold is not so general as the topological notion and symmetries play a crucial role. The homogenous spaces associated with pairs of groups and subgroups for which all points are metrically equivalent is a good example about the power of the algebraic approach made possible by maximal symmetries formulated by Klein as Erlangen program. In the construction of WCW geometry this approach seems to be the only possible one, and gives hopes that infinite-D geometric existence - and thus physics - is unique [K16].

   Standard sphere is this approach defined by condition $x^2 + y^2 + z^2 = R^2$ and makes sense in all number fields for rational values of R. Purely algebraic definition is especially suited for defining sub-varieties. Linear spaces and projective spaces are however definable as manifolds purely algebraically. The natural topology for algebraic varieties is so called Zariski topology [A109] in which closed sets correspond to lower-dimensional sub-varieties. TGD can be seen as sub-manifold gravity in $M^4 \times CP_2$ with space-time surfaces identified as preferred extremals characterized purely algebraically: this strongly favors algebraic approach. Algebraic definition of the imbedding space as a manifold and induction of space-time manifold structure from that for imbedding space is also necessary if one wants to define TGD so that it makes sense in all number fields (p-adic space-time sheets are interpreted as correlates for cognition, "thought bubbles").

   A correspondence between p-adics and reals is however required and this suggests that purely algebraic approach is not enough.

2. Second - extremely general - approach is topological but works as such nicely only in the real context. Manifolds are constructed by gluing together open n-balls. Here the inequality so dangerous in p-adic context enters the game: open ball consists of points with distance smaller than R from center. Real sphere in this approach is obtained by gluing two disks having overlap around equator.

   In p-adic context this approach fails since p-adic balls are either disjoint or nested. In fact, single point is open ball p-adically so that one can decompose a candidate for a p-adic manifold with p-adic coordinate charts to dust. It turns out that the replacement of p-adic norm with canonical identification resolves the problem and one can induce real topology to p-adic context by using canonical identification to define coordinate charts of the p-adic space-time surface as regions of real space-time surface. The essentially new elements are the use of real coordinate charts instead of p-adic ones and the notion of finite measurement resolution characterized by pinary cutoffs.

8.5.2 Could canonical identification allow construction of path connected topologies for p-adic manifolds?

The Berkovich approach [A211, A216] is an attempt to overcome the difficulty caused by the weird properties of p-adic balls by adding some points to p-adic balls so that its topology becomes path connected and the original p-adic ball is dense set in the Berkovich ball. Idea is same as in the completion of rationals to reals: new points make rationals a continuum and one can build calculus. I do not understand how Berkovich disks can be glued to manifolds - presumably the path connected topology implies that they can have overlaps without being identical or nested: the overlaps should be through the added points.

The problem of the Berkovich construction is that from physics point of view it looks rather complex: it is difficult to imagine physical realizations for the auxiliary spaces involved with the construction. Also giving up the p-adic topology seems strange since non-Archimedean topology has - to my opinion - a nice interpretation if one considers it as a correlate for cognition.

The Bruhat-Tits tree working for projective spaces does not seem to require completion. Path connectedness is implied by the tree having in well-defined sense projective space as boundary. Points of the p-adic projective space are represented by projective equivalence classes of lattices: this allows to connect the points of p-adic manifold by edge paths and even the notion of geodesic line can be defined.

In the following TGD inspired topological approach to the construction of p-adic manifolds is discussed. The proposal relies on the notion of canonical identification playing central role in TGD and means that one makes maps about p-adic preferred extremal using - not p-adic but *real coordinate charts* defined using canonical identification obeying the crucial triangle inequality. This approach allows also to make p-adic chart maps about real preferred extremals for some values of p-adic prime. The ultra-metric norms of Berkovich for formal power series are replaced by Archimedean norms defining coordinate functions and their information content is huge as compared to the Berkovich norms. The hierarchy of length scale resolutions gives rise to a hierarchy of canonical identifications in finite pinary resolution and preferred extremal property allows to complete the discrete image set consisting of rational points to a continuous surface. One can say that path-connectedness at the p-adic side is realized by using discretized paths using induced real topology defined by the canonical identification. This gives a resemblance with Bruhat-Tits tree.

Basic facts about canonical identification

In TGD framework one of the basic physical problems has been the connection between p-adic numbers and reals. Algebraic and topological approaches have been competing also here. The notion of canonical identification solves the conflict between algebra (in particular symmetries) and continuity. Canonical identification combined with the identification of common rationals in finite pinary resolution suggests also a natural replacement of p-adic topology with a path connected effective topology defined as real topology induced to p-adic context by canonical identification used to build real chart leafs.

1. In TGD inspired theory of consciousness canonical identification or some of its variants is a good candidate for defining cognitive representations as representations mapping real preferred extremals to p-adic preferred extremals as also for the realization of intentional action as a quantum jump replacing p-adic preferred extremal representing intention with a real preferred extremal representing action. Could these cognitive representations and their inverses actually define real coordinate charts for the p-adic "mind stuff" and vice versa?

2. In its basic form canonical identification I maps p-adic numbers $\sum x_n p^n$ to reals and is defined by the formula $I(x) = \sum x_n p^{-n}$. I is a continuous map from p-adic numbers to reals. Its inverse is also continuous but two-valued for a finite number of pinary digits since the pinary expansion of real number is not unique ($1 = .999999..$ is example of this in 10-adic case). For a real number with a finite number of pinary digits one can always choose the p-adic representative with a finite number of pinary digits.

3. Canonical identification has several variants. Assume that p-adic integers x are represented as expansion of powers of p^k as $x = p^{rk} \sum x_n p^{kn}$ with $x_0 \neq 0$. One can map p-adic rational number $p^{rk} m/n$ with m and n satisfying the analog of $x_0 \neq 0$ regarded as a p-adic number to a real number using $I^Q_{k,l}$: $I^Q_{k,l}(p^{rk} m/n) \equiv p^{-rk} I_{k,l}(m)/I_{k,l}(n)$.

   In this case canonical identification respects rationality but is ill-defined for p-adic irrationals. This is not a catastrophe if one has finite measurement resolution meaning that only rationals for which $m < p^l, n < p^l$ are mapped to the reals (real rationals actually). One can say that $I^Q_{k,l}$ identifies p-adic and real numbers along common rationals for p-adic numbers with a pinary cutoff defined by k and maps them to rationals for pinary cutoff defined by l. Discrete subset of rational points on p-adic side is mapped to a discrete subset of rational points on real side by this hybrid of canonical identification and identification along common rationals (see fig. `http://www.tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this http://www.tgdtheory.fi/appfigures/book.jpg, which is also). This form of canonical identification is the one needed in TGD framework.

4. Canonical identification does not commute with rational symmetries unless one uses the map $I^Q_{k,l}(p^{rk}m/n) = p^{-rk}I_{k,l}(m)/I_{k,l}(n)$ and also now only in finite resolution defined by k. For the large p-adic primes associated with elementary particles this is not a practical problem (electron corresponds to $M_{127} = 2^{127}-1$). The generalization to algebraic extensions makes also sense. Canonical identification breaks general coordinate invariance unless one uses group theoretically preferred coordinates for M^4 and CP_2 and subset of these for the space-time region considered.

The resolution of the conflict between symmetries and continuity

Consider now the resolution of the conflict between algebra and topology in more detail.

1. Algebraic approach suggests the identification of reals and various p-adic numbers along common rationals defined by $I^Q_{\infty,\infty}$ but this correspondence is completely dis-continuous. Therefore one must introduce a finite pinary cutoff p^k so that one maps only integers smaller than p^k to themselves. Since $I^Q_{k,l}$ does not make sense for p-adic irrationals, one must introduce also second pinary cutoff p^l and use $I^Q_{k,l}$ so that only a finite subset of rational points is mapped to their real counterparts.

2. Topological approach relies on canonical identification and its variants mapping p-adic numbers to reals in a continuous manner. $I_{k,\infty}$ applied to p-adics expressed as $x = p^k u$, $u = \sum x_n p^n$, where u has unit norm, defines such a correspondence. This correspondence does not however commute with the basic symmetries as correspondence along common rationals would do for subgroups of the symmetries represented in terms of rational matrices. Canonical identification fails also to commute with the field equations and the real image fails to be differentiable.

   Finite pinary cutoff ($I^Q_{k,\infty} \rightarrow I^Q_{k,l}$) saves the situation. Below the lower pinary cutoff p^k the pseudo-constants of p-adic differential equations would naturally relate to the identification of p-adics and reals along common rationals (plus common algebraics in the case of algebraic extensions).

The notion of finite measurement resolution allows therefore to find a compromise between the symmetries and continuity (that is, algebra and topology). $I^Q_{k,l}$ maps rationals to themselves only up to k pinary digits and the remaining points up to l digits are mapped to rationals but not to themselves. Canonical identification thus maps only a skeleton of manifold formed by discrete point set from real to p-adic context and the preferred extremals on both sides would contain this skeleton. There are many manners to select this rational skeleton, which can also define a decomposition of the real manifold to simplices or more general objects allowing to define homology theory in real context and to induce it to p-adic context so that real homology would be inherited to p-adic context.

Definition of p-adic manifold in terms of canonical identification with pinary cutoff

What is remarkable is that canonical identification can be seen as a continuous generalization of the p-adic norm defined as $N_p(x) \equiv I_{k,l}(x)$ having the highly desired Archimedean property. $I_{k,l}$ is the most natural variant of canonical identification for defining the chart maps from regions p-adic manifold to regions of corresponding real mani-fold (in particular, p-adic preferred extremals to their real counterparts).

1. As already mentioned, one must restrict the p-adic points mapped to real rationals since $I^Q_{k,l}(x)$ is not well-defined for p-adic irrationals having non-unique expression as ratios of p-adic integers. For the restriction to finite rationals the chart image on the real side would consist of rational points. The cutoff means that these rationals are not dense in the set of reals. Preferred extremal property could however allow to identify the chart leaf as a piece of preferred extremal containing the rational points in the measurement resolution used. This

would realize the dream of mapping p-adic p-adic preferred extremals to real ones playing a key role in number theoretical universality. When one cannot use preferred extremal property some other constraint would restrict the number of different chart leafs.

2. Canonical identification for the various coordinates defines a chart map mapping regions of p-adic manifold to R_+^n. That each coordinate is mapped to a norm $N_p(x)$ means that the real coordinates are always non-negative. If real spaces R_+^n would provide only chart maps, it is not necessary to require approximate commutativity with symmetries. Also Berkovich considers norms but for a space of formal power series assigned with the p-adic disk: in this case however the norms have extremely low information content.

3. $I_{k,l}^Q$ indeed defines the analog of Archimedean norm in the sense that one has $N_p^{k,l}(x+y) \leq N_p^{k,l}(x) + N_p^{k,l}(y)$. This follows immediately from the fact that the sum of pinary digits can vanish modulo p. The triangle inequality holds true also for the rational variant of I. $N_p^{k,l}(x)$ is however not multiplicative: only a milder condition $N_p^{k,l}(p^{nk}x) = N_p^{k,l}(p^{nk})N_p^{k,l}((x) = p^{-nk}N_p^{k,l}(x)$ holds true.

4. Archimedean property gives excellent hopes that p-adic space provided with chart maps for the coordinates defined by canonical identification inherits within pinary resolutions real topology and its path connectedness as a discretized version. In purely topological approach forgetting algebra and symmetries, a hierarchy of induced real topologies would be obtained as induced real topologies and characterized by various norms defined by $I_{k,\infty}$. When symmetries and algebra are brought in, $I_{k,l}^Q$ gives a correspondence discretizing the connecting paths. This would give a very close connection with physics.

5. The mapping of p-adic manifolds to real manifolds would make the construction of p-adic manifolds very concrete. For instance, one can map real preferred subset of rational points of a real preferred extremal to a p-adic one by the inverse of canonical identification by mapping the real points with finite number of pinary digits to p-adic points with a finite number of pinary digits. This does not of course guarantee that the p-adic preferred extremal is unique. One could however say that p-adic preferred extremals possesses the topological invariants of corresponding real preferred extremal.

6. The maps between different real charts would be induced by the p-adically analytic maps between the inverse images of these charts. At the real side the maps would be consistent with the p-adic maps only in the discretization below pinary cutoff and could be also smooth.

7. An objection against this approach is the loss of general coordinate invariance. One can however argue that one can require this only within the limits of finite measurement resolution. In TGD framework the symmetries of imbedding space provide a very narrow set of preferred coordinates.

The idea that the discretized version of preferred extremal could lead to preferred extremal by adding new points in iterative manner is not new. I have proposed assuming that preferred extremals can be also regarded as quaternionic surfaces (tangent spaces are in well-defined sense hyper-quaterionic sub-space of complexified octonionic space containing hyper-complex octonions as a preferred sub-space) [K89].

What about p-adic coordinate charts for a real preferred extremal and for p-adic extremal in different p-adic number field?

What is remarkable that one can also build p-adic coordinate charts about real preferred extremal using the inverse of the canonical identification assuming that finite rationals are mapped to finite rationals. There are actually good reasons to expect that coordinate charts make sense in both directions.

Furthermore, if real preferred extremal can be mapped to to p-adic extremals corresponding to two different primes p_1 and p_2, then p_1-adic preferred extremals serves as a chart for p_2-adic preferred extremal and vice versa (one can compose canonical identifications and their inverses to construct the chart maps).

Clearly, real and p-adic extremals define in this manner a category. Preferred extremals are the objects. The arrows are the composites of canonical identification and its inverses mapping to each other preferred extremals belonging to different number fields. This category would be very natural and have profound physical meaning: usually the notion of category tends to be quite too general for the needs of physicist. Category theoretical thinking suggests that full picture of physics is obtained only through this category: this is certainly the case if physics is extended to include physical correlates of cognition and intentionality.

Algebraic continuation from real to p-adic context is one good reason for p-adic chart maps. At the real side one can calculate the values of various integrals like Kähler action. This would favor p-adic regions as map leafs. One can require that Kähler action for Minkowskian and Euclidian regions (or their appropriate exponents) make sense p-adically and define the values of these functions for the p-adic preferred extremals by algebraic continuation. This could be very powerful criterion allowing to assign only very few p-adic primes to a given real space-time surface. This would also allow to define p-adic boundaries as images of real boundaries in finite measurement resolution. p-Adic path connectedness would be induced from real path-connectedness.

In the intersection of real and p-adic worlds the correspondence is certainly unique and means that one interprets the equations defining the p-adic space-time surface as real equations. The number of rational points (with cutoff) for the p-adic preferred extremal becomes a measure for how unique the chart map in the general case can be. For instance, for 2-D surfaces the surfaces $x^n + y^n = z^n$ allow no nontrivial rational solutions for $n > 2$ for finite real integers. This criterion does not distinguish between different p-adic primes and algebraic continuation is needed to make this distinction. The basic condition selecting preferred p-adic primes is that the value of real Kähler/Morse function or its real/imaginary exponent (or both) makes sense also p-adically in some finite-dimensional extension of p-adic numbers.

Some examples about chart maps of p-adic manifolds

The real map leafs must be mutually consistent so that there must be maps relating coordinates used in the overlapping regions of coordinate charts on both real and p-adic side. On p-adic side chart maps between real map leafs are naturally induced by identifying the canonical image points of identified p-adic points on the real side. For discrete chart maps $I_{k,l}^Q$ with finite pinary cutoffs one one must complete the real chart map to - say diffeomorphism. That this completion is not unique reflects the finite measurement resolution.

In TGD framework the situation is dramatically simpler. For sub-manifolds the manifold structure is induced from that of imbedding space and it is enough to construct the manifold structure $M^4 \times CP_2$ in a given measurement resolution (k, l). Due to the isometries of the factors of the imbedding space, the chart maps in both real and p-adic case are known in preferred imbedding space coordinates. As already discussed, this allows to achieve an almost complete general coordinate invariance by using subset of imbedding space coordinates for the space-time surface. The breaking of GCI has interpretation in terms of presence of cognition and selection of quantization axes.

For instance, in the case of Riemann sphere S^2 the holomorphism relating the complex coordinates in which rotations act as Möbius transformations and rotations around preferred axis act as phase multiplications - the coordinates z and w at Northern and Southern hemispheres are identified as $w = 1/z$ restricted to rational points at both side. For CP_2 one has three poles instead of two but the situation is otherwise essentially the same.

8.5.3 Could canonical identification make possible definition of integrals in p-adic context?

The notion of p-adic manifold using using real chart maps instead of p-adic ones allows an attractive approach also to p-adic integration and to the problem of defining p-adic version of differential forms and their integrals.

1. If one accepts the simplest form of canonical identification $I(x) : \sum_n x_n p^n \to \sum x_n p^{-n}$, the image of the p-adic surface is continuous but not differentiable and only integers $n < p$ are mapped to themselves. One can define integrals of real functions along images of the

p-adically analytic curves and define the values of their p-adic counterparts as their algebraic continuation when it exists.

In TGD framework this does not however work. If one wants to define induced quantities - such as metric and Kähler form - on the real side one encounters a problem since the image surface is not smooth and the presence of edges implies that these quantities containing derivatives of imbedding space coordinates possess delta function singularities. These singularities could be even dense in the integration region so that one would have no-where differentiable continuous functions and the real integrals would reduce to a sum which do not make sense.

2. In TGD framework finite measurement resolution realized in terms of pinary cutoffs saves the situation. $I_{k,l}^Q$ is a compromise between the direct identification along common rationals favored by algebra and symmetries but being totally discontinuous without the cutoff l. This cutoff breaks symmetries slightly but guarantees continuity in finite measurement resolution defined by the pinary cutoff l. Symmetry breaking can be made arbitrarily small and has interpretation in terms of finite measurement resolution. Due to the pinary cutoff the chart map applied to various p-adic coordinates takes discrete set of rationals to discrete set of rationals and preferred extremal property can be used to make a completion to a real space-time surface. Uniqueness is achieved only in finite measurement resolution and is indeed just what is needed. Also general coordinate invariance is broken in finite measurement resolution. In TGD framework it is however possible to find preferred coordinates in order to minimize this symmetry breaking.

3. The completion of the discrete image of p-adic preferred extremal under $I_{k,l}^Q$ to a real preferred extremal is very natural. This preferred extremal can be said to be unique apart from a finite measurement resolution represented by the pinary cutoffs k and l. All induced quantities are well defined on both sides.

   p-Adic integrals can be defined as pullbacks of real integrals by algebraic continuation when this is possible. The inverse image of the real integration region in canonical identification defines the p-adic integration region.

4. The integrals of p-adic differential forms can be defined as pullbacks of the real integrals. The integrals of closed forms, which are typically integers, would be the same integers but interpreted as p-adic integers.

It is interesting to study the algebraic continuation of Kähler action from real sector to p-adic sectors.

1. Kähler action for both Euclidian and Minkowskian regions reduces to the algebraic continuation of the integral of Chern-Simons-Kähler form over preferred 3-surfaces. The contributions from Euclidian and Minkowskian regions reduce to integrals of Chern-Simons form over 3-surfaces.

   The contribution from Euclidian regions defines Kähler function of WCW and the contribution from Minkowskian regions giving imaginary exponential of Kähler action has interpretation as Morse function, whose stationary points are expected to select special preferred extremals. One would expect that both functions have a continuous spectrum of values. In the case of Kähler function this is necessary since Kähler function defines the Kähler metric of WCW via its second derivatives in complex coordinates by the well-known formula.

2. The algebraic continuation of the exponent of Kähler function for a given p-adic prime is expected to require the proportionality to p^n so that not all preferred extremals are expected to allow a continuation to a given p-adic number field. This kind of assumption has been indeed made in the case of deformations of CP_2 type extremals in order to derive formula for the gravitational constant in terms of basic parameters of TGD but without real justification [K51].

3. The condition that the action exponential in the Minkowskian regions is a genuine phase factor implies that it reduces to a root of unity (one must have an algebraic extension of

p-adic numbers). Therefore the contribution to the imaginary exponent Kähler action from these regions for the p-adicizable preferred extremals should be of form $2\pi(k + m/n)$.

If all preferred real extremals allow p-adic counterpart, the value spectrum of the Morse function on the real side is discrete and could be forced by the preferred extremal property. If this were the case the stationary phase approximation around extrema of Kähler function on the real side would be replaced by sum with varying phase factors weighted by Kähler function.

An alternative conclusion is that the algebraic continuation of Kähler action to any p-adic number field is possible only for a subset of preferred extremals with a quantized spectrum of Morse function. One the real side stationary phase approximation would make sense. It however seems that the stationary phases must obey the above discussed quantization rule.

Also holomorphic forms allow algebraic continuation and one can require that also their integrals over cycles do so. An important example is provided by the holomorphic one-forms integrals over cycles of partonic 2-surface defining the Teichmueller parameters characterizing the conformal equivalence class of the partonic 2-surfaces as Riemann surface. The p-adic variants of these parameters exist if they allow an algebraic continuation to a p-adic number. The algebraic continuation from the real side to the p-adic side would be possible on for certain p-adic primes p if any: this would allow to assign p-adic prime or primes to a given real preferred extremal. This justifies the assumptions of p-adic mass calculations concerning the contribution of conformal modular degrees of freedom to mass squared [K17].

8.5.4 Canonical identification and the definition of p-adic counterparts of Lie groups

For Lie groups for which matrix elements satisfy algebraic equations, algebraic subgroups with rational matrix elements could be regarded as belonging to the intersection of real and p-adic worlds, and algebraic continuation by replacing rationals by reals or p-adics defines the real and p-adic counterparts of these algebraic groups. The challenge is to construct the canonical identification map between these groups: this map would identify the common rationals and possible common algebraic points on both sides and could be seen also a projection induced by finite measurement resolution.

A proposal for a construction of the p-adic variants of Lie groups was discussed in previous section. It was found that the p-adic variant of Lie group decomposes to a union of disjoint sets defined by a discrete subgroup G_0 multiplied by the p-adic counterpart $G_{p,n}$ of the continuous Lie group G. The representability of the discrete group requires an algebraic extension of p-adic numbers. The disturbing feature of the construction is that the p-adic cosets are disjoint. Canonical identification $I_{k,l}$ suggests a natural solution to the problem. The following is a rough sketch leaving a lot of details open.

1. Discrete p-adic subgroup G_0 corresponds as such to its real counterpart represented by matrices in algebraic extension of rationals. $G_{p,n}$ can be coordinatized separately by Lie algebra parameters for each element of G_0 and canonical identification maps each $G_{p,n}$ to a subset of real G. These subsets intersect and the chart-to-chart identification maps between Lie algebra coordinates associated with different elements of G_0 are defined by these intersections. This correspondence induces the correspondence in p-adic context by the inverse of canonical identification.

2. One should map the p-adic exponentials of Lie-group elements of $G_{p,n}$ to their real counterparts by some form of canonical identification.

    (a) Consider first the basic form $I = I_{1,\infty}$ of canonical identification mapping all p-adics to their real counterparts and maps only the p-adic integers $0 \leq k < p$ to themselves.

    The gluing maps between groups $G_{p,n}$ associated with elements g_m and g_n of G_0 would be defined by the condition $g_m I(exp(it_a T^a) = g_n I(exp(iv_a T^a)$. Here t_a and v_a are Lie-algebra coordinates for the groups at g_m and g_n. The delicacies related to the identification of p-adic analog of imaginary unit have been discussed in the previous section. It is important that Lie-algebra coordinates belong to the algebraic extension of p-adic numbers containing also the roots of unity needed to represent g_n. This condition allows to solve v_a in terms of t_a and $v_a = v_a(t_b)$ defines the chart map relating the two coordinate patches on the real side. The inverse of the canonical identification in turn defines the p-adic variant of the chart map in p-adic context. For I this map is not p-adically analytic as one might have guessed.

(b) The use of $I^Q_{k,l}$ instead of $I = I_{1,\infty}$ gives hopes about analytic chart-to chart maps on both sides. One must however restrict $I^Q_{k,l}$ to a subset of rational points (or generalized points in algebraic extension with generalized rational defined as ratio of generalized integers in the extension). Canonical identification respects group multiplication only if the integers defining the rationals m/n appearing in the matrix elements of group representation are below the cutoff p^k. The points satisfying this condition do not in general form a rational subgroup. The real images of rational points however generate a rational sub-group of the full Lie-group having a manifold completion to the real Lie-group.

One can define the real chart-to chart maps between the real images of $G_{p,k}$ at different points of G_0 using $I^Q_{k,l}(exp(iv_aT^a) = g_n^{-1}g_m \times I^Q_{k,l}(exp(it_aT^a)$. When real charts intersect, this correspondence should allow solutions v_a, t_b belonging to the algebraic extension and satisfying the cutoff condition. If the rational point at the other side does not correspond to a rational point it might be possible to perform pinary cutoff at the other side.

Real chart-to-chart maps induce via common rational points discrete p-adic chart-to-chart maps between $G_{p,k}$. This discrete correspondence should allow extension to a unique chart-to-chart map the p-adic side. The idea about algebraic continuation suggests that an analytic form for real chart-to-chart maps using rational functions makes sense also in the p-adic context.

3. p-Adic Lie-groups $G_{p,k}$ for an inclusion hierarchy with size characterized by p^{-k}. For large values of k the canonical image of $G_{p,k}$ for given point of G_0 can therefore intersect its copies only for a small number of neighboring points in G_0, whose size correlates with the size of the algebraic extension. If the algebraic extension has small dimension or if k becomes large for a given algebraic extension, the number of intersection points can vanish. Therefore it seems that in the situations, where chart-to-chart maps are possible, the power p^k and the dimension of algebraic extension must correlate. Very roughly, the order of magnitude for the minimum distance between elements of G_0 cannot be larger than p^{-k+1}. The interesting outcome is that the dimension of algebraic extension would correlate with the pinary cutoff analogous to the IR cutoff defining measurement resolution for four-momenta.

8.5.5 Cut and project construction of quasicrystals from TGD point of view

Cut and project [?] method is used to construct quasicrystals (QCs) in sub-spaces of a higher-dimensional linear space containing an ordinary space filling lattice, say cubic lattice. For instance, 2-D Penrose tiling is obtained as a projection of part of 5-D cubic lattice - known as Voronyi cell - around 2-D sub-space imbedded in five-dimensional space. The orientation of the 2-D sub-space must be chosen properly to get Penrose tiling. The nice feature of the construction is that it gives the entire 2-D QC. Using local matching rules the construction typically stops.

Sub-manifold gravity and generalization of cut and project method

The representation of space-time surfaces as sub-manifolds of 8-D $H = M^4 \times CP_2$ can be seen as a generalization of cut and project method.

1. The space-time surface is not anymore a linear 4-D sub-space as it would be in cut and project method but becomes curved and can have arbitrary topology. The imbedding space ceases to be linear $M^8 = M^4 \times E^4$ since E^4 is compactified to CP_2. Space-time surface is not a lattice but continuum.

2. The induction procedure geometrizing metric and gauge fields is nothing but projection for H metric and spinor connection at the continuum limit. Killing vectors for CP_2 isometries can be identified as classical gluon fields. The projections of the gamma matrices of H define induced gamma matrices at space-time surface. The spinors of H contain additional components allowing interpretation in terms of electroweak spin and hyper-charge.

Finite measurement resolution and construction of p-adic counterparts of preferred extremals forces "cut and project" via discretization

In finite measurement resolution realized as discretization by finite pinary cutoff one can expect to obtain the analog of cut and project since 8-D imbedding space is replaced with a lattice structure.

1. The p-adic/real manifold structure for space-time is induced from that for H so that the construction of p-adic manifold reduces to that for H.

2. The definition of the manifold structure for H in number theoretically universal manner requires for H discretization in terms of rational points in some finite region of M^4. Pinary cutoffs- two of them - imply that the manifold structures are parametrized by these cutoffs charactering measurement resolution. Second cutoff means that the lattice structure is piece of an infinite lattice. First cutoff means that only part of this piece is a direct imagine of real/p-adic lattice on p-adic/real side obtained by identifying common rationals (now integers) of real and p-adic number fields. The mapping of this kind lattice from real/p-adic side to p-adic/real side defines the discrete coordinate chart and the completion of this discrete structure to a preferred extremal gives a smooth space-time surface also in p-adic side if it is known on real side (and vice versa).

3. Cubic lattice structures with integer points are of course the simplest ones for the purposes of discretization and the most natural choice for M^4. For CP_2 the lattice is completely analogous to the finite lattices at sphere defined by orbits of discrete subgroups of rotation group and the analogs of Platonic solids emerge. Probably some mathematician has listed the Platonic solids in CP_2.

4. The important point is that this lattice like structure is defined at the level of the 8-D imbedding space rather than in space-time and the lattice structure at space-time level contains those points of the 8-D lattice like structure, which belong to the space-time surface. Finite measurement resolution suggests that all points of lattice, whose distance from space-time surface is below the measurement resolution for distance are projected to the space-time surface. Since space-time surface is curved, the lattice like structure at space-time level obtained by projection is more general than QC.

The lattice like structure results as a manifestation of finite measurement resolution both at real and p-adic sides and can be formally interpreted in terms of a generalization of cut and project but for a curved space-time surface rather than 4-D linear space, and for H rather than 8-D Minkowski space. It is of course far from clear whether one can obtain anything looking like say 3-D or 4-D version of Penrose tiling.

1. The size scale of CP_2 is so small (10^4 Planck lengths) that space-time surfaces with 4-D M^4 projection look like M^4 in an excellent first approximation and using M^4 coordinates the projected lattice looks like cubic lattice in M^4 except that the distances between points are not quite the M^4 distances but scaled by an amount determined by the difference between induced metric and M^4 metric. The effect is however very small if one believes on the general relativistic intuition.

   In TGD framework one however can have so called warped imbeddings of M^4 for which the component of the induced metric in some direction is scaled but curvature tensor and thus gravitational field vanishes. In time direction this scaling would imply anomalous time dilation in absence of gravitational fields. This would however cause only a the compression or expansion of M^4 lattice in some direction.

2. For Euclidian regions of space-time surface having interpretation as lines of generalized Feynman diagrams M^4 projection is 3-dimensional and at elementary particle level the scale associated with M^4 degrees of freedom is roughly the same as CP_2 scale. If CP_2 coordinates are used (very natural) one obtains deformation of a finite lattice-like structure in CP_2 analogous to a deformation of Platonic solid regarded as point set at sphere. Whether this lattice like structure could be seen as a subset of infinite lattice is not clear.

3. One can consider also string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ with 2-D M^4 projection and their deformations. In this case the projection of M^4 lattice to X^2 - having subset of two M^4 coordinates as coordinates - can differ considerably from a regular lattice since X^2 can be locally tilted with respect to M^4 lattice. This cannot however give rise to Penrose tiling requiring 5-D flat imbedding space. This argument applies also to 2-D string world sheets carrying spinor modes. In the idealized situation that string world sheet is plane in M^4 one might obtain an analog of Penrose tiling but with 4-D imbedding space.

The above quasi lattice like structures (QLs) are defined by a gravitational deformation of the cubic lattice of M^4. Is there any hope about the 4-D QLs in M^4 so that gravitation would give rise to the analogs of phason waves deforming them? Could cut and project method be generalized to give QL in M^4 as projection of 8-D cubic lattice in M^8?

$M^8 - H$ duality

Before considering an explicit proposal I try to describe what I call M^8-H duality ($H = M^4 \times CP_2$).

1. What I have christened $M^8 - H$ duality is a conjecture stating that TGD can be equivalently defined in M^8 or $M^4 \times CP_2$. This is the number theoretic counterpart of spontaneous compactification of string models but has nothing to do with dynamics: only two equivalent representations of dynamics would be in question.

2. Space-time surfaces (preferred extremals) in M^8 are postulated to be quaternionic sub-manifolds of M^8 possessing a fixed $M^2 \subset M^4 \subset M^8$ as sub-space of tangent space. "Quaternionic" means that the tangent space of M^4 is quaternionic and thus associative. Associativity conditions would thus determine classical dynamics. More generally, these subspaces $M^2 \subset M^8$ can form integrable distribution and they define tangent spaces of a 2-D sub-manifold of M^4. If this duality really holds true, space-time surfaces would define a lattice like structure projected from a cubic M^8 lattice. This of course does not guarantee anything: M^8-H duality itself suggests that these lattice like structures differ from regular M^4 crystals only by small gravitational effects.

3. The crucial point is that quaternionic sub-spaces are parametrized by CP_2. Quaternionic 4-surfaces of $M^8 = M^4 \times CP_2$ containing the fixed $M^2 \subset M^8$ can be mapped to those of $M^4 \times CP_2$ by defining M^4 coordinates as projections to preferred $M^4 \subset M^8$ and CP_2 coordinates as those specifying the tangent space of 4-surface at given point.

4. A second crucial point is that the preferred subspace $M^4 \subset M^8$ can be chosen in very many manners. This imbedding is a complete analog of the imbedding of lower-D subspace to higher-D one in cut and project method. M^4 can be identified as any 4-D subspace imbedded in M^4 and the group $SO(1,7)$ of 8-D Lorentz transformations defines different imbeddings of M^4 to M^8. The moduli space of different imbeddings of M^4 is the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and has dimension $D = 28 - 6 - 6 = 16$.

   When one fixes two coordinate axes as the real and one imaginary direction (physical interpretation is as an identification of rest system and spin quantization axes), one obtains $SO(1,7)/SO(2) \times SO(4)$ with higher dimension $D = 28 - 1 - 6 = 21$. When one requires also quaternionic structure one obtains the space $SO(1,7)/SU(1) \times SU(2)$ with dimension $D = 28 - 4 = 24$. Amusingly, this happens to be the number of physical degrees of freedom in bosonic string model.

How to obtain quasilattices and quasi-crystals in M^4?

Can one obtain quasi-lattice like structures (QLs) at space-time level in this framework? Consider first the space-time QLs possibly associated with the standard cubic lattice L^4_{st} of M^4 resulting as projections of the cubic lattice structure L^8_{st} of M^8.

1. Suppose that one fixes a cubic crystal lattice in M^8, call it L^8_{st}. Standard M^4 cubic lattice L^4_{st} is obtained as a projection to some M^4 sub-space of M^8 by simply putting 4 Euclidian coordinates for lattice points o constant. These sub-spaces are analogous to 2-D coordinate planes of E^3 in fixed Cartesian coordinates. There are $7!/3!4! = 35$ choices of this kind.

   One can consider also E_8 lattice is an interesting identification for the lattice of M^8 since E_8 is self-dual and defines the root lattice of the exceptional group E_8. E_8 is union of Z^8 and $(Z + 1/2)^8$ with the condition that the sum of all coordinates is an even integer. Therefore all lattice coordinates are either integers or half-integers. E_8 is a sub-lattice of 8-D cubic lattice with 8 generating vectors $e_i/2$, with e_i unit vector. Integral octonions are obtained from E_8 by scaling with factor 2. For this option one can imbed L^4_{st} as a sub-lattice to Z^8 or $(Z + 1/2)^8$.

2. Although $SO(1,3)$ leaves the imbedded 4-plane M^4 invariant, it transforms the 4-D crystal lattice non-trivially so that all 4-D Lorentz transforms are obtained and define different discretizations of M^4. These are however cubic lattices in the Lorentz transformed M^4 coordinates so that this brings nothing new. The QLs at space-time surface should be obtained as gravitational deformations of cubic lattice in M^4.

3. L^4_{st} indeed defines 4-D lattice at space-time surface apart from small gravitational effects in Minkowskian space-time regions. Elementary particles are identified in TGD a Euclidian space-time regions - deformed CP_2 type vacuum extremals. Also black-hole interiors are replaced with Euclidian regions: black-hole is like a line of a generalized Feynman diagram, elementary particle in some sense in the size scale of the black-hole. More generally, all physical objects, even in everyday scales, could possess a space-time sheet with Euclidian metric signature characterizing their size (AdS^5/CFT correspondence could inspire this idea). At these Euclidian space-time sheets gravitational fields are strong since even the signature of the induced metric is changed at their light-like boundary. Could it be that in this kind of situation lattice like structures, even QCs, could be formed purely gravitationally? Probably not: an interpretation as lattice vibrations for these deformations would be more natural.

It seems that QLs are needed *already at the level of* M^4. $M^8 - H$ duality indeed provides a natural manner to obtain them.

1. The point is that the projections of L^8_{str} to sub-spaces M^4 defined as the $SO(1,7)$ Lorentz transforms of L^4_{st} define generalized QLs parametrized by 16-D moduli space $SO(1,7)/SO(1,3)\times SO(4)$. These QLs include also QCs. Presumably QC is a QL possessing a non-trivial point group just like Penrose tiling has the isometry group of dodecagon as point group and 3-D analog of Penrose tiling has the isometries of icosahedron as point group.

   This would allow to conclude that the discretization at the level of M^8 required by the definition of p-adic variants of preferred extremals as cognitive representations of their real counterparts would make possible 4-D QCs. M^8 formulation of TGD would explain naturally the QL lattices as discretizations forced by finite measurement resolution and cognitive resolution.

   A strong number theoretical constraint on these discretizations come from the condition that the 4-D lattice like structure corresponds to an algebraic extension of rationals. Even more, if this algebraic extension is 8-D (perhaps un-necessarily strong condition), there are extremely strong constraints on the 22-parameters of the imbedding. Note that in p-adic context the algebraic extension dictates the maximal isometry group identified as subgroup of $SO(1,7)$ assignable to the imbedding as the discussion of p-adic icosahedron demonstrates.

2. What about the physical interpretation of these QLs/QCs? As such QLs define only natural discretizations rather than physical lattices. It is of course quite possible to have also physical

QLs/QCs such that the points - rather time like edge paths - of the discretization contain real particles. What about a "particle" localized to a point of 4-D lattice? In positive energy ontology there is no obvious answer to the question. In zero energy ontology the lattice point could correspond to a small causal diamond containing a zero energy state. In QFT context one would speak of quantum fluctuation. In p-adic context it would correspond to "though bubble" lasting for a finite time.

3. It is also possible to identify physical particles as edge paths of the 4-D QC, and one can consider time= constant snapshots as candidates for 3-D QCs. It is quite conceivable that the non-trivial point group of QCs favors them as physical QLs.

Expanding hyperbolic tesselations and quasi-tesselations obtained by imbedding $H^3 \subset M^4$ to $H^7 \subset M^8$

M^8-$M^4 \times CP_2$ duality and the discretization required by the notion of p-adic manifold relates in an interesting manner to expanding hyperbolic tesselations and quasi tesselations in $H^7 \subset M^8$, and possible expanding quasi-tesselations in obtained by imbedding $H^3 \subset M^4$ to $H^7 \subset M^8$

1. Euclidian lattices E_8, E_7, E_6

I have already considered E_8 lattice in M^8. The background space has however Minkowskian rather than Euclidian metric natural for the carrier space of the E_8 lattice. If one assigns some discrete subgroup of isometries to it, it is naturally subgroup of $SO(8)$ rather than $SO(1,7)$. Both these groups have $SO(7)$ as a subgroup meaning that preferred time direction is chosen as that associated with the real unit and considers a lattice formed from imaginary octonions.

E_8 lattice scaled up by a factor 2 to integer lattice allows octonionic integer multiplication besides sums of points so that the automorphism group of octonions: discrete subgroups of $G_2 \subset SO(7)$ would be the natural candidates for point groups crystals or lattice like structures.

If one assumes also fixed spatial direction identified as a preferred imaginary unit, G_2 reduces to $SU(3) \subset SO(6) = SU(4)$ identifiable physically as color group in TGD framework. From this one ends up with the idea about $M^8 - M^4 \times CP_2$ duality. Different imbeddings of $M^4 \subset M^8$ are quaternionic sub-spaces containing fixed M^2 are labelled by points of CP_2.

All this suggests that E_7 lattice in time=constant section of even E^6 lattice is a more natural object lattice to consider. Kind of symmetry breaking scenario $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow G_2 \rightarrow SU(3)$ is suggestive. This Euclidian lattice would be completely analogous to a slicing of 4-D space-time by 3-D lattices labelled by the value of time coordinate and is of course just what physical considerations suggest.

2. Hyperbolic tesselations

Besides crystals defined by a cubic lattice or associated with E_6 or E_7, one obtains an infinite number of hyperbolic tesselations in the case of M^8. These are much more natural in Minkowskian signature and could be also cosmologically very interesting. Quite generally, one can say that hyperbolic space is ideal for space-filling packings defined by hyperbolic manifolds H^n/Γ: they are completely analogous to space-filling packings of E^3 defined by discrete subgroups of translation group producing packings of E^3 by rhombohedra. One only replaces discrete translations with discrete Lorentz transformations. This is what makes these highly interesting from the point of view of quantum gravity.

1. In M^{n+1} one has tesselations of n-dimensional hyperboloid H^n defined by $t^2 - x1^2 - ... - x_n^2 = a^2 > 0$, where a defines Lorentz invariant which for $n = 4$ has interpretation as cosmic time in TGD framework. Any discrete subgroup Γ of the Lorentz group $SO(1,n)$ of M^{n+1} with suitable additional conditions (finite number of generators at least) allows a tesselation of H^n by basic unit H^n/Γ. These tesselations come as 1-parameter families labelled by the cosmic time parameter a. These 3-D tesselations participate cosmic expansion. Of course, also ordinary crystals are crystals only in spatial directions. One can of course discretize the values of a or some function of a in integer multiples of basic unit and assign to each copy of H^n/Γ a "center point" to obtain discretization of M^{n+1} needed for p-adicization.

2. For $n = 3$ one has M^4 and H^3, and this is very relevant in TGD cosmology. The parameter a defines a Lorentz invariant cosmic time for the imbeddings of Robertson-Walker cosmologies

to $M^4 \times CP_2$. The tesselations realized as physical lattices would have natural interpretation as expanding 3-D lattice like structures in cosmic scales. What is new is that discrete translations are replaced by discrete Lorentz boosts, which correspond to discrete velocities and observationally to discrete red shifts for distant objecst. Interestingly, it has been found that red shift is quantized along straight lines [E2]: "God's fingers" is the term used. I proposed for roughly two decades ago an explanation based on closed orbits of photons around cosmic strings [K20]. but explanation in terms of tesselations would also give rise to periodicity. A fascinating possibility is that these tesselation have defined macroscopically quantum coherent structures during the very early cosmology the the size scale of H^3/Γ was very small. One can also ask whether the macroscopic quantum coherence could still be there.

Hyperbolic manifold property has purely local signatures such as angle surplus: the very fact that there are infinite number of hyperbolic tesselations is in conflict with the the fact that we have Euclidian 3-geometry in every day length scales. In fact, for critical cosmologies, which allow a one-parameter family of imbeddings to $M^4 \times CP_2$ (parameter characterizes the duration of the cosmology) one obtains flat 3-space in cosmological scales. Also overcritical cosmologies for which $a = constant$ section is 3-sphere are possible but only with a finite duration. Many-sheeted space-time picture also leads to the view that astrophysical objects co-move but do not co-expand so that the geometry of time=constant snapshot is Euclidian in a good approximation.

3. Does the notion of hyperbolic quasi-tesselation make sense?

Can one construct something deserving to be called quasi tesselations (QTs)? For QCs translational invariance is broken but in some sense very weakly: given lattice point has still an infinite number of translated copies. In the recent case translations are replaced by Lorentz transformations and discrete Lorentz invariance should be broken in similar weak manner.

If cut and project generalizes, QTs would be obtained using suitably chosen non-standard imbedding $M^4 \subset M^8$. Depending on what one wants to assume, M^4 is now image of M^4_{st} by an element of $SO(1,7)$, $SO(7), SO(6)$ or G_2. The projection - call it P - must take place to M^4 sliced by scaled copies of H^3 from M^8_{st} sliced by scaled copies of H^7/Γ tesselation. The natural option is that P is directly from H^7 to $H^3 \subset H^7$ and is defined by a projecting along geodesic lines orthogonal to H^3. One can choose always the coordinates of M^4 and M^8 in such a manner that the coordinates of points of M^4 are $(t,x,y,z,0,0,0,0)$ with $t^2-r^2 = a_4^2$ whereas for a general point of H^7 the coordinates are $(t,x,y,z,x_4,...x_7)$ with $t^2-r^2-r_4^2 = a_8^2$ for $H^3 \subset H^7$. The projection is in this case simply $(t,x,y,z,x_4,...,x_7) \to (t,x,y,z,0,...,0)$. The projection is non-empty only if one has $a_4^2 - a_8^2 \geq 0$ and the 3-sphere S^3 with radius $r_4 = \sqrt{a_4^2 - a_8^2}$ is projected to single point. The images of points from different copies of H^7/Γ are identical if S^3 intersects both copies. For r_4 much larger than the size of the projection $P(H^7/\Gamma)$ of single copy overlaps certainly occurs. This brings strongly in mind the overlaps of the dodecagons of Penrose tiling and icosahedrons of 3-D icosahedral QC. The point group of tesselation would be Γ.

4. Does one obtain ordinary H^3 tesselations as limits of quasi tesselations?

Could one construct expanding 3-D hyperbolic tesselations H_3/Γ_3 from expanding 7-D hyperbolic tesselations having H^7/Γ_7 as a basic building brick? This seems indeed to be the outcome at at the limit $r_4 \to 0$. The only projected points are the points of H^3 itself in this case. The counterpart of the group $\Gamma_7 \subset SO(1,7)$ is the group obtained as the intersection $\Gamma_3 = \Gamma_7 \cap SO(1,3)$: this tells that the allowed discrete symmetries do not lead out from H^3. This seems to mean that the 3-D hyperbolic manifold is H^3/Γ_3, and one obtains a space-filling 3-tesselation in complete analogy for what one obtains by projecting cubic lattice of E^7 to E^3 imbedded in standard manner. Note that $\Gamma_3 = \Gamma_7 \cap SO(1,3)$, where $SO(1,3) \subset SO(1,7)$, depends on imbedding so that one obtains an infinite family of tesselations also from different imbeddings parametrized by the coset space $SO(1,7)/SO(1,3)$. Note that if Γ_3 contains only unit element $H^3 \subset H^7/\Gamma_7$ holds true and tesselation trivializes.

p-Adic variant of the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and Bruhat-Tits tree

p-Adicization requires also to consider the p-adic variants of the Grassmannian $SO(1,7)/SO(1,3)\times SO(4)$. Grassmannians define a generalization of projective spaces and appear in twistor Grassmannian program. According to the article [A129], the construction of the Bruhat-Tits tree generalizes for them. This gives excellent hopes for generalizing the twistor Grassmannian program to p-adic context. Bruhat-Tits tree for $S^2 = SO(3)/SO(2) = P^1(C)$ generalized to $P^1(K)$ (K is any algebraic extension of Q_p) is constructed in terms of projective equivalence classes of integer lattices in K^2 with inclusion relation defining the notion of edge path making possible path connectedness.

In the recent p-adic manifold structure forces 8-D lattices in K^8 and they seem to to take the role of the 2-D lattices K^2. Therefore TGD view about p-adic manifold structure might well be equivalent to the standard view in the case of Grassmannians. For the Grassmannian in question the projective equivalence is replaced with equivalence under $SO(1,3) \times SO(4)$. Therefore one expects that the generalization of Bruhat-Tits tree in 8-D case and its projections to sub-spaces assignable to algebraic extensions K of Q_p appear and correspond to discrete subgroups Γ of $SO(1,7)$. With some additional restrictions on Γ the spaces H^7/Γ define hyperbolic manifolds (H^7 is 7-D hyperboloid in M^8).

This argument makes sense if the counterpart of projective space $P^1(K)$ can be defined also as the analog of $SO(3)/SO(2)$. What looks like a problem is that the "Cartesian" dimension of this space is 2 whereas $P^1(K)$ is 1-D in this sense. The analog of $SO(3)/SO(2)$ can be indeed defined as the Grassmannian $SO(1,2)/Z_p^1 \times SO(2)$ with dimension 1. Z_p^1 denotes the group of p-adic integers with unit norm defining p-adic units analogous to complex phases: they have their inverse as conjugate. What this says that p-adic unit vector 1 is equivalent to any element of Z_p^1. In real context the group of units contains only the real unit so that one obtains Cartesian dimension 2.

8.6 What the notion of path connectedness could mean from quantum point of view?

The notions of open set and path connectedness express something physical but perhaps in a highly idealized form. Canonical identification for preferred extremals provides one promising approach to the challenge of defining path connected topology and at the same time achieving a compromise with symmetries and approximate correspondence via common rationals. The variant I_k^Q for the canonical identification with pinary cutoff can be used to map rational points of the real/p-adic preferred extremal to p-adic/real space-time points to define a skeleton completed to a preferred extremal, which of course need not be unique. In particular, real paths are mapped to p-adic paths in finite pinary cutoff so that the images are always discrete paths consisting of rational points so that the notion of finite pinary resolution is un-avoidable.

One could also try to formulate path connectedness more microscopically and physically using the tools of quantum physics.

1. The basic point is that there are correlations between different points or physical events associated with different points of manifold. Manifold is more like liquid than dust: one cannot pick up single point from it. In the idealistic description based on real topology one can pick up only open ball. This relates also to finite measurement resolution for lengths: it is not possible to specify single point.

2. Quantum physicist would formulate this in terms of physical correlations. The correlation functions for two fields defined in the manifold are non-vanishing even when the two fields are evaluated at different points.

If one takes the suggestion of quantum physicist seriously, one should reformulate the notion of manifold by bringing in quantum fields and their correlation functions. This approach is alternative to the formulation of p-adic (real) manifold based on real (p-adic) coordinate charts defined by canonical identification.

8.6.1 Could correlation functions for fermion fields code data about geometric objects?

Quantum TGD suggests another approach to the notion of path connectedness. What could the quantum fields needed to formulate the notion of manifold be in TGD framework? In TGD framework there are only very few choices to consider. Only the induced second quantized fermion fields can be considered in both real and p-adic context. Their correlation functions defined as vacuum expectations of bi-local bilinears are indeed well-defined in both real and p-adic context.

One can define classical bosonic correlation functions for the invariants formed from induce bosonic field but this requires integration over the space-time surface and this might be problematic in p-adic context unless one is able to algebraically continue the real correlation functions to p-adic context. Quantum ergodicity states that these correlation functions characterizing sub-manifold geometry statistically are identical for the space-time surfaces which can appear in the quantum superposition defining WCW spinor field.

1. One could perhaps say:

   Two points are "connected by path!" / have "edge connecting them" as Bruhat and Tits would say / belong to same space-time sheet/partonic 2-surface / belog to two distinct 3-surfaces forming part of a boundary of the same connected space-time surface $\leftrightarrow$ there are non-vanishing fermion-anti-fermion correlation functions for the point pair in question.

2. Note that one must consider separately pure right-handed neutrino modes and the remaining spinor modes. For the modified Dirac equation pure right-handed neutrino fields are covariantly constant in CP_2 degrees of freedom and de-localized along entire space-time sheet. In space-time interior the correlation functions for right-handed neutrinos should code for the geometry of the space-time sheet.

   The modes which do not represent pure right-handed neutrinos are restricted to 2-D string world sheets. The conformal correlation functions for the spinor fields restricted to string world sheets should code for the geometry of string world sheets.

3. Everything would reduce to fermionic correlation functions, which in principle are measurable in particle physics experiments. This is in accordance with the general vision of TGD that fermion fields provide all possible information about geometric objects. This would generalize the idea that one can hear the shape of the drum that is deduce the geometry of drum from the correlation functions for sound waves.

4. Real space-time topology would be only a highly idealized description of this physical connectedness, in more physical approach it would be described in terms of fermionic correlation functions allowing to decide whether two points belong to same geometric object or not.

8.6.2 p-Adic variant of WCW and M-matrix

In zero energy ontology (ZEO) the unitary U-matrix having non-unitary M-matrices are rows and allowing interpretation as "complex" square roots of hermitian density matrices are in key role. The unitary S-matrix appears as a "phase factor" of the "complex" square root and its modulus corresponds to Hermitian square roots of density matrix. What is essential is that M-matrices are multi-local functionals of 3-surfaces defining boundary components of connected space-time surface at the light-like boundaries of causal diamond.

By strong form of holography the information about three-surfaces reduces to data given at partonic 2-surfaces (and their tangent space data). The 3-D boundary components of space-time surface at the boundaries of CD define a coherent unit. The space-time surface takes the role of the path connecting two disjoint 3-surfaces in zero energy ontology and WCW is more like a space formed by multi-points (unions of several disjoint 3-surfaces). Hence the basic difficulty of p-adic manifold theory is circumvented.

Although WCW spinor fields are formally purely classical, the analogs of correlation functions as n-point functions in WCW make sense since the notion of 3-surface is generalized in the manner described above. M-matrix elements serve as building bricks of WCW spinor fields and they are are functionals about the data at partonic 2-surfaces at the boundaries of CD and could have an interpretation as correlation function in WCW giving rise to "path connectedness" in WCW in a number theoretically universal manner.

8.6.3 A possible analog for the space of Berkovich norms in the approach based on correlation functions

The idea about real preferred extremal as a coordinate chart for p-adic preferred extremal (and vice versa) suggest that canonical identification with cutoff could define naturally p-adic preferred extremal as a path connected space. It would also allow to map preferred real preferred extremals to their p-adic counterparts for some preferred primes and at the same time algebraically continue various quantities such as Kähler action. The hierarchies of pinary cutoffs and resolutions in phase degrees of freedom define a hierarchy of resolutions and the resulting Archimedean norms defined by the the hierarchy of canonical identifications define the analog of the norm space of Berkovich.

Also the idea about correlation functions as counterpart for path connectedness suggests that the ultra-metric norm of K-valued field needed to defined Berkovich disk might be replaced with fermionic correlation functions. Could the space of the Berkovich norms have as an analog in this more general approach? The notion of finite measurement resolution seems to lead naturally to this analog also for this option.

One can define the correlation functions in various resolutions. This means varying angle resolution and length scale resolution. Angle resolution -or rather phase resolution in p-adic context - means a hierarchy of algebraic extensions for p-adic number fields bringing in roots of unity $exp(i2\pi/n)$ with increasing values of n. Length scale resolution means increasing number of p-adic primes and CDs with scales given by integer multiples of CP_2 scale.

Fermionic Fock space defines a canonical example about hyper-finite factor of type II_1 (HFF) [K80] and the inclusions of HFFs having interpretation in terms of finite measurement resolution should be involved in the construction. The space of Berkovich norms is replaced with the correlation functions assignable to HFF having fractal structure containing infinite inclusion hierarchies of HFFs.

8.7 Appendix: Technical aspects of Bruhat-Tits tree and Berkovich disk

In the following more technical aspects of Bruhat-Tits tree and Berkovich disk are discussed.

8.7.1 Why notions like Bruhat-Tits tree and Berkovich disk?

The constructions like Bruhat-Tits tree and Berkovich disk remain totally incomprehensible unless one understands the underlying motivations. If I have understood correctly, the motivation behind all these strange and complicated looking structures is the attempt to generalize the notion of real manifold to p-adic context using topological approach based on p-adic coordinate maps to p-adic disks which must be completed to Berkovich disks ("disk" could quite well be replaced with "ball").

In the real context manifolds have open balls of R^n defining real topology as building bricks. One glues these balls together along their intersection suitably and obtains global differential structures with various topologies and manifold structures. For instance, sphere can be obtained by gluing two disks having overlap around equator.

In the p-adic context the topology is however totally disconnected meaning that single point is the smallest open set. One cannot build anything coherent from points: they are disjoint or identical unlike the open balls in the real case. More generally: two p-adic balls are either disjoint or either one is contained by another one! No gluing by overlap is possible!

This difficulty has stimulated various theories and Bruhat-Tits tree relates to the theory of Berkovich generalizing the notion of open ball to Berkovich disk [A211, A216] serving as a building brick of p-adic manifolds. The naive p-adic disk is contained as a dense subset to Berkovich disk so that this is like replacing rationals with reals and in this manner gluing them to continuum. Pragmatic physicist is not too enthusiastic about this kind of completions, especially so because the original p-adic topology is replaced with a new one in the completion.

8.7.2 Technical aspects of Bruhat-Tits tree

The construction of Bruhat-Tits tree for $P^1(Q_p)$ and its generalizations to algebraic extensions can be understood as follows.

1. One must be able to connect any pair of points of $P^1(Q_p)$ by and edge path. The basic building brick of edge path is single edge connecting nearby points of $P^1(Q_p)$. One can start from a simpler situtation first by considering Q_p^2 consisting of points (a,b). If one treats these points just as pairs of p-adic numbers, one cannot do anything. One must represent these pairs as geometric objects in order to define the notion of edge purely set theoretically. The Z_p lattice generated having the pair (a,b) as basis vectors is indeed an object labelled by the pair (a,b). If one wants projective space one must assume that the lattices different by scaling of (a,b) by a non-vanishing p-adic number are equivalent but this is not absolutely essential for the argument.

   Note: Also in TGD one has a space whose points are geometric objects. The geometric object is now 3-surface and the space is the "world of classical worlds" - the space formed by these 3-surfaces.

2. The projective space $P^1(C) = S^2$ has a representation as a coset space $PGl(2,C)/PGl(1,C) \times PGl(2,Z)$. This algebraic relation must generalize by replacing C with Q_p. This means that $PGl(2,Q_p)$ must act transitively in the set of the geometric objects associated with pairs (a,b). The action on lattices is indeed well-defined and transitive and one can generate all lattices from single lattice defined by the lattice characterized by $(a,b) = (1,1)$. One has a discrete analog of homogeneous space in the sense that its all points are geometrically equivalent because of the transitive action of $Gl(2,Q_p)$. This reduces the construction to single point, which is an enormous simplification.

   Note: Also the construction of the geometry of WCW [K16] in TGD relies on symmetric/homogeneous space property (actually the property of being a union of infinite-dimensional symmetric spaces) making the hopeless task managable by reducing the construction to that at single point of WCW and forcing infinite-dimensional symmetries (symplectic invariance imherited from the boundary of $CD \times CP_2$ and generalization of conformal invariance for light-like 3-surfaces and light-like boundaries of CD). Already in the case of loop spaces [A58] Kähler geometry exists only because of these infinite-dimensional symmetries and is also unique [A145]. One can say that infinite-dimensional Kähler geometric existence is unique.

3. The really important idea is that the internal structure of the point pairs (a,b) allows to define what the existence of "edge" between two nearby points of $P^1(Q_p)$ could mean. The definition is following. Two projective lattices $[M]$ and $[N]$ (projective equivalence classes of lattices) are connected by an edge if there exist representatives M and N such that $M \supset N \subset pM$. Note that this relation holds true only for some representatives, not all. It is also purely set-theoretic.

4. By reducing the situation to the simplest possible case $M \leftrightarrow (a,b) = (1,1)$ one can easily find the lattices N connected to M. The calculations reduce to the finite field F_p since the inclusion condition implies that $M/pM \supset N/pM \supset pM/pM = \{0\}$ and M/pM is just F_p^2. The allowed N correspond are in one-one correspondence with the F_p subspaces of F_p^2 and there are $p{+}1$ of them corresponding to space generated by F_p multiples of $(a,1)$, $a = 0, ...p{-}1$ and $(1,0)$. Therefore the point $(a,b) = (1,1)$ is connected to $p+1$ neighbours by single edge. By symmetric space property this is true for all points of $P^1(Q_p)$. The conclusion is that edge paths correspond to a regular tree with valence $p+1$.

5. $P^1(Q_p)$ is still totally disconnected in p-adic topology. The edge paths however provide $P^1(Q_p)$ with a path-connected topology. The example of Berkovich disk would suggest that one must add to $P^1(Q_p)$ something so that $P^1(Q_p)$ remains a dense subset of this larger structure. The situation would be same as for rationals: rationals become a path connected continuum if one adds all irrational numbers to obtain reals. Rationals define a dense subset of reals and numerics uses only them. In particular, integration becomes possible when irrationals are added. It is however not clear to me whether this kind of completion is needed.

One can wonder what must be added to the set of Z_p lattices in Q_p^2 or to the set of their projective equivalence classes to build the global differentiable structure. The answer perhaps comes from the observation that the ends of Bruhat-Tits tree correspond to *K-rationals* expressible as ratios of two K-integers - something that numerics can catch at least in real case. Could the completion mean adding also the ends which are K-irrationals? If so then the situation would be very similar to that in TGD inspired definition of p-adic manifolds.

6. Every pair of points in the completion $P^1(Q_p)$ is connected by an edge path consisting of some minimal number n_{min} of edges and this edge path defines the analog of geodesic with length n_{min}. This number is p-adic integer and could be infinite as a real integer for the completion of the p-adic manifold to a path connected manifold. Here the canonical identification $\sum x_n p^n \to z_n p^{-n}$ mapping p-adic integers to real numbers and playing a key role in p-adic mass calculations could come into play and allow to obtain a real valued finite distance measure. Real distances have continuous spectrum in the interval $[0, p)$. The objection is that this definition is not consistent with the idea of algebraic continuation of integrals from real context.

This construction generalizes to algebraic extensions K of Q_p and also to higher-dimensional projective spaces and symmetric spaces. In particular, the construction of the p-adic counterpart of CP_2 becomes possible. Now one replaces Q_p^2 with Q_p^3 or K^3 allowing the action of some discrete subgroup of the isometry group $SU(3)$ of CP_2. Lattices in K^3 replace the points of Q_p^3 and defines the counterpart of Bruhat-Tits tree in exactly the same manner as for $P^1(K)$.

Physically the highly interesting point is that only a discrete subgroup of CP_2 can be represented in the algebraic extension so that symmetry breaking to discrete subgroup is un-avoidable. In TGD framework the interpretation is in terms of finite measurement resolution forcing discretization and therefore also symmetry breaking. This symmetry breaking is quite different from that defined by Higgs mechanism or symmetry breaking taking place for the solutions of field equations for a variational principle characterized by the unbroken symmetry group.

8.7.3 The lattice construction of Bruhat-Tits tree does not work for K^n but works for $P^n(K)$: something deep?

The naive expectation is that the construction of Bruhat-Tits tree should work also in the simplest possible case that one can imagine: for p-adic numbers Q_p themselves. The naive guess is that the tree for p-adic numbers with norm bounded by p^n the tree is just the p+1-valent tree with trunk and representing all possible pinary expansions of these p-adic numbers. The lattice construction does not however give this correspondence.

Z_p lattices M in Q_p are parameterized by non-vanishing elements a of Q_p in this case. The multiplication by p-adic integer n of unit norm does not affect a given lattice M a since one has $nka = k_1 a$ where n, k, k_1 are p-adic integers. Therefore these lattices are not in one-one correspondence with Q_p but with powers p^n: $|q|_p \leq p^n$ for a given lattice. Therefore the lattice construction fails. It is essential that one considers projective space $P^1(Q)$ instead of Q_p. For Q_p^2 the construction however seems to work.

Note: The condition $M \supset N \supset pM$ for the existence of an edge between two lattices allows only two solutions: the trivial solution $N = M$ and the solution $N = pM$. The counterpart of Bruhat-Tits tree is now 1-valent tree with edges labelled by powers of p.

Also in the case of Q_p^n the correspondence between lattices and points of Q_p^n is 1-to-many since the multiplication by an element of Z_p with unit norm does not affect the lattice. As a matter fact, all elements of Qp^n related by $Sl(n, Q_p)$ correspond to same lattice. Hence the replacement of points with lattices must be restricted to the case of projective spaces.

Physicist might argue that the use of lattices is un-natural and quite too complicated from the point of view of practical physics. I am not sure: it might be that the lattices have some nice physical interpretation and perhaps the outcome - the tree - is more important than the lattices used to achieve it. The fact is that p-adic projective spaces have this kind of "skeleton", and one might well argue that there is no need for the ugly looking completion to a bigger space with path connected and non-ultra-metric topology.

In TGD framework the p-adic variants of S^2 and CP_2 are central and the existence of the "skeleton" might be of fundamental significance from the point of view of p-adic TGD and number theoretical universality. Note that S^2 emerges naturally for the light-cone boundary in the case of M^4 ($\delta M^4_+ = S^2 \times R_+$, where R_+ represents light-like radial direction). For M^n, $n \neq 4$, one obtains $S^{k=n-2}$, $k \neq 2$, and this space is not projective space. Also in twistor Grassmannian approach to scattering amplitudes utilizing residue integrals in projective spaces $Gl(n,C)/Gl(n-m,C) \times Gl(m,C)$ this property for the p-adic counterparts of these spaces might be of primary importance.

8.7.4 Some technicalities about Berkovich disk

Berkovich disk is a p-adic generalization of open ball and meant to serve as a building brick of p-adic manifolds in the same manner as open ball is the building brick of real manifolds. The first guess is that ordinary open ball for p-adic numbers defined by $|x-a| < r$ could work. As a matter fact, p-adic distance is quantized: $|x-a| = p^n$ holds true. The basic outcome of total disconnectedness of the ultrametic topology is that two p-adic balls are either disjoint of the other one is contained by another one. One cannot build manifolds by taking p-adic balls and allowing them to partially overlap to get global differentiable structures and various topologies.

The construction of Berkovich disk - call it B - is motivated by the need to generalize the standard approach to the construction of real manifolds. I do not know whether it is equivalent with the approach based on Bruhat-Tits tree. The explicit realization of Berkovich disk as a completion of ultra-metric unit disk is something which one cannot guess easily but when one has understood that the basic premises are satisfied for it, it begins to look less artificial.

I try to explain this construction described briefly in the lecture notes Buildings and Berkovich spaces [A216] by Annette Werner. I neglect all technical issues (I do not even understand them properly!). The basic idea is to imbed ultra-metric unit disk as a dense subset to some space possessing path connected topology. The challenge is to guess what this space is.

1. One starts from p-adic unit disk D: $|x|_p \leq 1$, which one wants to complete to Berkovich disk B containing D as a dense subset and possessing path connected topology. One could also replace Q_p with Q_p^n or K^n, where K is any algebraic extension Q_p. In the explanation provided in the lecture notes one considers for simplicity K, which is algebraically complete: this requires an algebraic extension allowing containing all algebraic numbers. This is unrealistic but the construction is possible also for general K but involves more technicalities.

2. One introduces the space of formal K-valued power series $f(z) = \sum f_n z^n$ in $D(0,1) \equiv D$. One can define for the an ultra-metric norm as $||f|| = Max\{|f_n|_K\}$. This is actually the supremum of p-adic norm $|f(x)|_K$ in $D(0,1)$. The p-adically largest coefficient f_n defines the norm known as Gauss norm. This norm is multiplicative. For constant functions, which are in one-one correspondence with points of K, this norm reduces to K-norm.

3. One considers also more general norms. In fact, the space of norms with attributes ultra-metric, bounded, and multiplicative and reducing for constant functions to K-norm $||_K$ defines the Berkovich unit disk B, which turns out to be a completion of the unit disk D containing D as a dense subset. Furthermore, B turns out to have have path connected topology as required making possible global differentiable structure and even hopes about p-adic integration.

4. Berkovich manages to construct these norms explicitly. The simplest norms of this kind are defined by points a of D. The norm is simply $|f(a)|_K$. These norms are in one-one correspondence with points of D and should define a dense subset of the entire space of norms. The points of K are therefore mapped to subspace of the space of norms: this is absolutely essential.

5. There are also other multiplicative, ultra-metric norms reducing to $||_K$ for constant functions in D. They are defined in terms of disks $|x-a|_K \leq r \leq 1$. The Gauss norm corresponds to $r = 1$ and the norm described in previous item to $r = 0$. These norms are analogous to irrationals numbers in the case of completion of rationals to reals. The Berkovich disk B contains points of four different types.

   - Points of type 1: $|f_a| = |f(a)|_K$ (imbedding of D to Berkovich disk B.

- Points of type 2: $|f|_{a,r} = sup|f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \in |K*|$, the value spectrum of K-norms (powers of p for Q_p). The Gauss norm corresponds to $r = 1$.
- Points of type 3: $|f|_{a,r} = sup|f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \notin |K*|$. There is a delicate difference between types 2 and 3 which I fail to understand.
- Points of type 4: $|f|_{a,r} = lim_{n\rightarrow\infty}|f|_{a_n,r_n}$ for a nested sequence $D(a_1,r_1) \supset D(a_2,r_2)....$ of closed disks in $D(0,1)$.

6. The topology in Berkovich disk is defined by a pointwise convergence of the norm in the space of functions f in D. This topology makes Berkovich disk path connected.

The above construction is rather complicated although and also assumes algebraic completeness. For finite-dimensional algebraic extensions the construction is expected to be even more complicated. I do not understand the possible connection between Bruhat-Tits tree and Berkovich construction: does Bruhat-Tits tree follow from Berkovich construction or not?

8.7.5 Could the construction of Berkovich disk have a physical meaning?

For the physicist the obvious question is whether the function space associated with the K-disk D could have some some physical interpretation? And what about the interpretation of the space of bounded multiplicative ultra-metric norms for this function space? Could these norms have some physical interpretation?

Consider first basic criticism what might be represented by a physicist.

1. The ultra-metric multiplicative norms in the function space carry extremely scarce information about the functions. Just the norm of the value of the function at single point. If one wants information in several points on must have a manifold consisting of large minimal number of Berkovich disks. An alternative manner to get information about the function space is to combine the information about all norms.

2. Physicists could also wonder what these K-valued functions are physically. Are they physical fields perhaps? If so, why not consider p-adic variants of correlation functions instead of p-adic norms scalars formed from these fields at single point. This forces however to ask whether the non-vanishing of these physical correlation functions for these fields could code for the existence of "connections" between points of the p-adic manifold so that there would be no need for the completion to Berkovich disk after all. Could the solution of the problem be achieved by bringing quantum physics a part of the definition of the manifold structure.

   It seems that in TGD framework there is no natural counterpart for the K-valued formal power series and their norms. One must perform a stronger generalization and this leads to the use of canonical identification mapping p-adic coordinate variables to their Archimedean norms defined by canonical identification and serving as real coordinates. Another, very speculative approach would be based on correlation functions of fermion fields as a possible manner to code the physical counterpart of path connectedness.

Part III

MISCELLANEOUS TOPICS

Chapter 9

Riemann Hypothesis and Physics

9.1 Introduction

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the critical line $Re(s) = 1/2$. Since Riemann zeta function allows a formal interpretation as thermodynamical partition function for a quantum field theoretical system consisting of bosons labeled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. The complex value of temperature is not however consistent with thermodynamics. In zero energy ontology one obtains quantum theory as a square root of thermodynamics and this objection can be circumvented and a nice argument allowing to interpret RH physically emerges.

Conformal invariance leads to a beautiful generalization of Hilbert-Polya conjecture allowing to interpret RH in terms of coherent states rather than energy eigenstates of a Hamiltonian. In zero energy ontology the interpretation is that the coherent states in question represent Bose-Einstein condensation at criticality. Zeros of zeta correspond to coherent states orthogonal to the coherent state characterized by $s = 0$, which has finite norm, and therefore does not represent Bose-Einstein condensation.

Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true. These considerations are however highly speculative and are represented at the end of the chapter.

9.1.1 Super-conformal invariance and generalization of Hilbert-Polya hypothesis

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

9.1.2 Zero energy ontology and RH

A further approach to RH is based on zero energy ontology and is consistent with the approach based on the notion of coherent state. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

9.1.3 Miscellaneous ideas

During years I have also considered several ideas about Riemann hypothesis which I would not call miscellaneous. I have moved them to the end of the chapter because of the highly speculative nature.

Logarithmic waves for zeros of zeta as complex algebraic numbers?

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [B3] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2+iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-symplectic weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

Universality Principle

A second strategy is based on, what I call, Universality Principle. The function, that I refer to as $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension Q_∞ of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-z})$ appearing in the product formula have value in Q_∞. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in Q_∞, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle implies the following stronger variant about sharpened form of the Riemann hypothesis: the real part of the phase p^{-iy} is rational for an infinite number of primes for zeros of ζ. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of the Riemann hypothesis becomes however extremely implausible. An important outcome of this approach is the realization that super-conformal invariance is a natural symmetry associated with ζ (not surprisingly, since the symmetry group of complex analysis is in question!).

These approaches reflect the evolution of the vision about TGD based physics as a generalized number theory. Two new realizations of the super-conformal algebra result and the second realization has direct application to the modelling of $1/f$ noise. The zeros of ζ would code for the states of an arithmetic quantum field theory coded also by infinite primes: also the hierarchical structure of the many-sheeted space-time would be coded.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19].

9.2 General vision

Quantum TGD has inspired several strategies of proof of the Riemann hypothesis. The first strategy is based on the modification of Hilbert Polya hypothesis by requiring that the physical system in question has super-conformal transformations as its symmetries. Second strategy is based on considerations based on TGD inspired quantum theory of cognition and a generalization of the number concept inspired by it. Together with some physical inputs one ends up to a hypothesis that Riemann Zeta is well defined in all number fields near its zeros provided finite-dimensional extensions of p-adic numbers are allowed. This hypothesis generalizes the earlier hypothesis assuming that the extensions are trivial or at most algebraic. Third strategy is based on, what I call, Universality Principle.

There are also strong physical motivations to say something explicit about the spectrum of zeros and here p-adicization program inspires the hypothesis the numbers q^{iy}, q prime, belong to a finite algebraic extension of p-adic number field R_p for every prime p. The findings about the correlations of the spectrum of zeros inspire very concrete hypothesis about the spectrum of zeros as a union of translates of the same basic spectrum and this hypothesis is supported by the physical identification of the zeros of Zeta as super-symplectic conformal weights.

9.2.1 Generalization of the number concept and Riemann hypothesis

The hypothesis about p-adic physics as physics of cognition leads to a generalization of the notion of number obtained by gluing reals and various p-adic number fields together along rational numbers common to all of them. This structure is visualizable as a book like structure with pages represented by the number fields and the rim of the book represented by rationals. Even this structure can be generalized by allowing all finite-dimensional extensions of p-adic numbers including also those containing transcendental numbers and performing similar identification. Kind of fractal book might serve as a visualization of this structure.

In TGD inspired theory of consciousness intentions are assumed to correspond to quantum jumps involving the transformation of p-adic space-time sheets to real ones. An intuitive expectation is p-adic and real space-time sheets to each other must have a maximum number of common rational points. The building of idealized model for this transformation leads to the problem of defining functions having Taylor series with rational coefficients and continuable to both real and p-adic functions from a subset of rational numbers (or points of space-time sheet with rational coordinates). In this manner one ends up with the hypothesis that p-adic space-time sheets correspond to finite-dimensional extensions of p-adic numbers, which can involve also transcendental numbers such as e. This leads to a series of number theoretic conjectures.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [B3] suggesting strongly that e and its $p-1$ powers at least should belong to extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u=n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2+iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. A hierarchy of number theoretical conjectures stating that a finite number of iterated logarithms about transcendentals appearing in the extension forms a closed system under the operation of taking logarithms. Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the complex numbers p^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of p and any zero $1/2+iy$ of ζ.

9.2.2 Modified form of Hilbert-Polya hypothesis

Super-conformal invariance inspires a strategy for proving (not a proof of, as was the first over-optimistic belief) the Riemann hypothesis. The vanishing of Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of super-conformal algebra. The eigenfunctions of D^+ are analogous to the so called coherent states and in general not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line and possibly those having $Re[s] > 1/2$.

A possible proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the states corresponding to zeros of ζ span a space with a hermitian metric. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal invariance in appropriate sense implies Riemann hypothesis. Indeed, a rather rigorous proof of Riemann hypothesis results from the observation that certain generator of conformal algebra permutes the two zeros located symmetrically with respect to the critical line. If the action of this generator exponentiates, Riemann hypothesis follows since exponentiation would imply the existence of infinite number of zeros along a line parallel to $Re[s]$-axis. One can formulate this argument rigorously using first order differential equation, and if one forgets all the preceding refined philosophical arguments, one can prove Riemann hypothesis using twenty line long analytic argument! Perhaps Ramajunan could have made this!

As already noticed, the state space metric can be made positive definite provided Riemann hypothesis holds true. Thus the system in question might quite well serve as a concrete physical model for quantum critical systems possessing super-conformal invariance as both dynamical and gauge symmetry.

9.2.3 Riemann hypothesis in zero energy ontology

Zeta reduces to a product $\zeta(s) = \prod_p Z_p(s)$ of partition functions $Z_p(s) = 1/[1-p^{-s}]$ over particles labelled by primes p. This relates very closely also to infinite primes and one can talk about Riemann gas with particle momenta/energies given by $log(p)$. s is in general complex number and for the zeros of the zeta one has $s = 1/2 + iy$. The imaginary part y is non-rational number. At $s = 1$ zeta diverges and for $Re(s) \leq 1$ the definition of zeta as product fails. Physicist would interpret this as a phase transition taking place at the critical line $s = 1$ so that one cannot anymore talk about Riemann gas. Should one talk about Riemann liquid? Or - anticipating what follows- about quantum liquid? What the vanishing of zeta could mean physically? Certainly the thermodynamical interpretation as sum of something interpretable as thermodynamical probabilities apart from normalization fails.

The basic problem with this interpretation is that it is only formal since the temperature parameter is complex. How could one overcome this problem?

A possible answer emerged as I read the interview.

1. One could interpret zeta function in the framework of TGD - or rather in zero energy ontology (ZEO) - in terms of square root of thermodynamics! This would make possible the complex analog of temperature. Thermodynamical probabilities would be replaced with probability amplitudes.

2. Thermodynamical probabilities would be replaced with complex probability amplitudes, and Riemann zeta would be the analog of vacuum functional of TGD which is product of exponent of Kähler function - Kähler action for Euclidian regions of space-time surface - and exponent of imaginary Kähler action coming from Minkowskian regions of space-time surface and defining Morse function. In QFT picture taking into account only the Minkowskian regions of space-time would have only the exponent of this Morse function: the problem is that path integral does not exist mathematically. In thermodynamics picture taking into account only the Euclidian regions of space-time one would only the exponent of Kähler function and would lose interference effects fundamental for QFT type systems. In quantum TGD both Kähler and Morse are present. With rather general assumptions the imaginary part and real part of exponent of vacuum functional are proportional to each other and to sum over the values of Chern-Simons action for 3-D wormhole throats and for space-like 3-surfaces at the ends of CD. This is non-trivial.

3. Zeros of zeta would in this case correspond to a situation in which the integral of the vacuum functional over the "world of classical worlds" (WCW) vanishes. The pole of ζ at $s = 1$ would correspond to divergence fo the integral for the modulus squared of Kähler function.

What the vanishing of the zeta could mean if one accepts the interpretation quantum theory as a square root of thermodynamics?

1. What could the infinite value of zeta at $s = 1$ mean? The The interpretation in terms of square root of thermodynamics implied following. In zero energy ontology zeta function function decomposition to $\prod_p Z_p$ corresponds to a product of single particle partition functions for which one can assigns probabilities $p^{-s}/Z_p(s)$ to single particle states. This does not make sense physically for complex values of s.

2. In ZEO one can however assume that the complex number p^{-sn} define the entanglement coefficients for positive and negative energy states with energies $nlog(p)$ and -$nlog(p)$: n bosons with energy log(p) just as for black body radiation. The sum over amplitudes over over all combinations of these states with some bosons labelled by primes p gives Riemann zeta which vanishes at critical line if RH holds.

3. One can also look for the values of thermodynamical probabilities given by $|p^{-ns}|^2 = p{-}n$ at critical line irrespective of zero. The sum over these gives for given p the factor $p/(p-1)$ and the product of all these factors gives $\zeta(1) = \infty$. Thermodynamical partition function diverges. The physical interpretation is in terms of Bose-Einstein condensation.

4. What the vanishing of the trace for the matrix coding for zeros of zeta defined by the amplitudes is physically analogous to the statement $\int \Psi dV = 0$ and is indeed true for many systems such as hydrogen atom. But what this means? Does it say that the zero energy state is orthogonal to vacuum state defined by unit matrix between positive and negative energy states? In any case, zeros and the pole of zeta would be aspects of one and same thing in this

interpretation. This is an something genuinely new and an encouraging sign. Note that in TGD based proposal for a strategy for proving Riemann hypothesis, similar condition states that coherent state is orthogonal to "false" tachyonic vacuum.

5. RH would state in this framework that all zeros of ζ correspond to zero energy states for which thermodynamical partition function diverges. Another manner to say this is that the system is critical. (Maximal) Quantum Criticality is indeed the key postulate about TGD Universe and fixes the Kähler coupling strength characterizing the theory uniquely (plus possible other free parameters). Quantum Criticality guarantees that the Universe is maximally complex. Physics as generalized number theory would suggest that also number theory is quantum critical! When the sum over numbers proportional to probabilities diverges, the probabilities are considerably different from zero for infinite number of states. At criticality the presence of fluctuations in all scales implying fractality indeed implies this. A more precise interpretation is in terms of Bose-Eisntein condensation.

6. The postulate that all zero energy states for Riemann system are zeros of zeta and critical in the sense being non-normalizable (Bose-Einstein condensation) combined with the fact that $s = 1$ is the only pole of ζ implies that the all zeros of ζ correspond to $Re(s) = 1/2$ so that RH follows from purely physical assumptions. The behavior at $s = 1$ would be an essential element of the argument. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$, which has finite norm and does not represent Bose-Einstein condensation. This would give a connection for the proposal for the strategy for proving Riemann Hypothesis by replacing eigenstates of energy with coherent states.

9.3 Riemann hypothesis and super-conformal invariance

Hilbert and Polya [A136] conjectured a long time ago that the non-trivial zeroes of Riemann Zeta function could have spectral interpretation in terms of the eigenvalues of a suitable self-adjoint differential operator H such that the eigenvalues of this operator correspond to the imaginary parts of the nontrivial zeros $z = x + iy$ of ζ. One can however consider a variant of this hypothesis stating that the eigenvalue spectrum of a non-hermitian operator D^+ contains the non-trivial zeros of ζ. The eigen states in question are eigen states of an annihilation operator type operator D^+ and analogous to the so called coherent states encountered in quantum physics [A170] . In particular, the eigenfunctions are in general non-orthogonal and this is a quintessential element of the the proposed strategy of proof.

In the following an explicit operator having as its eigenvalues the non-trivial zeros of ζ is constructed.

1. The construction relies crucially on the interpretation of the vanishing of ζ as an orthogonality condition in a hermitian metric which is is a priori more general than Hilbert space inner product.

2. Second basic element is the scaling invariance motivated by the belief that ζ is associated with a physical system which has super-conformal transformations [A126] as its symmetries.

The core elements of the construction are following.

1. All complex numbers are candidates for the eigenvalues of D^+ (formal hermitian conjugate of D) and genuine eigenvalues are selected by the requirement that the condition $D^\dagger = D^+$ holds true in the set of the genuine eigenfunctions. This condition is equivalent with the hermiticity of the metric defined by a function proportional to ζ.

2. The eigenvalues turn out to consist of $z = 0$ and the non-trivial zeros of ζ and only the eigenfunctions corresponding to the zeros with $Re[s] = 1/2$ define a subspace possessing a

hermitian metric. The vanishing of ζ tells that the 'physical' positive norm eigenfunctions (in general *not* orthogonal to each other), are orthogonal to the 'un-physical' negative norm eigenfunction associated with the eigenvalue $z = 0$.

The proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the space $\mathcal{V}$ spanned by the states corresponding to the zeros of ζ inside the critical strip has a hermitian induced metric. Riemann hypothesis follows also from the requirement that the induced metric in the spaces subspaces $\mathcal{V}_s$ of $\mathcal{V}$ spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$ does not possess negative eigenvalues: this condition is equivalent with the positive definiteness of the metric in $\mathcal{V}$. Conformal invariance in the sense of gauge invariance allows only the states belonging to $\mathcal{V}$. Riemann hypothesis follows also from a restricted form of a dynamical conformal invariance in $\mathcal{V}$. This allows the reduction of the proof to a standard analytic argument used in Lie-group theory.

9.3.1 Modified form of the Hilbert-Polya conjecture

One can modify the Hilbert-Polya conjecture by assuming scaling invariance and giving up the hermiticity of the Hilbert-Polya operator. This means introduction of the non-hermitian operators D^+ and D which are hermitian conjugates of each other such that D^+ has the nontrivial zeros of ζ as its complex eigenvalues

$$D^+\Psi = z\Psi. \tag{9.3.1}$$

The counterparts of the so called coherent states [A170] are in question and the eigenfunctions of D^+ are not expected to be orthogonal in general. The following construction is based on the idea that D^+ also allows the eigenvalue $z = 0$ and that the vanishing of ζ at z expresses the orthogonality of the states with eigenvalue $z = x + iy \neq 0$ and the state with eigenvalue $z = 0$ which turns out to have a negative norm.

The trial

$$\begin{aligned} D &= L_0 + V, \quad D^+ = -L_0 + V \\ L_0 &= t\tfrac{d}{dt}, \qquad V = \tfrac{dlog(F)}{d(log(t))} = t\tfrac{dF}{dt}\tfrac{1}{F} \end{aligned} \tag{9.3.2}$$

is motivated by the requirement of invariance with respect to scalings $t \to \lambda t$ and $F \to \lambda F$. The range of variation for the variable t consists of non-negative real numbers $t \geq 0$. The scaling invariance implying conformal invariance (Virasoro generator L_0 represents scaling which plays a fundamental role in the super-conformal theories [A126]) is motivated by the belief that ζ codes for the physics of a quantum critical system having, not only super-symmetries [A119] , but also super-conformal transformations as its basic symmetries.

9.3.2 Formal solution of the eigenvalue equation for operator D^+

One can formally solve the eigenvalue equation

$$D^+\Psi_z = \left[-t\frac{d}{dt} + t\frac{dF}{dt}\frac{1}{F}\right]\Psi_z = z\Psi_z. \tag{9.3.3}$$

for D^+ by factoring the eigenfunction to a product:

$$\Psi_z = f_z F. \tag{9.3.4}$$

The substitution into the eigenvalue equation gives

$$L_0 f_z = t\frac{d}{dt}f_z = -zf_z \tag{9.3.5}$$

allowing as its solution the functions

$$f_z(t) = t^z. \tag{9.3.6}$$

These functions are nothing but eigenfunctions of the scaling operator L_0 of the super-conformal algebra analogous to the eigen states of a translation operator. A priori all complex numbers z are candidates for the eigenvalues of D^+ and one must select the genuine eigenvalues by applying the requirement $D^\dagger = D^+$ in the space spanned by the genuine eigenfunctions.

It must be emphasized that Ψ_z is *not* an eigenfunction of D. Indeed, one has

$$D\Psi_z = -D^+\Psi_z + 2V\Psi_z = z\Psi_z + 2V\Psi_z. \tag{9.3.7}$$

This is in accordance with the analogy with the coherent states which are eigen states of annihilation operator but not those of creation operator.

9.3.3 $D^+ = D^\dagger$ condition and hermitian form

The requirement that D^+ is indeed the hermitian conjugate of D implies that the hermitian form satisfies

$$\langle f|D^+g\rangle = \langle Df|g\rangle. \tag{9.3.8}$$

This condition implies

$$\langle \Psi_{z_1}|D^+\Psi_{z_2}\rangle = \langle D\Psi_{z_1}|\Psi_{z_2}\rangle. \tag{9.3.9}$$

The first (not quite correct) guess is that the hermitian form is defined as an integral of the product $\overline{\Psi}_{z_1}\Psi_{z_2}$ of the eigenfunctions of the operator D over the non-negative real axis using a suitable integration measure. The hermitian form can be defined by continuing the integrand from the non-negative real axis to the entire complex t-plane and noticing that it has a cut along the non-negative real axis. This suggests the definition of the hermitian form, not as a mere integral over the non-negative real axis, but as a contour integral along curve C defined so that it encloses the non-negative real axis, that is C

1. traverses the non-negative real axis along the line $Im[t] = 0_-$ from $t = \infty + i0_-$ to $t = 0_+ + i0_-$,
2. encircles the origin around a small circle from $t = 0_+ + i0_-$ to $t = 0_+ + i0_+$,
3. traverses the non-negative real axis along the line $Im[t] = 0_+$ from $t = 0_+ + i0_+$ to $t = \infty + i0_+$.

   Here $0_\pm$ signifies taking the limit $x = \pm\epsilon$, $\epsilon > 0$, $\epsilon \to 0$.

C is the correct choice if the integrand defining the inner product approaches zero sufficiently fast at the limit $Re[t] \to \infty$. Otherwise one must assume that the integration contour continues along the circle S_R of radius $R \to \infty$ back to $t = \infty + i0_-$ to form a closed contour. It however turns out that this is not necessary. One can deform the integration contour rather freely: the only constraint is that the deformed integration contour does not cross over any cut or pole associated with the analytic continuation of the integrand from the non-negative real axis to the entire complex plane.

Scaling invariance dictates the form of the integration measure appearing in the hermitian form uniquely to be dt/t. The hermitian form thus obtained also makes possible to satisfy the crucial $D^+ = D^\dagger$ condition. The hermitian form is thus defined as

$$\langle \Psi_{z_1}|\Psi_{z_2}\rangle = -\frac{K(z_{12})}{2\pi i}\int_C \overline{\Psi_{z_1}}\Psi_{z_2}\frac{dt}{t}. \tag{9.3.10}$$

$K(z_{12})$ is real from the hermiticity requirement and the behavior as a function of $z_{12} = z_1 + \overline{z}_2$ by the requirement that the resulting Hermitian form defines a positive definite inner product. The value of $K(1)$ can can be fixed by requiring that the states corresponding to the zeros of ζ at the critical line have unit norm: with this choice the vacuum state corresponding to $z = 0$ has negative norm. Physical intuition suggests that $K(z_{12})$ is responsible for the Gaussian overlaps of the coherent states and this suggests the behavior

$$K(z_{12}) = exp(-\alpha|z_{12}|^2), \qquad (9.3.11)$$

for which overlaps between states at critical line are proportional to $exp(-\alpha(y_1 - y_2)^2)$ so that for $\alpha > 0$ Schwartz inequalities are certainly satisfied for large values of $|y_{12}|$. Small values of y_{12} are dangerous in this respect but since the matrix elements of the metric decrease for small values of y_{12} even for $K(z_{12}) = 1$, it is possible to satisfy Schwartz inequalities for sufficiently large value of α. It must be emphasized that the detailed behavior of K is *not* crucial for the arguments relating to Riemann hypothesis.

The possibility to deform the shape of C in wide limits realizes conformal invariance stating that the change of the shape of the integration contour induced by a conformal transformation, which is nonsingular inside the integration contour, leaves the value of the contour integral of an analytic function unchanged. This scaling invariant hermitian form is indeed a correct guess. By applying partial integration one can write

$$\langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = \langle D\Psi_{z_1} | \Psi_{z_2} \rangle - \frac{K(z_{12})}{2\pi i} \int_C dt \frac{d}{dt} \left[\overline{\Psi}_{z_1}(t) \Psi_{z_2}(t) \right] . \qquad (9.3.12)$$

The integral of a total differential comes from the operator $L_0 = td/dt$ and must vanish. For a non-closed integration contour C the boundary terms from the partial integration could spoil the $D^+ = D^\dagger$ condition unless the eigenfunctions vanish at the end points of the integration contour ($t = \infty + i0_\pm$).

The explicit expression of the hermitian form is given by

$$\begin{aligned} \langle \Psi_{z_1} | \Psi_{z_2} \rangle &= -\frac{K(z_{12})}{2\pi i} \int_C \frac{dt}{t} F^2(t) t^{z_{12}} , \\ z_{12} &= \overline{z}_1 + z_2 . \end{aligned} \qquad (9.3.12)$$

It must be emphasized that it is $\overline{\Psi}_{z_1} \Psi_{z_2}$ rather than eigenfunctions which is continued from the non-negative real axis to the complex t-plane: therefore one indeed obtains an analytic function as a result.

An essential role in the argument claimed to prove the Riemann hypothesis is played by the crossing symmetry

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \langle \Psi_0 | \Psi_{\overline{z}_1 + z_2} \rangle \qquad (9.3.13)$$

of the hermitian form. This symmetry is analogous to the crossing symmetry of particle physics stating that the S-matrix is symmetric with respect to the replacement of the particles in the initial state with their antiparticles in the final state or vice versa [A170] .

The hermiticity of the hermitian form implies

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle} . \qquad (9.3.14)$$

This condition, which is *not* trivially satisfied, in fact determines the eigenvalue spectrum.

9.3.4 How to choose the function F?

The remaining task is to choose the function F in such a manner that the orthogonality conditions for the solutions Ψ_0 and Ψ_z reduce to the condition that ζ or some function proportional to ζ vanishes at the point $-z$. The definition of ζ based on analytical continuation performed by Riemann suggests how to proceed. Recall that the expression of ζ converging in the region $Re[s] > 1$ following from the basic definition of ζ and elementary properties of Γ function [A214] reads as

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t)]}t^s . \tag{9.3.15}$$

One can analytically continue this expression to a function defined in the entire complex plane by noticing that the integrand is discontinuous along the cut extending from $t=0$ to $t=\infty$. Following Riemann it is however more convenient to consider the discontinuity for a function obtained by multiplying the integrand with the factor

$$(-1)^s \equiv exp(-i\pi s) .$$

The discontinuity $Disc(f) \equiv f(t) - f(texp(i2\pi))$ of the resulting function is given by

$$Disc\left[\frac{exp(-t)}{[1-exp(-t)]}(-t)^{s-1}\right] = -2isin(\pi s)\frac{exp(-t)}{[1-exp(-t)]}t^{s-1} . \tag{9.3.16}$$

The discontinuity vanishes at the limit $t \to 0$ for $Re[s] > 1$. Hence one can define ζ by modifying the integration contour from the non-negative real axis to an integration contour C enclosing non-negative real axis defined in the previous section.

This amounts to writing the analytical continuation of $\zeta(s)$ in the form

$$-2i\Gamma(s)\zeta(s)sin(\pi s) = \int_C \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t)]}(-t)^{s-1} . \tag{9.3.17}$$

This expression equals to $\zeta(s)$ for $Re[s] > 1$ and defines $\zeta(s)$ in the entire complex plane since the integral around the origin eliminates the singularity.

The crucial observation is that the integrand on the righthand side of Eq. 9.3.17 has precisely the same general form as that appearing in the hermitian form defined in Eq. 9.3.12 defined using the same integration contour C. The integration measure is dt/t, the factor t^s is of the same form as the factor $t^{\overline{z}_1+z_2}$ appearing in the hermitian form, and the function $F^2(t)$ is given by

$$F^2(t) = \frac{exp(-t)}{1-exp(-t)} .$$

Therefore one can make the identification

$$F(t) = \left[\frac{exp(-t)}{1-exp(-t)}\right]^{1/2} . \tag{9.3.18}$$

Note that the argument of the square root is non-negative on the non-negative real axis and that $F(t)$ decays exponentially on the non-negative real axis and has $1/\sqrt{t}$ type singularity at origin. From this it follows that the eigenfunctions $\Psi_z(t)$ approach zero exponentially at the limit $Re[t] \to \infty$ so that one can use the non-closed integration contour C.

With this assumption, the hermitian form reduces to the expression

$$\begin{aligned}\langle \Psi_{z_1}|\Psi_{z_2}\rangle &= -\frac{K(z_{12})}{2\pi i}\int_C \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t]}(-t)^{z_{12}} \\ &= \frac{K(z_{12})}{\pi}sin(\pi z_{12})\Gamma(z_{12})\zeta(z_{12}) .\end{aligned} \tag{9.3.17}$$

Recall that the definition $z_{12} = \overline{z}_1 + z_2$ is adopted. Thus the orthogonality of the eigenfunctions is equivalent to the vanishing of $\zeta(z_{12})$ if $K(z_{12})$ is positive definite.

9.3.5 Study of the hermiticity condition

In order to derive information about the spectrum one must explicitly study what the statement that $D^\dagger$ is hermitian conjugate of D means. The defining equation is just the generalization of the equation

$$A^\dagger_{mn} = \overline{A}_{nm} . \tag{9.3.18}$$

defining the notion of hermiticity for matrices. Now indices m and n correspond to the eigenfunctions Ψ_{z_i}, and one obtains

$$\langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = z_2 \langle \Psi_{z_1} | \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | D \Psi_{z_1} \rangle} = \overline{\langle D^+ \Psi_{z_2} | \Psi_{z_1} \rangle} = z_2 \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle} .$$

Thus one has

$$\begin{aligned} G(z_{12}) &= \overline{G(z_{21})} = \overline{G(\overline{z}_{12})} \\ G(z_{12}) &\equiv \langle \Psi_{z_1} | \Psi_{z_2} \rangle . \end{aligned} \tag{9.3.18}$$

The condition states that the hermitian form defined by the contour integral is indeed hermitian. This is *not* trivially true. Hermiticity condition obviously determines the spectrum of the eigenvalues of D^+.

To see the implications of the hermiticity condition, one must study the behavior of the function $G(z_{12})$ under complex conjugation of both the argument and the value of the function itself. To achieve this one must write the integral

$$G(z_{12}) = -\frac{K(z_{12})}{2\pi i} \int_C \frac{dt}{t} \frac{exp(-t)}{[1 - exp(-t)]} (-t)^{z_{12}}$$

in a form from which one can easily deduce the behavior of this function under complex conjugation. To achieve this, one must perform the change $t \to u = log(exp(-i\pi)t)$ of the integration variable giving

$$G(z_{12}) = -\frac{K(z_{12})}{2\pi i} \int_D du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12} u) . \tag{9.3.18}$$

Here D denotes the image of the integration contour C under $t \to u = log(-t)$. D is a fork-like contour which

1. traverses the line $Im[u] = i\pi$ from $u = \infty + i\pi$ to $u = -\infty + i\pi$,
2. continues from $-\infty + i\pi$ to $-\infty - i\pi$ along the imaginary u-axis (it is easy to see that the contribution from this part of the contour vanishes),
3. traverses the real u-axis from $u = -\infty - i\pi$ to $u = \infty - i\pi$.

The integrand differs on the line $Im[u] = \pm i\pi$ from that on the line $Im[u] = 0$ by the factor $exp(\mp i\pi z_{12})$ so that one can write $G(z_{12})$ as integral over real u-axis

$$G(z_{12}) = -\frac{K(z_{12})}{\pi} sin(\pi z_{12}) \int_{-\infty}^{\infty} du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12} u) . \tag{9.3.18}$$

From this form the effect of the transformation $G(z) \to \overline{G(\overline{z})}$ can be deduced. Since the integral is along the real u-axis, complex conjugation amounts only to the replacement $z_{21} \to z_{12}$, and one has

$$\begin{aligned}
\overline{G(\overline{z}_{12})} &= -\frac{\overline{K(z_{21})}}{\pi} \times \overline{sin(\pi z_{21})} \int_{-\infty}^{\infty} du \frac{exp(-exp(u))}{[1-exp(-(exp(u)))]} exp(z_{12}u) \\
&= \frac{\overline{K(z_{21})}}{K(z_{12})} \times \frac{\overline{sin(\pi z_{21})}}{sin(\pi z_{12})} G(z_{12}) . \qquad (9.3.18)
\end{aligned}$$

Thus the hermiticity condition reduces to the condition

$$G(z_{12}) = \frac{\overline{K(z_{21})}}{K(z_{12})} \times \frac{\overline{sin(\pi z_{21})}}{sin(\pi z_{12})} \times G(z_{12}) . \qquad (9.3.19)$$

The reality of $K(z_{12})$ guarantees that the diagonal matrix elements of the metric are real.

For non-diagonal matrix elements there are two manners to satisfy the hermiticity condition.

1. The condition

$$G(z_{12}) = 0 \qquad (9.3.20)$$

   is the only manner to satisfy the hermiticity condition for $x_1 + x_2 \neq n$, $y_1 - y_2 \neq 0$. This implies the vanishing of ζ:

$$\zeta(z_{12}) = 0 \text{ for } 0 < x_1 + x_2 < 1 . \qquad (9.3.21)$$

   In particular, this condition must be true for $z_1 = 0$ and $z_2 = 1/2 + iy$. Hence the physical states with the eigenvalue $z = 1/2 + iy$ must correspond to the zeros of ζ.

2. For the non-diagonal matrix elements of the metric the condition

$$exp(i\pi(x_1 + x_2)) = \pm 1 \qquad (9.3.22)$$

   guarantees the reality of $sin(\pi z_{12})$ factors. This requires

$$x_1 + x_2 = n . \qquad (9.3.23)$$

   The highly non-trivial implication is that the the vacuum state Ψ_0 and the zeros of ζ at the critical line span a space having a hermitian inner product. Note that for $x_1 = x_2 = n/2$, $n \neq 1$, the diagonal matrix elements of the metric vanish.

3. The metric is positive definite only if the function $K(z_{12})$ decays sufficiently fast: this is due to the exponential increase of the moduli of the matrix elements $G(1/2 + iy_1, 1/2 + iy_2)$ for $K(z_{12}) = 1$ and for large values of $|y_1 - y_2|$ (basically due to the $sinh\,[\pi\,(y_1 - y_2)]$-factor in the metric) implying the failure of the Schwartz inequality for $|y_1 - y_2| \to \infty$. Unitarity, guaranteeing probability interpretation in quantum theory, thus requires that the parameter α characterizing the Gaussian decay of $K(z_{12}) = exp(-\alpha|z_{12}|^2)$ is above some minimum value.

9.3.6 Various assumptions implying Riemann hypothesis

As found, the general strategy for proving the Riemann hypothesis, originally inspired by super-conformal invariance, leads to the construction of a set of eigen states for an operator D^+, which is effectively an annihilation operator acting in the space of complex-valued functions defined on the real half-line. Physically the states are analogous to coherent states and are not orthogonal to each other. The quantization of the eigenvalues for the operator D^+ follows from the requirement that the metric, which is defined by the integral defining the analytical continuation of ζ, and thus proportional to ζ ($\langle s_1, s_2\rangle \propto \zeta(\overline{s}_1 + s_2)$), is hermitian in the space of the physical states.

The nontrivial zeros of ζ are known to belong to the critical strip defined by $0 < Re[s] < 1$. Indeed, the theorem of Hadamard and de la Vallee Poussin [A7] states the non-vanishing of ζ on the line $Re[s] = 1$. If s is a zero of ζ inside the critical strip, then also $1 - \overline{s}$ as well as $\overline{s}$ and $1 - s$ are zeros. If Hilbert space inner product property is not required so that the eigenvalues of the metric tensor can be also negative in this subspace. There could be also un-physical zeros of ζ outside the critical line $Re[s] = 1/2$ but inside the critical strip $0 < Re[s] < 1$. The problem is to find whether the zeros outside the critical line are excluded, not only by the hermiticity but also by the positive definiteness of the metric necessary for the physical interpretation, and perhaps also by conformal invariance posed in some sense as a dynamical symmetry. This turns out to be the case.

Before continuing it is convenient to introduce some notations. Denote by $\mathcal{V}$ the subspace spanned by Ψ_s corresponding to the zeros s of ζ inside the critical strip, by $\mathcal{V}_{crit}$ the subspace corresponding to the zeros of ζ at the critical strip, and by $\mathcal{V}_s$ the space spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$. The basic idea behind the following proposals is that the basic objects of study are the spaces $\mathcal{V}$, $\mathcal{V}_{crit}$ and $\mathcal{V}_s$.

How to restrict the metric to $\mathcal{V}$?

One should somehow restrict the metric defined in the space spanned by the states Ψ_s labeled by a continuous complex eigenvalue s to the space $\mathcal{V}$ inside the critical strip spanned by a basis labeled by discrete eigenvalues. Very naively, one could try to do this by simply putting all other components of the metric to zero so that the states outside $\mathcal{V}$ correspond to gauge degrees of freedom. This is consistent with the interpretation of $\mathcal{V}$ as a coset space formed by identifying states which differ from each other by the addition of a superposition of states which do not correspond to zeros of ζ.

An more elegant manner to realize the restriction of the metric to $\mathcal{V}$ is to Fourier expand states in the basis labeled by a complex number s and define the metric in $\mathcal{V}$ using double Fourier integral over the complex plane and Dirac delta function restricting the labels of both states to the set of zeros inside the critical strip:

$$\begin{aligned}
\langle \Psi^{1)} | \Psi^{2)} \rangle &= \int d\mu(s_1) \int d\mu(s_2) \overline{\Psi}^{1)}_{s_1} \Psi^{2)}_{s_2} G(s_2 + \overline{s}_1) \delta(\zeta(s_1)) \delta(\zeta(s_2)) \\
&= \sum_{\zeta(s_1)=0, \zeta(s_2)=0} \overline{\Psi}^{1)}_{s_1} \Psi^{2)}_{s_2} G(s_2 + \overline{s}_1) \frac{1}{\sqrt{det(s_2) det(\overline{s}_1)}} , \\
d\mu(s) &= ds d\overline{s}, \qquad det(s) = \frac{\partial(Re\,[\zeta(s)]\,, Im\,[\zeta(s)])}{\partial(Re\,[s]\,, Im\,[s])} . \qquad (9.3.21)
\end{aligned}$$

Here the integrations are over the critical strip. $det(s)$ is the Jacobian for the map $s \to \zeta(s)$ at s. The appearance of the determinants might be crucial for the absence of negative norm states. The result means that the metric $G_{\mathcal{V}}$ in $\mathcal{V}$ effectively reduces to a product

$$\begin{aligned}
G_{\mathcal{V}} &= \overline{D} G D , \\
D(s_i, s_j) &= D(s_i) \delta(s_i, s_j) , \\
\overline{D}(s_i, s_j) &= D(\overline{s_i}) \delta(s_i, s_j) \\
D(s) &= \frac{1}{\sqrt{det(s)}} . \qquad (9.3.19)
\end{aligned}$$

In the sequel the metric G will be called reduced metric whereas $G_{\mathcal{V}}$ will be called the full metric. In fact, the symmetry $D(s) = D(\overline{s})$ holds true by the basic symmetries of ζ so that one has $D = \overline{D}$ and $G_{\mathcal{V}} = DGD$. This means that Schwartz inequalities for the eigen states of D^+ are not affected in the replacement of $G_{\mathcal{V}}$ with G. The two metrics can be in fact transformed to each other by a mere scaling of the eigen states and are in this sense equivalent.

Riemann hypothesis from the hermicity of the metric in $\mathcal{V}$

The mere requirement that the metric is hermitian in $\mathcal{V}$ implies the Riemann hypothesis. This can be seen in the simplest manner as follows. Besides the zeros at the critical line $Re[s] = 1/2$ also the symmetrically related zeros inside critical strip have positive norm squared but they do not have hermitian inner products with the states at the critical line unless one assumes that the inner product vanishes. The assumption that the inner products between the states at critical line and outside it vanish, implies additional zeros of ζ and, by repeating the argument again and again, one can fill the entire critical interval $(0, 1)$ with the zeros of ζ so that a reductio ad absurdum proof for the Riemann hypothesis results. Thus the metric gives for the states corresponding to the zeros of the Riemann Zeta at the critical line a special status as what might be called physical states.

It should be noticed that the states in $\mathcal{V}_s$ and $\mathcal{V}_{\overline{s}}$ have non-hermitian inner products for $Re[s] \neq 1/2$ unless these inner products vanish: for $Re[s] > 1/2$ this however implies that ζ has a zero for $Re[s] > 1$.

Riemann hypothesis from the requirement that the metric in $\mathcal{V}$ is positive definite

With a suitable choice of $K(z_{12})$ the metric is positive definite between states having $y_1 \neq y_2$. For s and $1 - \overline{s}$ one has $y_1 = y_2$ implying $K(z_{12}) = 1$ in $\mathcal{V}_s$. Thus the positive definiteness of the metric in $\mathcal{V}$ reduces to that for the induced metric in the spaces $\mathcal{V}_s$. This requirement implies also Riemann hypothesis as following argument shows.

The explicit expression for the norm of a $Re[s] = 1/2$ state with respect to the full metric $G_{\mathcal{V}}^{ind}$ reads as

$$\begin{aligned} G_{\mathcal{V}}^{ind}(1/2 + iy_n, 1/2 + iy_n) &= D^2(1/2 + iy) G^{ind}(1/2 + iy_n, 1/2 + iy_n), \\ G^{ind}(1/2 + iy_n, 1/2 + iy_n) &= -\frac{K(z_{12})}{\pi} sin(\pi)\Gamma(1)\zeta(1). \end{aligned} \tag{9.3.19}$$

Here G^{ind} is the metric in $\mathcal{V}_s$ induced from the reduced metric G. This expression involves formally a product of vanishing and infinite factors and the value of expression must be defined as a limit by taking in $Im[z_{12}]$ to zero. The requirement that the norm squared defined by G^{ind} equals to one fixes the value of $K(1)$:

$$K(1) = -\frac{\pi}{sin(\pi)\zeta(1)} = 1. \tag{9.3.20}$$

The components G^{ind} in $\mathcal{V}_s$ are given by

$$\begin{aligned} G^{ind}(s, s) &= -\frac{sin(2\pi Re[s])\Gamma(2Re[s])\zeta(2Re[s])}{\pi}, \\ G^{ind}(1 - \overline{s}, 1 - \overline{s}) &= -\frac{sin(2\pi(1 - Re[s]))\Gamma(2 - 2Re[s])\zeta(2(1 - [Re[s]))}{\pi}, \\ G^{ind}(s, 1 - \overline{s}) &= G^{ind}(1 - \overline{s}, s) = 1. \end{aligned} \tag{9.3.19}$$

The determinant of the metric $G_{\mathcal{V}}^{ind}$ induced from the full metric reduces to the product

$$Det(G_{\mathcal{V}}^{ind}) = D^2(s))D^2(1 - \overline{s}) \times Det(G^{ind}). \tag{9.3.20}$$

Since the first factor is positive definite, it suffices to study the determinant of G^{ind}. At the limit $Re[s] = 1/2$ G^{ind} formally reduces to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

This reflects the fact that the states Ψ_s and $\Psi_{1-\overline{s}}$ are identical. The actual metric is of course positive definite. For $Re[s] = 0$ the G^{ind} is of the form

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} .$$

The determinant of G^{ind} is negative so that the eigenvalues of both the full metric and reduced metric are of opposite sign. The eigenvalues for G^{ind} are given by $(-1 \pm \sqrt{5})/2$.

The determinant of G^{ind} in $\mathcal{V}_s$ as a function of $Re[s]$ is symmetric with respect to $Re[s] = 1/2$, equals to -1 at the end points $Re[s] = 0$ and $Re[s] = 1$, and vanishes at $Re[s] = 1/2$. Numerical calculation shows that the sign of the determinant of G^{ind} inside the interval $(0,1)$ is negative for $Re[s] \neq 1/2$. Thus the diagonalized form of the induced metric has the signature $(1,-1)$ except at the limit $Re[s] = 1/2$, when the signature formally reduces to $(1,0)$. Thus Riemann hypothesis follows if one can show that the metric induced to $\mathcal{V}_s$ does not allow physical states with a negative norm squared. This requirement is physically very natural. In fact, when the factor $K(z_{12})$ represents sufficiently rapidly vanishing Gaussian, this guarantees the metric to $\mathcal{V}_{crit}$ has only non-negative eigenvalues. Hence the positive-definiteness of the metric, natural if there is real quantum system behind the model, implies Riemann hypothesis.

Riemann hypothesis and conformal invariance

The basic strategy for proving Riemann hypothesis has been based on the attempt to reduce Riemann hypothesis to invariance under conformal algebra or some subalgebra of the conformal algebra in $\mathcal{V}$ or $\mathcal{V}_s$. That this kind of algebra should act as a gauge symmetry associated with ζ is very natural idea since conformal invariance is in a well-defined sense the basic symmetry group of complex analysis.

A physicist aware of the fundamental role of conformal invariance in modern fundamental physics could even consider the possibility of giving for the conformal invariance the status of an axiom of number theory and complex analysis in the sense that the spectrum of zeros of ζ is postulated to be conformally invariant in some sense to be specified. This vision resonates with the vision about physics as generalized number theory.

Consider now one particular strategy based on conformal invariance in the space of the eigen states of D^+.

1. Realization of conformal algebra as a spectrum generating algebra

The conformal generators are realized as operators

$$L_z = t^z D^+ \tag{9.3.21}$$

act in the eigen space of D^+ and obey the standard conformal algebra without central extension [A126] . D^+ itself corresponds to the conformal generator L_0 acting as a scaling. Conformal generators obviously act as dynamical symmetries transforming eigen states of D^+ to each other. What is new is that now conformal weights z have all possible complex values unlike in the standard case in which only integer values are possible. The vacuum state Ψ_0 having negative norm squared is annihilated by the conformal algebra so that the states orthogonal to it (non-trivial zeros of ζ inside the critical strip) form naturally another subspace which should be conformally invariant in some sense. Conformal algebra could act as gauge algebra and some subalgebra of the conformal algebra could act as a dynamical symmetry.

2. Realization of conformal algebra as gauge symmetries?

The definition of the metric in $\mathcal{V}$ involves in an essential manner the mapping $s \to \zeta(s)$. This suggests that one should define the gauge action of the conformal algebra as

$$\begin{aligned}
\Psi_s &\rightarrow \Psi_{\zeta(s)} \rightarrow L_z\Psi_{\zeta(s)} = \zeta(s)\Psi_{\zeta(s)+z} \\
&\rightarrow \zeta(s)\Psi_{\zeta^{-1}(\zeta(s)+z)} .
\end{aligned} \quad (9.3.21)$$

Clearly, the action involves a map of the conformal weight s to $\zeta(s)$, the action of the conformal algebra to $\zeta(s)$, and the mapping of the transformed conformal weight $z+\zeta(s)$ back to the complex plane by the inverse of ζ. For s zero of zeta the action maps Ψ_s to zero.

The inverse image is in general non-unique but in case of $\mathcal{V}$ this does not matter since the action annihilates automatically all states in $\mathcal{V}$. Thus conformal algebra indeed acts as a gauge symmetry. This symmetry does not however force Riemann hypothesis. It would only select zeros of zeta as special conformal weights.

3. Realization of conformal algebra as dynamical symmetries

One can also study the action of the conformal algebra or its suitable sub-algebra in $\mathcal{V}_s$ as a dynamical (as opposed to gauge) symmetry realized as

$$\Psi_s \rightarrow L_z\Psi_s = s\Psi_{s+z} . \quad (9.3.22)$$

Note that this symmetry is different from the above described gauge symmetry. One could regard this dynamical conformal symmetry as a basic axiom about the zeros of zeta and also about number theory.

The states Ψ_s and $\Psi_{1-\overline{s}}$ in $\mathcal{V}_s$ have non-vanishing norms and are obtained from each other by the conformal generators $L_{1-2Re[s]}$ and $L_{2Re[s]-1}$. For $Re[s] \neq 1/2$ the generators $L_{1-2Re[s]}$, $L_{2Re[s]-1}$, and L_0 generate $SL(2,R)$ algebra which is non-compact and generates infinite number of states from the states of $\mathcal{V}_s$. At the critical line this algebra reduces to the abelian algebra spanned by L_0. The requirement that the algebra naturally associated with $\mathcal{V}_s$ is a dynamical symmetry and thus generates only zeros of ζ leads to the conclusion that all points $s + n(1 - 2Re[s])$, n integer, must be zeros of ζ. Clearly, $Re[s] = 1/2$ is the only possibility so that Riemann hypothesis follows. In this case the dynamical symmetry indeed reduces to a gauge symmetry.

There is clearly a connection with the argument based on the requirement that the induced metric in $\mathcal{V}_s$ does not possess negative eigenvalues. Since $SL(2,R)$ algebra acts as the isometries of the induced metric for the zeros having $Re[s] \neq 1/2$, the signature of the induced metric must be $(1,-1)$.

4. Riemann hypothesis from the requirement that infinitesimal isometries exponentiate

One could even try to prove that the entire subalgebra of the conformal algebra spanned by the generators with conformal weights $n(1 - 2Re[s])$ acts as a symmetry generating new zeros of ζ so that corresponding states are annihilated by gauge conformal algebra. If this holds, $Re[s] = 1/2$ is the only possibility so that Riemann hypothesis follows. In this case the dynamical conformal symmetry indeed reduces to a gauge symmetry.

Since $L_{1-2Re[s]}$ acts as an infinitesimal isometry leaving the matrix element $\langle\Psi_0|\Psi_s\rangle = 0$ invariant, one can in spirit of Lie group theory argue that also the exponentiated transformations $exp(tL_{1-2Re[s]})$ have the same property for all values of t. The exponential action leaves Ψ_0 invariant and generates from Ψ_s a superposition of states with conformal weights $s + n(1 - 2Re[s])$, which all must be orthogonal to Ψ_0 since t is arbitrary. Since all zeros are inside the critical strip, $Re[s] = 1/2$ is the only possibility.

A more explicit formulation of this idea is based on a first order differential equation for the integral representation of ζ. One can write the matrix element of the metric using the analytical continuation of $\zeta(s)$:

$$\begin{aligned}
G(s) &= -2i\Gamma(s)\zeta(s)sin(\pi s) = H(s,a)_{|a=0} , \\
H(s,a) &= \int_C \frac{dt}{t}\frac{exp(-t+a(-t)^{1-2x})}{[1-exp(-t)]}(-t)^{x+iy-1} .
\end{aligned} \quad (9.3.22)$$

If $s = x+iy$ is zero of ζ then also $1-x+iy$ is zero of ζ and its is trivial to see that this means the both $H(x+iy,a)$ and its first derivative vanishes at $a=0$:

$$
\begin{aligned}
&H(s,a)_{|a=0} = 0 , \\
&\frac{d}{da}H(s,a)_{|a=0} = 0 .
\end{aligned}
\qquad (9.3.22)
$$

Suppose that $H(s,a)$ satisfies a differential equation of form

$$
\frac{d}{da}H(x+iy,a) = I(x,H(x+iy,a)) , \qquad (9.3.23)
$$

where $I(x,H)$ is some function having no explicit dependence on a so that the differential equation defines an autonomous flow. If the initial conditions of Eq. 9.3.22 are satisfied, this differential equation implies that all derivatives of H vanish which in turn, as it is easy to see, implies that the points $s+m(1-2x)$ are zeros of ζ. This leaves only the possibility $x=1/2$ so that Riemann hypothesis is proven. If I is function of also a, that is $I=I(a,x,H)$, this argument breaks down.

The following argument shows that the system is autonomous. One can solve a as function $a=a(x,H)$ from the Taylor series of H with respect to a by using implicit function theorem, substitute this series to the Taylor series of dH/da with respect to a, and by re-organizing the summation obtain a Taylor series with respect to H with coefficients which depend only on x so that one has $I=I(x,H)$.

5. Conclusions

To sum up, Riemann hypothesis follows from the requirement that the states in $\mathcal{V}$ can be assigned with a conformally invariant physical quantum system. This condition reduces to three mutually equivalent conditions: the metric induced to $\mathcal{V}$ is hermitian; positive definite; allows conformal symmetries as isometries. The hermiticity and positive definiteness properties reduce to the requirement that the dynamical conformal algebra naturally spanned by the states in $\mathcal{V}_s$ reduces to the abelian algebra defined by $L_0=D^+$. If the infinitesimal isometries for the matrix elements $\langle\Psi_0|\Psi_s\rangle=0$ generated by $L_{1-2Re[s]}$ can be exponentiated to isometries as Lie group theory based argument strongly suggests, then Riemann hypothesis follows.

It must be emphasized that all the arguments of this chapter produce Riemann hypothesis from some physically natural looking assumption rather than proving it. Conformal invariance form the spectrum of zeros of zeta in the proposed sense reducing to gauge invariance is perhaps the most natural axiom implying Riemann hypothesis.

9.3.7 Does the Hermitian form define inner product?

Before considering the question whether the Hermitian form defined by G or $G_{\mathcal{V}}$ defines a positive definite Hilbert space inner product, a couple of comments concerning the general properties of the Hermitian form G are in order.

1. The Hermitian form is proportional to the factor

$$
sin(i\pi(y_2-y_1)) \ ,
$$

which vanishes for $y_1=y_2$. For $y_1=y_2$ and $x_1+x_2=1$ ($x_1+x_2=0$) the diverging factor $\zeta(1)$ ($\zeta(0)$) compensates the vanishing of this factor. Therefore the norms of the eigenfunctions Ψ_z with $z=1/2+iy$ must be calculated explicitly from the defining integral. Since the contribution from the cut vanishes in this case, one obtains only an integral along a small circle around the origin. This gives the result

$$
\langle\Psi_{z_1}|\Psi_{z_1}\rangle = K \ \text{ for } \ z_1=\frac{1}{2}+iy \ , \quad \langle\Psi_0|\Psi_0\rangle = -K \ . \qquad (9.3.24)
$$

Thus the norms of the eigenfunctions are finite. For $K=1$ the norms of $z=1/2+iy$ eigenfunctions are equal to one. Ψ_0 has however negative norm -1 so that the Hermitian form in question is not a genuine inner product in the space containing Ψ_0.

2. For $x_1 = x_2 = 1/2$ and $y_1 \neq y_2$ the factor is non-vanishing and one has

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = -\frac{1}{\pi i} \zeta(1 + i(y_2 - y_1)) \Gamma(1 + i(y_2 - y_1)) sinh(\pi(y_2 - y_1)) \ . \tag{9.3.24}$$

The nontrivial zeros of ζ are known to belong to the critical strip defined by $0 < Re[s] < 1$. Indeed, the theorem of Hadamard and de la Vallee Poussin [A7] states the non-vanishing of ζ on the line $Re[s] = 1$. Since the non-trivial zeros of ζ are located symmetrically with respect to the line $Re[s] = 1/2$, this implies that the line $Re[s] = 0$ cannot contain zeros of ζ. This result implies that the states $\Psi_{z=1/2+y}$ are non-orthogonal unless $\Gamma(1 + i(y_2 - y_1))$ vanishes for some pair of eigenfunctions.

It is not at all obvious that the Hermitian form in question defines an inner product in the space spanned by the states Ψ_z, $z = 1/2 + iy$ having real and positive norm. Besides Hermiticity, a necessary condition for this is that Schwartz inequality

$$|\langle \Psi_{z_1} | \Psi_{z_2} \rangle| \leq |\Psi_{z_1}||\Psi_{z_2}|$$

holds true. In case of eigen states of D^+ this condition is not affected by the determinant factors and one can apply it to the metric G. This gives

$$\frac{1}{\pi} |\zeta(1 + iy_{12})| \times |\Gamma(1 + iy_{12})| \times |sin(i\pi y_{12})| \leq 1 \ , \tag{9.3.25}$$

where the shorthand notation $y_{12} = y_2 - y_1$ has been used.

Numerical computation suggests that $\zeta(1 + iy_{12})$ varies in a finite range of values for large values of y_{12} and that $\Gamma(1 + iy)$ behaves essentially as $exp(-\pi y/2)$ asymptotically so that the left hand side increases faster than $exp(\pi y_{12}/2)$ so that Schwartz inequality fails for the eigen states. It took a considerable time do realize that the solution to this difficulty is trivial: the only thing that is needed is to multiply the metric with the factor $K(z_{12})$ introduced already earlier. $K(z_{12})$ is expected to behave like a sufficiently narrow Gaussian on basis of the intuition about the behavior of coherent states.

Possible problems are also caused by the small values of y_{12} for which one might have $|G(1 + iy_{12})| > 1$ implying the failure of the Schwartz inequality

$$|\langle \Psi_{z_1} | \Psi_{z_2} \rangle| \quad \leq \quad |\Psi_{z_1}||\Psi_{z_2}| \tag{9.3.26}$$

characterizing positive definite metric. The direct calculation of $G(1 + iy)$ at the limit $y \to 0$ by using $\zeta(1 + iy) \simeq 1/iy$ however gives

$$G(1) = 1 \ . \tag{9.3.27}$$

By a straightforward calculation one can also verify that $z = 1$ is a local maximum of $|G(z)|$. Note that the Jacobians do not affect the required inequality at all in case of eigen states.

It is easy to see that arbitrary small values of y_{12} are unavoidable. The estimate of Riemann for the number of the zeros of ζ in the interval $Im[s] \in [0, T]$ along the line $Re[s] = 1/2$ reads as

$$N(T) \quad \simeq \quad \frac{T}{2\pi} \left[log(\frac{T}{2\pi}) - 1 \right] \ , \tag{9.3.28}$$

and allows to estimate the average density dN_T/dy of the zeros and to deduce an upper limit for the minimum distance y_{12}^{min} between two zeros in the interval T:

$$\begin{aligned} \frac{dN_T}{dy} &\simeq \frac{1}{2\pi}\left[log(\frac{T}{2\pi})-1\right] \ , \\ y_{12}^{min} &\leq \frac{1}{\frac{dN_T}{dy}} = \frac{2\pi}{\left[log(\frac{T}{2\pi})-1\right]} \rightarrow 0 \ \text{ for } \ T\rightarrow \infty \ . \end{aligned} \tag{9.3.28}$$

This implies that arbitrary small values of y_{12} are unavoidable.

9.3.8 Super-conformal symmetry

Before considering super-conformal symmetry it is good to summarize the basic results obtained hitherto.

1. Conformal invariance as a gauge symmetry is possible only in the space $\mathcal{V}$ spanned by the eigen states associated with the zeros of ζ.
2. The hermiticity of the metric in the space spanned by the eigen states associated with the zeros of ζ is possible only if the zeros are on the critical line.
3. The requirement that the algebra spanned by the generators $L_{2Re[s]-1}$, $L_{1-2Re[s]}$ act as a dynamical symmetry algebra generating new zeros of ζ, forces the zeros to be on the critical line: in this case the generators in question reduce to L_0 and the dynamical symmetry reduces to a gauge symmetry.

One can say that the relationship of the conformal invariance to Riemann hypothesis is understood. Although super-conformal invariance does not seem to bring in anything new in this respect, it is still interesting to look whether conformal symmetry could be generalized to super-conformal symmetry. Certainly the basic idea about the action as gauge symmetry remains the same as well as the manner how subalgebra of conformal algebra acts as a dynamical symmetry algebra.

In the following various approaches to the problem of finding a super-conformal generalization of the dynamical system associated with the Riemann Zeta are discussed.

Simplest variant of the super-conformal symmetry

One can indeed identify a conformal algebra naturally associated with the proposed dynamical system. Note first that the generators of the ordinary conformal algebra

$$L_z = \Psi_z D^+ \tag{9.3.29}$$

generate conformal algebra with commutation relations ($[A,B]\equiv AB-BA$)

$$[L_{z_1}, L_{z_2}] \ = \ (z_2-z_1)L_{z_1+z_2} \ . \tag{9.3.30}$$

Fermionic generators G_z satisfy the following anti-commutation and commutation relations:

$$\{G_{z_1}, G_{z_2}\} = L_{z_1+z_2} \ , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} \ , \ . \tag{9.3.30}$$

An explicit representation for the generators of the algebra extended to a super-algebra is obtained by introducing besides the bosonic coordinate t an anti-commuting coordinate θ. This means that the ordinary complex function algebra is replaced by the function algebra consisting of functions $f(t)+\theta g(t)$.

It is easy to verify that the generators defined as

$$L_z = t^z(D^+ + z\theta d_\theta) \ , \quad G_z = \frac{1}{\sqrt{2}}t^z(d_\theta + \theta D^+) \ . \tag{9.3.30}$$

satisfy the defining commutation and anti-commutation relations of the super conformal algebra. Notice that the definition of the operator $D^+ = L_0$ is not affected at all by the generalization and the eigenfunctions of D^+ come as doubly degenerate pairs consisting of a bosonic state Ψ_z and its fermionic partner $\Psi_z\theta$. Vacuum state however corresponds to the bosonic state since L_z and G_z do not annihilate the fermionic partner of the vacuum state.

The representation of this algebra as a gauge algebra is achieved in exactly the same manner as in the case of the ordinary conformal algebra. The gauge conditions for L_z are satisfied only by the bosonic eigen states so that actually nothing new seems to emerge from this generalization. The counterpart of the algebra generated by $L_{1-2Re[s]}$, $L_{2Re[s]-1}$ and L_0 is obtained by adding the generator G_0. Since any L_z commutes with G_0 the algebra closes. The requirement that this algebra acts as a symmetry in $\mathcal{V}$ implies Riemann hypothesis since the algebra reduces to that generated by L_0 and G_0 on the critical line. The super-symmetric variant of the theory is clearly somewhat disappointing exercise since it does not seem to bring anything genuinely new: even the space of the conformally invariant states remains the same.

Second quantized version of super-conformal symmetry

The following much more complex construction is essentially a construction of a second-quantized super-conformal quantum field theory for the super-symmetric system associated with D^+. It must be emphasized that this construction contains un-necessary complexities. In particular, the introduction of Kac Moody symmetry can be criticized since Kac Moody generators cannot annihilate physical states in the representation of the super-conformal symmetries as gauge symmetries in the space $\mathcal{V}$. It is however perhaps wise to keep also this option since it turn out to be of some value.

The extension of this algebra to super-conformal algebra requires the introduction of the fermionic generators G_z and $G_z^\dagger$. To avoid confusions it must be emphasized that following convention concerning Hermitian conjugation is adopted to make notation more fluent:

$$(O_w)^\dagger = O^\dagger_{\overline{w}} \quad . \tag{9.3.31}$$

Fermionic generators G_z and $G_z^\dagger$ satisfy the following anti-commutation and commutation relations:

$$\{G_{z_1}, G^\dagger_{z_2}\} = L_{z_1+z_2} \quad , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} \quad , \quad \left[L_{z_1}, G^\dagger_{z_2}\right] = -z_2 G^\dagger_{z_1+z_2} \quad . \tag{9.3.31}$$

This definition differs from that used in the standard approach [A126] in that generators G_z and $G_z^\dagger$ are introduced separately. Usually one introduces only the the generators G_n and assumes Hermiticity condition $G_{-n} = G_n^\dagger$. The anti-commutation relations of G_z contain usually also central extension term. Now this term is not present as will be found.

Conformal algebras are accompanied by Kac Moody algebra which results as a central extension of the algebra of the local gauge transformations for some Lie group on circle or line [A126] . In the standard approach Kac Moody generators are Hermitian in the sense that one has $T_{-n} = T_n^\dagger$ [A126] . Now this condition is dropped and one introduces also the generators $T_z^\dagger$. In present case the counterparts for the generators $T_z^\dagger$ of the local gauge transformations act as translations $z_1 \to z_1+z$ in the index space labeling eigenfunctions and geometrically correspond to the multiplication of Ψ_{z_1} with the function t^z

$$T^\dagger_{z_1}\Psi_{z_2} = t^{z_1}\Psi_{z_2} = \Psi_{z_1+z_2} \quad . \tag{9.3.32}$$

These transformations correspond to the isometries of the Hermitian form defined by $G(z_{12})$ and are therefore natural symmetries at the level of the entire space of the eigenfunctions.

The commutation relations with the conformal generators follow from this definition and are given by

$$[L_{z_1}, T_{z_2}] = z_2 T_{z_1+z_2} \quad , \quad \left[L_{z_1}, T^\dagger_{z_2}\right] = -z_2 T^\dagger_{z_1+z_2} \quad , \tag{9.3.33}$$

The central extension making this commutative algebra to Kac-Moody algebra is proportional to the Hermitian metric

$$[T_{z_1}, T_{z_2}] = 0 \ , \quad \left[T^{\dagger}_{z_1}, T^{\dagger}_{z_2}\right] = 0 \ , \quad \left[T^{\dagger}_{z_1}, T_{z_2}\right] = (z_1 - z_2)G(z_1 + z_2) \ . \tag{9.3.34}$$

One could also consider the central extension $\left[T^{\dagger}_{z_1}, T_{z_2}\right] = G(z_1 + z_2)$, which is however not the standard Kac-Moody central extension.

One can extend Kac Moody algebra to a super Kac Moody algebra by adding the fermionic generators Q_z and $Q^{\dagger}_z$ obeying the anti-commutation relations ($\{A, B\} \equiv AB + BA$)

$$\{Q_{z_1}, Q_{z_2}\} = 0 \ , \quad \{Q^{\dagger}_{z_1}, Q^{\dagger}_{z_2}\} = 0 \ , \{Q_{z_1}, Q^{\dagger}_{z_2}\} = G(z_1 + z_2) \ . \tag{9.3.35}$$

Note that also Q_0 has a Hermitian conjugate $Q^{\dagger}_0$, and one has

$$\{Q_0, Q^{\dagger}_0\} \quad = \quad G(0) = -\frac{1}{2} \tag{9.3.36}$$

implying that also the fermionic counterpart of Ψ_0 has negative norm. One can identify the fermionic generators as the gamma matrices of the infinite-dimensional Hermitian space spanned by the eigenfunctions Ψ_z. By their very definition, the complexified gamma matrices $\Gamma_{\overline{z}_1}$ and Γ_{z_2} anti-commute to the Hermitian metric $\langle \Psi_{z_1} | \Psi_{z_2} \rangle = G(\overline{z}_1 + z_2)$.

The commutation relations of the conformal and Kac Moody generators with the fermionic generators are given by

$$\begin{array}{ll} [L_{z_1}, Q_{z_2}] = z_2 Q_{z_1+z_2} \ , & \left[L_{z_1}, Q^{\dagger}_{z_2}\right] = -z_2 Q^{\dagger}_{z_1+z_2} \ , \\ \left[T_{z_1}, Q^{\dagger}_{z_2}\right] = 0 \ , & [T_{z_1}, Q_{z_2}] = 0 \ . \end{array} \tag{9.3.37}$$

The non-vanishing commutation relations of T_z with G_z and non-vanishing anticomutation relations of Q_z with G_z are given by

$$\begin{array}{ll} \left[G_{z_1}, T^{\dagger}_{z_2}\right] = Q_{z_1+z_2} \ , & \left[G^{\dagger}_{z_1}, T_{z_2}\right] = -Q^{\dagger}_{z_1+z_2} \ , \\ \{G_{z_1}, Q^{\dagger}_{z_2}\} = T_{z_1+z_2} \ , & \{G^{\dagger}_{z_1}, Q_{z_2}\} = T^{\dagger}_{z_1+z_2} \ . \end{array} \tag{9.3.38}$$

Super-conformal generators clearly transform bosonic and fermionic Super Kac-Moody generators to each other.

The final step is to construct an explicit representation for the generators G_z and L_z in terms of the Super Kac Moody algebra generators as a generalization of the Sugawara representation [A126] . To achieve this, one must introduce the inverse $G^{-1}(z_a z_b)$ of the metric tensor $G(z_a z_b) \equiv \langle \Psi_{z_a} | \Psi_{z_b} \rangle$, which geometrically corresponds to the contravariant form of the Hermitian metric defined by G. Adopting these notations, one can write the generalization for the Sugawara representation of the super-conformal generators as

$$\begin{aligned} G_z &= \sum_{z_a} T_{z+z_a} G^{z_a z_b} Q^{\dagger}_{z_b} \ , \\ G^{\dagger}_z &= \sum_{z_a} T^{\dagger}_{z+z_a} G^{z_a z_b} Q_{z_b} \ . \end{aligned} \tag{9.3.38}$$

One can easily verify that the commutation and anti-commutation relations with the super Kac-Moody generators are indeed correct. The generators L_z are obtained as the anti-commutators of the generators G_z and $G^{\dagger}_z$. Due to the introduction of the generators T_z, $T^{\dagger}_z$ and G_z, $G^{\dagger}_z$, the anti-commutators $\{G_{z_1}, G^{\dagger}_{z_2}\}$ do not contain any central extension terms. The expressions for the anti-commutators however contains terms of form $T^{\dagger}TQ^{\dagger}Q$ whereas the generators in the usual Sugawara representation contain only bilinears of type $T^{\dagger}T$ and $Q^{\dagger}Q$. The inspiration for introducing the generators T_z,G_z and $T^{\dagger}_z$, $G^{\dagger}_z$ separately comes from the construction of the physical states as generalized super-conformal representations in quantum TGD [K39] . The proposed algebra differs from the standard super-conformal algebra [A126] also in that the indices z are now complex numbers rather than half-integers or integers as in the case of the ordinary super-conformal algebras [A126] . It must be emphasized that one could also consider the commutation relations $\left[T_{z_1}^\dagger, T_{z_2}\right] = iG(z_1+z_2)$ and they might be more the physical choice since $z_2 - z_1$ is now a complex number unlike for ordinary super-conformal representations. It is not however clear how and whether one could construct the counterpart of the Sugawara representation in this case.

Imitating the standard procedure used in the construction of the representations of the super-conformal algebras [A126] , one can assume that the vacuum state is annihilated by *all* generators L_z irrespective of the value of z:

$$L_z|0\rangle = 0 \ , \quad G_z|0\rangle = 0 \ . \tag{9.3.39}$$

That all generators L_z annihilate the vacuum state follows from the representation $L_z = \Psi_z D_+$ because D_+ annihilates Ψ_0. If G_0 annihilates vacuum then also $G_z \propto [L_z, G_0]$ does the same.

The action of $T_z^\dagger$ on an eigenfunction is simply a multiplication by t^z: therefore one cannot require that T_z annihilates the vacuum state as is usually done [A126] . The action of T_0 is multiplication by $t^0 = 1$ so that T^0 and $T_0^\dagger$ act as unit operators in the space of the physical states. In particular,

$$T_0|0\rangle = T_0^\dagger|0\rangle = |0\rangle \ . \tag{9.3.40}$$

This implies the condition

$$\left[T_0, T_z^\dagger\right] = izG(z) = 0 \tag{9.3.41}$$

in the space of the physical states so that physical states must correspond to the zeros of ζ and possibly to $z = 0$. Thus one can generate the physical states from vacuum by acting using operators $Q_z^\dagger$ and $T_z^\dagger$ with $\zeta(z) = 0$. If one requires that the physical states also have real and positive norm squared, only the zeros of ζ on the line $Re[s] = 1/2$ are allowed. Hence the requirement that a unitary representation of the super-conformal algebra is in question, forces Riemann hypothesis.

It is important to notice that $T_z^\dagger$ and $Q\dagger_z$ cannot annihilate the vacuum: this would lead to the condition $G(z_1 + z_2) = 0$ implying the vanishing of $\zeta(z_1 + z_2)$ for any pair $z_1 + z_2$. One can however assume that Q_z annihilates the vacuum state

$$Q_z|0\rangle = 0 \ . \tag{9.3.42}$$

The realization of these conditions in case of super-conformal algebra is achieved by mapping the eigen states Ψ_s to $\Psi_{\zeta(s)}$, acting to these states by the generators of the algebra and mapping the resulting state (which vanishes for zeros of ζ) back to a state proportional to $\Psi_{\zeta^{-1}(\zeta(s)+z)}$. It must be however emphasized that for Kac Moody generators not annihilating the vacuum state the action is not well-defined.

This inspires the hypothesis that only the generators with conformal weights $z = 1/2 + iy$ generate physical states from vacuum realizable in the space of the eigenfunctions Ψ_z and their fermionic counterparts. This means that the action of the bosonic generators $T_{1/2+iy}^\dagger$ and fermionic generators $Q_0^\dagger$ and $Q_{1/2+iy}^\dagger$, as well as the action of the corresponding super-conformal generators $G_{1/2+iy}^\dagger$, generates bosonic and fermionic states with conformal weight $z = 1/2 + iy$ from the vacuum state:

$$|1/2+iy\rangle_B \equiv T_{1/2+iy}^\dagger|0\rangle \ , \quad |1/2+iy\rangle_F \equiv Q_{1/2+iy}^\dagger|0\rangle \ . \tag{9.3.43}$$

One can identify the states generated by the Kac Moody generators $T_z^\dagger$ from the vacuum as the eigenfunctions Ψ_z. The system as a whole represents a second quantized super-symmetric version of the bosonic system defined by the eigenvalue equation for D^+ obtained by assigning to each eigenfunction a fermionic counterpart and performing second quantization as a free quantum field theory.

It should be noticed that the ordinary Super Kac-Moody and super-conformal algebras with generators O_n labeled by integers $n>0$ generate zero norm states from any state $|z\rangle$ with $Re[z]=0$ or $Re[z]=1/2$ ($G(n_1+n_2)=0$). Thus ordinary super-conformal invariance holds true as gauge invariance. It is possible (although perhaps not absolutely necessary) to restrict the real parts of the conformal weights of the generators to be non-negative.

Is the proof of the Riemann hypothesis by reductio ad absurdum possible using second quantized super-conformal invariance?

Riemann hypothesis is proven if all eigenfunctions for which the Riemann Zeta function vanishes, correspond to the states having a real and positive norm squared. The expectation is that super-conformal invariance realized in some sense excludes all zeros of ζ except those on the line $Re[s]=1/2$. The problem is to define precisely what one means with super-conformal invariance and one can generate large number of reduction ad absurdum type proofs depending on how super-conformal invariance is assumed to be realized. The following considerations are completely independent of the already described and more recent realization of the super-conformal gauge invariance by applying ζ and its inverse to the conformal weights of the eigen states. I have kept this material because I feel that it might be unwise to to throw it way yet.

The most conservative option is that super-conformal invariance is realized in the standard sense. The action of the ordinary super-conformal generators L_n, and G_n, $n\neq 0$ on the vacuum states $|0\rangle_{B/F}$ or on any state $|1/2+iy\rangle_{B/F}$ indeed creates zero norm states as is obvious from the vanishing of the factor $sin(\pi z_{12}) = sin(\pi(x_1+x_2))$ associated with the inner inner products of these states. Thus the zeros of ζ define an infinite family of ground states for the representations of the ordinary super-conformal algebra. A generalization of this hypothesis is that the action of L_n and G_n, $n\neq 0$, on any state $|w\rangle_{B/F}$, $\zeta(w)=0$, creates states which are orthogonal zero norm states. This implies $\zeta(n+2Re[w])=0$ for all values of $n\neq 0$ and, since the real axis contains zeros of ζ only at the points $Re[s]=-2n$, $n>0$, leads to a reductio ad absurdum unless one has $Re[w]=1/2$. Thus the proof of the Riemann hypothesis would reduce to showing that the action of the ordinary super-conformal algebra generates mutually orthogonal zero norm states from any state $|w\rangle_{B/F}$ with $\zeta(w)=0$. The proof of this physically plausible hypothesis is not obvious.

One can imagine also other strategies. The minimal requirement is certainly that some subalgebra of the super-conformal algebra generates a space of states satisfying the Hermiticity condition. The quantity

$$\Delta(\overline{w}_1+w_2) \equiv \langle w_1|w_2\rangle - \overline{\langle w_2|w_1\rangle} = G(\overline{w}_1+w_2) - \overline{G(\overline{w}_2+w_1)} \tag{9.3.44}$$

must define the conformal invariant in question since this quantity must vanish in the space of the physical states for which the metric is Hermitian. This requirement does not however imply anything nontrivial for the ordinary conformal algebra having generators L_n and G_n: for $Re[w]\neq 1/2$ the condition is indeed satisfied because $G(n+2Re[w])$ does *not* satisfy the Hermiticity condition for any value of n.

One can try to abstract some property of the states associated with the zeros of ζ on the line $Re[s]=1/2$. The generators $L_{1/2-iy}$ and $G_{1/2-iy}$ generate zero norm states from the states $|1/2+iy\rangle_{B/F}$, when $1/2+iy$ corresponds to the zero of ζ on the line $Re[s]=1/2$. One can try to generalize this observation so that it applies to an arbitrary state $|w\rangle_{B/F}$, $\zeta(w)=0$. The generators $L_{1-\overline{w}}$ and $G_{1-\overline{w}}$ certainly generate zero norm states from the states $|w\rangle_{B/F}$. Also the Hermiticity condition holds true identically and does not have nontrivial implications. One can however consider alternative generalizations by assuming that

1. either the generators $L_{\overline{w}}$ and $G_{\overline{w}}$ or

2. $L_{1/2+iy}$ and $G_{1/2+iy}$ generate from the states $|w\rangle_{B/F}$, $\zeta(w)=0$ states satisfying the Hermiticity condition.

These two hypothesis lead to two versions of a reductio ad absurdum argument. Suppose that w is a zero of ζ. This means that the inner product of the states $Q_0^\dagger|0\rangle$ and $Q_w^\dagger|0\rangle$ and thus also $\Delta(w)$ vanishes:

$$\langle 0|Q_0 Q_w^\dagger|0\rangle = 0 \ , \quad \Delta(w) = 0 \ . \tag{9.3.45}$$

1. By acting on this matrix element by the conformal algebra generator $L_{\overline{w}}$ (which acts like derivative operator on the arguments of the should-be Hermitian form), and using the fact that $L_{\overline{w}}$ annihilates the vacuum state, one obtains

   $$\langle 0|Q_0 Q_{\overline{w}+w}^\dagger|0\rangle = G(w+\overline{w}) \ . \tag{9.3.46}$$

   The requirement $\Delta(w+\overline{w}) = 0$ implies the reality of $G(w+\overline{w})$ and thus the condition $Re[w] = 1/2$ leading to the Riemann hypothesis. Note that the argument implying the reality of $G(w+\overline{w})$ assumes only that L_w annihilates vacuum.

   If this line of approach is correct, the basic challenge would be to show on the basis of the super-conformal invariance alone that the condition $\zeta(w) = 0$ implies that the generators $L_{\overline{w}}$ and $G_{\overline{w}}$ generate new ground states satisfying the Hermiticity condition.

2. An alternative line of argument uses only the invariance under the generators $L_{1/2+iy}$ associated with the zeros of ζ, and thus certainly belonging to the conformal algebra associated with the physical states. By applying the generators $L_{1/2+iy_i}$ to the the matrix element $\langle 0|Q_0 Q_w^\dagger|0\rangle = 0$ and requiring that Hermiticity is respected, one can deduce that $G(w+1/2+iy_i)$ satisfies the Hermiticity condition. Hence the line $Re[s] = Re[w]+1/2$, and by the reflection symmetry also the line $Re[s] = 1/2 - Re[w]$, contain an infinite number of zeros of ζ if one has $Re[w] \neq 1/2$. By repeating this process once for the zeros on the line $Re[s] = 1/2 - Re[w]$, one finds that the lines $Re[s] = 1 - Re[w]$ and $Re[s] = Re[w]$ contain infinite number of the zeros of ζ of form $w_{ij} = w + i(y_i + y_j)$, where y_i and y_j are associated with the zeros of ζ on the line $Re[s] = 1/2$. By applying this two-step procedure repeatedly, one can fill the lines $Re[s] = Re[w]$,$1 - Re[w]$, $1/2 - Re[w]$, $1/2 + Re[w]$ with the zeros of ζ.

9.3.9 What about p-adic version of the modified Hilbert-Polya hypothesis?

The definition of a p-adic counterpart of Riemann zeta function is far from non-trivial. Both the summands n^{-s} of the product representation and factors $(1-p^{-s})^{-1}$ product representation fail to make sense p-adically unless s is integer.

p-Adic analog of Riemann Zeta or more generally p-adic L-function (see http://en.wikipedia.org/wiki/P-adic_L-function) defined as the analog of L-function (see http://en.wikipedia.org/wiki/L-function) can be however defined.

Dirichlet's L-function is defined as a meromorphic analytic continuation to entire complex plane of Dirichlet's L-series defined as a generalization of Riemann zeta:

$$L(n,\chi_N) = \sum_{n=1}^{\infty} \frac{\chi_N(n)}{n^s} \ . \tag{9.3.47}$$

Here χ_N is Dirichlet character (see http://en.wikipedia.org/wiki/Dirichlet_character). Dirichlet characters are multiplicative functions of integer arguments, which are periodic with period N, and satisfy $\chi_N(1) = 1$ and $\chi_N(0) = 0$ except for the trivial characterχ_1 for which one has $\chi_1(0) = 1$. For any a having no common divisors with N $\chi(a)$ is $\Phi(N)$:th root of unity, where $\Phi(N)$ is the totient function (see http://en.wikipedia.org/wiki/Totient_function) counting the number of integers $1 \leq n \leq N$ having no common divisors with N.

Two Dirichlet characters are equivalent if they induce the same Dirichlet character: induction is possible when M divides N and means simply the interpretation of a character χ_M as a character χ_N. Dirichlet characters form a character group. The smallest integer M defining Dirichlet character in a given equivalence class of Dirichlet characters is called conductor.

There are two definitions of the p-adic Riemann zeta due originally to Kubota and Leopoldt and Iwasawa respectively and having quite different origins but they have been shown to be more or less equivalent and only the simpler definition based on algebraic continuation will be summarized below.

Tomio Kubota and Heinrich-Wolfgang Leopoldt gave the first construction of p-adic zeta function by algebraic continuation of the ordinary zeta function from its values for odd negative integers. These values are expressible as in terms of generalized Bernoulli numbers, which are rationals and thus make sense p-adically. The formula is

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n} \ . \tag{9.3.48}$$

Here $B_{n,\chi}$ are the a generalized Bernoulli numbers (see http://en.wikipedia.org/wiki/Generalized_Bernoulli_number) defined by

$$\sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!} = -\sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1} \ . \tag{9.3.49}$$

Here integer f is the above defined conductor.

KubotaLeopoldt p-adic L-function $L_p(s,\chi)$ interpolates the Dirichlet L-function with the Euler factor at p removed so that for positive integers n divisible by $p-1$, one has

$$L_p(1-n,\chi) = (1-\chi(p)p^{n-1})L(1-n,\chi) \ . \tag{9.3.50}$$

The removal of Euler factor at p is necessary to achieve p-adic continuity as is clear from the fact that for $n = n_0 + rp^k$, $0 < r < p-1$, $k \to \infty$, the factor p^{n-1} approaches to zero rather than p^{n_0}.

When n is not divisible by $p-1$, one has a more complex formula

$$L_p(1-n,\chi) = (1-\chi(p)\omega^{-n}p^{n-1})L(1-n,\chi) \ . \tag{9.3.51}$$

$\omega(p)$ is so called Teichmueller character, (see http://en.wikipedia.org/wiki/Teichmueller_character) which is the p-adic analog of Dirichlet character.

p-Adic L-functions define measures (p-adic distributions (on profinite Galois groups with totally disconnected topology (p-adic topology is also totally disconnected)) and are therefore interesting also from TGD point view and might play key role in the construction of p-adic counterparts of quantum states.

Riemann hypothesis - or rather Hilbert-Polya conjecture generalizes to p-adic zeta in the sense that the zeros of p-adic L-function can be regarded as eigenvalues of an operator (see http://en.wikipedia.org/wiki/Riemann_hypothesis). An interesting question whether the coherent state hypothesis generalizes also: in other words, can one regard the vanishing values of p-adic zeta function as vanishing inner products between coherent states labelled by zero of zeta and tachyonic ground state. This would require a definition of p-adic variant of inner product.

9.3.10 Riemann Hypothesis and quasicrystals

Freeman Dyson has represented a highly interesting speculation related to Riemann hypothesis and 1-dimensional quasicrystals (QCs). He discusses QCs and Riemann hypothesis briefly in his Einstein lecture (see http://www.ams.org/notices/200902/rtx090200212p.pdf) [A135].

Dyson begins from the defining property of QC as discrete set of points of Euclidian space for which the spectrum of wave vectors associated with the Fourier transform is also discrete. What this says is that quasicrystal as also ordinary crystal creates discrete diffraction spectrum. This presumably holds true also in higher dimensions than $D = 1$ although Dyson considers mostly $D = 1$ case. Thus QC and its dual would correspond to discrete points sets. I will consider the consequences in TGD framework below.

Dyson considers first QCs at general level. Dyson claims that QCs are possible only in dimensions $D = 1, 2, 3$. I do not know whether this is really the case. In dimension $D = 3$ the known QCs have icosahedral symmetry and there are only very few of them. In 2-D case (Penrose tilings) there is n-fold symmetry, roughly one kind of QC associated with any regular polygon. Penrose tilings correspond to $n = 5$. In 1-D case there is no point group (subgroup of rotation group) and this explains why the number of QCs is infinite. For instance, so called PV numbers identified as algebraic integers, which are roots of any polynomial with integer coefficients such that all other roots have modulus smaller than unity. 1-D QCs is at least as rich a structure as PV numbers and probably much richer.

Dyson suggests that Riemann hypothesis and its generalisations might be proved by studying 1-D quasi-crystals.

1. If Riemann Hypothesis is true, the spectrum for the Fourier transform of the distribution of zeros of Riemann zeta is discrete. The calculations of Andrew Odlycko indeed demonstrate this numerically, which is of course not a proof. From Dyson's explanation I understand that it consists of sums of integer multiples $nlog(p)$ of logarithms of primes meaning that the non-vanishing Fourier components are apart from overall delta function (number of zeros) proportional to

$$F(n) = \sum_{s_k} n^{-is_k} = \hat{\zeta}(is_k) \ , \ s_k = 1/2 + iy_k \ ,$$

where s_k are zeros of Zeta. $\hat{\zeta}$ could be called the dual of zeta with summation over integers replaced with summation over zeros. For other "energies" than $E = log(n)$ the Fourier transform would vanish. One can say that the zeros of Riemann Zeta and primes (or p-adic "energy" spectrum) are dual. Dyson conjectures that each generalized zeta function (or rather, L-function) corresponds to one particular 1-D QC and that Riemann zeta corresponds to one very special 1-D QC.

There are also intriguing connections with TGD which inspire quaternionic generalization of Riemann Zeta and Riemann hypothesis.

1. What is interesting that the same "energy" spectrum (logarithms of positive integers) appears in an arithmetic quantum field theory assignable to what I call infinite primes. An infinite hierarchy of second quantizations of ordinary arithmetic QFT is involved. A the lowest level the Fourier transform of the spectrum of the arithmetic QFT would consist of zeros of zeta rotated by $\pi/2$! The algebraic extensions of rationals and the algebraic integers associated with them define an infinite series of infinite primes and also generalized zeta functions obtained by the generalization of the sum formula. This would suggest a very deep connection with zeta functions, quantum physics, and quasicrystals. These zeta functions could correspond to 1-D QCs.

2. The definition of p-adic manifold (in TGD framework) [K95] forces a discretisation of $M^4 \times CP_2$ having interpretation in terms of finite measurement resolution. This discretization induces also dicretization of space-time surfaces by induction of the manifold structure. The discretisation of M^4 (or E^3) is achieved by crystal lattices, by QCs, and perhaps also by more general discrete structures. Could lattices and QCs be forced by the condition that the lattice like structures defines a discrete distributions with discrete spectrum? But why this?

3. There is also another problem. Integration is a problematic notion in p-adic context and it has turned out that discretization is unavoidable and also natural in finite measurement resolution. The inverse of the Fourier transform however involves integration unless the spectrum of the Fourier transform is discrete so that in both E^3 and corresponding momentum space integration reduces to a summation. This would be achieved if discretisation is by lattice or QC so that one would obtain the desired constraint on discretizations. Thus Riemann hypothesis has excellent mathematical motivations to be true in TGD Universe!

4. What could be the counterpart of Riemann Zeta in the quaternionic case? Quaternionic analog of Zeta suggests itself: formally one can define quaternionic zeta using the same formula as for Riemann zeta.

   (a) Rieman zeta characterizes ordinary integers and s is in this case complex number, extension of reals by adding a imaginary unit. A naive generalization would be that quaternionic zeta characterizes Gaussian integers so that s in the sum $\zeta(s) = \sum n^{-s}$ should be replaced with quaternion and n by Gaussian integer. In octonionic zeta s should be replaced with octonion and n with a quaternionic integer. The sum is well-defined despite the non-commutativity of quaternions (non-associativity of octonions) if the powers n^{-s} are well-defined. Also the analytic continuation to entire quaternion/octonion plane should make sense.

   (b) Could the zeros s_k of quaternionic zeta $\zeta_H(s)$ reside at the 3-D hyper-plane $Re(q) = 1/2$, where $Re(q)$ corresponds to E^4 time coordinate (one must also be able to continue to M^4)? Could the duals of zeros in turn correspond to logarithms $ilog(n)$, n Gaussian integer. The Fourier transform of the 3-D distribution defined by the zeros would in turn be proportional to the dual of $\hat{\zeta}_H(is_k)$ of ζ_H. Same applies to the octonionic zeta.

   (c) The assumption that n is *ordinary* integer in ζ_H trivializes the situation. One obtains the distribution of zeros of ordinary Riemann zeta at each line $s = 1/2 + yI$, I any quaternionic unit and the loci of zeros would correspond to entire 2-spheres. The Fourier spectrum would not be discrete since only the magnitudes of the magnitudes of the quaternionic imaginary parts of "momenta" would be imaginary parts of zeros of Riemann zeta but the direction of momentum would be free. One would not avoid integration in the definition of inverse Fourier transform although the integrand would be constant in angular degrees of freedom.

9.4 Miscellaneous ideas about Riemann hypothesis

This section contains ideas about Riemann hypothesis which I regard as miscellaneous. I took them rather seriously for about more than decade ago but seeing them now makes me blush. I do not however have heart to throw away all these pieces of text away so that "miscellaneous" is a good attribute serving as a warning for the reader.

9.4.1 Universality Principle

The function, what I call $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-s})$ appearing in the product formula have value in the algebraic extension. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in the algebraic extension of the rationals, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle generalizes the original sharpened form of the Riemann hypothesis: the real parts of the phases p^{-iy} are rational. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of Riemann hypothesis becomes however extremely implausible and one could consider the possibility of regarding Riemann Hypothesis as an axiom.

9.4.2 How to understand Riemann hypothesis

The considerations of the preceding subsection lead to the requirement that the logarithmic waves $e^{iKlog(u)}$ exist in all number fields for $u = n$ (and thus for any rational value of u) implying number theoretical quantization of the scaling momenta K. Since the logarithmic waves appear also in Riemann Zeta as the basic building blocks, there is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$.

I have applied two basic strategies in my attempts to understand Riemann hypothesis. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

Some approaches to RH

It is appropriate to list various approaches to RH that I have considered during years.

1. Coherent state approach to RH

In this approach (see the preprint in [L1] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [H2]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

2. The approach based on number theoretical universality

The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what "some" could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1-p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations that one should try to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also rational roots of e. This would imply that also the quantities n^{iy} make sense for all number fields and one can develop Zeta into a p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta would therefore be defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

If $log(p) = q_1 exp(q_2)/\pi$ holds true, then $y = q(y)\pi$ should hold true for the zeros of ζ. In this case one would have

$$p^{iy} = exp\left[iq(y)q_1(p)exp\left(q_2(p)\right)\right] \quad .$$

This quantity exists p-adically if the exponent has p-adic norm smaller than one. $q_1(p)$ is divisible by finite number of primes p_1 so that p^{iy} does not exist in a finite-dimensional extension of R_{p_1} unless $q(y)$ is proportional to a positive power of p_1. Also in this case the multiplication of y by the units defined by infinite primes (to be discussed later) would save the day and would be completely invisible operation in real context.

3. Logarithmic plane waves and Hilbert-Polya conjecture

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

1. At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than

$v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short ($k < -1/2$) or long distances ($k > -1/2$).

2. Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.

3. The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

4. The approach based on zero energy ontology

The approach based on zero energy ontology is the newest one and generalizes the thermodynamical approach by replacing thermodynamics with its square root. The amplitudes p^s define quantities proportional to time-like entanglement coefficients between positive and negative energy parts of a zero energy state having opposite energies given by $\pm log(p)$. The hypothesis that the sum over moduli squared for the coefficients diverges states that the zero energy state is not normalizable and has a physical interpretation as a critical state representing Bose-Einstein condensation. The additional condition that zero of zeta is in question is analogous to the condition $\int \Psi dV = 0$ and should be given a better physical justification. The interpretation as a zero energy counterpart of a coherent state seems to makes sense also now. Note that in ZEO coherent state property is in accordance with energy conservation. In the case of coherent states of Cooper pairs same applies to fermion number conservation. With this interpretation the condition would state orthogonality with respect to the coherent zero energy state characterized by $s = 0$.

Connection with the conjecture of Berry and Keating

The idea that the imaginary parts y for the zeros of Riemann zeta function correspond to eigenvalues of some Hermitian operator H is not new. Berry and Keating [A119] however proposed quite recently that the Hamilton in question is super-symmetric and given by

$$H = xp - \frac{i}{2} . \tag{9.4.1}$$

Here the momentum operator p is defined as $p = -id/dx$ and x has non-negative real values.

H can be indeed expressed as a square $H = Q^2$ of a Hermitian super symmetry generator Q:

$$\begin{aligned}
Q &= \sqrt{i}\,[ix\sigma_1 + p\sigma_2] + \sqrt{\frac{i}{2}}\sigma_3 \ , \\
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \\
\sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\end{aligned} \tag{9.4.-1}$$

By a direct calculation one finds that the following relationship holds true:

$$Q^2 = \begin{pmatrix} xp + \frac{i}{2} & 0 \\ 0 & xp - \frac{i}{2} \end{pmatrix} .$$

The eigen spinors of Q can be written as

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} = x^{-iy} \begin{pmatrix} x^{1/2} \\ \sqrt{\frac{y}{i}} x^{-1/2} \end{pmatrix} .$$

The eigenvalues of Q are $q = \sqrt{y}$. For $y \geq 0$ the eigenvalues are real so that Q is Hermitian when inner product is defined appropriately. Obviously y is eigenvalue of Hamiltonian.

Orthogonality requirement for the solutions of the Dirac equation requires that the inner product reduces to the inner product for plane waves $exp(iu)$, $u = log(x)$. This is achieved if inner product for spinors $\psi_i = (u_i, v_i)$ is defined as

$$\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \frac{dx}{x} \left[\overline{u}_1 v_2 + \overline{v}_1 u_2 \right] . \quad (9.4.-2)$$

In the basis formed by solutions of Dirac equation this inner product is indeed positive definite as one finds by a direct calculation.

The actual spectrum assumed to give the zeros of the Riemann Zeta function however remains open without additional hypothesis. An attractive hypothesis motivated by previous considerations is that the sharpened form of Riemann hypothesis stating that n^{iy} exists for any number field provided finite-dimensional extensions are allowed for the zeros of Riemann zeta function, holds true. This implies that x^{iy} satisfies the same condition for any rational value of x. $x^{\pm 1/2}$ in turn belongs to the infinite-dimensional algebraic extension Q_C^∞ of complex rationals, when x is rational. Therefore the solutions of Dirac equation, being of form $x^{iy} x^{\pm 1/2}$, exist for all number fields for rational values of argument x.

Connection with arithmetic quantum field theory and quantization of time

There is also a very interesting connection with arithmetic quantum field theory and sharpened form of Riemann hypothesis. The Hamiltonian for a bosonic/fermionic arithmetic quantum field theory is given by

$$H = \sum_p log(p) a_p^\dagger a_p . \quad (9.4.-1)$$

where $a_p^\dagger$ and a_p satisfy standard bosonic/fermionic anti-commutation relations

$$\{a_{p_1}^\dagger, a_{p_2}\}_\pm = \delta(p_1, p_2) . \quad (9.4.0)$$

Here $\pm$ refers to anti-commutator/commutator. The sum of Hamiltonians defines super-symmetric arithmetic QFT. The states of the bosonic QFT are in one-one correspondence with non-negative integers and the decomposition of a non-negative integer to powers or prime corresponds to the decomposition of state to many boson states corresponding to various modes p. Analogous statement holds true for fermionic QFT.

The matrix element for the time development operator $U(t) \equiv exp(iHt)$ between states $|m\rangle$ and $|n\rangle$ can be written as

$$\langle m | U(t) | n \rangle = \delta(m, n) n^{it} . \quad (9.4.1)$$

Same form holds true both in bosonic and fermionic QFT:s. These matrix elements are defined for all number fields allowing finite-dimensional extensions if this holds true for n^{it} so that the allowed values of t corresponds to zeros of Riemann Zeta if one accepts Universality Principle. Similar statement holds in the case of fermionic QFT. One can say that the durations for the time evolutions are quantized in a well defined sense and allowed values of time coordinate correspond to the zeros of Riemann zeta function!

The result is very interesting from the point of view of quantum TGD since it would mean that $U(t)$ allows for the preferred values of the time parameter p-adicization ($p \ mod \ 4 = 3$) obtained by mapping the diagonal phases to their p-adic counterparts by phase preserving canonical identification. For phases this map means only the re-interpretation of the rational phase factor as a complexified p-adic number. For these quantized values of the time parameter time evolution operator of the arithmetic quantum field theory makes sense in all p-adic number fields besides complex numbers.

In the case of Berry's super-symmetric Hamiltonian the assumption that p^{iy} exists in all number fields with finite extensions allowed and the requirement that same holds true for the time evolution operator implies that allowed time durations for time evolution are given by $t = log(n)$. This means that there is nice duality between Berry's theory and arithmetic QFT. The allowed time durations (energies) in Berry's theory correspond to energies (allowed time durations) in arithmetic QFT.

9.4.3 Stronger variants for the sharpened form of the Riemann hypothesis

The previous form of the sharpened form of Riemann hypothesis was preceded by conjectures, which were much stronger. The strongest variant of the sharpening is that the phases p^{iy} are complex rational numbers for all primes and for all zeros ζ. A weaker form assumes that these phases belong to the square root allowing infinite-dimensional extension of rationals. Although these conjectures are probably unrealistic, they deserve a brief discussion.

Could the phases p^{iy} exist as complex rationals for the zeros of ζ?

The set $z = n/2 + iy$, $n > 0$ such that p^{-iy} is Pythagorean phase, is the set in which both real Riemann zeta function and the p-adic counterparts of Z_p exist for $p \ mod \ 4 = 3$. They exists also for $p \ mod \ 4 = 1$, if one *defines* $exp(ix) \equiv cos(x) + \sqrt{-1}sin(x)$: $\sqrt{-1}$ would be ordinary p-adic number for $p \ mod \ 4 = 1$. One could also allow phase factors in square root allowing algebraic extension of p-adics.

What is important that $x = 1/2$ is the smallest value of x for which the p-adic counterpart of $Z_B(p, x_p)$ exists. Already Riemann showed that the nontrivial zeros of Riemann Zeta function lie symmetrically around the line $x = 1/2$ in the interval $0 \leq x \leq 1$.

If one assumes that the zeros of Riemann zeta belong to the set at which the p-adic counterparts of Riemann zeta are defined, Riemann hypothesis follows in sharpened form.

1. Sharpened form of Riemann hypothesis does not necessarily exclude zeros with $x = 0$ or $x = 1$ as zeros of Riemann zeta unless they are explicitly excluded. It is however known that the lines $x = 0$ and $x = 1$ do not contains zeros of Riemann Zeta so that sharpened form implies also Riemann hypothesis.

2. The sharpening of the Riemann hypothesis following from p-adic considerations implies that the phases p^{iy} exist as rational complex phases for all values of $p \ mod \ 4 = 3$ when y corresponds to a zero of Riemann Zeta. Obviously the rational phases p^{iy} form a group with respect to multiplication isomorphic with the group of integers in case that y does not vanish. The same is also true for the phases corresponding to integers continuing only powers of primes $p \ mod \ 4 = 3$ phase factor.

3. A stronger form of sharpened hypothesis is that all primes p and all integers are allowed. This would mean that each zero of the Riemann Zeta would generate naturally group isomorphic with the group of integers. Pythagorean phases form a group and should contain this group as a subgroup. It might be that very simple number theoretic considerations exclude this possibility. If not, one would have infinite number of conditions on each zero of Riemann function and much sharper form of Riemann hypothesis which could fix the zeros of Riemann zeta completely:

   The zeros of Riemann Zeta function lie on axis $x = 1/2$ and correspond to values of y such that the phase factor p^{iy} is rational complex number for all values of prime $p \ mod \ 4 = 3$ or perhaps even for all primes p.

Of course, the proposed condition might be quite too strong. A milder condition is that $U_p(x_p)$ is rational for single value of p only: this would mean that the zeros of Riemann Zeta would correspond to Pythagorean angles labeled by primes. One can consider also the possibility that p^{iy} is rational for all y but for some primes only and that these preferred primes correspond to the p-adic primes characterizing the effective p-adic topologies realized in the physical world.

4. If this hypothesis is correct then each zero defines a subgroup of Pythagorean phases and also zeros have a natural group structure. Pythagorean phases contain an infinite number of subgroups generated by integer powers of phase. Each such subgroup has some number N of generators such that the subgroup is generated as products of these phases. From the fact that Pythagorean phases are in a one-one correspondence with rationals, it is obvious that there exists large number of subgroups of this kind. Every zero defines infinite number of Pythagorean phases and there are infinite number of zeros. The entire group generated by the phases is in one-one correspondence with the pairs (p, y).

5. If n^{iy} are rational numbers, there must exist imbedding map f: $(n, y) \rightarrow (r, s)$ from the set of phases n^{iy} to Pythagorean phases characterized by rationals $q = r/s$:

$$(r, s) = (f_1(n, y), f_2(n, y)) \ .$$

The multiplication of Pythagorean phases corresponds to certain map g

$$\begin{aligned}(r_1, s_1) \circ (r_2, s_2) &= [g_1(r_1, s_1; r_2, s_2),\ g_2(r_1, s_1; r_2, s_2)] \\ &= (r_1 r_2 - s_1 s_2, r_1 s_2 + r_2 s_1) \equiv (r, s)\end{aligned}$$

such that the values of r and s associated with the product can be calculated. Thus the product operation rise to functional equations giving constraints on the functional form of the map f.

i) Multiplication of n^{iy_1} and n^{iy_2} gives rise to a condition

$$f(n, y_1) \circ f(n, y_2) = f(n, y_1 + y_2) \ .$$

ii) Multiplication of n_1^{iy} and n_2^{iy} gives rise to a condition

$$f(n_1, y) \circ f(n_2, y) = f(n_1 n_2, y) \ .$$

This variant of the sharpened form of the Riemann hypothesis has turned out to be unnecessarily strong. Universality Principle requires only that the real parts of the factors $p^{-x}p^{-iy}$ are rational numbers: this means that allowed phases correspond to triangles whose two sides have integer-valued length squared whereas the third side has integer-valued length.

Sharpened form of Riemann hypothesis and infinite-dimensional algebraic extension of rationals

The proposed variant for the sharpened form of Riemann hypothesis states that the zeros of Riemann zeta are on the line $x = 1/2$ and that p^{iy}, where p is prime, are complex rational (Pythagorean) phases for zeros. Furthermore, Riemann hypothesis is equivalent with the corresponding statement for the fermionic partition function Z_F. If the sharpened form of Riemann hypothesis holds true, the value of $Z_F(z)$ in the set of zeros $z = 1/2 + iy$ of Z_F can be interpreted as a complex (vanishing) image of certain function $Z_F^{\infty}(1/2 + iy)$ having values in the infinite-dimensional algebraic extension of rationals defined by adding the square roots of all primes to the set of rational numbers.

1. The general element q of the infinite-dimensional extension Q_C^{∞} of complex rationals Q_C can be written as

$$
\begin{aligned}
q &= \sum_U q_U e_U \ , \\
e_U &= \prod_{i \in U} \sqrt{p}_i \ .
\end{aligned} \tag{9.4.1}
$$

Here q_U are complex rational numbers, U runs over the subsets of primes and e_U are the units of the algebraic extension analogous to the imaginary unit. One can map the elements of Q_C^∞ to reals by interpreting the generating units $\sqrt{p}$ as real numbers. The real images $(e_U)_R$ of e_U are thus real numbers:

$$
e_U \rightarrow [e_U]_R = \prod_i \sqrt{p_i} \ .
$$

2. The value of $Z_F(z)$ at $z = 1/2 + iy$ can be written as

$$
Z_F(z = 1/2 + iy) = \sum_U \left[\frac{1}{e_U}\right]_R \times (e_U^2)^{-iy} \ . \tag{9.4.2}
$$

Here $(e_U)_R$ means that e_U are interpreted as real numbers.

3. If one restricts the set of values of $z = 1/2 + iy$ to such values of y that p^{iy} is complex rational for every value of p, then the value of $Z_F(1/2 + iy)$ can be also interpreted as the real image of the value of a function $Z_F(Q_\infty|z = 1/2 + iy)$ restricted to the set of zeros of Riemann zeta and having values at Q_C^∞:

$$
\begin{aligned}
Z_F(1/2 + iy) &= [Z_F(Q_\infty|1/2 + iy)]_R \ , \\
Z_F(Q_\infty|1/2 + iy) &\equiv \sum_U \frac{1}{e_U} \times (e_U^2)^{-iy} \ .
\end{aligned} \tag{9.4.2}
$$

Note that $Z_F(Q_\infty|z = 1/2 + iy)$ cannot vanish as element of Q_∞. One can also define the Q_C^∞ valued counterparts of the partition functions $Z_F(p, 1/2 + iy)$

$$
\begin{aligned}
Z_F(Q_\infty|1/2 + iy) &= \prod_p Z_F(Q_\infty|p, z = 1/2 + iy) \ , \\
Z_F(Q_\infty|1/2 + iy) &\equiv 1 + p^{-1/2}p^{-iy} \ , \\
Z_F(p, 1/2 + iy) &= [Z_F(Q_\infty|p, 1/2 + iy)]_R \ .
\end{aligned} \tag{9.4.1}
$$

$Z_F(Q_\infty|1/2 + iy)$ and $Z_F(Q_\infty|p, 1/2 + iy)$ belong to Q_C^∞ only provided p^{iy} is Pythagorean phase.

4. The requirement that p^{iy} is rational does not yet imply Riemann hypothesis. One can however strengthen this condition. The simplest condition is that the real image of $Z_F(Q_\infty|1/2 + iy)$ is complex rational number for any value of Z_F. A stronger condition is that the complex images of the functions

$$
\frac{Z_F^\infty}{\prod_{p \in U} Z_p^\infty}
$$

are complex rational and U is finite set of primes. The complex counterparts of these functions are given by

$$\left[\frac{Z_F^{\infty}}{\prod_{p\in U} Z_p^{\infty}}\right]_R = \frac{Z_F}{\prod_{p\in U} Z_F(p,..)} \ . \tag{9.4.2}$$

Obviously these conditions can be true only provided that $Z_F(1/2+iy)$ vanishes identically for allowed values of y. This implies that sharpened form of Riemann hypothesis is true. "Physically" this means that the fermionic partition function restricted to any subset of integers not divisible by some finite set of primes, has real counterpart which is complex rational valued.

9.4.4 Are the imaginary parts of the zeros of Zeta linearly independent of not?

Concerning the structure of the weight space of super-symplectic algebra the crucial question is whether the imaginary parts of the zeros of Zeta are linearly independent or not. If they are independent, the space of conformal weights is infinite-dimensional lattice. Otherwise points of this lattice must be identified. The model of the scalar propagator identified as a suitable partition function in the super-symplectic algebra for which the generators have zeros of Riemann Zeta as conformal weights demonstrates that the assumption of linear independence leads to physically unrealistic results and the the propagator does not exist mathematically for the entire super-symplectic algebra. Also the findings about the distribution of zeros of Zeta favor a hypothesis about the structure of zeros implying a linear dependence.

Imaginary parts of non-trivial zeros as additive counterparts of primes?

The natural looking (and probably wrong) working hypothesis is that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent. This would mean that y_i define play the role of primes but with respect to addition instead of multiplication. If there exists no relationship of form $y_i = n2\pi + y_j$, the exponents e^{iy_i} define a multiplicative representation of the additive group, and these factors satisfy the defining condition for primeness in the conventional sense. The inverses e^{-iy_i} are analogous to the inverses of ordinary primes, and the products of the phases are analogous to rational numbers.

There would exist an algebra homomorphism from $\{y_i\}$ to ordinary primes ordered in the obvious manner and defined as the map as $y_i \leftrightarrow p_i$. The beauty of this identification would be that the hierarchies of p-adic cutoffs identifiable in terms of the p-adic length scale hierarchy and y-cutoffs identifiable in terms p-adic phase resolution (the higher the p-adic phase resolution, the higher-dimensional extension of p-adic numbers is needed) would be closely related. The identification would allow to see Riemann Zeta as a function relating two kinds of primes to each other.

A rather general assumption is that the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$\begin{aligned}
p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \ , \\
e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2+s^2} \ , \ r = r(p,y) \ , \ s = s(p,y) \ , \\
e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \ , \ m = m(p,y) \ , \ n = n(p,y) \ .
\end{aligned} \tag{9.4.1}$$

If the Pythagorean phases associated with two different zeros of zeta are different a linear independence over integers follows as a consequence.

Pythagorean phases form a multiplicative group having "prime" phases, which are in one-one correspondence with the squares of Gaussian primes, as its generators and Gaussian primes which are in many-to-one correspondence with primes $p_1 \ mod \ 4 = 1$. If p^{iy} is a product of algebraic phase and Pythagorean phase for any prime p, one should be able to decompose any zero y into two parts $y = y_1(p) + y_P(p)$ such that one has

$$log(p)y_1(p) = \frac{m2\pi}{n} \ , \ log(p)y_P(p) = \Phi_P = arctan\left[\frac{2rs}{r^2+s^2}\right] \ . \tag{9.4.2}$$

Note that the decomposition is not unique without additional conditions. The integers appearing in the formula of course depend on p.

Does the space of zeros factorize to a direct sum of multiples Pythagorean prime phase angles and algebraic phase angles?

As already noticed, the linear independence of the y_i follows if the Pythagorean prime phases associated with different zeros are different. The reverse of this implication holds also true. Suppose that there are two zeros $log(p)y_{1i} = \Phi_{P_1} + q_{1i}2\pi$, $i = a, b$ and two zeros $log(p)y_{2i} = \Phi_{P_2} + q_{2i}2\pi$, $i = a, b$, where q_{ij} are rational numbers. Then the linear combinations $n_1y_{1a} + n_2y_{2a}$ and $n_1y_{1b} + n_2y_{2b}$ represent same zeros if one has $n_1/n_2 = (q_{2a} - q_{2b})/(q_{1b} - q_{1a})$.

One can of course consider the possibility that linear independence holds true only in the weaker sense that one cannot express any zero of zeta as a linear combination of other zeros. For instance, this guarantees that the super-symplectic algebra generated by generators labeled by the zeros has indeed these generates as a minimal set of generating elements.

For instance, one can imagine the possibility that for any prime p a given Pythagorean phase angle $log(p)y_{P_k}$ corresponds to a set of zeros by adding to $\Phi_{P_k} = log(p)y_{P_k}$ rational multiples $q_{k,i}2\pi$ of 2π, where $Q_p(k) = \{q_{k,i}|i = 1, 2, ..\}$ is a subset of rationals so that one obtains subset $\{\Phi_{P_k} + q_{k,i}2\pi|q_{k,i} \in Q_p(k)\}$. Note that the definition of y_P involves an integer multiple of 2π which must be chosen judiciously: for instance, if y_P is taken to be minimal possible (that is in the range $(0, \pi/2)$, one obviously ends up with a contradiction. The same is true if $q_{k,i} < 1$ is assumed. Needless to say, the existence of this kind of decomposition for every prime p is extremely strong number theoretic condition.

The facts that Pythagorean phases are linearly independent and not expressible as a rational multiple of 2π imply that no zero is expressible as a linear combination of other zeros whereas the linear independence fails in a more general sense as already found. An especially interesting situation results if the set $Q_p(k)$ for given p does not depend on the Pythagorean phase so that one can write $Q_p(k) = Q_p$. In this case the set of zeros of Zeta would be obtained as a union of translates of the set Q_p by a subset of Pythagorean phase angles and approximate translational invariance realized in a statistical sense would result. Note that the Pythagorean phases need not correspond to Pythagorean prime phases: what is needed is that a multiple of the same prime phase appears only once.

An attractive interpretation for the existence of this decomposition to Pythagorean and algebraic phases factors for every prime is in terms of the p-adic length scale evolution. The possibility to express the zeros of Zeta in an infinite number of manners labeled by primes could be seen as a number theoretic realization of the renormalization group symmetry of quantum field theories. Primes p define kind of length scale resolution and in each length scale resolution the decomposition of the phases makes sense. This assumption implies the following relationship between the phases associated with y:

$$\frac{\left[\Phi_{P(p_1)} + q(p_1)2\pi\right]}{log(p_1)} = \frac{\left[\Phi_{P(p_2)} + q(p_2)2\pi\right]}{log(p_2)} \ . \tag{9.4.3}$$

In accordance with earlier number theoretical speculations, assume that $log(p_2)/log(p_1) \equiv Q(p_2, p_1)$ is rational. This condition allows to deduce how the phases p_1^{iy} transform in $p_1 \rightarrow p_2$ transformation. Let $p_1^{iy} = U_{P,p_1,y}U_{q,p_1,y}$ be the representation of p_1^{iy} as a product of Pythagorean and algebraic phases. Using the previous equation, one can write

$$p_2^{iy} = U_{P,p_2,y}U_{q,p_2,y} = U_{P,p_1,y}^{Q(p_2,p_1)}U_{q,p_1,y}^{Q(p_2,p_1)} \ . \tag{9.4.4}$$

This means that the phases are mapped to rational powers of phases. In the case of Pythagorean phases this means that Pythagorean phase becomes a product of some Pythagorean and an algebraic phase whereas algebraic phases are mapped to algebraic phases. The requirement that the set of phases p_2^{iy} is same as the set of phases p_1^{iy} implies that the rational power $U_{P,p_1,y}^{Q(p_2,p_1)}$ is proportional to some Pythagorean phase U_{P,p_1,y_1} times algebraic phase U_q such that the the product of $U_q U_{q,p_1,y}^{Q(p_2,p_1)}$ gives an allowed algebraic phase. The map $U_{P,p_1,y} \rightarrow U_{P,p_1,y_1}$ from Pythagorean phases to Pythagorean phases induced in this manner must be one-to one must be the map between algebraic phases. Thus it seems that in principle the hypothesis might make sense.

The basic question is why the phases q^{iy} should exist p-adically in some finite-dimensional extension of R_p for every p. Obviously some function coding for the zeros of Zeta should exist p-adically. The factors $G_q = 1/(1-q^{-iy-1/2})$ of the product representation of Zeta obviously exist if this assumption is made for every prime p but the product is not expected to converge p-adically.

Also the logarithmic derivative of Zeta codes for the zeros and can be written as

$$\frac{\zeta'}{\zeta} = -\sum_q log(q) \frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{9.4.5}$$

As such this function does not exist p-adically but dividing by $log(p)$ one obtains

$$\frac{1}{log(p)} \frac{\zeta'}{\zeta} = -\sum_q Q(q,p) \frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{9.4.6}$$

This function exists if the the p-adic norms rational numbers $Q(q,p)$ approach to zero for $q \rightarrow \infty$: $|Q(q,p)|_p \rightarrow 0$ for $q \rightarrow \infty$. The p-adic existence of the logarithmic derivative would thus give hopes of universal coding for the zeros of Zeta and also give strong constraints to the behavior of the factors $Q(q,p)$. The simplest guess would be $Q(q,p) \propto p^q$ for $q \rightarrow \infty$.

Correlation functions for the spectrum of zeros favors the factorization of the space of zeros

The idea that the imaginary parts of the zeros of Zeta are linearly independent is a very attractive but must be tested against what is known about the distribution of the zeros of Zeta.

There exists numerical evidence for the linear independence of y_i as well as for the hypothesis that the zeros correspond to a union of translates of a basic set Q_1 by subset of Pythagorean phase angles. Lu and Sridhar have studied the correlation among the zeros of ζ [A183] . They consider the correlation functions for the fluctuating part of the spectral function of zeros smoothed out from a sum of delta functions to a sum of Lorentzian peaks. The correlation function between two zeros with a constant distance $K_2 - K_1 + s$ with the first zero in the interval $[K_1, K_1 + \Delta]$ and second zero in the interval $[K_2, K_2 + \Delta]$ is studied. The choice $K_1 = K_2$ assigns a correlation function for single interval at K_1 as a function of distance s between the zeros.

1. The first interesting finding, made already by Berry and Keating, is that the peaks for the negative values of the correlation function correspond to the lowest zeros of Riemann Zeta (only those contained in the interval Δ can appear as minima of correlation function). This phenomenon observed already by Berry and Keating is known as resurgence. That the anti-correlation is maximal when the distance of two zeros corresponds to a low lying zero of zeta can be understood if linear combinations of the zeros of Zeta are the least probable candidates for zeros. Stating it differently, large zeros tend to avoid the points which represent linear combinations of the smaller zeros.

2. Direct numerical support the hypothesis that the correlation function is approximately translationally invariant, which means that it depends on $K_2 - K_1 + s$ only. Correlation function is also independent of the width of the spectral window Δ. In the special $K_1 = K_2$ the finding means that correlation function does not depend at all on the position K_1 of the window and depends only on the variable s. Prophecy means that the correlation function between the interval $[K, K + \Delta]$ and its mirror image $[-K - \Delta, -K]$ is the correlation function for

the interval $[2K + \Delta]$ and depends only on the variable $2K + s$ allowing to allows to deduce information about the distribution of zeros outside the range $[-K, K]$. This property obviously follows from the proposed hypothesis implying that the spectral function is a sum of translates of a basic distribution by a subset of Pythagorean prime phase angles.

This hypothesis is consistent with the properties of the the smoothed out spectral density for the zeros given by

$$\langle\rho(k)\rangle = \frac{1}{2\pi}log(\frac{k}{2\pi}) \ . \tag{9.4.7}$$

This implies that the smoothed out number of zeros y smaller than Y is given by

$$N(Y) = \frac{Y}{2\pi}(log(\frac{Y}{2\pi}) - 1) \ . \tag{9.4.8}$$

$N(Y)$ increases faster than linearly, which is consistent with the assumption that the distribution of zeros with positive imaginary part is sum over translates of a single spectral function ρ_{Q_0} for the rational multiples q_iX_p, $X_p = 2\pi/log(p)$, $q_i \in Q_p$, for every prime p.

If the smoothed out spectral function for $q_i \in Q_p$ is constant:

$$\rho_{Q_p} = \frac{1}{K_p 2\pi} \ , \ K_p > 0 \ , \tag{9.4.9}$$

the number $N_P(Y, p)$ of Pythagorean prime phases increases as

$$N_P(Y|p) = K_p(log(\frac{Y}{2\pi}) - 1) \ , \tag{9.4.10}$$

so that the smoothed out spectral function associated with $N_P(Y|p)$ is given by the function

$$\rho_P(k|p) = \frac{K_p}{k} \tag{9.4.11}$$

for sufficiently large values of k. Therefore the distances between subsequent zeros could quite well correspond to the same Pythagorean phase for a given p and thus should allow to deduce information about the spectral function ρ_{Q_0}. A convenient parameterization of K_p is as $K = K_{p,0}/4\pi^2$ since the points of Q_p are of form $q_i 2\pi = (n(q_i) + q_1(q_i))2\pi$, $q_1 < 1$, and $n(q_i)$ must in the average sense form an evenly spaced subset of reals.

9.5 Universality Principle and Riemann hypothesis

The basic definition of $\zeta(s = x + iy)$ based on the product formula does not converge for $Re[s] \leq 1$. One can however define 'universal' ζ, call it $\hat{\zeta}$, as the product of the partition functions $Z_{p_1}(s) = 1/(1 - p^{-s})$, in the subset of complex plane, where the factors Z_{p_i} are complex algebraic numbers. The idea is to regard the value of $\hat{\zeta}$ as an element of an infinite-dimensional algebraic extension of the rationals containing all roots of primes. $\hat{\zeta}$ can be regarded as a vector with infinite number of components and is completely well defined despite the fact that the product expansion does not converge as an ordinary complex number unless one somehow specifies how the 'producting' is done.

In case that the factors $|Z_{p_1}|^2$ of the partition functions $Z_{p_1} = 1/(1 - p^{-z})$ are complex rationals, one can rewrite the product formula by applying adelic formula to the norm squared $|Z_{p_1}|^2$ appearing in the product formula. The basic hypothesis is that the product of the p-adic norms of the complex norm squared of the function $\hat{\zeta}$ defined by the product formula obtained by changing the order of producting gives the norm squared of the analytically continued ζ in the region $(Re[s] < 1 \ , \ Im[s] \neq 0)$ at the points, where the factors $|Z_{p_1}|^2$ are algebraic numbers:

$|\hat{\zeta}|^2 = \prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$. A milder version of this hypothesis is that the product of the p-adic norms squared of $|\hat{\zeta}|^2$ converges to some function proportional to $|\zeta|^2$.

If this hypothesis is correct, the following vision giving good hopes about the proof of the Riemann hypothesis, suggests itself.

1. $|\hat{\zeta}|^2$ is a number in an infinite-dimensional algebraic extension of rationals and can vanish only if it contains a rational factor which vanishes. The vanishing of this factor is possible if it is a product of an infinite number of moduli squared $|Z_{p_1}(z)|^2$ having a rational value. For the values of y for which this is true on the line $Re[s] = n + 1/2$ correspond to the phases p_1^{-iy} having the following general form.

$$\begin{array}{l} p^{-iy} = U_1 U = \frac{(r_1 + is_1\sqrt{k(p_1,y)})}{\sqrt{p_1}} \times \frac{(r+is\sqrt{k(p_1,y)})}{n_1} \quad , \\ r_1^2 + s_1^2 k(p_1,y) = p_1 \quad , \\ r^2 + s^2 k(p_1,y) = n_1^2 \quad . \end{array}$$

   $r_1^2+s_1^2k(p_1,y) = p_1$ condition is solved by $k(p_1,y) = \sqrt{p_1 - m^2}$, $m < \sqrt{p}$. $r^2+s^2k(p_1,y) = n_1^2$ condition is satisfied if U is a product of even powers of the phases of type U_1. Unless $k(p_1,y)$ is not square, the phases correspond to orthogonal triangles with one short side having integer valued length and the other sides having integer valued length squared.

2. If y defines rational value of $|Z_{p_1}(z)|^2$, also its integer multiples ny do the same. If the values of integers $k(p_1,y)$ do not depend on the value of y, the allowed values of y generate an additive group having integers as a coefficient ring. Even powers of the phases guaranteeing the rationality of $|Z_{p_1}(z)|^2$ on the line $Re[s] = 1/2$, guarantee rationality on the lines $Re[s] = n$.

3. Especially important subset of these phases correspond to the choice $k_1 = 1$. These phases correspond to Gaussian primes having the form $G = r_1 + is_1$, $r_1^2 + s_1^2 = p_1$, $p_1 \ mod \ 4 = 1$, and can compensate the irrationality of the $p_1^{-n-1/2}$ factor only in this case. The products of the squares of Gaussian primes define Pythagorean triangles and the corresponding phases are rational. Rather interestingly, the linear superpositions $y = n_1y_2 + n_2y_2$ of only *two* Pythagorean values of y_i form a dense subset of reals. Eisenstein primes having the general form r_1+s_1w, $w = -1/2\pm\sqrt{3/2}$, $r_1^2+s_1^2-r_1s_1 = p_1$, $p_1 \ mod \ 3 = 1$, are second, probably very important class of complex primes. They can compensate the irrationality of the $p_1^{-n-1/2}$ factor for $p_1 \ mod \ 3 = 1$ (note that the 1/2 is not relevant for the phase). Also other phases are needed since for primes satisfying $p_1 \ mod \ 4 = 3$ and $p_1 \ mod \ 3 = 2$ simultaneously neither Gaussian nor Eisenstein primes can compensate the irrationality of the $p_1^{-1/2}p_1^{-iy}$ factor.

4. The lines on which the real parts for an infinite number of factors Z_{p_1} can be rational, correspond to the lines $Re[s] = n/2$. This in turns leads to the conclusion that the norm squared of $\hat{\zeta}$ can vanish only on the lines $Re[s] = n/2$. If the norm squared of the $\hat{\zeta}$ coincides with the norm squared of the analytically continued ζ, Riemann hypothesis follows since it is known that the lines $Re[s] = n/2$, $n \neq 1$ do not contain zeros of ζ.

In the following this vision is developed in detail and it is shown that it survives the basic tests.

9.5.1 Detailed realization of the Universality Principle

Universality Principle states that ζ vanishes only if $|\hat{\zeta}|^2$ understood as a number in an infinite-dimensional algebraic extension of rationals vanishes and hence must contain a rational factor resulting from an infinite number of rational factors Z_{p_1}. This hypothesis alone makes Riemann hypothesis very plausible. In this section an attempt to reduce the Universality Principle to something more concrete is made. Adelic formula and the hypothesis that the norm of $|\hat{\zeta}|^2$ defined by the modified adelic formula equals to $|\zeta|^2$ are described and shown to imply Universality Principle if the modified adelic formula defines a norm in the infinite-dimensional algebraic extension of rationals. The conditions guaranteeing the rationality and the reduction of the p-adic norm of $|Z_{p_1}|^2$ are derived, and the connection between Pythagorean phases and basic facts about Gaussian and Eisenstein primes are summarized.

Modified adelic formula and Universality Principle

Although the product representation of ζ does not converge absolutely for $Re[s] \leq 1$, one can consider the possibility that the convergence of the function $\hat{\zeta}$ defined by the product representation occurs in some exceptional points in some natural sense. The points at which the value of $\hat{\zeta}$ belongs to the infinite-dimensional algebraic extension of rationals are obviously excellent candidates for these points. $\hat{\zeta}$ identified as an element of this algebraic extension certainly exists mathematically as a vector with an infinite number of components. The convergence in the strong sense would mean that the interpretation of the algebraic numbers of the algebraic extension as real numbers in the expression of $\hat{\zeta}$ gives the analytically continued ζ somehow. In the weak sense the convergence would mean that the complex norm squared for $\hat{\zeta}$, if defined in a suitable sense, equals or is proportional, to the norm squared of the analytically continued ζ.

1. Modified Adelic formula and Universality Principle

The fact that the product formula for ζ at rational points converges only conditionally, suggests that one should be able to device a natural method of 'producting' giving rise to the norm squared of the analytically continued ζ. Adelic formula provides very attractive approach to this problem (the appearance of the norm squared instead of norm is motivated by the Adelic formula).

The adelic formula expresses the real norm of a rational number as a product of the inverses of the p-adic norms

$$\frac{1}{|x|_R} = \prod_p |x|_p \ . \tag{9.5.1}$$

This formula generalizes also to the norms of the complex rationals. How to generalize this formula to the infinite-dimensional algebraic extension of rationals? The simplest possibility is to write the complex norm squared as vector in the infinite-dimensional extension having rational coefficients and to apply adelic formula to each factor separately.

$$\begin{aligned} |x|_R &= \sum_k e_R^{k)} \prod_p |\frac{1}{x_k}|_p \ , \\ |x| &= \sum_k e^{k)} x_k \ . \end{aligned} \tag{9.5.1}$$

Here $e^{k)}$ denote the units of the infinite-dimensional algebraic extension (products of roots of primes and analogous to imaginary unit) and $e_R^{k)}$ denote the evaluations of these units identified as real numbers. The resulting norm is indeed equal to the real norm when the resulting number is interpreted as a real number.

In the case that the factors Z_{p_1} of ζ are complex rationals, one can write the real norm of the real ζ for $Re[s] > 1$ as a product

$$|\zeta(z)|^2 = \ = \prod_{p_1} \left[\prod_p N_p(|\frac{1}{Z_{p_1}(z)}|^2) \right] \equiv \prod_{p_1} \left[\prod_p N_p(|Z_{p_1}^{p)}(z)|^2) \right] \ . \tag{9.5.2}$$

Here $N_p(x)$ denotes the p-adic norm of number x. This formula explains why one must define the p-adic zeta as an arithmetic inverse of the real ζ. The generalization of this formula to the case that $\hat{\zeta}^2$ has values in the set of the complex rationals is straightforward.

The problem with this representation is that the product over primes p_1 does not converge in an absolute sense for $Re[s] \leq 1$. By a suitable rearrangement of a conditionally convergent product a convergence to any number can be achieved. This suggests that one could find some unique manner to rearrange the terms to a convergent expression converging to $|\zeta|^2$. A unique definition indeed suggests itself: the analytic continuation of ζ from the region $Re[s] > 1$ might be equivalent with the exchange of the order of 'producting' in the expression of ζ:

$$
\begin{aligned}
|\hat{\zeta}(z)|^2 &= \prod_p N_p(|\frac{1}{\zeta(z)}|^2 = \prod_p \left[\prod_{p_1} N_p(|\frac{1}{Z_{p_1}(z)}|)\right] \\
&= \prod_p N_p(|\frac{1}{\zeta}|^2) = \prod_p N_p(|\zeta^{p)}|)^2 \quad . \qquad (9.5.2)
\end{aligned}
$$

The minimal working hypothesis is that $|\hat{\zeta}|^2$ defined as the product its p-adic norms equals to $|\zeta|^2$ at points, where its values are *rational*:

$$
\prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2 \quad . \qquad (9.5.3)
$$

The generalization to the algebraic extension of rationals is straightforward since the p-adic norm squared is sum over the p-adic norms of the components of the algebraic extension with various units $e^{k)}$ of the algebraic extension multiplying them interpreted as real numbers $e_R^{k)}$

$$
\begin{aligned}
\prod_p N_p(|\hat{\zeta}|^2) &= \sum_k e_R^{k)} \prod_p N_p(\frac{1}{|\hat{\zeta}|_k^2}) = |\zeta|^2 \quad , \\
|\hat{\zeta}|^2 &= \sum_k e^{k)} |\zeta|_k^2 \quad . \qquad (9.5.3)
\end{aligned}
$$

From this formula Universality Principle follows automatically. Since $|\hat{\zeta}|^2$ can be regarded as a vector having infinite number of components, the only manner to achieve the vanishing of $\prod_p N_p(|\hat{\zeta}|^2)$ is to require that it contains a vanishing rational factor. As will be found, the points at which infinite number of the factors of $|\hat{\zeta}|^2$ can be rational, very probably belong to the lines $Re(s) = n/2$. Thus the Universality Principle, and as it seems, also Riemann hypothesis, reduces to the statement that the modified Adelic formula defines a genuine norm which vanishes only when the vector is a null vector and is equal to $|\zeta|^2$. Of course, one could consider also the possibility that this norm is proportional to $|\zeta|^2$.

The conditions guaranteeing the rationality of the factors $|Z_{p_1}|^2$

Universality Principle states that zeros of ζ correspond to zeros of $|\hat{\zeta}|^2$. This quantity, when well-defined, belongs to an infinite-dimensional real algebraic extension of rationals, and its vanishing is possible if it contains a vanishing rational factor which is product of an infinite number of factors Z_{p_1} which are rational. $|\hat{\zeta}|^2$ is the product of the factors

$$
\frac{1}{Z_{p_1}(x+iy)Z_{p_1}(x-iy)} = 1 - 2p_1^{-x} Re[p_1^{iy}] + p_1^{-2x} \quad . \qquad (9.5.4)
$$

This expression equals to a rational number q, if one has

$$
Re[p_1^{iy}] = \frac{qp_1^x - p_1^{-x}}{2} \quad . \qquad (9.5.5)
$$

In this case the integer multiples ny do not satisfy the rationality condition, to say nothing about the superpositions of different values of y. It is also implausible that this condition would hold true for an infinite number of primes p_1 required by the vanishing of a rational factor of $\hat{\zeta}$.

An alternative manner to achieve rationality is by requiring that the two terms are separately rational. p_1^{-2x} factor is rational only if one has $x = n/2$. To achieve rationality $Re[p_1^{iy}]$ should contain a factor compensating the irrationality of the $p_1^{-n/2}$ factor somehow. On the lines $Re[s] = x = n/2$ one has

$$\frac{1}{Z_{p_1}(n/2+iy)Z_{p_1}(n/2-iy)} = 1-2p_1^{-n/2}Re[p_1^{iy}]+p_1^{-n} \ .$$

It is of crucial importance that the moduli squared depend on the real part of p_1^{iy} only. If this is rational, rationality is achieved for even values of n.

On the lines $Re[s]=n+1/2$ rationality is achieved provided that p_1^{iy} factors contain the phase factor $(r_1+is_1\sqrt{k})/\sqrt{p_1}$ compensating the $p_1^{-1/2}$ factor and multiplying a factor which of the same type:

$$\begin{aligned} p_1^{iy} &= U_1U = \frac{(r_1+is_1\sqrt{k})}{\sqrt{p_1}}\times\frac{(r+is\sqrt{k})^2}{r^2+s^2k} \ , \\ r_1^2+s_1^2k_1 &= p_1 \ . \end{aligned} \qquad (9.5.5)$$

The latter equation is satisfied if one has

$$k = \sqrt{p_1-m^2} \ , \ 0<m<\sqrt{p} \ . \qquad (9.5.6)$$

On the lines $Re[s]=n$ one must have

$$p_1^{iy} = \frac{(r+is\sqrt{k})^2}{r^2+s^2k} \ . \qquad (9.5.7)$$

The overall conclusions are following.

1. The vanishing of $|\hat{\zeta}|^2$ requires only the rationality of the *real parts* of Z_{p_1} for infinite number of values of p_1. The basic ansatz allows rationality only on the lines $Re[s]=n/2$ and my subjective feeling is that it is extremely implausible that exceptional ansatz gives rise to the rationality of an infinite number of $|Z_{p_1}|^2$ factors. That this is really the case might turn out to be difficult part in attempts to prove Riemann hypothesis even if one has proved the identity $\prod_p N_p(|\hat{\zeta}|^2)=|\zeta|^2$ and that this product defines a norm.

2. Rationality requirement allows p_1^{-iy} to consist of the products of the phases of very general algebraic numbers $r+is\sqrt{k}$. The products of these numbers are always of same form and their norm squared is r^2+s^2k. Geometrically these numbers correspond to orthogonal triangles with one or two sides having integer valued length and remaining side having integer valued length squared.

3. For given value of y all integer multiples ny of y provide a solution of the rationality conditions. It is not necessary to require that the algebraic extensions $r+is\sqrt{k(p_1,y_i)}$ associated with y_1 and y_2 satisfying the condition, are same for given value of p_1: that is, one can have

$$k(p_1,y_1)\neq k(p_1,y_2) \ .$$

   For $k(p_1,y_1)=k(p_1,y_2)$ also the linear combinations $m_1y_1+n_1y_2$ satisfy rationality conditions. For the minimal solution to the rationality conditions, only multiples of each y solve the rationality conditions. For the maximal solution all solutions y_i correspond to the same algebraic extension for given p_1 and unrestricted linear superposition of the y_i holds true.

4. For p mod $4=1$ rational phase factors p_1^{-iy} defined by the powers of the Gaussian primes provide the minimal manner to achieve rationality such that unrestricted superposition of solutions holds true. For p_1 mod $4=3$ and p_1 mod $3=1$ the minimal manner to achieve compensation is by using Eisenstein primes. For the primes p_1 mod $4=3$ and p_1 mod $3=1$ one cannot compensate $\sqrt{p_1}$ factor using Gaussian or Eisenstein primes and a more general algebraic extension of integers is necessary. For given prime p_1 there is finite number of possible algebraic extensions.

The conditions guaranteeing the reduction of the p-adic norm

The term p_1^{-iy} appearing in the factors $1/Z_{p_1}$ is inversely proportional to integers and thus have p-adic norm which is larger than one for the primes appearing as factors of the integer n_1. Some mechanism guaranteeing the reduction of the p-adic norm must be at work and this mechanism gives strong conditions on the allowed phases p_1^{iy}.

The condition guaranteeing the reduction is very general. What is required is the reduction of the p-adic norm

$$|X\overline{X}|_p \quad , \quad X = 1 - Up_1^{iy} \quad , \quad U = (\epsilon p_1)^{-n/2} \quad . \tag{9.5.8}$$

Here one has $\epsilon = 1$ for even values of n whereas for for odd values of n one has $\epsilon = \pm 1$ depending on whether the square root exists or not p-adically: the sole purpose of this factor is to take care that the p-adic counterpart of U is an ordinary p-adic number.

By writing

$$p_1^{-iy} \equiv cos(\phi) + isin(\phi) \quad ,$$

one obtains

$$|X\overline{X}|_p = |1 - 2Ucos(\phi) + U^2|_p \quad .$$

Not surprisingly, the vanishing of the norm modulo p implies in modulo p accuracy

$$U = cos(\phi) - \sqrt{-1} sin(\phi) \quad .$$

Since U must be real, the only possible manner to satisfy the condition is to require that

$$sin(\phi) = 0 \ \ mod \ \ p \quad , \quad cos(\phi) = 1 \ \ mod \ \ p \quad . \tag{9.5.9}$$

Clearly, ϕ must correspond to angle 0 or π in modulo p accuracy. What this condition says is that partition functions Z_{p_1} are real in order p. This is very natural condition on the line $Re[s] = 1/2$ where the ζ is indeed real.

The condition $cos^2(\phi) = 1 \ mod \ p$ implies

$$p_1^n \ mod \ p = 1 \quad . \tag{9.5.10}$$

p_1 can be always written as a power $p_1 = a^k$ of a primitive root a satisfying $a^{p-1} = 1$ modulo p such that k divides $p-1$. Thus $p_1^n \ mod \ p = 1$ holds true only only if $n \ mod \ (p-1)/k = 0$ is satisfied.

The conditions guaranteeing modulo p reality of Z_{p_1} for prime p dividing the denominator of p_1^{-iy}, when written explicitly, give

$$\begin{array}{lll}
Re[s] = n \ : & r^2 - s^2k = r^2 + s^2k \ , & \frac{2rs}{r^2+s^2k} = 0 \ , \\
Re[s] = n + \frac{1}{2} \ : & (r^2 - s^2k)r_1 - 2rss_1k = r^2 + s^2k \ , & \frac{2rsr_1 + (r^2-s^2k)s_1}{r^2+s^2k} = 0 \ .
\end{array} \tag{9.5.10}$$

In the case of Gaussian primes ($k = 1$) also second option is possible since the multiplication with $\pm i$ yields new rational phase factor: this option corresponds simply the exchange of $r^2 - s^2$ and $2rs$ factors in the formula above.

Rather general solution to the conditions can be written rather immediately. In both cases the conditions

$$s \ mod \ p^2 = 0 \quad , \quad r \ mod \ p = 0 \tag{9.5.11}$$

are satisfied. Note that $s\ mod\ p^2 = 0$ is necessary since $r^2 + s^2k\ mod\ p = 0$ holds true. Besides this the conditions

$$\begin{aligned} & r_1^2 + s_1^2 k\ mod\ p = 1 && \text{for } Re[s] = n \ , \\ & s_1\ mod\ p = 0\ \ \&\ \ r_1\ mod\ p = 1 && \text{for } Re[s] = n + \tfrac{1}{2} \ , \end{aligned} \tag{9.5.12}$$

are satisfied.

If p_1^{-iy} is inversely proportional to integer containing as factors powers of a prime p larger than p_1, the reduction of the norm cannot occur for $Re[s] = 1/2$ but is possible for sufficiently large values of $Re[s] = n/2$. For $p_1 = 2$ and $p_1 = 3$ factors the reduction of the norm is certainly not possible on the line $Re[s] = 1/2$ since the condition $2p + 1 \leq p_1$ cannot be satisfied for any prime in these cases. The reduction of the p-adic norm of the ζ suggests strongly that the condition $2p_i + 1 \leq p_1$ is satisfied for large primes p_1 and odd primes p_i. The condition is satisfied always for $p_i = 2$ and $p_1 \geq 3$. If it is satisfied completely generally, the phase factors associated with Z_3 must be of the general form

$$3^{-iy} = \frac{(\pm 1 \pm \sqrt{2}i)}{\sqrt{3}} \times \frac{(r(y) + i\sqrt{2}s(y))^2}{r^2(y) + 2s^2(y)} \ , \quad r^2(y) + 2s^2(y) = 3^k \text{ or } 2 \times 3^k \ .$$

This condition and similar conditions associated with larger primes give very strong constraints on the zeros.

The general conclusions are following.

1. The reduction of the p-adic norm and the related modulo p reality of Z_{p_1} is the p-adic counterpart for the reality of ζ on the critical line which suggests that it might occur completely generally. It requires that $p_1^n\ mod\ p = 1$ holds true for all primes appearing as factors of the denominator n_1 of the rational part of the phase p_1^{-iy}.

2. If the denominator of p_1^{-iy} is square-free integer, the p-adic norm of Z_{p_1} is never larger than unity except possibly in the diagonal case $p = p_1$.

3. In the diagonal case the norm grows like p_1^{n+1} for $Re[s] = n + 1/2$ and p_1^n for $Re[s] = n$. This conforms with the fact that ζ has no zeros for $Re[s] \geq 1$ but has zeros for $Re[s] = -2n$.

4. If rational points of ζ obey linear superposition, then the rational points on the lines $Re[s] = n$ contain an even number of y_i:s needed to achieve the rationality of $Re[p^{-iy}]$. Hence the denominator tends to have larger p-adic norm than it can have on the line $Re[s] = 1/2$. This means that the line $Re[s] = 1/2$ is optimal as far as zeros of $|\hat{\zeta}|^2$ are considered. It can however happen that in the product $p_1^{iy_1} p_1^{iy_2}$ complex conjugates of factor phases can compensate each other so that the p-adic norm of $p_1^{i(y_1+y_2)}$ is not always larger than the norms of the factors. In particular, the factors $(r_1 + is_1\sqrt{k})/\sqrt{p_1}$ could cancel in the product $p_1^{iy_1} p_1^{-iy_2}$ This mechanism could imply the emergence small values of ζ for $y_{ij} = y_i - y_j$ on the line $Re[s] = 1$ required by the inner product property of the Hermitian form defined by the super-conformal model for the zeros of ζ.

Gaussian primes and Eisenstein primes

The general manner to satisfy the rationality requirement is to assume that the phases p_1^{iy} correspond to orthogonal triangles with one or two sides with an integer valued length and one side with integer valued length squared. A rather general and mathematically highly interesting manner to realize the rationality of the the phases $p_1^{-n/2} p_1^{iy}$ is by choosing the phases to be products of Gaussian or Eisenstein primes [A160] .

Gaussian primes consist of complex integers $e_i \in \{\pm 1, \pm i+\}$, ordinary primes $p\ mod\ 4 = 3$ multiplied by the units e_i to give four different primes, and complex Gaussian primes $r \pm is$ multiplied by the units e_i to give 8 primes with the same modulus squared equal to prime $p\ mod\ 4 = 1$. Every prime $p\ mod\ 4 = 1$ gives rise to 8 non-degenerate Gaussian primes. Pythagorean phases correspond to the phases of the squares of complex Gaussian integers $m + in$ expressible as products of even powers of Gaussian primes $G_p = r + is$:

$$G_p = r + is \ , \quad G\overline{G} = r^2 + s^2 = p \ , \quad p \text{ prime } \& \ p \ mod \ 4 = 1 \ . \tag{9.5.13}$$

The general expression of a Pythagorean phase expressible as a product of even number of Gaussian primes is

$$U = \frac{r^2 - s^2 + i2rs}{r^2 + s^2} \ . \tag{9.5.14}$$

By multiplying this expression by a Gaussian prime i, one obtains second type of Pythagorean phase

$$U = \frac{2rs + i(r^2 - s^2)}{r^2 + s^2} \ . \tag{9.5.15}$$

Gaussian primes allow to achieve rationality of $p_1^{-n+1/2}p^{-iy}$ factor for $p_1 \ mod \ 4 = 1$. The generality of the mechanism suggests that Gaussian primes should be very important. For $Re[s] \neq n/2$ it is not possible to achieve complex rationality with any decomposition of p_1^{iy} to Gaussian primes.

Besides Gaussian primes also so called Eisenstein primes are known to exist [A160] and the fact that only the rationality of the real parts of $1/Z_{p_1}$ factors is necessary for the rationality of $|Z_{p_1}|^2$ means that they are also possible. Note however that now the multiplication the phase by $\pm i$ makes the real part of the phase irrational, and is thus not allowed. Thus only four-fold degeneracy is present now for ζ.

Whereas Gaussian primes rely on modulo 4 arithmetics for primes, Eisenstein primes rely on modulo 3 arithmetics. Let $w = exp(i\phi)$, $\phi = \pm 2\pi/3$, denote a nontrivial third root of unity. The number 1-w and its associates obtained by multiplying this number by ± 1 and $\pm i$; the rational primes $p \ mod \ 3 = 2$ and its associates; and the factors $r + sw$ of primes $p \ mod \ 3 = 1$ together with their associates, are Eisenstein primes. One can write Eisenstein prime in the form

$$w = r - \frac{s}{2} + is\frac{\sqrt{3}}{2} \ . \tag{9.5.16}$$

What might be called Eisenstein triangles correspond to the products of powers of the squares of Eisenstein primes and have integer-valued long side. The sides of the orthogonal triangle associated with a square of Eisenstein prime E_p have lengths

$$(r^2 - rs - \frac{3s^2}{2} \ , \ s\frac{\sqrt{3}}{2} \ , \ p = r^2 + s^2 - rs) \ .$$

Eisenstein primes clearly span the ring of the complex integers having the general form $z = (r + i\sqrt{3}s)/2$, r and s integers.

One can use Eisenstein prime E_p to achieve the replacement of the $p_1^{-1/2}$-factor with $1/p_1$-factor in the partition functions Z_{p_1} the same effect for $p_1 \ mod \ 4 = 1$ and $p_1 \ mod \ 3 = 1$ with the net result that $i\sqrt{3}$ term appears. This trick does not work for $p_1 \ mod \ 4 = 3$ and $p_1 \ mod \ 3 = 2$. Note that the presence of *both* Gaussian and Eisenstein primes in the same factor Z_{p_1} is not allowed since in this case also the real part of Z_{p_1} would contain $\sqrt{3}$. This suggests that quite generally $p \ mod \ 4 = 1$ *resp.* $p \ mod \ 4 = 3 \wedge p \ mod \ 3 = 1$ parts of $\hat{\zeta}$ could correspond to Gaussian *resp.* Eisenstein primes.

For the factors Z_{p_1} satisfying $p_1 \ mod \ 4 = 3$ & $p_1 \ mod \ 3 = 2$ simultaneously, neither Gaussian nor Eisenstein primes can affect the rationalization of $p^{-n+1/2-iy}$ factor, and in this case more general algebraic extension of complex numbers is necessary as already found.

The algebraic extensions of rational numbers allow the notion of algebraic integer and prime quite generally [A140] . In the general case however the decomposition of an algebraic integer into primes is not unique. In case of complex extensions of form $r + \sqrt{-d}s$ unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [A140] . $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$, which perhaps makes Gaussian and Eisenstein primes special.

9.5.2 Tests for the $|\hat{\zeta}|^2 = |\zeta|^2$ hypothesis

The fact that the phases p_1^{iy} correspond to non-vanishing values of y, suggests that $|\hat{\zeta}|^2 = |\zeta|^2$ equality holds on the real axis only in the sense of a limiting procedure $y \to 0$. If the the values of y giving rise to allowed phases obey linear superposition (that is $k_1(p_1, y)$ defining the algebraic extension does not depend on y), the allowed values of y form a dense set of the real axis, since arbitrarily small differences $y_i - y_j$ are possible for the zeros of ζ. Hence the limiting procedure $y \to 0$ should be well-defined and give the expected answer if the basic hypothesis is correct.

What happens on the real axis?

The simplest test for the basic hypothesis is to look what happens on the real axis at the points $s = n$. Real ζ diverges at $s = 1$ and $s = 0$ and has trivial zeros are at the points $s = -2n$. The norm of $\hat{\zeta}$ is given by

$$|\hat{\zeta}(n)|_R = \prod_p \left[\prod_{p_1} |1 - p_1^{-n}|_p \right] . \tag{9.5.17}$$

For $n = 0$ a straightforward substitution to the formula implies that $|\hat{\zeta}(0)|$ vanishes. For $n > 0$ one has

$$|\hat{\zeta}(n)|_R = \prod_p \left[\prod_{p_1} |\frac{p_1^n - 1}{p_1^n}|_p \right] = \prod_p p^n \left[\prod_k \prod_{p_1^n \ mod \ p^k = 1} p^{-k} \right] . \tag{9.5.18}$$

Since the number of primes p_1 satisfying the condition $p_1^n \ mod \ p^k = 1$ is infinite, the norm vanishes for all values $n > 0$. For $s = -n < 0$ one has,

$$|\hat{\zeta}(n)|_R = \prod_p \left[\prod_{p_1} |1 - p_1^n|_p \right] . \tag{9.5.19}$$

and also this product vanishes always.

How to understand these results?

1. The results are consistent with the view that $|\zeta|_R$ on the real axis should be estimated by taking the limit $y \to 0$. Since the values of y in question involve necessarily differences of very large values of y, it is conceivable that the limiting procedure does not yield zero. That the limiting procedure can give zero for $Re[s] < 0$ could be partially due to the fact that for $Re[s] = -n < 0$ one has for the diagonal $p_1 = p$ contribution $|Z_p(-n + iy)|_p = 1$ whereas for $Re[s] = n > 0$ one has $|Z_p(n + iy)|_p > 1$ in general. Furthermore, for $Re[s] = -n$ only $p_1^n \ mod \ p^k = 1$ condition leads to the reduction of the p-adic norm of $Z_{p_1 \neq p}$ whereas for $Re[s] = -2n$ also $p_1^n \ mod \ p^k = -1$ condition has the same effect.

2. One cannot exclude the possibility that only the proportionality $|\hat{\zeta}|^2 \propto |\zeta|^2$ holds true. For instance, in the super-conformal model predicting that the physical states of the model correspond to the zeros of ζ on the critical line, the Hermitian form defining the 'inner product' is proportional to the product of $sin(i\pi z)\Gamma(z)\zeta(z)$. This function vanishes for $Re[s] \notin \{0, 1\}$ and the coefficient function of ζ is finite in the critical strip. For $s = 0$ this function however has the value $-1/2$ and for $s = 1$ the value is 1, whereas the naively evaluated value of $|\hat{\zeta}|$ vanishes identically at these points. Thus something else is necessarily involved.

3. It could also be that the product representation for the norm squared of $\hat{\zeta}$ as a product of its p-adic norms converges only in a restricted region. It would not be surprising if the negative values of y were excluded from the region of convergence for the representation of $|\hat{\zeta}|^2$ as a product of its p-adic norms. Concerning the proof of the Riemann hypothesis, the minimal requirement is that the region $[1/2 \leq Re[s] \leq 1 \ , \ y \neq 0]$ is included in the region of convergence.

One might think that $|\zeta|^2 = |\hat{\zeta}|^2$ hypothesis is testable simply by comparing the norm squared of the real zeta with the product of the p-adic norms of $|\hat{\zeta}|^2$. The problems are that the value for the product of p-adic norms is extremely sensitive to numerical errors since the p-adic norm of Pythagorean triangles phases fluctuates wildly as a function of the phase angle, and that one does not actually know what the values of p_1^{iy} actually are. One testable prediction, also following from the super-conformal model of the Riemann Zeta, is that the superpositions of the zeros are probably small values or minima of $|\zeta|_R$ on the lines $Re[s] = n/2$. More precisely, it is the function $G(1 + iy_{12})$ which should have values smaller than one if the metric defined by G is Hermitian. One could also try to understand whether the the norm of $\hat{\zeta}$ allows a continuation to a continuous function of the complex argument identifiable as a modulus of an analytic function.

Can the imaginary part of $\hat{\zeta}$ vanish on the critical line?

Riemann Zeta is real on the critical line $Re[s] = 1/2$. A natural question is whether also $\hat{\zeta}$ has a vanishing imaginary part on this line. This is certainly not necessary since $\hat{\zeta}$ has values in the infinite-dimensional algebraic extension of rationals. It would be however highly desirable if this condition would hold true.

One cannot formulate the vanishing condition for the imaginary part in terms of the norm squared of any quantity defined by using the generalization of the adelic formula. The vanishing of the imaginary part of $\hat{\zeta}$ is however consistent with the Universality Principle. One can see this by expanding the factors $Z_{p_1} = 1/(1 - p_1^{-1/2-iy})$ to a geometric series in powers of the irrational imaginary part of $p_1^{-1/2-iy}$. Each odd term in this series is proportional to $\sqrt{k(p_1, y)}$. One can combine the product of all these geometric series with the same value $k(p_1, y) = k$ to a sum of a rational part and an irrational part proportional to $\sqrt{k}$. If the irrational parts vanish separately for all allowed values of k, the imaginary part of $\hat{\zeta}$ indeed vanishes. This requires that the same value of $k(p_1, y) = k$ is associated with an infinite number of factors Z_{p_1}.

What is interesting is that the terms appearing in the sum over primes p_1 with the same value of k are proportional to $1/p_1^n$, $n \geq 1$: $n = 1$ terms are on the borderline at which the absolute convergence fails. If the number of primes p_1 with the same value of k is sufficiently small, also the sum over $n = 1$ terms with given k converges. The allowed values of k are given by $k = \sqrt{p_1 - m^2}$, $m \leq \sqrt{p_1}$ and the simplest hypothesis is that each value of k appears with same probability so that for a given prime p_1 the probability for a $k(p_1, y) = k$ is $P(k) \sim 1/\sqrt{p_1}$. This would suggest that the lowest terms in the sum defining the imaginary part behaves as $1/p_1^{3/2}$ so that convergence is indeed achieved. Note that convergence requirement does not support the special role of Gaussian or Eisenstein primes in the set of algebraic numbers appearing in the expansion of $\hat{\zeta}$.

The general algebraic properties of $\hat{\zeta}$ must be consistent with the vanishing of $Im[\zeta]$ on the critical line. The reality of ζ on the critical line follows from the symmetry with respect to the critical line reducing on the critical line to the condition $\zeta(s) = \zeta(1 - s)$ implying the reality of $\zeta(s)\zeta(1 - s)$. This condition makes sense also for $\hat{\zeta}$. In general case, one has

$$\hat{\zeta}(s)\hat{\zeta}(1-s) = \prod_{p_1} Z_{p_1}(x+iy)Z_{p_1}(1-x-iy) = \prod_{p_1} \frac{1}{\left[1 - p_1^{-x}p_1^{-iy} - p_1^{-1+x}p_1^{iy} + \frac{1}{p_1}\right]} \quad .$$

Due to the presence of p^{-x} terms, the moduli squared for these factors are complex irrational numbers.

On the line $Re[s] = 1/2$, the product representation for this function reduces to the product of *real* factors

$$\frac{1}{Z_{p_1}(1/2+iy)Z_{p_1}(1/2-iy)} = 1 - p_1^{-1/2}(p_1^{iy} + p_1^{-iy}) + \frac{1}{p_1} \tag{9.5.20}$$

in the algebraic extension of rationals. Thus the reality and rationality of the function $\hat{\zeta}(s)\hat{\zeta}(1-s)$ on the critical line corresponds in a very transparent manner the reality of ζ on the critical line. Note also that the modulo p reality of the factors Z_{p_1} implied by the reduction of the p-adic norm can be regarded as the p-adic counterpart for the reality of ζ on the critical line.

What about non-algebraic zeros of ζ?

In principle real ζ could also have non-algebraic zeros. The following argument however demonstrates that they do not pose a problem. If Universality Principles holds true, and if the norm squared of $\hat{\zeta}$ defined as a product of its p-adic norms indeed equals to the norm squared of the real ζ in the set of of complex plane in which the factors $1/(1-p^{-s})$ are algebraic numbers, one obtains strict bounds for the norm of the real ζ excluding the zeros in the dense set inside the critical strip. The continuity of the real ζ in turn implies that it cannot vanish except on the critical line.

9.6 Could local zeta functions take the role of Riemann Zeta in TGD framework?

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.

2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.

3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights or perhaps more naturally, to complex square roots of real conformal weights [K25] . The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

It was also found that there are good reasons for expecting that the zetas in question should have only a finite number zeros. In the same section the self-referentiality hypothesis for ζ was proposed on basis of physical arguments. In this section (written before the emergence of self-referentiality hypothesis) the situation will be discussed from different view point.

9.6.1 Local zeta functions and Weil conjectures

Riemann Zeta is not the only zeta [A1, A110]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [A104] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s)\times\zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [A104] .

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

9.6.2 Local zeta functions and TGD

The local zetas are associated with single prime p, they satisfy functional equation, their zeros lie at the critical lines, and they are rational functions of p^{-s}. These features are highly desirable from the TGD point of view.

Why local zeta functions could be natural in TGD framework?

In TGD framework modified Dirac equation assigns to a partonic 2-surface a p-adic prime p and inverse of the zeta defines local conformal weight. The intersection of the real and corresponding p-adic parton 2-surface is the set containing the points that one is interested in. Hence local zeta sharing the basic properties of Riemann zeta is highly desirable and natural. In particular, if the local zeta is a rational function then the inverse images of rational points of the geodesic sphere are algebraic numbers. Of course, one might consider a stronger constraint that the inverse image is rational. Note that one must still require that p^{-s} as well as s are algebraic numbers for the zeros of the local zeta (conditions 1) and 2) listed in the beginning) if one wants the number theoretical universality.

Ifthe modified Dirac operator indeed assigns to a given partonic 2-surface a p-adic prime p, one can ask whether the inverse $\zeta_p^{-1}(z)$ of some kind of local zeta directly coding data about partonic 2-surface could define the generalized eigenvalues of the modified Dirac operator and radial super-symplectic conformal weights so that the conjectures about Riemann Zeta would not be needed at all.

The eigenvalues of the mass squared assignable to the modes of the modified Dirac operator, whose ground state part codes information about four-surface [K89] could in a holographic manner code for information about partonic 2-surface. This kind of algebraic geometric data are absolutely relevant for TGD since U-matrix and probably also S-matrix must be formulated in terms of the data related to the intersection of real and partonic 2-surfaces (number theoretic braids) obeying same algebraic equations and consisting of algebraic points in the appropriate algebraic extension of p-adic numbers. Note that the hierarchy of algebraic extensions of p-adic number fields would give rise to a hierarchy of zetas so that the algebraic extension used would directly reflect itself in the eigenvalue spectrum of the modified Dirac operator and super-symplectic conformal weights. This is highly desirable but not achieved if one uses Riemann Zeta.

One must of course leave open the possibility that for real-real transitions the inverse of the zeta defined as a product of the local zetas (very much analogous to Riemann Zeta defines the conformal weights. This kind of picture would conform with the idea about real physics as a kind of adele formed from p-adic physics.

Finite field hierarchy is not natural in TGD context

That local zeta functions are assigned with a hierarchy of finite field extensions do not look natural in TGD context. The reason is that these extensions are regarded as abstract extensions of $G(p, k)$ as opposed to a large number of algebraic extensions isomorphic with finite fields as abstract number fields and induced from the extensions of p-adic number fields. Sub-field property is clearly highly relevant in TGD framework just as the sub-manifold property is crucial for geometrizing also other interactions than gravitation in TGD framework.

The $O(p^n)$ hierarchy for the p-adic cutoffs would naturally replace the hierarchy of finite fields. This hierarchy is quite different from the hierarchy of finite fields since one expects that the number of solutions becomes constant at the limit of large n and also at the limit of large p so that powers in the function G coding for the numbers of solutions of algebraic equations as function of n should not increase but approach constant N_∞. The possibility to factorize $exp(G)$ to a product $exp(G_0)exp(G_\infty)$ would mean a reduction to a product of a rational function and factor(s) $\zeta_p(s) = 1/(1-p^{-s_1})$ associated with Riemann Zeta with argument s shifted to $s_1 = s - log_p(N_\infty)$.

What data local zetas could code?

The next question is what data the local zeta functions could code.

1. It is not at clear whether it is useful to code global data such as the numbers of points of partonic 2-surface modulo p^n. The notion of number theoretic braid occurring in the proposed

approach to S-matrix suggests that the zeta at an algebraic point z of the geodesic sphere S^2 of CP_2 or of light-cone boundary should code purely local data such as the numbers N_n of points which project to z as function of p-adic cutoff p^n. In the generic case this number would be finite for non-vacuum extremals with 2-D S^2 projection. The n^{th} coefficient $g_n = N_n/n$ of the function G_p would code the number N_n of these points in the approximation $O(p^{n+1}) = 0$ for the algebraic equations defining the p-adic counterpart of the partonic 2-surface.

2. In a region of partonic 2-surface where the numbers N_n of these points remain constant, $\zeta(s)$ would have constant functional form and therefore the information in this discrete set of algebraic points would allow to deduce deduce information about the numbers N_n. Both the algebraic points and generalized eigenvalues would carry the algebraic information.

3. A rather fascinating self referentiality would result: the generalized eigen values of the modified Dirac operator expressible in terms of inverse of zeta would code data for a sequence of approximations for the p-adic variant of the partonic 2-surface. This would be natural since second quantized induced spinor fields are correlates for logical thought in TGD inspired theory of consciousness. Even more, the data would be given at points $\zeta(s)$, s a rational value of a super-symplectic conformal weight or a value of generalized eigenvalue of modified Dirac operator (which is essentially function $s = \zeta_p^{-1}(z)$ at geodesic sphere of CP_2 or of light-cone boundary).

9.6.3 Galois groups, Jones inclusions, and infinite primes

Langlands program [A56, A150] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [A217] . In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on WCW-spinor fields. One can also speak about WCW spinor s invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the

primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \to K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the biologically highly interesting p-adic length scale range 10 nm-5 μm contains as many as four scaled up electron Compton lengths $L_e(k) = \sqrt{5}L(k)$ associated with Gaussian Mersennes ($M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$), which suggests that the emergence of living matter means an improved cognitive resolution.

Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \to \sum x_n z^n$ [A61] . Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [L9] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of imbedding space point allows to represent WCW (the world of classical worlds associated with the light-cone of a given point of H) and WCW spinor fields emerges naturally [L9] .

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [L11] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on WCW spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. WCW spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

9.7 Elementary particle vacuum functionals for dark matter

One of the open questions is how dark matter hierarchy reflects itself in the properties of the elementary particles. The basic questions are how the quantum phase $q = ep(2i\pi/n)$ makes itself visible in the solution spectrum of the modified Dirac operator D and how elementary particle vacuum functionals depend on q. Considerable understanding of these questions emerged recently. One can generalize modular invariance to fractional modular invariance for Riemann surfaces possessing Z_n symmetry and perform a similar generalization for theta functions and elementary particle vacuum functionals. In particular, without any further assumptions $n = 2$ dark fermions have only three families. The existence of space-time correlate for fermionic 2-valuedness suggests that fermions indeed correspond to $n = 2$, or more generally to even values of n, so that this result would hold quite generally. Elementary bosons (actually exotic particles) would correspond to $n = 1$, and more generally odd values of n, and could have also higher families.

9.7.1 Connection between Hurwitz zetas, quantum groups, and hierarchy of Planck constants?

The action of modular group SL(2,Z) on Riemann zeta [A84] is induced by its action on theta function [A93] . The action of the generator $\tau \to -1/\tau$ on theta function is essential in providing the functional equation for Riemann Zeta. Usually the action of the generator $\tau \to \tau+1$ on Zeta is not considered explicitly. The surprise was that the action of the generator $\tau \to \tau+1$ on Riemann Zeta does not give back Riemann zeta but a more general function known as Hurwitz zeta $\zeta(s,z)$ for $z = 1/2$. One finds that Hurwitz zetas for certain rational values of argument define in a well defined sense representations of fractional modular group to which quantum group can be assigned naturally. This could allow to code the value of the quantum phase $q = exp(i2\pi/n)$ to the solution spectrum of the modified Dirac operator D.

Hurwitz zetas

Hurwitz zeta is obtained by replacing integers m with $m + z$ in the defining sum formula for Riemann Zeta:

$$\zeta(s,z) = \sum_m (m+z)^{-s} \quad . \tag{9.7.1}$$

Riemann zeta results for $z = n$.
Hurwitz zeta obeys the following functional equation for rational $z = m/n$ of the second argument [A44] :

$$\zeta(1-s,\frac{m}{n}) = \frac{2\Gamma(s)}{2\pi n}^s \sum_{k=1}^{n} cos(\frac{\pi s}{2} - \frac{2\pi km}{n})\zeta(s,\frac{k}{n}) \quad . \tag{9.7.2}$$

The representation of Hurwitz zeta in terms of θ [A44] is given by the equation

$$\int_0^\infty [\theta(z,it)-1]\, t^{s/2}\frac{dt}{t} = \pi^{(1-s)/2}\Gamma(\frac{1-s}{2})\, [\zeta(1-s,z)+\zeta(1-s,1-z)] \quad . \tag{9.7.3}$$

By the periodicity of theta function this gives for $z = n$ Riemann zeta.

The action of $\tau \to \tau+1$ transforms $\zeta(s,0)$ to $\zeta(s,1/2)$

The action of the transformations $\tau \to \tau+1$ on the integral representation of Riemann Zeta [A84] in terms of θ function [A93]

$$\theta(z;\tau) - 1 = 2\sum_{n=1}^{\infty} [exp(i\pi\tau)]^{n^2} cos(2\pi nz) \tag{9.7.4}$$

is given by

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty \left[\theta(0;it) - 1\right] t^{s/2}\frac{dt}{t} \quad . \tag{9.7.5}$$

Using the first formula one finds that the shift $\tau = it \to \tau + 1$ in the argument θ induces the shift $\theta(0;\tau) \to \theta(1/2;\tau)$. Hence the result is Hurwitz zeta $\zeta(s,1/2)$. For $\tau \to \tau+2$ one obtains Riemann Zeta.

Thus $\zeta(s,0)$ and $\zeta(s,1/2)$ behave like a doublet under modular transformations. Under the subgroup of modular group obtained by replacing $\tau \to \tau + 1$ with $\tau \to \tau + 2$ Riemann Zeta forms a singlet. The functional equation for Hurwitz zeta relates $\zeta(1-s,1/2)$ to $\zeta(s,1/2)$ and $\zeta(s,1) = \zeta(s,0)$ so that also now one obtains a doublet, which is not surprising since the functional equations directly reflects the modular transformation properties of theta functions. This doublet might be the proper object to study instead of singlet if one considers full modular invariance.

Hurwitz zetas form n-plets closed under the action of fractional modular group

The inspection of the functional equation for Hurwitz zeta given above demonstrates that $\zeta(s,m/n)$, $m = 0,1,...,n$, form in a well-defined sense an n-plet under fractional modular transformations obtained by using generators $\tau \to -1/\tau$ and $\tau \to \tau + 2/n$. The latter corresponds to the unimodular matrix $(a,b;c,d) = (1,2/n;0,1)$. These matrices obviously form a group. Note that Riemann zeta is always one member of the multiplet containing n Hurwitz zetas.

These observations bring in mind fractionization of quantum numbers, quantum groups corresponding to the quantum phase $q = exp(i2\pi/n)$, and the inclusions for hyper-finite factors of type II_1 partially characterized by these quantum phases. Fractional modular group obtained using generator $\tau \to \tau+2/n$ and Hurwitz zetas $\zeta(s,k/n)$ could very naturally relate to these and related structures.

9.7.2 Could Hurwitz zetas relate to dark matter?

These observations suggest a speculative application to quantum TGD.

Basic vision about dark matter

1. In TGD framework inclusions of HFFs of type II_1 are directly related to the hierarchy of Planck constants involving a generalization of the notion of imbedding space obtained by gluing together copies of 8-D $H = M^4 \times CP_2$ with a discrete bundle structure $H \to H/Z_{n_a} \times Z_{n_b}$ together along the 4-D intersections of the associated base spaces [K25] . A book like structure results and various levels of dark matter correspond to the pages of this book. One can say that elementary particles proper are maximally quantum critical and live in the 4-D intersection of these imbedding spaces whereas their "field bodies" reside at the pages of the Big Book. Note that analogous book like structures results when real and various p-adic variants of the imbedding space are glued together along common algebraic points.

2. The integers n_a and n_b give Planck constant as $\hbar/\hbar_0 = n_a/n_b$, whose most general value is a rational number. In Platonic spirit one can argue that number theoretically simple integers involving only powers of 2 and Fermat primes are favored physically. Phase transitions between different matters occur at the intersection.

3. The inclusions $\mathcal{N} \subset \mathcal{M}$ of HFFs relate also to quantum measurement theory with finite measurement resolution with $\mathcal{N}$ defining the measurement resolution so that N-rays replace complex rays in the projection postulate and quantum space $\mathcal{M}/\mathcal{N}$ having fractional dimension effectively replaces $\mathcal{M}$.

4. Geometrically the fractional modular invariance would naturally relate to the fact that Riemann surface (partonic 2-surface) can be seen as an $n_a \times n_b$-fold covering of its projection to the base space of H: fractional modular transformations corresponding to n_a and n_b would

relate points at different sheets of the covering of M^4 and CP_2. This means $Z_{n_a n_b} = Z_{n_a} \times Z_{n_b}$ conformal symmetry. This suggests that the fractionization could be a completely general phenomenon happening also for more general zeta functions.

What about exceptional cases $n = 1$ and $n = 2$?

Also $n = 1$ and $n = 2$ are present in the hierarchy of Hurwitz zetas (singlet and doublet). They do not correspond to allowed Jones inclusion since one has $n > 2$ for them. What could this mean?

1. It would seem that the fractionization of modular group relates to Jones inclusions ($n > 2$) giving rise to fractional statistics. $n = 2$ corresponding to the full modular group Sl(2,Z) could relate to the very special role of 2-valued logic, to the degeneracy of $n = 2$ polygon in plane, to the very special role played by 2-component spinors playing exceptional role in Riemann geometry with spinor structure, and to the canonical representation of HFFs of type II_1 as fermionic Fock space (spinors in the world of classical worlds). Note also that SU(2) defines the building block of compact non-commutative Lie groups and one can obtain Lie-algebra generators of Lie groups from n copies of SU(2) triplets and posing relations which distinguish the resulting algebra from a direct sum of SU(2) algebras.

2. Also $n =$ 2-fold coverings $M^4 \to M^4/Z_2$ and $CP_2 \to CP_2/Z_2$ seem to make sense. One can argue that by quantum classical correspondence the spin half property of imbedding space spinors should have space-time correlate. Could $n = 2$ coverings allow to define the space-time correlates for particles having half odd integer spin or weak isospin? If so, bosons would correspond to $n = 1$ and fermions to $n = 2$. One could of course counter argue that induced spinor fields already represent fermions at space-time level and there is no need for the doubling of the representation.

   The trivial group Z_1 and Z_2 are exceptional since Z_1 does not define any quantization axis and Z_2 allows any quantization axis orthogonal to the line connecting two points. For $n \geq 3$ Z_n fixes the direction of quantization axis uniquely. This obviously correlates with $n \geq 3$ for Jones inclusions.

Dark elementary particle functionals

One might wonder what might be the dark counterparts of elementary particle vacuum functionals. Theta functions $\theta_{[a,b]}(z,\Omega)$ with characteristic $[a,b]$ for Riemann surface of genus g as functions of z and Teichmueller parameters Ω are the basic building blocks of modular invariant vacuum functionals defined in the finite-dimensional moduli space whose points characterize the conformal equivalence class of the induced metric of the partonic 2-surface. Obviously, kind of spinorial variants of theta functions are in question with $g+g$ spinor indices for genus g.

The recent case corresponds to $g = 1$ Riemann surface (torus) so that a and b are $g = 1$-component vectors having values 0 or 1/2 and Hurwitz zeta corresponds to $\theta_{[0,1/2]}$. The four Jacobi theta functions listed in Wikipedia [A93] correspond to these thetas for torus. The values for a and b are 0 and 1 for them but this is a mere convention.

The extensions of modular group to fractional modular groups obtained by replacing integers with integers shifted by multiples of $1/n$ suggest the existence of new kind of q-theta functions with characteristics $[a,b]$ with a and b being g-component vectors having fractional values k/n, $k = 0, 1...n-1$. There exists also a definition of q-theta functions working for $0 \leq |q| < 1$ but not for roots of unity [A73] . The q-theta functions assigned to roots of unity would be associated with Riemann surfaces with additional Z_n conformal symmetry but not with generic Riemann surfaces and obtained by simply replacing the value range of characteristics $[a,b]$ with the new value range in the defining formula

$$\Theta[a,b](z|\Omega) \quad = \quad \sum_n exp\left[i\pi(n+a)\cdot\Omega\cdot(n+a) + i2\pi(n+a)\cdot(z+b)\right] \quad . \tag{9.7.5}$$

for theta functions. If Z_n conformal symmetry is relevant for the definition of fractional thetas it is probably so because it would make the generalized theta functions sections in a bundle with a finite fiber having Z_n action.

This hierarchy would correspond to the hierarchy of quantum groups for roots of unity and Jones inclusions and one could probably define also corresponding zeta function multiplets. These theta functions would be building blocks of the elementary particle vacuum functionals for dark variants of elementary particles invariant under fractional modular group. They would also define a hierarchy of fractal variants of number theoretic functions: it would be interesting to see what this means from the point of view of Langlands program [A56] discussed also in TGD framework [K35] involving ordinary modular invariance in an essential manner.

This hierarchy would correspond to the hierarchy of quantum groups for roots of unity and Jones inclusions and one could probably define also corresponding zeta function multiplets. These theta functions would be building blocks of the elementary particle vacuum functionals for dark variants of elementary particles invariant under fractional modular group.

Hierarchy of Planck constants defines a hierarchy of quantum critical systems

Dark matter hierarchy corresponds to a hierarchy of conformal symmetries Z_n of partonic 2-surfaces with genus $g \geq 1$ such that factors of n define subgroups of conformal symmetries of Z_n. By the decomposition $Z_n = \prod_{p|n} Z_p$, where $p|n$ tells that p divides n, this hierarchy corresponds to an hierarchy of increasingly quantum critical systems in modular degrees of freedom. For a given prime p one has a sub-hierarchy Z_p, $Z_{p^2} = Z_p \times Z_p$, etc... such that the moduli at n+1:th level are contained by n:th level. In the similar manner the moduli of Z_n are sub-moduli for each prime factor of n. This mapping of integers to quantum critical systems conforms nicely with the general vision that biological evolution corresponds to the increase of quantum criticality as Planck constant increases.

The group of conformal symmetries could be also non-commutative discrete group having Z_n as a subgroup. This inspires a very short-lived conjecture that only the discrete subgroups of SU(2) allowed by Jones inclusions are possible as conformal symmetries of Riemann surfaces having $g \geq 1$. Besides Z_n one could have tetrahedral and icosahedral groups plus cyclic group Z_{2n} with reflection added but not Z_{2n+1} nor the symmetry group of cube. The conjecture is wrong. Consider the orbit of the subgroup of rotational group on standard sphere of E^3, put a handle at one of the orbits such that it is invariant under rotations around the axis going through the point, and apply the elements of subgroup. You obtain a Riemann surface having the subgroup as its isometries. Hence all discrete subgroups of SU(2) can act even as isometries for some value of g.

The number theoretically simple ruler-and-compass integers having as factors only first powers of Fermat primes and power of 2 would define a physically preferred sub-hierarchy of quantum criticality for which subsequent levels would correspond to powers of 2: a connection with p-adic length scale hypothesis suggests itself.

Spherical topology is exceptional since in this case the space of conformal moduli is trivial and conformal symmetries correspond to the entire $SL(2,C)$. This would suggest that only the fermions of lowest generation corresponding to the spherical topology are maximally quantum critical. This brings in mind Jones inclusions for which the defining subgroup equals to $SU(2)$ and Jones index equals to $\mathcal{M}/\mathcal{N} = 4$. In this case all discrete subgroups of $SU(2)$ label the inclusions. These inclusions would correspond to fiber space $CP_2 \rightarrow CP_2/U(2)$ consisting of geodesic spheres of CP_2. In this case the discrete subgroup might correspond to a selection of a subgroup of $SU(2) \subset SU(3)$ acting non-trivially on the geodesic sphere. Cosmic strings $X^2 \times Y^2 \subset M^4 \times CP_2$ having geodesic spheres of CP_2 as their ends could correspond to this phase dominating the very early cosmology.

Fermions in TGD Universe allow only three families

What is nice that if fermions correspond to $n = 2$ dark matter with Z_2 conformal symmetry as strong quantum classical correspondence suggests, the number of ordinary fermion families is three without any further assumptions. To see this suppose that also the sectors corresponding to $M^4 \rightarrow M^4/Z_2$ and $CP_2 \rightarrow CP_2/Z_2$ coverings are possible. Z_2 conformal symmetry implies that partonic Riemann surfaces are hyper-elliptic. For genera $g > 2$ this means that some theta functions of $\theta_{[a,b]}$ appearing in the product of theta functions defining the vacuum functional vanish. Hence fermionic elementary particle vacuum functionals would vanish for $g > 2$ and only 3 fermion families would be possible for $n = 2$ dark matter.

This results can be strengthened. The existence of space-time correlate for the fermionic 2-valuedness suggests that fermions quite generally to even values of n, so that this result would hold for all fermions. Elementary bosons (actually exotic particles belonging to Kac-Moody type representations) would correspond to odd values of n, and could possess also higher families. There is a nice argument supporting this hypothesis. n-fold discretization provided by covering associated with H corresponds to discretization for angular momentum eigenstates. Minimal discretization for $2j + 1$ states corresponds to $n = 2j + 1$. $j = 1/2$ requires $n = 2$ at least, $j = 1$ requires $n = 3$ at least, and so on. $n = 2j + 1$ allows spins $j \leq n - 1/2$. This spin-quantum phase connection at the level of space-time correlates has counterpart for the representations of quantum SU(2).

These rules would hold only for genuinely elementary particles corresponding to single partonic component and all bosonic particles of this kind are exotics (excitations in only "vibrational" degrees of freedom of partonic 2-surface with modular invariance eliminating quite a number of them.

Acknowledgements

I am grateful for my son Timo for stimulating discussions relating to p-adic numbers. I would like also to thank Carlos Castro for interesting email conversations relating to Riemann Zeta and for bringing in my attention the possible connection with Golden Mean and for informing me about Hilbert-Polya hypothesis. I want to express my gratitude to Matthew Watkins for providing me information about connections between Golden Mean and Riemann Zeta and for generous help, in particular for pointing out several inaccuracies in the proof of Riemann hypothesis. Also Prof. Masud Chaichian and Doc. Claus Montonen are acknowledged for encouragement and help.

Chapter 10

Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness

10.1 Introduction

Goro Kato has proposed an ontology of consciousness relying on category theory [A175, A176] . Physicist friendly summary of the basic concepts of category theory can be found in [A169]) whereas the books [A189, A205] provide more mathematically oriented representations. Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. To mention only one example, C. J. Isham [A169] has proposed that topos theory could provide a new approach to quantum gravity in which space-time points would be replaced by regions of space-time and that category theory could geometrize and dynamicize even logic by replacing the standard Boolean logic with a dynamical logic dictated by the structure of the fundamental category purely geometrically [A156] .

Although I am an innocent novice in this field and know nothing about the horrible technicalities of the field, I have a strong gut feeling that category theory might provide the desired systematic approach to quantum TGD proper, the general theory of consciousness, and the theory of cognitive representations [K50] .

10.1.1 Category theory as a purely technical tool

Category theory could help to disentangle the enormous technical complexities of the quantum TGD and to organize the existing bundle of ideas into a coherent conceptual framework. The construction of the geometry of the configuration space ("world of classical worlds") [K8, K7, K33, K16] , of classical configuration space spinor fields [K15] , and of S-matrix [K18] using a generalization of the quantum holography principle are especially natural applications. Category theory might also help in formulating the new TGD inspired view about number system as a structure obtained by "gluing together" real and p-adic number fields and TGD as a quantum theory based on this generalized notion of number [K8, K7, K71, K72, K70] .

10.1.2 Category theory based formulation of the ontology of TGD Universe

It is interesting to find whether also the ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences [K76] could be expressed elegantly using the language of the category theory.

There are indeed natural and non-trivial categories involved with many-sheeted space-time and the geometry of the configuration space ("the world of classical worlds"); with configuration space spinor fields; and with the notions of quantum jump, self and self hierarchy. Functors between these categories could express more precisely the quantum classical correspondences and self-referentiality of quantum states allowing them to express information about quantum jump sequence.

1. Self hierarchy has a structure of category and corresponds functorially to the hierarchical structure of the many-sheeted space-time.

2. Quantum jump sequence has a structure of category and corresponds functorially to the category formed by a sequence of maximally deterministic regions of space-time sheet. Even the quantum jump could have space-time correlates made possible by the generalization of the Boolean logic to what might be space-time correlate of quantum logic and allowing to identify space-time correlate for the notion of quantum superposition.

3. The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

4. In zero energy ontology (ZEO), which emerged many years after writing the first version of this chapter, causal diamonds (CDs) defined in terms of intersection of future and past directed light-cones form a category with arrow identified as inclusion.

5. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of imbedding space respects associativity (co-associativity) [K72]. The duality would allow to construct new preferred extremals of Kähler action.

10.1.3 Other applications

One can imagine also other applications.

1. Categories posses inherent logic [A156] based on the notion of sieves relying on the notion of presheaf which generalizes Boolean logic based on inclusion. In TGD framework inclusion is naturally replaced by topological condensation and this leads to a two-valued logic realizing space-time correlate of quantum logic based on the notions of quantum sieve and quantum topos.

   This suggests the possibility to geometrize the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through logic in which the law of excluded middle is not true. Interestingly, the p-adic logic of cognition is naturally 2-valued whereas the real number based logic of sensory experience allows excluded middle (is the person at the door in or out, in and out, or neither in nor out?). The quantum logic naturally associated with spinors (in the "world of classical worlds") is consistent with the logic based on quantum sieves.

2. Simple Boolean logic of right and wrong does not seem to be ideal for understanding moral rules. Same applies to the beauty-ugly logic of aesthetic experience. The logic based on quantum sieves would perhaps provide a more flexible framework.

3. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience. Here the new elements associated with the ontology of space-time due to the generalization of number concept would be central. Category theory could be also helpful in the modelling of conscious communications, in particular the telepathic communications based on sharing of mental images involving the same mechanism which makes possible space-time correlates of quantum logic and quantum superposition.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19].

10.2 What categories are?

In the following the basic notions of category theory are introduced and the notion of presheaf and category induced logic are discussed.

10.2.1 Basic concepts

Categories [A189, A205, A169] are roughly collections of objects A, B, C... and morphisms $f(A \to B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Topological/linear spaces form a category with continuous/linear maps acting as morphisms. Also algebraic structures of a given type form a category: morphisms are now homomorphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can can be very general: for instance, partial ordering $a \leq b$ can define morphism $f(A \to B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Group representation is example of this kind of a functor: now group action in group is mapped to a linear action at the level of the representations. Commuting square is an easy visual manner to understand the basic properties of a functor, see Fig. 10.2.1.

The product $C = AB$ for objects of categories is defined by the requirement that there are projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \to A)$ and $g(D \to B)$ there exist morphism $h(D \to C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically (see Fig. 10.2.1) this corresponds to a square diagram in which pairs A,B and C,D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC.

Examples of product categories are Cartesian products of topological and linear spaces, of differentiable manifolds, groups, etc. Also tensor products of linear spaces satisfies these axioms. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

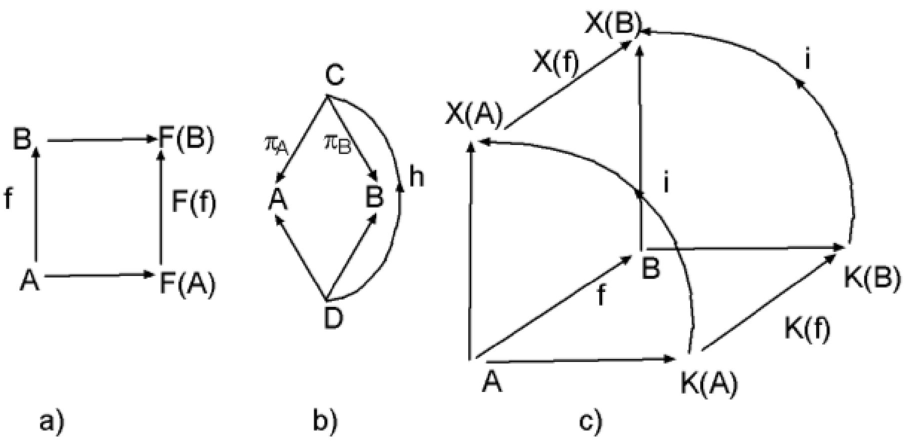

Figure 10.1: Commuting diagram associated with the definition of a) functor, b) product of objects of category, c) presheaf K as sub-object of presheaf X ("two pages of book".)

10.2.2 Presheaf as a generalization for the notion of set

Presheafs can be regarded as a generalization for the notion of set. Presheaf is a functor X that assigns to any object of a category **C** an object in the category **Set** (category of sets) and maps morphisms to morphisms (maps between sets for **C**). In order to have a category of presheafs, also morphisms between presheafs are needed. These morphisms are called natural transformations $N : X(A) \to Y(A)$ between the images $X(A)$ and $Y(A)$ of object A of **C**. They are assumed to obey the commutativity property $N(B)X(f) = Y(f)N(A)$ which is best visualized as a commutative square diagram. Set theoretic inclusion $i : X(A) \subset Y(A)$ is obviously a natural transformation.

An easy manner to understand and remember this definition is commuting diagram consisting of two pages of book with arrows of natural transformation connecting the corners of the pages: see Fig. 10.2.1.

As noticed, presheafs are generalizations of sets and a generalization for the notion of subset to a sub-object of presheaf is needed and this leads to the notion of topos [A156, A169] . In the classical set theory a subset of given sets X can be characterized by a mapping from set X to the set $\Omega = \{true, false\}$ of Boolean statements. Ω itself belongs to the category **C**. This idea generalizes to sub-objects whose objects are collections of sets: Ω is only replaced with its Cartesian power. It can be shown that in the case of presheafs associated with category **C** the sub-object classifier Ω can be replaced with a more general algebra, so called Heyting algebra [A156, A169] possessing the same basic operations as Boolean algebra (and, or, implication arrow, and negation) but is not in general equivalent with any Boolean algebra. What is important is that this generalized logic is inherent to the category **C** so that many-valued logic ceases to be an ad hoc construct in category theory.

In the theory of presheafs sub-object classifier Ω, which belongs to **Set**, is defined as a particular presheaf. Ω is defined by the structure of category **C** itself so that one has a geometrization of the notion of logic implied by the properties of category. The notion of sieve is essential here. A sieve for an object A of category **C** is defined as a collection of arrows $f(A \to ...)$ with the property that if $f(A \to B)$ is an arrow in sieve and if $g(B \to C)$ is any arrow then $gf(A \to C)$ belongs to sieve.

In the case that morphism corresponds to a set theoretic inclusion the sieve is just either empty set or the set of all sets of category containing set A so that there are only two sieves corresponding to Boolean logic. In the case of a poset (partially ordered set) sieves are sets for which all elements are larger than some element.

10.2.3 Generalized logic defined by category

The presheaf $\Omega : \mathbf{C} \to \mathbf{Set}$ defining sub-object classifier and a generalization of Boolean logic is defined as the map assigning to a given object A the set of all sieves on A. The generalization of maps $X \to \Omega$ defining subsets is based on the the notion of sub-object K. K is sub-object of presheaf X in the category of presheaves if there exist natural transformation $i : K \to X$ such that for each A one has $K(A) \subset X(A)$ (so that sub-object property is reduced to subset property).

The generalization of the map $X \to \Omega$ defining subset is achieved as follows. Let K be a sub-object of X. Then there is an associated characteristic arrow $\chi^K : X \to \Omega$ generalizing the characteristic Boolean valued map defining subset, whose components $\chi^K_A : X(A) \to \Omega(A)$ in **C** is defined as

$$\chi^K_A(x) = \{f(A \to B) | X(f)(x) \in K(B)\} \ .$$

By using the diagrammatic representation of Fig. 10.2.1 for the natural transformation i defining sub-object, it is not difficult to see that by the basic properties of the presheaf K $\chi^K_A(x)$ is a sieve. When morphisms f are inclusions in category Set, only two sheaves corresponding to all sets containing X and empty sheaf result. Thus binary valued maps are replaced with sieve-valued maps and sieves take the role of possible truth values. What is also new that truths and logic are in principle context dependent since each object A of **C** serves as a context and defines its own collection of sieves.

The generalization for the notion of point of set X exists also and corresponds to a selection of single element γ_A in the set $X(A)$ for each A object of **C**. This selection must be consistent with the action of morphisms $f(A \to B)$ in the sense that the matching condition $X(f)(\gamma_A) = \gamma_B$ is satisfied. It can happen that category of presheaves has no points at all since the matching condition need not be satisfied globally.

It turns out that TGD based notion of subsystem leads naturally to what might be called quantal versions of topos, presheaves, sieves and logic.

10.3 More precise characterization of the basic categories and possible applications

In the following the categories associated with self and quantum jump are discussed in more precise manner and applications to communications and cognition are considered.

10.3.1 Intuitive picture about the category formed by the geometric correlates of selves

Space-time surface $X^4(X^3)$ decomposes into regions obeying either real or p-adic topology and each region of this kind corresponds to an unentangled subsystem or self lasting at least one quantum jump. By the localization in the zero modes these decompositions are equivalent for all 3-surfaces X^3 in the quantum superposition defined by the prepared WCW spinor fields resulting in quantum jumps. There is a hierarchy of selves since selves can contain sub-selves. The entire space-time surface $X^4(X^3)$ represents the highest level of the self hierarchy.

This structure defines in a natural manner a category. Objects are all possible sub-selves contained in the self hierarchy: sub-self is set consisting of lower level sub-selves, which in turn have a further decomposition to sub-selves, etc... The naive expectation is that geometrically sub-self belongs to a self as a subset and this defines an inclusion map acting as a natural morphism in this category. This expectation is not quite correct. More natural morphisms are the arrows telling that self as a set of sub-selves contains sub-self as an element. These arrows define a structure analogous to a composite of hierarchy trees.

To be more precise, for a single space-time surface $X^4(X^3)$ this hierarchy corresponds to a subjective time slice of the self hierarchy defined by a single quantum jump. The sequence of hierarchies associated with a sequence of quantum jumps is a natural geometric correlate for the self hierarchy. This means that the objects are now sequences of submoments of consciousness. Sequences are not arbitrary. Self must survive its lifetime although sub-selves at various levels can disappear and reappear (generation and disappearance of mental images). Geometrically this means typically a phase transition transforming real or p_1-adic to p_2-adic space-time region with same topology as the environment. Also sub-selves can fuse to single sub-self. The constraints on self sequences must be such that it takes these processes into account. Note that these constraints emerge naturally from the fact that quantum jumps sequences define the sequences of surfaces $X^4(X^3)$.

By the rich anatomy of the quantum jump there is large number of quantum jumps leading from a given initial quantum history to a given final quantum history. One could envisage quantum jump also as a discrete path in the space of WCW spinor fields leading from the initial state to the final state. In particular, for given self there is an infinite number of closed elementary paths leading from the initial quantum history back to the initial quantum history and these paths in principle give all possible conscious information about a given quantum history/idea: kind of self morphisms are in question (analogous to, say, group automorhisms). Information about point of space is obtained only by moving around and coming back to the point, that is by studying the surroundings of the point. Self in turn can be seen as a composite of elementary paths defined by the quantum jumps. Selves can define arbitrarily complex composite closed paths giving information about a given quantum history.

10.3.2 Categories related to self and quantum jump

The categories defined by moments of consciousness and the notion of self

Since quantum jump involves state reduction and the sequence of self measurement reducing all entanglement except bound state entanglement, it defines a hierarchy of unentangled subsystems allowing interpretation as objects of a category. Arrows correspond to subsystem-system relationship and the two subsystems resulting in self measurement to the system. What subsystem corresponds mathematically is however not at all trivial and the naive description as a tensor factor does not work. Rather, a definition relying on the notion of p-adic length scale cutoff identified as a fundamental aspect of nature and consciousness is needed.

It is not clear what the statement that self corresponds to a subsystem which remains unentangled in subsequent quantum jump means concretely since subsystem can certainly change in some limits. What is clear that bound state entanglement between selves means a loss of consciousness. Category theory suggests that there should exist a functor between categories defined by two subsequent moments of consciousness. This functor maps submoments of consciousness to submoments of consciousness and arrows to arrows. Two subsequent submoments of consciousness belong to same sub-self is the functor maps the first one to the latter one. Thus category theory would play essential role in the precise definition of the notion of self.

The sequences of moments of consciousness form a larger category containing sub-selves as sequences of unentangled subsystems mapped to each other by functor arrows functoring subsequent quantum jumps to each other.

What might then be the ultimate characterizer of the self-identity? The theory of infinite primes suggests that space-time surface decomposes into regions labelled by finite p-adic primes. These primes must label also real regions rather than only p-adic ones, and one could understand this as resulting from a resonant transformation of intention to action. A p-adic space-time region characterized by prime p can transform to a real one or vice versa in quantum jump if the sizes of real and p-adic regions are characterized by the p-adic length scale L_p (or n-ary p-adic length scale $L_p(n)$. One can also consider the possibility that real region is accompanied by a p-adic region characterized by a definite prime p and providing a cognitive self-representation of the real region.

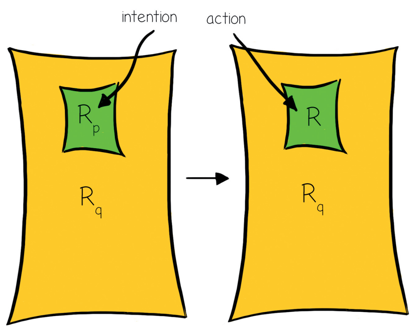

Figure 10.2: Transformation of intention to action as real-to-p-adc transformation and realization of intention as action as its reversal.

If this view is correct, the p-adic prime characterizing a given real or p-adic space-time sheet could be one characterizer of the self-identity. Self identity is lost in bound state entanglement with another space-time sheet (at least when a space-time sheet with smaller value of the p-adic prime joins by join along boundaries bond to a one with a higher value of the p-adic prime). Self identity is also lost if a space-time sheet characterized by a given p-adic prime disappears in quantum jump.

The category associated with quantum jump sequences

There are several similarities between the ontologies and epistemologies of TGD and of category theory. Conscious experience is always determined by the discrete paths in the space of configuration space spinor fields defined by a quantum jump connecting two quantum histories (states) and is never determined by single quantum history as such (quantum states are unconscious). Also category theory is about relations between objects, not about objects directly: self-morphisms give information about the object of category (in case of group composite paths would correspond to products of group automorphisms). Analogously closed paths determined by quantum jump sequences give information about single quantum history. The point is however that it is impossible to have direct knowledge about the quantum histories: they are not conscious.

One can indeed define a natural category, call it **QSelf**, applying to this situation. The objects of the category **QSelf** are initial quantum histories of quantum jumps and correspond to prepared quantum states. The discrete path defining quantum jump can be regarded as an elementary morphism. Selves are composites of elementary morphisms of the initial quantum history defined by quantum jumps: one can characterize the morphisms by the number of the elementary morphisms in the product. Trivial self contains no quantum jumps and corresponds to the identity morphism, null path. Thus the collection of all possible sequences of quantum jumps, that is collections of selves allows a description in terms of category theory although the category in question is not a subcategory of the category **Set**.

Category **QSelf** does not possess terminal and initial elements (for terminal (initial) element T there is exactly one arrow $A \rightarrow T$ ($T \rightarrow A$) for every A: now there are always many paths between quantum histories involved).

10.3.3 Communications in TGD framework

Goro Kato identifies communications between conscious entities as natural maps between them whereas in TGD natural maps bind submoments of consciousness to selves. In TGD framework quantum measurement and the sharing of mental images are the basic candidates for communications. The problem is that the identification of communications as sharing of mental images is not consistent with the naive view about subsystem as a tensor factor. Many-sheeted space-time however forces length scale dependent notion of subsystem at space-time level and this saves the situation.

What communications are?

Communication is essentially generation of desired mental images/sub-selves in receiver. Communication between selves need not be directly conscious: in this case communication would generate mental images at some lower level of self hierarchy of receiver: for instance generate large number of sub-sub-selves of similar type. This is like communications between organizations. Communication can be also vertical: self can generate somehow sub-self in some sub-sub....sub-self or sub-sub...sub-self can generate sub-self of self somehow. This is communication from boss to the lower levels organization or vice versa.

These communications should have direct topological counterparts. For instance, the communication between selves could correspond to an exchange of mental image represented as a space-time region of different topology inside sender self space-time sheet. The sender self would simply throw this space-time region to a receiver self like a ball. This mechanism applies also to vertical communications since the ball could be also thrown from a boss to sub...sub-self at some lower level of hierarchy and vice versa.

The sequence of space-time surfaces provides a direct topological counterpart for communication as throwing balls representing sub-selves. Quantum jump sequence contains space-time surfaces in which the regions corresponding to receiver and sender selves are connected by a join along boundaries bond (perhaps massless extremal) representing classically the communication: during the communication the receiver and sender would form single self. The cartoon vision about rays connecting the eyes of communication persons would make sense quite concretely.

More refined means of communication would generate sub-selves of desired type directly at the end of receiver. In this case it is not so obvious how the sequence $X^(X^3)$ of space-time surfaces could represent communication. Of course, one can question whether communication is really what happens in this kind of situation. For instance, sender can affect the environment of receiver to be such that receiver gets irritated (computer virus is good manner to achieve this!) but one can wonder whether this is real communication.

Communication as quantum measurement?

Quantum measurement generates one-one map between the states of the entangled systems resulting in quantum measurement. Both state function reduction and self measurement give rise to this kind of map. This map could perhaps be interpreted as quantum communication between unentangled subsystems resulting in quantum measurement. For the state reduction process the space-time correlates are the values of zero modes. For state preparation the space-time correlates should correspond to classical spinor field modes correlating for the two subsystems generated in self measurement.

Communication as sharing of mental images

It has become clear that the sharing of mental images induced by quantum entanglement of sub-selves of two separate selves represents genuine conscious communication which is analogous telepathy and provides general mechanism of remote mental interactions making possible even molecular recognition mechanisms.

1. The sharing of mental images is not possible unless one assumes that self hierarchy is defined by using the notion of length scale resolution defined by p-adic length scale. The notion of scale of resolution is indeed fundamental for all quantum field theories (renormalization group invariance) for all quantum field theories and without it the practical modelling of physics would not be possible. The notion reflects directly the length scale resolution of conscious experience. For a given sub-self of self the resolution is given by the p-adic length scale associated with the sub-self space-time sheet.

2. Length scale resolution emerges naturally from the fact that sub-self space-time sheets having Minkowskian signature of metric are separated from the one representing self by wormhole contacts with Euclidian signature of metric. The signature of the induced metric changes from Minkowskian signature to Euclidian signature at 'elementary particle horizons' surrounding the throats of the wormhole contacts and having degenerate induced metric. Elementary particle horizons are thus metrically two-dimensional light like surfaces analogous to the boundary of the light cone and allow conformal invariance. Elementary particle horizons act as causal horizons. Topologically condensed space-time sheets are analogous to black hole interiors and due to the lack of the causal connectedness the standard description of sub-selves as tensor factors of the state space corresponding to self is not appropriate.

   Hence systems correspond, not to the space-time sheets plus entire hierarchy of space-time sheets condensed to it, but rather, to space-time sheets with holes resulting when the space-time sheets representing subsystems are spliced off along the elementary particle horizons around wormhole contacts. This does not mean that all information about subsystem is lost: subsystem space-time sheet is only replaced by the elementary particle horizon. In analogy with the description of the black hole, some parameters (mass, charges,...) characterizing the classical fields created by the sub-self space-time sheet characterize sub-self.

   One can say that the state space of the system contains 'holes'. There is a hierarchy of state spaces labelled by p-adic primes defining length scale resolutions. This picture resolves a longstanding puzzle relating to the interpretation of the fact that particle is characterize by both classical and quantum charges. Particle cannot couple simultaneously to both and this is achieved if quantum charge is associated with the lowest level description of the particle as CP_2 extremal and classical charges to its description at higher levels of hierarchy.

3. The immediate implication indeed is that it is possible to have a situation in which two selves are unentangled although their sub-selves (mental images) are entangled. This corresponds to the fusion and sharing of mental images. The sharing of the mental images means that union of disjoint hierarchy trees with levels labelled by p-adic primes p is replaced by a union of hierarchy threes with horizontal lines connecting subsystems at the same level of hierarchy. Thus the classical correspondence defines a category of presheaves with both vertical arrows replaced by sub-self-self relationship, horizontal arrows representing sharing of mental images, and natural maps representing binding of submoments of consciousness to selves.

Comparison with Goro Kato's approach

It is of interest to compare Goro Kato's approach with TGD approach. The following correspondence suggests itself.

1. In TGD each quantum jumps defines a category analogous to the Goro Kato's category of open sets of some topological space but set theoretic inclusion replaced by topological condensation. The category defined by a moment of consciousness is dynamical whereas the category of open sets is non-dynamical.

2. The assignment of a 3-surface acting as a causal determinant to each unentangled subsystem defined by a moment of consciousness defines a unique "quantum presheaf" which is the counterpart of the presheaf in Goro Kato's theory. The conscious entity of Kato's theory corresponds to the classical correlate for a moment of consciousness.

3. Natural maps between the causal determinants correspond to the space-time correlates for the functor arrows defining the threads connecting submoments of consciousness to selves. In Goro Kato's theory natural maps are interpreted as communications between conscious entities. The sharing of mental images by quantum entanglement between subsystems of unentangled systems defines horizontal bi-directional arrows between subsystems associated with same moment of consciousness and is counterpart of communication in TGD framework. It replaces the union of disjoint hierarchy trees associated with various unentangled subsystems with hierarchy trees having horizontal connections defining the bi-directional arrows. The sharing of mental images is not possible if subsystem is identified as a tensor factor and thus without taking into account length scale resolution.

10.3.4 Cognizing about cognition

There are close connections with basic facts about cognition.

1. Categorization means classification and abstraction of common features in the class formed by the objects of a category. Already quantum jump defines category with hierarchical structure and can be regarded as consciously experienced analysis in which totally entangled entire universe $U\Psi_i$ decomposes to a product of maximally unentangled subsystems. The sub-selves of self are like elements of set and are experienced as separate objects whereas sub-sub-selves of sub-self self experiences as an average: they belong to a class or category formed by the sub-self. This kind of averaging occurs also for the contributions of quantum jumps to conscious experience of self.

2. The notions of category theory might be useful in an attempt to construct a theory of cognitive structures since cognition is indeed to high degree classification and abstraction process. The sub-selves of a real self indeed have p-adic space-time sheets as geometric correlates and thus correspond to cognitive sub-selves, thoughts. A meditative experience of empty mind means in case of real self the total absence of thoughts.

3. Predicate logic provides a formalization of the natural language and relies heavily on the notion of n-ary relation. Binary relations $R(a,b)$ corresponds formally to the subset of the product set $A \times B$. For instance, statements like 'A does something to B' can be expressed as a binary relation, particular kind of arrow and morphism ($A \leq B$ is a standard example). For sub-selves this relation would correspond to a dynamical evolution at space-time level modelling the interaction between A and B. The dynamical path defined by a sequence of quantum jumps is able to describe this kind of relationships too at level of conscious experience. For instance, 'A touches B' would involve the temporary fusion of sub-selves A and B to sub-self C.

10.4 Logic and category theory

Category theory allows naturally more general than Boolean logics inherent to the notion of topos associated with any category. Basic question is whether the ordinary notion of topos algebra based on set theoretic inclusion or the notion of quantum topos based on topological condensation is physically appropriate. Starting from the quasi-Boolean algebra of open sets one ends up to the conclusion that quantum logic is more natural. Also WCW spinor fields lead naturally to the notion of quantum logic.

10.4.1 Is the logic of conscious experience based on set theoretic inclusion or topological condensation?

The algebra of open sets with intersections and unions and complement defined as the interior of the complement defines a modification of Boolean algebra having the peculiar feature that the points at the boundary of the closure of open set cannot be said to belong to neither interior of open set or of its complement. There are two options concerning the interpretation.

1. 3-valued logic could be in question. It is however not possible to understand this three-valuedness if one defines the quasi-Boolean algebra of open sets as Heyting algebra. The resulting logic is two-valued and the points at boundaries of the closure do not correspond neither to the statement or its negation. In p-adic context the situation changes since p-adic open sets are also closed so that the logic is strictly Boolean. That our ordinary cognitive mind is Boolean provides a further good reason for why cognition is p-adic.

2. These points at the boundary of the closure belong to both interior and exterior in which case a two-valued "quantum logic" allowing superposition of opposite truth values is in question. The situation is indeed exactly the same as in the case of space-time sheet having wormhole contacts to several space-time sheets.

The quantum logic brings in mind Zen consciousness [J3] (which I became fascinated of while reading Hofstadter's book "Gödel, Escher,Bach" [A166]) and one can wonder whether selves having real space-time sheets as geometric correlates and able to live simultaneously in many parallel worlds correspond to Zen consciousness and Zen logic. Zen logic would be also logic of sensory experience whereas cognition would obey strictly Boolean logic.

The causal determinants associated with space-time sheets correspond to light like 3-surfaces which could elementary particle horizons or space-time boundaries and possibly also 3-surfaces separating two maximal deterministic regions of a space-time sheet. These surfaces act as 3-dimensional quantum holograms and have the strange Zen property that they are neither space-like nor time-like so that they represent both the state and the process. In the TGD based model for topological quantum computation (TQC) light-like boundaries code for the computation so that TQC program code would be equivalent with the running program [K78] .

10.4.2 Do WCW spinor fields define quantum logic and quantum topos

I have proposed already earlier that WCW spinor fields define what might be called quantum logic. One can wonder whether WCW spinor s could also naturally define what might be called quantum topos since the category underlying topos defines the logic appropriate to the topos. This question remains unanswered in the following: I just describe the line of though generalizing ordinary Boolean logic.

Finite-dimensional spinors define quantum logic

Spinors at a point of an $2N$-dimensional space span 2^N-dimensional space and spinor basis is in one-one correspondence with Boolean algebra with N different truth values (N bits). 2N=2-dimensional case is simple: Spin up spinor= true and spin-dow spinor=false. The spinors for $2N$-dimensional space are obtained as an N-fold tensor product of 2-dimensional spinors (spin up,spin down): just like in the case of Cartesian power of Ω.

Boolean spinors in a given basis are eigen states for a set N mutually commuting sigma matrices providing a representation for the tangent space group acting as rotations. Boolean spinors define N Boolean statements in the set Ω^N so that one can in a natural manner assign a set with a Boolean spinor. In the real case this group is $SO(2N)$ and reduces to $SU(N)$ for Kähler manifolds. For pseudo-euclidian metric some non-compact variant of the tangent space group is involved. The selections of N mutually commuting generators are labelled by the flag-manifold $SO(2N)/SO(2)^N$ in real context and by the flag-manifold $U(N)/U(1)^N$ in the complex case. The selection of these generators defines a collection of N 2-dimensional linear subspaces of the tangent space.

Spinors are in general complex superpositions of spinor basis which can be taken as the product spinors. The quantum measurement of N spins representing the Cartan algebra of $SO(2N)$

$(SU(N))$ leads to a state representing a definite Boolean statement. This suggests that quantum jumps as moments of consciousness quite generally make universe classical, not only in geometric but also in logical sense. This is indeed what the state preparation process for WCW spinor field seems to do.

Quantum logic for finite-dimensional spinor fields

One can generalize the idea of the spinor logic also to the case of spinor fields. For a given choice of the local spinor basis (which is unique only modular local gauge rotation) spinor field assigns to each point of finite-dimensional space a quantum superposition of Boolean statements decomposing into product of N statements.

Also now one can ask whether it is possible to find a gauge in which each point corresponds to definite Boolean statement and is thus an eigen state of a maximal number of mutually commuting rotation generators Σ_{ij}. This is not trivial if one requires that Dirac equation is satisfied. In the case of flat space this is certainly true and constant spinors multiplied by functions which solve d'Alembert equation provide a global basis.

The solutions of Dirac equation in a curved finite-dimensional space do not usually possess a definite spin direction globally since spinor curvature means the presence of magnetic spin-flipping interaction and since there need not exist a global gauge transformation leading to an eigen state of the local Cartan algebra everywhere. What might happen is that the local gauge transformation becomes singular at some point: for instance, the direction of spin would be radial around given point and become ill defined at the point. This is much like the singularities for vector fields on sphere. The spinor field having this kind of singularity should vanish at singularity but the local gauge rotation rotating spin in same direction everywhere is necessarily ill-defined at the singularity.

In fact, this can be expressed using the language of category theory. The category in question corresponds to a presheaf which assigns to the points of the base space the fiber space of the spinor bundle. The presence of singularity means that there are no global section for this presheaf, that is a continuous choice of a non-vanishing spinor at each point of the base space. The so called Kochen-Specker theorem discussed in [A169] is closely related to a completely analogous phenomenon involving non-existence of global sections and thus non-existence of a global truth value.

Thus in case of curved spaces is not necessarily possible to have spinor field basis representing globally Boolean statements and only the notion of locally Boolean logic makes sense. Indeed, one can select the basis to be eigen state of maximal set of mutually commuting rotation generators in single point of the compact space. Any such choice does.

Quantum logic and quantum topos defined by the prepared WCW spinor fields

The prepared WCW spinor fields occurring as initial and final states of quantum jumps are the natural candidates for defining quantum logic. The outcomes of the quantum jumps resulting in the state preparation process are maximally unentangled states and are as close to Boolean states as possible.

WCW spinors correspond to fermionic Fock states created by infinite number of fermionic (leptonic and quarklike) creation and annihilation operators. The spin degeneracy is replaced by the double-fold degeneracy associated with a given fermion mode: given state either contains fermion or not and these two states represent true and false now. If WCW were flat, the Fock state basis with definite fermion and anti-fermion numbers in each mode would be in one-one correspondence with Boolean algebra.

Situation is however not so simple. Finite-dimensional curved space is replaced with the fiber degrees of freedom of WCW in which the metric is non-vanishing. The precise analogy with the finite-dimensional case suggests that if the curvature form of the WCW spinor connection is nontrivial, it is impossible to diagonalize even the prepared maximally unentangled WCW spinor fields Ψ_i in the entire fiber of WCW (quantum fluctuating degrees of freedom) for given values of the zero modes. Local singularities at which the spin quantum numbers of the diagonalized but vanishing WCW spinor field become ill-defined are possible also now.

In the infinite-dimensional context the presence of the fermion-anti-fermion pairs in the state means that it does not represent a definite Boolean statement unless one defines a more general basis of WCW spinor s for which pairs are present in the states of the state basis: this generalization is indeed possible. The sigma matrices of the WCW appearing in the spinor connection term of the Dirac operator of WCW indeed create fermion-fermion pairs. What is decisive, is not the absence of fermion-anti-fermion pairs, but the possibility that the spinor field basis cannot be reduced to eigen states of the local Cartan algebra in fiber degrees of freedom globally.

Also for bound states of fermions (say leptons and quarks) it is impossible to reduce the state to a definite Boolean statement even locally. This would suggest that fermionic logic does not reduce to a completely Boolean logic even in the case of the prepared states.

Thus WCW spinor fields could have interpretation in terms of non-Boolean quantum logic possessing Boolean logics only as sub-logics and define what might be called quantum topos. Instead of Ω^N-valued maps the values for the maps are complex valued quantum superpositions of truth values in Ω^N.

An objection against the notion of quantum logic is that Boolean algebra operations AND and OR do not preserve fermion number so that quantum jump sequences leading from the product state defined by operands to the state representing the result of operation are therefore not possible. One manner to circumvent the objection is to consider the sub-algebra spanned by fermion and anti-fermion pairs for given mode so that fermion number conservation is not a problem. The objection can be also circumvented for pairs of space-time sheets with opposite time orientations and thus opposite signs of energies for particles. One can construct the algebra in question as pairs of many fermion states consisting of positive energy fermion and negative energy anti-fermion so that all states have vanishing fermion number and logical operations become possible. Pairs of MEs with opposite time orientations are excellent candidates for carries of these fermion-anti-fermion pairs.

Quantum classical correspondence and quantum logic

The intuitive idea is that the global Boolean statements correspond to sections of Z^2 bundle. Möbius band is a prototype example here. The failure of a global statement would reduce to the non-existence of global section so that true would transforms to false as one goes around full 2π rotation.

One can ask whether fermionic quantum realization of Boolean logic could have space-time counterpart in terms of Z_2 fiber bundle structure. This would give some hopes of having some connection between category theoretical and fermionic realizations of logic. The following argument stimulated by email discussion with Diego Lucio Rapoport suggests that this might be the case.

1. The hierarchy of Planck constants realized using the notion of generalized imbedding space involves only groups $Z_{n_a} \times Z_{n_b}$, $n_a, n_b \neq 2$ if one takes Jones inclusions as starting point. There is however no obvious reason for excluding the values $n_a = 2$ and $n_b = 2$ and the question concerns physical interpretation. Even if one allows only $n_i \geq 3$ one can ask for the physical interpretation for the factorization $Z_{2n} = Z_2 \times Z_n$. Could it perhaps relate to a space-time correlates for Boolean two-valuedness?

2. An important implication of fiber bundle structure is that the partonic 2-surfaces have $Z_{n_a} \times Z_{n_b} = Z_{n_a n_b}$ as a group of conformal symmetries. I have proposed that n_a or n_b is even for fermions so that Z_2 acts as a conformal symmetry of the partonic 2-surface. Both n_a and n_b would be odd for truly elementary bosons. Note that this hypothesis makes sense also for $n_i \geq 3$.

3. Z_2 conformal symmetry for fermions would imply that all partonic 2-surfaces associated with fermions are hyper-elliptic. As a consequence elementary particle vacuum functionals defined in modular degrees of freedom would vanish for fermions for genus $g > 2$ so that only three fermion families would be possible in accordance with experimental facts. Since gauge bosons and Higgs correspond to pairs of partonic 2-surfaces (the throats of the wormhole contact) one has 9 gauge boson states labelled by the pairs (g_1, g_2) which can be grouped to SU(3) singlet and octet. Singlet corresponds to ordinary gauge bosons.

super-symplectic bosons are truly elementary bosons in the sense that they do not consist of fermion-anti-fermion pairs. For them both n_a and n_b should be odd if the correspondence is taken seriously and all genera would be possible. The super-conformal partners of these bosons have the quantum numbers of right handed neutrino. Since both spin directions are possible, one can ask whether Boolean Z_2 must be present also now. This need not be the case, ν_R generates only super-symmetries and does not define a family of fermionic oscillator operators. The electro-weak spin of ν_R is frozen and it does not couple at all to electro-weak intersections. Perhaps (only) odd values of n_i are possible in this case.

4. If fermionic Boolean logic has a space-time correlate, one can wonder whether the fermionic Z_2 conformal symmetry might correspond to a space-time correlate for the Boolean true-false dichotomy. If the partonic 2-surface contains points which are fixed points of Z_2 symmetry, there exists no everywhere non-vanishing sections. Furthermore, induced spinor fields should vanish at the fixed points of Z_2 symmetry since they correspond to singular orbifold points so that one could not actually have a situation in which true and false are true simultaneously. Global sections could however fail to exist since CP_2 spinor bundle is non-trivial.

10.4.3 Category theory and the modelling of aesthetic and ethical judgements

Consciousness theory should allow to model model the logics of ethics and aesthetics. Evolution (representable as p-adic evolution in TGD framework) is regarded as something positive and is a good candidate for defining universal ethics in TGD framework. Good deeds are such that they support this evolution occurring in statistical sense in any case. Moral provides a practical model for what good deeds are and moral right-wrong statements are analogous to logical statements. Often however the two-valued right-wrong logic seems to be too simplistic in case of moral statements. Same applies to aesthetic judgements. A possible application of the generalized logics defined by the inherent structure of categories relates to the understanding of the dilemmas associated with the moral and aesthetic rules.

As already found, quantum versions of sieves provide a formal generalization of Boolean truth values as a characteristic of a given category. Generalized moral rules could perhaps be seen as sieve valued statements about deeds. Deeds are either right or wrong in what might be called Boolean moral code. One can also consider Zen moral in which some deeds can be said to be right and wrong simultaneously. Some deeds could also be such that there simply exists no globally consistent moral rule: this would correspond to the non-existence of what is called global section assigning to each object of the category consisting of the pairs formed by a moral agents and given deed) a sieve simultaneously.

10.5 Platonism, Constructivism, and Quantum Platonism

During years I have been trying to understand how Category Theory and Set Theory relate to quantum TGD inspired view about fundamentals of mathematics and the outcome section is added to this chapter several years after its first writing. I hope that reader does not experience too unpleasant discontinuity. I managed to clarify my thoughts about what these theories are by reading the article Structuralism, Category Theory and Philosophy of Mathematics by Richard Stefanik [A208] . Blog discussions and email correspondence with Sampo Vesterinen have been very stimulating and inspired the attempt to represent TGD based vision about the unification of mathematics, physics, and consciousness theory in a more systematic manner.

Before continuing I want to summarize the basic ideas behind TGD vision. One cannot understand mathematics without understanding mathematical consciousness. Mathematical consciousness and its evolution must have direct quantum physical correlates and by quantum classical correspondence these correlates must appear also at space-time level. Quantum physics must allow to realize number as a conscious experience analogous to a sensory quale. In TGD based ontology there is no need to postulate physical world behind the quantum states as mathematical entities (theory is the reality). Hence number cannot be any physical object, but can be identified as a quantum state or its label and its number theoretical anatomy is revealed by the conscious experiences induced by the number theoretic variants of particle reactions. Mathematical systems and their axiomatics are dynamical evolving systems and physics is number theoretically universal selecting rationals and their extensions in a special role as numbers, which can can be regarded elements of several number fields simultaneously.

10.5.1 Platonism and structuralism

There are basically two philosophies of mathematics.

1. Platonism assumes that mathematical objects and structures have independent existence. Natural numbers would be the most fundamental objects of this kind. For instance, each natural number has its own number-theoretical anatomy decomposing into a product of prime numbers defining the elementary particles of Platonia. For quantum physicist this vision is attractive, and even more so if one accepts that elementary particles are labelled by primes (as I do)! The problematic aspects of this vision relate to the physical realization of the Platonia. Neither Minkowski space-time nor its curved variants understood in the sense of set theory have no room for Platonia and physical laws (as we know them) do not seem to allow the realization of all imaginable internally consistent mathematical structures.

2. Structuralist believes that the properties of natural numbers result from their relations to other natural numbers so that it is not possible to speak about number theoretical anatomy in the Platonic sense. Numbers as such are structureless and their relationships to other numbers provide them with their apparent structure. According to [A208] structuralism is however not enough for the purposes of number theory: in combinatorics it is much more natural to use intensional definition for integers by providing them with inherent properties such as decomposition into primes. I am not competent to take any strong attitudes on this statement but my physicist's intuition tells that numbers have number theoretic anatomy and that this anatomy can be only revealed by the morphisms or something more general which must have physical counterparts. I would like to regard numbers are analogous to bound states of elementary particles. Just as the decays of bound states reveal their inner structure, the generalizations of morphisms would reveal to the mathematician the inherent number theoretic anatomy of integers.

10.5.2 Structuralism

Set theory and category theory represent two basic variants of structuralism and before continuing I want to clarify to myself the basic ideas of structuralism: the reader can skip this section if it looks too boring.

Set theory

Structuralism has many variants. In set theory [A86] the elements of set are treated as structureless points and sets with the same cardinality are equivalent. In number theory additional structure must be introduced. In the case of natural numbers one introduces the notion of successor and induction axiom and defines the basic arithmetic operations using these. Set theoretic realization is not unique. For instance, one can start from empty set Φ identified as 0, identify 1 as $\{\Phi\}$, 2 as $\{0,1\}$ and so on. One can also identify 0 as Φ, 1 as $\{0\}$, 2 as $\{\{0\}\}$,.... For both physicist and consciousness theorist these formal definitions look rather weird.

The non-uniqueness of the identification of natural numbers as a set could be seen as a problem. The structuralist's approach is based on an extensional definition meaning that two objects are regarded as identical if one cannot find any property distinguishing them: object is a representative for the equivalence class of similar objects. This brings in mind gauge fixing to the mind of physicists.

Category theory

Category theory [A15] represents a second form of structuralism. Category theorist does not worry about the ontological problems and dreams that all properties of objects could be reduced to the arrows and formally one could identify even objects as identity morphisms (looks like a trick to me). The great idea is that functors between categories respecting the structure defined by morphisms provide information about categories. Second basic concept is natural transformation which maps functors to functors in a structure preserving manner. Also functors define a category so that one can construct endless hierarchy of categories. This approach has enormous unifying power since functors and natural maps systemize the process of generalization. There is no doubt that category theory forms a huge piece of mathematics but I find difficult to believe that arrows can catch all of it.

The notion of category can be extended to that of n-category: in [L4] I described a geometric realization of this hierarchy in which one defines 1-morphisms by parallel translations, 2-morphisms by parallel translations of parallel translations, and so on. In infinite-dimensional space this hierarchy would be infinite. Abstractions about abstractions about.., thoughts about thoughts about, statements about statements about..., is the basic idea behind this interpretation. Also the hierarchy of logics of various orders corresponds to this hierarchy. This encourages to see category theoretic thinking as being analogous to higher level self reflection which must be distinguished from the direct sensory experience.

In the case of natural numbers category theoretician would identify successor function as the arrow binding natural numbers to an infinitely long string with 0 as its end. If this approach would work, the properties of numbers would reflect the properties of the successor function.

10.5.3 The view about mathematics inspired by TGD and TGD inspired theory of consciousness

TGD based view might be called quantum Platonism. It is inspired by the requirement that both quantum states and quantum jumps between them are able to represent number theory and that all quantum notions have also space-time correlates so that Platonia should in some sense exist also at the level of space-time. Here I provide a brief summary of this view as it is now. The articles "TGD" [L6] and "TGD inspired theory of consciousness" [L7] provide an overview about TGD and TGD inspired theory of consciousness.

Physics is fixed from the uniqueness of infinite-D existence and number theoretic universality

1. The basic philosophy of quantum TGD relies on the geometrization of physics in terms of infinite-dimensional Kähler geometry of WCW, whose uniqueness is forced by the mere mathematical existence. Space-time dimension and imbedding space $H = M^4 \times CP_2$ are fixed among other things by this condition and allow interpretation in terms of classical number fields. Physical states correspond to WCW spinor fields with WCW spinor s having interpretation as Fock states. Rather remarkably, WCW Clifford algebra defines standard representation of so called hyper finite factor of II_1, perhaps the most fascinating von Neumann algebra.

2. Number theoretic universality states that all number fields are in a democratic position. This vision can be realized by requiring generalization of notions of imbedding space by gluing together real and p-adic variants of imbedding space along common algebraic numbers. All algebraic extensions of p-adic numbers are allowed. Real and p-adic space-time sheets intersect along common algebraics. The identification of the p-adic space-time sheets as correlates of cognition and intentionality explains why cognitive representations at space-time level are always discrete. Only space-time points belonging to an algebraic extension of rationals associated contribute to the data defining S-matrix. These points define what I call number theoretic braids. The interpretation in of algebraic discreteness terms of a physical realization of axiom of choice is highly suggestive. The axiom of choice would be dynamical and evolving quantum jump by quantum jump as the algebraic complexity of quantum states increases.

Holy trinity of existence

In TGD framework one would have 3-levelled ontology numbers should have representations at all these levels [L7] .

1. Subjective existence as a sequence of quantum jumps giving conscious sensory representations for numbers and various geometric structures would be the first level.

2. Quantum states would correspond to Platonia of mathematical ideas and mathematician- or if one is unwilling to use this practical illusion- conscious experiences about mathematic ideas, would be in quantum jumps. The quantum jumps between quantum states respecting the symmetries characterizing the mathematical structure would provide conscious information about the mathematical ideas not directly accessible to conscious experience. Mathematician would live in Plato's cave. There is no need to assume any independent physical reality behind quantum states as mathematical entities since quantum jumps between these states give rise to conscious experience. Theory-reality dualism disappears since the theory is reality or more poetically: painting is the landscape.

3. The third level of ontology would be represented by classical physics at the space-time level essential for quantum measurement theory. By quantum classical correspondence space-time physics would be like a written language providing symbolic representations for both quantum states and changes of them (by the failure of complete classical determinism of the fundamental variational principle). This would involve both real and p-adic space-time sheets corresponding to sensory and cognitive representations of mathematical concepts. This representation makes possible the feedback analogous to formulas written by mathematician crucial for the ability of becoming conscious about what one was conscious of and the dynamical character of this process allows to explain the self-referentiality of consciousness without paradox.

This ontology releases a deep Platonistic sigh of relief. Since there are no physical objects, there is no need to reduce mathematical notions to objects of the physical world. There are only quantum states identified as mathematical entities labelled naturally by integer valued quantum numbers; conscious experiences, which must represent sensations giving information about the number theoretical anatomy of a given quantum number; and space-time surfaces providing space-time correlates for quantum physics and therefore also for number theory and mathematical structures in general.

Factorization of integers as a direct sensory perception?

Both physicist and consciousness theorist would argue that the set theoretic construction of natural numbers could not be farther away from how we experience integers. Personally I feel that neither structuralist's approach nor Platonism as it is understood usually are enough. Mathematics is a conscious activity and this suggests that quantum theory of consciousness must be included if one wants to build more satisfactory view about fundamentals of mathematics.

Oliver Sack's book *The man who mistook his wife for a hat* [J2] (see also [K62]) contains fascinating stories about those aspects of brain and consciousness which are more or less mysterious from the view point of neuroscience. Sacks tells in his book also a story about twins who were classified as idiots but had amazing number theoretical abilities. I feel that this story reveals something very important about the real character of mathematical consciousness.

The twins had absolutely no idea about mathematical concepts such as the notion of primeness but they could factorize huge numbers and tell whether they are primes. Their eyes rolled wildly during the process and suddenly their face started to glow of happiness and they reported a discovery of a factor. One could not avoid the feeling that they quite concretely saw the factorization process. The failure to detect the factorization served for them as the definition of primeness. For them the factorization was not a process based on some rules but a direct sensory perception.

The simplest explanation for the abilities of twins would in terms of a model of integers represented as string like structures consisting of identical basic units. This string can decay to strings. If string containing n units decaying into $m > 1$ identical pieces is not perceived, the conclusion is that a prime is in question. It could also be that decay to units smaller than 2 was forbidden in this dynamics. The necessary connection between written representations of numbers and representative strings is easy to build as associations.

This kind theory might help to understand marvellous feats of mathematicians like Ramanujan who represents a diametrical opposite of Groethendienck as a mathematician (when Groethendienck was asked to give an example about prime, he mentioned 57 which became known as Groethendienck prime!).

The lesson would be that one very fundamental representation of integers would be, not as objects, but conscious experiences. Primeness would be like the quale of redness. This of course does not exclude also other representations.

Experience of integers in TGD inspired quantum theory of consciousness

In quantum physics integers appear very naturally as quantum numbers. In quantal axiomatization or interpretation of mathematics same should hold true.

1. In TGD inspired theory of consciousness [L7] quantum jump is identified as a moment of consciousness. There is actually an entire fractal hierarchy of quantum jumps consisting of quantum jumps and this correlates directly with the corresponding hierarchy of physical states and dark matter hierarchy. This means that the experience of integer should be reducible to a certain kind of quantum jump. The possible changes of state in the quantum jump would characterize the sensory representation of integer.

2. The quantum state as such does not give conscious information about the number theoretic anatomy of the integer labelling it: the change of the quantum state is required. The above geometric model translated to quantum case would suggest that integer represents a multiplicatively conserved quantum number. Decays of this this state into states labelled by integers n_i such that one has $n = \prod_i n_i$ would provide the fundamental conscious representation for the number theoretic anatomy of the integer. At the level of sensory perception based the space-time correlates a string-like bound state of basic particles representing n=1.

3. This picture is consistent with the Platonist view about integers represented as structured objects, now labels of quantum states. It would also conform with the view of category theorist in the sense that the arrows of category theorist replaced with quantum jumps are necessary to gain conscious information about the structure of the integer.

Infinite primes and arithmetic consciousness

Infinite primes [K70] were the first mathematical fruit of TGD inspired theory of consciousness and the inspiration for writing this posting came from the observation that the infinite primes at the lowest level of hierarchy provide a representation of algebraic numbers as Fock states of a super-symmetric arithmetic QFT so that it becomes possible to realize quantum jumps revealing the number theoretic anatomy of integers, rationals, and perhaps even that of algebraic numbers.

1. Infinite primes have a representation as Fock states of super-symmetric arithmetic QFT and at the lowest level of hierarchy they provide representations for primes, integers, rationals and algebraic numbers in the sense that at the lowest level of hierarchy of second quantizations the simplest infinite primes are naturally mapped to rationals whereas more complex infinite primes having interpretation as bound states can be mapped to algebraic numbers. Conscious experience of number can be assigned to the quantum jumps between these quantum states revealing information about the number theoretic anatomy of the number represented. It would be wrong to say that rationals only label these states: rather, these states represent rationals and since primes label the particles of these states.

2. More concretely, the conservation of number theoretic energy defined by the logarithm of the rational assignable with the Fock state implies that the allowed decays of the state to a product of infinite integers are such that the rational can decompose only into a product of rationals. These decays could provide for the above discussed fundamental realization of multiplicative aspects of arithmetic consciousness. Also additive aspects are represented

since the exponents k in the powers p^k appearing in the decomposition are conserved so that only the partitions $k = \sum_i k_i$ are representable. Thus both product decompositions and partitions, the basic operations of number theorist, are represented.

3. The higher levels of the hierarchy represent a hierarchy of abstractions about abstractions bringing strongly in mind the hierarchy of n-categories and various similar constructions including n:th order logic. It also seems that the n+1:th level of hierarchy provides a quantum representation for the n:th level. Ordinary primes, integers, rationals, and algebraic numbers would be the lowest level, -the initial object- of the hierarchy representing nothing at low level. Higher levels could be reduced to them by the analog of category theoretic reductionism in the sense that there is arrow between n:th and n+1:th level representing the second quantization at this level. On can also say that these levels represent higher reflective level of mathematical consciousness and the fundamental sensory perception corresponds the lowest level.

4. Infinite primes have also space-time correlates. The decomposition of particle into partons can be interpreted as a infinite prime and this gives geometric representations of infinite primes and also rationals. The finite primes appearing in the decomposition of infinite prime correspond to bosonic or fermionic partonic 2-surfaces. Many-sheeted space-time provides a representation for the hierarchy of second quantizations: one physical prediction is that many particle bound state associated with space-time sheet behaves exactly like a boson or fermion. Nuclear string model is one concrete application of this idea: it replaces nucleon reductionism with reductionism occurs first to strings consisting of $A \leq 4$ nuclei and which in turn are strings consisting of nucleons. A further more speculative representation of infinite rationals as space-time surfaces is based on their mapping to rational functions.

Number theoretic Brahman=Atman identity

The notion of infinite primes leads to the notion of algebraic holography in which space-time points possess infinitely rich number-theoretic anatomy. This anatomy would be due to the existence of infinite number of real units defined as ratios of infinite integers which reduce to unit in the real sense and various p-adic senses. This anatomy is not visible in real physics but can contribute directly to mathematical consciousness [K70] .

The anatomies of single space-time point could represent the entire world of classical worlds and quantum states of universe: the number theoretic anatomy is of course not visible in the structure of these these states. Therefore the basic building brick of mathematics - point- would become the Platonia able to represent all of the mathematics consistent with the laws of quantum physics. Space-time points would evolve, becoming more and more complex quantum jump by quantum jump. WCW and quantum states would be represented by the anatomies of space-time points. Some space-time points are more "civilized" than others so that space-time decomposes into "civilizations" at different levels of mathematical evolution.

Paths between space-time points represent processes analogous to parallel translations affecting the structure of the point and one can also define n-parallel translations up to $n = 4$ at level of space-time and $n = 8$ at level of imbedding space. At level of world of classical worlds whose points are representable as number theoretical anatomies arbitrary high values of n can be realized.

It is fair to say that the number theoretical anatomy of the space-time point makes it possible self-reference loop to close so that structured points are able to represent the physics of associated with with the structures constructed from structureless points. Hence one can speak about algebraic holography or number theoretic Brahman=Atman identity.

Finite measurement resolution, Jones inclusions, and number theoretic braids

In the history of physics and mathematics the realization of various limitations have been the royal road to a deeper understanding (Uncertainty Principle, Gödel's theorem). The precision of quantum measurement, sensory perception, and cognition are always finite. In standard quantum measurement theory this limitation is not taken into account but forms a corner stone of TGD based vision about quantum physics and of mathematics too as I want to argue in the following.

The finite resolutions has representation both at classical and quantum level.

1. At the level of quantum states finite resolution is represented in terms of Jones inclusions N subset M of hyper-finite factors of type II_1 (HFFs) [K25] . N represents measurement resolution in the sense that the states related by the action of N cannot be distinguished in the measurement considered. Complex rays are replaced by N rays. This brings in non-commutativity via quantum groups [K9] . Non-commutativity in TGD Universe would be therefore due to a finite measurement resolution rather than something exotic emerging in the Planck length scale. Same applies to p-adic physics: p-adic space-time sheets have literally infinite size in real topology!

2. At the space-time level discretization implied by the number theoretic universality could be seen as being due to the finite resolution with common algebraic points of real and p-adic variant of the partonic 3-surface chosen as representatives for regions of the surface. The solutions of modified Dirac equation are characterized by the prime in question so that the preferred prime makes itself visible at the level of quantum dynamics and characterizes the p-adic length scale fixing the values of coupling constants. Discretization could be also understood as effective non-commutativity of imbedding space points due to the finite resolution implying that second quantized spinor fields anti-commute only at a discrete set of points rather than along stringy curve.

In this framework it is easy to imagine physical representations of number theoretical and other mathematical structures.

1. Every compact group corresponds to a hierarchy of Jones inclusions corresponding to various representations for the quantum variants of the group labelled by roots of unity. I would be surprised if non-compact groups would not allow similar representation since HFF can be regarded as infinite tensor power of n-dimensional complex matrix algebra for any value of n. Somewhat paradoxically, the finite measurement resolution would make possible to represent Lie group theory physically [K25] .

2. There is a strong temptation to identify the Galois groups of algebraic numbers as the infinite permutation group S_∞ consisting of permutations of finite number of objects, whose projective representations give rise to an infinite braid group B_∞. The group algebras of these groups are HFFs besides the representation provided by the spinors of the world of classical worlds having physical identification as fermionic Fock states. Therefore physical states would provide a direct representation also for the more abstract features of number theory [K35] .

3. Number theoretical braids crucial for the construction of S-matrix provide naturally representations for the Galois groups G associated with the algebraic extensions of rationals as diagonal imbeddings $G \times G \times ....$ to the completion of S_∞ representable also as the action on the completion of spinors in the world of classical worlds so that the core of number theory would be represented physically [K35] . At the space-time level number theoretic braid having G as symmetries would represent the G. These representations are analogous to global gauge transformations. The elements of S_∞ are analogous to local gauge transformations having a natural identification as a universal number theoretical gauge symmetry group leaving physical states invariant.

Hierarchy of Planck constants and the generalization of imbedding space

Jones inclusions inspire a further generalization of the notion of imbedding space obtained by gluing together copies of the imbedding space H regarded as coverings $H \to H/G_a \times G_b$. In the simplest scenario $G_a \times G_b$ leaves invariant the choice of quantization axis and thus this hierarchy provides imbedding space correlate for the choice of quantization axes inducing these correlates also at space-time level and at the level of world of classical worlds [K25] .

Dark matter hierarchy is identified in terms of different sectors of H glued together along common points of base spaces and thus forming a book like structure. For the simplest option elementary particles proper correspond to maximally quantum critical systems in the intersection of all pages. The field bodies of elementary particles are in the interiors of the pages of this "book".

One can assign to Jones inclusions quantum phase $q = exp(i2\pi/n)$ and the groups Z_n acts as exact symmetries both at level of M^4 and CP_2. In the case of M^4 this means that space-time sheets have exact Z_n rotational symmetry. This suggests that the algebraic numbers q^m could have geometric representation at the level of sensory perception as Z_n symmetric objects. We need not be conscious of this representation in the ordinary wake-up consciousness dominated by sensory perception of ordinary matter with $q = 1$. This would make possible the idea about transcendentals like π, which do not appear in any finite-dimensional extension of even p-adic numbers (p-adic numbers allow finite-dimensional extension by since e^p is ordinary p-adic number). Quantum jumps in which state suffers an action of the generating element of Z_n could also provide a sensory realization of these groups and numbers $exp(i2\pi/n)$.

Planck constant is identified as the ratio n_a/n_b of integers associated with M^4 and CP_2 degrees of freedom so that a representation of rationals emerge again. The so called ruler and compass rationals whose definition involves only a repeated square root operation applied on rationals are cognitively the simplest ones and should appear first in the evolution of mathematical consciousness. The successful [K22] quantum model for EEG is only one of the applications providing support for their preferred role. Other applications are to Bohr quantization of planetary orbits interpreted as being induced by the presence of macroscopically quantum coherent dark matter [K64] .

10.5.4 Farey sequences, Riemann hypothesis, tangles, and TGD

Farey sequences allow an alternative formulation of Riemann Hypothesis and subsequent pairs in Farey sequence characterize so called rational 2-tangles. In TGD framework Farey sequences relate very closely to dark matter hierarchy, which inspires *"Platonia as the best possible world in the sense that cognitive representations are optimal"* as the basic variational principle of mathematics. This variational principle supports RH.

Possible TGD realizations of tangles, which are considerably more general objects than braids, are considered. One can assign to a given rational tangle a rational number a/b and the tangles labelled by a/b and c/d are equivalent if $ad - bc = \pm 1$ holds true. This means that the rationals in question are neighboring members of Farey sequence. Very light-hearted guesses about possible generalization of these invariants to the case of general N-tangles are made.

Farey sequences

Some basic facts about Farey sequences [A32] demonstrate that they are very interesting also from TGD point of view.

1. Farey sequence F_N is defined as the set of rationals $0 \leq q = m/n \leq 1$ satisfying the conditions $n \leq N$ ordered in an increasing sequence.

2. Two subsequent terms a/b and c/d in F_N satisfy the condition $ad - bc = 1$ and thus define and element of the modular group $SL(2, Z)$.

3. The number $|F(N)|$ of terms in Farey sequence is given by

$$|F(N)| = |F(N-1)| + \phi(N-1) \ . \tag{10.5.1}$$

   Here $\phi(n)$ is Euler's totient function giving the number of divisors of n. For primes one has $\phi(p) = 1$ so that in the transition from p to $p+1$ the length of Farey sequence increases by one unit by the addition of $q = 1/(p+1)$ to the sequence.

   The members of Farey sequence F_N are in one-one correspondence with the set of quantum phases $q_n = exp(i2\pi/n)$, $0 \leq n \leq N$. This suggests a close connection with the hierarchy of Jones inclusions, quantum groups, and in TGD context with quantum measurement theory with finite measurement resolution and the hierarchy of Planck constants involving the generalization of the imbedding space. Also the recent TGD inspired ideas about the hierarchy of subgroups of the rational modular group with subgroups labelled by integers N and in direct correspondence with the hierarchy of quantum critical phases [K19] would naturally relate to the Farey sequence.

Riemann Hypothesis and Farey sequences

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_N are $a_{n,N}$, $0 < n \leq |F_N|$. Define

$$d_{n,N} = a_{n,N} - \frac{n}{|F_N|} \quad .$$

In other words, $d_{n,N}$ is the difference between the n:th term of the N:th Farey sequence, and the n:th member of a set of the same number of points, distributed evenly on the unit interval. Franel and Landau proved that both of the following statements

$$\begin{aligned} \sum_{n=1,...,|F_N|} |d_{n,N}| &= O(N^r) \text{ for any } r > 1/2 \ , \\ \sum_{n=1,...,|F_N|} d_{n,N}^2 &= O(N^r) \text{ for any } r > 1 \ . \end{aligned} \tag{10.5.1}$$

are equivalent with Riemann hypothesis.

One could say that RH would guarantee that the numbers of Farey sequence provide the best possible approximate representation for the evenly distributed rational numbers $n/|F_N|$.

Farey sequences and TGD

Farey sequences seem to relate very closely to TGD.

1. The rationals in the Farey sequence can be mapped to the roots of unity by the map $q \rightarrow exp(i2\pi q)$. The numbers $1/|F_N|$ are in turn mapped to the numbers $exp(i2\pi/|F_N|)$, which are also roots of unity. The statement would be that the algebraic phases defined by Farey sequence give the best possible approximate representation for the phases $exp(in2\pi/|F_N|)$ with evenly distributed phase angle.

2. In TGD framework the phase factors defined by F_N corresponds to the set of quantum phases corresponding to Jones inclusions labelled by $q = exp(i2\pi/n)$, $n \leq N$, and thus to the N lowest levels of dark matter hierarchy. There are actually two hierarchies corresponding to M^4 and CP_2 degrees of freedom and the Planck constant appearing in Schrödinger equation corresponds to the ratio n_a/n_b defining quantum phases in these degrees of freedom. $Z_{n_a \times n_b}$ appears as a conformal symmetry of "dark" partonic 2-surfaces and with very general assumptions this implies that there are only in TGD Universe [K19, K17] .

3. The fusion of physics associated with various number fields to single coherent whole requires algebraic universality. In particular, the roots of unity, which are complex algebraic numbers, should define approximations to continuum of phase factors. At least the S-matrix associated with p-adic-to-real transitions and more generally $p_1 \rightarrow p_2$ transitions between states for which the partonic space-time sheets are p_1- resp. p_2-adic can involve only this kind of algebraic phases. One can also say that cognitive representations can involve only algebraic phases and algebraic numbers in general. For real-to-real transitions and real-to-padic transitions U-matrix might be non-algebraic or obtained by analytic continuation of algebraic U-matrix. S-matrix is by definition diagonal with respect to number field and similar continuation principle might apply also in this case.

4. The subgroups of the hierarchy of subgroups of the modular group with rational matrix elements are labelled by integer N and relate naturally to the hierarchy of Farey sequences. The hierarchy of quantum critical phases is labelled by integers N with quantum phase transitions occurring only between phases for which the smaller integer divides the larger one [K19] .

Interpretation of RH in TGD framework

Number theoretic universality of physics suggests an interpretation for the Riemann hypothesis in TGD framework. RH would be equivalent to the statement that the Farey numbers provide best possible approximation to the set of rationals $k/|F_N|$ or to the statement that the roots of unity contained by F_N define the best possible approximation for the roots of unity defined as $exp(ik2\pi/|F_N|)$ with evenly spaced phase angles. The roots of unity allowed by the lowest N levels of the dark matter hierarchy allows the best possible approximate representation for algebraic phases represented exactly at $|F_N|$:th level of hierarchy.

A stronger statement would be that the Platonia, where RH holds true would be the best possible world in the sense that algebraic physics behind the cognitive representations would allow the best possible approximation hierarchy for the continuum physics (both for numbers in unit interval and for phases on unit circle). Platonia with RH would be cognitive paradise.

One could see this also from different view point. "Platonia as the cognitively best possible world" could be taken as the "axiom of all axioms": a kind of fundamental variational principle of mathematics. Among other things it would allow to conclude that RH is true: RH must hold true either as a theorem following from some axiomatics or as an axiom in itself.

Could rational N-tangles exist in some sense?

The article of Kauffman and Lambropoulou [A177] about rational 2-tangles having commutative sum and product allowing to map them to rationals is very interesting from TGD point of view. The illustrations of the article are beautiful and make it easy to get the gist of various ideas. The theorem of the article states that equivalent rational tangles giving trivial tangle in the product correspond to subsequent Farey numbers a/b and c/d satisfying $ad - bc = \pm 1$ so that the pair defines element of the modular group SL(2,Z).

1. Rational 2-tangles

1. The basic observation is that 2-tangles are 2-tangles in both "s- and t-channels". Product and sum can be defined for all tangles but only in the case of 2-tangles the sum, which in this case reduces to product in t-channel obtained by putting tangles in series, gives 2-tangle. The so called rational tangles are 2-tangles constructible by using addition of $\pm[1]$ on left or right of tangle and multiplication by $\pm[1]$ on top or bottom. Product and sum are commutative for rational 2-tangles but the outcome is not a rational 2-tangle in the general case. One can also assign to rational 2-tangle its negative and inverse. One can map 2-tangle to a number which is rational for rational tangles. The tangles $[0]$, $[\infty]$, $\pm[1]$, $\pm 1/[1]$, $\pm[2]$, $\pm[1/2]$ define so called elementary rational 2-tangles.

2. In the general case the sum of $M-$ and $N-$tangles is $M+N-2$-tangle and combines various $N-$tangles to a monoidal structure. Tensor product like operation giving $M+N$-tangle looks to me physically more natural than the sum.

3. The reason why general 2-tangles are non-commutative although 2-braids obviously commute is that 2-tangles can be regarded as sequences of $N-$tangles with 2-tangles appearing only as the initial and final state: N is actually even for intermediate states. Since $N > 2$-braid groups are non-commutative, non-commutativity results. It would be interesting to know whether braid group representations have been used to construct representations of $N-$tangles.

2. Does generalization to $N >> 2$ case exist?

One can wonder whether the notion of rational tangle and the basic result of the article about equivalence of tangles might somehow generalize to the $N > 2$ case.

1. Could the commutativity of tangle product allow to characterize the $N > 2$ generalizations of rational 2-tangles. The commutativity of product would be a space-time correlate for the commutativity of the S-matrices defining time like entanglement between the initial and final quantum states assignable to the N-tangle. For 2-tangles commutativity of the sum

would have an analogous interpretation. Sum is not a very natural operation for N-tangles for $N > 2$. Commutativity means that the representation matrices defined as products of braid group actions associated with the various intermediate states and acting in the same representation space commute. Only in very special cases one can expect commutativity for tangles since commutativity is lost already for braids.

2. The representations of 2-tangles should involve the subgroups of N-braid groups of intermediate braids identifiable as Galois groups of N:th order polynomials in the realization as number theoretic tangles. Could non-commutative 2-tangles be characterized by algebraic numbers in the extensions to which the Galois groups are associated? Could the non-commutativity reflect directly the non-commutativity of Galois groups involved? Quite generally one can ask whether the invariants should be expressible using algebraic numbers in the extensions of rationals associated with the intermediate braids.

3. Rational 2-tangles can be characterized by a rational number obtained by a projective identification $[a,b]^T \rightarrow a/b$ from a rational 2-spinor $[a,b]^T$ to which SL(2(N-1),Z) acts. Equivalence means that the columns $[a,b]^T$ and $[c,d]^T$ combine to form element of SL(2,Z) and thus defining a modular transformation. Could more general 2-tangles have a similar representation but in terms of algebraic integers?

4. Could N-tangles be characterized by $N-1$ $2(N-1)$-component projective column-spinors $[a_i^1, a_i^2, .., a_i^{2(N-1)}]^T$, $i = 1,...N-1$ so that only the ratios $a_i^k/a_i^{2(N-1)} \leq 1$ matter? Could equivalence for them mean that the $N-1$ spinors combine to form $N-1+N-1$ columns of $SL(2(N-1),Z)$ matrix. Could N-tangles quite generally correspond to collections of projective $N-1$ spinors having as components algebraic integers and could $ad-bc = \pm 1$ criterion generalize? Note that the modular group for surfaces of genus g is SL(2g,Z) so that $N-1$ would be analogous to g and $1 \leq N \geq 3$- braids would correspond to $g \leq 2$ Riemann surfaces.

5. Dark matter hierarchy leads naturally to a hierarchy of modular sub-groups of $SL(2,Q)$ labelled by N (the generator $\tau \rightarrow \tau+2$ of modular group is replaced with $\tau \rightarrow \tau + 2/N$). What might be the role of these subgroups and corresponding subgroups of $SL(2(N-1),Q)$. Could they arise in "anyonization" when one considers quantum group representations of 2-tangles with twist operation represented by an N:th root of unity instead of phase U satisfying $U^2 = 1$?

How tangles could be realized in TGD Universe?

The article of Kauffman and Lambropoulou stimulated the question in what senses N-tangles could be be realized in TGD Universe as fundamental structures.

1. Tangles as number theoretic braids?

The strands of number theoretical $N-$braids correspond to roots of N:th order polynomial and if one allows time evolutions of partonic 2-surface leading to the disappearance or appearance of real roots $N-$tangles become possible. This however means continuous evolution of roots so that the coefficients of polynomials defining the partonic 2-surface can be rational only in initial and final state but not in all intermediate "virtual" states.

2. Tangles as tangled partonic 2-surfaces?

Tangles could appear in TGD also in second manner.

1. Partonic 2-surfaces are sub-manifolds of a 3-D section of space-time surface. If partonic 2-surfaces have genus $g > 0$ the handles can become knotted and linked and one obtains besides ordinary knots and links more general knots and links in which circle is replaced by figure eight and its generalizations obtained by adding more circles (eyeglasses for $N-$eyed creatures).

2. Since these 2-surfaces are space-like, the resulting structures are indeed tangles rather than only braids. Tangles made of strands with fixed ends would result by allowing spherical

partons elongate to long strands with fixed ends. DNA tangles would the basic example, and are discussed also in the article. DNA sequences to which I have speculatively assigned invisible (dark) braid structures might be seen in this context as space-like "written language representations" of genetic programs represented as number theoretic braids.

10.6 Quantum Quandaries

John Baez's [A115] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to imbedding space would conform with category theoretic thinking.

10.6.1 The *-category of Hilbert spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the imbedding space in TGD.

10.6.2 The monoidal *-category of Hilbert spaces and its counterpart at the level of nCob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups [K9] too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional imbedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

10.6.3 TQFT as a functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

10.6.4 The situation is in TGD framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naive idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A198] only the exotic diffeo-structures modify the situation in 4-D case.

Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of $2\rightarrow 2$ reaction open string is pinched to a point at vertex. $1\rightarrow 2$ vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy.

Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. There is U-matrix between zero energy states. This is expected to be rather trivial but very important from the point of view of description of intentional actions as transitions transforming p-adic partonic 3-surfaces to their real counterparts.

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics.

10.7 How to represent algebraic numbers as geometric objects?

Physics blogs are also interesting because they allow to get some grasp about very different styles of thinking of a mathematician and physicist. For mathematician it is very important that the result is obtained by a strict use of axioms and deduction rules. Physicist is a cognitive opportunist: it does not matter how the result is obtained by moving along axiomatically allowed paths or not, and the new result is often more like a discovery of a new axiom and physicist is ever-grateful for Gödel for giving justification for what sometimes admittedly degenerates to a creative hand-waving. For physicist ideas form a kind of bio-sphere and the fate of the individual idea depends on its ability to survive, which is determined by its ability to become generalized, its consistency with other ideas, and ability to interact with other ideas to produce new ideas.

10.7.1 Can one define complex numbers as cardinalities of sets?

During few days before writing this we have had in Kea's blog a little bit of discussion inspired by the problem related to the categorification of basic number theoretical structures. I have learned that sum and product are natural operations for the objects of category. For instance, one can define sum as in terms of union of sets or direct sum of vector spaces and product as Cartesian product of sets and tensor product of vector spaces: rigs [A111] are example of categories for which natural numbers define sum and product.

Subtraction and division are however problematic operations. Negative numbers and inverses of integers do not have a realization as a number of elements for any set or as dimension of vector space. The naive physicist inside me asks immediately: why not go from statics to dynamics and take operations (arrows with direction) as objects: couldn't this allow to define subtraction and division? Is the problem that the axiomatization of group theory requires something which purest categorification does not give? Or aren't the numbers representable in terms of operations of finite groups not enough? In any case cyclic groups would allow to realize roots of unity as operations (Z_2 would give -1).

One could also wonder why the algebraic numbers might not somehow result via the representations of permutation group of infinite number of elements containing all finite groups and thus Galois groups of algebraic extensions as subgroups? Why not take the elements of this group as objects of the basic category and continue by building group algebra and hyper-finite factors of type II_1 isomorphic to spinors of world of classical worlds, and so on.

After having written the first half of the section, I learned that something similar to the transition from statics to dynamics is actually carried out but by manner which is by many orders of magnitudes more refined than the proposal above and that I had never been able to imagine. The article *Objects of categories as complex numbers* of Marcelo Fiore and Tom Leinster [A111] describes a fascinating idea summarized also by John Baez [A102] about how one can assign to the objects of a category complex numbers as roots of a polynomial $Z = P(Z)$ defining an isomorphism of object. Z is the element of a category called rig, which differs from ring in that integers are replaced with natural numbers. One can replace Z with a complex number $|Z|$ defined as a root of polynomial. $|Z|$ is interpreted formally as the cardinality of the object. It is essential to have natural numbers and thus only product and sum are defined. This means a restriction: for instance, only complex algebraic numbers associated with polynomials having natural numbers as coefficients are obtained. Something is still missing.

Note that this correspondence assumes the existence of complex numbers and one cannot say that complex numbers are categorified. Maybe basic number fields must be left outside categorification. One can however require that all of them have a concrete set theoretic representation rather than only formal interpretation as cardinality so that one still encounters the problem how to represent algebraic complex number as a concrete cardinality of a set.

10.7.2 In what sense a set can have cardinality -1?

The discussion in Kea's blog led me to ask what the situation is in the case of p-adic numbers. Could it be possible to represent the negative and inverse of p-adic integer, and in fact any p-adic number, as a geometric object? In other words, does a set with -1 or $1/n$ or even $\sqrt{-1}$ elements exist? If this were in some sense true for all p-adic number fields, then all this wisdom combined together might provide something analogous to the adelic representation for the norm of a rational number as product of its p-adic norms. As will be found, alternative interpretations of complex algebraic numbers as p-adic numbers representing cardinalities of p-adic fractals emerge. The fractal defines the manner how one must do an infinite sum to get an infinite real number but finite p-adic number.

Of course, this representation might not help to define p-adics or reals categorically but might help to understand how p-adic cognitive representations defined as subsets for rational intersections of real and p-adic space-time sheets could represent p-adic number as the number of points of p-adic fractal having infinite number of points in real sense but finite in the p-adic sense. This would also give a fundamental cognitive role for p-adic fractals as cognitive representations of numbers.

How to construct a set with -1 elements?

The basic observation is that p-adic -1 has the representation

$$-1 = (p-1)/(1-p) = (p-1)(1+p+p^2+p^3....)$$

As a real number this number is infinite or -1 but as a p-adic number the series converges and has p-adic norm equal to 1. One can also map this number to a real number by canonical identification taking the powers of p to their inverses: one obtains p in this particular case. As a matter fact, any rational with p-adic norm equal to 1 has similar power series representation.

The idea would be to represent a given p-adic number as the infinite number of points (in real sense) of a p-adic fractal such that p-adic topology is natural for this fractal. This kind of fractals can be constructed in a simple manner: from this more below. This construction allows to represent any p-adic number as a fractal and code the arithmetic operations to geometric operations for these fractals.

These representations - interpreted as cognitive representations defined by intersections of real and p-adic space-time sheets - are in practice approximate if real space-time sheets are assumed to have a finite size: this is due to the finite p-adic cutoff implied by this assumption and the meaning a finite resolution. One can however say that the p-adic space-time itself could by its necessarily infinite size represent the *idea* of given p-adic number faithfully.

This representation applies also to the p-adic counterparts of algebraic numbers in case that they exist. For instance, roughly one half of p-adic numbers have square root as ordinary p-adic number and quite generally algebraic operations on p-adic numbers can give rise to p-adic numbers so that also these could have set theoretic representation. For $p\ mod\ 4 = 1$ also $\sqrt{(}-1)$ exists: for instance, for $p = 5$: $2^2 = 4 = -1\ mod\ 5$ guarantees this so that also imaginary unit and complex numbers would have a fractal representation. Also many transcendentals possess this kind of representation. For instance $exp(xp)$ exists as a p-adic number if x has p-adic norm not larger than 1: also $log(1 + xp)$ does so.

Hence a quite impressive repertoire of p-adic counterparts of real numbers would have representation as a p-adic fractal for some values of p. Adelic vision would suggest that combining these representations one might be able to represent quite a many real numbers. In the case of π I do not find any obvious p-adic representation (for instance $sin(\pi/6) = 1/2$ does not help since the p-adic variant of the Taylor expansion of $\pi/6 = arcsin(1/2)$ does not converge p-adically for any value of p). It might be that there are very many transcendentals not allowing fractal representation for any value of p.

Conditions on the fractal representations of p-adic numbers

Consider now the construction of the fractal representations in terms of rational intersections of real real and p-adic space-time sheets. The question is what conditions are natural for this representation if it corresponds to a cognitive representation is realized in the rational intersection of real and p-adic space-time sheets obeying same algebraic equations.

1. Pinary cutoff is the analog of the decimal cutoff but is obtained by dropping away high positive rather than negative powers of p to get a finite real number: example of pinary cutoff is $-1 = (p-1)(1 + p + p^2 + ...) \rightarrow (p-1)(1 + p + p^2)$. This cutoff must reduce to a fractal cutoff meaning a finite resolution due to a finite size for the real space-time sheet. In the real sense the p-adic fractal cutoff means not forgetting details below some scale but cutting out all above some length scale. Physical analog would be forgetting all frequencies below some cutoff frequency in Fourier expansion.

   The motivation comes from the fact that TGD inspired consciousness assigns to a given biological body there is associated a field body or magnetic body containing dark matter with large $\hbar$ and quantum controlling the behavior of biological body and so strongly identifying with it so as to belief that this all ends up to a biological death. This field body has an onion like fractal structure and a size of at least order of light-life. Of course, also larger onion layers could be present and would represent those levels of cognitive consciousness not depending on the sensory input on biological body: some altered states of consciousness could relate to these levels. In any case, the larger the magnetic body, the better the numerical skills of the p-adic mathematician.

2. Lowest pinary digits of $x = x_0 + x_1p + x_2p^2 + ...$, $x_n \leq p$ must have the most reliable representation since they are the most significant ones. The representation must be also highly redundant to guarantee reliability. This requires repetitions and periodicity. This is

guaranteed if the representation is hologram like with segments of length p^n with digit x_n represented again and again in all segments of length p^m, $m > n$.

3. The TGD based physical constraint is that the representation must be realizable in terms of induced classical fields assignable to the field body hierarchy of an intelligent system interested in artistic expression of p-adic numbers using its own field body as instrument. As a matter, sensory and cognitive representations are realized at field body in TGD Universe and EEG is in a fundamental role in building this representation. By p-adic fractality fractal wavelets are the most natural candidate. The fundamental wavelet should represent the p different pinary digits and its scaled up variants would correspond to various powers of p so that the representation would reduce to a Fourier expansion of a classical field.

Concrete representation

Consider now a concrete candidate for a representation satisfying these constraints.

1. Consider a p-adic number

$$y = p^{n_0}x, \quad x = \sum x_n p^n \quad , \quad n \geq n_0 = 0 \quad .$$

If one has a representation for a p-adic unit x the representation of is by a purely geometric fractal scaling of the representation by p^n. Hence one can restrict the consideration to p-adic units.

2. To construct the representation take a real line starting from origin and divide it into segments with lengths $1, p, p^2, ...$ In TGD framework this scalings come actually as powers of $p^{1/2}$ but this is just a technical detail.

3. It is natural to realize the representation in terms of periodic field patterns. One can use wavelets with fractal spectrum $p^n\lambda_0$ of "wavelet lengths", where λ_0 is the fundamental wavelength. Fundamental wavelet should have p different patterns correspond to the p values of pinary digit as its structures. Periodicity guarantees the hologram like character enabling to pick n:th digit by studying the field pattern in scale p^n anywhere inside the field body.

4. Periodicity guarantees also that the intersections of p-adic and real space-time sheets can represent the values of pinary digits. For instance, wavelets could be such that in a given p-adic scale the number of rational points in the intersection of the real and p-adic space-time sheet equals to x_n. This would give in the limit of an infinite pinary expansion a set theoretic realization of any p-adic number in which each pinary digit x_n corresponds to infinite copies of a set with x_n elements and fractal cutoff due to the finite size of real space-time sheet would bring in a finite precision. Note however that p-adic space-time sheet necessarily has an infinite size and it is only real world realization of the representation which has finite accuracy.

5. A concrete realization for this object would be as an infinite tree with $x_n + 1 \leq p$ branches in each node at level n ($x_n + 1$ is needed in order to avoid the splitting tree at $x_n = 0$). In 2-adic case -1 would be represented by an infinite pinary tree. Negative powers of p correspond to the of the tree extending to a finite depth in ground.

10.7.3 Generalization of the notion of rig by replacing naturals with p-adic integers

Previous considerations do not relate directly to category theoretical problem of assigning complex numbers to objects. It however turns out that p-adic approach allows to generalize the proposal of [A111] by replacing natural numbers with p-adic integers in the definition of rig so that any algebraic complex number can define cardinality of an object of category allowing multiplication and sum and that these complex numbers can be replaced with p-adic numbers if they make sense as such so that previous arguments provide a concrete geometric representation of the cardinality.

The road to the realization this simple generalization required a visit to the John Baez's Weekly Finds (Week 102) [A102] .

The outcome was the realization that the notion of rig used to categorify the subset of algebraic numbers obtained as roots of polynomials with *natural number* valued coefficients generalizes trivially by replacing natural numbers by *p-adic integers*. As a consequence one obtains beautiful p-adicization of the generating function F(x) of structure as a function which converges p-adically for any rational $x = q$ for which it has prime p as a positive power divisor.

Effectively this generalization means the replacement of natural numbers as coefficients of the polynomial defining the rig with all rationals, also negative, and *all* complex algebraic numbers find a category theoretical representation as "cardinalities". These cardinalities have a dual interpretation as p-adic integers which in general correspond to infinite real numbers but are mappable to real numbers by canonical identification and have a geometric representation as fractals.

Mapping of objects to complex numbers and the notion of rig

The idea of rig approach is to categorify the notion of cardinality in such a manner that one obtains a *subset* of algebraic complex numbers as cardinalities in the category-theoretical sense. One can assign to an object a polynomial with coefficients, which are *natural numbers* and the condition $Z = P(Z)$ says that $P(Z)$ acts as an isomorphism of the object. One can interpret the equation also in terms of complex numbers. Hence the object is mapped to a complex number Z defining a root of the polynomial interpreted as an ordinary polynomial: it does not matter which root is chosen. The complex number Z is interpreted as the "cardinality" of the object but I do not really understand the motivation for this. The deep further result is that also more general polynomial equations $R(|Z|) = Q(|Z|)$ satisfied by the generalized cardinality Z imply $R(Z) = Q(Z)$ as isomorphism.

I try to reproduce what looks the most essential in the explanation of John Baez and relate it to my own ideas but take this as my talk to myself and visit This Week's Finds [A102] , one of the many classics of Baez, to learn of this fascinating idea.

1. Baez considers first the ways of putting a given structure to n-element set. The set of these structures is denoted by F_n and the number of them by $|F_n|$. The generating function $|F|(x) = \sum_n |F_n|x^n$ packs all this information to a single function.

   For instance, if the structure is binary tree, this function is given by $T(x) = \sum_n C_{n-1}x^n$, where C_{n-1} are Catalan numbers and n¿0 holds true. One can show that T satisfies the formula

   $$T = X + T^2 \quad ,$$

   since any binary tree is either trivial or decomposes to a product of binary trees, where two trees emanate from the root. One can solve this second order polynomial equation and the power expansion gives the generating function.

2. The great insight is that one can also work directly with structures. For instance, by starting from the isomorphism $T = 1 + T^2$ applying to an object with cardinality 1 and substituting T^2 with $(1 + T^2)^2$ repeatedly, one can deduce the amazing formula $T^7(1) = T(1)$ mentioned by Kea, and this identity can be interpreted as an isomorphism of binary trees.

3. This result can be generalized using the notion of rig category [A111] . In rig category one can add and multiply but negatives are not defined as in the case of ring. The lack of subtraction and division is still the problem and as I suggested in previous posting p-adic integers might resolve the problem.

   Whenever Z is object of a rig category, one can equip it with an isomorphism $Z = P(Z)$ where $P(Z)$ is polynomial with *natural numbers* as coefficients and one can assign to object "cardinality" as any root of the equation $Z = P(Z)$. Note that set with n elements corresponds to $P(|Z|) = n$. Thus subset of algebraic complex numbers receive formal identification as cardinalities of sets. Furthermore, if the cardinality satisfies another equation $Q(|Z|) = R(|Z|)$ such that neither polynomial is constant, then one can construct an isomorphism $Q(Z) = R(Z)$. Isomorphisms correspond to equations!

4. This is indeed nice that there is something which is not so beautiful as it could be: why should we restrict ourselves to *natural numbers* as coefficients of $P(Z)$? Could it be possible to replace them with integers to obtain *all complex algebraic numbers* as cardinalities? Could it be possible to replace natural numbers by p-adic integers?

p-Adic rigs and Golden Object as p-adic fractal

The notions of generating function and rig generalize to the p-adic context.

1. The generating function $F(x)$ defining isomorphism Z in the rig formulation converges p-adically for any p-adic number containing p as a factor so that the idea that all structures have p-adic counterparts is natural. In the real context the generating function typically diverges and must be defined by analytic continuation. Hence one might even argue that p-adic numbers are more natural in the description of structures assignable to finite sets than reals.

2. For rig one considers only polynomials $P(Z)$ (Z corresponds to the generating function F) with coefficients which are natural numbers. Any p-adic integer can be however interpreted as a non-negative integer: natural number if it is finite and "super-natural" number if it is infinite. Hence can generalize the notion of rig by replacing natural numbers by p-adic integers. The rig formalism would thus generalize to arbitrary polynomials with integer valued coefficients so that all complex algebraic numbers could appear as cardinalities of category theoretical objects. Even rational coefficients are allowed. This is highly natural number theoretically.

3. For instance, in the case of binary trees the solutions to the isomorphism condition $T = p+T^2$ giving $T = [1 \pm (1-4p)^{1/2}]/2$ and T would be complex number $[p \pm (1-4p)^{1/2}]/2$. $T(p)$ can be interpreted also as a p-adic number by performing power expansion of square root in case that the p-adic square root exists: this super-natural number can be mapped to a real number by the canonical identification and one obtains also the set theoretic representations of the category theoretical object $T(p)$ as a p-adic fractal. This interpretation of cardinality is much more natural than the purely formal interpretation as a complex number. This argument applies completely generally. The case $x = 1$ discussed by Baez gives $T = [1 \pm (-3)^{1/2}]/2$ allows p-adic representation if $-3 == p - 3$ is square mod p. This is the case for $p = 7$ for instance.

4. John Baez [A102] poses also the question about the category theoretic realization of "Golden Object", his big dream. In this case one would have $Z = G = -1 + G^2 = P(Z)$. The polynomial on the right hand side does not conform with the notion of rig since -1 is not a natural number. If one allows p-adic rigs, $x = -1$ can be interpreted as a p-adic integer $(p-1)(1+p+...)$, positive and infinite and "super-natural", actually largest possible p-adic integer in a well defined sense.

   A further condition is that Golden Mean converges as a p-adic number: this requires that $\sqrt{5}$ must exist as a p-adic number: $(5 = 1 + 4)^{1/2}$ certainly converges as power series for $p = 2$ so that Golden Object exists 2-adically. By using [A74] of Euler, one finds that 5 is square mod p only if p is square mod 5. To decide whether given p is Golden it is enough to look whether p mod 5 is 1 or 4. For instance, $p = 11, 19, 29, 31$ ($=M_5$) are Golden. Mersennes M_k, $k = 3, 7, 127$ and Fermat primes are not Golden. One representation of Golden Object as p-adic fractal is the p-adic series expansion of $[1/2 \pm 5^{1/2}]/2$ representable geometrically as a binary tree such that there are $0 \leq x_n + 1 \leq p$ branches at each node at height n if n:th p-adic coefficient is x_n. The "cognitive" p-adic representation in terms of wavelet spectrum of classical fields is discussed in the previous posting.

5. It would be interesting to know how quantum dimensions of quantum groups assignable to Jones inclusions [K80, K25, K9] relate to the generalized cardinalities. The root of unity property of quantum phase ($q^{n+1} = q$) suggests $Q = Q^{n+1} = P(Q)$ as the relevant isomorphism. For Jones inclusions the cardinality $q = exp(i2\pi/n)$ would not be however equal to quantum dimension $D(n) = 4cos^2(\pi/n)$.

Is there a connection with infinite integers?

Infinite primes [K70] correspond to Fock states of a super-symmetric arithmetic quantum field theory and there is entire infinite hierarchy of them corresponding to repeated second quantization. Also infinite primes and rationals make sense. Besides free Fock states spectrum contains at each level also what might be identified as bound states. All these states can be mapped to polynomials. Since the roots of polynomials represent complex algebraic numbers and as they seem to characterize objects of categories, there are reasons to expect that infinite rationals might allow also interpretation in terms of say rig categories or their generalization. Also the possibility to identify space-time coordinate as isomorphism of a category might be highly interesting concerning the interpretation of quantum classical correspondence.

10.8 Gerbes and TGD

The notion of gerbes has gained much attention during last years in theoretical physics and there is an abundant gerbe-related literature in hep-th archives. Personally I learned about gerbes from the excellent article of Jouko Mickelson [A186] (Jouko was my opponent in PhD dissertation for more than two decades ago: so the time flows!).

I have already applied the notion of bundle gerbe in TGD framework in the construction of the Dirac determinant which I have proposed to define the Kähler function for the WCW (see [K15]). The insights provided by the general results about bundle gerbes discussed in [A186] led, not only to a justification for the hypothesis that Dirac determinant exists for the modified Dirac action, but also to an elegant solution of the conceptual problems related to the construction of Dirac determinant in the presence of chiral symmetry. Furthermore, on basis of the special properties of the modified Dirac operator there are good reasons to hope that the determinant exists even without zeta function regularization. The construction also leads to the conclusion that the space-time sheets serving as causal determinants must be geodesic sub-manifolds (presumably light like boundary components or "elementary particle horizons"). Quantum gravitational holography is realized since the exponent of Kähler function is expressible as a Dirac determinant determined by the local data at causal determinants and there would be no need to find absolute minima of Kähler action explicitly.

In the sequel the emergence of 2-gerbes at the space-time level in TGD framework is discussed and shown to lead to a geometric interpretation of the somewhat mysterious cocycle conditions for a wide class of gerbes generated via the $\wedge d$ products of connections associated with 0-gerbes. The resulting conjecture is that gerbes form a graded-commutative Grassmman algebra like structure generated by -1- and 0-gerbes. 2-gerbes provide also a beautiful topological characterization of space-time sheets as structures carrying Chern-Simons charges at boundary components and the 2-gerbe variant of Bohm-Aharonov effect occurs for perhaps the most interesting asymptotic solutions of field equations especially relevant for anyonics systems, quantum Hall effect, and living matter [K78] .

10.8.1 What gerbes roughly are?

Very roughly and differential geometrically, gerbes can be regarded as a generalization of connection. Instead of connection 1-form (0-gerbe) one considers a connection $n+1$-form defining n-gerbe. The curvature of n-gerbe is closed $n+2$-form and its integral defines an analog of magnetic charge. The notion of holonomy generalizes: instead of integrating n-gerbe connection over curve one integrates its connection form over n+1-dimensional closed surface and can transform it to the analog of magnetic flux.

There are some puzzling features associated with gerbes. Ordinary $U(1)$-bundles are defined in terms of open sets U_α with gauge transformations $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ defined in $U_\alpha \cap U_\beta$ relating the connection forms in the patch U_β to that in patch U_α. The 3-cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \tag{10.8.1}$$

makes it possible to glue the patches to a bundle structure.

In the case of 1-gerbes the transition functions are replaced with the transition functions $g_{\alpha\beta\gamma} = g^{-1}_{\gamma\beta\alpha}$ defined in triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and 3-cocycle must be replaced with 4-cocycle:

$$g_{\alpha\beta\gamma}g_{\beta\gamma\delta}g_{\gamma\delta\alpha}g_{\delta\alpha\beta} = 1 \ . \tag{10.8.2}$$

The generalizations of these conditions to n-gerbes is obvious.

In the case of 2-intersections one can build a bundle structure naturally but in the case of 3-intersections this is not possible. Hence the geometric interpretation of the higher gerbes is far from obvious. One possible interpretation of non-trivial 1-gerbe is as an obstruction for lifting projective bundles with fiber space CP_n to vector bundles with fiber space C^{n+1} [A186] . This involves the lifting of the holomorphic transition functions g_α defined in the projective linear group $PGL(n+1, C)$ to $GL(n+1, C)$. When the 3-cocycle condition for the lifted transition functions $\overline{g}_{\alpha\beta}$ fails it can be replaced with 4-cocycle and one obtains 1-gerbe.

10.8.2 How do 2-gerbes emerge in TGD?

Gerbes seem to be interesting also from the point of view of TGD, and TGD approach allows a geometric interpretation of the cocycle conditions for a rather wide class of gerbes.

Recall that the Kähler form J of CP_2 defines a non-trivial magnetically charged and self-dual $U(1)$-connection A. The Chern-Simons form $\omega = A \wedge J = A \wedge dA$ having CP_2 Abelian instanton density $J \wedge J$ as its curvature form and can thus be regarded as a 3-connection form of a 2-gerbe. This 2-gerbe is induced by 0-gerbe.

The coordinate patches U_α are same as for $U(1)$ connection. In the transition between patches A and ω transform as

$$\begin{array}{rcl} A & \rightarrow & A + d\phi \ , \\ \omega & \rightarrow & \omega + dA_2 \ , \\ A_2 & = & \phi \wedge J \ . \end{array} \tag{10.8.0}$$

The transformation formula is induced by the transformation formula for $U(1)$ bundle. Somewhat mysteriously, there is no need to define anything in the intersections of U_α in the recent case.

The connection form of the 2-gerbe can be regarded as a second $\wedge d$ power of Kähler connection:

$$A_3 \equiv A \wedge dA \ . \tag{10.8.1}$$

The generalization of this observation allows to develop a different view about n-gerbes generated as $\wedge d$ products of 0-gerbes.

The hierarchy of gerbes generated by 0-gerbes

Consider a collection of $U(1)$ connections $A^{i)}$. They generate entire hierarchy of gerbe-connections via the $\wedge d$ product

$$A_3 = A^{1)} \wedge dA^{2)} \tag{10.8.2}$$

defining 2-gerbe having a closed curvature 4-form

$$F_4 = dA^{1)} \wedge dA^{2)} \ . \tag{10.8.3}$$

$\wedge d$ product is commutative apart from a gauge transformation and the curvature forms of $A^{1)} \wedge dA^{2)}$ and $A^{2)} \wedge dA^{1)}$ are the same.

Quite generally, the connections A_m of $m-1$ gerbe and A_n of $n-1$-gerbe define $m+n+1$ connection form and the closed curvature form of $m+n$-gerbe as

$$
\begin{aligned}
A_{m+n+1} &= A_m^{1)} \wedge dA_n^{2)} \ , \\
F_{m+n+2} &= dA_m^{1)} \wedge dA_n^{2)} \ .
\end{aligned}
\tag{10.8.3}
$$

The sequence of gerbes extends up to $n = D - 2$, where D is the dimension of the underlying manifold. These gerbes are not the most general ones since one starts from 0-gerbes. One can of course start from $n > 0$-gerbes too.

The generalization of the $\wedge d$ product to the non-Abelian situation is not obvious. The problems stem from the that the Lie-algebra valued connection forms $A^{1)}$ and $A^{2)}$ appearing in the covariant version $D = d + A$ do not commute.

10.8.3 How to understand the replacement of 3-cycles with n-cycles?

If n-gerbes are generated from 0-gerbes it is possible to understand how the intersections of the open sets emerge. Consider the product of 0-gerbes as the simplest possible case. The crucial observation is that the coverings U_α for $A^{1)}$ and V_β for $A^{2)}$ need not be same (for CP_2 this was the case). One can form a new covering consisting of sets $U_\alpha \cap V_{\alpha_1}$. Just by increasing the index range one can replace V with U and one has covering by $U_\alpha \cap U_{\alpha_1} \equiv U_{\alpha\alpha_1}$.

The transition functions are defined in the intersections $U_{\alpha\alpha_1} \cap U_{\beta\beta_1} \equiv U_{\alpha\alpha_1\beta\beta_1}$ and cocycle conditions must be formulated using instead of intersections $U_{\alpha\beta\gamma}$ the intersections $U_{\alpha\alpha_1\beta\beta_1\gamma\gamma_1}$. Hence the transition functions can be written as $g_{\alpha\alpha_1\beta\beta_1}$ and the 3-cocycle are replaced with 5-cocycle conditions since the minimal co-cycle corresponds to a sequence of 6 steps instead of 4:

$$
U_{\alpha\alpha_1\beta\beta_1} \rightarrow U_{\alpha_1\beta\beta_1\gamma} \rightarrow U_{\beta\beta_1\gamma\gamma_1} \rightarrow U_{\beta_1\gamma\gamma_1\alpha} \rightarrow U_{\gamma\gamma_1\alpha\alpha_1} \ .
$$

The emergence of higher co-cycles is thus forced by the modification of the bundle covering necessary when gerbe is formed as a product of lower gerbes. The conjecture is that any even gerbe is expressible as a product of 0-gerbes.

An interesting application of the product structure is at the level of WCW ("world of classical worlds"). The Kähler form of WCW defines a connection 1-form and this generates infinite hierarchy of connection $2n+1$-forms associated with $2n$-gerbes.

10.8.4 Gerbes as graded-commutative algebra: can one express all gerbes as products of -1 and 0-gerbes?

If one starts from, say 1-gerbes, the previous argument providing a geometric understanding of gerbes is not applicable as such. One might however hope that it is possible to represent the connection 2-form of any 1-gerbe as a $\wedge d$ product of a connection 0-form ϕ of "-1"-gerbe and connection 1-form A of 0-gerbe:

$$
A_2 = \phi dA \equiv A \wedge d\phi \ ,
$$

with different coverings for ϕ and A. The interpretation as an obstruction for the modification of the underlying bundle structure is consistent with this interpretation.

The notion of -1-gerbe is not well-defined unless one can define the notion of -1 form precisely. The simplest possibility that 0-form transforms trivially in the change of patch is not consistent. One could identify contravariant n-tensors as $-n$-forms and d for them as divergence and d^2 as the antisymmetrized double divergence giving zero. ϕ would change in a gauge transformation by a divergence of a vector field. The integral of a divergence over closed M vanishes identically so that if the integral of ϕ over M is non-vanishing it corresponds to a non-trivial 0-connection. This interpretation of course requires the introduction of metric.

The requirement that the minimal intersections of the patches for 1-gerbes are of form $U_{\alpha\beta\gamma}$ would be achieved if the intersections patches can be restricted to the intersections $U_{\alpha\beta\gamma}$ defined by $U_\alpha \cap V_\gamma$ and $U_\beta \cap V_\gamma$ (instead of $U_\beta \cap V_\delta$), where the patches V_γ would be most naturally associated with -1-gerbe. It is not clear why one could make this restriction. The general conjecture is that any gerbe decomposes into a multiple $\wedge d$ product of -1 and 0-gerbes just like integers decompose into primes. The $\wedge d$ product of two odd gerbes is anti-commutative so that there is also an analogy with the decomposition of the physical state into fermions and bosons, and gerbes for a graded-commutative super-algebra generalizing the Grassmann algebra of manifold to a Grassmann algebra of gerbe structures for manifold.

10.8.5 The physical interpretation of 2-gerbes in TGD framework

2-gerbes could provide some insight to how to characterize the topological structure of the many-sheeted space-time.

1. The cohomology group H^4 is obviously crucial in characterizing 2-gerbe. In TGD framework many-sheetedness means that different space-time sheets with induced metric having Minkowski signature are separated by elementary particle horizons which are light like 3-surfaces at which the induced metric becomes degenerate. Also the time orientation of the space-time sheet can change at these surfaces since the determinant of the induced metric vanishes.

   This justifies the term elementary particle horizon and also the idea that one should treat different space-time sheets as generating independent direct summands in the homology group of the space-time surface: as if the space-time sheets not connected by join along boundaries bonds were disjoint. Thus the homology group H^4 and 2-gerbes defining instanton numbers would become important topological characteristics of the many-sheeted space-time.

2. The asymptotic behavior of the general solutions of field equations can be classified by the dimension D of the CP_2 projection of the space-time sheet. For $D = 4$ the instanton density defining the curvature form of 2-gerbe is non-vanishing and instanton number defines a topological charge. Also the values of the Chern-Simons invariants associated with the boundary components of the space-time sheet define topological quantum numbers characterizing the space-time sheet and their sum equals to the instanton charge. CP_2 type extremals represent a basic example of this kind of situation. From the physical view point $D = 4$ asymptotic solutions correspond to what might be regarded chaotic phase for the flow lines of the Kähler magnetic field. Kähler current vanishes so that empty space Maxwell's equations are satisfied.

3. For $D = 3$ situation is more subtle when boundaries are present so that the higher-dimensional analog of Aharonov-Bohm effect becomes possible. In this case instanton density vanishes but the Chern-Simons invariants associated with the boundary components can be non-vanishing. Their sum obviously vanishes. The space-time sheet can be said to be a neutral C-S multipole. Separate space-time sheets can become connected by join along boundaries bonds in a quantum jump replacing a space-time surface with a new one. This means that the cohomology group H^4 as well as instanton charges and C-S charges of the system change.

Concerning the asymptotic dynamics of the Kähler magnetic field, $D = 3$ phase corresponds to an extremely complex but highly organized phase serving as an excellent candidate for the modelling of living matter. Both the TGD based description of anyons and quantum Hall effect and the model for topological quantum computation based on the braiding of magnetic flux tubes rely heavily on the properties $D = 3$ phase [K78] .

The non-vanishing of the C-S form implies that the flow lines of the Kähler magnetic are highly entangled and have as an analog mixing hydrodynamical flow. In particular, one cannot define non-trivial order parameters, say phase factors, which would be constant along the lines. The interpretation in terms of broken super-conductivity suggests itself. Kähler current can be non-vanishing so that there is no counterpart for this phase at the level of Maxwell's equations.

10.9 Appendix: Category theory and construction of S-matrix

The construction of WCW geometry, spinor structure and of S-matrix involve difficult technical and conceptual problems and category theory might be of help here. As already found, the application of category theory to the construction of WCW geometry allows to understand how the arrow of psychological time emerges.

The construction of the S-matrix involves several difficult conceptual and technical problems in which category theory might help. The incoming states of the theory are what might be called free states and are constructed as products of the WCW spinor fields. One can effectively regard them as being defined in the Cartesian power of WCW divided by an appropriate permutation group. Interacting states in turn are defined in the WCW.

Cartesian power of WCW of 3-surfaces is however in geometrical sense more or less identical with WCW since the disjoint union of N 3-surfaces is itself a 3-surface in WCW. Actually it differs from WCW itself only in that the 3-surfaces of many particle state can intersect each other and if one allows this, one has paradoxical self-referential identification $CH = \overline{CH^2}/S_2 = ... = \overline{CH^N}/S_N...$, where over-line signifies that intersecting 3-surfaces have been dropped from the product.

Note that arbitrarily small deformation can remove the intersections between 3-surfaces and four-dimensional general coordinate invariance allows always to use non-intersecting representatives. In case of the spinor structure of the Cartesian power this identification means that the tensor powers SCH^N of the WCW spinor structure are in some sense identical with the spinor structure SCH of the WCW. Certainly the oscillator operators of the tensor factors must be assumed to be mutually anti-commuting.

The identities $CH = \overline{CH^2}/S_2 = ..$ and corresponding identities $SCH = SCH^2 = ...$ for the space SCH of WCW spinor fields might imply very deep constraints on S-matrix. What comes into mind are counterparts for the Schwinger-Dyson equations of perturbative quantum field theory providing defining equations for the n-point functions of the theory [A170] . The isomorphism between SCH^2 and SCH is actually what is needed to calculate the S-matrix elements. Category theory might help to understand at a general level what these self-referential and somewhat paradoxical looking identities really imply and perhaps even develop TGD counterparts of Schwinger-Dyson equations.

There is also the issue of bound states. The interacting states contain also bound states not belonging to the space of free states and category theory might help also here. It would seem that the state space must be constructed by taking into account also the bound states as additional 'free' states in the decomposition of states to product states.

A category naturally involved with the construction of the S-matrix (or U-matrix) is the space of the absolute minima $X^4(X^3)$ of the Kähler action which might be called interacting category. The canonical transformations acting as isometries of the configuration space geometry act naturally as the morphisms of this category. The group $Diff^4$ of general coordinate transformations in turn acts as gauge symmetries.

S-matrix relates free and interacting states and is induced by the classical long range interactions induced by the criticality of the preferred extremals in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes S-matrix elements are essentially Glebch-Gordan coefficients relating the states in the tensor power of the interacting super-symplectic representation with the interacting super-symplectic representation itself. More concretely, N-particle free states can be seen as WCW spinor fields in CH^N obtained as tensor products of ordinary CH spinor fields. Free states correspond classically to the unions of space-time surfaces associated with the 3-surfaces representing incoming particles whereas interacting states correspond classically to the space-time surfaces associated with the unions of the 3-surfaces defining incoming states. These two states define what might be called free and interacting categories with canonical transformations acting as morphisms.

The classical interaction is represented by a functor $S : \overline{CH^N}/S_N \to CH$ mapping the classical free many particle states, that is objects of the product category defined by $\overline{CH^N}/S_N$ to the interacting category CH. This functor assigns to the union $\cup_i X^4(X_i^3)$ of the absolute minima $X^4(X_i^3)$ of Kähler action associated with the incoming, free states X_i^3 the absolute minimum $X^4(\cup X_i^3)$ associated with the union of three-surfaces representing the outgoing interacting state. At quantum level this functor maps the state space SCH^N associated with $\cup_i X^4(X_i^3)$ to SCH in a unitary manner. An important constraint on S-matrix is that it acts effectively as a flow in zero modes correlating the quantum numbers in fiber degrees of freedom in one-to-one manner with the values of zero modes so that quantum jump $U\Psi_i \to \Psi_0...$ gives rise to a quantum measurement.

Chapter 11

Non-Standard Numbers and TGD

11.1 Introduction

This chapter represents some comments on articles of Elemer E. Rosinger as a physicist from the point of view of Topological Geometrodynamics. To a large extent a comparison of two possible generalizations of reals is in question: the surreal numbers introduced originally by Robinson [?]nd infinite primes and corresponding generalization of reals inspired by TGD approach [?] The articles which have inspired the comments below are following:

- How Far Should the Principle of Relativity Go?
- Quantum Foundations: Is Probability Ontological?
- Group Invariant Entanglements in Generalized Tensor Products
- Heisenberg Uncertainty in Reduced Power Algebras
- Surprising Properties of Non-Archimedean Field Extensions of the Real Numbers
- No-Cloning in Reduced Power Algebras

I have a rather rudimentary knowledge about non-standard numbers and my comments are very subjective and TGD centered. I however hope that they might tell also something about Rosinger's work [?] My interpretation of the message of articles relies on associations with my own physics inspired ideas related to the notion of number. I divide the articles to physics related and purely mathematical ones. About the latter aspects I am not able to say much.

The construction of ultrapower fields (generalized scalars) is explained using concepts familar to physicist using the close analogies with gauge theories, gauge invariance, and with the singularities of classical fields. Some questions related to the physical applications of non-standard numbers are discussed including interpretational problems and the problems related to the notion of definite integral. The non-Archimedean character of generalized scalars is discussed and compared with that of p-adic numbers. Rosinger considers several physical ideas inspired by ultrapower fields including the generalization of general covariance to include the independence of the formulation of physics on the choice of generalized scalars, the question whether generalized scalars might allow to understand the infinities of quantum field theories, and the question whether the notion of measurement precision could realized in terms of scale hierarchy with levels related by infinite scalings. These ideas are commented in the article by comparison to p-adic variants of these ideas.

Non-standard numbers are compared with the numbers generated by infinite primes. It is found that the construction of infinite primes, integers, and rationals has a close similarity with construction of the generalized scalars. The construction replaces at the lowest level the index set $\Lambda = \mathbb{N}$ of natural numbers with algebraic numbers $\mathbb{A}$, Frechet filter of $\mathbb{N}$ with that of $\mathbb{A}$, and $\mathbb{R}$ with unit circle S^1 represented as complex numbers of unit magnitude. At higher levels of the hierarchy generalized -possibly infinite and infinitesimal- algebraic numbers emerge. This correspondence maps a given set in the dual of Frechet filter of $\mathbb{A}$ to a phase factor characterizing infinite rational algebraically so that correspondence is like representation of algebra. The basic difference between two approaches to infinite numbers is that the counterpart of infinitesimals is infinitude of real units with complex number theoretic anatomy: one might loosely say that these real units are exponentials of infinitesimals.

With motivations coming from quantum computation, Rosinger discusses also a possible generalization of the notion of entanglement [?]llowing to define it also for what could be regarded as classical systems. Entanglement is also number theoretically very interesting notion. For instance, for infinite primes and integers the notion of number theoretical entanglement emerges and relates to the physical interpretation of infinite primes as many particles states of second quantized supersymmetry arithmetic QFT. What is intriguing that the algebraic extension of rationals induces de-entanglement. The de-entanglement corresponds directly to the replacement of a polynomial with rational coefficients with a product of the monomials with algebraic roots in general.

There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

11.2 Could the generalized scalars be useful in physics?

The basic question is whether the generalized scalars could replace reals in theoretical physics. It is best to proceed by making questions.

11.2.1 Are reals somehow special and where to stop?

The following questions relate to the interpretation of generalized scalars.

1. Why reals should be so special? The possible answer is that reals, complex numbers and quaternions form associative continua. Classical number fields are indeed in central role in TGD [K72] , [L12] . Already p-adic number fields consist of disconnected pieces in the sense that one cannot connect two arbitrary points by a continuous curve (p-adic norm of point must change discontinuously at some point of curve is the norms of end points are different).

2. What -if anything physical- it means to replace temperature at space-time point with a function of a natural number? Doesn't this mean the replacement of real numbers with $R \times N$ and replacement of Minkowski space with $M^4 \times N^4$?

3. What is the physical meaning of generalized scalar understood as an equivalence class of real functions of natural number modulo functions vanishing in some set belonging to a filter (possibly ultrafilter)? What could be the physical meaning of filter? Could the quotient construction be interpreted as some sort of gauge invariance or could it just realize the idea "almost-everywhere is everywhere physically"?

4. Can one stop if the step replacing reals with generalized scalars is taken? Recall that quantization means replacement of the WCW with the function space associated with it. Second quantization brings in function space associated with this space and so on. This hierarchy of quantizations is involved with the construction of infinite primes (and rationals) in TGD framework [K70] , [L14] and in this case one has a concrete physical interpretation in terms of many-sheeted space-time.

   Should one replace natural numbers with the power set of natural numbers consisting of finite subsets of natural numbers (dual of the Frechet filter for $\mathbb{N}$) at the next step and perform similar construction. This could be continued ad infinitum. Does one obtain an infinite hierarchy of increasingly surreal numbers in this manner? One can imagine also other kinds of constructions but it is this construction with would be analogous to that for the hierarchy of infinite primes.

11.2.2 Can one generalize calculus?

The obvious question of physicist is whether one can generalize differential and integral calculus - necessary for physics as we know it. Surreals were actually introduced to justify the notion of infinitesimal so that differential calculus should not be a problem. The notion of integral function is neither a problem but definite integral might be due to the loss of Archimedean property. One could try to define the notion of integral in terms of the imbedding of real numbers as constant functions and define definite integral algebraically as a substitution of the integral function between real limits. For arbitrarily limits one cannot order the limits and it seems that one should restrict the considerations to real limits.

What might also pose a problem is the definition of numerical integration - in terms of Riemann sum in its simplest form. One should divide the integration range to short ordered pieces and approximate the integral with sum. But there exists infinite number of paths connecting two functions to each other and one cannot order the pieces in general. Should one generalize complex analyticity so that functions of surreals would be expressible as power series of function and the integrals would not depend on integration path unless the surreal analytic function has singularities such as poles? Does this mean that one can choose one particular path which corresponds a path restricted to real axis so that the integral would reduce to the ordinary real integral.

In p-adic context non-Archimedean property implies that the notion of definite integral is indeed problematic [K47] . The basic problem is that one cannot in general tell which one of the two p-adic numbers with the same norm is the larger one and therefore one cannot define the notion boundary essential in variational calculus. One could use algebraic definition of definite integral as a substitution of integral function and in complex case residue calculus could help. One could use the ordering of rational numbers imbedded to p-adic numbers fields to induce the ordering of p-adic rationals. The p-adic existence of the integral function poses additional conditions encountered already for the integrals of rational functions which can give logarithms of rationals leading out from the realm of rationals. These difficulties have served as a key guiding principle in the attempts to fuse real and p-adic physics to a larger structure.

11.2.3 Generalizing general covariance

What happens to the notion general covariance (or Principle of Relativity in the terminology used by Rosinger, see the article How Far Should the Principle of Relativity Go? [A194])? Here I would like to do some nitpicking by distinguishing between Principle of Relativity which refers to the isometries of Minkowski space and General Coordinate Invariance analogous to gauge symmetry. Various symmetry groups make sense also in the surreal context since they are defined algebraically. A generalization of General Coordinate Invariance meaning that the formulation of physics becomes independent of the choice of generalized scalars is proposed by Rosinger. This notion could be interpreted as a form invariance or as the condition that the physics is indeed the same irrespective of what number field is used in which case the introduction of generalize scalars would not bring in anything new.

Rosinger chooses the non-trivial option which means that the formulation of the laws of physics should make sense irrespective of the number field chosen and considers various examples as applications of the generalized view. He shows that no-cloning theorem of quantum computation holds true also for generalized scalars because the theorem depends on the linearity of quantum theory alone (cloning would map state to two of its copies, something essentially nonlinear).

In TGD framework the notion Number Theoretical Universality interpreted as number field independent formulation of physics seems to relate closely to this principle.

1. All constructions making sense in real context should makes sense also in the p-adic context [K71] , [L13] . Real and p-adic physics meet in the intersection of real and p-adic worlds and result from each other by a kind of algebraic continuation. Simplifying somewhat, at the level of space-time surfaces the intersection would correspond to rational points in some preferred coordinates shared by real and p-adic surfaces and at the level of "world of classical worlds" (WCW) to surfaces expressible in terms of rational functions expressible using polynomials with rational coefficients so that real and p-adic variants of this kind of surfaces are can be identified.

2. Number Theoretic Universality leads to extremely powerful conditions on the geometry of WCW since both its real and p-adic sectors should exist and integrate to a larger structure [K26] . Rationals defining the intersection of reals and various p-adics play a key role and one ends up with a generalization of number concept obtained by gluing reals and p-adics as well as their algebraic extensions to single book like structure [K71] , [L13] .

3. One is also forced to adopt a more refined view about General Coordinate Invariance since the coordinate transformations must respect the algebraic extensions of p-adic numbers used. This brings also non-uniqueness: there are several choices of coordinate frames not transformable to each other. The interpretation would be that that they serve as correlates of cognition. Mathematician is not an outsider and the choice of coordinate system affects the reality albeit in very delicate manner.

This allows to see a relationship between TGD inspired fusion of real and p-adic physics and Rosingers's proposal as roughly following correspondence.

Reals and p-adic number fields resp. rationals defining the intersection of reals and p-adic worlds $\leftrightarrow$ *various generalized scalars resp. reals defining the intersection of various surreals worlds.*

The independence on the choice of generalized scalars might give powerful constraints on the formulation of the theory.

If surreal number fields are important for theoretical physics, physical systems must be characterized by the generalized scalars. What determines this number field or algebra? Can one speak about some kind of quantal evolution in which physical systems evolve more and more complex number theoretically. Could the field of generalized scalars be replaced with a new one in quantum jump taking place via reals common to different generalized scalars?

The attempt to fuse real physics as physics of matter and p-adic physics as physics of cognition one ends up with this kind of picture and one can say that the prime characterizing p-adic number field and the algebraic numbers defining its extension (say roots of unity) characterize its evolutionary level. During evolution the algebraic complexity of the systems steadily increases.

11.2.4 The notion of precision and generalized scalars

Rosinger proposes [A197] that the notion of precision of experiment could be assigned to the self-similar structure of the generalized scalars meaning a hierarchy of scales which differ from each other by infinite scale factors if real norm is used as a measure for the scale. There would be infinite hierarchy of precisions and what looks infinitesimal, finite, or infinite would depend on the precision used and characterized by what generalized scalars are used. Thus one can speak about relative precision.

That one could have units of (say length) differing by infinite scaling in real sense looks rather weird idea. In TGD framework one interpretation for the hierarchy of infinite primes would be that there is infinite hierarchy of variants of Minkowski space such that at the given level of the hierarchy lower levels represent infinitesimals. This would mean fractal cosmology in which the conscious entities above us in the hierarchy would be literally God like as compared to us. No hopes about testing this at LHC!

In p-adic context similar notion emerges but the infinities at different levels are not related by infinite scalings with respect to the p-adic measure for size. Given walkable world correspond in p-adic context to p-adic numbers with fixed norm and in this operational sense p-adic primes with larger norm are infinite. p-Adic prime p indeed characterizes length scale resolution and the roots of unitary used in algebraic extension of p-adics characterize the angle resolution.

Even more, if one accepts that p-adic space-time surfaces serve as correlates for cognition one is forced to conclude that cognition cannot be localized in a finite space-time volume and that "thought bubbles" have actually the size of the entire Universe. Only cognitive representations defined by rational intersections of real and p-adic space-time surfaces would be localized to a finite real volume. Maybe the infinite hierarchy of Rosinger could be assigned to the levels of existence that we are used to assign with cognition and matter corresponds to the lowest level.

11.2.5 Further questions about physical interpretation

Rosinger raises further interesting questions about physical interpretation.

1. In the article *Does Heisenberg Uncertainty Principle make sense in reduced power algebras?* [A197] Rosenberg shows that the answer to the question of the title is affirmative. Rosinger asks in the same article whether the values of fundamental constants like c and $\hbar$ depend on the choice of generalized scalars. For instance, could $\hbar$ be infinitesimal for some generalized scalars? Could c have a well-defined infinite value for some generalized scalars.

   In the case of c one could argue that it is just a conversion factor so that one can put $c = 1$ always by a suitable choice of units. Most physicists would argue that the same is true for $\hbar$. I have however proposed a different vision explaining some strange findings in both astrophysics and biology.

2. Could the fact that infinitesimal and infinite numbers have precise meaning for generalized scalars allow to resolve the problems caused by the infinities of local quantum field theories? Rosinger argues that this might be the case [A197] . The notion of infinity is relative one for generalized scalars and one could replace reals with some other generalized scalars and this could make infinite finite. As a matter fact, in p-adic context for a given p-adic number all p-adic numbers with larger norm represent an operational infinity in the sense that they cannot be reached by walks consisting of integer valued steps. As p-adic numbers they are however finite. It seems that one must be very careful how one defines the infinite: does one use norm or does on use reachability by integer valued steps as the criterion.

   One can counter argue that reals can be distinguished uniquely by their topological properties just like rationals can be distinguished by their number theoretic properties uniquely. Skeptic might say that the situation would become even worse since one would had infinite number of different kind of infinities. The infinities would be completely well-defined functions with finite number of poles but what it means to replaces temperature at space-time point with a function of natural number? Doesn't this mean that space-time point is replaced with natural numbers.

   I have myself considered the possibility that p-adic mathematics for which integers infinite in real sense can make sense p-adically and have norm not larger than unity could allow to resolve the problem of infinities. In particular ultra-metric topology implies that the sum of n numbers is never larger than the maximum of the largest number involved -this is just what walkable universe expresses- raises optimism. It turned however that these ideas did not work in my hands.

11.3 How generalized scalars and infinite primes relate?

The comparison of Rosinger's ideas with the number theoretic ideas of TGD inspires further questions.

1. Classical number fields play a key role in the formulation of quantum TGD. Do the notions of sur-complex, sur-quaternion and and sur-octonion make sense as one might expect?

2. What happens if one replaces real functions define in Λ (say natural numbers) with p-adic valued functions. One obtains algebra also now and one can define ideals and use quotient construction using ultrafilter. Does the notion of sur-p-adic make sense?

3. In TGD framework one ends up with the notion of infinite prime having direct connection with repeated second quantization of super-symmetric arithmetic quantum field theory with fermions and bosons labelled by primes- finite primes at the lowest level of hierarchy. This notion of infinity is essentially number theoretical and implies that the number theoretic anatomy of numbers and space-time points becomes an essential aspect of physics. Can one assign number theoretic anatomy also to non-standard numbers or does the real topology wipe it out?

4. How does the hierarchy of infinite primes relate to the possibly existing hierarchy of reals, surreals, sursurreals,... obtained by replacing real number valued function with surreal number valued functions replaced in turn with ....?

The last question deserves a more detailed consideration since it could provide an improved understanding of infinite primes. Consider first the construction of infinite primes [K70] , [L14] .

1. Infinite primes at the lowest level of hierarchy can be generated from two fermionic vacuum states $P_{\pm} = X \pm 1$, where X is defined as a product of all finite primes having p-adic norm less than one for all finite primes p. X is analogous to Dirac sea with all negative energy states filled. Simple infinite primes are of form $mX/n+rn$, where m and n have no common divisors and r consists of same primes as n. $m = \prod p_i^{k_i}$ corresponds to many boson state with k_i bosons with "momentum" p_i. In fermionic sector the square free integer n has interpretation as many-fermion state with single fermion in the modes involved. r corresponds to many-boson states in these modes. Simple infinite primes are clearly analogous to many particle states obtained by kicking fermions from sea to get positive energy holes and adding bosons whose number is arbitrary in a given mode labelled by finite prime. Simple infinite primes have unit p-adic norm so that "infinite" is a relative notion.

2. More complex infinite primes are infinite integers obtained as sums of products of infinite primes. The interpretation is in terms of bound many-particle states.

3. In zero energy ontology (ZEO) an attractive interpretation for infinite rationals is as zero energy states with numerator and denominator representing positive and negative energy parts of the state.

4. One can continue the construction indefinitely. At the next level X is replaced with the product of all infinite primes at the first level of the hierarchy and the process is repeated. The physical interpretation would be that at the next level many particle states of previous level take the role of single particle states and one constructs free and bound many particle states of these. The many-sheeted space-time of TGD suggests a concrete realization of this process and I have indeed proposed a concrete physical interpretation of standard model quantum numbers in terms of what I call (hyper-)octonionic primes, which would generate a structure analogous to infinite primes.

Generalized scalars define a function algebra and this inspires the question is whether one could somehow assign a function algebra also to infinite primes and in this manner to see what is common features these very different looking notions might have. Infinite primes can be indeed mapped to polynomial primes as the following argument shows.

1. Simple infinite primes are characterized by two integers which have no common divisors and can be thus mapped in a natural manner to rationals $q = rn^2/m$. They can can be also mapped to monomials $x - q$, $q = rn^2/m$, where X could be seen as a particular value of x. Complex infinite primes constructed as products of simple infinite primes can be mapped to products of these monomials and sums of their products to sums of these so that on obtains a mapping to polynomial primes at the lowest level of the hierarchy. Vacua are mapped to rationals 1 and -1. One can decompose the polynomials to products of monomials $x - r$, where r is a finite algebraic number, and the interpretation would be that one considers primes in an algebraic extension of rationals and this representation applies to infinite prime when x is substituted with X.

2. This mapping makes sense also at the next level of hierarchy at least formally. Call the product of finite and infinite primes at the first level X_1 and corresponding formal variable x_1. Infinite rationals correspond now to rational functions of x_1 and x defined as ratios of polynomials $P_k(x_1, x)$ for which the highest power of x_1 is by definition x_1^k. The roots in the product representation of polynomials are obtained by the substitution $x \to X$ in the expressions of the roots as functions of x. The roots are generalized algebraic numbers which can be infinite or vanish as real numbers. This kind of mapping makes also sense at the higher levels of hierarchy. The roots of polynomial at the n:th level of the hierarchy are obtained by substituting to their expressions as algebraic functions $x_m = X_m$, $m < n$.

3. What one obtains is a map to polynomials so that one can indeed map infinite primes and also integers and rationals to a function algebra consisting of polynomials. Ideals correspond now to polynomial ideals consisting of polynomials proportional to some polynomial prime. There are no divisors of zero so that quotient construction is not needed now.

This construction leads to intriguing observations relating the construction of infinite primes to the construction of generalized scalars and suggesting that infinite primes represent a generalization of the concept of sur-complex numbers by identifying ultrafilter in terms of complements of finite subsets of algebraic numbers (Frechet filter actually). The heuristic argument goes as follows.

1. The hierarchy of subsets of algebraic numbers defined by the infinite primes at the lowest level of hierarchy defines *complement of Frechet filter* $\mathcal{CF}$ with the following defining properties. $\mathcal{CF}$ contains empty set and all finite subsets of Λ, unions of sets of $\mathcal{CF}$ belong to $\mathcal{CF}$, and subsets of a set belonging to $\mathcal{CF}$ belong to $\mathcal{CF}$.

   Note that powers of infinite primes define the same set in $\mathcal{CF}$ as infinite prime itself so that the correspondence does not seem to be many-to-one. It is not clear whether fermionic statistics could be used as a physical excuse to exclude these powers and more generally products of infinite primes for which same finite prime appears in more than one different infinite primes. Also subsets of genuinely algebraic numbers could correspond to several infinite integers and rationals.

   If one restricts the consideration to square free integers defined by the fermionic parts of infinite primes then the sets of natural numbers assignable to infinite primes correspond to finite subsets of square free natural numbers defining a Frechet filter for them.

2. $\Lambda = \mathbb{N}$ is replaced with algebraic numbers $\mathbb{A}$ so that the function space defining generalized scalars would consist of functions $f : \mathbb{A} \to \mathbb{C}$. It is not however clear what kind of functions one should consider.

   (a) The first guess is that the quantum states of supersymmetric arithmetic QFT (SAQFT) correspond to functions non-vanishing only in some finite set belonging to $\mathcal{CF}$. They would map to zero in the quotient construction of ultrapower field. The functions which do not map to zero would correspond to non-vanishing elements of the ultrapower field and would have no physical interpretation. This does not sound sensible physically.

   (b) The many-particle states of arithmetic QFT could more naturally correspond to functions having values on circle S^1 -rather than $\mathbb{C}$- identified as complex numbers with unit magnitude. The value of this kind of functions would be constant - most naturally 1 - for given infinite set of $\mathcal{U}$ and root of unity in the complement of $\mathcal{U}$ defined by infinite integer or rational.

   These functions would be analogous to plane waves having modulus equal to 1 and if they correspond to roots of unity they would make sense also for algebraic extensions of p-adic numbers. This conforms with the fact that p-adic norms of infinite primes and rationals are equal to unity. This would lead to a rather astonishing conclusion: there are no infinite numbers nor infinitesimals in the field generated by infinite primes in the sense of generalized scalars!

   Note that functions which reduce to phases in the set of algebraic numbers are also natural in the sense that there are hopes of defining for them inner product as sum over algebraic numbers. The inner product should be consistent with the inner product induced by that for Fock states and it might be better to start directly from this inner product.

   (c) It is important to realize that the complements of infinite rationals do not define support for functions but the functions themselves so that the analogy with the ultrapower construction fails.

3. The higher levels in the hierarchy of infinite primes are also present and require a further generalization of the construction. At the second level of the hierarchy algebraic numbers are replaced with the power set consisting of all finite subsets of algebraic numbers and dual of Frechet filter with that consisting of all finite subsets of this power set. Higher levels of the hierarchy would correspond a repeated replacement of the set with its power set.

4. Mathematical skeptic reader might wonder why this infinite hierarchy of constructions? Does it even lead outside the realm of algebraic numbers? What is however remarkable is that it generalizes the physics by replacing the first two quantizations with an infinite hierarchy of quantizations.

11.3.1 Explicit realization for the function algebra associated with infinite rationals

Consider now an explicit realizations of this algebra as a function algebra. The idea is to assigns to a given infinite rational a unique phase representing and that the algebraic structure defined by multiplication is preserved. This is like mapping rationals $q = m/n$ to phases $exp(i2\pi q)$ so that products are mapped to products. One can start from the observation that simple infinite primes can be mapped to rationals. More complex infinite primes, integers, and rationals can be mapped to collections of algebraic numbers representing the roots of corresponding polynomial primes.

1. The simplest option is that the value of the complex valued function of algebraic numbers assigned to simple infinite prime characterized by rational q is equal to $exp(i2\pi q)$ for rational q and to 1 for other algebraic numbers. The product of simple infinite integers os mapped to the product of these functions assigned to the factors. The ratio of two simple infinite integers is mapped to the ratio of corresponding functions.

2. By utilizing the decomposition the map to polynomial or rational function and its decomposition into monomials with possibly algebraic roots one could map the polynomials of rational function to factors $\prod_i exp(2\pi r_i)$ for a given infinite rational in its polynomial representation decompose to a product of monomials. This representation would map products (ratios) of infinite integers to products (ratios) but sums would not be mapped to sums but products in algebraic extension of rationals. That the images would be always non-vanishing functions would conform with the basic properties of infinite primes and with non-existence of infinitesimals and infinite numbers in the sense of the usual ultrapower construction.

3. One would have functions in the set of algebraic numbers at the first level of hierarchy. At the next level of hierarchy one would have complex complex defined in the set of generalized rationals constructed from infinite integers. These phases are actually well defined since the infinite rational appearing in the exponent can be decomposed to a sum of terms. Only those terms which are finite contribute to the phase so that one obtains a well-defined outcome. This hierarchy would continue ad infinitum. Similar hierarchy can be associated with generalized scalars.

4. Primes are replaced with prime ideals in a more abstract approach to number theory. One could also assign to the rationals assigned to simple infinite primes the prime ideal of real or complex valued functions with value equal to one for all rationals except the selected rational. The product of simple infinite primes would correspond to the ideal consisting of functions which differ from unity for the rationals appearing in the product. The sum of simple infinite primes would in turn correspond to similar functions but differing from unity also for algebraic numbers. This would give a hierarchy of ideals with particular ideal defined in terms of functions whose value is larger than integer n for most rationals and algebraic numbers.

11.3.2 Generalization of the notion of real by bringing in infinite number of real units

Infinite rationals lead also to a generalization of the real numbers in the sense that given real number is replaced with infinitude of numbers having the same magnitude by multiplying it by real units which differ number theoretically [K70] , [L14] . There exists infinite number of rationals constructed as ratios of infinite integers at various levels of the hierarchy which as real numbers are equal to real unit but have arbitrarily complex number theoretical anatomy. Single point of real line is replaced with infinitely complex infinite-dimensional structure defined by the space of real units. This generalization applies also to other classical number fields. The role of infinitesimals would be taken by the infinitude of real units and this would extend real numbers.

This has inspired the ontological proposal that the quantum states of Universe (and even the world of classical worlds (or its sub-world defined associated with 4-surfaces inside $CD \times CP_2$) could be imbedded to this space. A less wild statement is that at least the quantum states and sub-WCW assignable to the so called causal diamond identified as the intersection of future and past directed light-cones and defining the basic structural unit in zero energy ontology can be realized in terms of the number theoretic anatomy of single space-time point.

Real units (and their generalizations to octonionic context) are analogous to quantum states. Their sum is analogous to a quantum superposition and gives a real unit by using a simple normalization. Real units are also analogous to zero energy states. By writing each infinite prime P_i at a given level of hierarchy in the form $P_i = Q_i(X_n - 1)$ (note that P_i is infinitesimal as compared to X_n), one finds that real unit condition implies that the total numbers of X_n:s in the numerator and denominator of a real unit must be same. One can apply the same procedure for the factor

$$\frac{\prod_{num} Q_i}{\prod_{den} Q_i} \quad (\text{"}num\text{" and "}den\text{" denote numerator and denominator of infinite prime})$$

to conclude that it must contain same number of X_{n-1}:s in its numerator and denominator. At the lowest level one finds that one obtains ratio of integers expressed as products of powers of finite primes p_i which must be equal to unity. The interpretation in positive energy ontology is that the total number theoretic momentum coming as integer multiple of $log(p_i)$ is same for the positive and negative energy parts of the state and therefore conserved for each finite prime p_i separately (the numbers $log(p_i)$ are algebraically independent). Conservation is indeed what one expects in arithmetic QFT.

$M^4 \times CP_2$ with structured space-time points could be able to represent all the structures of quantum theory having otherwise somewhat questionable ontological status. A given mathematical structure would "really" exist if it allows imbedding to generalized $M^4 \times CP_2$, which itself has interpretation in terms of classical number fields. Accordingly, one could talk about number theoretic Brahman=Atman identity or algebraic holography.

The above considerations suggest that the hierarchy of infinite primes and hierarchy of generalized scalars cannot be identified. It is not clear clear whether could consider the fusion of these notions. Also the fusion of real and p-adic number fields to a book like structure and of generalized scalars could be considered.

11.3.3 Finding the roots of polynomials defined by infinite primes

Infinite primes identifiable as analogs of bound states correspond at n:th level of the hierarchy to irreducible polynomials in the variable X_n which corresponds to the product of all primes at the previous level of hierarchy. At the first level of hierarchy the roots of this polynomial are ordinary algebraic numbers but at higher levels they correspond to infinite algebraic numbers which are somewhat weird looking creatures. These numbers however exist p-adically for all primes at the previous levels because one one can develop the roots of the polynomial in question as powers series in X_{n-1} and this series converges p-adically. This of course requires that infinite-p p-adicity makes sense. Note that all higher terms in series are p-adically infinitesimal at higher levels of the hierarchy. Roots are also infinitesimal in the scale defined X_n. Power series expansion allows to construct the roots explicitly at given level of the hierarchy as the following induction argument demonstrates.

1. At the first level of the hierarchy the roots of the polynomial of X_1 are ordinary algebraic numbers and irreducible polynomials correspond to infinite primes. Induction hypothesis states that the roots can be solved at n:th level of the hierarchy.

2. At $n+1$:th level of the hierarchy infinite primes correspond to irreducible polynomials

$$P_m(X_{n+1}) = \sum_{s=0,\ldots,m} p_s X_{n+1}^s \quad .$$

The roots R are given by the condition

$$P_m(R) = 0 \ .$$

The ansatz for a given root R of the polynomial is as a Taylor series in X_n:

$$R = \sum r_k X_n^k \ ,$$

which indeed converges p-adically for all primes of the previous level. Note that R is infinitesimal at $n+1$:th level. This gives

$$P_m(R) = \sum_{s=0,...,m} p_s(\sum r_k X_n^k)^s = 0 \ .$$

(a) The polynomial contains constant term (zeroth power of X_{n+1} given by

$$P_m(r_0) = \sum_{s=0,...,m} p_r r_0^s \ .$$

The vanishing of this term determines the value of r_0. Although r_0 is infinite number the condition makes sense by induction hypothesis.

One can indeed interpret the vanishing condition

$$P_{m\times m_1}(r_0) = 0$$

as a vanishing of a polynomial at the n:th level of hierarchy having coefficients at $n-1$:th level. Here m_1 is determined by the dependence on infinite primes of lower level expressible in terms of rational functions. One can continue the process down to the lowest level of hierarchy obtaining $m \times m_1 ... \times m_k$:th order polynomial at k:th step. At the lowest level of the hierarchy one obtains just ordinary polynomial equation having ordinary algebraic numbers as roots.

One can expand the infinite primes as a Taylor expansion in variables X_i and the resulting number differs from an ordinary algebraic number by an infinitesimal in the multi-P infinite-P p-adic topology defined by any choice of n-plet of infinite-P p-adic primes $(P_1, ..., P_n)$ from subsequent levels of the hierarchy appearing in the expansion. In this sense the resulting number is infinitely near to an ordinary algebraic number and the structure is analogous to a completion of algebraic numbers to reals. Could one regard this structure as a possible alternative view about reals remains an open question. If so, then also reals could be said to have number theoretic anatomy.

(b) If one has found the values of r_0 one can solve the coefficients r_s, $s > 0$ as linear expressions of the coefficients r_t, $t < s$ and thus in terms of r_0.

(c) The naive expectation is that the fundamental theorem of algebra generalizes so that that the number of different roots r_0 would be equal to m in the irreducible case. This seems to be the case. Suppose that one has constructed a root R of P_m. One can write $P_m(X_{n+1})$ in the form

$$P_m(X_{n+1}) = (X_{n+1} - R) \times P_{m-1}(X_{n+1}) \ ,$$

and solve P_{m-1} by expanding P_m as Taylor polynomial with respect to $X_{n+1} - R$. This is achieved by calculating the derivatives of both sides with respect to X_{m+1}. The derivatives are completely well-defined since purely algebraic operations are in question. For instance, at the first step one obtains $P_{m-1}(R) = (dP_m/dX_{n+1})(R)$. The process stops at m:th step so that m roots are obtained.

What is remarkable that the construction of the roots at the first level of the hierarchy forces the introduction of p-adic number fields and that at higher levels also infinite-p p-adic number fields must be introduced. Therefore infinite primes provide a higher level concept implying real and p-adic number fields. If one allows all levels of the hierarchy, a new number X_n must be introduced at each level of the hierarchy. About this number one knows all of its lower level p-adic norms and infinite real norm but cannot say anything more about them. The conjectured correspondence of real units built as ratios of infinite integers and zero energy states however means that these infinite primes would be represented as building blocks of quantum states and that the points of imbedding space would have infinitely complex number theoretical anatomy able to represent zero energy states and perhaps even the world of classical worlds associated with a given causal diamond.

11.4 Further comments about physics related articles

In the following I represent comments on the physics related articles of Rosinger not directly related to generalized scalars. I have not commented the purely mathematics related more technical articles since I do not have the competence to say anything interesting about them.

11.4.1 Quantum Foundations: Is Probability Ontological

In this highly interesting article [A195] Rosinger poses the question whether the notion of probability is ontological or only epistemic. Are probabilities basic aspect of existence or are they are "a useful construct of mind only". My own very first reaction is a counter question. Can one speak about "mere construct of mind"? "Mind" is a part of existence and the future physics must include it to its world order. If mind is able to construct a notion like probability this notion could have some quantal correlate.

Rosinger introduces the notions of deterministic (classical typically) and non-deterministic systems and distinguishes probabilistic, fuzzy and chaotic systems as special cases of non-deterministic systems. For fuzzy and chaotic systems probability is clearly a fictive but useful notion. For probabilistic systems, in particular quantum systems the situation is not clear at all.

As a mathematician Rosinger raises purely mathematical objections against the ontological status of probability. Rosinger mentions the technical difficulties with the description of stochastic processes with continuous time and objections against axiomatizations -say in terms of Kolmogorov axioms. Rosinger mentions also frequency interpretation and somewhat fuzzy propensity interpretation of probabilities and that the notion of infinity is unavoidable also now. I cannot say much about these technical aspects and can only represent the comments based on my own physics inspired belief system.

To my very subjective view the situation is far from settled from the point of view of theoretical physics and one can consider several deformations of the notion of probability.

1. Khrennikov [A181] has formulated the notion of p-adic valued probability and also I have considered p-adic thermodynamics based model for particle masses (see the first part of [K46]) whose predictions, which are basically due to number theoretic existence constraints- are mapped to real numbers by a canonical correspondence between reals and p-adics.

2. Also the notion of quantum spinors related in TGD framework to the description of finite measurement resolution [K80] raises the possibility that the probability itself becomes observable instead of spin (by the finite precision associated with the determination of quantization axes) and has a universal spectrum.

3. The findings of Russian biologist Shnoll [K5] , [E1] , [E1] suggesting that the expected single peaked distributions for fluctuations of various process described by probability distributions for integer valued observable are replaced by many-peaked distributions encourage to think that the time scale of experiment is essential and the usual idea about smooth approach to probabilities as the duration of experiment increases is not correct. I have proposed an explanation of these findings in terms of the deformations of probability distributions depending on rational valued parameters so that they make sense also p-adically. This predicts precise and universal deviations which can be tested.

Rosinger relates [A195] the famous Bohr-Einstein debate to the ontological status of probability concept. The divisor line between Bohr and Einstein was the attitude towards non-determinism. Neither of them could accept the idea that the determinism of Schrödinger equation could fail temporarily. Bohr was ready to give up the notion of objective reality altogether whereas Einstein refused to accept state function reduction since it would have meant giving up also the deterministic dynamics of the space-time geometry. According to Rosinger, Copenhagenist would regard probability and probability amplitudes as a fundamental aspect of existence whereas Einstein would have given for probability only episthemic role.

To my opinion both Einstein and Bohr were both right and wrong. If one accepts the view that quantum states actually correspond to superpositions of deterministic histories (generalized Bohr orbits) -as suggested also by holography principle- the problem disappears. Quantum jump recreates the quantum state as quantum superposition of entire deterministic time evolution rather than tinkering with a particular time evolution. There is no contradiction between the determinism of field equation and non-determinism of quantum jump and genuine evolution emerges as a by-product.

In this framework one also ends up with the identification of theory as a mathematical objects with the reality itself. There is no need to assume reality behind the quantum states as mathematical objects. Reality is its mathematical description as quantum state and therefore nothing but this "construct of mind". Probability amplitudes receive a firm ontological status and in TGD framework correspond to what I call spinors fields of WCE having purely geometric interpretation. Whether probabilities defined in terms of density matrix have independent ontological status is not quite clear. In quantum theory continuous stochastic process would not really occur and could be seen as a mere idealization of a process which takes as discrete quantum jumps. The technical difficulties in their description would not represent argument against the ontological status of probability amplitudes.

Thermodynamical probability is usually regarded as having only episthemic status but in zero energy ontology - one characteristic aspect of TGD quantum - positive energy quantum states are replaced with zero energy states which can be regarded mathematically as complex square roots of density matrices -which I call M-matrices- decomposable to diagonal matrix representing square roots of probabilities and unitary S-matrix. M-matrices can be organized to orthogonal rows of unitary U-matrix defining the theory. Does this mean thermodynamical holography in the sense that single particle states are able to represent the mathematics of thermodynamical ensembles in terms of their quantum states?

11.4.2 Group Invariant Entanglements in Generalized Tensor Products

Rosinger proposes [A196] a generalization of the notion of entanglement from Hilbert space context to much more general context. The motivation is that it might allow quantum computation like operations even in classical physics context so that the problems caused by the fragility of quantum entanglement could be circumvented.

Recall that ordinary quantization leads from Cartesian product to tensor product as one replaces the points of Cartesian factors with quantum states localized at these points and forms all possible tensor products and also their superpositions. In quantum theory entanglement would emerge at the level of the function space associated with Cartesian space. Already ordinary functions of several variables allow entanglement in this sense. Un-entangled functions of several variables correspond to products of functions of single variable and the sums of these products are in general entangled. Quite generally the special functions of mathematical physics emerges as separable/un-entangled solutions of linear partial differential equations and non-linearity typically implies entanglement in this sense.

The goal of Rosinger is to generalize this framework that is to find spaces - which he calls non-Cartesian spaces- containing Cartesian product as a sub-space with the points in the complement of Cartesian product identified entangled states. Rosinger defines what he calls group invariant entanglement for a Cartesian product and shows that group operations respect the property of being entangled. As an example sequences of point pairs of Cartesian product with algebraic operation analogous to tensor product defined by convolution are considered.

The notion of entanglement has turned out to be highly interesting and non-trivial also in TGD framework.

1. A rather abstract view about entanglement is in terms of correlations. In TGD framework quantum classical correspondence realized as holography defines a very abstract form of entanglement. In this case, the quantum states assignable to the partonic 2-surfaces plus 4-D tangent space-data correspond to classical physics in the interior of space-time surface so that one obtains entanglement through this correlation. This kind of entanglement would give rise to quantum classical correspondence.

2. For infinite primes [K70] , [L14] the notion of entanglement emerges naturally from number theory. This is not so surprising because they can be interpreted in terms of Fock state basis for second quantized arithmetic quantum field theory. The point is that the sum of infinite integers cannot be done by using fingers since we do not possess infinite number of fingers. Therefore the sum of infinite integers is just as it is written: one cannot in general eliminate the plus from the expression unless one leaves the realm of rationals in which case one can decompose the infinite integer to a product of infinite primes. The sums of infinite integers are like superpositions of quantum states and one cannot indeed use reals as field multiplying the infinite primes. Since the products of infinite primes at the lowest level of hierarchy involve parts which can be organized to a polynomial in powers of the variable X defined by the product of finite primes identifiable formally as a variable of polynomial , one can find the expansion of infinite integer as sums over products of infinite primes and this representation is very much like the representation of entangled state.

   What is interesting is that a decomposition into unentangled state product state is obtained if one allows algebraic extension of rationals and the question is whether something like this could be achieved also for quantum states quite generally by some extension of state space concept.

Entanglement has also other number theoretic aspects.

1. One could speak about irreducible entanglement in a given extension of rationals or p-adic numbers in the sense that entanglement is reducible only if the diagonalization of the density matrix is possible in the number field considered.

2. Shannon entropy has also infinite number of number theoretic variants of entanglement probabilities are rational and even algebraic numbers [K42] . The number theoretic Shannon entropy is obtained by replacing the probabilities p_i in the argument of $log(p_i)$ with their p-adic norms and changing the overall sign in the definition of Shannon entropy. The resulting entanglement negentropy can be negative and achieves negative minimum for a unique prime. This means a possibility of information carrying entanglement conjecture to characterize the difference between living and inanimate matter identified as something residing in the intersection of real and p-adic worlds. Negentropy Maximization Principle [K42] stating that state function reduction reduces entanglement entropy would indeed make this kind of entanglement stable under state function reduction.

3. The stability of entanglement could also follow from the hypothesis that physical systems are ordered with respect to the hierarchy of algebraic extensions of rationals assigned with them if one believes on number theoretically irreducib le entanglement. The hierarchy of Planck constants with arbitrarily large values of Planck constants [K25] would provide a further stabilization mechanism since quantum time scales typically scale like $\hbar$. The implications for quantum computation for which the fragility of entanglement is the basic obstacle are obvious.

4. A further aspect is related to finite measurement resolution which I have suggested to be realized in terms of inclusions of hyper-finite factors [K80] . The basic idea is that complex rays of state space are replaced with the orbits of included algebra characterizing measurement resolution. This leads to the replacement of complex numbers with non-commutative algebra as generalized scalars and generalizes the proposal of Rosinger in another direction. In this framework quantum spinors appear as finite-dimensional non-commutative spinors characterized by fractal dimension and probability becomes the observable instead of spin. One can speak also about quantum entanglement in given measurement resolution defined by the included algebra.

Chapter 12

Infinite Primes and Motives

12.1 Introduction

The construction of twistor amplitudes has led to the realization that the work of Grothendieck related to motivic cohomology simplifies enormously the calculation of the integrals of holomorphic forms over sub-varieties of the projective spaces involved. What one obtains are integrals of multi-valued functions known as Grassmannian poly-logarithms generalizing the notion of poly-logarithm [B18] and Goncharov has given a simple formula for these integrals [B24] using methods of motivic cohomology [A63] in terms of classical polylogarithms $Li_k(x)$, $k = 1, 2, 3, ...$ This suggests that motivic cohomology might have applications in quantum physics also as a a conceptual tool. One could even hope that quantum physics could provide fresh insights algebraic geometry and topology.

Ordinary theoretical physicist probably does not encounter the notions of homotopy, homology, and cohomology in his daily work and Grothendieck's work looks to him (or at least me!) like a horrible abstraction going completely over the head. Perhaps it is after all good to at least try to understand what this all is about. The association of new ideas with TGD is for me the most effective manner to gain at least the impression that I have managed to understand something and I will apply this method also now. If anything else, this strategy makes the learning of new concepts an intellectual adventure producing genuine surprises, reckless speculations, and in some cases perhaps even genuine output. I do not pretend of being a real mathematician and I present my humble apologies for all misunderstandings unavoidable in this kind enterprise. One should take the summary about the basics of cohomology theory just as a summary of a journalist. I still hope that these scribblings could stimulate mathematical imagination of a real mathematician.

While trying to understand Wikipedia summaries about the notions related to the motivic cohomology I was surprised in discovering how similar the goals and basic ideas about how to achieve them of quantum TGD and motive theory are despite the fact that we work at totally different levels of mathematical abstraction and technicality. I am however convinced that TGD as a physical theory represents similar high level of abstraction and therefore dare hope that the interaction of the these ideas might produce something useful. As a matter fact, I was also surprised that TGD indeed provides a radically new approach to the problem of constructing topological invariants for algebraic and even more general surfaces.

12.1.1 What are the deep problems?

In motivic cohomology one wants to relate and unify various cohomologies defined for a given number field and its extensions and even for different number fields if I have understood correctly. In TGD one would like to fuse together real and various p-adic physics and this would suggest that one must relate also the cohomology theories defined in different number fields. Number theoretical universality [K71] allowing to relate physics in different number fields is one of the key ideas involved.

Why the generalization of homology [A42] and cohomology [A20] to p-adic context is so non-trivial? Is it the failure of the notion of boundary does not allow to define homology in geometric sense in p-adic context using geometric approach. The lack of definite integral in turn does not allow to define p-adic counterparts of forms except as a purely local notion so that one cannot speak about values of forms for sub-varieties. Residue calculus provides one way out and various cohomology theories defined in finite and p-adic number fields actually define integration for forms over closed surfaces (so that the troublesome boundaries are not needed), which is however much less than genuine integration. In twistor approach to scattering amplitudes one indeed encounters integrals of forms for varieties in projective spaces.

Galois group [A38] is defined as the group leaving invariant the rational functions of roots of polynomial having values in the original field. A modern definition is as the automorphism group of the algebraic extension of number field generated by roots with the property that it acts trivially in the original field.

1. Some examples Galois group in the field or rationals are in order. The simplest example is second order polynomial in the field of rationals for which the group is Z_2 if roots are not rational numbers. Second example is $P(x) = x^n - 1$ for which the group is cyclic group $S(n)$ permuting the roots of unity which appear in the elementary symmetric functions of the roots which are rational. When the roots are such that all their products except the product of all roots are irrational numbers, the situation is same since all symmetric functions appearing in the polynomial must be rational valued. Group is smaller if the product for two or more subsets of roots is real. Galois group generalizes to the situation when one has a polynomial of many variables: in this case one obtains for the first variable ordinary roots but polynomials appearing as arguments. Now one must consider algebraic functions as extension of the algebra of polynomial functions with rational coefficients.

2. Galois group permutes branches of the graph $x = (P_n^{-1})(y, ...)$ of the inverse function of the polynomial analogous to the group permuting sheets of the covering space. Galois group is therefore analogous to first homotopy group. Since Galois group is subgroup of permutation group, since permutation group can be lifted to braid group acting as the first homotopy group on plane with punctures, and since the homotopies of plane can be induced by flows, this analogy can be made more precise and leads to a connection with topological quantum field theories for braid groups.

3. Galois group makes sense also in padic context and for finite fields and its abelianization by mapping commutator group to unit element gives rise to the analog of homology group and by Poincare duality to cohomology group. One can also construct p-adic and finite field representations of Galois groups.

These observations motivate the following questions. Could Galois group be generalized to so that they would give rise to the analogs of homotopy groups and homology and cohomology groups as their abelianizations? Could one find a geometric representation for boundary operation making sense also in p-adic context?

12.1.2 TGD background

The visions about physics as geometry and physics as generalized number theory suggest that number theoretical formulation of homotopy-, homology-, and cohomology groups might be possible in terms of a generalization of the notion of Galois group, which is the unifying notion of number theory. Already the observations of Andre Weil suggesting a deep connection between topological characteristics of a variety and its number theoretic properties indicate this kind of connection and this is what seems to emerge and led to Weil cohomology formulated. The notion of motivic Galois group is an attempt to realize this idea.

Physics as a generalized number theory involves three threads.

1. The fusion of real and p-adic number fields to a larger structure requires number theoretical universality in some sense and leads to a generalization of the notion of number by fusion reals and p-adic number fields together along common rationals (roughly) [K71].

2. There are good hopes that the classical number fields could allow to understand standard model symmetries and there are good hopes of understanding $M^4 \times CP_2$ and the classical dynamics of space-time number theoretically [K72].

3. The construction of infinite primes having interpretation as a repeated second quantization of an supersymmetric arithmetic QFT having very direct connections with physics is the third thread [K70]. The hierarchy has many interpretations: as a hierarchy of space-time sheets for many-sheeted space with each level of hierarchy giving rise to elementary fermions and bosons as bound states of lower level bosons and fermions, hierarchy of logics of various orders realized as statements about statements about..., or a hierarchy of polynomials of several variables with a natural ordering of the arguments.

   This approach leads also to a generalization of the notion of number by giving it an infinitely complex number theoretical anatomy implied by the existence of real units defined by the ratios of infinite primes reducing to real units in real topology. Depending one one's tastes one can speak about number theoretic Brahman=Atman identity or algebraic holography. This picture generalizes to the level of quaternionic and octonionic primes and leads to the proposal that standard model quantum numbers could be understand number theoretically. The proposal is that the number theoretic anatomy could allow to represent the "world of classical worlds" (WCW) as sub-manifolds of the infinite-dimensional space of units assignable to single point of space-time and also WCW spinor fields as quantum superpositions of the units. One also ends up with he idea that there is an evolution associated with the points of the imbedding space as an increase of number theoretical complexity. One could perhaps say that this space represents "Platonia".

12.1.3 Homology and cohomology theories based on groups algebras for a hierarchy of Galois groups assigned to polynomials defined by infinite primes

The basic philosophy is that the elements of homology and cohomology should have interpretation as states of supersymmetric quantum field theory just as the infinite primes do have. Even more, TGD as almost topological QFT requires that these groups should define quantum states in the Universe predicted by quantum TGD. The basic ideas of the proposal are simple.

1. One can assign to infinite prime at n:th level of hierarchy of second quantizations a rational function and solve its polynomial roots by restricting the rational function to the planes $x_n, ...x_k = 0$. At the lowest level one obtains ordinary roots as algebraic number. At each level one can assign Galois group and to this hierarchy of Galois groups one wants to assign homology and cohomology theories. Geometrically boundary operation would correspond to the restriction to the plane $x_k = 0$. Different permutations for the restrictions would define non-equivalent sequences of Galois groups and the physical picture suggests that all these are needed to characterize the algebraic variety in question.

2. The boundary operation applied to G_k gives element in the commutator subgroup $[G_{k-2}, G_{k-2}]$. In abelianization this element goes to zero and one obtains ordinary homology theory. Therefore one has the algebraic analog of homotopy theory,

3. In order to obtain both homotopy and cohomotopy and cohomology and homology as their abelizations plus a resemblance with ordinary cohomology one must replace Galois groups by their group algebras. The elements of the group algebras have a natural interpretation as bosonic wave functions. The dual of group algebra defines naturally cohomotopy and cohomology theories. One expects that there is a large number of boundary homomorphisms and the assumption is that these homomorphisms satisfy anti-commutation relations with anti-commutator equal to an element of commutator subgroup $[G_{k-2}, G_{k-2},]$ so that in abelianization one obtains ordinary anti-commutation relations. The interpretation for the boundary and coboundary operators would be in terms of fermionic annihilation (creation) operators is suggestive so that homology and cohomology would represent quantum states of super-symmetric QFT. Poincare duality would correspond to hermitian conjugation mapping fermionic creation operators to annihilation operators and vice versa. It however turns out that the analogy with Dolbeault cohomology with several exterior derivatives is more appropriate.

4. In quantum TGD states are realized as many-fermion states assignable to intersections of braids with partonic 2-surfaces. Braid picture is implied by the finite measurement resolution implying discretization at space-time level. Symplectic transformations in turn act as fundamental symmetries of quantum TGD and given sector of WCW corresponds to symplectic group as far as quantum fluctuating degrees of freedom are considered. This encourages the hypothesis that the hierarchy of Galois groups assignable to infinite prime (integer/rational) having interpretation in terms of repeated second quantization can be mapped to a braid of braids of .... The Galois group elements lifted to braid group elements would be realized as symplectic flows and boundary homomorphism would correspond to symplectic flow induced at given level in the interior of sub-braids and inducing action of braid group. In this framework the braided Galois group cohhomology would correspond to the states of WCW spinor fields in "orbital" degrees of freedom in finite measurement resolution realized in terms of number theoretical discretization.

If this vision is correct, the construction of quantum states in finite measurement resolution would have purely number theoretic interpretation and would conform with the interpretation of quantum TGD as almost topological QFT. That the groups characterize algebraic geometry than mere topology would give a concrete content to the overall important "almost" and would be in accordance with physics as infinite-dimensional geometry vision.

12.1.4 p-Adic integration and cohomology

This picture leads also to a proposal how p-adic integrals could be defined in TGD framework.

1. The calculation of twistorial amplitudes reduces to multi-dimensional residue calculus. Motivic integration gives excellent hopes for the p-adic existence of this calculus and braid representation would give space-time representation for the residue integrals in terms of the braid points representing poles of the integrand: this would conform with quantum classical correspondence. The power of 2π appearing in multiple residue integral is problematic unless it disappears from scattering amplitudes. Otherwise one must allow an extension of p-adic numbers to a ring containing powers of 2π.

2. Weak form of electric-magnetic duality and the general solution ansatz for preferred extremals reduce the Kähler action defining the Kähler function for WCW to the integral of Chern-Simons 3-form. Hence the reduction to cohomology takes places at space-time level and since p-adic cohomology exists there are excellent hopes about the existence of p-adic variant of Kähler action. The existence of the exponent of Kähler gives additional powerful constraints on the value of the Kähler fuction in the intersection of real and p-adic worlds consisting of algebraic partonic 2-surfaces and allows to guess the general form of the Kähler action in p-adic context.

3. One also should define p-adic integration for vacuum functional at the level of WCW. p-Adic thermodynamics serves as a guideline leading to the condition that in p-adic sector exponent of Kähler action is of form $(m/n)^r$, where m/n is divisible by a positive power of p-adic prime p. This implies that one has sum over contributions coming as powers of p and the challenge is to calculate the integral for K= constant surfaces using the integration measure defined by an infinite power of Kähler form of WCW reducing the integral to cohomology which should make sense also p-adically. The p-adicization of the WCW integrals has been discussed already earlier using an approach based on harmonic analysis in symmetric spaces and these two approaches should be equivalent. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $n + p - 2$ powers of $e^{1/n}$ can be considered.

4. If the Galois group algebras indeed define a representation for WCW spinor fields in finite measurement resolution, also WCW integration would reduce to summations over the Galois groups involved so that integrals would be well-defined in all number fields.

12.1.5 Topics related to TGD-string theory correspondence

Although M-theory has not been successful as a physical theory it has led to a creation of enormously powerful mathematics and there are all reasons to expect that this mathematics applies also in TGD framework.

Floer homology, Gromov-Witten invariants, and TGD

Floer homology defines a generalization of Morse theory allowing to deduce symplectic homology groups by studying Morse theory in loop space of the symplectic manifold. Since the symplectic transformations of the boundary of $\delta M^4_\pm \times CP_2$ define isometry group of WCW, it is very natural to expect that Kähler action defines a generalization of the Floer homology allowing to understand the symplectic aspects of quantum TGD. The hierarchy of Planck constants implied by the one-to-many correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates leads naturally to singular coverings of the imbedding space and the resulting symplectic Morse theory could characterize the homology of these coverings.

One ends up to a more precise definition of vacuum functional: Kähler action reduces Chern-Simons terms (imaginary in Minkowskian regions and real in Euclidian regions) so that it has both phase and real exponent which makes the functional integral well-defined. Both the phase factor and its conjugate must be allowed and the resulting degeneracy of ground state could allow to understand qualitatively the delicacies of CP breaking and its sensitivity to the parameters of the system. The critical points with respect to zero modes correspond to those for Kähler function. The critical points with respect to complex coordinates associated with quantum fluctuating degrees of freedom are not allowed by the positive definiteness of Kähler metric of WCW. One can say that Kähler and Morse functions define the real and imaginary parts of the exponent of vacuum functional.

The generalization of Floer homology inspires several new insights. In particular, space-time surface as hyper-quaternionic surface could define the 4-D counterpart for pseudo-holomorphic 2-surfaces in Floer homology. Holomorphic partonic 2-surfaces could in turn correspond to the extrema of Kähler function with respect to zero modes and holomorphy would be accompanied by super-symmetry.

Gromov-Witten invariants appear in Floer homology and topological string theories and this inspires the attempt to build an overall view about their role in TGD. Generalization of topological string theories of type A and B to TGD framework is proposed. The TGD counterpart of the mirror symmetry would be the equivalence of formulations of TGD in $H = M^4 \times CP_2$ and in $CP_3 \times CP_3$ with space-time surfaces replaced with 6-D sphere bundles.

K-theory, branes, and TGD

K-theory and its generalizations play a fundamental role in super-string models and M-theory since they allow a topological classification of branes. After representing some physical objections against the notion of brane more technical problems of this approach are discussed briefly and it is proposed how TGD allows to overcome these problems. A more precise formulation of the weak form of electric-magnetic duality emerges: the original formulation was not quite correct for space-time regions with Euclidian signature of the induced metric. The question about possible TGD counterparts of R-R and NS-NS fields and S, T, and U dualities is discussed.

12.1.6 p-Adic space-time sheets as correlates for Boolean cognition

p-Adic physics is interpreted as physical correlate for cognition. The so called Stone spaces are in one-one correspondence with Boolean algebras and have typically 2-adic topologies. A generalization to p-adic case with the interpretation of p pinary digits as physically representable Boolean statements of a Boolean algebra with $2^n > p > p^{n-1}$ statements is encouraged by p-adic length scale hypothesis. Stone spaces are synonymous with profinite spaces about which both finite and infinite Galois groups represent basic examples. This provides a strong support for the connection between Boolean cognition and p-adic space-time physics. The Stone space character of Galois groups suggests also a deep connection between number theory and cognition and some arguments providing support for this vision are discussed.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- Infinite primes [L21]

12.2 Some backgbround about homology and cohomology

Before representing layman's summary about the motivations for the motivic cohomology it is good to introduce some basic ideas of algebraic geometry [A142].

12.2.1 Basic ideas of algebraic geometry

In algebraic geometry one considers surfaces defined as common zero locus for some number $m \leq n$ of functions in n-dimensional space and therefore having dimension $n-m$ in the generic case and one wants to find homotopy invariants for these surfaces: the notion of variety is more precise concept in algebraic geometry than surface. The goal is to classify algebraic surfaces represented as zero loci of collections of polynomials.

The properties of the graph of the map $y = P(x)$ in (x,y)-plane serve as an elementary example. Physicists is basically interested on the number of roots x for a given value of y. For polynomials one can solve the roots easily using computer and the resulting numbers are in the generic case algebraic numbers. Galois group is the basic object and permutes the roots with each other. It is analogous to the first homotopy group permuting the points of the covering space of graph having various branches of the many-valued inverse function $x = P^{-1}(y)$ its sheets. Clearly, Galois group has topological meaning but the topology is that of the imbedding or immersion.

There are invariants related to the internal topology of the surface as well as invariants related to the external topology such as Galois group. The generalization of the Galois group for polynomials of single variable to polynomials of several variables looks like an attractive idea. This would require an assignment of sequence of sub-varieties to a given variety. One can assign algebraic extensions also to polynomials and it would seem that these groups must be involved. For instance, the absolute Galois group associated with the algebraic closure of polynomials in algebraically closed field is free group of rank equal to the cardinality of the field (rank is the cardinality of the minimal generating set).

Homotopy [A43], homology [A43], and cohomology [A43] characterize algebraically the shape of the surface as invariant not affected by continuous transformations and by homotopies. The notion of continuity depends on context and in the most general case there is no need to restrict the consideration to rational functions or polynomials or make restrictions on the coefficient field of these functions. For algebraic surfaces one poses restrictions on coefficient field of polynomials and the ordinary real number based topology is replaced with much rougher Zariski topology for which algebraic surfaces define closed sets. Physicists might see homology and cohomology theories as linearizations of nonlinear notions of manifold and surface obtained by gluing together linear manifolds. This linearization allows to gain information about the topology of manifolds in terms of linear spaces assignable to surfaces of various dimensions.

In homology one considers formal sums for these surfaces with coefficients in some field and basically algebraizes the statement that boundary has no boundary. Cohomology is kind of dual of homology and in differential geometry based cohomology forms having values as their integrals over surfaces of various dimensions realize this notion.

Betti cohomology or singular cohomology [A12] defined in terms of simplicial complexes is probably familiar for physicists and even more so the de Rham cohomology [A22] defined by n-forms as also the Dolbeault cohomology [A27] using forms characterized by m holomorphic and n antiholomorphic indices. In this case the role of continuous maps is taken by holomorphic maps. For instance, the classification of the moduli of 2-D Riemann surfaces involves in an essential manner the periods of one forms on 2-surfaces and plays important role in the TGD based explanation of family replication phenomenon [K17].

In category theoretical framework homology theory can be seen as a http://en.wikipedia.org/wiki/functorfuncto [A35] that assigns to a variety (or manifold) a sequence of homology groups characterized by the dimension of corresponding sub-manifolds. One considers formal sums of surfaces. The basic operation is that of taking boundary which has operation δ as algebraic counterpart. One identifies cycles as those sums of surfaces for which algebraic boundary vanishes. This is identically true for exact cycles defined as a boundaries of cycles since boundary of boundary is empty. Only those cycles with are not exact matter and the homology group is defines as the coset space of the kernel at n:th level with respect to the image of the $n+1$:th level two spaces. Cohomology groups can be defined in a formally similar manner and for de Rham cohomology Poincare duality maps homology group H_k to H^{n-k}. The correspondence between covariant with vanishing exterior derivative and contravariant antisymmetric tensors with vanishing divergence is the counterpart of homology-cohomology correspondence in Riemann manifolds.

The calculation of homology and cohomology groups relies on general theorems which are often raised to the status of axioms in generalizations of cohomology theory.

1. Exact sequences [A30] of Abelian groups define an important calculational tool. So called short exact sequence $0 \to B \to C \to 0$ of chain complexes gives rise to long exact sequence $H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$....

   One example of short exact sequence is $0 \to H \to G \to G/H \to 0$ holding true when H is normal subgroup so that also G/H is group. This condition allows to express the homology groups of G as direct sums of those for H and G/H. In relative cohomology inclusion and δ define exact sequences allowing to express relative cohomology groups [A80] $H_n(X, A \subset X)$ in terms of those for X and A. Mayer- Vietoris sequence relates the cohomologies of sets A, B and $X = A \cup B$.

2. Künneth theorem [A54] allows to calculated homology groups for Cartesian product as convolution of those for the factors with respect to direct sum.

Steenrod- Eilenberg axioms [A88] axiomatize cohomology theory in the category of topological spaces: cohomology theory in this category is a functor to graded abelian groups, satisfying the Eilenberg-Steenrod axioms: functoriality, naturality of the boundary homomorphism, long exact sequence, homotopy invariance, and excision. In algebraic cohomology the category is much more restricted: algebraic varieties defined in terms of polynomial equations and these axioms are not enough. In this case Weil cohomology [A103] defines a possible axiomatization consisting of finite generation, vanishing outside the range $[0, dim(X)]$, Poincare duality, Künneth product formula, a cycle class map, and the weak and strong Lefschetz axioms.

In p-adic context sets do not have boundaries since p-adic numbers are not well-ordered so that the statement that boundary has vanishing boundary should be formulated using purely algebraic language. Also cohomology is problematic since definite integral is ill-defined for the same reason. This forces to question either the notion of cohomology and homology groups or the definition of geometric boundary operation and inspires the question whether Galois groups might be a more appropriate notion.

Perhaps it is partially due to the lack of a geometric realization of the boundary operation in the case of general number field that there are very many cohomology theories: the brief summary by Andreas Holmstrom written when he started to work with his thesis, gives some idea about how many!

12.2.2 Algebraization of intersections and unions of varieties

There are several rather abstract notions involved with cohomology theories: categories, functoriality, sheaves, schemes, abelian rings. Abelian ring is essentially the ring of polynomial functions generated by the coordinates in the open subset of the variety.

1. The spectrum of ring consists of its proper prime ideals of this function algebra. Ideal is subset of functions s closed under sum and multiplication by any element of the algebra and proper ideal is subspace of the entire algebra. In the case of the abelian ring defined on algebraic variety maximal ideals correspond to functions vanishing at some point. Prime ideals correspond to functions vanishing in some sub-variety, which does not reduce to a union

of sub-varieties (meaning that one has product of two functions of ring which can separately vanish). Thus the points in spectrum correspond to sub-varieties and product of functions correspond to a union of sub-varieties.

2. What is extremely nice that the product of functions represents in general union of disjoint surfaces: for physicist this brings in mind many boson states created by bosonic creation operators with particles identified as surfaces. Therefore union corresponds to a product of ideals defining a non-prime ideal. The notion of ideal is needed since there is enormous gauge invariance involved in the sense that one can multiply the function defining the surface by any everywhere non-vanishing function.

3. The intersection of varieties in turn corresponds to the condition that the functions defining the varieties vanish separately. If one requires that all sums of the functions belonging to the corresponding ideals vanish one obtains the same condition so that one can say that intersection corresponds to vanishing condition for the sum for ideals. The product of co-homology elements corresponds by Poincare duality [A70] the intersection of corresponding homology elements interpreted as algebraic cycles so that a beautiful geometric interpretation is possible in real context at least.

Remark: For fermionic statistics the functions would be anti-commutative and this would prevent automatically the powers of ideals. In fact, the possibility of multiple roots for polynomials of several variables implying what is known as ramification [A78] represents a non-generic situation and one of the technical problems of algebraic geometry. For ordinary integers ramification means that integer contains in its composition to primes a power of prime which is higher than one. For the extensions of rationals this means that rational prime is product of primes of extension with some roots having multiplicity larger than one. One can of course ask whether higher multiplicity could be interpreted in terms of many-boson state becoming possible at criticality: in quantum physics bosonic excitations (Goldstone bosons) indeed emerge at criticality and give rise to long range interactions. In fact, for infinite primes allowing interpretation in terms of quantum states of arithmetic QFT boson many particle states corresponds to powers of primes so that the analogy is precise.

12.2.3 Motivations for motives

In the following I try to clarify for myself the motivations for the motivic cohomology which as a general theory is still only partially existent. There is of course no attempt to say anything about the horrible technicalities involved. I just try to translate the general ideas as I have understood (or misunderstood) them to the simple language of mathematically simple minded physicist.

Grothendieck has carried out a monumental work in algebraizing cohomology which only mathematician can appreciate enough. The outcome is a powerful vision and mathematical tools allowing to develop among other things the algebraic variant of de Rham cohomology, etale cohomology having values in p-adic fields different from the p-adic field defining the values of cohomology, and crystalline cohomology [A21].

As the grand unifier of mathematics Grothendieck posed the question whether there good exists a more general theory allowing to deduce various cohomologies from single grand cohomology. These cohomology theories would be like variations of the same them having some fundamental core element -motive- in common.

Category theory [A16] and the notion of scheme [A85], which assigns to open sets of manifold abelian rings - roughly algebras of polynomial functions- consistent with the algebra of open sets, provide the backbone for this approach. To the mind of physicist the notion of scheme brings abelian gauge theory with non-trivial bundle structure requiring several patches and gauge transformations between them. A basic challenge is to relate to each other the cohomologies associated with algebraic varieties with given number field k manifolds. Category theory is the basic starting point: cohomology theory assigns to each category of varieties category of corresponding cohomologies and functors between these categories allow to map the cohomologies to each other and compare different cohomology theories.

One of the basic ideas underlying the motivic cohomology seems is that one should be able perform a local lifting of a scheme from characteristic p (algebraic variety in p-adic number field or its algebraic extension) to that in characteristic 0 (characteristic is the integer n for which the sum of n units is zero, for rational numbers, p-adic number fields and their extensions characteristic is zero and p for finite fields) that is real or complex algebraic variety, to calculate various cohomologies here as algebraic de Rham cohomology and using the lifting to induce the cohomology to p-adic context. One expects that the ring in which cohomology has naturally values consists of ordinary or p-adic integers or extension of p-adic integers. In the case of crystalline cohomology this is however not enough.

The lifting of the scheme is far from trivial since number fields are different and real cohomology has naturally Z or Q as coefficient ring whereas p-adic cohomology has p-adic integers as coefficient ring. This lift must bring in analytic continuation which is lacking at p-adic side since n particular in p-adic topology two spheres with same radius are either non-intersecting or identical. Analytical continuation using a net of overlapping open sets is not possible.

One could even dream of relating the cohomologies associated with different number fields. I do not know to what extend this challenge is taken or whether it is regarded as sensible at all. In TGD framework this kind of map is needed and leads ot the generalization of the number field obtained by glueing together reals and p-adic numbers among rationals and common algebraic numbers. This glueing together makes sense also for the space of surfaces by identifying the surfaces which correspond to zero loci of rational functions with rational coefficients. Similar glueing makes sense for the spaces of polynomials and rational functions.

Remarks::

1. The possibility of p-adic pseudo-constants in the solutions of p-adic differential and p-adic differential equations reflects this difficulty. This lifting should remove this non-uniqueness in analytical continuation. One can of course ask whether the idea is good: maybe the p-adic pseudo constants have some deep meaning. A possible interpretation would be in terms of non-deterministic character of cognition for which p-adic space-time sheets would be correlates. The p-adic space-time sheets would represent intentions which can be transformed to actions in quantum jumps. If one works in the intersection of real and p-adic worlds in which one allows only rational functions with coefficients in the field or rationals or possibly in some algebraic extension of rationals situation changes and non-uniqueness disappears in the intersection of real and p-adic worlds and one might argue that it is here where the universal cohomology applies or that real and p-adic cohomologies are obtained by some kind of algebraic continuation from this cohomology.

2. The universal cohomology theory brings in mind the challenge encountered in the construction of quantum TGD. The goal is to fuse real physics and various p-adic physics to single coherent whole so that one would have kind of algebraic universality. To achieve this I have been forced to introduce a heuristic generalization of number field by fusing together reals and various p-adic number fields among rationals and common algebraic numbers. The notion of infinite primes is second key notion. The hierarchy of Planck constants involving extensions of p-adic numbers by roots of unity is closely related to p-adic length scale hierarchy and seems to be an essential part of the number theoretical vision.

12.3 Examples of cohomologies

In the following some examples of cohomologies are briefly discussed in hope of giving some idea about the problems involved. Probably the discussion reflects the gaps in my understanding rather than my understanding.

12.3.1 Etale cohomology and l-adic cohomology

Etale cohomology [A29] is defined for algebraic varieties as analogues of ordinary cohomology groups of topological space. They are defined purely algebraically and make sense also for finite fields. The notion of definite integral fails in p-adic context so that also the notion of form makes sense only locally but not as a map assigning numbers to surfaces. This is cohomological counterpart for the non-existence of boundaries in p-adic realm. Etale cohomology allows to define cohomology groups also in p-adic context as l-adic cohomology groups.

In Zariski topology closed sets correspond to surfaces defined as zero loci for polynomials in given field. The number of functions is restricted only by the dimension of the space. In the real case this topology is much rougher than real topology. In etale cohomology Zariski topology is too rough. One needs more open sets but one does not want to give up Zariski topology.

The category of etale maps is the structure needed and actually generalizes the notion of topology. Instead of open sets one considers maps to the space and effectively replaces the open sets with their inverse images in another space. Etale maps -idempotent are essentially projections from coverings of the variety to variety. One can say that open sets are replaced with open sets for the covering of the space and mapping is replaced with a correspondence (for algebraic surfaces X and Y the correspondence is given by algebraic equations in $X \times Y$) which in general is multi-valued and this leads to the notion of etale topology. The etale condition is formulated in the Wikipedia article in a rather tricky manner telling not much to a physicist trying to assign some meaning to this word. Etale requirement is the condition that would allow one to apply the implicit function theorem if it were true in algebraic geometry: it is not true since the inverse of rational map is not in general rational map except in the ase of birational maps to which one assigns birational geometry [A13].

Remarks:

1. In TGD framework field as a map from M^4 to some target space is replaced with a surface in space $M^4 \times CP_2$ and the roles of fields and space are permuted for the regions of space-time representing lines of generalized Feynman diagrams. Therefore the relation between M^4 and CP_2 coordinates is given by correspondence. Many-sheeted space-time is locally a many-sheeted covering of Minkowski space.

2. Also the hierarchy of Planck constant involving hierarchy of coverings defined by same values of canonical momentum densities but different values of time derivatives of imbedding space coordinates. The enormous vacuum degeneracy of Kähler action is responsible for this many-valuedness.

3. Implicit function theorem indeed gives several values for time derivatives of imbedding space coordinates as roots to the conditions fixing the values of canonical momentum densities.

The second heuristic idea is that certain basic cases corresponding to dimensions 0 and 1 and abelian varieties which are also algebraic groups obeying group law defined by regular (analytic and single valued) functions are special and same results should follow in these cases.

Etale cohomologies satisfy Poincare duality and Künneth formula stating that homology groups for Cartesian product are convolutions of homology groups with respect to tensor product. l-adic cohomology groups have values in the ring of l-adic integers and are acted on by the absolute Galois group of rational numbers for which no direct description is known.

12.3.2 Crystalline cohomology

Crystalline cohomology represents such level of technicality that it is very difficult for physicists without the needed background to understand what is in question. I however make a brave attempt by comparing with analogous problems encountered in the realization of number theoretic universality in TGD framework. The problem is however something like follows.

1. For an algebraically closed field with characteristic p it is not possible to have a cohomology in the ring Z_p of p-adic integers. This relates to the fact that the equation for $x^n = x$ in finite field has only complex roots of unity as its solutions when n is not divisible by p whereas for he integers n divisible by p are exceptional due to the fact that $x^p = x$ holds true for all elements of finite field $G(p)$. This implies that $x^p = x$ has p solutions which are ordinary p-adic numbers rather than numbers in an algebraic extension by a root of unity. p-Adic numbers indeed contain n:th cyclotomic field only if n divides $p-1$. On the other hand, any finite field has order $q = p^n$ and can be obtained as an algebraic extension of finite field $G(p)$ with p elements. Its elements satisfy the Frobenius condition $x^{q=p^n} = x$. This condition cannot be satisfied if the extension contains p:th root of unity satisgying $u^p = 1$ since one would have $(xu)^{p^n} = x \neq xu$. Therefore finite fields do not allow an algebraic

extensions allowing p :th root of unity so the extension of p-adic numbers containing p:th root of unity cannot be not induced by the extension of $G(p)$. As a consequence one cannot lift cohomology in finite field $G(p^n)$ to p-adic cohomology.

2. Also in TGD inspired vision about integration $p-1$:th and possibly also p:th roots are problematic. p-Adic cohomology is about integration of forms and the reason why integration necessitates various roots of unity can be understood as follows in TGD framework. The idea is to reduce integration to Fourier analysis which makes sense even for the p-adic variant of the space in the case that it is symmetric space. The only reasonable definition of Fourier analysis is in terms of discrete plane waves which come as powers of n:th root of unity. This notion makes sense if n is not divisible by p. This leads to a construction of p-adic variants of symmetric spaces G/H obtained by discretizing the groups to some algebraic subgroup and replacing the discretized points by p-adic continuum. Certainly the n:th roots of unity with n dividing $p-1$ are problematic since they do not corresponds to phase factors. It seems however clear that one can construct an extension of p-adic numbers containing p:th roots of unity. If it is however necesssary to assume that the extension of p-adic numbers is induced by that for a finite field, situation changes. Only roots of unity for n not divisible by factors of $p-1$ and possibly also by p can appear in the discretizations. There is infinite number extensions and the interpretation is in terms of a varying finite measurement resolution.

3. In TGD framework one ends up with roots of unity also when one wants to realize p-adic variants of various finite group representations. The simplest case is p-adic representations of angular momentum eigenstates and plane waves. In the construction of p-adic variants of symmetric spaces one is also forced to introduce roots of unity. One obtains a hierarchy of extensions involving increasing number of roots of unity and the interpretation is in terms of number theoretic evolution of cognition involving both the increase of maximal value of n and the largest prime involved. Witt ring could be seen as an idealization in which all roots of unity possible are present.

For $l=p$ l-adic cohomology fails for characteristic p. Crystalline cohomology fills in this gap. Roughly speaking crystalline cohomology is de Rham cohomology of a smooth lift of X over a field k with with characteristic p to a variety so called ring of Witt vectors with characteristic 0 consisting of infinite sequences of the elements of k while de Rham cohomology of X is the crystalline cohomology reduced modulo p.

The ring of Witt vectors for characteristic p is particular example of ring of Witt vectors [A106] assignable to any ring as infinite sequences of elements of ring. For finite field G_p the Witt vectors define the ring of p-adic integers. For extensions of finite field one has extensions of p-adic numbres. The algebraically closed extension of finite field contains n:th roots of unity for all n not divisible by p so that one has algebraic closure of finite field with p elements. For maximal extension of the finite field G_p the Witt ring is thus a completion of the maximal unramified extension of p-adic integers and contains n:th roots of unity for n not divisible by p. "Unramified" [A78] means that p defining prime for p-adic integers splits in extension to primes in such a manner that each prime of extension occurs only once: the analogy is a polynomial whose roots have multiplicity one. This ring is much larger than the ring of p-adic integers. The algebraic variety is lifted to a variety in Witt ring with characteristic 0 and one calculates de Rham cohomology using Witt ring as a coefficient field.

12.3.3 Motivic cohomology

Motivic cohomology is a attempt to unify various cohomologies as variations of the same motive common to all of them. In motivic cohomology [A63] one encounters pure motives and mixed motives. Pure motives is a category associated with algebraic varieties in a given number field k with a contravariant functor from varieties to the category assigning to the variety its cohomology groups. Only smooth projective varieties are considered. For mixed motives more general varieties are allowed. For instance, the condition that projective variety meaning that one considers only homogenous polynomials is given up.

Chow motives [A64] is an example of this kind of cohomology theory and relies on very geometric notion of Chow ring with equivalence of algebraic varieties understood as rational equivalence. One can replace rational equivalence with many variants: birational, algebraic, homological, numerical, etc...

The vision about rationals as common points of reals and p-adic number fields leads to ask whether the intersection of these cohomologies corresponds to the cohomology associated with varieties defined by rational functions with rational coefficients. In both p-adic and real cases the number of varieties is larger but the equivalences are stronger than in the intersection. For a non-professional it is impossible to say whether the idea about rational cohomology in the intersection of these cohomologies makes sense.

Homology and cohomology theories rely in an essential manner to the idea of regarding varieties with same shape equivalent. This inspires the idea that the polynomials or rational functions with rational coefficients could correspond to something analogous to a gauge choice without losing relevant information or bringing in information which is irrelevant. If this gauge choice is correct then real and p-adic cohomologies and homologies would be equivalent apart from modifications coming from the different topology for the real and p-adic integers.

12.4 Infinite rationals define rational functions of several variables: a possible number theoretic generalization for the notions of homotopy, homology, and cohomology

This section represents my modest proposal for how the generalization of number theory based on infinite integers might contribute to the construction of topological and number theoretic invariants of varieties. I can represent only the primitive formulation using the language of second year math student. The construction is motivated by the notion of infinite prime but applies to ordinary polynomials in which case however the motivation is not so obvious. The visions about TGD as almost topological QFT, about TGD as generalized number theory, and about TGD as infinite-dimensional geometry serve as the main guidelines and allow to resolve the problems that plagued the first version of the theory.

12.4.1 Infinite rationals and rational functions of several variables

Infinite rationals correspond in natural manner to rational functions of several variables.

1. If the number of variables is 1 one has infinite primes at the first level of the hierarchy as formal rational functions of variable X having as its value as product of all finite primes and one can decompose the polynomial to prime polynomial factors. This amounts to solving the roots of the polynomial by obtained by replacing X with formal variable x which is real variable for ordinary rationals. For Gaussian rationals one can use complex variable.

2. If the roots are not rationals one has infinite prime. Physically this state is the analog of bound state whereas first order polynomials correspond to free many-particle states of supersymmetric arithmetic QFT.

3. Galois group permuting the roots has geometric interpretation as the analog of the group of deck transformations permuting the roots of the covering of the graph of the polynomial y=f(x) at origin. Galois group is analogous to fundamental group whose abelianization obtained as a coset group by dividing with the commutator group gives first homology group. The finiteness of the Galois group does not conform with the view about cohomology and homology, which suggests that it is the group algebra of Galois group which is the correct mathematical structure to consider.

One can find the roots also at the higher levels of the hierarchy of infinite primes. One proceeds by finding the roots at the highest level as roots which are algebraic functions. In other words finds the decomposition

$$P(x_n,...) = \prod_k (x_n - R_k(x_{n-1},...)$$

with R_k expanded in powers series with respect to x_{n-1}. This expansion is the only manner to make sense about the root if x_{n-1} corresponds to infinite prime. At the next step one puts $x_n = 0$ and obtains a product of R_k and performs the same procedure for x_{n-1} and continues down to $n = 1$ giving ordinary algebraic numbers as roots. One therefore obtains a sequence of sub-varieties by restricting the polynomial to various planes $x_i = 0$, $i = k, ..., n$ of dimension $k - 1$. The invariants associated with the intersections with these planes define the Galois groups characterizing the polynomial and therefore also infinite prime itself.

1. The process takes place in a sequential manner. One interprets first the infinite primes at level n+1 as as polynomial function in the variable X_{n+1} with coefficients depending on X_k, $k < n + 1$. One expands the roots R in power series in the variable X_n. In p-adic topology this series converges for all primes of the previous levels and the deviation from the value at $X_n = 0$ is infinitesimal in infinite-P p-adic topology.

2. What is new as compared to the ordinary situation is that the necessity of Taylor expansion, which might not even make sense for ordinary polynomials. One can find the roots and one can assign a Galois group to them.

3. One obtains a hierarchy of Galois groups permuting the roots and at the lowest level on obtains roots as ordinary algebraic numbers and can assign ordinary Galois group to them. The Galois group assigned to the collection of roots is direct sum of the Galois groups associated with the individual roots. The roots can be regarded as a power series in the variables X and the deviation from algebraic number is infinitesimal in infinite-p p-adic topology.

4. The interesting possibility is that the infinitesimal deformations of algebraic numbers could be interpreted as a generalization of real numbers. In the construction of motivic cohomology the idea is to lift varieties defined for surfaces in field of characteristic p (finite fields and their extensions) to surfaces in characteristic 0 field (p-adic numbers) in some sense to infinitesimal thickenings of their characteristic 0 counterparts. Something analogous is encountered in the proposed scenario since the roots of the polynomials are algebraic numbers plus multi-p p-adic expansion in terms of infinite-p p-adic numbers representing infinitesimal in infinite-p p-adic topology.

12.4.2 Galois groups as non-commutative analogs of homotopy groups

What one obtains is a hierarchy of Galois groups and varieties of $n + 1$-dimensional space with dimensions $n, n - 1, ..., 1, 0$.

1. A suggestive geometric interpretation would be as an analog of first homotopy group permuting the roots which are now surfaces of given dimension k on one hand and as a higher homotopy group π_k on the on the other hand. This and the analogy with ordinary homology groups suggests the replacement of Galois group with their group algebras. Homology groups would be obtained by abelianization of the analogs of homotopy groups with the square of the boundary homomorphism mapping the group element to commutator sub-group. Group algebra allows also definition of cohomotopy and cohomology groups by assigning them to the dual of the group algebra.

2. The boundary operation is very probably not unique and the natural proposal inspired by physical intuition is that the boundary operations form an anti-commutative algebra having interpretation in terms of fermionic creation (say) operators. Cohomology would in turn correspond to annilation operators. Poincare duality would be hermitian conjugation mapping fermionic creation operators to annihilation operators and vice versa. Number theoretic vision combined with the braid representation of the infinite primes in turn suggests that the construction actually reduces the construction of quantum TGD to the construction of these homology and cohomology theories.

3. The Galois analogs of homotopy groups and their duals up to the dimension of the algebraic surface would be obtained but not the higher ones. Note that for ordinary homotopy groups

all homotopy group π_n, $n > 1$ are Abelian so that the analogy is not complete. The abelianizations of these Galois groups could in turn give rise to higher homology groups. Since the rational functions involved make sense in all number fields this could provide a possible solution to the challenge of constructing universal cohomology theory.

The hierarchy of infinite primes and the hierarchy of Galois groups associated with the corresponding polynomials have as an obvious analogy the hierarchy of loop groups and corresponding homotopy groups.

1. The construction brings in mind the reduction of n-dimensional homotopy to a 1-D homotopy of n-1-D homotopy. Intuitively n-dimensional homotopy indeed looks like a 1-D homotopy of n-1-D homotopy so that everything should reduce to iterated 1-dimensional homotopies by replacing the original space with the space of maps to it.

2. The hierarchical ordering of the variables plays an essential role. The ordering brings strongly in mind loop groups. Loop group $L(X^m, G)$ defined by the maps from space X^n to group G can be also regarded as a loop group from space X^m to the loop group $L(X^{n-m}, G)$ and one obtains $L(X^n, G) = L(X^1, L(X^{n-1})$.

   The homotopy equivalence classes of these maps define homotopy groups using the spaces X^n instead of spheres. Infinite primes at level n would correspond to $L(X^n, G)$. Locally the fundamental loop group is defined by $X = S^1$ which would suggest that homotopy theory using tori might be more natural then the one using spheres. Naively one might hope that this kind of groups could code for all homotopic information about space. As a matter fact, even more general identity $L(X \times Y, G) = L(X, L(Y, G))$ seems to hold true.

3. Note that one can consider also many variants of homotopy theories since one can replace the image of the sphere in manifold with the image of any manyfold and construct corresponding homotopy theory. Sphere and tori define only the simplest homotopy theories.

12.4.3 Generalization of the boundary operation

The algebraic realization of boundary operation should have a geometric counterpart at least in real case and it would be even better if this were the case also p-adically and even for finite fields.

1. The geometric analog of the boundary operation would replace the k-dimensional variety with its intersection with $x_k = 0$ hyperplane producing a union of $k-1$-dimensional varieties. This operation would make sense in all number fields. The components in the union of the surface would be very much analogous to the lower-dimensional edges of k-simplex so that boundary operation might make sense. What comes in mind is relative homology $H(X, A)$ in which the intersection of X with $A \subset X$ is equivalent with boundary so that its boundary vanishes. Maybe one should interpret the homology groups as being associated with the sequence of relative homologies defined by the sequence of varieties involved as $A_0 \subset A_1 \subset ..$ and relativizing for each pair in the sequence. The ordinary geometric boundary operation is ill-defined in p-adic context but its analog defined in this manner would be number theoretically universal notion making sense also for finite fields.

2. The geometric idea about boundary of boundary as empty set should be realized somehow- at least in the real context. If the boundary operation is consistent with the ordinary homology, it should give rise to a surface which as an element of H_{n-2} is homologically trivial. In relative homology interpretation this is indeed the case. In real context the condition is satisfied if the intersection of the n-dimensional surface with the $x_{n-1} = 0$ hyper-plane consists of closed surface so that the boundary indeed vanishes. This is indeed the case as simplest visualizations in 3-D case demonstrate. Therefore the key geometric idea would be that that the intersection of the surface defined by zeros of polynomial with lower dimensional plane is a closed surface in real context and that this generalizes to p-adic context as algebraic statement at the level of homology.

3. The sequence of slicings could be defined by any permutation of coordinates. The question is whether the permutations lead to identical homologies and cohomologies. The physical

interpretation does not encourage this expection so that different permutation would all be needed to characterize the variety using the proposed homology groups.

12.4.4 Could Galois groups lead to number theoretical generalizations of homology and cohomology groups?

My own humble proposal for a number theoretic approach to algebraic topology is motivated by the above questions. The notion of infinite primes leads to a proposal of how one might assign to a variety a sequence of Galois group [A38] algebras defining analogs of homotopy groups assignable to the algebraic extensions of polynomials of many variables obtained by putting the variables of a polynomial of n-variable polynomial one by one to zero and finding the Galois groups of the resulting lower dimensional varieties as Galois groups of corresponding extensions of polynomial fields. The construction of the roots is discussed in detail [K44], where infinite primes are compared with non-standard numbers. The earlier idea about the possibility to lift Galois groups to braid groups is also essential and implies a connection with several key notions of quantum TGD.

1. One can assign to infinite primes at the n:th level of hierarchy (n is the number of second quantizations) polynomials of n variables with variables ordered according to the level of the hierarchy by replacing the products $X_k = \pi_i P_i$ of all primes at k:th level with formal variables x_n to obtain polynomial in x_n with coefficients which are rational functions of x_k, $k < n$. Note that X_k is finite in p-adic topologies and infinitesimal in their infinite-P variants.

2. One can construct the root decomposition of infinite prime at n:th level as the decomposition of the corresponding polynomial to a product of roots which are algebraic functions in the extensions of polynomials. One starts from highest level and derives the decomposition by expanding the roots as powers series with respect to x_n. The process can be done without ever mentioning infinite primes. After this one puts $x_n = 0$ to obtain a product of roots at $x_n = 0$ expressible as rational functions of remaining variables. One performs the decomposition with respect to x_{n-1} for all the roots and continues down to $n = 1$ to obtain ordinary algebraic numbers.

3. One obtains a collection of varieties in n-dimensional space. At the highest level one obtains $n-1$-D variety referred to as divisor in the standard terminology, $n-2$-D variety in $x_n = 0$ hyperplane, $n-3$-D surface in $(x_n, x_{n-1}) = (0, 0)$ plane and so on. To each root at given level one can assign polynomial Galois group permuting the polynomial roots at various levels of the hierarchy of infinite primes in correspondence with the branches of surfaces of a many-valued map. At the lowest level one obtains ordinary Galois group relating the roots of an ordinary polynomial. The outcome is a collection of sequences of Galois groups $\{(G_n, G_{n,i}, G_{n,i,j}...)\}$ corresponding to all sequences of roots from $k = n$ to $k = 1$.

   One can also say that at given level one has just one Galois group which is Cartesian product of the Galois groups associated with the roots. Similar situation is encountered when one has a product of irreducible polynomials so that one has two independent sets of roots.

The next question is how to induce the boundary operation. The boundary operation for the analogs of homology groups should be be induced in some sense by the projection map putting one of the coordinates x_k to zero. This suggests a geometric interpretation in terms of a hierarchy of relative homologies $H_k(S_k, S_{k-1})$ defined by the hierarchy of surfaces S_k. Boundary map would map S_k to is intersection at $(x_n = 0, ..., x_k = 0)$ plane. This map makes sense also p-adically. The square of boundary operation would produce an intersection of this surface in $x_{k-1} = 0$ plane and this should correspond to boundary sense for Galois groups.

Algebraic representation of boundary operations in terms of group homomorphisms

The challenge is to find algebraic realizations for the boundary operation or operations in terms of group homomorphisms $G_k \rightarrow G_{k-1}$. One can end up with the final proposal through heuristic ideas and counter arguments and relying on the idea that algebraic geometry should have interpretation in terms of quantum physics as it is described by TGD as almost topological QFT.

1. n-dimensional Galois group is somewhat like a fundamental group acting in the space of n-1-dimensional homotopies so that Grothendieck's intuition that 1-D homotopies are somehow fundamental is realized. The abelianizations of these Galois groups would define excellent candidates for homology groups and Poincare duality would give cohomology groups. The homotopy aspects becomes clearer if one interprets Galois group for n:th order polynomials as subgroup of permutation group and lifts the Galois group to a subgroup of corresponding braid group. Galois groups are also stable againt small changes of the coefficients of the polynomial so that topological invariance is guaranteed.

2. Non-abelian boundary operations $G_k \rightarrow G_{k-1}$ must reduce to their abelian counterparts in abelianization so that they their squares defining homomorphisms from level k to $k-2$ must be maps of G_k to the commutator subgroup $[G_{k-2}, G_{k-2}]$.

3. There is however a grave objection. Finite abelianized Galois groups contain only elements with finite order so that in this sense the analogy with ordinary homotopy and homology groups fails. On the other hand, if Galois group is replaced with its group algebra and group algebra is defined by (say) integer valued maps, one obtains something very much analogous to homotopy and homology groups. Also group algebras in other rings or fields can be considered. This replacement would provide the basis of the homotopy and homology groups with an additional multiplicative structure induced by group operation allowing the interpretation as representations of Galois group acting as symmetry groups. The tentative physical interpretation would in terms of quantum states defined by wave functions in groups. Coboundary operation in the dual of group algebra would be induced by the action of boundary operation in group algebra. Homotopy and homology would be associated with the group algebra and and cohomotopy and cohomology with its dual.

4. A further grave objection against the analog of homology theory is there is no reason to expect that the boundary homomorphism is unique. For instance, one can always have a trivial solution mapping G_k to unit element of G_{k-1}. Isomorphism theorem [A52] implies that the image of the group G_k in G_{k-1} under homomorphism h_k is $G_k/ker(h_k)$, where $ker(h_k)$ is a normal subgroup of G_k as is easy to see. One must have $h_{k-1}(G_k/ker(h_k)) \subset [G_{k-2}, G_{k-2}]$, which is also a normal subgroup.

   The only reasonable option is to accept all boundary homomorphisms. This collection of boundary homomorphisms would satisfy anti-commutation relations inducing similar anti-commutation relations in cohomology. Putting all together, one would would obtain the analog of fermionic oscillator algebra. In particular, Poincare duality would correspond to the mapping exchanging fermionic creation and annihilation operators. It however turns out that tis interpretation fails. Rather, braided Galois homology could represent the states of WCW spinor fields in "orbital" degrees of freedom of WCW in finite measurement resolution. A better analogy for braided Galois cohomology is provided by Dolbeault cohomology which also allows complex conjugation.

If this picture makes sense, one would clearly have what category theorist would have suggested from the beginning. TGD as almost topological QFT indeed suggests strongly the interpretation of quantum states in terms of homology and cohomology theories.

Lift of Galois groups to braid groups and induction of braidings by symplectic flows

One can build a tighter connection with quantum TGD by developing the idea about the analogy between homotopy groups and Galois groups.

1. The only homotopy groups [A43], which are non-commutative are first homotopy groups π_1 and plane with punctures provides the minimal realization for them. The lift of permutation groups to `http://en.wikipedia.org/wiki/Braid_group`braid groups [A14] by giving up the condition that the squares of generating permutations satisfy $s_i^2 = 1$ defines a projective representation for them and should apply also now. There is also analogy with Wilson loops. This leads to topological QFTs for knots and braids [A201, A217].

2. In TGD framework light-like 3-surfaces (and also space-like at the ends of causal diamonds) carry braids beginning at partonic 2-surfaces and ending at partonic 2-surfaces at the boundaries of causal diamonds. This realization is highly suggestive now. This also conforms with the general TGD inspired vision about absolute Galois group of rationals as permutation group S_∞ lifted to braiding groups such that its representation always reduce to finite-dimensional ones [K35]. This also conforms with the view about the role of hyper-finite factors of type II_1 and the idea about finite measurement resolution and one would obtain a new connection between various mathematical structure of TGD.

3. The physical interpretation of infinite primes represented by polynomials as bound states suggests that infinite prime at level n corresponds to a braid of braids of ... braids such that at given level of hierarchy braid group acting on the physical states is associated with covering group realized as subgroup of the permutation group for the objects whose number is the number of roots. This gives also a connection with the the notion of operad [A67, A182, A130] which involves also a hierarchy of discrete structures with the action of permutation group inside each and appears also in quantum TGD as a natural notion [K14, K18].

4. The assumption that the braidings are induced by flows of the partonic 2-surface could glue the actions of different Galois groups to single coherent whole was originally motivated by the hope that boundary homomorphism could be made unique in this manner. This restriction is however un-necessary and the physical picture does not support it. The basic motivation for the braid representation indeed comes from TGD as an almost topological QFT vision.

5. The role of symplectic transformations in TGD suggests the identification of flows as symplectic flows induced by those of $\delta M^2 \times CP_2$. These flows should map the area enclosed by the sub-braid (of braids) to itself and corresponding Hamiltonian should be constant at the boundary of the area and induce a flow horizontal to the boundary and also continuous at the boundary. The flow would in general be non-trivial inside the area and induce the braiding of the sub-braid of braids. One could assign "Galois spin" to the sub-braids with respect to the higher Galois group and boundary homomorphism would realize unitary action of G_k as spin rotation at k_1:th level. At k_2:th level the "Galois spin" rotation would reduce to that in commutator subgroup and in homology theory would become trivial. The interpretation of the commutator group as the analog of gauge group might make sense. This would conform with an old idea of quantum TGD that the commutator subgroup of symplectic group acts as gauge transformations.

6. It is not necessary to assign the braids at various level of the hierarchy to the same partonic 2-surface. Since the symplectic transformations act on $\delta M^4_\pm \times CP_2$, one can consider also the projections of the braids to the homologically non-trivial 2-sphere of CP_2 or to the 2-sphere at light-cone boundary: both of these spheres play important part in the formulation of quantum TGD and I have indeed assigned the braidings to these surfaces [K34].

7. The representation of the hierarchy of Galois groups acting on the braid of braids of... can be understood in terms of the replacement of symplectic group of $\delta M^4_\pm \times CP_2$ -call it G- permuting the points of the braids with its discrete subgroup obtained as a factor group G/H, where H is a normal subgroup of G leaving the endpoints of braids fixed. One must also consider subgroups of the permutation group for the points of the triangulation since Galois group for n:th order polynomial is in general subgroup of S_n One can also consider flows with these properties to get braided variant of G/H.

The braid group representation works also for ordinary polynomials with continuous coefficients in all number fields as also finite fields. One therefore achieves number theoretical universality. The values of the variables x_i appearing in the polynomials can belong to any numer field and the representation spaces of the Galois groups correspond to any number field. Since the Galois groups are stable against small perturbations of coefficients one obtains topological invariance in both real and p-adic sense. Also the representation in all number fields are possible for the Galois groups.

The construction is universal but infinite primes provide the motivation for it and can be regarded as a representation of the generalized cohomology group for surfaces which belong to the intersection of real and p-adic worlds (rational coefficients). In particular, the expansion of the roots in powers series is the only manner to make sense about the roots when x_n is identified with X_n so that convergence takes place if some of the lower level infinite primes appearing in the product defining X_n is interpreted as infinite p-adic prime. All higher powers are infinitesimal in infinite-P p-adic norm. At the lowest level one obtains expansion in X_1 for which X_1^n has norm p^{-n} with respect to any prime p. The value of the product of primes different from p is however not well-defined for given p-adic topology. If it makes sense to speak about multi-p p-adic expansion all powers X_1^n, $n > 0$ would be infinitesimal.

What can one say about the lifting to braid groups?

The generators of symmetry group are given by permutations s_i permuting i:th and $i+1$:th element of n-element set. The permutations s_i and s_j obviously commute for $|i-j| > 2$. It is also easy to see that the identity $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ holds true. Besides this the identity $s_i^2 = 1$ holds true.

Braid group B_n [A14] is obtained by dropping the condition $s_i^2 = 1$ and can be regarded as an infinite covering group of the permutation group. For instance, for the simplest non-trivial case $n = 3$ the braid group is universal central extension of the modular group $PSL(2,Z)$. In the general case the braid group is isomorphic to the mapping class group of a punctured disk with n punctures and the realization of the braidings as a symplectic transformations would mean additional restriction to the allowed isotopies inducing the braid group action.

One can decompose any element of braid group B_n to a product of element of symmetric group S_n and of pure braid group P_n consisting of braidings which correspond to trivial permutations. P_n is a normal subgroup of braid group and the following short exact sequence $1 \rightarrow F_{n-1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow 1$ allows to decompose P_n to a product of image of free group F_{n-1} and of the image of P_n in P_{n-1}. This leads to a decomposition to a representation of P_n as an iterated semidirect product of free groups.

Concerning the lifting of Galois groups to subgroups of braid groups following observations are relevant.

1. For n:th order polynomial of single variable Galois group can be regarded as a subgroup of permutation group S_n. The identification is probably not completely unique (at least inner automorphisms make the identification non-unique) but I am unable to say whether this has significance in the recent context.

2. The natural lifting of Galois group to its braided version is as a product of corresponding subgroup of S_n with with pure braid group of n braids so that pure braidings would allow also braidings of all permutations as intermediate stages. Pure braid group is normal subgroup trivially. Whether also more restricted braidings are possible is not clear to me. Braid group has a subgroup obtained by coloring braid strands with a finite number of colors and allowing only the braidings which induce permutations of braids of same color. Clearly this group is a good candidate for the minimal group decomposable to a product of subgroups of symmetric subgroups containing braided Galois group. Different colors would correspond to the decomposition of S_n to a product of permutation groups. Note that one can have cyclic subgroups of permutation sub-groups.

One might hope that it is enough to lift the boundary homomorphisms between Galois groups G_k and G_{k-1} to homomorphisms between corresponding braided groups. Life does not look so simple.

1. The group algebra of Galois group is replaced with an infinite-dimensional group algebra of braid groups so that the number of physical states is expected to become much larger and the interpretation could be in terms of many-boson states.

2. The square of the boundary homomorphism must map braided Galois group $B(G_k)$ to $[B(G_{k-2}), B(G_{k-2})]$. The obvious question is whether this conditions reduces to corresponding conditions for Galois group and pure braided groups. In other words, does the braiding commute with the formation of commutator sub-group: $[B(G_k), B(G_k)] = B([G_k, G_k])$? In this case the decomposition of the braided Galois group to a product of Galois group and

pure braid group would allow to realize the braided counterpart of boundary homomorphism as a product of Galois group homomorphism and homomorphism acting on the pure braid group. Direct calculation however shows that this is not the case so that the problem is considerably more complicated.

More detailed view about braided Galois homology

Consider next a more detailed view about the braided Galois homology.

1. One can wonder whether the application of only single boundary operator creates a state which represents gauge degree of freedom or whether boundaries correspond to "full" boundaries obtained by applying maximum number of boundary operations, which k:th level is k. "Full boundary" would correspond to what one obtains by applying at most k boundary operators to the state, and many combinations are possible if the number of boundary homomorphisms is larger than k. The physical states as elements of homology group would be analogous many-fermion states bu would differ from them in the sense that they would be annihilated by all fermionic creation operators. In particular, full Fermi spheres at k:th level would represent gauge degrees of freedom.

   Homologically non-trivial states are expected to be rather rare, especially so if already single boundary operation creates gauge degree of freedom. Certainly the existence of constraints is natural since infinite primes corresponding to irreducible polynomials of degree higher are interpreted as bound states. Homological non-triviality would most naturally express bound state property in bosonic degrees of freedom. In any case, one can argue that fermionic analogy is not complete and that a more natural interpretation is as an analog of cohomology with several exterior derivatives.

2. The analogy with fermionic oscillator algebra makes also the realization of bosonic oscillator operator algebra suggestive. Pointwise multiplication of group algebra elements regarded as functions in group looks the most plausible option since for continuous groups like $U(1)$ this implies additivity of quantum numbers. Many boson states for given mode would correspond to powers of group algebra element with respect to pointwise multiplication. If the commutator for the analogs of the bosonic oscillator operators is defined as

   $$[B_1,B_2] \equiv \sum_{g_1,g_2} B_1(g_1)B_2(g_2)[g_1,g_2] \quad , [g_1,g_2] \equiv g_1g_2g_1^{-1}g_2^{-1} \quad ,$$

   it is automatically in the commutator sub-group. This condition is not consistent with fermionic anti-commutation relations. The consistency requires that the commutator is defined as

   $$[B_1,B_2] \equiv \sum_{g_1,g_2} (B_1(g_1)B_2(g_2)[[g_1,g_2] \quad , [g_1,g_2] \equiv g_1g_2 - g_2g_1 \quad . \tag{12.4.1}$$

   The commutator must belong to the group algebra of the commutator subgroup. In this case the commutativity conditions are non-trivial. Bosonic commutation relations would put further constraints on the homology.

   A delicacy related to commutation and anti-commutation relations should be noticed. One could fermionic creation (annihilation) operators as elements in the dual of group algebra. If group algebra and its dual are not identified (this might not be possible) then the anti-commutator is element of the field of ring in which group algebra elements have values. In the bosonic case the conjugate of the bosonic group algebra element should be treated in the same manner as a pointwise multiplication operator instead of an exterior derivative like operator.

3. One could perhaps interpret the commutation and anti-commutation relations modulo commutator subgroup in terms of finite measurement resolution realized by the transition to homology implying that observables commute in the standard sense. The connection of finite measurement resolution with inclusions of hyper-finite factors of type II_1 implying a connection with quantum groups and non-commutative geometry conforms also with the vision that finite measurement resolution means commutativity modulo commutator group.

4. The alert reader has probably already asked why one could not define also diagonal homology for G_k via diagonal boundary operators $\delta_k : G_k \rightarrow H_k$, where H_k is subgroup of G_k. The above argument would suggest interpretation for this cohomology in terms of finite measurement resolution. If one allows this the Galois cohomology groups would be labelled by two integers. Similar situation is encountered in motivic cohomology [A63].

Some remarks

Some remarks about the proposal are in order.

1. The proposal makes as such sense if the polynomials with rational coefficients define a subset of more general function space able to catch the non-commutative homotopy and homology and their duals terms of Galois groups associated with rational functions with coefficients. One could however abstract the construction so that it applies to polynomials with coefficients in real and p-adic fields and forget infinite primes altogether. One can even consider the replacement of algebraic surfaces with more general surfaces as along as the notion of Galois group makes sense since braiding makes sense also in more general situation. This picture would conform with the idea of number theoretical universality based on algebraic continuation from rationals to various number fields. In this case infinite primes would characterize the rational sector in the intersection of real and p-adic worlds.

2. The above discussion is for the rational primes only. Each algebraic extension of rationals however gives rise to its own primes. In particular, one obtains also complex integers and Gaussian primes. Each algebraic extension gives to its own notion of infinite prime. One can also consider quaternionic and octonionic primes and their generalization to infinite primes and this generalization is indeed one of the key ideas of the number theoretic vision [K70]. Note that already for quaternions Galois group defined by the automorphisms of the arithmetics is continuous Lie group.

3. The decomposition of infinite primes to primes in extension of rational or polynomials is analogous to the decomposition of hadron to quarks in higher resolution and suggests that reduction of the quantum system to its basic building bricks could correspond number theoretically to the introduction of higher algebraic extensions of various kinds of number fields. The emergence of higher extensions would mean emergence of algebraic complexity and have interpretation as evolution of cognition in TGD inspired theory of consciousness.

This picture conforms with the basic visions of quantum TGD about physics as infinite-dimensional geometry on one hand and physics as generalized number theory on one hand implying that algebraic geometry reduces in some sense to number theory and one can also regard quantum states as representations of algebraic geometric invariants in accordance with the vision about TGD as almost topological QFT.

Infinite primes form a discrete set since all the coefficients are rational (unless one allows even algebraic extensions of infinite rationals). Physically infinite primes correspond to elementary particle like states so that elementary particle property corresponds to number theoretic primeness. Infinite integers define unions of sub-varieties identifiable physically as many particle states. Rational functions are in turn interpreted in zero energy ontology as surfaces assignable to initial and final states of physical event such that positive energy states correspond to the numerator and negative energy states to the denominator of the polynomial. One also poses the additional condition that the ratio equals to real unit in real sense so that real units in this sense are able to represent zero energy state and the number theoretic anatomy of single space-time point might be able to represent arbitrary complex quantum states.

The generalization of the notion of real point has been already mentioned as also the fact that the number theoretic anatomy could in principle allow to code for zero energy states if they correspond to infinite rationals reducing to unit in real sense. Also space-time surfaces could by quantum classical correspondence represent in terms of this anatomy as I have proposed. Single space-time point could code in its structure not only the basic algebraic structure of topology as proposed but represent Platonia. If the above arguments really maks sense then this number theoretic Brahman=Atman identify would not be a mere beautiful philosophical vision but would have also practical consequences for mathematics.

12.4.5 What is the physical interpretation of the braided Galois homology

The resulting cohomology suggests either the interpretation in terms of many-fermion states or as a generalization of de Rham cohomology involving several exterior derivative operators. The arguments below show that fermionic interpretation does not make sense and that the more plausible interpretation in concordance with finite measurement resolution is in terms of "orbital" WCW degrees of freedom represented by the symplectic group assignable to the product of light-cone boundary and CP_2.

What the restriction to the plane $x_k = 0$ could correspond physically?

The best manner to gain a more detailed connection between physics and homology is through an attempt to understand what operation putting $x_k = 0$ could mean physically.

1. Given infinite prime at level n corresponds to single particle state characterized by Galois group G_n. The fermionic part of the state corresponds to its small part and purely bosonic part multiplies X_{n-1} factors as powers of primes not dividing the fermionic part of the state. Therefore the finite part of the state contains information about fermions and bosons labelled by fermionic primes. When one puts $x_n = 0$, the information about the bosonic part is lost.

   One can of course divide the polynomial by a suitable infinite integer of previous level so that its highest term is just power of X_n with a unit coefficient. Bosonic part appears in this case in the denominator of the finite part of the infinite prime and does not contribute to zeros of the resulting rational function at $n-1$:th level: it of course affects the zeros at n:th level. Hence the information about bosons at $n-1$:th level is lost also now unless one considers also the Galois groups assignable to the poles of the resulting rational function at $n-1$:th level.

2. What could this loss of information about bosons correspond geometrically and physically? To answer this question must understand how the polynomial of many variables can be represented physically in TGD Universe.

   The proposal has been that a union of hierarchically ordered partonic 2-surfaces gives rise to a local representation of n-fold Cartesian power for a piece of complex plane. A more concrete realization would be in terms of wormhole throats at the end of causal diamond at 3-surfaces topologically condensed at each other. The operation $x_n = 0$ would corresponding to the basic reductionistic step destroying the bound state by removing the largest space-time sheets so that one would have many-particle state rather than elementary particle at the lower level of the hierarchy of space-time sheet. This loss of information would be unavoidable outcome of the reductionistic analysis.

One can consider two alternative geometric interpretations depending on whether one identifies to infinite primes connected 3-surfaces or connected 2-surfaces.

1. If infinite primes correspond to connected 3-surfaces having hierarchical structure of topological condensate the disappearing bosons could correspond to the wormhole throats connecting smaller space-time sheet to the largest space-time sheet involved. Wormhole throats would carry bosonic quantum numbers and would be removed when the largest space-time sheet disappears. Many-fermion state at highest level represented by the "finite" part of the infinite

prime would correspond to "half" wormhole throats- CP_2 type vacuum extremals topological condensed at smaller space-time sheets but not at the highest one.

2. If elementary particles/infinite primes correspond to connected partonic 2-surfaces (this is not quite not the case since tangent space data about partonic 2-surfaces matters), one must replace 3-D topological condensation by its 2-dimensional version. Infinite prime would correspond to single wormhole throat asa partonic 2-surface at which smaller wormhole throats would have suffered topological condensation. Topological condensation would correspond to a formation of a connection by flux tube like structure between the 2-surfaces considered. The disappearance of this highest level would mean decay to a many particle state containing several wormhole contacts. The formation of anyonic many-particle states could be interpreted in terms of build-up of higher level infinite primes.

3. What ever the interpretation is, it should be consistent with the idea that braiding as induced by symplectic flow. If the symplectic flow is defined by the inherent symplectic structure of the partonic 2-surface only the latter option works. If the symplectic flow acts at the level of the imbedding space - as is natural to assume- both interpretations make sense.

The restriction to $x_k = 0$ plane cannot correspond to homological boundary operation

Can one model the restriction to $x_k = 0$ plane as boundary operation in the sense of generalized homology? There are several objections.

1. There are probably several homological boundary operations δ_i at given level whereas the restriction $x_k = 0$ is a unique operation (recall however the possibility to permute the arguments in the case of polynomial).

2. The homology is expected to contain large number of generators whereas the state defined by infinite prime is unique as are also the states resulting via restriction operations.

3. It is not possible to assign fermion number to $x_k = 0$ operation since fermion number is not affected: this would not allow to assign fermion number to the homological boundary operators.

Although the interpretation as many-fermion states does not make sense, one must notice that the structure of homology is highly analogous to the space of states of super-symmetric QFT and of the set of infinite primes. Only the infinite primes $X_n \pm 1$, where X_n is the product of all primes at level n, correspond to states containing no fermions and have interpretation as Dirac sea and vacuum state. In the same manner the elements of braided Galois homology in general are obtained by applying the analogs of fermionic annihilation (creation) operators to a full Fermi sphere (Fock vacuum). Also the identification of *all* physical states as many-fermion states in quantum TGD where all known elementary bosons are identified as fermion pairs conforms with this picture.

A more natural interpretation of the restriction operation is as an operation making possible to assign to a given state in fermionic sector the space of possible states in WCW degrees of freedom characterized in terms of Galois cohomology represented in terms of the symplectic group of acting as isometries of WCW. The transition from Lie algebra description natural for continuum situation to discrete subgroup is natural due to the discretization realizing the finite measurement resolution.

One cannot however avoid a nasty question. What about the lower level bosonic primes associated with the infinite prime? What is their interpretation if they do not correspond to WCW degrees of freedom? Maybe one could identify the bosonic parts of infinite prime as super-partners of fermions behaving like bosons. The addition of a right handed neutrino to a given quantum state could represent this supersymmetry.

Braided Galois group homology and construction of quantum states in WCW degrees of freedom in finite measurement resolution

The above arguments fix the physical interpretation of infinite primes and corresponding group cohomology to quite high degree.

1. From above it is clear that the restriction operation cannot correspond directly to homological boundary operation. Single infinite prime corresponds to an entire spectrum of states. Hence the assignment of fermion number to the boundary operators is not correct thing to do and one must interpret the coboundary operations as analogs of exterior derivatives and various states as bosonic excitations of a given state analogous to states assignable to closed forms of various degrees in topological or conformal quantum field theories.

2. The natural interpretation of Galois homology is as a homology assignable to a discrete sub-group hierarchy of the symplectic group acting as isometries of WCW and therefore as the space of wave functions in WCW degrees of freedom in finite measurement resolution. Infinite primes would code for fermionic degrees of freedom identifiable as spinor degrees of freedom at the level of WCW.

3. The connection between infinite primes and braided Galois homology would basically reflect the supersymmetry relating these degrees of freedom at the level of WCW geometry where WCW Hamiltonians correspond to bosonic generators and contractions of WCW gamma matrices with symplectic currents to the fermionic generators of the super-symmetry algebra. If this identification is correct, it would solve the problem of constructing the modes of WCW spinor fields in finite measurement resolution. An especially well-come feature would be the reduction of WCW integration to summations in braided Galois group algebra allowing an easy realization of number theoretical universality. If the picture is correct it should also have connections to the realization of finite measurement resolution in terms of inclusions of hyper-finite factors of type II_1 [K25] for which fermionic oscillator algebra provides the basic realization.

4. Of course, it is far from clear whether it is really possible to reduce spin, color and electroweak quantum numbers to number theoretic characteristics of infinite primes and it might well be that the proposed construction does not apply to center of mass degrees of freedom of the partonic 2-surface. I have considered these questions for the octonionic generalization of infinite primes and suggested how standard model quantum numbers could be understood in terms of subset of infinite octonionic primes [K70].

12.4.6 Is there a connection with the motivic Galois group?

The proposed generalized of Galois group brings in mind he notion of motivic Galois group, which is one possible generalization for the notion of zero-dimensional Galois group associated with algebraic extensions of number fields to the level of algebraic varieties.

One of the many technical challenges of the motivic cohomology theory is the non-uniqueness of the imbedding of the algebraic extension as a subfield in the algebraic closure of k. The number of these imbeddings is however finite and absolute Galois group associated with the algebraic closure of k acts in the set of the imbeddings. Which of them one should choose?

Quantum physicist would solve this problem by saying that there is no need to choose: one could introduce quantum superpositions of different choices and "Galois spin" regarding the different imbeddings as analogs of different spin components. Absolute Galois group would act on the quantum states regarded as superpositions of different imbeddings by permuting them. In TGD framework this kind of representation could emerge in p-adic context raise Galois group to a role of symmetry group acting on quantum states: indeed absolute Galois group is very natural notion in TGD framework. I have proposed this kind of interpretation for some years ago in a chapter [K35] about Langlands program [A149, A56, A150, A148].

If I have understood correctly, the idea of the motivic Galois theory is to generalize this correspondence so that the varieties in field k are replaced the varieties in the extension of k imbedded to the algebraic closure of k, the number of which is finite. Whether the number of the lifts for varieties is finite seems to depends on the situation.

1. If the imbedding is assumed to be same for all points of the variety the situation seems to reduce to the imbeddings of k to the algebraic completion of rationals and one would have quantum superposition of varieties in the union of finite number of representatives of the algebraic extension to which the absolute Galois group acts.

2. Physicist could however ask whether the invariance under the action of Galois group could be local in some sense. The selection of separable extension could indeed be only pseudo-constant in p-adic case and thus depend on finite number of pinary digits of the k-valued coordinates of the point of the algebraic variety. Local gauge invariance would say that any pseudo constant element of local absolute Galois group acts as a symmetry. This would suggest that one can introduce Galois connection. Since Lie algebra is not defined now one should introduce the connection as parallel translations by Galois group element for paths in the algebraic variety.

One key result is that pure motives using numerical equivalence are equivalent with the category of representations of an algebraic group called motivic Galois group which has Lie algebra and is thus looks like a continuous group.

1. Lie algebra structure for something apparently discrete indeed makes sense for profinite groups (synonymous to Stone spaces). Spaces with p-adic topology are basic examples of this kind of spaces. For instance, 2-adic integers is a Stone space obtained as the set of all bit sequences allowed to contain infinite number of non-vanishing digits. This implies that real discreteness transforms to p-adic continuity and the notion of Lie algebra makes sense. For polynomials this would correspond to polynomials with strictly infinite degree unless one considers the absolute Galois group associated with the algebraic extension of rationals associated with an ordinary polynomial. For infinite primes this would correspond to many-fermion states containing infinite number of fermions kicked out from the Dirac sea and from the point of view of physics would look like an idealization.

2. Motivic Galois group does not obviously correspond to the Galois groups as they are introduced above. Absolute Galois group for the extension of say rationals however emerges if one performs the lift to the algebraic completion and this might be how one ends up with motivic Galois group and also with p-adic physics. One can perhaos say that the Galois groups as introduced above make sense in the intersection of real and p-adic worlds.

3. The choice of algebraic extension might be encountered also in the construction of roots for the polynomials associated with infinite primes and since this choice is not unique it seems that one must use quantum superposition of the different choices and must introduce the action of an appropriate absolute Galois group. This group would be absolute Galois group for algebraic extension of polynomials of n variables at n:th level and ordinary Galois group at the lowest level of hierarchy which should be or less the same as the Galois group introduced above. This could bring in additional spin like degrees of freedom in which the absoltey Galois group acts.

   The fascinating question is whether one could regard not only the degrees of freedom associated with the finite Galois groups but even those associated with the absolute Galois group as physical. Physically the analogs of color quantum numbers whose net values vanish for confined states would be in question. To sum up, it seems that number theory could contain implicitly an incredible rich spectrum of physics.

12.5 Motives and twistor approach applied to TGD

Motivic cohomology has turned out to pop up in the calculations of the twistorial amplitudes using Grassmannian approach [B18, B24]. The amplitudes reduce to multiple residue integrals over smooth projective sub-varieties of projective spaces. Therefore they represent the simplest kind of algebraic geometry for which cohomology theory exists. Also in Grothendieck's vision about motivic cohomology [A187] projective spaces are fundamental as spaces to which more general spaces can be mapped in the construction of the cohomology groups (factorization).

12.5.1 Number theoretic universality, residue integrals, and symplectic symmetry

A key challenge in the realization of the number theoretic universality is the definition of p-adic definite integral. In twistor approach integration reduces to the calculation of multiple residue integrals over closed varieties. These could exist also for p-adic number fields. Even more general integrals identifiable as integrals of forms can be defined in terms of motivic cohomology.

Yangian symmetry [A108], [B27] is the symmetry behind the successes of twistor Grassmannian approach [B19] and has a very natural realization in zero energy ontology [K81]. Also the basic prerequisites for twistorialization are satisfied. Even more, it is possible to have massive states as bound states of massless ones and one can circumvent the IR difficulties of massless gauge theories. Even UV divergences are tamed since virtual particles consist of massless wormhole throats without bound state condition on masses. Space-like momentum exchanges correspond to pairs of throats with opposite sign of energy.

Algebraic universality could be realized if the calculation of the scattering amplitudes reduces to multiple residue integrals just as in twistor Grassmannian approach. This is because also p-adic integrals could be defined as residue integrals. For rational functions with rational coefficients field the outcome would be an algebraic number apart from power of 2π, which in p-adic framework is a nuisance unless it is possible to get rid of it by a proper normalization or unless one can accepts the infinite-dimensional transcendental extension defined by 2π. It could also happen that physical predictions do not contain the power of 2π.

Motivic cohomology defines much more general approach allowing to calculate analogs of integrals of forms over closed varieties for arbitrary number fields. In motivic integration [A162] - to be discussed below - the basic idea is to replace integrals as real numbers with elements of so called scissor group whose elements are geometric objects. In the recent case one could consider the possibility that $(2\pi)^n$ is interpreted as torus $(S^1)^n$ regarded as an element of scissor group which is free group formed by formal sums of varieties modulo certain natural relations meaning.

Motivic cohomology allows to realize integrals of forms over cycles also in p-adic context. Symplectic transformations are transformation leaving areas invariant. Symplectic form and its exterior powers define natural volume measures as elements of cohomology and p-adic variant of integrals over closed and even surfaces with boundary might make sense. In TGD framework symplectic transformations indeed define a fundamental symmetry and quantum fluctuating degrees of freedom reduce to a symplectic group assignable to $\delta M^4 \pm \times CP_2$ in well-defined sense [K16]. One might hope that they could allow to define scissor group with very simple canonical representatives- perhaps even polygons- so that integrals could be defined purely algebraically using elementary area (volume) formulas and allowing continuation to real and p-adic number fields. The basic argument could be that varieties with rational symplectic volumes form a dense set of all varieties involved.

12.5.2 How to define the p-adic variant for the exponent of Kähler action?

The exponent of Kähler function defined by the Kähler action (integral of Maxwell action for induced Kähler form) is central for quantum at least in the real sector of WCW. The question is whether this exponent could have p-adic counterpart and if so, how it should be defined.

In the real context the replacement of the exponent with power of p changes nothing but in the p-adic context the interpretation is affected in a dramatic manner. Physical intuition provided by p-adic thermodynamics [K39] suggest that the exponent of Kähler function is analogous to Bolzmann weight replaced in the p-adic context with non-negative power of p in order to achieve convergence of the series defining the partition function not possible for the exponent function in p-adic context.

1. The quantization of Kähler function as $K = rlog(m/n)$, where r is integer, $m > n$ is divisible by a positive power of p and n is indivisible by a power of p, implies that the exponent of Kähler function is of form $(m/n)^r$ and therefore exists also p-adically. This would guarantee the p-adic existence of the vacuum functional for any prime dividing m and for a given prime p would select a restricted set of p-adic space-time sheets (or partonic 2-surfaces) in the intersection of real and p-adic worlds. It would be possible to assign several p-adic primes to a given space-time sheet (or partonic 2-surface). In elementary particle physics a possible interpretation is that elementary particle can correspond to several p-adic mass scales differing by a power of two [K43]. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by

p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $p + n - 2$ powers of $e^{1/n}$ can be considered.

2. The natural continuation to p-adic sector would be the replacement of integer coefficient r with a p-adic integer. For p-adic integers not reducing to finite integers the p-adic norm of the vacuum functional would however vanish and their contribution to the transition amplitude vanish unless the number of these space-time sheets increases with an exponential rate making the net contribution proportional to a finite positive power of p. This situation would correspond to a critical situation analogous to that encountered in string models as the temperature approaches Hagedorn temperature [B17] and the number states with given energy increases as fast as the Boltzmann weight. Hagedorn temperature is essentially due to the extended nature of particles identified as strings. Therefore this kind of non-perturbative situation might be encountered also now.

3. Rational numbers m/n with n not divisible by p are also infinite as real integers. They are somewhat problematic. Does it make sense to speak about algebraic extensions of p-adic numbers generated by $p^{1/n}$ and giving $n - 1$ fractional powers of p in the extension or does this extension reduce to something equivalent with the original p-adic number field when one redefines the p-adic norm as $|x|_p \rightarrow |x|^{1/n}$? Physically this kind of extension could have a well defined meaning. If this does not make sense, it seems that one must treat p-adic rationals as infinite real integers so that the exponent would vanish p-adically.

4. If one wants that Kähler action exists p-adically a transcendental extension of rational numbers allowing all powers of $log(p)$ and $log(k)$, where $k < p$ is primitive $p - 1$:th root of unity in $G(p)$. A weaker condition would be an extension to a ring with containing only $log(p)$ and $log(k)$ but not their powers. That only single $k < p$ is needed is clear from the identity $log(k^r) = rlog(k)$, from primitive root property, and from the possibility to expand $log(k^r + pn)$, where n is p-adic integer, to powers series with respect to p. If the exponent of Kähler function is the quantity coding for physics and naturally required to be ordinary p-adic number, one could allow $log(p)$ and $log(k)$ to exists only in symbolic sense or in the extension of p-adic numbers to a ring with minimal dimension.

   Remark: One can get rid of the extension by $log(p)$ and $log(k)$ if one accepts the definition of p-adic logarithm as $log(x) = log(p^{-k}x/x_0)$ for $x = p^k(x_0 + py)$, $|y|_p < 1$. To me this definition looks somewhat artificial since this function is not strictly speaking the inverse of exponent function but might have a deeper justification.

5. What happens in the real sector? The quantization of Kähler action cannot take place for all real surfaces since a discrete value set for Kähler function would mean that WCW metric is not defined. Hence the most natural interpretation is that the quantization takes place only in the intersection of real and p-adic worlds, that is for surfaces which are algebraic surfaces in some sense. What this actually means is not quite clear. Are partonic 2-surfaces and their tangent space data algebraic in some preferred coordinates? Can one find a universal identification for the preferred coordinates- say as subset of imbedding space coordinates selected by isometries?

If this picture inspired by p-adic thermodynamics holds true, p-adic integration at the level of WCW would give analog of partition function with Boltzmann weight replaced by a power of p reducing a sum over contributions corresponding to different powers of p with WCW integra.l over space-time sheets with this value of Kähler action defining the analog for the degeneracy of states with a given value of energy. The integral over space-time sheets corresponding to fixed value of Kähler action should allow definition in terms of a symplectic form defined in the p-adic variant of WCW. In finite-dimensional case one could worry about odd dimension of this sub-manifold but in infinite-dimensional case this need not be a problem. Kähler function could defines one particular zero mode of WCW Kähler metric possessing an infinite number of zero modes.

One should also give a meaning to the p-adic integral of Kähler action over space-time surface assumed to be quantized as multiples of $log(m/n)$.

1. The key observation is that Kähler action for preferred extremals reduces to 3-D Chern-Simons form by the weak form of electric-magnetic duality. Therefore the reduction to

cohomology takes place and the existing p-adic cohomology gives excellent hopes about the existence of the p-adic variant of Kähler action. Therefore the reduction of TGD to almost topological QFT would be an essential aspect of number theoretical universality.

2. This integral should have a clear meaning also in the intersection of real and p-adic world. Why the integrals in the intersection would be quantized as multiple of $log(m/n)$, m/n divisible by a positive power of p? Could $log(m/n)$ relate to the integral of $\int_1^p dx/x$, which brings in mind $\oint dz/z$ in residue calculus. Could the integration range $[1, m/n]$ be analogous to the integration range $[0, 2\pi]$. Both multiples of 2π and logarithms of rationals indeed emerge from definite integrals of rational functions with rational coefficients and allowing rational valued limits and in both cases $1/z$ is the rational function responsible for this.

3. $log(m/n)$ would play a role similar to 2π in the approach based on motivic integration where integral has geometric objects as its values. In the case of 2π the value would be circle. In the case of $log(m/n)$ the value could be the arc between the points $r = m/n > 1$ and $r = 1$ with r identified the radial coordinate of light-cone boundary with conformally invariant length measures dr/r. One can also consider the idea that $log(m/n)$ is the hyperbolic angle analogous to 2π so that these two integrals could correspond to hyper-complex and complex residue calculus respectively.

4. TGD as almost topological QFT means that for preferred extremals the Kähler action reduces to 3-D Chern-Simons action, which is indeed 3-form as cohomology interpretation requires, and one could consider the possibility that the integration giving $log(m/n)$ factor to Kähler action is associated with the integral of Chern-Simons action density in time direction along light-like 3-surface and that the integral over the transversal degrees of freedom could be reduced to the flux of the induced CP_2 Kähler form. The logarithmic quantization of the effective distance between the braid end points the in metric defined by modified gamma matrices has been proposed earlier [K26].

Since p-adic objects do not possess boundaries, one could argue that only the integrals over closed varieties make sense. Hence the basic premise of cohomology would fail when one has p-adic integral over braid strand since it does not represent closed curve. The question is whether one could identify the end points of braid in some sense so that one would have a closed curve effectively or alternatively relative cohomology. Periodic boundary conditions is certainly one prerequisite for this kind of identification.

1. In one of the many cohomologies known as quantum cohomology [A75, A155] one indeed assumes that the intersection of varieties is fuzzy in the sense that two surfaces for which points are connected by what is called pseudo-holomorphic curve can be said to intersect at these points. As a special case pseudo-holomorphic curve reduce to holomorphic curve defined by a holomorphic map of 2-D Kähler manifold to complex manifold with Kähler structure. The question arises what "pseudoholomorphic curve connects points" really means. In the recent case a natural analog would be 2-D string world sheets or partonic 2-surfaces so that complex numbers are replaced by hyper-complex numbers effectively. The boundaries of string world sheets would be 1-D braid strands at wormhole throats and at the end of space-time sheet at boundaries of CD. In spirit of algebraic geometry one could also call the 1-D braid strands holomorphic curves connecting points of the partonic 2-surfaces at the two light-like boundaries of CD. In the similar manner space-like braid strands would connect points of partonic 2-surface at same end of CD.

2. In the construction of the solutions of the modified Dirac equation one assumes periodic boundary conditions so that in physical sense these points are identified [K89]. This assumption actually reduces the locus of solutions of the modified Dirac equation to a union of braids at light-like 3-surfaces so that finite measurement resolution for which discretization defines space-time correlates becomes an inherent property of the dynamics. The coordinate varying along the braid strands is light-like so that the distance in the induced metric vanishes between its end points (unlike the distance in the effective metric defined by the modified gamma matrices): therefore also in metric sense the end points represent intersection point. Also the effective 2-dimensionality means are effectively one and same point.

3. The effective metric 2-dimensionality of the light-like 2-surfaces implies the counterpart of conformal invariance with the light-like coordinate varying along braid strands so that it might make sense to say that braid strands are pseudo-holomorphic curves. Note also that the end points of a braid along light-like 3-surface are not causally independent: this is why M-matrix in zero energy ontology is non-trivial. Maybe the causal dependence together with periodic boundary conditions, light-likeness, and pseudo-holomorphy could imply a variant of quantum cohomology and justify the p-adic integration over the braid strands.

12.5.3 Motivic integration

While doing web searches related to motivic cohomology I encountered also the notion of motivic measure [A162] proposed first by Kontsevich. Motivic integration is a purely algebraic procedure in the sense that assigns to the symbol defining the variety for which one wants to calculate measure. The measure is not real valued but takes values in so called scissor group, which is a free group with group operation defined by a formal sum of varieties subject to relations. Motivic measure is number theoretical universal in the sense that it is independent of number field but can be given a value in particular number field via a homomorphism of motivic group to the number field with respect to sum operation.

Some examples are in order.

1. A simple example about scissor group is scissor group consisting operations needed in the algorithm transforming plane polygon to a rectangle with unit edge. Polygon is triangulated; triangles are transformed to rectangle using scissors; long rectangles are folded in one half; rectangles are rescaled to give an unit edge (say in horizontal direction); finally the resulting rectangles with unit edge are stacked over each other so that the height of the stack gives the area of the polygon. Polygons which can be transformed to each other using the basic area preserving building bricks of this algorithm are said to be congruent.

   The basic object is the free abelian group of polygons subject to two relations analogous to second homology group. If P is polygon which can be cut to two polygons P_1 and P_2 one has $[P] = [P_1] + [P_2]$. If P and P' are congruent polygons, one has $[P] = [P']$. For plane polygons the scissor group turns out to be the group of real numbers and the area of polygon is the area of the resulting rectangle. The value of the integral is obtained by mapping the element of scissor group to a real number by group homomorphism.

2. One can also consider symplectic transformations leaving areas invariant as allowed congruences besides the slicing to pieces as congruences appearing as parts of the algorithm leading to a standard representation. In this framework polygons would be replaced by a much larger space of varieties so that the outcome of the integral is variety and integration means finding a simple representative for this variety using the relations of the scissor group. One might hope that a symplectic transformations singular at the vertices of polygon combined with with scissor transformations could reduce arbitrary area bounded by a curve into polygon.

3. One can identify also for discrete sets the analog of scissor group. In this case the integral could be simply the number of points. Even more abstractly: one can consider algebraic formulas defining algebraic varieties and define scissor operations defining scissor congruences and scissor group as sums of the formulas modulo scissor relations. This would obviously abstract the analytic calculation algorithm for integral. Integration would mean that transformation of the formula to a formula stating the outcome of the integral. Free group for formulas with disjunction of formulas is the additive operation [A187]. Congruence must correspond to equivalence of some kind. For finite fields it could be bijection between solutions of the formulas. The outcome of the integration is the scissor group element associated with the formula defining the variety.

4. For residue integrals the free group would be generated as formal sums of even-dimensional complex integration contours. Two contours would be equivalent if they can be deformed to each other without going through poles. The standard form of variety consists of arbitrary small circles surrounding the poles of the integrand multiplied by the residues which are algebraic numbers for rational functions. This generalizes to rational functions with both

real and p-adic coefficients if one accepts the identification of integral as a variety modulo the described equivalence so that $(2\pi)^n$ corresponds to torus $(S^1)^n$. One can replace torus with 2π if one accepts an infinite-dimensional algebraic extension of p-adic numbers by powers of 2π. A weaker condition is that one allows ring containing only the positive powers of 2π.

5. The Grassmannian twistor approach for two-loop hexagon Wilson gives dilogarithm functions $L_k(s)$ [B24]. General polylogarithm is defined by obey the recursion formula:

$$Li_{s+1}(z) = \int_0^z Li_s(t)\frac{dt}{t} \quad .$$

   Ordinary logarithm $Li_1(z) = -log(1-z)$ exists p-adically and generates a hierarchy containing dilogarithm, trilogarithm, and so on, which each exist p-adically for $|x| < 1$as is easy to see. If one accepts the general definition of logarithhs one finds that the entire function series exists p-adically for integer values of s. An interesting question is how strong constraints p-adic existence gives to the the twistor loop integrals and to the underlying QFT.

6. The ring having p-adic numbers as coefficients and spanned by transcendentals $log(k)$ and $log(p)$, where k is primitive root of unity in $G(p)$ emerges in the proposed p-adicization of vacuum functional as exponent of Kähler action. The action for the preferred extremals reducing to 3-D Chern-Simons action for space-time surfaces in the intersection of real and p-adic worlds would be expressible p-adically as a linear combination of $log(p)$ and $log(k)$. $log(m/n)$ expressible in this manner p-adically would be the symbolic outcome of p-adic integral $\int dx/x$ between rational points. x could be identified as a preferred coordinate along braid strand. A possible identification for x earlier would be as the length in the effective metric defined by modified gamma matrices appearing in the modified Dirac equation [K26] .

12.5.4 How could one calculate p-adic integrals numerically?

Riemann sum gives the simplest numerical approach to the calculation of real integrals. Also p-adic integrals should allow a numerical approach and very probably such approaches already exist and "motivic integration" presumably is the proper word to google. The attempts of an average physicist to dig out this kind of wisdom from the vastness of mathematical literature however lead to a depression and deep feeling of inferiority. The only manner to avoid the painful question "To whom should I blame for ever imagining that I could become a real mathematical physicist some day?" is a humble attempt to extrapolate real common sense to p-adic realm. One must believe that the almost trivial Riemann integral must have an almost trivial p-adic generalization although this looks far from obvious.

A proposal for p-adic numerical integration

The physical picture provided by quantum TGD gives strong constraints on the notion of p-adic integral.

1. The most important integrals should be over partonic 2-surfaces. Also p-adic variants of 3-surfaces and 4-surfaces can be considered. The p-adic variant of Kähler action would be an especially interesting integral and reduces to Chern-Simons terms over 3-surfaces for preferred extremals. One should use this definition also in the p-adic context since the reduction of a total divergence to boundary term is not expected to take place in numerical approach if one begins from a 4-dimensional Kähler action since in p-adic context topological boundaries do not exist. The reduction to Chern-Simons term means also a reduction to cohomology and p-adic cohomology indeed exists.

   At the first step one could restrict the consideration to algebraic varieties - in other words zero loci for a set of polynomials $P_i(x)$ at the boundary of causal diamond consisting of pieces of $\delta M^4_\pm \times CP_2$. 5 equations are needed. The simplest integral would be the p-adic volume of the partonic 2-surface.

2. The numerics must somehow rely on the p-adic topology meaning that very large powers p^n are very small in p-adic sense. In the p-adic context Riemann sum makes no sense since the sum never has p-adic norm larger than the maximum p-adic norm for summands so that the limit would give just zero. Finite measurement resolution suggests that the analog for the limit $\Delta x \to 0$ is pinary cutoff $O(p^n) = 0$, $n \to \infty$, for the function f to be integrated. In the spirit of algebraic geometry one must asume at least power series expansion if not even the representability as a polynomial or rational function with rational or p-adic coefficients.

3. Number theoretic approach suggests that the calculation of the volume $vol(V)$ of a p-adic algebraic variety V as integral should reduce to the counting of numbers for the solutions for the equations $f_i(x) = 0$ defining the variety. Together with the finite pinary cutoff this would mean counting of numbers for the solutions of equations $f_i(x)\ mod\ p^n = 0$. The p-adic volume $Vol(V, n)$ of the variety in the measurement resolution $O(p^n) = 0$ would be simply the number of p-adic solutions to the equations $f_i(x)\ mod\ p^n = 0$. Although this number is expected to become infinite as a real number at the limit $n \to \infty$, its p-adic norm is never larger than one. In the case that the limit is a well-defined as p-adic integer, one can say that the variety has a well-defined p-adic valued volume at the limit of infinite measurement resolution. The volume $Vol(V, n)$ could behave like n_p^n and exist as a well defined p-adic number only if n_p is divisible by p.

4. The generalization of the formula for the volume to an integral of a function over the volume is straightforward. Let f be the function to be integrated. One considers solutions to the conditions $f(x) = y$, where y is p-adic number in resolution $O(p^n) = 0$, and therefore has only a finite number of values. The condition $f(x) - y = 0$ defines a codimension 1 sub-variety V_y of the original variety and the integral is defined as the weighted sum $\sum_y y \times vol(V_y)$, where y denotes the point in the finite set of allowed values of $f(x)$ so that calculation reduces to the calculation of volumes also now.

General coordinate invariance

From the point of view of physics general coordinate invariance of the volume integral and more general integrals is of utmost importance.

1. The general coordinate invariance with respect to the internal coordinates of surface is achieved by using a subset of imbedding space-coordinates as preferred coordinates for the surface. This is of also required if one works in algebraic geometric setting. In the case of projective spaces and similar standard imbedding spaces of algebraic varieties natural preferred coordinates exist. In TGD framework the isometries of $M^4 \times CP_2$ define natural preferred coordinate systems.

2. The question whether the formula can give rise to a something proportional to the volume in the induced metric in the intersection of real and rational worlds interesting. One could argue that one must include the square root of the determinant of the induced metric to the definition of volume in preferred coordinates but this might not be necessary. In fact, p-adic integration is genuine summation whereas the determinant of metric corresponds density of volume and need not make no sense in p-adic context. Could the fact that the preferred coordinates transform in simple manner under isometries of the imbedding space (linearly under maximal subgroup) alone guarantee that the information about the imbedding space metric is conveyed to the formula?

3. Indeed, since the volume is defined as the number of p-adic points, the proposed formula should be invariant at least under coordinate transformations mediated by bijections of the preferred coordinates expressible in terms of rational functions. In fact, even more general bijections mapping p-adic numbers to p-adic numbers could be allowed since they effectively mean the introduction of new summation indices. Since the determinant of metric changes in coordinate transformations this requires that the metric determinant is not present at all. Thus summation is what allows to achieve the p-adic variant of general coordinate invariance.

4. This definition of volume and more general integrals amounts to solving the remaining coordinates of imbedding space as (in general) many-valued functions of these coordinates. In the integral those branches contribute to the integral for which the solution is p-adic number or belongs to the extension of p-adic numbers in question. By p-adic continuity the number of p-adic value solutions is locally constant. In the case that one integrates function over the surface one obtains effectively many-valued function of the preferred coordinates and can perform separate integrals over the branches.

Numerical iteration procedure

A convenient iteration procedure is based on the representation of integrand f as sum $\sum_k f_k$ of functions associated with different p-adic valued branches $z_k = z_k(x)$ for the surface in the coordinates chosen and identified as a subset of preferred imbedding space coordinates. The number of branches z_k contributing is by p-adic continuity locally constant.

The function f_k -call it g for simplicity - can in turn be decomposed into a sum of piecewise constant functions by introducing first the piecewise constant pinary cutoffs $g_n(x)$ obtained in the approximation $O(p^{n+1}) = 0$. One can write g as

$$g(x) = \sum h_n(x) \ , \ h_0(x) = g_0(x) \ , \ h_n = g_n(x) - g_{n-1}(x) \ \text{for } n > 0 \ .$$

Note that $h_n(x)$ is of form $g_n(x) = a_n(x)p^n$, $a_n(x) \in \{0, p-1\}$ so that the representation for integral as a sum of integrals for piecewise constant functions h_n converge rapidly. The technical problem is the determination of the boundaries of the regions inside which these functions contribute.

The integral reduces to the calculation of the number of points for given value of $h_n(x)$ and by the local constancy for the number of p-adic valued roots $z_k(x)$ the number of points for $N_0 \sum_{k\geq 0} p^k = N_0/(1-p)$, where N_0 is the number of points x with the property that not all points $y = x(1 + O(p))$ represent p-adic points $z(x)$. Hence a finite number of calculational steps is enough to determine completely the contribution of given value to the integral and the only approximation comes from the cutoff in n for $h_n(x)$.

Number theoretical universality

This picture looks nice but it is far from clear whether the resulting integral is that what physicist wants. It is not clear whether the limit $Vol(V, n)$, $n \to \infty$, exists or even should exist always.

1. In TGD Universe a rather natural condition is algebraic universality requiring that the p-adic integral is proportional to a real integral in the intersection of real and p-adic worlds defined by varieties identified as loci of polynomials with integer/rational coefficients. Number theoretical universality would require that the value of the p-adic integral is p-adic rational (or algebraic number for extensions of p-adic numbers) equal to the value of the real integral and in algebraic sense independent of the number field. In the eyes of physicist this condition looks highly non-trivial. For a mathematician it should be extremely easy to show that this condition cannot hold true. If true the equality would represent extremely profound number theoretic truth.

   The basic idea of the motivic approach to integration is to generalize integral formulas so that the same formula applies in any number field: the specialization of the formula to given number field would give the integral in that particular number field. This is of course nothing but number theoretical universality. Note that the existence of this kind of formula requires that in the intersection of the real and p-adic worlds real and p-adic integrals reduce to same rational or transcendentals (such as $log(1 + x)$ and polylogarithms).

2. If number theoretical universality holds true one can imagine that one just takes the real integral, expresses it as a function of the rational number valued parameters (continuable to real numbers) characterizing the integrand and the variety and algebraically continues this expression to p-adic number fields. This would give the universal formula which can be specified to any number field. But it is not at all clear whether this definition is consistent with the proposed numerical definition.

3. There is also an intuitive expectation in an apparent conflict with the number theoretic universality. The existence of the limit for a finite number p-adic primes could be interpreted as mathematical realization of the physical intuition suggesting that one can assign to a given partonic 2-surface only a finite number of p-adic primes [K26]. Indeed, quantum classical correspondence combined with the p-adic mass calculations suggests that the partonic 2-surfaces assignable to a given elementary particle in the intersection of real and p-adic worlds corresponds to a finite number of p-adic primes somehow coded by the geometry of the partonic 2-surface.

   One way out of the difficulty is that the functions - say polynomials - defining the surface have as coefficients powers of e^n. For given prime p only the powers of e^p exist p-adically so that only the primes p dividing n would be allowed. The transcendentals of form $log(1+px)$ and their polylogarithmic generalizations resulting from integrals in the intersection of real and p-adic worlds would have the same effect. Second way out of the difficulty would be based on the condition that the functional integral over WCW ("world of classical worlds") converges. There is a good argument stating that the exponent of Kähler action reduces to an exponent of integer n and since all powers of n appear the convergence is achieved only for p-adic primes dividing n.

Can number theoretical universality be consistent with the proposed numerical definition of the p-adic integral?

The equivalence of the proposed numerical integral with the algebraic definition of p-adic integral motivated by the algebraic formula in the real context expressed in terms of various parameters defining the variety and the integrand and continued to all number fields would be such a number theoretical miracle that it deserves italics around it:

For algebraic surfaces the real volume of the variety equals apart from constant C to the number of p-adic points of the variety in the case that the volume is expressible as p-adic integer.

The proportionality constant C can depend on p-adic number field , and the previous numerical argument suggests that the constant could be simply the factor $1/(1-p)$ resulting from the sum of p-adic points in p-adic scales so short that the number of the p-adic branches $z_k(x)$ is locally constant. This constant is indeed needed: without it the real integrals in the intersection of real and p-adic worlds giving integer valued result $I = m$ would correspond to functions for which the number of p-adic valued points is finite.

The statement generalizes also to the integrals of rational and perhaps even more general functions. The equivalence should be considered in a weak form by allowing the transcendentals contained by the formulas have different meanings in real and p-adic number fields. Already the integrals of rational functions contain this kind of transcendentals.

The basic objection that number of p-adic points without cannot give something proportional to real volume with an appropriate interpretation cannot hold true since real integral contains the determinant of the induced metric. As already noticed the preferred coordinates for the imbedding space are fixed by the isometries of the imbedding space and therefore the information about metric is actually present. For constant function the correspondence holds true and since the recipe for performing of the integral reduce to that for an infinite sum of constant functions, it might be that the miracle indeed happens.

The proposal can be tested in a very simple manner. The simplest possible algebraic variety is unit circle defined by the condition $x^2 + y^2 = 1$.

1. In the real context the circumference is 2π and p-adic transcendental requiring an infinite-dimensional algebraic extension defined in terms of powers of 2π. Does this mean that the number of p-adic points of circle at the limit $n \to \infty$ for the pinary cutoff $O(p^n) = 0$ is ill-defined? Should one define 2π as this integral and say that the motivic integral calculus based on manipulation of formulas reduces the integrals to a combination of p-adically existing numbers and 2π? In motivic integration the outcome of the integration is indeed formula rather than number and only a specialization gives it a value in a particular number field. Does 2π have a specialization to the original p-adic number field or should one introduce it via transcendental extension?

2. The rational points $(x, y) = (k/m, l/m)$ of the p-adic unit circle would correspond to Pythagorean triangles satisfying $k^2 + l^2 = m^2$ with the general solution $k = r^2 - s^2$, $l = 2rs$, $m = r^2 + s^2$. Besides this there is an infinite number of p-adic points satisfying the same equation: some of the integers k, l, m would be however infinite as real integers. These points can be solved by starting from $O(p) = 0$ approximation $(k, l, m) \to (k, l, m)\ mod\ p \equiv (k_0, l_0, m_0)$. One must assume that the equations are satisfied only modulo p so that Pythagorean triangles modulo p are the basic objects. Pythagorean triangles can be also degenerate modulo p so that either k_0, l_0 or even m_0 vanishes. Note that for surfaces $x^n + y^n = z^n$ no non-trivial solutions exists for $x^n, y^n, z^n < p$ for $n > 2$ and all p-adic points are infinite as real integers.

   The Pythagorean condition would give a constraint between higher powers in the expressions for k, l and m. The challenge would be to calculate the number of this kind of points. If one can choose the integers $k - (k\ mod\ p)$ and $l - (l\ mod\ p)$ freely and solve $m - (m\ mod\ p)$ from the quadratic equations uniquely, the number of points of the unit circle consisting of p-adic integers must be of form $N_0/(1 - p)$. At the limit $n \to \infty$ the p-adic length of the unit circle would be in p-adic topology equal to the number of modulo p Pythagorean triangles (r, s). The p-adic counterpart of 2π would be ordinary p-adic number depending on p. This definition of the length of unit circle as number of its modulo p Pythagorean points also Pythagoras would have agreed with since in the Pythagorean world view only rational triangles were accepted.

3. One can look the situation also directly solving y as $y = \pm\sqrt{1 - x^2}$. The p-adic square root exists always for $x = O(p^n)$, $n > 0$. The number of these points x is $2/(1 - p)$. For $x = O(p^0)$ the square root exist for roughly one half of the integers $n \in \{0, p - 1\}$. The number of integers $(x^2)_0$ is therefore roughly $(p - 1)/2$. The study of $p = 5$ cae suggests that the number of integers $(1 - (x^2)_0)_0 \in \{0, p - 1\}$ which are squares is about $(p - 1)/4$. Taking into account the $\pm$ sign the number of these points by $N_0 \simeq (p - 1)/2$. In this case the higher $O(p)$ contribution to x is arbitrary and one obtains total contribution $N_0/(1 - p)$. Altogether one would have $(N_0 + 2)/(1 - p)$ so that eliminating the proportionality factor the estimate for the p-adic counterpart of 2π would be $(p + 3)/2$.

4. One could also try a trick. Express the points of circle as $(x, y) = (cos(t), sin(t))$ such that t is any p-adic number with norm smaller than one in p-adic case. This unit circle is definitely not the same object as the one defined as algebraic variety in plane. One can however calculate the number of p-adic points at the limit $n \to \infty$. Besides $t = 0$, all p-adic numbers with norm larger than p^{-n} and smaller than 1 are acceptable and one obtains as a result $N(n) = 1 + p^{n-1}$, where "1" comes from overall important point $t = 0$. One has $N(n) \to 1$ in p-adic sense. If $t = 0$ is not allowed the length vanishes p-adically. The circumference of circle in p-adic context would have length equal to 1 in p-adic topology so that no problems would be encountered (numbers $exp(i2\pi/n)$ would require algebraic extension of p-adic numbers and would not exist as power series).

   The replacement of the coordinates (x, y) with coordinate t does not respect the rules of algebraic geometry since trigonometric functions are not algebraic functions. Should one allow also exponential and trigonometric functions and their inverses besides rational functions and define circle also in terms of these. Note that these functions are exceptional in that corresponding transcendental extensions -say that containing e and its powers- are finite-dimensional?

5. To make things more complicated, one could allow algebraic extensions of p-adic numbers containing roots $U_n = exp(i2\pi/n)$ of unity. This would affect the count too but give a well-defined answer if one accepts that the points of unit circle correspond to the Pythagorean points multiplied by the roots of unity.

p-Adic thermodynamics for measurement resolution?

The proposed definition is rather attractive number theoretically since everything would reduce to the counting of p-adic points of algebraic varieties. The approach generalizes also to algebraic extensions of p-adic numbers. Mathematicians and also physicists love partition functions, and one can indeed assign to the volume integral a partition function as p-adic valued power series in powers $Z(t) = \sum v_n t^n$ with the coefficients v_n giving the volume in $O(p^n) = 0$ cutoff. One can also define partition functions $Z_f(t) = \sum f_n t^n$, with f_n giving the integral of f in the same approximation.

Could this kind of partition functions have a physical interpretation as averages over physical measurements over different pinary cutoffs? p-Adic temperature can be identified as $t = p^{1/T}$, $T = 1/k$. For p-adically small temperatures the lowest terms corresponding to the worst measurement resolution dominate. At first this sounds counter-intuitive since usually low temperatures are thought to make possible good measurement resolution. One can however argue that one must excite p-adic short range degrees of freedom to get information about them. These degrees of freedom correspond to the higher pinary digits by p-adic length scale hypothesis and high energies by Uncertainty Principle. Hence high p-adic temperatures are needed. Also measurement resolution would be subject to p-adic thermodynamics rather than being freely fixed by the experimentalist.

12.5.5 Infinite rationals and multiple residue integrals as Galois invariants

In TGD framework one could consider also another kind of cohomological interpretation. The basic structures are braids at light-like 3-surfaces and space-like 3-surfaces at the ends of space-time surfaces. Braids intersects have common ends points at the partonic 2-surfaces at the light-like boundaries of a causal diamond. String world sheets define braid cobordism and in more general case 2-knot [K34]. Strong form of holography with finite measurement resolution would suggest that physics is coded by the data associated with the discrete set of points at partonic 2-surfaces. Cohomological interpretation would in turn would suggest that these points could be identified as intersections of string world sheets and partonic 2-surface defining dual descriptions of physics and would represent intersection form for string world sheets and partonic 2-surfaces.

Infinite rationals define rational functions and one can assign to them residue integrals if the variables x_n are interpreted as complex variables. These rational functions could be replaced with a hierarchy of sub-varieties defined by their poles of various dimensions. Just as the zeros allow realization as braids or braids also poles would allow a realization as braids of braids. Hence the n-fold residue integral could have a representation in terms of braids. Given level of the braid hierarchy with n levels would correspond to a level in the hierarchy of complex varieties with decreasing complex dimension.

One can assign also to the poles (zeros of polynomial in the denominator of rational function) Galois group and obtains a hierarchy of Galois groups in this manner. Also the braid representation would exists for these Galois groups and define even cohomology and homology if they do so for the zeros. The intersections of braids with of the partonic 2-surfaces would represent the poles in the preferred coordinates and various residue integrals would have representation in terms of products of complex points of partonic 2-surface in preferred coordinates. The interpretation would be in terms of quantum classical correspondence.

Galois groups transform the poles to each other and one can ask how much information they give about the residue integral. One would expect that the n-fold residue integral as a sum over residues expressible in terms of the poles is invariant under Galois group. This is the case for the simplest integrals in plane with n poles and probably quite generally. Physically the invariance under the hierarchy of Galois group would mean that Galois groups act as the symmetry group of quantum physics. This conforms with the number theoretic vision and one could justify the formula for the residue integral also as a *definition* motivated by the condition of Galois invariance. Of course, all symmetric functions of roots would be Galois invariants and would be expected to appear in the expressions for scattering amplitudes.

The Galois groups associated with zeros and poles of the infinite rational seem to have a clear physical significance. This can be understood in zero energy ontology if positive (negative) physical states are indeed identifiable as infinite integers and if zero energy states can be mapped to infinite rationals which as real numbers reduce to real units. The positive/negative energy part of the zero energy state would correspond to zeros/poles in this correspondence. An interesting question is how strong correlations the real unit property poses on the two Galois groups hierarchies. The asymmetry between positive and negative energy states would have interpretation in terms of the thermodynamic arrow of geometric time [K6] implied by the condition that either positive or negative energy states correspond to state function reduced/prepared states with well defined particle numbers and minimum amount of entanglement.

12.5.6 Twistors, hyperbolic 3-manifolds, and zero energy ontology

While performing web searches for twistors and motives I have begun to realize that Russian mathematicians have been building the mathematics needed by quantum TGD for decades by realizing the vision of Grothendieck. One of the findings was the article Volumes of hyperbolic manifolds and mixed Tate motives [A157] by Goncharov- one of the great Russian mathematicians involved with the drama- about volumes of hyperbolic n-manifolds and motivic integrals.

Hyperbolic n-manifolds [A46] are n-manifolds equipped with complete Riemann metric having constant sectional curvature equal to -1 (with suitable choice of length unit) and therefore obeying Einstein's equations with cosmological constant. They are obtained as coset spaces on proper-time constant hyperboloids of n+1-dimensional Minkowski space by dividing by the action of discrete subgroup of SO(n,1), whose action defines a lattice like structure on the hyperboloid. What is remarkable is that the volumes of these closed spaces are homotopy invariants in a well-define sense.

What is even more remarkable that hyperbolic 3-manifolds [A45] are completely exceptional in that there are very many of them. The complements of knots and links in 3-sphere are often cusped hyperbolic 3-manifolds (having therefore tori as boundaries). Also Haken manifolds are hyperbolic. According to Thurston's geometrization conjecture, proved by Perelman (whom we all know!), any closed, irreducible, atoroidal 3-manifold with infinite fundamental group is hyperbolic. There is an analogous statement for 3-manifolds with boundary. One can perhaps say that very many 3-manifolds are hyperbolic.

The geometrization conjecture of Thurston [A39] allows to see hyperbolic 3-manifolds in a wider framework. The theorem states that compact 3-manifolds can be decomposed canonically into sub-manifolds that have geometric structures. It was Perelman who sketched the proof of the conjecture. The prime decomposition with respect to connected sum reduces the problem to the classification of prime 3-manifolds and geometrization conjecture states that closed 3-manifold can be cut along tori such that the interior of each piece has a geometric structure with finite volume serving as a topological invariant. There are 8 possible geometric structures in dimension three and they are characterized by the isometry group of the geometry and the isotropy group of point.

Important is also the behavior under Ricci flow [A82] $\partial_t g_{ij} = -2R_{ij}$: here t is not space-time coordinate but a parameter of homotopy. If I have understood correctly, Ricci flow is a dissipative flow gradually polishing the metric for a particular region of 3-manifold to one of the 8 highly symmetric metrics defining topological invariants. This conforms with the general vision about dissipation as the source of maximal symmetries. For compact n-manifolds the normalized Ricci flow $\partial_t g_{ij} = -2R_{ij} + (2/n)Rg_{ij}$ preserving the volume makes sense. Interestingly, for $n = 4$ the right hand side is Einstein tensor so that the solutions of vacuum Einstein's equations in dimension four are fixed points of normalized Ricci flow. Ricci flow expands the negatively curved regions and contracts the positively curved regions of space-time time. Hyperbolic geometries represent one these 8 geometries and for the Ricci flow is expanding. The outcome is amazingly simple and gives also support for the idea that the preferred extremals of Kähler action could represent maximally symmetries 4-geometries defining topological or algebraic geometric invariants: the preferred extremals would be maximally symmetric representatives - kind of archetypes- for a given topology or algebraic geometry.

The volume spectrum for hyperbolic 3-manifolds forms a countable set which is however not discrete: some reader might understand what the statement that one can assign to them ordinal ω^ω could possibly mean for the man of the street. What comes into my simple mind is that p-adic integers and more generally, profinite spaces which are not finite, are something similar: one can enumerate them by infinitely long sequences of pinary digits so that they are countable (I do not know whether also infinite p-adic primes must be allowed). They are totally disconnected in real sense but do not form a discrete set since since can connect any two points by a p-adically continuous curve.

What makes twistor people excited is that the polylogarithms emerging from twistor integrals and making sense also p-adically seems to be expressible in terms of the volumes of hyperbolic manifolds. What fascinates me is that the moduli spaces for causal diamonds or rather for the double light-cones associated with their M^4 projections with second tip fixed are naturally lattices of the3-dimensional hyperbolic space defined by all positions of the second tip and 3-dimensional hyperbolici spaces are the most interesting ones! At least in the intersection of the real and p-adic worlds number theoretic discretization requires discretization and volume could be quantized in discrete manner.

For $n = 3$ the group defining the lattice is a discrete subgroup of the group of SO(3,1) which equals to $PSL(2, C)$ obtained by identifying $SL(2, C)$ matrices with opposite sign. The divisor group defining the lattice and hyperbolic spaces as its lattice cell is therefore a subgroup of $PSL(2, Z_c)$, where Z_c denotes complex integers. Recall that $PSL(2, Z_c)$ acts also in complex plane (and therefore on partonic 2-surfaces) as discrete Möbius transformations wheras $PSL(2, Z)$ correspond to 3-braid group. Reader is perhaps familiar with fractal like orbits of points under iterated Möbius transformations. The lattice cell of this lattice obtained by identifying symmetry related points defines hyperbolic 3-manifolds. Therefore zero energy ontology realizes directly the hyperbolic manifolds whose volumes should somehow represent the polylogarithms.

The volumes, which are topological invariants, are said to be highly transcendental. In the intersection of real and p-adic worlds only algebraic volumes are possible unless one allows extension by say finite number of roots of e (e^p is p-adic number). The p-adic existence of polylogarithms suggests that also p-adic variants of hyperbolic spaces make sense and that one can assign to them volume as topological invariance although the notion of ordinary volume integral is problematic. In fact, hyperbolic spaces are symmetric spaces and general arguments allow to imagine what the p-adic variants of real symmetric spaces could be.

12.6 Floer homology and TGD

Floer homology [A33] has provided considerable understanding of symplectic manifolds using physics based approach relying on 2-D variational principle called symplectic action. One variant of Floer theory has been applied also to deduce topological invariants of 3-manifolds in terms of SU(2) Chern-Simons action. The basics of Floer homology without recourse to quantum field theoretic approach are described at technical level in the lectures of Dietmar Salamon [A200]. The notion of quantum cohomology closely related to Floer homology and related approaches and involving also supersymmetry is described by Alexander Givental in [A155].

The quantum fluctuating degrees of freedom of TGD Universe are parameterized by symplectic group acting as isometries of WCW, which can be regarded as a union of symmetric spaces assignable to the symplectic group. Hence the optimistic hunch is that Floer homology might provide new insights about quantum TGD - in particular about the problem of understanding the preferred extremals of Kähler action. Especially interesting is the relationship of Floer homology to the proposed vision about braided Galois homology. The following considerations encourage this optimism. In particular, completely new insights about the role of Minkowskian and Euclidian regions emerge.

12.6.1 Trying to understand the basic ideas of Floer homology

I do not have competence to describe Floer's homology as a mathematician. Instead, I try just to outline the basic ideas as I have (possibly mis-)understood them as a physicist by reading the basic introduction to the theory [A33]. The motivation for the symplectic Floer homology came from Arnold's conjecture stating that for a closed symplectic manifold the number of fixed points for non-degenerate (isolated critical points) symplecto-morphisms has the sum of the Betti numbers as a lower bound. The equivalence of Floer's symplectic homology for closed symplectic manifolds with singular homology proves this conjecture. This means that symplectic Floer homology as such is not interesting from TGD view point of view.

Morse function in the loop space of the symplectic manifold

Recall that Morse function is a monotonically increasing real valued function in n-manifold for which critical points are isolated. Its level surfaces induce the slicing of the manifold $n-1$-dimensional surfaces. At the extrema the topology of the slice changes as is clear from a simple example provided by torus (standing on tangent plane orthogonal to the plane defined by the torus with Morse function identified as the height function defined by the coordinate orthogonal to the plane). There is minimum and maximum and two saddle points. Quite generally, the signature of the matrix defined by the second derivatives of the Morse function -Hessian- characterizes the properties of the critical point. Hessian allows to deduce information about the topology of the manifold and Morse theorem states that the number of critical points has a lower limit given by the sum of the Betti numbers defining the dimensions of various homology groups of the manifolds in singular homology.

Floer generalizes Morse theory from the level of symplectic manifold M with a Morse function defined by Hamiltonian to the level of the free loop space LM of M. This Morse function depends on preferred Hamiltonian and its cyclic time variation defining a loop in LM. Salamon represents the approach without recourse to the methods of topological quantum field theories [A200]. A very schematic representation -even more schematic than that in [A155] - using referring to quantum about what one does is attempted in following.

1. 2-dimensional action for an orbit of string in M replaces Morse function. The extrema of the action analogous to critical points of Morse function are crucial for calculating path integral in QFT approach using saddle point approximation. In topological QFTs path integral reduces to a well-defined finite dimensional integrals over moduli spaces. One constructs action principle in the form

$$S = \int_{-\infty}^{\infty}(||\partial_u m||^2 + ||\nabla f||^2)du \quad (12.6.1)$$

   where u can be seen regarded as a coordinate parallel to cylinder axes defined by the orbit of the loop of M and t could be regarded as an angle coordinate of the loop. f denotes the symplectic action functional of the loop defined by time dependent Hamiltonian H_t. ∇f is the functional gradient of f with respect to coordinates of m regarded as analogous to fields $S^1 \times R$. $||...||^2$ defines inner product in the space of maps $S^1 \to M$ involving integral over the circle parameterized by coordinate t. Note that this action introduces preferred parameterization of the cylinder meaning breaking of at least manifest general coordinate invariance.

2. Schematically the field equations read as

$$\partial_u^2 m = \nabla^2 f \ , \quad (12.6.2)$$

   where ∇^2 is functional d'Alembertian reducing to its analog at the level of M but depending on preferred Hamilton H_t. This condition states that the cylinder represents a harmonic map $S^1 \times R \to M$ with respect to the almost Kähler metric of M.

3. Assuming the analog of $\mathcal{N} = 2$ supersymmetry for the solution the above equation reduces to

$$\partial_u m = \pm\nabla f \ . \quad (12.6.3)$$

   This condition is just the condition saying that one has a wave packed moving to right or left and state the hyper-complex variant of holomorphy. These left and right moving solutions are in key role in string model. In Euclidian metric of $S^1 \times R$ the conditions have interpretation as the generalization of Cauchy-Riemann conditions stating that the map $S^1 \times R \to M$ commutes with complex conjugation: in other worlds the multiplication by imaginary unit in $S^1 \times R$ is equivalent with the tensor multiplication defined by the almost Kähler form in M. The tangent space of image is complex sub-space of tangent space of M. Depending on the sign on the right hand side one has pseudo-holomorphy or anti-pseudo-holomorphy.

4. The solutions with finite action become asymptotically independent of u so that one has $\nabla f = 0$. This states that the loop represents a cyclic solution of Hamilton's equations for Hamilton H. Hamilton could also depend on time in periodic manner so that for $t = 0$ and $t = 2\pi$ one has $H_t = H$.

5. One can consider also solutions which are independent of u and t asymptotically so that the circles reduce to critical points asymptotically. One can also consider solutions representing spheres with more than two critical points as marked points. Also solutions with higher genus can be considered These solutions are relate closely to the definition of Gromov-Witten invariants in quantum cohomology.

This approach generalizes also to Chern-Simons action by replacing f with Chern-Simons action for the 3-manifold X^3 and $R \times S^1$ with $R \times X^3$ to get space-time. The symplectic manifold is replaced with the space of Yang-Mills gauge potentials. In this case field equations from the variational principle are YM equations and instanton and anti-instanton equations are obtained in the super-symmetric case. Time independent solutions correspond asymptotically to static solutions describing magnetic monopoles. In this case the critical points of Morse function can be seen as points at which the topology of the slice of field space defined by the Morse function changes its topology. A good intuitive guideline is Morse function for torus.

About Witten's approach to Floer homology

Using the ideas discussed for the first time in Witten's classic work revealing a connection between supersymmetry and Morse theory [A167], one can extend M to a super-manifold. Witten defines $\mathcal{N} = 2$ SUSY algebra by introducing a parameter dependent deformation of the exterior algebra via $d_t = exp(-th)dexp(th)$ and its conjugate $d_t^* = exp(th)dexp(-th)$: for $t = 0$ one has $d_t = d_t^*$. h takes the role the role of Morse function. $Q_1 = d_t + d_t^*$ and $Q_2 = i(d_t - d_t^*)$ obey standard supersymmetry algebra $Q_1Q_2 + Q_2Q_1 = 0$ and $Q_1^2 = Q_2^2 \equiv H_t$. The solutions of $d_t\Psi = 0$ are differential forms of various degrees and correspond to zero energy solutions for which the supersymmetry is not broken. The deformed cohomology is equivalent with the original cohomology by $\Psi \rightarrow exp(th)\Psi$. This gives a direct connection between cohomology and supersymmetry whose existence is to be expected from the basic properties of exterior algebra.

The motivation for the deformation is that for degree p closed forms are localized around critical points of h with Hessian having p negative eigenvalues so that the correspondence between homology generators and critical points becomes manifest. There is indeed a natural mapping from de Rham cohomology to the critical points such that the degree of the form correspond to the number of negative eigenvalues of the Hessian.

Later Witten managed to expand his ideas about supersymmetric Morse theory so that it could be applied to Floer homology (1+1 case) and to the calculation of Donaldson invariants of 4-manifold (1+3 case). Recently Witten has been working with the applications to knot theory (1+2 case) for ordinary knots and for 2-knots and cobordisms of 1-knots (1+3 case) [A217, A100, A218].

Representation of loops with fixed based in terms of Hamiltonians with cyclic time dependence

As already noticed Floer - whose work preceded Witten's work - considered instead of the symplectic manifold M its free loop space LM. One begins with symplectic action identified as the sum of the symplectic area of the loop expressible as the value of the one-form defining the symplectic form over the loop and integral of the Hamiltonian H around the loop. The natural choice of the loop parameter is as the canonical conjugate of the symplectic potential so that the integrated quantity is analogous to the minimal substitution $p - eA$ of familiar from elementary quantum mechanics. The variational equations for the symplectic action are Hamiltonian equations of motion in the force field defined by the Hamiltonian H and one considers periodic orbits (recall that there is conserved energy associated with the orbits defined by the Hamiltonian). The counterparts of critical points are loops which correspond to the extrema of symplectic action.

One can also consider time dependent Hamiltonians H_t for which the initial and final value of the Hamiltonian is the same preferred Hamiltonian. This kind of Hamiltonians define via their time evolutions loops in the loop space LG of the symplectic group. At the level of LM the resulting map of M to itself is symplecto-morphism. Now however energy is not in general conserved. By periodicity the critical points of the Hamiltonian H correspond to cyclic orbits of periodically time varying Hamiltonian so that the homotopies of LM with base point defined by H are mapped to a collection of homotopies of M defined by the critical points of the Hamiltonian. For constant Hamiltonian $H_t = H$ the critical orbits reduce to a point and the need to obtain non-trivial elements of homotopy group of M explains why one needs Hamiltonians with cyclic time dependence. The homotopy group of LM is mapped to that of M by homomorphism.

One could consider also higher homotopy groups of the loop space. The first homotopy group would correspond to loops in loop space mapped to tori associated with the fixed points of the Hamiltonian. In this manner one would obtain analogs of homotopy groups defined by mappings from $(S^1)^n$ to loop space to M and also of homotopy groups. By taking the initial loop to be trivial so that initial Hamiltonian is constant Hamiltonian, one obtains the symplectic analogs of ordinary homotopy groups defined as a map from S^n to loop space to M. Also the condition that loops are contracted to points asymptotically gives rise to homotopy groups.

Representation of non-closed paths of LM as paths connecting critical points of M

In Floer homology one considers also paths of LM and M,which are not closed. These paths form the first homotopy groupoid of LM. Since the elements of $\pi_0(LM)$ (loops not deformable to each other) represented by Hamiltonians with cyclic time dependence are mapped to those of $\pi_1(M)$ at critical points, a good guess is that the elements of homotopy group $\pi_1(LM)$ can be mapped to elements of $\pi_2(M)$ connecting critical points of H. If the loops at the ends of cylinder reduce to points the images of $\pi_1(LM)$ are indeed elements of $\pi_2(M)$ containing two critical points. As noticed, the number critical points can be also higher.

To achieve the representation of first homotopy group one considers a path of LM parameterized by a parameter u defining a cylinder in M which should connect the critical points. This requires that the deformation becomes at the limit $u \to \pm\infty$ independent of u so that one obtains a cyclic deformation of H. The partial differential equations state that one has gradient flow defined by symplectic action in loop space. The equations (resulting from supersymmetry in QFT approach) pseudo-holomorphy or generalized Cauchy-Riemann conditions as

$$\partial_u m \pm L_{H_t}(m) = 0 \ ,$$

where $L_{H_t}(m) = 0$ denotes Hamiltonian equations for the coordinates m of M so that $L_{H_t}m$ is indeed the functional gradient of symplectic action. At the asymptotic limit $\partial_u m \to 0$ boundary conditions give just Hamiltonian equations.

As already found, one can assign to to these equations a supersymmetric action functional defined in terms of the almost Kähler metric defining the analog of energy. As a matter fact, the existence of almost complex structure in M is enough (transitions functions between coordinate patches need not be holomorphic in this case). The condition that the energy is finite requires asymptotic u-independence and super-symmetry condition since energy density is the sum of kinetic energy densities associated with the motion in u direction and of the square of the vector $L_{H_t}m$. Since the time evolution with respect to u is not energy conserving, the cylinders can connect different critical points of H. This motivates the term "connecting cylinder". From the point of view of physicist the role of the field equations is to perform a "gauge choice" selecting particular representative for homotopy.

The orbit of the loop as a pseudo-holomorphic surface

The cylinder defined by the loop defines a pseudo-holomorphic surface. The sub-spaces connected by pseudo-holomorphic surfaces intersect in quantum cohomology and Gromow-Witten invariant counts for the number of the pseudo-holomorphic surfaces connecting/intersecting given n surfaces. Stringy interpretation for the pseudo-holomorphic curves (holomorphic for Kähler manifolds) would be as string world sheets. There is an obvious connection with the vision about branes connected by string world sheets. If the asymptotic images of S^1 contract to points, they correspond to critical points (marked points). One can consider also more general solutions of field with n asymptotic circles containing n critical points as marked points.

The statement of quantum cohomology that two surfaces intersect in fuzzy sense when they are connected by pseudo-holomorphic curve would mean that that two surfaces intersect when they both have points common with the pseudo-holomorphic curve. The 2-dimensional mapping cylinders can be filled to 3-D objects by adding the 2-dimensional pseudo-holomorphic surface. From this the connection with Chern-Simons action and possibility to apply analogous construction to 3-D manifold topology becomes obvious. Chern-Simons action in turn implies connection to 4-D manifold topology.

The correspondence with the singular homology

Symplectic Floer homology for closed symplectic manifolds is equivalent with singular homology. This means that one has one-to-one map of the space spanned by the critical points to the singular homology. Critical points are classified by the signature of the Hessian of Hamiltonian so that there is natural ordering of the critical points, which should correspond to the ordering of the homology groups since signature varies from n (maximum of Morse function) to zero (minimum of Morse function). The study of the homology of torus defined in terms of critical points of height function h serves as a guide-line when one tries to guess the idea behind the correspondence.

To each critical point one can assign a tangent plane defined as the plane of negative signature of the Hessian of h. Its value equals to 0,1,1,2 for the critical points of h. The critical manifolds assigned with the negative signature tangent space at critical points can be identified as point, first homologically non-trivial circle, second homologically non-trivial circle, and the entire torus and correspond to the generators of the homology. In Floer homology the correspondence need not be as simple as this but one expect similar correspondence so that the value of grading of homology corresponds to the signature of the critical point. One must allow only the connections going to the direction of smaller energy and by a proper choices of signs the dynamics defined by the action defined gradient flow is indeed dissipative so that this condition is satisfied.

Quantum cup product and pseudo-holomorphic surfaces

As the analog of intersection product in ordinary cohomology homology, the cohomology associated with the symplectic Floer homology corresponds to the so called pair of pants product of quantum cohomology [A155] which is a deformed cup product having fuzzy intersection as its dual at the level of homology.

Ordinary cup product for two forms of degree n_1 and n_2 is a form which is characterized by its values for the elements of homology with co-dimension $n_1 + n_2$ so that $d - n_1 - n_2$ is the dimension of the intersection of the corresponding surfaces. The product is characterized by a coefficients $W(\alpha, \beta, \gamma)$ where the arguments represent homology equivalence classes identifiable as Gromov-Witten invariants assignable to sphere with three punctures. One can say that three representatives α, β, γ of homology give rise to a non-vanishing coefficient $W(\alpha, \beta, \gamma)$ if there is a pair of pants having non-empty intersections with α, β, γ. The coefficient $W(\alpha, \beta, \gamma)$ is analogous to a coupling constant associated with vertex with α, β, γ representing the particles entering to the vertex.

The factors of the cup product of quantum cohomology are associated with the two legs of the pants and the outcome of the product to the "waist". More abstractly, by conformal transformations the legs and "waist" can be reduced to 3 marked points and the number of marked points can be arbitrary and represent the intersection points for n manifolds connected by a pseudo-holomorphic surface with n marked points. One can indeed generalize the variational principle to allow besides cylinders also pseudo-holomorphic surfaces with arbitrary number holes whose boundaries are associated with loops containing critical point so that critical points would indeed represent marked points of a sphere with holes. When H_t reduces to H, loops and marked spheres reduce to point a so that ordinary cup product results.

12.6.2 Could Floer homology teach something new about Quantum TGD?

The understanding of both quantum TGD and its classical counterpart is still far from from comprehensive. For instance, skeptic could argue that the understanding of the preferred extremals of Kähler action is still just a bundle of ideas without a coherent overview. Also the physical roles of

Kähler actions for Euclidian and Minkowskian space-time regions is far from clear. Do they provide dual descriptions as suggested or are both needed? Kähler action for preferred extremal in Euclidian regions defines naturally Kähler function. Could Kähler action in Minkowskian regions- naturally imaginary by negative sign of metric determinant- give an imaginary contribution to the vacuum functional and define Morse function so that both Kähler and Morse would find a prominent role in the world order of TGD? One might hope that the mathematical insights from Floer homology combined with the physical picture and constraints from quantum classical correspondence could provide additional insights about the construction preferred extremals of Kähler action.

Basic picture about preferred extremals of Kähler action

It is useful to gather some basic ideas about construction of preferred extremals before the discussion of ideas inspired by Floer homology.

1. For the preferred extremals Kähler action reduces to Chern-Simons term at the light-like surfaces defining orbits of partonic 2-surfaces and space-like 3-surfaces the ends of the space-time sheets. These 3-surfaces are extremals of Chern-Simons action subject to the constraint force defined by the weak form of electric-magnetic duality implying that TGD does not reduce to a mere topological QFT. One has clearly two dynamics: one along light-like 3-surfaces and one along space-like 3-surfaces and their internal consistency is a powerful constraint.

2. The Chern-Simons contributions from Minkowskian region is imaginary and corresponds to almost topological QFT aspect of TGD. The argument reducing the action to Chern-Simons term has been discussed in detail only in Minkowskian regions and involves in an essential manner the notions of local polarization and light-like momentum direction: the latter one does not make sense in Euclidian regions. Note however that Laplace equation makes sense and local polarization and momentum directions are replaced by those for color quantum numbers. It will be found that internal consistency requires holography both in Minkowskian and Euclidian regions. In any case, the Euclidian contribution would give rise to the exponent of Kähler function and Minkowskian contribution to a phase factor appearing usually in path integral defining topological QFT. Exponent of Kähler function would guarantee that integration over WCW is mathematically well-defined.

3. How could one extend the 3-surfaces to 4-surfaces using strong form of holography? One could think of having for each time=constant collection of 2-D slices of the light-like 3-surfaces a space-like Chern-Simons dynamics connecting them to each other. One would have two dynamics-one time-like and one space-like as effective 2-dimensionality required by the strong form of holography requires. These dynamics should be mutually consistent and this should give consistency conditions. The time parameters for these two dynamics would correspond to the two coordinates of string world sheets involved.

4. The idea that one could assign Hamiltonians to the marked points of the partonic 2-surfaces as carriers is physically compelling. The Hamiltonians of $\delta M^4_\pm \times CP_2$ inducing Hamiltonians of WCW play essential role in quantum theory. Also the Hamiltonians at ends of braid strands should have classical counterparts at space-time level. Could braid strand obey Hamiltonian dynamics defined by Hamiltonian attached to it? This would give a constraint to the wormhole throat making itself visible also a properties of the space-time sheet. If so then braid strands would define a kind of the skeleton for the space-time sheet. This idea could be generalized so that one would have a skeleton of space-time consisting of string world sheets and finite measurement resolution would mean the restriction of consideration to this skeleton. Also the braid strands carrying fermion number (other than right handed neutrino number) should obey their own dynamics.

Braided Galois homology as counterpart of Floer homology?

The picture suggested by braided Galois homology seems to have natural correspondences with that provided by Floer homology.

1. The quantum fluctuating degrees of freedom correspond to the symplectic group of $\delta M^4_\pm \times CP_2$. Finite measurement resolution leads to the discretization. One considers the subgroup G of symplectic group of $\delta M^4_\pm \times CP_2$ permuting a given set of n points of the partonic 2-surface defining the end points of braids. Subgroup of S_n having interpretation as Galois group is in question. The normal subgroup H of symplecto-morphisms leaving these points invariant and the factor group G/H is the target of primary interest and expected to be discrete group. The braiding of this group is intuitively equivalent with the replacement of symplectic transformations with flows and the points can be interpreted as critical points of infinite number of Hamiltonian belonging to H. In Floer's theory one makes a gauge choice selecting a generic non-degenerate Hamiltonian. This choice -or a generalization of it- should have a definite physical meaning in TGD framework in terms of classical correlates for the quantum numbers of the zero energy state.

2. Preferred Hamiltonian acting and its time dependent deformation play a key role in Floer homology and represent homotopy in symplectic group. In the recent case braided Galois homology assigns to preferred extremals subgroup of symplectic flow in Minkowskian space-time regions and the braid points are invariant under its normal subgroup. The flow defined by time dependent deformation a Hamiltonian of subgroup defines a candidate for the flow defined by preferred Hamiltonian. The connecting flows in turn would correspond to the Galois group. The condition that the flow lines of the Hamilton along 3-surfaces poses a strong condition on the choice of Hamiltonian on one hand and on the preferred extremal on the other hand. The time evolution of Hamiltonian could be realized by the slicing of imbedding space by light-cone boundaries parallel to the lower or upper boundary of CD.

3. For braided Galois homology the generators d_i representing boundary homomorphisms whose square maps to commutator subgroup and to zero after abelianization define candidates for the algebra of SUSY generators. Parameter dependent deformation of these generators would make sense also now and give rise a homology analogous to that of Witten. The generators of the cohomology would correspond to supersymmetric ground states and one would expect that cohomology is non-trivial for the critical points of Morse function. This super-symmetry, which need not have anything to do with the standard notion of supersymmetry, would be assigned to Minkowskian regions of space-time. One cannot of course exclude purely fermionic representations of braided Galois homology and number theoretic quantization of fermions would pose a powerful constraint on the spectrum of fermionic modes.

Kähler function as Kähler action in Euclidian regions and Morse function as Kähler action in Minkowskian regions?

The role of Kähler action in the Floer like aspects of TGD has been already briefly discussed.

1. Symplectic Floer homology for imbedding space gives just the homology groups of $S^2 \times CP_2$. This homology is crucial for the interpretation of TGD but much more detailed information is required. The analog of Floer homology must be associated with WCW for which quantum fluctuating degrees of freedom are parametrized by symplectic group of $\delta M^4_\pm \times CP_2$ or symmetric space associated with it. In finite measurement resolution one would have discrete subgroup defined as a factor group of subgroup permuting braid points and normal subgroup leaving them invariant identifiable in terms of a hierarchy of Galois groups. Flows must be considered in order to have braiding. The flows could also correspond to parameter dependent Hamiltonians with the parameter varying along light-like wormhole throat or space-like 3-surface at the end of CD.

2. In the case of Chern-Simons action the critical points correspond to flat connections and define the generators of the homology for the space of connections. For YM action instanton solutions play similar role. In the recent case the space of 3-surfaces associated with given CD seems to be natural object of study.

   Kähler function - to be distinguished from Kähler action - would be the first guess for the Morse function in WCW and the analog of Floer homology would be formally defined by the sums of the 3-surfaces which correspond to the extrema of Kähler function. This idea fails.

Kähler metric must be positive definite. Therefore the Hessian of the Kähler function in holomorphic quantum fluctuating degrees of freedom characterized by complex coordinates of WCW should have only non-negative or non-positive eigen values.

One could try to circumvent the difficulty by assuming that the allowed extrema with varying signature of Hessian are associated with the zero modes. Therefore the analog of Floer homology based on Kähler function would not however tell anything about symplectic degrees of freedom -at least those assignable to the Euclidian regions.

Remark: One can wonder how the Kähler function can escape the implications of Morse theorem. In the case of CP_2 the degeneracy of Kähler function - meaning that it depends on single $U(2)$ invariant CP_2 coordinate only - takes care of the problem. Also now infinite-dimensional symmetries of WCW are expected to allow to circumvent the Morse theorem.

3. The only manner to save this idea is that the Euclidian regions defined by the generalized Feynman graphs define Kähler function and Minkowskian regions the analog of the action defining path integral. The earlier proposed duality states that the formulation TGD is possible either as a functional integral or a path integral. If duality holds true, its effect would be analogous to that of Wick rotation. The alternative approach would assign physical significance to both contributions. The Kähler action in Minkowskian regions could serve as Morse function. This identification is rather natural since the determinant of the induced metric appearing in the action indeed gives imaginary unit in Minkowskian regions. If this were the case interference effects would result already at the level of action and the connection with quantum field theories would be much tighter than previously thought.

   Euclidian regions would guarantee the convergence of the functional integral and one would have a mathematically well-defined theory. The analog of Floer homology would represent quantum superpositions of critical points identifiable a ground states defined by the extrema of Kähler action for Minkowskian regions. Perturbative approach to quantum TGD would rely on functional integrals around the extrema of Kähler function.

4. Should one assume that the reduction to Chern-Simons terms occurs for the preferred extremals in *both* Minkowskian and Euclidian regions or only in Minkowskian regions?

   (a) All arguments for this have been represented for Minkowskian regions [K26] involve local light-like momentum direction which does not make sense in the Euclidian regions. This does not however kill the argument: one can have non-trivial solutions of Laplacian equation in the region of CP_2 bounded by wormhole throats: for CP_2 itself only covariantly constant right-handed neutrino represents this kind of solution and at the same time supersymmetry. In the general case solutions of Laplacian represent broken super-symmetries and should be in one-one correspondences with the solutions of the modified Dirac equation. The interpretation for the counterparts of momentum and polarization would be in terms of classical representation of color quantum numbers.

   (b) If the reduction occurs in Euclidian regions, it gives in the case of CP_2 two 3-D terms corresponding to two 3-D gluing regions for three coordinate patches needed to define coordinates and spinor connection for CP_2 so that one would have two Chern-Simons terms. I have earlier claimed that without any other contributions the first term would be identical with that from Minkowskian region apart from imaginary unit and different coefficient. This statement is wrong since the space-like parts of the corresponding 3-surfaces are discjoint for Euclidian and Minkowskian regions.

   (c) There is also a very beautiful argument stating that Dirac determinant for Kähler-Dirac action equals to Kähler function, which would be lost if Euclidian regions would not obey holography. The argument obviously generalizes and applies to both Morse and Kähler function which are definitely not proportional to each other.

5. The preferred extremal of Kähler action itself would connect 3-surfaces at the opposite boundaries of CD just as the action for Floer theory connects two loops assignable to critical points. In zero energy ontology the unions of 3-surfaces at the ends of CD is the basic unit and correspond to the critical points of Morse function. The question is whether objects can be

mapped to a set of critical points of the preferred Hamiltonian in a natural manner. Braided Galois homology with preferred Hamiltonian defining the braids as its flow lines gives hopes about this.

6. In Floer theory the homology of LM is mapped to homology of M. The homology of the WCW cannot be mapped to that of the imbedding space. The hierarchy of Planck constants [K25] assigned to the multi-valued correspondence between canonical momentum densities of Kähler action and time derivatives of imbedding space coordinates leads to the introduction of singular covering spaces of the imbedding space with the number of sheets of covering depending on space-time region. The homology of WCW might be mapped homomorphically to the homology of this space.

   In the case of loop space $H_0(LM)$ is mapped to $H_1(M)$. Something similar should take place now since all odd homology groups of WCW must vanish if it is Kähler manifold whereas zeroth homology could be non-trivial. In zero energy ontology 3-surfaces having disjoint components at the ends of CD indeed correspond naturally to paths of connected 3-surface so that this condition might be realized.

On basis of these arguments it seems that the general structure of Floer homology fits rather nicely the structure of quantum TGD.

TGD counterparts for pseudo-holomorphic surfaces

If the Morse function exists as Kähler action for preferred extremal in the Minkowskian regions of the space-time, there are good hopes of obtaining the analog of Floer homology in TGD framework. Consider first pseudo-holomorphic surfaces.

1. The analogy with Floer homology would suggest that the analogs of pseudo-holomorphic surfaces assignable to the critical points of Morse function correspond to 3-surfaces at the ends of CD are 3-surface defined by the simultaneous vanishing of two holomorphic rational functions of the complex coordinates of $S^2 \subset \delta M^4_\pm$ and of CP_2 depending parametrically on the light-like radial coordinate of $\delta M^\pm$ giving $7-4=3$ conditions. The effective metric 2-dimensionality implied by the strong form of holography is expected to pose conditions on the radial dependence of these functions.

2. Pseudo-holomorphic closed string world sheets with punctures provide a beautiful geometric realization of quantum cohomology. If positive and negative energy parts of zero energy states can be regarded as elements of homology, space-time sheets could take a similar role. In finite measurement resolution string world sheets would perform the same function so that closed strings would be replaced with open ones as connectors in TGD based quantum cohomology. Signature is not a problem: in string theories the hypercomplex variant of holomorphy is allowed. String world sheets would connect partonic two surfaces at the given end of partonic CD and also at different ends of CD. String world sheets could branch but the mechanism would be the decay of open string creating new partonic 2-surfaces meeting at TGD counterpart of Feynman vertex. Note that also in Witten's approach to Floer theory and Donaldson theory the signature of string world sheets is Minkowskian.

   Remarks:

   (a) One can imagine an extremely simple definition for the intersection for partonic 2-surfaces at opposite boundaries of CD proposed actually earlier. One could identify the opposite boundaries of CD given by pieces $\delta M^4_\pm \times CP_2$ by identifying δM^4_+ and δM^4_- in an obvious manner. This definition is however a natural dynamical counterpart for intersection in classical sense obtained by identifying the boundaries of CD.

   (b) So called massless extremals represent one example about the analogs of right and left moving solutions in TGD framework [K10]. They distinguish sharply between classical TGD and Maxwell's hydrodynamics. There are arguments suggesting that quite generally the preferred extremals in Minkowskian regions representable as graphs of maps $M^4 \times CP_2$ decompose to regions characterized by local directions of momentum and polarization representing propagation of massless waves. This would be the classical space-time correlate for the decomposition of radiation to massless quanta.

3. Partonic 2-surfaces with particles at the ends of braid strands would define basic objects and would naturally correspond to holomorphic surfaces for the critical points of Morse function defined by the contribution of Minkowskian regions to Kähler action. The hyper-complex string world sheets and hyper-quaternionicity are however necessary for the $M^4 \times CP_2 - M^8$ correspondence suggested by physics as generalized number theory vision. The finite dimensions of the moduli spaces would not be a problem since holomorphy would characterize only the critical points. The connection between super-symmetry and cohomology plays a key role in TQFT and pseudo-holomorphy is an excellent candidate for the geometric correlate of supersymmetry of some kind.

The natural question is whether pseudo-holomorphy could generalize in 4-D context to its quaternionic analog.

1. One of the basic conjectures of TGD is that preferred extremals of Kähler action can be regarded as associative (co-associative) sub-manifolds. The tangent spaces of space-time surfaces would define hyper-quaternionic sub-spaces of complexified octonions with imaginary units of quaternions would be multiplied by commuting imaginary unit.

2. The tangent spaces of space-time surface would also contain a preferred hyper-complex plane or more generally, a hyper-complex plane which depends on position so that these planes integrate to string world sheet. This would allow to regard space-time surfaces either as surfaces in $M^4 \times CP_2$ or in hyper-octonionic subspace M^8 [K72]. Integrable distributions of the hyper-complex sub-manifolds would define string world sheets analogous with hypercomplex sub-manifolds. The physical interpretation would be in terms of local preferred planes of un-physical polarizations. The philosophical motivation of hyper-quaternionicity would be that associativity for space-time surfaces and commutativity for string world sheets could define a number theoretical variational principle.

3. The role of pseudo-holomorphy suggests that hyper-quaternionicity could characterize the critical points of Morse function defined by Kähler action in Minkowskian regions of space-time. If all preferred extremals are hyper-quaternionic, this property cannot imply holomorphy of the partonic surfaces.

4. It was already mentioned that finite measurement resolution defines a skeleton of space-time surface realized in terms of string world sheets. This skeleton would generalize a curve of complex plane at which holomorphic function defining a complex coordinate is real to hyper-complex sub-manifold of hyper-quaternionic space-time surface. Given this skeleton, the construction of space-time surface would be analogous to an analytic continuation from hyper-complex realm to hyper-quaternionic realm.

Hierarchy of Planck constants, singular coverings of the imbedding space, and homology of WCW

1. As already noticed, the homology groups of imbedding space are certainly too simple to be of interest from the point of physics and quantum TGD. Physically interesting analogs of homology groups could be associated with the space-time surface itself or with the singular covering of imbedding space allowing to describe the many-valued correspondence between canonical momentum densities and time derivatives of imbedding space coordinates. This would allow to interpret the resulting non-trivial homology as a property of either space-time surface or of effective imbedding space. In any case, one should add to the homology the constraint that the elements of homology are representable as sub-varieties for the preferred extremals of Kähler action. This might allow to code physics using the formalism of homology theory. Floer like theory would also define a homomorphism mapping the homology $H_n(WCW)$ to the homology group H_{m+1} of the singular covering of the imbedding space.

2. The recent interpretation for the effective hierarchy of Planck constants coming as integer multiples of ordinary Planck constants has interpretation in terms of effective coverings of space-time surface implied by the 1-to-many character of the map assigning to canonical momentum densities of Kähler action time derivatives of imbedding space coordinates. The

strange sounding proposal is that at partonic two surfaces branching occurs in the sense that the various branchings of the many-valued function involved with this correspondence co-incide. Branching would however occur both in the direction of the light-like 3-surface and space-like 3-surface at the end of CD. Branching could occur at both ends of given CD or only at single end if the branching is taken as a space-time correlate for dissipation and arrow of time, and perhaps even for quantum superposition as will be discussed below.

3. This branching brings in mind the emergence of homologically non-trivial curves from the critical points in Floer cohomology and possibility of several curves connecting two critical points (torus serves as a good illustration also now). The analogy would be more convincing if one could assign to the branches a sign factor analogous to the sign of the eigenvalue of Hessian as physical signature. One possibility is that the sign factor tells whether the line is incoming or outgoing. Also the sign of energy in the case of virtual particles could appear in the sign factor.

How detailed quantum classical correspondence can be?

The gradient dynamics is quite essential for the super-symmetric solutions of Floer theory and typically gradient dynamics is dissipative leading to fixed points of the function function involved. Dissipative dynamics allows to order critical points in terms of the energy defined by Hamilton and also connect different critical points. Physicist would obviously ask whether this aspect of the dynamics is only an artefact of the model or whether it has a much deeper physical significance. If it does not, the following considerations can be taken only as a proposal for how the quantum correlates could be represented at space-time level and how detailed they can be.

Can the dynamics defined by preferred extremals of Kähler action be dissipative in some sense? The generation of the arrow of time has a nice realization in zero energy ontology as a choice of well-defined particle numbers and other quantum numbers at the "lower" end of CD. By quantum classical correspondence this should have a space-time correlate. Gradient dynamics is a highly phenomenological realization of the dissipative dynamics and one must try to identify a microscopic variant of dissipation in terms of entropy growth of some kind. If the arrow of time and dissipation has space-time correlate, there are hopes about the identification of this kind of correlate.

Quantum classical correspondence has been perhaps the most useful guiding principle in the construction of quantum TGD. What is says that not only quantum numbers but also quantum jump sequences should have space-time correlates: about this the failure of strict determinism of Kähler action gives good hopes. Even the quantum superposition- at least for certain situations -might have space-time correlates.

1. Measurement interaction term in the modified Dirac action at the ends of CD indeed defines a coupling of quantala dynamics to the classical dynamics [K26]. The interpretation of TGD as square root of thermodynamics suggests that measurement interaction terms are completely analogous to the Lagrange multiplier terms fixing the values of observables in thermodynamics. Now the classical conserved charges would be fixed to their quantal values for the space-time surfaces appearing in quantum superposition. These Lagrange multiplier terms would also give to Kähler-Dirac action 3-D boundary terms. By the localization spinor modes to space-time sheets these boundary terms are effectively one-dimension and localized to stringy curves.

   This kind of measurement interaction is indeed basic element of quantum TGD. Also the color and charges and angular momentum associated with the Hamiltonians at point of braids could couple to the dynamics via the boundary conditions.

2. The braid strand with a given Hamiltonian could obey Hamiltonian equations of motion: this would give rise to a skeleton of space-time defined by braid strands possibly continued to string world sheets and would provided different realization of quantum classical correspondence.

3. Quantum TGD can be regarded as a square root of thermodynamics in well-defined sense. Could it be possible to couple the Hermitian square root of density matrix appearing in M-matrix and characterizing zero energy state thermally to the geometry of space-time sheets by

coupling it to the classical dynamical via boundary conditions depending on its eigenvalues? This is indeed the case if one accepts the description of the equality of classical and quantum charges in terms of Lagrange multipliers.

The necessity to choose single eigenvalue would give a representation for single measurement outcome. One can achieve a representation of the ensemble at space-time level consisting of space-time sheets representing various outcomes of measurement. This ensemble would be realized as ensemble of sub-CDs for a given CD.

4. One can as whther a quantum superposition of WCW spinor fields could have a space-time correlate in the sense that all space-time surfaces in the superposition would carry information about the superposition itself? Obviously this would mean self-referentiality via quantum-classical feedback.

The following discussion concentrates on possible space-time correlates for the quantum superposition of WCW spinor fields and for the arrow of time.

1. It seems difficult to imagine space-time correlate for the quantum superposition of final states with varying quantum numbers since these states correspond to quantum superpositions of different space-time surfaces. How could one code information about quantum superposition of space-time surfaces to the space-time surfaces appearing in the superposition? This kind of self-referentiality seems to be necessary if one requires that various quantum numbers characterizing the superposition (say momentum) couple via boundary conditions to the space-time dynamics.

2. The failure of non-determinism of quantum dynamics is behind dissipation and strict determinism fails for Kähler action. This gives hopes that the dynamics induces also arrow of time. Energy non-conservation is of course excluded and one should be able to identify a measure of entropy and the analog of second law of thermodynamics telling what happens at for preferred extremals when the situation becomes non-deterministic. The vertices of generalized Feynman graphs are natural places were non-determinism emerges as are also sub-CDs. Naive physical intuition would suggest that dissipation means generation of entropy: the vertices would favor decay of particles rather than their spontaneous assembly. The analog of blackhole entropy assignable to partonic 2-surfaces might allow to characterize this quantatively. The symplectic area of partonic 2-surface could be a symplectic invariant of this kind.

3. Could the mysterious branching of partonic 2-surfaces -obviously analogous to even more mysterious branching of quantum state in many worlds interpretation of quantum mechanics-assigned to the multi-valued character of the correspondence between canonical momentum densities and time derivatives of H coordinates allow to understand how the arrow of time is represented at space-time level?

   (a) This branching would effectively replace CD with its singular covering with number of branches depending on space-time region. The relative homology with respect to the upper boundary of CD (so that the branches of the trees would effectively meet there) could define the analog of Floer homology with various paths defined by the orbits of partonic 2-surfaces along lines of generalize Feynman diagram defining the first homology group. Typically tree like structures would be involved with the ends of the tree at the upper boundary of CD effectively identified.

   (b) This branching could serve as a representation for the branching of quantum state to a superposition of eigenstates of measured quantum observables. If this is the case, the various branches to which partonic 2-surface decays at partonic 2-surface would more or less relate to quantum superposition of final states in particle reaction. The number of branches would be finite by finite measurement resolution. For a given choice of the arrow of geometric time the partonic surface would not fuse back at the upper end of CD.

(c) Rather paradoxically, the space-time correlate for the dissipation would reduce the dissipation by increasing the effective value of $\hbar$: the interpretation would be however in terms of dark matter identified in terms of large $\hbar$ phase. In the same manner dissipation would be accompanied by evolution since the increase of $\hbar$ naturally implies formation of macroscopically quantum coherent states. The space-time representation of dissipation would compensate the increase of entropy at the ensemble level.

(d) The geometric representation of quantum superposition might take place only in the intersection of real and p-adic worlds and have interpretation in terms of cognitive representations. In the intersection one can also have a generalization of second law [K42] in which the generation of genuine negentropy in some space-time regions via the build up of cognitive representation compensated by the generation of entropy at other space-time regions. The entropy generating behavior of living matter conforms with this modification of the second law. The negentropy measure in question relies on the replacement of logarithms of probabilities with logarithms of their p-adic norms and works for rational probabilities and also their algebraic variants for finite-dimensional algebraic extensions of rationals.

(e) Each state in the superposition of WCW quantum states would contain this representation as its space-time correlate realizing self-referentiality at quantum level in the intersection of real and p-adic worlds. Also the state function reduced members of ensemble could contain this cognitive representation at space-time level. Essentially quantum memory making possible self-referential linguistic representation of quantum state in terms of space-time geometry and topology would be in question. The formulas written by mathematicians would define similar map from quantum level to the space-time level making possible to "see" one's thoughts.

12.7 Could Gromov-Witten invariants and braided Galois homology together allow to construct WCW spinor fields?

The challenge of TGD is to understand the structure of WCW spinor fields both in the zero modes which correspond to symplectically invariant degrees of freedom not contributing to the WCW Kähler metric and in quantum fluctuating degrees of freedom parametrized by the symplectic group of $\delta M^4_\pm \times CP_2$. The following arguments suggest that an appropriate generalization of Gromov-Witten invariants to covariants combined with braid Galois homology could allow do construct WCW spinor fields and at the same time M-matrices defining the rows of the unitary U-matrix between zero energy states.

12.7.1 Gromov-Witten invariants

Gromov- Witten invariants [A41] are rational numbers $GW^{X,A}_{g,n}$, which in a loose sense count the number of pseudo-holomorphic curves of genus g and n marked points and homology equivalence class A in symplectic space X meeting n surfaces of X with given homology equivalence classes. These invariants can distinguish between different symplectic manifolds. Since also the proposed generalized homology groups would define symplectic invariants if the realization of braided Galois groups as symplectic flows works, the attempt to understand the relation of Gromov-Witten invariants of TGD is well-motivated.

Let X be a symplectic manifold with almost complex structure J (the transition functions are not holomorphic) and C be an algebraic variety in X of genus g and with complex structure j having n marked points x_1,...x_n, which are points of X. Pseudo-homolomorphic maps of C to X are by definition maps, whose Jacobian map commutes with the multiplication of the tangent space vectors with the antisymmetric tensor representing imaginary unit $J \circ df = df \circ j$. If the symplectic manifold allows Kähler structure, one can say that pseudohomolomorphic maps commute with the multiplication by imaginary unit so that tangent plane of complex 2-manifold is mapped to a complex tangent plane of X.

The moduli space $M_{g,n}(X)$ of the pseudoholomorphic maps is finite-dimensional. One considers also its subspaces $M_{g,n}(X,A)$ of $M_{g,n}(X)$, where A represents a fixed homology equivalence class A for the image of C in X. The so called evaluation map from $M_{g,n}(X,A)$ to $M_{g,n}(X)) \times X^n$ defined by $(C, x_1, x_2, ...x_n, f) \rightarrow (st(C, x_1, x_2, ...x_n); f(x_1), ..., f(x_n))$. Here $st(C, x_1, x_2, ...x_n)$ denotes so called stabilization of $(C, x_1, ...x_n)$ defined in the following manner. A smooth component of Riemann surface is said to be stable if the number of automorphisms (conformal transformations) leaving the marked and nodal (double) points invariant is finite. Stabilization is obtained by dropping away the unstable components from the domain of C.

The image of the fundamental class of the moduli space $M_{g,n}(X)$ defines a homology class in $M_{g,n}(X)) \times X^n$. Since the homology groups of $M_{g,n}(X)) \times X^n$ are by Künneth theorem expressible as convolutions of homology groups of $M_{g,n}(X)$ and n copies of X, this homology class can be expressed as a sum

$$\sum_{\beta,\alpha_i} GW^{X,A}_{g,n} \beta \times \alpha_1 ... \times \alpha_n \quad .$$

The coefficients, which in the general case are rational valued, define Gromov-Witten invariants. One can roughly say that these rational numbers count the number of surfaces C intersecting the n homology classes α_i of X. n surfaces intersect when there is a surface of genus g with n marked points intersection the surfaces at marked points and Gromov-Witten invariant counts the number of homologically non-equivalent pseudo-holomorphic 2-surfaces of this kind [A155].

Branes connected by closed strings would represent a basic example about quantum intersections. Also in Floer homology [A200] and quantum cohomology [A75] this kind of fuzzy intersection is encountered. The fundamental Gromov-coefficients $W(\alpha, \beta, \gamma)$ are for three homology generators α, β, γ and connecting surface correspond to pseudo-holomorphic spheres (or higher genus surfaces) with three marked points obtained by contracting the outgoing three strings of stringy trouser vertex to point.

12.7.2 Gromov-Witten invariants and topological string theory of type A

Gromow-Witten invariants appear in topological string theory of type A [A96] for which the scattering amplitudes depend on Kähler structure of X only. The target space X of this theory is 6-dimensional symplectic manifold. X can correspond to 6-dimensional Calabi-Yau manifold. Twistor space is one particular example of this kind of manifold and one can indeed relate twistor amplitudes to those of topological string theory in twistor space.

Type A topological string theory contains both fundamental string orbits, which are 2-surfaces wrapping over 2-real-D holomorphic curves in X and D2 branes, whose 3-D "orbits" in X wrap over Lagrangian manifolds having by definition a vanishing induced symplectic form. There are also strings connecting the branes. C corresponds now to the world sheet of string with n marked points representing emitted particles. Gromov-Witten invariants are defined as integrals over the moduli spaces $M_{g,n}(X)$ and provide a rigorous definition for path integral and the free energy at given genus g is the generating function for Gromov-Witten invariants.

Witten introduced the formulation of the topological string theories in terms of topological sigma models [A95]. The formulation involves the analog of BRST symmetry encountered in gauge fixing meaning that one replaces target space with super-space by assigning to target space-coordinates anti-commuting partners which do not however represent genuine fermionic degrees of freedom. One also replaces string world sheet with a super-manifold $\mathcal{N} = (2,2)$ SUSY and spinors are world sheet spinors and Lorentz transformations act on string world sheet. Topological string models are characterized by continuous R-symmetries and the mixing of rotational and R-symmetries takes place. The R-symmetry associated with 2-D world sheet Lorentz transformation compensates for the spin rotation so that one indeed obtains a BRST charge Q (for elementary introduction to BRST symmetry see [B34]), which is scalar and the condition $Q^2 = 0$ is satisfied identically so that cohomology is obtained.

12.7.3 Gromov-Witten invariants and WCW spinor fields in zero mode degrees of freedom

One can ask whether Gromow-Witten invariants of something more general could emerge naturally in TGD framework.

1. Gromov-Witten invariants modified so that closed string orbits are replaced by open string world sheets with boundaries identifiable as braid strands relate to the braided Galois homology. Both the geometric interpretation these invariants in terms of fuzzy quantum intersection induced by connecting string world sheets and the discussion of the Floer homology like aspects of quantum TGD support this idea.

2. Another interpretation is that Gromov-Witten invariants or their generalizations emerge in the construction of WCW spinor fields in zero mode degrees of freedom, which do not contribute to the line element of WCW Kähler metric. Contrary to the first hopes there is no convincing support for this view.

Comparison of the basic geometric frameworks

The basic geometric frameworks are sufficiently similar to encourage the idea that Gromov-Witten type invariants might make sense in TGD framework.

1. In the standard formulation of TGD the 6-dimensional symplectic manifold is replaced with the metrically 6-dimensional manifold $\delta M^4_\pm \times CP_2$ having degenerate symplectic and Kähler structure and reducing effectively (metrically) to the symplectic manifold $S^2 \times CP_2$. Partonic 2-surfaces at the light-like boundaries of CD identifiable as wormhole throats define the counterparts of fundamental string like object of topological string theory of type A. The n marked points of Gromov-Witten theory could correspond to the ends of braid strands carrying purely bosonic quantum numbers characterized by the attached $\delta M^4_\pm \times CP_2$ Hamiltonians with well defined angular momentum and color quantum numbers. One must distinguish these braid strands from the braid strands carrying fermion quantum numbers.

2. There are also differences. One assigns 3-D surfaces to the boundaries of CD and partonic 2-surfaces at CD are connected with are interpreted as strings so that partonic 2-surfaces have also brane like character. One can identify 3-D surfaces for which induced Kähler forms of CP_2 and $\delta M^4_\pm$ vanish (any surface with 1-D projection to $\delta M^4_\pm$ and 2-D CP_2 projection with Lagrangian manifold would define counterpart of brane) but it is not natural to raise these objects to a special role.

3. I have proposed that quantum TGD is analogous to a physical analog of Turing machine in the sense that the inclusions of HFFs could allow to emulate any QFT with almost gauge group assignable to the included algebra [K25]. The representation of these gauge groups as subgroups of symplectic transformations leaving the marked points of the partonic 2-surfaces invariant gives hopes of realizing this idea mathematically. Symplectic groups are indeed completely exceptional because of their representative power [A91] and used already in classical mechanics and field theory to represent symmetries. An interesting question is whether the symplectic group associated with $\delta M^4_\pm \times CP_2$ could be universal in the sense that any gauge group of this kind allows a faithful homomorphism to this group.

One should understand what pseudo-holomorphy means in TGD framework. One must consider both the identification of pseudo-holomorphic surfaces as string world sheets or as partonic 2-surfaces. Consider first the interpretation of pseudo-holomorphic 2-surfaces as string world sheets assignable to the space-time sheets.

1. String world sheets would not represent closed strings and their ends would define braid strands at light-like 3-surfaces and at the space-like 3-surfaces defining the ends of space-time. This is not a problem: also the standard picture about pseudo-holomorphic surfaces as spheres with punctures is obtained by idealizing the holes of closed string with punctures [A200]. Open string world sheet be seen as a string containing holes defined by the boundary

braid strands. Disjoint partonic two surfaces at the ends of braid strands would intersect in quantum sense. The interpretation for the fuzzy intersection would be in terms of causal dependence of the quantum state at the ends of CD so that the assignment of Gromov-Witten invariants to them would be natural.

2. This option looks very natural from TGD point of view since the moduli space is expected to be finite-dimensional and have interpretation in terms of the preferred extremal property. For a given partonic 2-surfaces and tangent space data at them the moduli would be fixed more or less uniquely and the variation of the tangent space data would vary the moduli.

Also the identification of pseudo-holomorphic surfaces as partonic 2-surfaces can be considered. It would apparently conform with the canonical identification of pseudo-holomorphic surfaces but the interpretation as connectors in fuzzy cup product can be challenged.

1. Since the moduli space of pseudo-holomorphic surfaces is finite-dimensional, only a very restricted set of partonic 2-surfaces satisfies pseudo-holomorphy condition. The induced metric of the partonic 2-surface defines a unique complex structure. Pseudo-holomorphy states that Jacobian takes the complex tangent place of partonic 2-surface to a comlex plane of the tangent space of $\delta M^4_\pm \times CP_2$. Pseudo-holomorphy is implied by holomorphy stating that both CP_2 coordinates and S^2 coordinates as functions of the complex coordinate of the partonic 2-surface are holomorphic functions implying that the induced metric as the standard $ds^2 = g_{z\overline{z}}dzd\overline{z}$. Holomorphy is also implied if one can express as a variety using functions which are holomorphic functions of $\delta M^4_\pm$ and CP_2 complex coordinates and analytic functions of the radial coordinate r. These surfaces are characterized by the homology-equivalence classes of their projections in $\delta M^4_\pm$ (3-D Euclidian space with puncture at origin) and in CP_2. Both are characterized by integer. These surfaces obviously define a subset of partonic 2-surfaces and one can actually assign to the string-like objects as cartesian products of string world sheets satisfying minimal surface equations and of 2-D complex sub-manifolds of CP_2.

2. The first objection is that partonic two-surfaces do not represent time-evolution so punctures associated with them cannot be regarded as causally dependent. From physics point of view it does not make sense to speak about fuzzy intersection except in terms of finite measurement resolution implying that second quantized induced spinor fields have finite number of modes so that they do not anti-commute at partonic 2-surfaces anymore.

3. Second objection is that there is nothing physically interesting that partonic 2-surfaces could connect!

4. The third counter argument is that pseudo-holomorphy condition allows only finite-dimensional moduli space whereas the space of partonic 2-surfaces is infinite-dimensional. Two explanations suggest itself.

   (a) The finite-measurement resolution might imply an effective reduction of the space of partonic 2-surfaces to this moduli space? Finite measurement resolution could be understood also as a kind of gauge invariance when realized in terms of inclusion of hyper-finite factors of type II_1 (HFFs) with the action of sub-factor having no effect on its observable properties. Holomorphy would serve as a gauge fixing condition.

   (b) If TGD as almost topological QFT can be formulated as an analog of Floer's theory relying on action principle, the natural proposal is that holomorphic partonic 2-surfaces correspond to critical values for the Kähler action assignable to the Minkowskian regions of the preferred extremal.

It seems relatively safe to conclude that only the string world sheets have a natural interpretation as connectors the deformed interwection product in TGD framework.

Could an analog of topological string theory make sense in TGD framework

The observations of previous paragraphs motivate the question whether an analog of type A topological string theory could emerge in the construction of WCW spinor fields. The basic problem is to understand how the WCW spinor fields depend on symplectic invariants, which however need not correspond to zero modes which should be expressible in terms of symplectic fluxes alone. One might hope that topological string theory of some kind could allow to construct this kind of symplectic invariants.

1. The encouraging symptom is that the n-point functions of both A and B type topological string theories are non-trivial only in dimension $D = 6$, which is the metric dimension of $\delta M^4_\pm \times CP_2$. Since the n-point functions of type A topological string theory depend only on the Kähler structure associated now by CP_2 and $\delta M^4_\pm$ Kähler forms they could code for the physics associated with the zero modes representing non-quantum fluctuating degrees of freedom. Since type B topological string theory requires vanishing of the first Chern class implying Calabi-Yau property, this theory is not possible in the standard formulation of TGD.

   The emergence of the topological string theory of type A seems to be in conflict with what twistorialization suggests. Witten suggested in his classic article [B39] boosting the twistor revolution, that the Fourier transforms of the sattering amplitudes from momentum space to twistor space scattering amplitudes for perturbative $\mathcal{N} = 4$ SUSY could be interpreted in terms of D-instanton expansion of topological string theory of type B defined in twistor space CP_3.

2. One can identify the marked points as the end points of both space-like and time-like braids but it is not natural to assign them fermionic quantum numbers except those of covariantly constant right-handed neutrino spinor with the points of symplectic triangulation. This is well-motivated since symplectic algebra extends to super-symplectic algebra with covariantly constant right handed neutrino spinor defining the super-symmetry. One can assign the values of Hamiltonians of $\delta M^4_\pm \times CP_2$ to the marked points belonging to the irreducible representations of rotation group and color group such that the total quantum numbers vanish by the symplectic invariance. n-point functions would be correlation functions for Hamiltonians. In a well-defined sense one would have color and angular momentum confinement in WCW degrees of freedom.

   The vanishing of net quantum numbers need not hold true for single connected partonic 2-surface. Also it could hold true only for a collection of partonic 2-surfaces associated with same 3-surface at either end of CD. The most general condition would be that the total color and spin numbers of positive and negative energy parts of the state sum up to zero in symplectic degrees of freedom.

3. The generating function for Gromov-Witten invariants is defined for a connected pseudo-holomorphic 2-surface with a fixed genus g as such is not general enough if one allows partonic 2-surfaces with several components. The generalization would provide information about the preferred extremal of Kähler action and about the topology of space-time surface. The generalization of the Gromov-Witten partition function would define as its inverse the normalization factor for zero energy state identifiable as M-matrix defined as a positive diagonal square root of density matrix multiplied by S-matrix for which initial partons possess fixed genus and which contains superposition over braids with arbitrary number of strands. The intuition from ordinary thermodynamics suggests that this partition function is in a reasonable approximation expressible as convolution for n-points functions for individual partonic 2-surfaces allowing the set of marked points to carry net $\delta M^4_\pm$ angular momentum and color quantum numbers.

Description of super-symmetries in TGD framework

It is interesting to see whether the formulation of super-symmetries in the framework of topological sigma models giving rise to Gromov-Witten invariants [A95] has any reasonable relation to TGD where the notion of super-space does not look natural as a fundamental notion although it might be very useful as a formal tool in the formulation of SUSY QFT limit [K27] and even quantum TGD itself.

1. Almost topological QFT property means that Kähler action for the preferred extremals reduces to Chern-Simons action assuming the weak form of electric magnetic duality. In the fermionic sector one must use modified gamma matrices defined as contractions of the canonical momentum densities for Kähler action (Kähler-Chern-Simon action) with imbedding space gamma matrices in the counterpart of Dirac action in the interior of space-time sheet and at 3-D wormhole throats. The modified gamma matrices define effective metric quadratic in canonical momentum densities which is typically highly degenerate. It contains information about the induced metric. Therefore one cannot expect that topological sigma model approach could work as such in TGD framework.

2. In TGD framework supersymmetries are generated by right-handed covariantly constant neutrinos and antineutrinos with both spin directions. These spinors are imbedding space spinors rather than world sheet spinors but one can say that the induction of the spinor structure makes them world sheet spinors. Since the momentum of the spinors is vanishing, one can assign all possible spin directions to the neutrinos.

3. Covariantly constant right-handed neutrino and antineutrino can have all possible spin directions and for fixed choice of quantization axes two spin directions are possible. Therefore one could say that rotation group acts as non-Abelian group of R-symmetries. TGD formulation need not be based on sigma model so that it is not all clear whether a twisted Lorenz transformations are needed. If so, the most obvious guess is that space-time rotations are accompanied by R-symmetry rotation of right-handed neutrino spinors compensating the ordinary rotation it as in the case of topological sigma model originally introduced by Witten.

It is interesting to look the situation also from the point of view of the breaking of SUSY for supergravity defined in dimension 8 by using the table listing super-gravities in various dimensions [B10].

1. One can assign to the causal diamond a fixed direction as a WCW correlate for the fixing of spin quantization axis and this direction corresponds to a particular modulus. The preferred time directiond defined by the line connecting the tips of CD and this direction define a plane of non-physical polarizations having in number theoretical approach as a preferred hypercomplex plane of hyper-octonions [K72]. Hence it would seem that by the symmetry breaking by the choice of quantization axes allows only two spin directions the right handed neutrino and antineutrino and that different choices of the quantization axes correspond to different values for the moduli space of CDs.

2. Since imbedding space spinors are involved, the sugra counterpart of TGD is $\mathcal{N}=2$ super gravity in dimension 8 for which super charges are Dirac spinors and their hermitian conjugates with $U(2)$ acting as R-symmetries. Note that the supersymmetry does not require Majorana spinors unlike $N=1$ supersymmetry does in string model and fixes the target space dimension to $D=10$ or $D=11$. Just like $D=11$ of M-theory is the unique maximal dimension if one requires fundamental Majorana spinors (for which there is no empirical support), $D=8$ of TGD is the unique maximal dimension if one allows only Dirac spinors.

3. In dimensional reduction to $D=6$, which is the metric dimension of the boundary of δCD a breaking of $\mathcal{N}=8$ sugra $\mathcal{N}=(2,2)$ sugra occurs, and one obtains decomposition into pseudoreal representations with supercharges in representations (4,0) and (0,4) of $R=Sp(2)\times Sp(2)$ ($Sp(2)=Sl(2,R)$ corresponds to 2-D symplectic transformations identifiable also as Lorentz group SO(1,2)). (4,0) and (0,4) could correspond to left and right handed neutrinos with both directions of helicities and thus potentially massive. CP_2 geometry breaks this supersymmetry.

4. The reduction to the level of right handed neutrinos requires a further symmetry breaking and $D=5$ sugra indeed contains supercharges Q and their conjugates in 4-D pseudoreal representation of $R=Sp(4)$. Note that this group corresponds to 2×2 quaternionic matrices.

A possible interpretation would be as a reduction in CP_2 degrees freedom to $U(2) \times U(1)$ invariant sphere.

5. The R-symmetries mixing neutrinos and antineutrinos are pysically questionable so that a breaking of R-symmetry to $Sp(2) \times Sp(2)$ to $SU(2) \times SU(2)$ or even $SU(2)$ should take place. A further reduction to homologically non-trivial geodesic sphere of CP_2 might reduce the action of $CP_2(2)$ holonomies to those generated by electric charge and weak isospin and thus leaving right-handed neutrinos invariant. Fixing the quantization axis of spin would reduce R-symmetry to $U(1)$. The inverse imaged of this geodesic sphere is identified as string world sheet [K34].

How braided Galois homology and Gromov-Witten type homology and WCW spinor fields could relate?

One can distinguish between WCW "orbital" degrees of freedom and fermionic degrees of freedom and in the case of WCW degrees of freedom also between zero modes expressible in terms of Kähler fluxes and quantum fluctuating degrees of freedom expressible using wave functions in symplectic group.

1. Quantum fluctuating degrees of freedom

As far as quantum number are considered, quantum fluctuating degrees of freedom correspond to the symplectic algebra in the basis defined by Hamiltonians belonging to the irreps of rotation group and color group.

1. At the level of partonic 2-surfaces finite measurement resolution leads to discretization in terms of braid ends and symplectic triangulation. At the level of WCW discretization replaces symplectic group with its discrete subgroup. This discrete subgroup must result as a coset space defined by the subgroup of symplectic group acting as Galois group in the set of braid points and its normal subgroup leaving them invariant. The group algebra of this discrete subgroup of symplectic group would have interpretation in terms of braided Galois cohomology. This picture provides an elegant realization for finite measurement resolutions and there is also a connection with the realization of finite measurement resolution using categorification [A114], [K14].

2. The proposed generalized homology theory involving braided Galois group and symplectic group of $\delta M^4_\pm \times CP_2$ would realize the "almost" in TGD as almost topological QFT in finite measurement resolution replacing symplectic group with its discretized version. This algebra would relate to the quantum fluctuating degrees of freedom. The braids would carry only fermion number and there would be no Hamiltonians attached with them. The braided Galois homology could define in the more general situation invariants of symplectic isotopies.

3. The generalization of Gromov-Witten invariants to n-point functions defined by Hamiltonians of $\delta M^4_\pm \times CP_2$ are symplectic invariants if net $\delta M^4_\pm \times CP_2$ quantum numbers vanish. As As a special case one obtains Gromove-Witten invariants. The most general definition assumes that the vanishing of quantum numbers occurs only for zero energy states having disjoint unions of partonic 2-surfaces at the boundaries of CDs as geometric correlate. Since Hamiltonians correspond to quantum fluctuating degrees of freedom the interpretation in terms of zero modes is not not possible. The comparison of Floer homology with quantum TGD encourages to think that the generalizations of Gromov-Witten invariants can be assigned to the braided Galois homology.

4. One should also add four-momenta and twistors to this picture. The separation of dynamical fermionic and sup-symplectic degrees of freedom suggests that the Fourier transforms for amplitudes containing the fermionic braid end points as arguments define twistorial amplitudes. The representations of light-like momenta using twistors would lead to a generalization of the twistor formalism. At zero momentum limit one would obtain symplectic QFT with states characterized by collections of Hamiltonians and their super-counterparts.

2. Zero modes

WCW spinor field depends also on zero modes and the challenge is to identify the appropriate variables coding for this information in accordance with quantum classical correspondence. The best that one could achieve would be a basis for the parts of WCW spinor fields in these degrees of freedom. Zero modes correspond essentially to the non-local symplectic invariants assignable to the projections of the $\delta M^4_\pm$ and CP_2 symplectic forms to the space-time surface and expressible in terms of symplectic fluxes only. The appropriate symplectic fluxes should be determined by the information about the quantum state in quantum fluctuating degrees of freedom by quantum classical correspondence.

1. The exponent of Kähler action for preferred extremal- by above proposal real in Euclidian regions and imaginary in Minkowskian regions and reducing to Chern-Simons action at both sides - contains also information about zero modes and would code implicitly the vacuum functional in zero modes. What would be needed is an explicit representation for this part of vacuum functional. The identification of zero modes as classical variables requires entanglement between zero modes and quantum fluctuating degrees of freedom and one-one correspondence analogous to that between the states of the measurement apparatus and the outcome of quantum measurement is expected. This duality would express quantum holography and quantum classical correspondence crucial for quantum measurement theory.

2. Could the generating function for appropriately generalized Gromow-Witten invariants define a candidate for what might be regarded as a vacuum functional in zero modes separating into a factor in WCW spinor field? The first thing to notice is that symplectic invariance is not equivalent with zero mode property. In Floer homology there is a preferred Hamiltonian interpreted in TGD framework in terms of the braiding defining braided Galois homology. Neither Floer homology, Gromov-Witten invariants nor braided Galois homology do depend on the details of the Hamiltonian. Does this mean that the TGD counterparts of Gromov-Witten invariants might could be interpreted as zero modes and generating function for these invariants as vacuum functional in zero modes? Or does the fact that Hamiltonian flow is involved mean that information about quantum fluctuating degrees of freedom is present?

Symplectic QFT [K14] provides a more promising approach to the description of zero modes in terms of symplectic fluxes.

1. The earlier proposal [K14] for symplectic QFT defined as a generalization of conformal QFT coding for these degrees of freedom assigns to the partonic 2-surface collections of marked points defining its division to 2-polygons carrying Kähler magnetic flux together with the signed area defined by R^3_+ symplectic form (essentially solid angle assignable to partonic 2-surface or its portion with respect to the tip of light-cone). A given assignment of marked points defines symplectic fusion algebra and these algebras integrate to an operad with a product defined by the product of fusion algebras.

2. Symplectic triangulation would define symplectic invariants. The nodes of the symplectic triangulation could be identified as the ends of braid strands assignable to string world sheets. If the information about quantum state can be used to fix the edges of the triangulation, the phases defined by the fluxes associated with the triangles define physically interesting symplectic invariants. If one assumes that each Hamiltonian assignable to the partonic 2-surface defines its own symplectic triangulation, the Hamiltonian equations associated with the Hamiltonian would naturally define the edges of the triangulation. Symplectic triangulation would characterize a Bose-Einstein condensate like state assignable to single Hamiltonian. The total magnetic flux for the triangulation would characterize the Hamiltonian. If only single Hamiltonian is involved the orbit should be a closed orbit connecting the node to itself and also now could assign to it a symplectic area.

3. Symplectic triangulation would add additional pieces to the proposed skeleton of the space-time surface. If the symplectic triangulation can be continued from partonic 2-surfaces to the interior of space-time in both time and spatial direction it would provide space-time with a web string world sheets connected by sheets assignable to the edges of the symplectic triangulation.

12.8 K-theory, branes, and TGD

K-theory has played important role in brane classification in super string models and M-theory. The excellent lectures by Harah Evslin with title *What doesn't K-theory classify?* [B21] make it possible to learn the basic motivations for the classification, what kind of classifications are possible, and what are the failures. Also the Wikipedia article [B4] gives a bird's eye of view about problems. As a by-product one learns something about the basic ideas of K-theory and about possible mathematical and physical problems of string theories and M-theory.

In the sequel I will discuss critically the basic assumptions of brane world scenario, sum up my understanding about the problems related to the topological classification of branes and also to the notion itself, ask what goes wrong with branes and demonstrate how the problems are avoided in TGD framework, and conclude with a proposal for a natural generalization of K-theory to include also the division of bundles inspired by the generalization of Feynman diagrammatics in quantum TGD, by zero energy ontology, and by the notion of finite measurement resolution.

12.8.1 Brane world scenario

The brane world scenario looks attractive from the mathematical point of view ine one is able to get accustomed with the idea that basic geometric objects have varying dimensions. Even accepting the varying dimensions, the basic physical assumptions behind this scenario are vulnerable to criticism.

1. Branes are geometric objects of varying dimension in the 10-/11-dimensional space-time -call it M- of superstring theory/M-theory. In M-theory the fundamental strings are replaced with M-branes, which are 2-D membranes with 3-dimensional orbit having as its magnetic dual 6-D M5-brane. Branes are thought to emerge non-perturbatively from fundamental 2-branes but what this really means is not understood. One has D-p-branes with Dirichlet boundary conditions fixing a $p+1$-dimensional surface of M as brane orbit: one of the dimensions corresponds to time. Also S-branes localized in time have been proposed.

2. In the description of the classical limit branes interact with the classical fields of the target space by the generalization of the minimal coupling of charged point-like particle to electromagnetic gauge potential. The coupling is simply the integral of the gauge potential over the world-line - the value of 1-form for the word-line. Point like particle represents 0-brane and in the case of p-brane the generalization is obtained by replacing the gauge potential represented by a 1-from with $p+1$-form. The exterior derivative of this $p+1$-form is $p+2$-form representing the analog of electromagnetic field. Complete dimensional democracy strongly suggests that string world sheets should be regarded as 1-branes.

3. From TGD point of view the introduction of branes looks a rather ad hoc trick. By generalizing the coupling of electromagnetic gauge potential to the word line of point like particle one could introduce extended objects of various dimensions also in the ordinary 4-D Maxwell theory but they would be always interpreted as idealizations for the carriers of 4- currents. Therefore the crucial step leading to branes involves classical idealization in conflict with Uncertainty Principle and the genuine quantal description in terms of fields coupled to gauge potentials.

   My view is that the most natural interpretation for what is behind branes is in terms of currents in D=10 or D= 11 space-time. In this scheme branes have role only as semi-classical idealizations making sense only above some scale. Both the reduction of string theories to quantum field theories by holography and the dynamical character of the metric of the target space conforms with super-gravity interpretation. Internal consistency requires also the identification of strings as branes so that superstring theories and M-theory would reduce to an idealization to 10-/11-dimensional quantum gravity.

In this framework the brave brane world episode would have been a very useful Odysseia. The possibility to interpret various geometric objects physically has proved to be an extremely powerful tool for building provable conjectures and has produced lots of immensely beautiful mathematics. As a fundamental theory this kind of approach does not look convincing to me.

12.8.2 The basic challenge: classify the conserved brane charges associated with branes

One can of course forget these critical arguments and look whether this general picture works. The first thing that one can do is to classify the branes topologically. I made the same question about 32 years ago in TGD framework: I thought that cobordism for 3-manifolds might give highly interesting topological conservation laws. I was disappointed. The results of Thom's classical article about manifold cobordism demonstrated that there is no hope for really interesting conservation laws. The assumption of Lorentz cobordism meaning the existence of global time-like vector field would make the situation more interesting but this condition looked too strong and I could not see a real justification for it. In generalized Feynman diagrammatics there is no need for this kind of condition.

There are many alternative approaches to the classification problem. One can use homotopy, homology, cohomology and their relative and other variants, topological or algebraic K-theory, twisted K-theory, and variants of K-theory not yet existing but to be proposed within next years. The list is probably endless unless something like motivic cohomology brings in enlightment.

1. First of all one must decide whether one classifies p-dimensional time=constant sections of p-branes or their $p+1$-dimensional orbits. Both approaches have been applied although the first one is natural in the standard view about spontaneous compactification. For the first option topological invariants could be seen as conserved charges: homotopy invariants and homological and cohomological characteristics of branes provide this kind of invariants. For the latter option the invariants would be analogous to instanton number characterizing the change of magnetic charge.

2. Purely topological invariants come first in mind. Homotopy groups of the brane are invariants inherent to the brane (the brane topology can however change). Homological and cohomological characteristics of branes in singular homology characterize the imbedding to the target space. There are also more delicate differential topological invariants such as de Rham cohomology defining invariants analogous to magnetic charges. Dolbeault cohomology emerges naturally for even-dimensional branes with complex structure.

3. Gauge theories - both abelian and non-Abelian - define a standard approach to the construction of brane charges for the bundle structures assigned with branes. Chern-Simons classes are fundamental invariants of this kind. Also more delicate invariants associated with gauge potentials can be considered. Chern-Simons theory with vanishing field strengths for solutions of field equations provides a basic example about this. For instance, SU(2) Chern-Simons theory provides 3-D topological invariants and knot invariants.

4. More refined approaches involve K-theory -closely related to motivic cohomology - and its twisted version. The idea is to reduce the classification of branes to the classification of the bundle structures associated with them. This approach has had remarkable successes but has also its short-comings.

The challenge is to find the mathematical classification which suits best the physical intuitions (, which might be fatally wrong as already proposed) but is universal at the same time. This challenge has turned out to be tough. The Ramond-Ramond (RR) p-form fields of type II superstring theory are rather delicate objects and a source of most of the problems. The difficulties emerge also by the presence of Neveu-Schwartz 3-form $H = dB$ defining classical background field.

K-theory has emerged as a good candidate for the classification of branes. It leaves the confines of homology and uses bundle structures associated with branes and classifies these. There are many K-theories. In topological K-theory bundles form an algebraic structure with sum, difference, and multiplication. Sum is simply the direct sum for the fibers of the bundle with common base space. Product reduces to a tensor product for the fibers. The difference of bundles represents a more abstract notion. It is obtained by replacing bundles with pairs in much the same way as rationals can be thought of as pairs of integers with equivalence $(m, n) = (km, kn)$, k integer. Pairs $(n, 1)$ representing integers and pairs $(1, n)$ their inverses. In the recent case one replaces multiplication with sum and regards bundle pairs and (E, F) and $(E + G, F + G)$ equivalent. Although the pair as such remains a formal notion, each pair must have also a real world representativs. Therefore the sign for the bundle must have meaning and corresponds to the sign of the charges assigned to the bundle. The charges are analogous to winding of the brane and one can call brane with negative winding antibrane. The interpretation in terms of orientation looks rather natural. Later a TGD inspired concrete interpretation for the bundle sum, difference, product and also division will be proposed.

12.8.3 Problems

The classification of brane structures has some problems and some of them could be argued to be not only technical but reflect the fact that the physical picture is wrong.

Problems related to the existence of spinor structure

Many problems in the classification of brane charges relate to the existence of spinor structure. The existence of spinor structure is a problem already in general general relativity since ordinary spinor structure exists only if the second Stiefel- Whitney class [A89] of the manifold is non-vanishing: if the third Stiefel-Whitney class vanishes one can introduce so called spin^c structure. This kind of problems are encountered already in lattice QCD, where periodic boundary conditions imply non-uniqueness having interpretation in terms of 16 different spinor structures with no obvious physical interpretation. One the strengths of TGD is that the notion of induced spinor structure eliminates all problems of this kind completely. One can therefore find direct support for TGD based notion of spinor structure from the basic inconsistency of QCD lattice calculations!

1. Freed-Witten anomaly [B22] appearing in type II string theories represents one of the problems. Freed and Witten show that in the case of 2-branes for which the generalized gauge potential is 3-form so called spin^c structure is needed and exists if the third Stiefel-Whitney class w_3 related to second Stiefel Whitney class whose vanishing guarantees the existence of ordinary spin structure (in TGD framework spin^c structure for CP_2 is absolutely essential for obtaining standard model symmetries).

   It can however happen that w_3 is non-vanishing. In this case it is possible to modify the spin^c structure if the condition $w_3 + [H] = 0$ holds true. It can however happen that there is an obstruction for having this structure - in other words $w_3 + [H]$ does not vanish - known as Freed-Witten anomaly. In this case K-theory classification fails. Witten and Freed argue that physically the wrapping of cycle with non-vanishing $w_3 + [H]$ by a Dp-brane requires the presence of $D(p-2)$ brane cancelling the anomaly. If $D(p-2)$ brane ends to anti-Dp in which case charge conservation is lost. If there is not place for it to end one has semi-infinite brane with infinite mass, which is also problematic physically. Witten calls these branes baryons: these physically very dubious objects are not classified by K-theory.

2. The non-vanishing of $w_3 + [H] = 0$ forces to generalize K-theory to twisted K-theory [A98]. This means a modification of the exterior derivative to get twisted de Rham cohomology and twisted K-theory and the condition of closedness in this cohomology for certain form becomes the condition guaranteeing the existence of the modified spin^c structure. D-branes act as sources of these fields and the coupling is completely analogous to that in electrodynamics. In the presence of classical Neveu-Schwartz (NS-NS) 3-form field H associated with the background geometry the field strength $G^{p+1} = dC_p$ is not gauge invariant anymore. One must replace the exterior derivative with its twisted version to get twisted de Rham cohomology:

   $$d \rightarrow d + H\wedge \quad .$$

   There is a coupling between p- and p+2-forms together and gauge symmetries must be modified accordingly. The fluxes of twisted field strengths are not quantized but one can return to original p-forms which are quantized. The coupling to external sources also becomes more complicated and in the case of magnetic charges one obtains magnetically charged Dp-branes. Dp-brane serves as a source for $D(p-2)$- branes.

   This kind of twisted cohomology is known by mathematicians as Deligne cohomology. At the level of homology this means that if branes with dimension of p are presented then also branes with dimension $p + 2$ are there and serve as source of Dp-branes emanating from them or perhaps identifiable as their sub-manifolds. Ordinary homology fails in this kind of situation and the proposal is that so called twisted K-theory could allow to classify the brane charges.

3. A Lagrangian formulation of brane dynamics based on the notion of p-brane democracy [B37] due to Peter Townsend has been developed by various authors.

Ashoke Sen has proposed a grand vision for understanding the brane classification in terms of tachyon condensation in absence of NS-NS field H [B36]. The basic observation is that stacks of space-filling D- and anti D-branes are unstable against process called tachyon condensation which however means fusion of $p + 1$-D brane orbits rather than p-dimensional time slices of branes. These branes are however accompanied by lower-dimensional branes and the decay process cannot destroy these. Therefore the idea arises that suitable stacks of D9 branes and anti-D9-branes could code for all lower-dimensional brane configurations as the end products of the decay process.

This leads to a creation of lower-dimensional branes. All decay products of branes resulting in the decay cascade would be by definition equivalent. The basic step of the decay process is the fusion of D-branes in stack to single brane. In bundle theoretic language one can say that the D-branes and anti-D branes in the stack fuse together to single brane with bundle fiber which is direct sum of the fibers on the stack. This fusion process for the branes of stack would correspond in topological K-theory. The fusion of D-branes and anti-D branes would give rise to nothing since the fibers would have opposite sign. The classification would reduce to that for stacks of D9-branes and anti D9-branes.

Problems with Hodge duality and S-duality

The K-theory classification is plagued by problems all of which need not be only technical.

1. R-R fields are self dual and since metric is involved with the mapping taking forms to their duals one encounters a problem. Chern characters appearing in K-theory are rational valued but the presence of metric implies that the Chern characters for the duals need not be rational valued. Hence K-theory must be replaced with something less demanding.

   The geometric quantization inspired proposal of Diaconescu, Moore and Witten [B16] is based on the polarization using only one half of the forms to get rid of the proboem. This is like thinking the 10-D space-time as phase space and reducing it effectively to 5-D space: this brings strongly in mind the identification of space-time surfaces as hyper-quaternionic (associative) sub-manifolds of imbedding space with octonionic structure and one can ask whether the basic objects also in M-theory should be taken 5-dimensional if this line of thought is taken seriously. An alternative approach uses K-theory to classify the intersections of branes with 9-D space-time slice as has been porposed by Maldacena, Moore and Seiberg [B28].

2. There another problem related to classification of the brane charges. Witten, Moore and Diaconescu [B16] have shown that there are also homology cycles which are unstable against decay and this means that twisted K-theory is inconsistent with the S-duality of type IIB string theory. Also these cycles should be eliminated in an improved classification if one takes charge conservation as the basic condition and an hitherto un-known modification of cohomology theory is needed.

3. There is also the problem that K-theory for time slices classifies only the R-R field strengths. Also R-R gauge potentials carry information just as ordinary gauge potentials and this information is crucial in Chern-Simons type topological QFTs. K-theory for entire target space classifies D-branes as $p+1$-dimensional objects but in this case the classification of R-R field strengths is lost.

The existence of non-representable 7-D homology classes for targent space dimension $D > 9$

There is a further nasty problem which destroys the hopes that twisted K-theory could provide a satisfactory classification. Even worse, something might be wrong with the superstring theory itself.

The problem is that not all homology classes allow a representation as non-singular manifolds. The first dimension in which this happens is $D = 10$, the dimension of super-string models! Situation is of course the same in M-theory. The existence of the non-representables was demonstrated by Thom - the creator of catastrophe theory and of cobordism theory for manifolds- for a long time ago.

What happens is that there can exist 7-D cycles which allow only singular imbeddings. A good example would be the imbedding of twistor space CP_3, whose orbit would have conical singularity for which CP_3 would contract to a point at the "moment of big bang". Therefore homological classification not only allows but demands branes which are orbifolds. Should orbifolds be excluded as unphysical? If so then homology gives too many branes and the singular branes must be excluded by replacing the homology with something else. Could twisted K-theory exclude non-representable branes as unstable ones by having non-vanishing $w_3 + [H]$? The answer to the question is negative: D6-branes with $w_3 + [H] = 0$ exist for which K-theory charges can be both vanishing or non-vanishing.

One can argue that non-representability is not a problem in superstring models (M-theory) since spontaneous compactification leads to $M \times X_6$ ($M \times X_7$). On the other hand, Cartesian product topology is an approximation which is expected to fail in high enough length scale resolution and near big bang so that one could encounter the problem. Most importantly, if M-theory is theory of everything it cannot contain this kind of beauty spots.

12.8.4 What could go wrong with super string theory and how TGD circumvents the problems?

As a proponent of TGD I cannot avoid the temptation to suggest that at least two things could go wrong in the fundamental physical assumptions of superstrings and M-theory.

1. The basic failure would be the construction of quantum theory starting from semiclassical approximation assuming localization of currents of 10 - or 11-dimensional theory to lower-dimensional sub-manifolds. What should have been a generalization of QFT by replacing point-like particles with higher-dimensional objects would reduce to an approximation of 10- or 11-dimensional supergravity.

   This argument does not bite in TGD. 4-D space-time surfaces are indeed fundamental objects in TGD as also partonic 2-surfaces and braids. This role emerges purely number theoretically inspiring the conjecture that space-time surfaces are associative sub-manifolds of octonionic imbedding spaces, from the requirement of extended conformal invariance, and from the non-dynamical character of the imbedding space.

2. The condition that all homology equivalence classes are representable as manifolds excludes all dimensions $D > 9$ and thus super-strings and M-theory as a physical theory. This would be the case since branes are unavoidable in M-theory as is also the landscape of compactifications. In semiclassical supergravity interpretation this would not be catastrophe but if branes are fundamental objects this shortcoming is serious. If the condition of homological representability is accepted then target space must have dimension $D < 10$ and the arguments sequence leading to D=8 and TGD is rather short. The number theoretical vision provides the mathematical justification for TGD as the unique outcome.

3. The existence of spin structure is clearly the source of many problems related to R-R form. In TGD framework the induction of spin^c structure of the imbedding space resolves all problems associated with sub-manifold spin structures. For some reason the notion of induced spinor structure has not gained attention in super string approach.

4. Conservative experimental physicist might criticize the emergence of branes of various dimensions as something rather weird. In TGD framework electric-magnetic duality can be understood in terms of general coordinate invariance and holography and branes and their duals have dimension 2, 3, and 4 organize to sub-manifolds of space-time sheets. The TGD counterpart for the fundamental M-2-brane is light-like 3-surface. Its magnetic dual has dimension given by the general formula $p_{dual} = D - p - 4$, where D is the dimension of the target space [B20]. In TGD one has $D = 8$ giving $p_{dual} = 2$. The first interpretation is in

terms of self-duality. A more plausible interpretation relies on the identification of the duals of light-like 3-surfaces as space-like 3-surfaces at the light-like boundaries of CD. General Coordinate Invariance in strong sense implies this duality. For partonic 2-surface one would have $p = 1$ and $p_{dual} = 3$. The identification of the dual would be as space-time surface. The crucial distinction to M-theory would be that branes of different dimension would be sub-manifolds of space-time surface.

5. For $p = 0$ one would have $p_{dual} = 4$ assigning five-dimensional surface to orbits of point-like particles identifiable most naturally as braid strands. One cannot assign to it any direct physical meaning in TGD framework and gauge invariance for the analogs of brane gauge potentials indeed excludes even-dimensional branes in TGD since corresponding forms are proportional to Kähler gauge potential (so that they would be analogous to odd-dimensional branes allowed by type II_B superstrings).

   4-branes could be however mathematically useful by allowing to define Morse theory for the critical points of the Minkowskian part of Kähler action. While writing this I learned that Witten has proposed a 4-D gauge theory approach with $\mathcal{N} = 4$ SUSY to the classification of knots. Witten also ends up with a Morse theory using 5-D space-times in the category-theoretical formulation of the theory [A153]. For some time ago I also proposed that TGD as almost topological QFT defines a theory of knots, knot braidings, and of 2-knots in terms of string world sheets [K34]. Maybe the 4-branes could be useful for understanding of the extrema of TGD of the Minkowskian part of Kähler action which would take take the same role as Hamiltonian in Floer homology: the extrema of 5-D brane action would connect these extrema.

6. Light-like 3-surfaces could be seen as the analogs von Neuman branes for which the boundary conditions state that the ends of space-like 3-brane defined by the partonic 2-surfaces move with light-velocity. The interpretation of partonic 2-surfaces as space-like branes at the ends of CD would in turn make them D-branes so that one would have a duality between D-branes and N-brane interpretations. T-duality exchanges von Neumann and Dirichlet boundary conditions so that strong from of general coordinate invariance would correspond to both electric-magnetic and T-duality in TGD framework. Note that T-duality exchanges type II_A and type II_B super-strings with each other.

7. What about causal diamonds and their 7-D light-like boundaries? Could one regard the light-like boundaries of CDs as analogs of 6-branes with light-like direction defining time-like direction so that space-time surfaces would be seen as 3-branes connecting them? This brane would not have magnetic dual since the formula for the dimensions of brane and its magnetic dual allows positive brane dimension p only in the range (1,3).

12.8.5 Can one identify the counterparts of R-R and NS-NS fields in TGD?

R-R and NS-NS 3-forms are clearly in fundamental role in M-theory. Since in TGD partonic 2-surfaces define the analogs of fundamental M-2-branes, one can wonder whether these 3-forms could have TGD counterparts.

1. In TGD framework the 3-forms $G_{3,A} = dC_{2,A}$ defined as the exterior derivatives of the two-forms $C_{2,A}$ identified as products $C_{2,A} = H_A J$ of Hamiltonians H_A of $\delta M^4_\pm \times CP_2$ with Kähler forms of factors of $\delta M^4_\pm \times CP_2$ define an infinite family of closed 3-forms belonging to various irreducible representations of rotation group and color group. One can consider also the algebra generated by products $H_A A$, $H_A J$, $H_A A \wedge J$, $H_A J \wedge J$, where A *resp.* J denotes the Kähler gauge potential *resp.* Kähler form or either $\delta M^4_\pm$ or CP_2. A *resp.* Also the sum of Kähler potentials *resp.* forms of $\delta M^4_\pm$ and CP_2 can be considered.

2. One can define the counterparts of the fluxes $\int Adx$ as fluxes of $H_A A$ over braid strands, $H_A J$ over partonic 2-surfaces and string world sheets, $H_A A \wedge J$ over 3-surfaces, and $H_A J \wedge J$ over space-time sheets.Gauge invariance however suggests that for non-constant Hamiltonians one must exclude the fluxes assigned to odd dimensional surfaces so that only odd-dimensional

branes would be allowed. This would exclude 0-branes and the problematic 4-branes. These fluxes should be quantized for the critical values of the Minkowskian contributions and for the maxima with respect to zero modes for the Euclidian contributions to Kähler action. The interpretation would be in terms of Morse function and Kähler function if the proposed conjecture holds true. One could even hope that the charges in Cartan algebra are quantized for all preferred extremals and define charges in these irreducible representations for the isometry algebra of WCW. The quantization of electric fluxes for string world sheets would give rise to the familiar quantization of the rotation $\int E \cdot dl$ of electric field over a loop in time direction taking place in superconductivity.

3. Should one interpret these fluxes as the analogs of NS-NS-fluxes or R-R fluxes? The exterior derivatives of the forms G_3 vanish which is the analog for the vanishing of magnetic charge densities (it is however possible to have the analogs of homological magnetic charge). The self-duality of Ramond p-forms could be posed formally ($G_p =^* G_{8-p}$) but does not have any implications for $p < 4$ since the space-time projections vanish in this case identically for $p > 3$. For $p = 4$ the dual of the instanton density $J \wedge J$ is proportional to volume form if M^4 and is not of topological interest. The approach of Witten eliminating one half of self dual R-R-fluxes would mean that only the above discussed series of fluxes need to be considered so that one would have no troubles with non-rational values of the fluxes nor with the lack of higher dimensional objects assignable to them. An interesting question is whether the fluxes could define some kind of K-theory invariants.

4. In TGD imbedding space is non-dynamical and there seems to be no counterpart for the NS 3-form field $H = dB$. The only natural candidate would correspond to Hamiltonian $B = J$ giving $H = dB = 0$. At quantum level this might be understood in terms of bosonic emergence [K53] meaning that only Ramond representations for fermions are needed in the theory since bosons correspond to wormhole contacts with fermion and anti-fermions at opposite throats. Therefore twisted cohomology is not needed and there is no need to introduce the analogy of brane democracy and 4-D space-time surfaces containing the analogs of lower-dimensional brains as sub-manifolds are enough. The fluxes of these forms over partonic 2-surfaces and string world sheets defined non-abelian analogs of ordinary gauge fluxes reducing to rotations of vector potentials and suggested be crucial for understanding braidings of knots and 2-knots in TGD framework. [K34]. Note also that the unique dimension D=4 for space-time makes 4-D space-time surfaces homologically self-dual so that only they are needed.

12.8.6 What about counterparts of S and U dualities in TGD framework?

The natural question is what could be the TGD counterparts of $S-$, $T-$ and U-dualities. If one accepts the identification of U-duality as product $U = ST$ and the proposed counterpart of T duality as a strong form of general coordinate invariance, it remains to understand the TGD counterpart of S-duality - in other words electric-magnetic duality - relating the theories with gauge couplings g and $1/g$. Quantum criticality selects the preferred value of g_K: Kähler coupling strength is very near to fine structure constant at electron length scale and can be equal to it. Since there is no coupling constant evolution associated with α_K, it does not make sense to say that g_K becomes strong and is replaced with its inverse at some point. One should be able to formulate the counterpart of S-duality as an identity following from the weak form of electric-magnetic duality and the reduction of TGD to almost topological QFT. This seems to be the case.

1. For preferred extremals the interior parts of Kähler action reduces to a boundary term because the term $j^\mu A_\mu$ vanishes. The weak form of electric-magnetic duality requires that Kähler electric charge is proportional to Kähler magnetic charge, which implies reduction to abelian Chern-Simons term: the Kähler coupling strength does not appear at all in Chern-Simons term. The proportionality constant beween the electric and magnetic parts J_E and J_B of Kähler form however enters into the dynamics through the boundary conditions stating the weak form of electric-magnetic duality. At the Minkowskian side the proportionality constant must be proportional to g_K^2 to guarantee a correct value for the unit of Kähler electric charge

- equal to that for electric charge in electron length scale- from the assumption that electric charge is proportional to the topologically quantized magnetic charge. It has been assumed that

$$J_E = \alpha_K J_B$$

holds true at *both sides* of the wormhole throat but this is an un-necessarily strong assumption at the Euclidian side. In fact, the self-duality of CP_2 Kähler form stating

$$J_E = J_B$$

favours this boundary condition at the Euclidian side of the wormhole throat. Also the fact that one cannot distinguish between electric and magnetic charges in Euclidian region since all charges are magnetic can be used to argue in favor of this form. The same constraint arises from the condition that the action for CP_2 type vacuum extremal has the value required by the argument leading to a prediction for gravitational constant in terms of the square of CP_2 radius and α_K the effective replacement $g_K^2 \rightarrow 1$ would spoil the argument.

2. Minkowskian and Euclidian regions should correspond to a strongly/weakly interacting phase in which Kähler magnetic/electric charges provide the proper description. In Euclidian regions associated with CP_2 type extremals there is a natural interpretation of interactions between magnetic monopoles associated with the light-like throats: for CP_2 type vacuum extremal itself magnetic and electric charges are actually identical and cannot be distinguished from each other. Therefore the duality between strong and weak coupling phases seems to be trivially true in Euclidian regions if one has $J_B = J_E$ at Euclidian side of the wormhole throat. This is however an un-necessarily strong condition as the following argument shows.

3. In Minkowskian regions the interaction is via Kähler electric charges and elementary particles have vanishing total Kähler magnetic charge consisting of pairs of Kähler magnetic monopoles so that one has confinement characteristic for strongly interacting phase. Therefore Minkowskian regions naturally correspond to a weakly interacting phase for Kähler electric charges. One can write the action density at the Minkowskian side of the wormhole throat as

$$\frac{(J_E^2 - J_B^2)}{\alpha_K} = \alpha_K J_B^2 - \frac{J_B^2}{\alpha_K} \ .$$

The exchange $J_E \leftrightarrow J_B$ accompanied by $\alpha_K \rightarrow -1/\alpha_K$ leaves the action density invariant. Since only the behavior of the vacuum functional infinitesimally near to the wormhole throat matters by almost topological QFT property, the duality is realized. Note that the argument goes through also in Euclidian regions so that it does not allow to decide which is the correct form of weak form of electric-magnetic duality.

4. S-duality could correspond geometrically to the duality between partonic 2-surfaces responsible for magnetic fluxes and string worlds sheets responsible for electric fluxes as rotations of Kähler gauge potentials around them and would be very closely related with the counterpart of T-duality implied by the strong form of general coordinate invariance and saying that space-like 3-surfaces at the ends of space-time sheets are equivalent with light-like 3-surfaces connecting them.

The boundary condition $J_E = J_B$ at the Euclidian side of the wormhole throat inspires the question whether all Euclidian regions could be self-dual so that the density of Kähler action would be just the instanton density. Self-duality follows if the deformation of the metric induced by the deformation of the canonically imbedded CP_2 is such that in CP_2 coordinates for the Euclidian region the tensor $(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\mu\beta})/\sqrt{g}$ remains invariant. This is certainly the case for CP_2 type vacuum extremals since by the light-likeness of M^4 projection the metric remains invariant. Also conformal scalings of the induced metric would satisfy this condition. Conformal scaling is not consistent with the degeneracy of the 4-metric at the wormhole throat. Self-duality is indeed an un-necessarily strong condition.

Comparison with standard view about dualities

One can compare the proposed realization of T, S and U to the more general dualities defined by the modular group $SL(2,Z)$, which in QFT framework can hold true for the path integral over all possible gauge field configurations. In the resent case the dualities hold true for every preferred extremal separately and the functional integral is only over the space-time projections of fixed Kähler form of CP_2. Modular invariance for Maxwell action was discussed by E. Verlinde for Maxwell action with θ term for a general 4-D compact manifold with Euclidian signature of metric in [B38]. In this case one has path integral giving sum over infinite number of extrema characterized by the cohomological equivalence class of the Maxwell field the action exponential to a high degree. Modular invariance is broken for CP_2: one obtains invariance only for $\tau \rightarrow \tau + 2$ whereas S induces a phase factor to the path integral.

1. In the recent case these homology equivalence classes would correspond to homology equivalence classes of holomorphic partonic 2-surfaces associated with the critical points of Kähler function with respect to zero modes.

2. In the case that the Euclidian contribution to the Kähler action is expressible solely in terms of wormhole throat Chern-Simons terms, and one can neglect the measurement interaction terms fixing the values of some classical conserved quantities to be equal with their quantal counterparts for the space-time surfaces allowed in quantum superposition, the exponent of Kähler action can be expressed in terms of Chern-Simons action density as

$$
\begin{aligned}
L &= \tau L_{C-S} \ , \\
L_{C-S} &= J \wedge A \ , \\
\tau &= \frac{1}{g_K^2} + i\frac{k}{4\pi} \ , \quad k = 1 \ . \qquad (12.8.\text{-}1)
\end{aligned}
$$

Here the parameter τ transforms under full $SL(2,Z)$ group as

$$
\tau \rightarrow \frac{a\tau + b}{c\tau + d} \ . \qquad (12.8.0)
$$

The generators of $SL(2,Z)$ transformations are $T: \tau \rightarrow \tau + 1$, $S: \tau \rightarrow -1/\tau$. The imaginary part in the exponents corresponds to Kac-Moody central extension $k = 1$.

This form corresponds also to the general form of Maxwell action with CP breaking θ term given by

$$
L = \frac{1}{g_K^2} J \wedge^* J + i\frac{\theta}{8\pi^2} J \wedge J \ , \quad \theta = 2\pi \ . \qquad (12.8.1)
$$

Hence the Minkowskian part mimics the θ term but with a value of θ for which the term does not give rise to CP breaking in the case that the action is full action for CP_2 type vacuum extremal so that the phase equals to 2π and phase factor case is trivial. It would seem that the deviation from the full action for CP_2 due to the presence of wormhole throats reducing the value of the full Kähler action for CP_2 type vacuum extremal could give rise to CP breaking. One can visualize the excluded volume as homologically non-trivial geodesic spheres with some thickness in two transverse dimensions. At the limit of infinitely thin geodesic spheres CP breaking would vanish. The effect is exponentially sensitive to the volume deficit.

CP breaking and ground state degeneracy

Ground state degeneracy due to the possibility of having both signs for Minkowskian contribution to the exponent of vacuum functional provides a general view about the description of CP breaking in TGD framework.

1. In TGD framework path integral is replaced by inner product involving integral over WCV. The vacuum functional and its conjugate are associated with the states in the inner product so that the phases of vacuum functionals cancel if only one sign for the phase is allowed. Minkowskian contribution would have no physical significance. This of course cannot be the case. The ground state is actually degenerate corresponding to the phase factor and its complex conjugate since $\sqrt{g}$ can have two signs in Minkowskian regions. Therefore the inner products between states associated with the two ground states define 2×2 matrix and non-diagonal elements contain interference terms due to the presence of the phase factor. At the limit of full CP_2 type vacuum extremal the two ground states would reduce to each other and the determinant of the matrix would vanish.

2. A small mixing of the two ground states would give rise to CP breaking and the first principle description of CP breaking in systems like $K - \overline{K}$ and of CKM matrix should reduce to this mixing. K^0 mesons would be CP even and odd states in the first approximation and correspond to the sum and difference of the ground states. Small mixing would be present having exponential sensitivity to the actions of CP_2 type extremals representing wormhole throats. This might allow to understand qualitatively why the mixing is about 50 times larger than expected for B^0 mesons.

3. There is a strong temptation to assign the two ground states with two possible arrows of geometric time. At the level of M-matrix the two arrows would correspond to state preparation at either upper or lower boundary of CD. Do long- and short-lived neutral K mesons correspond to almost fifty-fifty orthogonal superpositions for the two arrow of geometric time or almost completely to a fixed arrow of time induced by environment? Is the dominant part of the arrow same for both or is it opposite for long and short-lived neutral mesons? Different lifetimes would suggest that the arrow must be the same and apart from small leakage that induced by environment. CP breaking would be induced by the fact that CP is performed only K^0 but not for the environment in the construction of states. One can probably imagine also alternative interpretations.

Remark: The proportionality of Minkowskian and Euclidian contributions to the same Chern-Simons term implies that the critical points with respect to zero modes appear for both the phase and modulus of vacuum functional. The Kähler function property does not allow extrema for vacuum functional as a function of complex coordinates of WCW since this would mean Kähler metric with non-Euclidian signature. If this were not the case. the stationary values of phase factor and extrema of modulus of the vacuum functional would correspond to different configurations.

12.8.7 Could one divide bundles?

TGD differs from string models in one important aspects: stringy diagrams do not have interpretation as analogs of vertices of Feynman diagrams: the stringy decay of partonic 2-surface to two pieces does not represent particle decay but a propagation along different paths for incoming particle. Particle reactions in turn are described by the vertices of generalized Feynman diagrams in which the ends of incoming and outgoing particles meet along partonic 2-surface. This suggests a generalization of K-theory for bundles assignable to the partonic 2-surfaces. It is good to start with a guess for the concrete geometric realization of the sum and product of bundles in TGD framework.

1. The analogs of string diagrams could represent the analog for direct sum. Difference between bundles could be defined geometrically in terms of trouser vertex $A + B \to C$. B would by definition represent $C - A$. Direct sum could make sense for single particle states and have as space-time correlate the conservation of braid strands.

2. A possible concretization in TGD framework for the tensor product is in terms of the vertices of generalized Feynman diagrams at which incoming light-like 3-D orbits of partons meet along their ends. The tensor product of incoming state spaces defined by fermionic oscillator algebras is naturally formed. Tensor product would have also now as a space-time correlate conservation of braid strands. This does not mean that the number of braid strands is conserved in reactions if also particular exchanges can carry the braid strands of particles coming to the vertex.

Why not define also division of bundles in terms of the division for tensor product? In terms of the 3-vertex for generalized Feynman diagrams $A \otimes B = C$ representing tensor product B would be by definition C/A. Therefore TGD would extend the K-theory algebra by introducing also division as a natural operation necessitated by the presence of the join along ends vertices not present in string theory. I would be surprised if some mathematician would not have published the idea in some exotic journal. Below I represent an argument that this notion could be also applied in the mathematical description of finite measurement resolution in TGD framework using inclusions of hyper-finite factor. Division could make possible a rigorous definition for for non-commutative quantum spaces.

Tensor division could have also other natural applications in TGD framework.

1. One could assign bundles M_+ and M_- to the upper and lower light-like boundaries of CD. The bundle M_+/M_- would be obtained by formally identifying the upper and lower light-like boundaries. More generally, one could assign to the boundaries of CD positive and negative energy parts of WCW spinor fields and corresponding bundle structures in "half WCW". Zero energy states could be seen as sections of the unit bundle just like infinite rationals reducing to real units as real numbers would represent zero energy states.

2. Finite measurement resolution would encourage tensor division since finite measurement resolution means essentially the loss of information about everything below measurement resolution represented as a tensor product factor. The notion of coset space formed by hyper-finite factor and included factor could be understood in terms of tensor division and give rise to quantum group like space with fractional quantum dimension in the case of Jones inclusions [K80]. Finite measurement resolution would therefore define infinite hierarchy of finite dimensional non-commutative spaces characterized by fractional quantum dimension. In this case the notion of tensor product would be somewhat more delicate since complex numbers are effectively replaced by the included algebra whose action creates states not distinguishable from each other [K80]. The action of algebra elements to the state $|B\rangle$ in the inner product $\langle A|B\rangle$ must be equivalent with the action of its hermitian conjugate to the state $\langle A|$. Note that zero energy states are in question so that the included algebra generates always modifications of states which keep it as a zero energy state.

12.9 A connection between cognition, number theory, algebraic geometry, topology, and quantum physics

I have had some discussions with Stephen King and Hitoshi Kitada in a closed discussion group about the idea that the duality between Boolean algebras and Stone spaces could be important for the understanding of consciousness, at least cognition. In this vision Boolean algebras would represent conscious mind and Stone spaces would represent the matter: space-time would emerge.

I am personally somewhat skeptic because I see consciousness and matter as totally different levels of existence. Consciousness (and information) is about something, matter just is. Consciousness involves always a change as we no from basic laws about perception. There is of course also the experience of free will and the associated non-determinism. Boolean algebra is a model for logic, not for conscious logical reasoning. There are also many other aspects of consciousness making it very difficult to take this kind of duality seriously.

I am also skeptic about the emergence of space-time say in the extremely foggy form as it used in entropic gravity arguments. Recent day physics poses really strong constraints on our view about space-time and one must take them very seriously.

This does not however mean that Stone spaces could not serve as geometrical correlates for Boolean consciousness. In fact, p-adic integers can be seen as a Stone space naturally assignable to Boolean algebra with infinite number of bits.

12.9.1 Innocent questions

I ende up with the innocent questions, as I was asked to act as some kind of mathematical consultant and explain what Stone spaces actually are and whether they could have a connection to p-adic numbers. Anyone can of course go to Wikipedia and read the article Stone's representation theorem for Boolean algebras. For a layman this article does not however tell too much.

Intuitively the content of the representation theorem looks rather obvious, at least at the first sight. As a matter fact, the connection looks so obvious that physicists often identify the Boolean algebra and its geometric representation without even realizing that two different things are in question. The subsets of given space- say Euclidian 3-space- with union and intersection as basic algebraic operations and inclusion of sets as ordering relation defined a Boolean algebra for the purposes of physicist. One can assign to each point of space a bit. The points for which the value of bit equals to one define the subset. Union of subsets corresponds to logical OR and intersection to AND. Logical implication B$\rightarrow$ A corresponds to A contains B.

When one goes to details problems begin to appear. One would like to have some non-trivial form of continuity.

1. For instance, if the sets are form open sets in real topology their complements representing negations of statements are closed, not open. This breaks the symmetry between statement and it negation unless the topology is such that closed sets are open. Stone's view about Boolean algebra assumes this. This would lead to discrete topology for which all sets would be open sets and one would lose connection with physics where continuity and differential structure are in key role.

2. Could one dare to disagree with Stone and allow both closed and open sets of E^3 in real topology and thus give up clopen assumption? Or could one tolerate the asymmetry between statements and their negations and give some special meaning for open or closet sets- say as kind of axiomatic statements holding true automatically. If so, one an also consider algebraic varieties of lower dimension as collections of bits which are equal to one. In Zariski topology used in algebraic geometry these sets are closed. Again the complements would be open. Could one regard the lower dimensional varieties as identically true statements so that the set of identically true statements would be rather scarce as compared to falsities? If one tolerates some quantum TGD, one could ask whether the 4-D quaternionic/associative varieties defining classical space-times and thus classical physics could be identified as the axiomatic truths. Associativity would be the basic truth inducing the identically true collections of bits.

12.9.2 Stone theorem and Stone spaces

For reasons which should be clear it is perhaps a good idea to consider in more detail what Stone duality says. Stone theorem states that Boolean algebras are dual with their Stone spaces. Logic and certain kind of geometry are dual. More precisely, any Boolean algebra is isomorphic to closed open subsets of some Stone space and vice versa. Stone theorem respects category theory. The homomorphisms between Boolean algebras A and B corresponds to homomorphism between Stone spaces $S(B)$ and $S(A)$: one has contravariant functor between categories of Boolean algebras and Stone spaces. In the following set theoretic realization of Boolean algebra provides the intuitive guidelines but one can of course forget the set theoretic picture altogether and consider just abstract Boolean algebra.

1. Stone space is defined as the space of homomorphisms from Boolean algebra to 2-element Boolean algebra. More general spaces are spaces of homomorphisms between two Boolean algebras. The analogy in the category of linear spaces would be the space of linear maps between two linear spaces. Homomorphism is in this case truth preserving map: $h(A\ AND\ B) = h(a)\ AND\ h(B)$, $h(\ OR\ B) = h(a)\ OR\ h(B)$ and so on.

2. For any Boolean algebra Stone space is compact, totally disconnected Hausdorff space. Conversely, for any topological space, the subsets, which are both closed and open define Boolean algebra. Note that for a real line this would give 2-element Boolean algebra. Set is closed and open simultaneously only if its boundary is empty and in p-adic context there are no boundaries. Therefore for p-adic numbers closed sets are open and the sets of p-adic numbers with p-adic norm above some lower bound and having some set of fixed pinary digits, define closed-open subsets.

3. Stone space dual to the Boolean algebra does not conform with the physicist's ideas about space-time. Stone space is a compact totally disconnected Hausdorff space. Disconnected space is representable as a union of two or more disjoint open sets. For totally disconnected space this is true for every subset. Path connectedness is stronger notion than connected and says that two points of the space can be always connected by a curve defined as a mapping of *real* unit interval to the space. Our physical space-time seems to be however connected in this sense.

4. The points of the Stone space $S(B)$ can be identified ultrafilters. Ultrafilter defines homomorphism of B to 2-element of Boolean algebra Boolean algebra. Set theoretic realization allows to understand what this means. Ultrafilter is a set of subsets with the property that intersections belong to it and if set belongs to it also sets containing it belong to it: this corresponds to the fact that set inclusion $A \supset B$ corresponds to logical implication. Either set or its complement belongs to the ultrafilter (either statement or its negation is true). Empty set does not. Ultrafilter obviously corresponds to a collection of statements which are simultaneously true without contradictions. The sets of ultrafilter correspond to the statements interpreted as collections of bits for which each bit equals to 1.

5. The subsets of B containing a fixed point b of Boolean algebra define an ultrafilter and imbedding of b to the Stone space by assigning to it this particular principal ultrafilter. b represents a statement which is always true, kind of axiom for this principal ultrafilter and ultrafilter is the set of all statements consistent with b.

   Actually any finite set in the Boolean algebra consisting of a collection of fixed bits b_i defines an ultrafilter as the set all subsets of Boolean algebra containing this subset. Therefore the space of all ultra-filters is in one-one correspondence with the space of subsets of Boolean statements. This set corresponds to the set of statements consistent with the truthness of b_i analogous to axioms.

12.9.3 2-adic integers and 2-adic numbers as Stone spaces

I was surprised to find that p-adic numbers are regarded as a totally disconnected space. The intuitive notion of connected is that one can have a continuous curve connecting two points and this is certainly true for p-adic numbers with curve parameter which is p-adic number but not for curves with real parameter which became obvious when I started to work with p-adic numbers and invented the notion of p-adic fractal. In other words, p-adic integers form a continuum in p-adic but not in real sense. This example shows how careful one must be with definitions. In any case, to my opinion the notion of path based on p-adic parameter is much more natural in p-adic case. For given p-adic integers one can find p-adic integers arbitrary near to it since at the limit $n \to \infty$ the p-adic norm of p^n approaches zero. Note also that most p-adic integers are infinite as real integers.

Disconnectedness in real sense means that 2-adic integers define an excellent candidate for a Stone space and the inverse of the Stone theorem allows indeed to realize this expectation. Also 2-adic numbers define this kind of candidate since 2-adic numbers with norm smaller than 2^n for any n can be mapped to 2-adic integers. One would have union of Boolean algebras labelled by the 2-adic norm of the 2-adic number. p-Adic integers for a general prime p define obviously a generalization of Stone space making sense for effectively p-valued logic: the interpretation will be discussed below.

Consider now a Boolean algebra consisting of all possible infinitely long bit sequences. This algebra corresponds naturally to 2-adic integers. The generating Boolean statements correspond to sequences with single non-vanishing bit: by taking the unions of these points one obtains all sets.

The natural topology is that for which the lowest bits are the most significant. 2-adic topology realizes this idea since n:th bit has norm 2^{-n}. 2-adic integers as an p-adic integers are as spaces totally disconnected.

That 2-adic integers and more generally, 2-adic variants of n-dimensional p-adic manifolds would define Stone bases assignable to Boolean algebras is consistent with the identification of p-adic space-time sheets as correlates of cognition. Each point of 2-adic space-time sheet would represent 8 bits as a point of 8-D imbedding space. In TGD framework WCW ("world of classical worlds") spinors correspond to Fock space for fermions and fermionic Fock space has natural identification as a Boolean algebra. Fermion present/not present in given mode would correspond to true/false. Spinors decompose to a tensor product of 2-spinors so that the labels for Boolean statements form a Boolean algebra too in this case. A possible interpretation is as statements about statements.

In TGD Universe life and thus cognition reside in the intersection of real and p-adic worlds. Therefore the intersections of real and p-adic partonic 2-surfaces represent the intersection of real and p-adic worlds, those Boolean statements which are expected to be accessible for conscious cognition. They correspond to rational numbers or possibly numbers in an algebraic extension of rationals. For rationals pinary expansion starts to repeat itself so that the number of bits is finite. This intersection is also always discrete and for finite real space-time regions finite so that the identification looks a very natural since our cognitive abilities seem to be rather limited. In TGD inspired physics magnetic bodies are the key players and have much larger size than the biological body so that their intersection with their p-adic counterparts can contain much more bits. This conforms with the interpretation that the evolution of cognition means the emergence of increasingly longer time scales. Dark matter hierarchy realized in terms of hierarchy of Planck constants realizes this.

12.9.4 What about p-adic integers with $p > 2$?

The natural generalization of Stone space would be to a geometric counterpart of p-adic logic which I discussed for some years ago. The representation of the statements of p-valued logic as sequences of pinary digits makes the correspondence trivial if one accepts the above represented arguments. The generalization of Stone space would consist of p-adic integers and imbedding of a p-valued analog of Boolean algebra would map the number with only n:th digit equal to $1, ..., p-1$ to corresponding p-adic number.

One should however understand what p-valued statements mean and why p-adic numbers near powers of 2 are important. What is clear that p-valued logic is too romantic to survive. At least our every-day cognition is firmly anchored to a reality where everything is experience to be true or false.

1. The most natural explanation for $p > 2$ adic logic is that all Boolean statements do not allow a physical representation and that this forces reduction of 2^n valued logic to $p < 2^n$-valued one. For instance, empty set in the set theoretical representation of Boolean logic has no physical representation. In the same manner, the state containing no fermions fails to represent anything physically. One can represent physically at most $2^n - 1$ one statements of n-bit Boolean algebra and one must be happy with $n-1$ completely represented digits. The remaining statements containing at least one non-vanishing digit would have some meaning, perhaps the last digit allowed could serve as a kind of parity check.

2. If this is accepted then p-adic primes near to power 2^n of 2 but below it and larger than the previous power 2^{n-1} can be accepted and provide a natural topology for the Boolean statements grouping the binary digits to p-valued digit which represents the allowed statements in 2^n valued Boolean algebra. Bit sequence as a unit would be represented as a sequence of physically realizable bits. This would represent evolution of cognition in which simple yes or not statements are replaced with sequences of this kind of statements just as working computer programs are fused as modules to give larger computer programs. Note that also for computers similar evolution is taking place: the earliest processors used byte length 8 and now 32, 64 and maybe even 128 are used.

3. Mersenne primes $M_n = 2^n - 1$ would be ideal for logic purposes and they indeed play a key role in quantum TGD. Mersenne primes define p-adic length scales characterize many elementary particles and also hadron physics. There is also evidence for p-adically scaled up variants of hadron physics (also lepto-hadron physics allowed by the TGD based notion of color predicting colored excitations of leptons). LHC will certainly show whether M_{89} hadron physics at TeV energy scale is realized and whether also leptons might have scaled up variants.

4. For instance, M_{127} assignable to electron secondary p-adic time scale is .1 seconds, the fundamental time scale of sensory perception. Thus cognition in .1 second time scale single pinary statement would contain 126 digits as I have proposed in the model of memetic code. Memetic codons would correspond to 126 digit patterns with duration of .1 seconds giving 126 bits of information about percept.

If this picture is correct, the interpretation of p-adic space-time sheets- or rather their intersections with real ones- would represent space-time correlates for Boolean algebra represented at quantum level by fermionic many particle states. In quantum TGD one assigns with these intersections braids- or number theoretic braids- and this would give a connection with topological quantum field theories (TGD can be regarded as almost topological quantum field theory).

12.9.5 One more road to TGD

The following arguments suggests one more manner to end up with TGD by requiring that fermionic Fock states identified as a Boolean algebra have their Stone space as space-time correlate required by quantum classical correspondence. Second idea is that space-time surfaces define the collections of binary digits which can be equal to one: kind of eternal truths. In number theoretical vision associativity condition in some sense would define these divine truths. Standard model symmetries are a must- at least as their p-adic variants -and simple arguments forces the completion of discrete lattice counterpart of M^4 to a continuum.

1. If one wants Poincare symmetries at least in p-adic sense then a 4-D lattice in M^4 with $SL(2,Z) \times T^4$, where T^4 is discrete translation group is a natural choice. $SL(2,Z)$ acts in discrete Minkowski space T^4 which is lattice. Poincare invariance would be discretized. Angles and relative velocities would be discretized, etc..

2. The p-adic variant of this group is obtained by replacing Z and T^4 by their p-adic counterparts: in other words Z is replaced with the group Z_p of p-adic integers. This group is p-adically continuous group and acts continuously in T^4 defining a p-adic variant of Minkowski space consisting of all bit sequences consisting of 4-tuples of bits. Only in real sense one would have discreteness: note also that most points would be at infinity in real sense. Therefore it is possible to speak about analytic functions, differential calculus, and to write partial differential equations and to solve them. One can construct group representations and talk about angular momentum, spin and 4-momentum as labels of quantum states.

3. If one wants standard model symmetries p-adically one must replace T^4 with $T^4 \times CP_2$. CP_2 would be now discrete version of CP_2 obtained from discrete complex space C^3 by identifying points different by a scaling by complex integer. Discrete versions of color and electroweak groups would be obtained.

The next step is to ask what are the laws of physics. TGD fan would answer immediately: they are of course logical statements which can be true identified as subsets of $T^4 \times CP_2$ just as subset in Boolean algebra of sets corresponds to bits which are true.

1. The collections of 8-bit sequences consisting of only 1:s would define define 4-D surfaces in discrete $T^4 \times CP_2$. Number theoretic vision would suggest that they are quaternionic surfaces so that one associativity be the physical law at geometric level. The conjecture is that preferred extremals of Kähler action are associative surfaces using the definition of associativity as that assignable to a 4-plane defined by modified gamma matrices at given point of space-time surface.

2. Induced gauge field and metric make sense for p-adic integers. p-Adically the field equations for Kähler action make also sense. These p-adic surfaces would represent the analog of Boolean algebra. They would be however something more general than Stone assumes since they are not closed-open in the 8-D p-adic topology.

One however encounters a problem.

1. Although the field equations associated with Kähler action make sense, Kähler action itself does not exists as integral nor does the genuine minimization make sense since p-adically numbers are not well ordered and one cannot in general say which of two numbers is the larger one. This is a real problem and suggests that p-adic field equations are not enough and must be accompanied by real ones. Of course, also the metric properties of p-adic space-time are in complete conflict with what we believe about them.

2. One could argue that for preferred extremals the integral defining Kähler action is expressible as an integral of 4-form whose value could be well-defined since integrals of forms for closed algebraic surfaces make sense in p-adic cohomology theory pioneered by Grothendieck. The idea would be to use the definition of Kähler action making sense for preferred extremals as its definition in p-adic context. I have indeed proposed that space-time surfaces define representatives for homology with inspiration coming from TGD as almost topological QFT. This would give powerful constraints on the theory in accordance with the interpretation as a generalized Bohr orbit.

3. This argument together with what we know about the topology of space-time on basis of everyday experience however more or less forces the conclusion that also real variant of $M^4 \times CP_2$ is there and defines the proper variational principle. The finite points (on real sense) of $T^4 \times CP_2$ (in discrete sense) would represent points common to real and p-adic worlds and the identification in terms of braid points makes sense if one accepts holography and restricts the consideration to partonic 2-surfaces at boundaries of causal diamond. These discrete common points would represent the intersection of cognition and matter and living systems and provide a representation for Boolean cognition.

4. Finite measurement resolution enters into the picture naturally. The proper time distance between the tips would be quantized in multiples of CP_2 length. There would be several choices for the discretized imbedding space corresponding to different distance between lattice points: the interpretation is in terms of finite measurement resolution.

It should be added that discretized variant of Minkowski space and its p-adic variant emerge in TGD also in different manner in zero energy ontology.

1. The discrete space $SL(2, Z) \times T^4$ would have also interpretation as acting in the moduli space for causal diamonds identified as intersections of future and past directed light-cones. T^4 would represent lattice for possible positions of the lower tip of CD and and $SL(2, Z)$ leaving lower tip invariant would act on hyperboloid defined by the position of the upper tip obtained by discrete Lorentz transformations. This leads to cosmological predictions (quantization of red shifts). CP_2 length defines a fundamental time scale and the number theoretically motivated assumption is that the proper time distances between the tips of CDs come as integer multiples of this distance.

2. The stronger condition explaining p-adic length scale hypothesis would be that only octaves of the basic scale are allowed. This option is not consistent with zero energy ontology. The reason is that for more general hypothesis the M-matrices of the theory for Kac-Moody type algebra with finite-dimensional Lie algebra replaced with an infinite-dimensional algebra representing hermitian square roots of density matrices and powers of the phase factor replaced with powers of S-matrix. All integer powers must be allowed to obtain generalized Kac-Moody structure, not only those which are powers of 2 and correspond naturally to integer valued proper time distance between the tips of CD. Zero energy states would define the symmetry Lie-algebra of S-matrix with generalized Yangian structure.

3. p-Adic length scale hypothesis would be an outcome of physics and it would not be surprising that primes near power of two are favored because they are optimal for Boolean cognition.

The outcome is TGD. Reader can of course imagine alternatives but remember the potential difficulties due to the fact that minimization in p-adic sense does not make sense and action defined as integral does not exist p-adically. Also the standard model symmetries and quantum classical correspondence are to my opinion "must":s.

12.9.6 A connection between cognition and algebraic geometry

Stone space is synonym for profinite space. The Galois groups associated with algebraic extensions of number fields represent an extremely general class of profinite group [A71]. Every profinite group appears in Galois theory of some field K. The most most interesting ones are inverse limits of $Gal(F_1/K)$ where F_1 varies over all intermediate fields. Profinite groups appear also as fundamental groups in algebraic geometry. In algebraic topology fundamental groups are in general not profinite. Profiniteness means that p-adic representations are especially natural for profinite groups.

There is a fascinating connection between infinite primes and algebraic geometry discussed above leads to the proposal that Galois groups - or rather their projective variants- can be represented as braid groups acting on 2-dimensional surfaces. These findings suggest a deep connection between space-time correlates of Boolean cognition, number theory, algebraic geometry, and quantum physics and TGD based vision about representations of Galois groups as groups lifted to braiding groups acting on the intersection of real and p-adic variants of partonic 2-surface conforms with this.

Fermat theorem serves as a good illustration between the connection between cognitive representations and algebraic geometry. A very general problem of algebraic geometry is to find rational points of an algebraic surface. These can be identified as common rational points of the real and p-adic variant of the surface. The interpretation in terms of consciousness theory would be as points defining cognitive representation as rational points common to real partonic 2-surface and and its p-adic variants. The mapping to polynomials given by their representation in terms of infinite primes to braids of braids of braids.... at partonic 2-surfaces would provide the mapping of n-dimensional problem to 2-dimensional one.

One considers the question whether there are integer solutions to the equation $x^n+y^n+z^n=1$. This equation defines 2-surfaces in both real and p-adic spaces. In p-adic context it is easy to construct solutions but they usually represent infinite integers in real sense. Only if the expansion in powers of p contains finite number of powers of p, one obtains real solution as finite integers.

The question is whether there are any real solutions at all. If they exist they correspond to the intersections of the real and p-adic variants of these surfaces. In other words p-adic surface contains cognitively representable points. For $n>2$ Fermat's theorem says that only single point $x=y=z=0$ exists so that only single p-adic multi-bit sequence $(0,0,0,...)$ would be cognitively representable.

This relates directly to our mathematical cognition. Linear and quadratic equations we can solve and in these cases the number in the intersection of p-adic and real surfaces is indeed very large. We learn the recipes already in school! For $n>2$ difficulties begin and there are no general recipes and it requires mathematician to discover the special cases: a direct reflection of the fact that the number of intersection points for real and p-adic surfaces involved contains very few points.

12.9.7 Quantum Mathematics

To my view the self referentiality of consciousness is the real "hard problem" of consciousness theories. The "hard problem" as it is usually understood is only a problem of dualistic approach. My own belief is that the understanding of self-referentiality requires completely new mathematics with explicitly built-in self-referentiality. One possible view about this new mathematics is described in [K86]: here I provide only a brief summary in a form of recipe. The basic idea could have been abstracted from algebraic holography: replace numbers by Hilbert spaces and basic arithmetic operations with their counterparts for Hilbert spaces. Repeat this procedure by replacing the points of Hilbert spaces with Hilbert spaces and continue this procedure ad infinitum. It is quite possible that this procedure analogous to second quantization is more or less equivalent with the construction of infinite primes [K70].

Construction recipe

The construction recipe is following.

1. The idea is to start from arithmetics: + and $\times$ for natural numbers and generalize it .

   (a) The key observation is that + and $\times$ have direct sum and tensor product for Hilbert spaces as complete analogs and natural number n has interpretation as Hilbert space dimension and can be mapped to n-dimensional Hilbert space.

   Replace natural numbers n with n-dimensional Hilbert spaces at the first abstraction step. $n+m$ and $n\times m$ go to direct sum $n\oplus m$ and tensor product $n\otimes m$ of Hilbert spaces. One would calculate with Hilbert spaces rather than numbers. This induces calculation also for Hilbert space states and sum and product are like 3-particle vertices.

   (b) At second step construct integers (also negative) as pairs of Hilbert spaces (m,n) identifying $(m\oplus r, n\oplus r)$ and (m,n). This gives what might be called negative dimensional Hilbert spaces! Then take these pairs and define rationals as Hilbert space pairs (m,n) of this kind with (m,n) equivalent to $(k\otimes m, k\otimes n)$. This gives rise to what might be called m/n-dimensional Hilbert spaces!

   (c) At the third step construct Hilbert space variants of algebraic extensions of rationals. Hilbert space with dimension $\sqrt{2}$ say: this is a really nice trick [K86]. The idea is to consider for n-dimensional extension n-tuples of Hilbert spaces and induce tensor product for them from the product for the numbers of extension. After that one can continue with p-adic number fields and even reals: one can indeed understand even what π-dimensional Hilbert space could be! These spaces could also have interpretation in term of hyper-finite factors for which Hilbert spaces which otherwise would have infinite-dimension have finite and continuous dimension [K80]. If Hilbert space infinite-dimensional in the usual sense has dimension 1 (say) in the sense that identity operator has trace equal to 1 then subspaces in general have continuous range of dimensions smaller than one.

   The direct sum decompositions and tensor products would have genuine meaning Hilbert spaces associated with transcendentals are finite-dimensional in the sense as it is defined here but infinite-dimensional in ordinary sense. These Hilbert spaces would have different decompositions and would not be equivalent. Also in quantum physics decompositions to tensor products and direct sums (say representations of symmetry group) have phyiscal meaning: abstract Hilbert space of infinite dimension is too rough a concept.

   A direct connection with the ideas about complexity emerges. Rationals correspond to pairs of pairs of finite-dimensional Hilbert spaces corresponding to integers. Algebraic numbers correspond to n-tuples of finite-dimensional Hilbert spaces. Transcendentals correspond to infinite-dimensional Hilbert spaces decomposing to direct sums of tensor products: for instance, pinary expansion could define this decomposition. This decomposition matters so that abstract infinite-dimensional Hilbert spaces are not in question. The additional structure due to tensor product and direct sum is present also in physical applications: for instance the decomposition to irreducible representations defines this kind of direct sum decomposition.

2. Do the same for complex numbers, quaternions, and octonions, imbedding space $M^4\times CP_2$, etc.. The objection is that the construction is not general coordinate invariant. In coordinates in which point corresponds to integer valued coordinate one has finite-D Hilbert space and in coordinates in which coordinates of point correspond to transcendentals one has infinite-D Hilbert space. This makes sense only if one interprets the situation in terms of cognitive representations for points. π is very difficult to represent cognitively since it has infinite number of digits for which one cannot give a formula. "2" in turn is very simple to represent. This suggests interpretation in terms of self-referentiality. The two worlds with different coordinatizations are not equivalent since they correspond to different cognitive contents.

3. Replace also the coordinates of points of Hilbert spaces with Hilbert spaces again and again!

   The second key observation is that one can do all this again but at new level. Replace the numbers defining vectors of the Hilbert spaces (number sequences) assigned to numbers with Hilbert spaces! Continue ad infinitum by replacing points with Hilbert spaces again and again.

   One obtains a sequence of abstractions, which would be analogous to a hierarchy of n:th order logics. At lowest levels would be just predicate calculus: statements like $4 = 2^2$. At second level abstractions like $y = x^2$. At next level collections of algebraic equations, etc....

   This construction is structurally very similar to - if not equivalent with - the construction of infinite primes which corresponds to repeated second quantization in quantum physics. There is also a close relationship to - maybe equivalence with - what I have called algebraic holography or number theoretic Brahman=Atman identity [K70]. Numbers have infinitely complex anatomy not visible for physicist but necessary for understanding the self referentiality of consciousness and allowing mathematical objects to be holograms coding for mathematics. Hilbert spaces would be the DNA of mathematics from which all mathematical structures would be built!

Generalized Feynman diagrams as mathematical formulas?

One can assign to direct sum and tensor product their co-operations [K86, K9] and sequences of mathematical operations are very much like generalized Feynman diagrams. Co-product for instance would assign to integer m superposition of all its factorizations to a product of two integers with some amplitude for each factorization. Same applies to co-sum. Operation and co-operation would together give meaning to number theoretical 3-particle vertices. The amplitudes for the different factorizations must satisfy consistency conditions: associativity and distributivity could give constraints to the couplings to different channels- as particle physicist might express it.

The proposal is that quantum TGD is indeed quantum arithmetics with product and sum and their co-operations. Perhaps even something more general since also quantum logics and quantum set theory could be included! Generalized Feynman diagrams would correspond to formulas and sequences of mathematical operations with stringy 3-vertex as fusion of 3 -surfaces corresponding to $\oplus$ and Feynmanian 3-vertex as gluing of 3-surfaces along their ends, which is partonic 2-surface, corresponding to $\otimes$! One implication is that all generalized Feynman diagrams would reduce to a canonical form without loops and incoming/outgoing legs could be permuted. This is actually a generalization of old fashioned string model duality symmetry that I proposed years ago but gave it up as too "romantic" [K9].

Chapter 13

Langlands Program and TGD

13.1 Introduction

Langlands program [?]s an attempt to unify number theory and representation theory of groups and as it seems all mathematics. About related topics I know frustratingly little at technical level. Zeta functions and theta functions [?, ?] and more generally modular forms [?]re the connecting notion appearing both in number theory and in the theory of automorphic representations of reductive Lie groups. The fact that zeta functions have a key role in TGD has been one of the reasons for my personal interest.

The vision about TGD as a generalized number theory [?]ives good motivations to learn the basic ideas of Langlands program. I hasten to admit that I am just a novice with no hope becoming a master of the horrible technicalities involved. I just try to find whether the TGD framework could allow new physics inspired insights to Langlands program and whether the more abstract number theory relying heavily on the representations of Galois groups could have a direct physical counterpart in TGD Universe and help to develop TGD as a generalized number theory vision. After these apologies I however dare to raise my head a little bit and say aloud that mathematicians might get inspiration from physics inspired new insights.

The basic vision is that Langlands program could relate very closely to the unification of physics as proposed in TGD framework [?] TGD can indeed be seen both as infinite-dimensional geometry, as a generalized number theory involving several generalizations of the number concept, and as an algebraic approach to physics relying on the unique properties of hyper finite factors of type II_1 so that unification of mathematics would obviously fit nicely into this framework. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and sub-factors, and the notion of infinite prime, inspired a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

13.1.1 Langlands program very briefly

Langlands program [?]tates that there exists a connection between number theory and automorphic representations of a very general class of Lie groups known as reductive groups (groups whose all representations are fully reducible). At the number theoretic side there are Galois groups characterizing extensions of number fields, say rationals or finite fields. Number theory involves also so called automorphic functions to which zeta functions carrying arithmetic information via their coefficients relate via so called Mellin transform $\sum_n a_n n^s \rightarrow \sum_n a_n z^n$ [?]

Automorphic functions, invariant under modular group $SL(2,Z)$ or subgroup $\Gamma_0(N) \subset SL(2,Z)$ consisting of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ , \ c \ mod \ N = 0 \ ,$$

emerge also via the representations of groups $GL(2,R)$. This generalizes also to higher dimensional groups $GL(n,R)$. The dream is that all number theoretic zeta functions could be understood in terms of representation theory of reductive groups. The highly non-trivial outcome would be possibility to deduce very intricate number theoretical information from the Taylor coefficients of these functions.

Langlands program relates also to Riemann hypothesis and its generalizations. For instance, the zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [?] in this case Riemann hypothesis has been proven.

It must be emphasized that algebraic 1-dimensionality is responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [?] In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

One might also conjecture that Langlands duality for Lie groups reflects some deep duality on physical side. For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in the framework of YM theories and string models. In particular, Witten proposes that electric-magnetic duality which indeed relates gauge group and its dual, provides a physical correlate for the Langlands duality for Lie groups and could be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [?] Interestingly, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT. In this chapter it will be proposed that super-symmetry might correspond to the Langlands duality in TGD framework.

13.1.2 Questions

Before representing in more detail the TGD based ideas related to Langlands correspondence it is good to summarize the basic questions which Langlands program stimulates.

Could one give more concrete content to the notion of Galois group of algebraic closure of rationals?

The notion of Galois group for algebraic closure of rationals $Gal(\overline{Q}/Q)$ is immensely abstract and one can wonder how to make it more explicit? Langlands program adopts the philosophy that this group could be defined only via its representations. The so called automorphic representations constructed in terms of adeles. The motivation comes from the observation that the subset of adeles consisting of Cartesian product of invertible p-adic integers is a structure isomorphic with the maximal abelian subgroup of $Gal(\overline{Q}/Q)$ obtained by dividing $Gal(\overline{Q}/Q)$ with its commutator subgroup. Representations of finite abelian Galois groups are obtained as homomorphisms mapping infinite abelian Galois group to its finite factor group. In this approach the group $Gal(\overline{Q}/Q)$ remains rather abstract and adeles seem to define a mere auxiliary technical tool although it is clear that so called l-adic representations for Galois groups are are natural also in TGD framework.

This raises some questions.

1. Could one make $Gal(\overline{Q}/Q)$ more concrete? For instance, could one identify it as an infinite symmetric group S_∞ consisting of finite permutations of infinite number of objects? Could one imagine some universal polynomial of infinite degree or a universal rational function resulting as ratio of polynomials of infinite degree giving as its roots the closure of rationals?

2. S_∞ has only single normal subgroup consisting of even permutations and corresponding factor group is maximal abelian group. Therefore finite non-abelian Galois groups cannot be represented via homomorphisms to factor groups. Furthermore, S_{infty} has only infinite-dimensional non-abelian irreducible unitary representations as a simple argument to be discussed later shows.

   What is highly non-trivial is that the group algebras of S_∞ and closely related braid group B_∞ define hyper-finite factors of type II_1 (HFF). Could sub-factors characterized by finite groups G allow to realize the representations of finite Galois groups as automorphisms p HFF? The interpretation would be in terms of "spontaneous symmetry breaking" $Gal(\overline{Q}/Q) \rightarrow G$. Could it be possible to get rid of adeles in this manner?

3. Could one find a concrete physical realization for the action of S_∞? Could the permuted objects be identified as strands of braid so that a braiding of Galois group to infinite braid group B_∞ would result? Could the outer automorphism action of Galois group on number theoretic braids defining the basic structure of quantum TGD allow to realize Galois groups physically as Galois groups of number theoretic braids associated with subset of algebraic points defined by the intersection of real and p-adic partonic 2-surface? The requirement that mathematics is able to represent itself physically would provide the reason for the fact that reality and various p-adicities intersect along subsets of rational and algebraic points only.

Could one understand the correspondences between the representations of finite Galois groups and reductive Lie groups?

Langlands correspondence involves a connection between the representations of finite-dimensional Galois groups and reductive Lie groups.

1. Could this correspondence result via an extension of the representations of finite groups in infinite dimensional Clifford algebra to those of reductive Lie groups identified for instance as groups defining sub-factors (any compact group can define a unique sub-factor)? If Galois groups and reductive groups indeed have a common representation space, it might be easier to understand Langlands correspondence.

2. Is there some deep difference between between general Langlands correspondence and that for $GL(2,F)$ and could this relate to the fact that subgroups of $SU(2)$ define sub-factors with quantized index $\mathcal{M}:\mathcal{N} \leq 4$.

3. McKay correspondence [**?**]elates finite subgroups of compact Lie groups to compact Lie group (say finite sub-groups of SU(2) to ADE type Lie-algebras or Kac-Moody algebras). TGD approach leads to a general heuristic explanation of this correspondence in terms of Jones inclusions and Connes tensor product. Could sub-factors allow to understand Langlands correspondence for general reductive Lie groups as both the fact that any compact Lie group can define a unique sub-factor and an argument inspired by McKay correspondence suggest.

Could one unify geometric and number theoretic Langlands programs?

There are two Langlands programs: algebraic Langlands program and geometric one [**?**]ne corresponding to ordinary number fields and function fields. The natural question is whether and how these approaches could be unified.

1. Could the discretization based on the notion of number theoretic braids induce the number theoretic Langlands from geometric Langlands so that the two programs could be unified by the generalization of the notion of number field obtained by gluing together reals with union of reals and various p-adic numbers fields and their extensions along common rationals and algebraics. Certainly the fusion of p-adics and reals to a generalized notion of number should be essential for the unification of mathematics.

2. Could the distinction between number fields and function fields correspond to two kinds of sub-factors corresponding to finite subgroups $G \subset SU(2)$ and SU(2) itself leaving invariant the elements of imbedded algebra? This would obviously generalize to imbeddings of Galois groups to arbitrary compact Lie group. Could gauge group algebras contra Kac Moody algebras be a possible physical interpretation for this. Could the two Langlands programs correspond to two kinds of ADE type hierarchies defined by Jones inclusions? Could minimal conformal field theories with finite number of primary fields correspond to algebraic Langlands and full string theory like conformal field theories with infinite number of primary fields to geometric Langlands? Could this difference correspond to sub-factors defined by disrete groups and Lie groups?

3. Could the notion of infinite rational [**?**]e involved with this unification? Infinite rationals are indeed mapped to elements of rational function fields (also algebraic extensions of them)

so that their interpretation as quantum states of a repeatedly second quantized arithmetic super-symmetric quantum field theory might provide totally new mathematical insights.

Is it really necessary to replace groups $GL(n,F)$ with their adelic counterparts?

If the group of invertible adeles is not needed or allowed then a definite deviation from Langlands program is implied. It would seem that multiplicative adeles (ideles) are not favored by TGD view about the role of p-adic number fields. The l-adic representations of p-adic Galois groups corresponding to single p-adic prime l emerge however naturally in TGD framework.

1. The 2×2 Clifford algebra could be easily replaced with its adelic version. A generalization of Clifford algebra would be in question and very much analogous to $GL(2,A)$ in fact. The interpretation would be that real numbers are replaced with adeles also at the level of imbedding space and space-time. This interpretation does not conform with the TGD based view about the relationship between real and p-adic degrees of freedom. The physical picture is that H is 8-D but has different kind of local topologies and that spinors are in some sense universal and independent of number field.

2. WCW spinors define a hyper-finite factor of type II_1. It is not clear if this interpretation continues to make sense if configuration space spinors (fermionic Fock space) are replaced with adelic spinors. Note that this generalization would require the replacement of the group algebra of S_{infty} with its adelic counterpart.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19].

13.2 Basic concepts and ideas related to the number theoretic Langlands program

The basic ideas of Langlands program are following.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL(2,A)$ and more generally $GL(n,A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [A62], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL(n,A)$.

2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL(n,A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n,R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k=1,..,n$ acting in group algebra of $Gl(n,R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

3. The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_L(F)$ of $G(F)$.

13.2.1 Correspondence between n-dimensional representations of $Gal(\overline{F}/F)$ and representations of $GL(n, A_F)$ in the space of functions in $GL(n, F)\backslash GL(n, A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^{\times}$, where $Z_p^{\times}$ consists of invertible p-adic integers [A150] .

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p\in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^{\times}\backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^{\times}\backslash A_F^{\times}$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(1, A_F)$ in the space of functions defined in $GL(1, F)\backslash GL(1, A_F)$.*

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(n, A_F)$ in the space of functions defined in $GL(n, F)\backslash GL(n, A_F)$.*

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

2. The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \to \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \to \infty$ from a diagonal imbedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \to \infty$ the imbedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

3. These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [A81] . In the case of elliptic curves [A150] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL(n, E)$.

   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative manners to state the same thing.

Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \rightarrow Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [A150] . The representations can be also l-adic being defined in $GL(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [A150] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p, 1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^{\times}$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^{\times}|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [A150] for the route from automorphic adelic representations of $GL(2, R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL(2, Q)$ are constructed in the space of smooth bounded functions $GL(2, Q)\backslash GL(2, A) \rightarrow C$ or equivalently in the space of $GL(2, Q)$ left-invariant functions in $GL(2, A)$. A denotes adeles and $GL(2, A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL(2, A)$. The motivating idea is the central role of double coset decompositions $G = K_1AK_2$, where K_i are compact subgroups and A denotes the space of double cosets K_1gK_2 in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.

   To my best understanding $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL(2, Z_p)$. For ramified primes K_p consists of $SL(2, Z_p)$ matrices with $c \in p^{n_p}Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2, R)$ with eigenvalue ρ so that irreducible representations of $gl(2, R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \quad ,$$

where one has

$$X_0 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix} .$$

3. The center $A^\times$ of $GL(2,A)$ consists of $A^\times$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^\times \to C$ is a character providing a multiplicative representation of $A^\times$.

4. Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g)du = 0$$

is satisfied [A150] . Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

The groups $gl(2,R)$, $O(2)$ and $GL(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL(2,A_F) \times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL(2,Q)\backslash GL(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is conductor [A150]. This means enormous simplification since one gets ride of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Cartesian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL(2,Z)$ consists of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad c \ mod \ N = 0.$$

$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL(2,Z)$ so that also $SL(2,Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \to \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL(2,R)$ to upper half-plane $H_u = SL(2,R)/SO(2)$

The representations of $(gl(2,C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A150] . For the discrete series representation π giving square integrable representation in $SL(2,R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ \ , \ \ X_+ v_\infty = 0 \ .$$

The latter module is in turn generated by the lowest weight vector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_\infty \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) \ = \ \sum_{n=0}^{\infty} a_n q^n \ . \tag{13.2.1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL(2,Z_p)$ bi-invariant functions on $GL(2,Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A150] .

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmueller parameters of algebraic Riemann surface are affected by absolute Galois group. This induces Sl(2g,Z) transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroup of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

13.2.2 Some remarks about the representations of $Gl(n)$ and of more general reductive groups

The simplest representations of $Gl(n,R)$ have the property that the Borel group B of upper diagonal matrices is mapped to diagonal matrices consisting of character ξ which decomposes to a product of characters χ_k associated with diagonal elements b_k of B defining homomorphism

$$b_k \to sgn(b)^{m(k)}|b_k|^{ia_k}$$

to unit circle if a_k is real. Also more general, non-unitary, characters can be allowed. The representation itself satisfies the condition $f(bg) = \chi(b)f(g)$. Thus n complex parameters a_k defining a reducible representation of $C^\times$ characterize the irreducible representation.

In the case of $GL(2,R)$ one can consider also genuinely two-dimensional discrete series representations characterized by only single continuous parameter and the previous example represented just this case. These representations are square integrable in the subgroup $SL(2,R)$. Their origin is related to the fact that the algebraic closure of R is 2-dimensional. The so called Weil group W_R which is semi-direct product of complex conjugation operation with $C^\times$ codes for this number theoretically. The 2-dimensional representations correspond to irreducible 2-dimensional representations of W_R in terms of diagonal matrices of $Gl(2,C)$.

In the case of $GL(n,R)$ the representation is characterized by integers n_k: $\sum n_k = n$ characterizing the dimensions $n_k = 1,2$ of the representations of W_R. For $Gl(n,C)$ one has $n_k = 1$ since Weil group W_C is obviously trivial in this case.

In the case of a general reductive Lie group G the homomorphisms of W_R to the Langlands dual G_L of G defined by replacing the roots of the root lattice with their duals characterize the automorphic representations of G.

The notion of Weil group allows also to understand the general structure of the representations of $GL(n,F)$ in $GL(n,K)$, where F is p-adic number field and K its extension. In this case Weil group is a semi-direct product of Galois group of $Gal(K/F)$ and multiplicative group $K^\times$. A very rich structure results since an infinite number of extensions exists and the dimensions of discrete series representations.

The deep property of the characterization of representations in terms of Weil group is functoriality. If one knows the homomorphisms $W_F \to G$ and $G \to H$ then the composite homomorphism defines an automorphic representation of H. This means that irreps of G can be passed to those of H by homomorphism [A149] .

13.3 TGD inspired view about Langlands program

In this section a general TGD inspired vision about Langlands program is described. If is of course just a bundle of physics inspired ideas represented in the hope that real professionals might find some inspiration. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II$_1$ and their sub-factors, and the notion of infinite prime, lead to a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

13.3.1 What is the Galois group of algebraic closure of rationals?

Galois group is essentially the permutation group for the roots of an irreducible polynomial. It is a subgroup of symmetric group S_n, where n is the degree of polynomial. One can also imagine the notion of Galois group $Gal(\overline{Q}/Q)$ for the algebraic closure of rationals but the concretization of this notion is not easy.

$Gal(\overline{Q}/Q)$ as infinite permutation group?

The maximal abelian subgroup of $Gal(\overline{Q}/Q)$, which is obtained by dividing with the normal subgroup of even permutations, is identifiable as a product of multiplicative groups $Z_p^\times$ of invertible p-adic integers $n = n_0+pZ$, $n_0 \in \{1,..p-1\}$ for all p-adic primes and can be understood reasonably via its isomorphism to the product $\hat{Z} = \prod_p Z_p$ of multiplicative groups Z_p of invertible p-adic integers, one factor for each prime p [A149, A56, A150] .

Adeles [A4] are identified as the subring of $(\hat{Z} \otimes_Z Q) \times R$ containing only elements for which the elements of Q_p belong to Z_p except for a finite number of primes so that the number obtained can be always represented as a product of element of $\hat{Z}$ and point of circle R/Z: $A = \hat{Z} \times R/Z$. Adeles define a multiplicative group $A^\times$ of ideles and $GL(1, A)$ allow to construct representations $Gal(Q^{ab}/Q)$.

It is much more difficult to get grasp on $Gal(\overline{Q}/Q)$. The basic idea of Langlands program is that one should try to understand $Gal(\overline{Q}/Q)$ through its representations rather than directly. The natural hope is that n-dimensional representations of $Gal(\overline{Q}/Q)$ could be realized in $GL(n, A)$.

1. $Gal(\overline{Q}/Q)$ as infinite symmetric group?

One could however be stubborn and try a different approach based on the direct identification $Gal(\overline{Q}/Q)$. The naive idea is that $Gal(\overline{Q}/Q)$ could in some sense be the Galois group of a polynomial of infinite degree. Of course, for mathematical reasons also a rational function defined as a ratio of this kind of polynomials could be considered so that the Galois group could be assigned to both zeros and poles of this function. In the generic case this group would be an infinite symmetric group S_∞ for an infinite number of objects containing only permutations for subsets containing a finite number of objects. This group could be seen as the first guess for $Gal(\overline{Q}/Q)$.

S_∞ can be defined by generators e_m representing permutation of $m^{th}$and $(m+1)^{th}$ object satisfying the conditions

$$\begin{aligned} e_m e_m &= e_n e_m \text{ for } |m-n| > 1 , \\ e_n e_{n+1} e_n &= e_n e_{n+1} e_n e_{n+1} \text{ for } n = 1, ..., n-2 \ , \\ e_n^2 &= 1 \ . \end{aligned} \tag{13.3.-1}$$

By the definition S_∞ can be expected to possess the basic properties of finite-dimensional permutation groups. Conjugacy classes, and thus also irreducible unitary representations, should be in one-one correspondence with partitions of n objects at the limit $n \to \infty$. Group algebra defined by complex functions in S_∞ gives rise to the unitary complex number based representations and the smallest dimensions of the irreducible representations are of order n and are thus infinite for S_∞. For representations based on real and p-adic number based variants of group algebra situation is not so simple but it is not clear whether finite dimensional representations are possible.

S_n and obviously also S_∞ allows an endless number of realizations since it can act as permutations of all kinds of objects. Factors of a Cartesian and tensor power are the most obvious possibilities for the objects in question. For instance, S_n allows a representation as elements of rotation group $SO(n)$ permuting orthonormalized unit vectors e_i with components $(e_i)^k = \delta_i^k$. This induces also a realization as spinor rotations in spinor space of dimension $D = 2^{d/2}$.

2. Group algebra of S_∞ as HFF

The highly non-trivial fact that the group algebra of S_∞ is hyper-finite factor of type II_1 (HFF) [A47] suggests a representation of permutations as permutations of tensor factors of HFF interpreted as an infinite power of finite-dimensional Clifford algebra. The minimal choice for the finite-dimensional Clifford algebra is $M^2(C)$. In fermionic Fock space representation of infinite-dimensional Clifford algebra e_i would induce the transformation $(b^\dagger_{m,i}, b^\dagger_{m,i+1}) \to (b^\dagger_{m,i+1}, b^\dagger_{m,i})$. If the index m is lacking, the representation would reduce to the exchange of fermions and representation would be abelian.

3. Projective representations of S_∞ as representations of braid group B_∞

S_n can be extended to braid group B_n by giving up the condition $e_i^2 = 1$ for the generating permutations of the symmetric group. Generating permutations are represented now as homotopies exchanging the neighboring strands of braid so that repeated exchange of neighboring strands induces a sequence of twists by π. Projective representations of S_∞ could be interpreted as representations of B_∞. Note that odd and even generators commute mutually and for unitary representations either of them can be diagonalized and are represented as phases $exp(i\phi)$ for braid group. If $exp(i\phi)$ is not a root of unity this gives effectively a polynomial algebra and the polynomials subalgebras of these phases might provide representations for the Hecke operators also forming commutative polynomial algebras.

The additional flexibility brought in by braiding would transform Galois group to a group analogous to homotopy group and could provide a connection with knot and link theory [A217] and topological quantum field theories in general [A201] . Finite quantum Galois groups would generate braidings and a connection with the geometric Langlands program where Galois groups are replaced with homotopy groups becomes suggestive [A150, A148] .

4. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is an entire infinity of infinities in number theoretical sense. After all, any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

The group algebra of Galois group of algebraic closure of rationals as hyper-finite factor of type II$_1$

The most natural framework for constructing unitary irreducible representations of Galois group is its group algebra. In the recent case this group algebra would be that for S_∞ or B_∞ if braids are allowed. What puts bells ringing is that the group algebra of S_∞ is a hyper-finite factor of type II$_1$ isomorphic as a von Neumann algebra to the infinite-dimensional Clifford algebra [A47] , which in turn is the basic structures of quantum TGD whose localized version might imply entire quantum TGD. The very close relationship with the braid group makes it obvious that same holds true for corresponding braid group B_∞. Indeed, the group algebra of an infinite discrete group defines under very general conditions HFF. One of these conditions is so called amenability [A6] . This correspondence gives hopes of understanding the Langlands correspondence between representations of discrete Galois groups and the representations of $GL(n, F)$ (more generally representations of reductive groups).

Thus it seems that WCW spinor s (fermionic Fock space) could naturally define a finite-dimensional spinor representation of finite-dimensional Galois groups associated with the number theoretical braids. Inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors realize the notion of finite measurement resolution and give rise to finite dimensional representations of finite groups G leaving elements of $\mathcal{N}$ invariant. An attractive idea is that these groups are identifiable as Galois groups.

The identification of the action of G on $\mathcal{M}$ as homomorphism $G \rightarrow Aut(\mathcal{M})$ poses strong conditions on it. This is discussed in the thesis of Jones [C2] which introduces three algebraic invariants for the actions of finite group in hyperfinite-factors of type II$_1$, denoted by $\mathcal{M}$ in the sequel. In general the action reduces to inner automorphism of $\mathcal{M}$ for some normal subgroup $H \subset G$: this group is one of the three invariants of G action. In general one has projective representation for H so that one has $u_{h_1}u_{h_2} = \mu(h_1, h_2)u_{h_1h_2}$, where $\mu(h_1)$ is a phase factor which satisfies cocycle conditions coming from associativity.

1. The simplest action is just a unitary group representation for which $g \in G$ is mapped to a unitary operator u_g in $\mathcal{M}$ acting in $\mathcal{M}$ via adjoint action $m \rightarrow u_g m u_g^\dagger = Ad(u_g)m$. In this case one has $H = G$. In this case the fixed point algebra does not however define a factor and there is no natural reduction of the representations of $Gal(\overline{Q}/Q)$ to a finite subgroup.

2. The exact opposite of this situation outer action of G mean $H = \{e\}$. All these actions are conjugate to each other. This gives gives rise to two kinds of sub-factors and two kinds of representations of G. Both actions of Galois group could be realized either in the group or braid algebra of $Gal(\overline{Q}/Q)$ or in infinite dimensional Clifford algebra. In neither case the action be inner automorphic action $u \rightarrow gug^\dagger$ as one might have naively expected. This is crucial for circumventing the difficulty caused by the fact that $Gal(\overline{Q}/Q)$ identified as S_∞ allows no finite-dimensional complex representation.

3. The first sub-factor is $\mathcal{M}^G \subset \mathcal{M}$ corresponding, where the action of G on $\mathcal{M}$ is outer. Outer action defines a fixed point algebra for all finite groups G. For $D = \mathcal{M} : \mathcal{N} < 4$ only finite subgroups $G \subset SU(2)$ would be represented in this manner. The index identifiable as the fractal dimension of quantum Clifford algebra having $\mathcal{N}$ as non-abelian coefficients is $D = 4cos^2(\pi/n)$. One can speak about quantal representation of Galois group. The image of Galois group would be a finite subgroup of $SU(2)$ acting as spinor rotations of quantum Clifford algebra (and quantum spinors) regarded as a module with respect to the included algebra invariant under inner automorphisms. These representations would naturally correspond to 2-dimensional representations having very special role for the simple reason that the algebraic closure of reals is 2-dimensional.

4. Second sub-factor is isomorphic to $\mathcal{M}^G \subset (\mathcal{M} \otimes L(H))^G$. Here $L(H)$ is the space of linear operators acting in a finite-dimensional representation space H of a unitary irreducible representation of G. The action of G is a tensor product of outer action and adjoint action. The index of the inclusion is $dim(H)^2 \geq 1$ [A202] so that the representation of Galois group can be said to be classical (non-fractal).

5. The obvious question is whether and in what sense the outer automorphisms represent Galois subgroups. According to [C2] the automorphisms belong to the completion of the group of inner automorphisms of HFF. Identifying HFF as group algebra of S_∞, the interpretation would be that outer automorphisms are obtained as diagonal embeddings of Galois group to $S_n \times S_n \times ....$ If one includes only a finite number of these factors the outcome is an inner automorphisms so that for all finite approximations inner automorphisms are in question. At the limit one obtains an automorphisms which does not belong to S_∞ since it contains only finite permutations. This identification is consistent with the identification of the outer automorphisms as diagonal embedding of G to an infinite tensor power of sub-Clifford algebra of Cl_∞.

This picture is physically very appealing since it means that the ordering of the strands of braid does not matter in this picture. Also the reduction of the braid to a finite number theoretical braid at space-time level could be interpreted in terms of the periodicity at quantum level. From the point of view of physicist this symmetry breaking would be analogous to a spontaneous symmetry breaking above some length scale L. The cutoff length scale L would correspond to the number N of braids to which finite Galois group G acts and corresponds also to some p-adic length scale.

One might hope that the emergence of finite groups in the inclusions of hyper-finite factors could throw light into the mysterious looking finding that the representations of finite Galois groups and unitary infinite-dimensional automorphic representations of $GL(n, R)$ are correlated by the connection between the eigenvalues of Frobenius element Fr_p on Galois side and eigenvalues of commuting Hecke operators on automorphic side. The challenge would be to show that the action of Fr_p as outer automorphism of group algebra of S_∞ or B_∞ corresponds to Hecke algebra action on configuration space spinor fields or in modular degrees of freedom associated with partonic 2-surface.

Could there exist a universal rational function having $Gal(\overline{Q}/Q)$ as the Galois group of its zeros/poles?

The reader who is not fascinated by the rather speculative idea about a universal rational function having $Gal(\overline{Q}/Q)$ as a permutation group of its zeros and poles can safely skip this subsection since it will not be needed anywhere else in this chapter.

1. Taking the idea about permutation group of roots of a polynomial of infinite order seriously, one could require that the analytic function defining the Galois group should behave like a polynomial or a rational function with rational coefficients in the sense that the function should have an everywhere converging expansion in terms of products over an infinite number of factors $z - z_i$ corresponding to the zeros of the numerator and possible denominator of a rational function. The roots z_i would define an extension of rationals giving rise to the entire algebraic closure of rationals. This is a tall order and the function in question should be number theoretically very special.

2. One can speculate even further. TGD has inspired the conjecture that the non-trivial zeros $s_n = 1/2+iy_n$ of Riemann zeta [A110] (assuming Riemann hypothesis) are algebraic numbers and that also the numbers p^{s_n}, where p is any prime, and thus local zeta functions serving as multiplicative building blocks of ζ have the same property [K61] . The story would be perfect if these algebraic numbers would span the algebraic closure of rationals.

   The symmetrized version of Riemann zeta defined as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfying the functional equation $\xi(s) = \xi(1-s)$ and having only the trivial zeros could appear as a building block of the rational function in question. The function

$$f(s) = \frac{\xi(s)}{\xi(s+1)} \times \frac{s-1}{s}$$

   has non-trivial zeros s_n of ζ as zeros and their negatives as $-s_n$ as poles. There are no other zeros since trivial zeros as well as the zeros at $s=0$ and $s=1$ are eliminated. Using Stirling formula one finds that $\xi(s)$ grows as s^s for real values of $s \rightarrow \infty$. The growths of the numerator and denominator compensate each other at this limit so that the function approaches constant equal to one for $Re(s) \rightarrow \infty$.

   If $f(s)$ indeed behaves as a rational function whose product expansion converges everywhere it can be expressed in terms of its zeros and poles as

$$\begin{aligned} f(s) &= \prod_{n>0} A_n(s) \ , \\ A_n &= \frac{(s-s_n)(s-\overline{s}_n)}{(1+s-s_n)(1+s-\overline{s}_n)} \ . \end{aligned} \tag{13.3.-1}$$

   The product expansion seems to converge for any finite value of s since the terms A_n approach unity for large values of $|s_n| = |1/2+iy_n|$. $f(s)$ has $s_n = 1/2+iy_n$ indeed has zeros and $s_n = -1/2+iy_n$ as poles.

3. This proposal might of course be quite too simplistic. For instance, one might argue that the phase factors p^{iy} associated with the non-trivial zeros give only roots of unity multiplied by Gaussian integers. One can however imagine more complex functions obtained by forming products of $f(s)$ with its shifted variants $f(s+\Delta)$ with algebraic shift Δ in, say, the interval $[-1/2,1/2]$. Some kind of limiting procedure using a product of this kind of functions might give the desired universal function.

13.3.2 Physical representations of Galois groups

It would be highly desirable to have concrete physical realizations for the action of finite Galois groups. TGD indeed provides two kinds of realizations of this kind. For both options there are good hopes about the unification of number theoretical and geometric Galois programs obtained by replacing permutations with braiding homotopies and by discretization of continuous situation to a finite number theoretic braids having finite Galois groups as automorphisms.

Number theoretical braids and the representations of finite Galois groups as outer automorphisms of braid group algebra

Number theoretical braids [K19, K18, K71] are in a central role in the formulation of quantum TGD based on general philosophical ideas which might apply to both physics and mathematical cognition and, one might hope, also to a good mathematics.

An attractive idea inspired by the notion of the number theoretical braid is that the symmetric group S_n might act on roots of a polynomial represented by the strands of braid and could thus be replaced by braid group.

The basic philosophy underlying quantum TGD is the notion of finite resolution, both the finite resolution of quantum measurement and finite cognitive resolution [K19, K18] . The basic implication is discretization at space-time level and finite-dimensionality of all mathematical structures which can be represented in the physical world. At space-time level the discretization means that the data involved with the definition of S-matrix comes from a subset of a discrete set of points in the intersection of real and p-adic variants of partonic 2-surface obeying same algebraic equations. Note that a finite number of braids could be enough to code for the information needed to reconstruct the entire partonic 2-surface if it is given by polynomial or rational function having coefficients as algebraic numbers. Entire WCW of 3-surfaces would be discretized in this picture. Also the reduction of the infinite braid to a finite one would conform with the spontaneous symmetry breaking S_∞ to diagonally imbedded finite Galois group imbedded diagonally.

1. Two objections

Langlands correspondence assumes the existence of finite-dimensional representations of $Gal(\overline{Q}/Q)$. In the recent situation this encourages the idea that the restrictions of mathematical cognition allow to realize only the representations of $Gal(\overline{Q}/Q)$ reducing in some sense to representations for finite Galois groups. There are two counter arguments against the idea.

1. It is good to start from a simple abelian situation. The abelianization of $G(\overline{A}/Q)$ must give rise to multiplicative group of adeles defined as $\hat{Z} = \prod_p Z_p^\times$ where $Z_p^\times$ corresponds to the multiplicative group of invertible p-adic integers consisting of p-adic integers having p-adic norm equal to one. This group results as the inverse limit containing the information about subgroup inclusion hierarchies resulting as sequences $Z^\times/(1+pZ)^\times \subset Z^\times/(1+p^2Z)^\times \subset ..$ and expressed in terms factor groups of multiplicative group of invertible p-adic integers. Z_∞/A_∞ must give the group $\prod_p Z_p^\times$ as maximal abelian subgroup of Galois group. All smaller abelian subgroups of S_∞ would correspond to the products of subgroups of $\hat{Z}^\times$ coming as $Z_p^\times/(1+p^nZ)^\times$. Representations of finite cyclic Galois groups would be obtained by representing trivially the product of a commutator group with a subgroup of $\hat{Z}$. Thus one would obtain finite subgroups of the maximal abelian Galois group at the level of representations as effective Galois groups. The representations would be of course one-dimensional.

   One might hope that the representations of finite Galois groups could result by a reduction of the representations of S_∞ to $G = S_\infty/H$ where H is normal subgroup of S_∞. Schreier-Ulam theorem [A190] the non-trivial normal subgroups are fintary alternating subgroup A_∞ and finitary symmetric group consisting if finitary permutations. Since the braid group B_∞ as a special case reduces to S_∞ there is no hope of obtaining finite-dimensional representations except abelian ones.

2. The identification of $Gal(\overline{Q}/Q) = S_\infty$ is not consistent with the finite-dimensionality in the case of complex representations. The irreducible unitary representations of S_n are in one-one correspondence with partitions of n objects. The direct numerical inspection based on the formula for the dimension of the irreducible representation of S_n in terms of Yang tableau [A107] suggests that the partitions for which the number r of summands differs from $r = 1$ or $r = n$ (1-dimensional representations) quite generally have dimensions which are at least of order n. If d-dimensional representations corresponds to representations in $GL(d, C)$, this means that important representations correspond to dimensions $d \to \infty$ for S_∞.

Both these arguments would suggest that Langlands program is consistent with the identification $Gal(\overline{F}, F) = S_\infty$ only if the representations of $Gal(\overline{Q}, Q)$ reduce to those for finite Galois subgroups via some kind of symmetry breaking.

2. Diagonal imbedding of finite Galois group to S_∞ as a solution of problems

The idea is to imbed the Galois group acting as inner automorphisms diagonally to the m-fold Cartesian power of S_n imbedded to S_∞. The limit $m \to \infty$ gives rise to outer automorphic action since the resulting group would not be contained in S_∞. Physicist might prefer to speak about number theoretic symmetry breaking $Gal(\overline{Q}/Q) \to G$ implying that the representations are irreducible only in finite Galois subgroups of $Gal(\overline{Q}/Q)$. The action of finite Galois group G is indeed analogous to that of global gauge transformation group which belongs to the completion of the group of local gauge transformations. Note that G is necessarily finite.

About the detailed definition of number theoretic braids

The work with hyper-finite factors of type II_1 (HFFs) combined with experimental input led to the notion of hierarchy of Planck constants interpreted in terms of dark matter [K25] . The hierarchy is realized via a generalization of the notion of imbedding space obtained by gluing infinite number of its variants along common lower-dimensional quantum critical sub-manifolds. These variants of imbedding space are characterized by discrete subgroups of SU(2) acting in M^4 and CP_2 degrees of freedom as either symmetry groups or homotopy groups of covering. Among other things this picture implies a general model of fractional quantum Hall effect.

The identification of number theoretic braids

To specify number theoretical criticality one must specify some physically preferred coordinates for $M^4 \times CP_2$ or at least $\delta M^4_\pm \times CP_2$. Number theoretical criticality requires that braid belongs to the algebraic intersection of real and p-adic variants of the partonic 2-surface so that number theoretical criticality reduces to a finite number of conditions. This is however not strong enough condition and one must specify further physical conditions.

1. What are the preferred coordinates for H?

What are the preferred coordinates of M^4 and CP_2 in which algebraicity of the points is required is not completely clear. The isometries of these spaces must be involved in the identification as well as the choice of quantization axes for given CD. In [K47] I have discussed the natural preferred coordinates of M^4 and CP_2.

1. For M^4 linear M^4 coordinates chosen in such manner that $M^2 \times E^2$ decomposition fixing quantization axes is respected are very natural. This restricts the allowed Lorentz transformations to Lorentz boosts in M^2 and rotations in E^2 and the identification of M^2 as hyper-complex plane fixes time coordinate uniquely. E^2 coordinates are fixed apart from the action of $SO(2)$ rotation. The rationalization of trigonometric functions of angle variables allows angles associated with Pythagorean triangles as number theoretically simplest ones.

2. The case of CP_2 is not so easy. The most obvious guess in the case of CP_2 the coordinates corresponds to complex coordinates of CP_2 transforming linearly under $U(2)$. The condition that color isospin rotations act as phase multiplications fixes the complex coordinates uniquely. Also the complex coordinates transforming linearly under $SO(3)$ rotations are natural choice for S^2 ($r_M = constant$ sphere at $\delta M^4_\pm$).

3. Another manner to deal with CP_2 is to apply number M^8-H duality. In M^8 CP_2 corresponds to E^4 and the situation reduces to linear one and $SO(4)$ isometries help to fix preferred coordinate axis by decomposing E^4 as $E^4 = E^2 \times E^2$. Coordinates are fixed apart the action of the commuting $SO(2)$ sub-groups acting in the planes E^2. It is not clear whether the images of algebraic points of E^4 at space-time surface are mapped to algebraic points of CP_2.

2. The identification of number theoretic braids

The identification of number theoretic braids is not by no means a trivial task [K15, K54] . As a matter fact, there are several alternative identifications and it seems that all of them are needed. Consider first just braids without the attribute 'number theoretical'.

1. Braids could be identified as lifts of the projections of X^3_l to the quantum critical sub-manifolds M^2 or S^2_I, $i = I, II$, and in the generic case consist of 1-dimensional strands in X^3_l These sub-manifolds are obviously in the same role as the plane to which the braid is projected to obtain a braid diagram. This requires that a unique identification of the slicing of space-time surfaces by 3-surfaces.

2. Braid points are always quantum critical against the change of Planck constant so that TQFT like theory characterizes the freedom remaining intact at quantum criticality. Quantum criticality in this sense need not have anything to do with the quantum criticality in the sense

that the second variation of Kähler action vanishes -at least for the variations representing dynamical symmetries in the sense that only the inner product $\int(\partial L_D/\partial h^k_\alpha)\delta h^k d^4x$ (L_D denotes modified Dirac Lagrangian) without the vanishing of the integrand. This criticality leads to a generalization of the conceptual framework of Thom's catastrophe theory [K15] .

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$.

3. It is not clear whether these three braids form some kind of trinity so that one of them is enough to formulate the theory or whether all of them are needed. Note also that one has quantum superposition over CDs corresponding to different choices of M^2 and the pair formed by S^2_I and S^2_{II} (note that the spheres are not independent if both appear). Quantum measurement however selects one of these choices since it defines the choice of quantization axes.

4. One can consider also more general definition. The extrema of Kähler magnetic field strength defined as coordinate invariant $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define in natural manner a discrete set of points defining the nodes of symplectic triangulation: note that this involves division with metric determinant in preferred coordinates. This set of extremals is same for all deformations of X^3_l allowed in the functional integral over symplectic group although the positions of points change. For preferred symplectically invariant light-like coordinate of X^3_l braid results. Also now geodesic spheres and M^2 would define the counterpart of the plane to which the braids are projected.

5. A physically attractive realization of the braids - and more generally- of slicings of space-time surface by 3-surfaces and string world sheets, is discussed in [K34] by starting from the observation that TGD defines an almost topological QFT of braids, braid cobordisms, and 2-knots. The boundaries of the string world sheets at the space-like 3-surfaces at boundaries of CDs and wormhole throats would define space-like and time-like braids uniquely.

The idea relies on a rather direct translation of the notions of singular surfaces and surface operators used in gauge theory approach to knots [A218] to TGD framework. It leads to the identification of slicing by three-surfaces as that induced by the inverse images of $r = constant$ surfaces of CP_2, where r is $U(2)$ invariant radial coordinate of CP_2 playing the role of Higgs field vacuum expectation value in gauge theories. $r = \infty$ surfaces correspond to geodesic spheres and define analogs of fractionally magnetically charged Dirac strings identifiable as preferred string world sheets. The union of these sheets labelled by subgroups $U(2) \subset SU(3)$ would define the slicing of space-time surface by string world sheets. The choice of $U(2)$ relates directly to the choice of quantization axes for color quantum numbers characterizing CD and would have the choice of braids and string world sheets as a space-time correlate. $r = \infty$ points correspond to three homologically non-trivial geodesic spheres S^2 analogous to North and South poles of CP_2 and the projections to M^4 and S^2 define braid projections. Braid strands could be interpreted as orbits of Kähler charged particle in Kähler magnetic field and enclosing fractional Kähler flux.

The beauty of this identification is that one starts from braids at the ends of space-time surface partonic 2-surfaces at boundaries of CD and from intersection of braid points and determines space-time surface and string world sheets from this data in accordance with holography and quantum classical correspondence. This picture conforms also with the recent view about modified Dirac equation for which the construction of solutions leads to the notion of braid too.

Number theoretic braids would be braids which are number theoretically critical. This means that the points of braid in preferred coordinates are algebraic points so that they can be regarded as being shared by real partonic 2-surface and its p-adic counterpart obeying same algebraic equations. The phase transitions between number fields would mean leakage via these 2-surfaces playing the role of back of a book along which real and p-adic physics representing the pages of a book are glued together. The transformation of intention to action would represent basic example of this kind of leakage and number theoretic criticality could be decisive feature of living matter. For number theoretic braids at X_l^3 whose real and p-adic variants obey same algebraic equations, only subset of algebraic points is common to real and p-adic pages of the book so that discretization of braid strand is unavoidable.

Representation of finite Galois groups as outer automorphism groups of HFFs

Any finite group G has a representation as outer automorphisms of a hyper-finite factor of type II_1 (briefly HFF in the sequel) and this automorphism defines sub-factor $\mathcal{N} \subset \mathcal{M}$ with a finite value of index $\mathcal{M} : \mathcal{N}$ [A144] . Hence a promising idea is that finite Galois groups act as outer automorphisms of the associated hyper-finite factor of type II_1.

More precisely, sub-factors (containing Jones inclusions as a special case) $\mathcal{N} \subset \mathcal{M}$ are characterized by finite groups G acting on elements of $\mathcal{M}$ as outer automorphisms and leave the elements of $\mathcal{N}$ invariant whereas finite Galois group associated with the field extension K/L act as automorphisms of K and leave elements of L invariant. For finite groups the action as outer automorphisms is unique apart from a conjugation in von Neumann algebra. Hence the natural idea is that the finite subgroups of $Gal(\overline{Q}/Q)$ have outer automorphism action in group algebra of $Gal(\overline{Q}/Q)$ and that the hierarchies of inclusions provide a representation for the hierarchies of algebraic extensions. Amusingly, the notion of Jones inclusion was originally inspired by the analogy with field extensions [A144]!

It must be emphasized that the groups defining sub-factors can be extremely general and can represent much more than number theoretical information understood in the narrow sense of the word. Even if one requires that the inclusion is determined by outer automorphism action of group G uniquely, one finds that any amenable, in particular compact [A6] , group defines a unique sub-factor by outer action [A144] . It seems that practically any group works if uniqueness condition is given up.

The TGD inspired physical interpretation is that compact groups would serve as effective gauge groups defining measurement resolution by determining the measured quantum numbers. Hence the physical states differing by the action of $\mathcal{N}$ elements which are G singlets would not be indistinguishable from each other in the resolution used. The physical states would transform according to the finite-dimensional representations in the resolution defined by G.

The possibility of Lie groups as groups defining inclusions raises the question whether hyper-finite factors of type II_1 could mimic any gauge theory and one might think of interpreting gauge groups as Galois groups of the algebraic structure of this kind of theories. Also Kac-Moody algebras emerge naturally in this framework as will be discussed, and could also have an interpretation as Galois algebras for number theoretical dynamical systems obeying dynamics dictated by conformal field theory. The infinite hierarchy of infinite rationals in turn suggests a hierarchy of groups S_∞ so that even algebraic variants of Lie groups could be interpreted as Galois groups. These arguments would suggest that HFFs might be kind of Universal Math Machines able to mimic any respectable mathematical structure.

Number theoretic braids and unification of geometric and number theoretic Langlands programs

The notion of number theoretic braid has become central in the attempts to fuse real physics and p-adic physics to single coherent whole. Number theoretic braid leads to the discretization of quantum physics by replacing the stringy amplitudes defined over curves of partonic 2-surface with amplitudes involving only data coded by points of number theoretic braid. The discretization of quantum physics could have counterpart at the level of geometric Langlands [B33] [A150, A174] , whose discrete version would correspond to number theoretic Galois groups associated with the points of number theoretic braid. The extension to braid group would mean that the global homotopic information is not lost.

1. *Number theoretic braids belong to the intersection of real and p-adic partonic surface*

The points of number theoretic braid belong to the intersection of the real and p-adic variant of partonic 2-surface consisting of rationals and algebraic points in the extension used for p-adic numbers. The points of braid have same projection on an algebraic point of the geodesic sphere of $S^2 \subset CP_2$ belonging to the algebraic extension of rationals considered (the reader willing to understand the details can consult [K19]).

The points of braid are obtained as solutions of polynomial equation and thus one can assign to them a Galois group permuting the points of the braid. In this case finite Galois group could be realized as left or right translation or conjugation in S_∞ or in braid group.

To make the notion of number theoretic braid more concrete, suppose that the complex coordinate w of $\delta M^4_\pm$ is expressible as a polynomial of the complex coordinate z of CP_2 geodesic sphere and the radial light-like coordinate r of $\delta M^4_\pm$ is obtained as a solution of polynomial equation $P(r,z,w)=0$. By substituting w as a polynomial $w=Q(z,r)$ of z and r this gives polynomial equation $P(r,z,Q(z,r))=0$ for r for a given value of z. Only real roots can be accepted. Local Galois group (in a sense different as it is used normally in literature) associated with the algebraic point of S^2 defining the number theoretical braid is thus well defined.

If the partonic 2-surface involves all roots of an irreducible polynomial, one indeed obtains a braid for each point of the geodesic sphere $S^2 \subset CP_2$. In this case the action of Galois group is naturally a braid group action realized as the action on induced spinor fields and WCW spinor s.

The choice of the points of braid as points common to the real and p-adic partonic 2-surfaces would be unique so that the obstacle created by the fact that the finite Galois group as function of point of S^2 fluctuates wildly (when some roots become rational Galois group changes dramatically: the simplest example is provided by $y-x^2=0$ for which Galois group is Z_2 when y is not a square of rational and trivial group if y is rational).

2. Modified Dirac operator assigns to partonic 2-surface a unique prime p which could define l-adic representations of Galois group

The overall scaling of the eigenvalue spectrum of the modified Dirac operator assigns to the partonic surface a unique p-adic prime p which physically corresponds to the p-adic length scale which appears in the discrete coupling constant evolution [K19, K4] . One can solve the roots of the the resulting polynomial also in the p-adic number field associated with the partonic 2-surface by the modified Dirac equation and find the Galois group of the extension involved. The p-adic Galois group, known as local Galois group in literature, could be assigned to the p-adic variant of partonic surface and would have naturally l-adic representation, most naturally in the p-adic variant of the group algebra of S_∞ or B_∞ or equivalently in the p-adic variant of infinite-dimensional Clifford algebra. There are however physical reasons to believe that infinite-dimensional Clifford algebra does not depend on number field. Restriction to an algebraic number based group algebra therefore suggests itself. Hence, if one requires that the representations involve only algebraic numbers, these representation spaces might be regarded as equivalent.

3. Problems

There are however problems.

1. The triviality of the action of Galois group on the entire partonic 2-surface seems to destroy the hopes about genuine representations of Galois group.
2. For a given partonic 2-surface there are several number theoretic braids since there are several algebraic points of geodesic sphere S^2 at which braids are projected. What happens if the Galois groups are different? What Galois group should one choose?

A possible solution to both problems is to assign to each braid its own piece X^2_k of the partonic 2-surface X^2 such that the deformations X^2 can be non-trivial only in X^2_k. This means separation of modular degrees of freedom to those assignable to X^2_k and to "center of mass" modular degrees of freedom assignable to the boundaries between X^2_k. Only the piece X^2_k associated with the k^{th} braid would be affected non-trivially by the Galois group of braid. The modular invariance of the conformal field theory however requires that the entire quantum state is modular invariant under the modular group of X^2. The analog of color confinement would take place in modular degrees of freedom. Note that the region containing braid must contain single handle at least in order to allow representations of $SL(2,C)$ (or $Sp(2g,Z)$ for genus g).

As already explained, in the general case only the invariance under the subgroup $\Gamma_0(N)$ [A62] of the modular group $SL(2,Z)$ can be assumed for automorphic representations of $GL(2,R)$ [A149, A150, A79] . This is due to the fact that there is a finite set of primes (prime ideals in the algebra of integers), which are ramified [A79] . Ramification means that their decomposition to a product of prime ideals of the algebraic extension of Q contains higher powers of these prime ideals: $p \rightarrow (\prod_k P_k)^e$ with $e > 1$. The congruence group is fixed by the integer $N = \prod_k p^{n_k}$ known as conductor coding the set of exceptional primes which are ramified.

The construction of modular forms in terms of representations of $SL(2,R)$ suggests that it is possible to replace $\Gamma_0(N)$ by the congruence subgroup $\Gamma(N)$, which is normal subgroup of $SL(2,R)$ so that $G_1 = SL(2,Z)/\Gamma$ is group. This would allow to assign to individual braid regions carrying single handle well-defined G_1 quantum numbers in such a manner that entire state would be G_1 singlet.

Physically this means that the separate regions of the partonic 2-surface each containing one braid strand cannot correspond to quantum states with full modular invariance. Elementary particle vacuum functionals [K17] defined in the moduli space of conformal equivalence classes of partonic 2-surface must however be modular invariant, and the analog of color confinement in modular degrees of freedom would take place.

Hierarchy of Planck constants and dark matter and generalization of imbedding space

Second hierarchy of candidates for Galois groups is based on the generalization of the notion of the imbedding space $H = M^4 \times CP_2$, or rather the spaces $H_\pm = M^4_\pm \times CP_2$ defining future and past light-cones inside H [K25] . This generalization is inspired by the quantization of Planck constant explaining dark matter as a hierarchy of macroscopically quantum coherent phases and by the requirement that sub-factors have a geometric representation at the level of the imbedding space and space-time (quantum-classical correspondence).

Galois groups could also correspond to finite groups $G_a \times G_b \subset SU(2) \times SU(2) \subset$SL(2,C)$\times SU(3)$. These groups act as covering symmetries for the sectors of the imbedding space, which can be regarded as singular $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm/G_a \times G_b$ bundles containing orbifold points (fixed points of $G_a \times G_b$ or either of them. The copies of H with same G_a or G_b are glued together along $M^4_\pm$ or CP_2 factor and along common orbifold points left fixed by G_b or G_a. The group $G_a \times G_b$ plays both the role of both Galois group and homotopy group.

There are good reasons to expect that both these Galois groups and those associated with number theoretic braids play a profound role in quantum TGD based description of dark matter as macroscopically quantum coherent phases. For instance, G_a would appear as symmetry group of dark matter part of bio-molecules in TGD inspired biology [L8] .

Question about representations of finite groups

John Baez made an interesting question in n-Category-Cafe [A116] . The question reads as follows:

Is every representation of every finite group definable on the field Q^{ab} obtained by taking the field Q of rational numbers and by adding all possible roots of unity?

Since every finite group can appear as Galois group the question translates to the question whether one can represent all possible Galois groups using matrices with elements in Q^{ab}.

This form of question has an interesting relation to Langlands program. By Langlands conjecture the representations of the Galois group of algebraic closure of rationals can be realized in the space of functions defined in $GL(n,F)\backslash GL(n,Gal(Q^{ab}/Q))$, where $Gal(Q^{ab}/Q)$ is the maximal Abelian subgroup of the Galois group of the algebraic closure of rationals. Thus one has group algebra associated with the matrix group for which matrix elements have values in $Gal(Q^{ab}/Q)$. Something by several orders of more complex than matrices having values in Q^{ab}.

Suppose that Galois group of algebraic numbers can be regarded as the permutation group S_∞ of infinite number of objects generated by permutations for finite numbers of objects and that its physically interesting representations reduce to the representations of finite Galois groups G with element $g \in G$ represented as infinite product $g \times g \times ...$ belonging to the completion of S_∞ and thus to the completion of its group algebra identifiable as hyper-finite factor of type II_1. This would mean number theoretic local gauge invariance in the sense that all elements of S_∞ would leave physical states invariant whereas G would correspond to global gauge transformations. These tensor factors would have as space-time correlates number theoretical braids allowing to represent the action of G.

What this has then to do with John's question and Langlands program? S_∞ contains any finite group G as a subgroup. If all the representations of finite-dimensional Galois groups could be realized as representations in $Gl(n,Q^{ab})$, same would hold true also for the proposed symmetry breaking representations of the completion of S_∞ reducing to the representations of finite Galois groups. There would be an obvious analogy with Langlands program using functions defined in the space $Gl(n,Q)\backslash Gl(n,Gal(Q^{ab}/Q))$. Be as it may, mathematicians are able to work with incredibly abstract objects! A highly respectful sigh is in order!

13.3.3 What could be the TGD counterpart for the automorphic representations?

The key question in the following is whether quantum TGD could act as a general math machine allowing to realize any finite-dimensional manifold and corresponding function space in terms of configuration space spinor fields and whether also braided representations of Galois groups accompanying the braiding could be associated naturally with this kind of representations.

Some general remarks

Before getting to the basic idea some general remarks are in order.

1. WCW spinor fields would certainly transform according to a finite-dimensional and therefore non-unitary representation of $SL(2,C)$ which is certainly the most natural group involved and should relate to the fact that Galois groups representable as subgroups of SU(2) acting as rotations of 3-dimensional space correspond to sub-factors with $\mathcal{M}:\mathcal{N}\leq 4$.

2. Also larger Lie groups can be considered and diagonal imbeddings of Galois groups would be naturally accompanied by diagonal imbeddings of compact and also non-compact groups acting on the decomposition of infinite-dimensional Clifford algebra Cl_∞ to an infinite tensor power of finite-dimensional sub-Clifford algebra of form $M(2,C)^n$.

3. The basic difference between Galois group representation and corresponding Lie group representations is that the automorphisms in the case of discrete groups are automorphisms of S_∞ or B_∞ whereas for Lie groups the automorphisms are in general automorphisms of group algebra of S_∞ or B_∞. This could allow to understand the correspondence between discrete groups and Lie groups naturally.

4. Unitary automorphic representations are infinite-dimensional and require group algebra of $GL(n,F)$. Therefore WCW spinors - to be distinguished from WCW spinor fields- cannot realize them. WCW spinor field might allow the realization of these infinite-dimensional representations if groups themselves allow a finite-dimensional geometric realization of groups. Are this kind of realizations possible? This is the key question.

Could TGD Universe act as a universal math machine?

The questions are following. Could one find a representations of both Lie groups and their linear and non-linear representation spaces -and even more - of any manifold representable as a sub-manifold of some linear space in terms of braid points at partonic 2-surfaces X^2? What about various kinds of projective spaces and coset spaces? Can one construct representations of corresponding function spaces in terms of WCW spinor fields? Can one build representations of parameter groups of Lie groups as braided representations defined by the orbits of braid points in X_l^3? Note that this would assign to the representations of closed paths in the group manifold a representation of braid group and Galois group of the braid and might make it easier to understand the Langlands correspondence.

A professional mathematician - if she still continues reading - might regard the following argument as rather pathetic poor man's argument but I want to be honest and demonstrate my stupidity openly.

1. The n braid points represent points of $\delta H=\delta M^4_\pm\times CP_2$ so that braid points represent a point of $7n$-dimensional space $\delta H^n/S_n$. $\delta M^4_\pm$ corresponds to E^3 with origin removed but

$E^{2n}/S_n = C^n/S_n$ can be represented as a sub-manifold of $\delta M^4_\pm$. This allows to almost-represent both real and complex linear spaces. E^2 has a unique identification based on $M^4 = M^2 \times E_2$ decomposition required by the choice of quantization axis. One can also represent the spaces $(CP_2)^n/S_n$ in this manner.

2. The first - and really serious - problem is caused by the identification of the points obtained by permuting the n coordinates: this is of course what makes possible the braiding since braid group is the fundamental group of $(X^2)^n$. Could the quantum numbers at the braid points act as markers distinguishing between them so that one would effectively have E^{2n}? Could the fact that the representing points are those of imbedding space rather than X^2 be of significance? Second - less serious - problem is that the finite size of CD allows to represent only a finite region of E^2. On the other hand, ideal mathematician is a non-existing species and even non-ideal mathematician can imagine the limit at which the size of CD becomes infinite.

3. Matrix groups can be represented as sub-manifolds of linear spaces defined by the general linear group $Gl(n,R)$ and $Gl(n,C)$. In the p-adic pages of the imbedding space one can realize also the p-adic variants of general linear groups. Hence it is possible to imbed any real (complex) Lie group to E^{2n} (C^n), if n is chosen large enough.

4. WCW spinor fields restricted to the linear representations spaces or to the group itself represented in this manner would allow to realize as a special case various function spaces, in particular groups algebras. If WCW spinor fields satisfy additional symmetries, projective spaces and various coset spaces can be realized as effective spaces. For instance CP_2 could be realized effectively as $SU(3)/U(2)$ by requiring $U(2)$ invariance of the WCW spinor fields in $SU(3)$ or as C^3/Z by requiring that WCW spinor field is scale invariant. Projective spaces might be also realized more concretely as imbeddings to $(CP_2)^n$.

5. The action of group element $g = exp(Xt)$ belonging to a one-parameter sub-group of a non-compact linear group in a real (complex) linear representation space of dimension m could be realized in a subspace of E^{2n}, $m < 2n$ (C^n, $m \leq n$), as a flow in X^3_l taking the initial configuration of points of representation space to the final configuration. Braid strands - the orbits of points p_i defining the point p of the representation manifold under the action of one-parameter subgroup- would correspond to the points $exp(Xu)(p)$, $0 \leq u \leq t$. Similar representation would work also in the group itself represented in a similar manner.

6. Braiding in X^3_l would induce a braided representation for the action of the one parameter subgroup. This representation is not quite the same thing as the automorphic representation since braiding is involved. Also trivial braid group representation is possible if the representation can be selected freely rather than being determined by the transformation properties of fermionic oscillator operator basis in the braiding.

7. An important prerequisite for math machine property is that the wave function in the space of light-like 3-surfaces with fixed ends can be chosen freely. This is the case since the degrees of freedom associate with the interior of light-like 3-surface X^3_l correspond to zero modes assignable to Kac-Moody symmetries [K16, K71] . Dicretization seems however necessary since functional integral in these degrees of freedom is not-well defined even in the real sense and even less so p-adically. This conforms with the fact that real world mathematical representations are always discrete. Quantum classical correspondence suggests the dynamics represented by X^3_l correlates with the quantum numbers assigned with X^2 so that Boolean statements represented in terms of Fermionic Fock states would be in one-one correspondence with these wave functions.

Besides representing mathematical structures this kind of math machine would be able to perform mathematical deductions. The fermionic part of the state zero energy state could be interpreted as a quantum super-position of Boolean statement $A_i \rightarrow B_i$ representing various instances of the general rule $A \rightarrow B$. Only the statements consistent with fundamental conservation laws would be possible. Quantum measurements performed for both positive and negative energy parts of the state would produce statements. Performing the measurement of the observable $O(A \rightarrow B)$ would produce from a given state a zero energy state representing statement $A \to B$. If the measurement of observable $O(C \to D)$ affects this state then the statement $(A \to B) \to (C \to D)$ cannot hold true. For $A = B$ the situation reduces to simpler logic where one tests truth value of statements of form $A \to B$. By increasing the number of instances in the quantum states generalizations of the rule can be tested.

13.3.4 Super-conformal invariance, modular invariance, and Langlands program

The geometric Langlands program [A150, A148] deals with function fields, in particular the field of complex rational analytic functions on 2-dimensional surfaces. The sheaves in the moduli spaces of conformal blocks characterizing the n-point functions of conformal field theory replaces automorphic functions coding both arithmetic data and characterizing the modular representations of $GL(n)$ in number theoretic Langlands program [A150] . These moduli spaces are labelled both by moduli characterizing the conformal equivalence class of 2-surface, in particular the positions of punctures, in TGD framework the positions of strands of number theoretic braids, as well as the moduli related to the Kac-Moody group involved.

Transition to function fields in TGD framework

According to [A150] conformal field theories provide a very promising framework for understanding geometric Langlands correspondence.

1. That the function fields on 2-D complex surfaces would be in a completely unique role mathematically fits nicely with the 2-dimensionality of partons and well-defined stringy character of anti-commutation relations for induced spinor fields. According to [A150] there are not even conjectures about higher dimensional function fields.

2. There are very direct connections between hyper-finite factors of type II_1 and topological QFTs [A201, A217] , and conformal field theories. For instance, according to the review [H2] [A144] Ocneacnu has show that Jones inclusions correspond in one-one manner to topological quantum field theories and TGD can indeed be regarded as almost topological quantum field theory (metric is brought in by the light-likeness of partonic 3-surfaces). Furthermore, Connes has shown that the decomposition of the hierarchies of tensor powers $\mathcal{M} \otimes_{\mathcal{N}} .... \otimes_{\mathcal{N}} M$ as left and right modules to representations of lower tensor powers directly to fusion rules expressible in terms of 4-point functions of conformal field theories [A144] .

In TGD framework the transition from number fields to function fields would not be very dramatic.

1. Suppose that the representations of $SL(n, R)$ occurring in number theoretic Langlands program can indeed be realized in the moduli space for conformal equivalence classes of partonic 2-surface (or, by previous arguments, moduli space for regions of them with fixed boundaries). This means that representations of local Galois groups associated with number theoretic braids would involve global data about entire partonic 2-surface. This is physically very important since it otherwise discretization would lead to a loss of the information about dimension of partonic 2-surfaces.

2. In the case of geometric Langlands program this moduli space would be extended to the moduli space for n-point functions of conformal field theory defined at these 2-surfaces containing the original moduli space as a subspace. Of course, the extension could be present also in the number theoretic case. Thus it seems that number theoretic and geometric Langlands programs would utilize basic structures and would differ only in the sense that single braid would be replaced by several braids in the geometric case.

3. In TGD Kac-Moody algebras would be also present as well as the so called super-symplectic algebra [K19] related to the isometries of "the world of classical worlds" (the space of light-like 3-surfaces) with generators transforming according to the irreducible representations of rotation group $SO(3)$ and color group $SU(3)$. It must be emphasized that TGD view about

conformal symmetry generalizes that of string models since light-like 3-surfaces (orbits of partons) are the basic dynamical objects [K19] .

What about more general reductive groups?

Langlands correspondence is conjectured to apply to all reductive Lie groups. The question is whether there is room for them in TGD Universe. There are good hopes.

1. Pairs formed by finite Galois groups and Lie groups containing them and defining sub-factors

Any amenable (in particular compact Lie) group acting as outer automorphism of $\mathcal{M}$ defines a unique sub-factor $\mathcal{N} \subset \mathcal{M}$ as a group leaving the elements of $\mathcal{N}$ invariant. The representations of discrete subgroups of compact groups extended to representations of the latter would define natural candidates for Langlands correspondence and would expand the repertoire of the Galois groups representable in terms of unique factors. If one gives up the uniqueness condition for the sub-factor, one can expect that almost any Lie group can define a sub-factor.

2. McKay correspondences and inclusions

The so called McKay correspondence assigns to the finite subgroups of SU(2) extended Dynkin diagrams of ADE type Kac-Moody algebras. McKay correspondence also generalizes to the discrete subgroups of other compact Lie groups q [A172]. The obvious question is how closely this correspondence between finite groups and Lie groups relates with Langlands correspondence.

The principal graphs representing concisely the fusion rules for Connes tensor products of $\mathcal{M}$ regarded as $\mathcal{N}$ bi-module are represented by the Dynkin diagrams of ADE type Lie groups for $\mathcal{M} : \mathcal{N} < 4$ (not all of them appear). For index $\mathcal{M} : \mathcal{N} = 4$ extended ADE type Dynkin diagrams labelling Kac-Moody algebras are assigned with these representations.

I have proposed that TGD Universe is able to emulate almost any ADE type gauge theory and conformal field theory involving ADE type Kac-Moody symmetry and represented somewhat misty ideas about how to construct representations of ADE type gauge groups and Kac-Moody groups using many particle states at the sheets of multiple coverings $H \rightarrow H/G_a \times G_b$ realizing the idea about hierarchy of dark matters already mentioned. Also vertex operator construction also distinguishes ADE type Kac-Moody algebras in a special position.

It is possible to considerably refine this conjecture picture by starting from the observation that the set of generating elements for Lie algebra corresponds to a union of triplets $\{J_i^{\pm}, J_i^3\}$, $i = 1, ..., n$ generating SU(2) sub-algebras. Here n is the dimension of the Cartan sub-algebra. The non-commutativity of quantum Clifford algebra suggests that Connes tensor product can induce deformations of algebraic structures so that ADE Lie algebra could result as a kind of deformation of a direct sum of commuting SU(2) Lie (Kac-Moody) algebras associated with a Connes tensor product. The physical interpretation might in terms of a formation of a bound state. The finite depth of $\mathcal{N}$ would mean that this mechanism leads to ADE Lie algebra for an n-fold tensor power, which then becomes a repetitive structure in tensor powers. The repetitive structure would conform with the diagonal imbedding of Galois groups giving rise to a representation in terms of outer automorphisms.

This picture encourages the guess that it is possible to represent the action of Galois groups on number theoretic braids as action of subgroups of dynamically generated ADE type groups on configuration space spinors. The connection between the representations of finite groups and reductive Lie groups would result from the natural extension of the representations of finite groups to those of Lie groups.

*3. What about Langlands correspondence for Kac-Moody groups?*vm

The appearance of also Kac-Moody algebras raises the question whether Langlands correspondence could generalize also to the level of Kac-Moody groups or algebras and whether it could be easier to understand the Langlands correspondence for function fields in terms of Kac-Moody groups as the transition from global to local occurring in both cases suggests.

Could Langlands duality for groups reduce to super-symmetry?

Langlands program involves dualities and the general structure of TGD suggests that there is a wide spectrum of these dualities.

1. A very fundamental duality would be between infinite-dimensional Clifford algebra and group algebra of S_∞ or of braid group B_∞. For instance, one can ask could it be possible to map this group algebra to the union of the moduli spaces of conformal equivalence classes of partonic 2-surfaces. HFFs consists of bounded operators of a separable Hilbert space. Therefore they are expected to have very many avatars: for instance there is an infinite number sub-factors isomorphic to the factor. This seems to mean infinite number of manners to represent Galois groups reflected as dualities.

2. Langlands program involves the duality between reducible Lie groups G and its Langlands dual having dual root lattices. The interpretation for this duality in terms of electric-magnetic duality is suggested by Witten [A174] . TGD suggests an alternative interpretation. The super symmetry aspect of super-conformal symmetry suggests that bosonic and fermionic representations of Galois groups could be very closely related. In particular, the representations in terms of WCW spinor s and in terms of modular degrees of freedom of partonic 2-surface could be in some sense dual to each other. Rotation groups have a natural action on WCW spinor s whereas symplectic groups have a natural action in the moduli spaces of partonic 2-surfaces of given genus possessing symplectic and Kähler structure. Langlands correspondence indeed relates $SO(2g+1,R)$ realized as rotations of WCW spinor s and $Sp(2g,C)$ realized as transformations in modular degrees of freedom. Hence one might indeed wonder whether super-symmetry could be behind the Langlands correspondence.

13.3.5 What is the role of infinite primes?

Infinite primes primes at the lowest level of the hierarchy can be represented as polynomials and as rational functions at higher levels. These in turn define rational function fields. Physical states correspond in general to infinite rationals which reduce to unit in real sense but have arbitrarily complex number theoretical anatomy [K70] , [L3, L11] .

Does infinite prime characterize the l-adic representation of Galois group associated with given partonic 2-surface

Consider first the lowest level of hierarchy of infinite primes [K70] . Infinite primes at the lowest level of hierarchy are in a well-defined sense composites of finite primes and correspond to states of super-symmetric arithmetic quantum field theory. The physical interpretation of primes appearing as composites of infinite prime is as characterizing of the p-adic prime p assigned by the modified Dirac action to partonic 2-surfaces associated with a given 3-surface [K15, K19] .

This p-adic prime could naturally correspond to the possible prime associated with so called l-adic representations of the Galois group(s) associated with the p-adic counterpart of the partonic 2-surface. Also the Galois groups associated with the real partonic 2-surface could be represented in this manner. The generalization of moduli space of conformal equivalence classes must be generalized to its p-adic variant. I have proposed this generalization in context of p-adic mass calculations [K17] .

It should be possible to identify WCW spinor s associated with real and p-adic sectors if anti-commutations relations for the fermionic oscillator operators make sense in any number field (that is involve only rational or algebraic numbers). Physically this seems to be the only sensible option.

Could one assign Galois groups to the extensions of infinite rationals?

A natural question is whether one could generalize the intuitions from finite number theory to the level of infinite primes, integers, and rationals and construct Galois groups and there representations for them. This might allow alternative very number theoretical approach to the geometric Langlands duality.

1. The notion of infinite prime suggests that there is entire hierarchy of infinite permutation groups such that the N_∞ at given level is defined as the product of all infinite integers at that level. Any group is a permutation group in formal sense. Could this mean that the hierarchy of infinite primes could allow to interpret the infinite algebraic sub-groups of Lie groups as Galois groups? If so one would have a unification of group theory and number theory.

2. An interesting question concerns the interpretation of the counterpart of hyper-finite factors of type II_1 at the the higher levels of hierarchy of infinite primes. Could they relate to a hierarchy of local algebras defined by HFF? Could these local algebras be interpreted in terms of direct integrals of HFFs so that nothing essentially new would result from von Neumann algebra point of view? Would this be a correlate for the fact that finite primes would be the irreducible building block of all infinite primes at the higher levels of the hierarchy?

3. The transition from number fields to function fields is very much analogous to the replacement of group with a local gauge group or algebra with local algebra. I have proposed that this kind of local variant based on multiplication by of HFF by hyper-octonion algebra could be the fundamental algebraic structure from which quantum TGD emerges. The connection with infinite primes would suggest that there is infinite hierarchy of localizations corresponding to the hierarchy of space-time sheets.

4. Perhaps it is worth of mentioning that the order of S_∞ is formally $N_\infty = \lim_{n\to\infty} n!$. This integer is very large in real sense but zero in p-adic sense for all primes. Interestingly, the numbers $N_\infty/n+n$ behave like normal integers in p-adic sense and also number theoretically whereas the numbers $N_\infty/n+1$ behave as primes for all values of n. Could this have some deeper meaning?

Could infinite rationals allow representations of Galois groups?

One can also ask whether infinite primes could provide representations for Galois groups. For instance, the decomposition of infinite prime to primes (or prime ideals) assignable to the extension of rationals is expected to make sense and would have clear physical interpretation. Also (hyper-)quaternionic and (hyper-)octonionic primes can be considered and I have proposed explicit number theoretic interpretation of the symmetries of standard model in terms of these primes. The decomposition of partonic primes to hyper-octonionic primes could relate to the decomposition of parton to regions, one for each number theoretic braid.

There are arguments supporting the view that infinite primes label the ground states of super-conformal representations [K19, K70] . The question is whether infinite primes could allow to realize the action of Galois groups. Rationality of infinite primes would imply that the invariance of ground states of super-conformal representations under the braid realization of $Gal(\overline{Q}/Q)$ of finite Galois groups. The infinite prime as a whole could indeed be invariant but the primes in the decomposition to a product of primes in algebraic extension of rationals need not be so. This kind of decompositions of infinite prime characterizing parton could correspond to the above described decomposition of partonic 2-surface to regions X^2_k at which Galois groups act non-trivially. It could also be that only infinite integers are rational whereas the infinite primes decomposing them are hyper-octonionic. This would physically correspond to the decomposition of color singlet hadron to colored partons [K70] .

13.3.6 Could Langlands correspondence, McKay correspondence and Jones inclusions relate to each other?

The understanding of Langlands correspondence for general reductive Lie groups in TGD framework seems to require some physical mechanism allowing the emergence of these groups in TGD based physics. The physical idea would be that quantum dynamics of TGD is able to emulate the dynamics of any gauge theory or even stringy dynamics of conformal field theory having Kac-Moody type symmetry and that this emulation relies on quantum deformations induced by finite measurement resolution described in terms of Jones inclusions of sub-factors characterized by group G leaving elements of sub-factor invariant. Finite measurement resolution would would result simply from the fact that only quantum numbers defined by the Cartan algebra of G are measured.

There are good reasons to expect that infinite Clifford algebra has the capacity needed to realize representations of an arbitrary Lie group. It is indeed known that that any quantum group characterized by quantum parameter which is root of unity or positive real number can be assigned to Jones inclusion [A144] . For $q = 1$ this would gives ordinary Lie groups. In fact, all amenable groups define unique sub-factor and compact Lie groups are amenable ones.

It was so called McKay correspondence q [A172] which originally stimulated the idea about TGD as an analog of Universal Turing machine able to mimic both ADE type gauge theories and theories with ADE type Kac-Moody symmetry algebra. This correspondence and its generalization might also provide understanding about how general reductive groups emerge. In the following I try to cheat the reader to believe that the tensor product of representations of SU(2) Lie algebras for Connes tensor powers of $\mathcal{M}$ could induce ADE type Lie algebras as quantum deformations for the direct sum of n copies of $SU(2)$ algebras This argument generalizes also to the case of other compact Lie groups.

About McKay correspondence

McKay correspondence q [A172] relates discrete finite subgroups of $SU(2)$ ADE groups. A simple description of the correspondences is as follows q [A172].

1. Consider the irreps of a discrete subgroup $G \subset SU(2)$ which correspond to irreps of G and can be obtained by restricting irreducible representations of $SU(2)$ to those of G. The irreducible representations of $SU(2)$ define the nodes of the graph.

2. Define the lines of graph by forming a tensor product of any of the representations appearing in the diagram with a doublet representation which is always present unless the subgroup is 2-element group. The tensor product regarded as that for $SU(2)$ representations gives representations $j-1/2$, and $j+1/2$ which one can decompose to irreducibles of G so that a branching of the graph can occur. Only branching to two branches occurs for subgroups yielding extended ADE diagrams. For the linear portions of the diagram the spins of corresponding SU(2) representations increase linearly as $.., j, j+1/2, j+1, ...$

   One obtains extended Dynkin diagrams of ADE series representing also Kac-Moody algebras giving A_n, D_n, E_6, E_7, E_8. Also A_∞ and $A_{-\infty,\infty}$ are obtained in case that subgroups are infinite. The Dynkin diagrams of non-simply laced groups B_n $(SO(2n+1))$, C_n (symplectic group $Sp(2n)$ and quaternionic group $Sp(n)$), and exceptional groups G_2 and F_4 are not obtained.

ADE Dynkin diagrams labelling Lie groups instead of Kac-Moody algebras and having one node less, do not appear in this context but appear in the classification of Jones inclusions for $\mathcal{M} : \mathcal{N} < 4$. As a matter fact, ADE type Dynkin diagrams appear in very many contexts as one can learn from John Baez's This Week's Finds [A94] .

1. The classification of integral lattices in Rn having a basis of vectors whose length squared equals 2

2. The classification of simply laced semisimple Lie groups.

3. The classification of finite sub-groups of the 3-dimensional rotation group.

4. The classification of simple singularities . In TGD framework these singularities could be assigned to origin for orbifold CP_2/G, $G \subset SU(2)$.

5. The classification of tame quivers.

Principal graphs for Connes tensor powers $\mathcal{M}$

The thought provoking findings are following.

1. The so called principal graphs characterizing $\mathcal{M} : \mathcal{N} = 4$ Jones inclusions for $G = SU(2)$ are extended Dynkin diagrams characterizing ADE type affine (Kac-Moody) algebras. D_n is possible only for $n \geq 4$.

2. $\mathcal{M} : \mathcal{N} < 4$ Jones inclusions correspond to ordinary ADE type diagrams for a subset of simply laced Lie groups (all roots have same length) A_n $(SU(n))$, D_{2n} $(SO(2n))$, and E_6 and E_8. Thus D_{2n+1} $(SO(2n+2))$ and E_7 are not allowed. For instance, for $G = S_3$ the principal graph is not D_3 Dynkin diagram.

The conceptual background behind principal diagram is necessary if one wants to understand the relationship with McKay correspondence.

1. The hierarchy of higher commutations defines an invariant of Jones inclusion $\mathcal{N} \subset \mathcal{M}$. Denoting by $\mathcal{N}'$ the commutant of $\mathcal{N}$ one has $\mathcal{N}' \cap \mathcal{N} \subset \mathcal{N}' \cap \mathcal{M} \subset \mathcal{N}' \cap \mathcal{M}^1 \subset ...$ and $C = \mathcal{M}' \cap \mathcal{M} \subset \mathcal{M}' \cap \mathcal{M}^1 \subset ....$ There is also a sequence of vertical inclusions $\mathcal{M}' \cap \mathcal{M}^k \subset \mathcal{N}' \cap \mathcal{M}^k$. This hierarchy defines a hierarchy of Temperley-Lieb algebras Templieb assignable to a finite hierarchy of braids. The commutants in the hierarchy are direct sums of finite-dimensional matrix algebras (irreducible representations) and the inclusion hierarchy can be described in terms of decomposition of irreps of k^{th} level to irreps of $(k-1)^{th}$ level irreps. These decomposition can be described in terms of Bratteli diagrams [A122] .

2. The information provided by infinite Bratteli diagram can be coded by a much simpler bipartite diagram having a preferred vertex. For instance, the number of $2k$-loops starting from it tells the dimension of k^{th} level algebra. This diagram is known as principal graph.

Principal graph emerges also as a concise description of the fusion rules for Connes tensor powers of $\mathcal{M}$.

1. It is natural to decompose the Connes tensor powers q [A172] $\mathcal{M}_k = \mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$ to irreducible $\mathcal{M} - \mathcal{M}$, $\mathcal{N} - \mathcal{M}$, $\mathcal{M} - \mathcal{N}$, or $\mathcal{N} - \mathcal{N}$ bi-modules. If $\mathcal{M} : \mathcal{N}$ is finite this decomposition involves only finite number of terms. The graphical representation of these decompositions gives rise to Bratteli diagram.

2. If $\mathcal{N}$ has finite depth the information provided by Bratteli diagram can be represented in nutshell using principal graph. The edges of this bipartite graph connect $\mathcal{M} - \mathcal{N}$ vertices to vertices describing irreducible $\mathcal{N} - \mathcal{N}$ representations resulting in the decomposition of $\mathcal{M} - \mathcal{N}$ irreducibles. If this graph is finite, $\mathcal{N}$ is said to have finite depth.

A mechanism assigning to tensor powers Jones inclusions ADE type gauge groups and Kac-Moody algebras

The proposal made for the first time in [K25] is that in $\mathcal{M} : \mathcal{N} < 4$ case it is possible to construct ADE representations of gauge groups or quantum groups and in $\mathcal{M} : \mathcal{N} = 4$ using the additional degeneracy of states implied by the multiple-sheeted cover $H \rightarrow H/G_a \times G_b$ associated with space-time correlates of Jones inclusions. Either G_a or G_b would correspond to G. In the following this mechanism is articulated in a more refined manner by utilizing the general properties of generators of Lie-algebras understood now as a minimal set of elements of algebra from which the entire algebra can be obtained by repeated commutation operator (I have often used " Lie algebra generator" as an synonym for "Lie algebra element"). This set is finite also for Kac-Moody algebras.

1. Two observations

The explanation to be discussed relies on two observations.

1. McKay correspondence for subgroups of G ($\mathcal{M} : \mathcal{N} = 4$) *resp.* its variants ($\mathcal{M} : \mathcal{N} < 4$) and its counterpart for Jones inclusions means that finite-dimensional irreducible representations of allowed $G \subset SU(2)$ label both the Cartan algebra generators and the Lie (Kac-Moody) algebra generators of t_+ and t_- in the decomposition $g = h \oplus t_+ \oplus t_-$, where h is the Lie algebra of maximal compact subgroup.

2. Second observation is related to the generators of Lie-algebras and their quantum counterparts (see Appendix for the explicit formulas for the generators of various algebras considered). The observation is that each Cartan algebra generator of Lie- and quantum group algebras, corresponds to a triplet of generators defining an SU(2) sub-algebra. The Cartan algebra of affine algebra contains besides Lie group Cartan algebra also a derivation d identifiable as an infinitesimal scaling operator L_0 measuring the conformal weight of the Kac-Moody generators. d is exceptional in that it does not give rise to a triplet. It corresponds to the preferred node added to the Dynkin diagram to get the extended Dynkin diagram.

2. Is ADE algebra generated as a quantum deformation of tensor powers of $SU(2)$ Lie algebras representations?

The ADE type symmetry groups could result as an effect of finite quantum resolution described by inclusions of HFFs in TGD inspired quantum measurement theory.

1. The description of finite resolution typically leads to quantization since complex rays of state space are replaced as $\mathcal{N}$ rays. Hence operators, which would commute for an ideal resolution cease to do so. Therefore the algebra $SU(2) \otimes ... \otimes SU(2)$ characterized by n mutually commuting triplets, where n is the number of copies of $SU(2)$ algebra in the original situation and identifiable as quantum algebra appearing in $\mathcal{M}$ tensor powers with $\mathcal{M}$ interpreted as $\mathcal{N}$ module, could suffer quantum deformation to a simple Lie algebra with $3n$ Cartan algebra generators. Also a deformation to a quantum group could occur as a consequence.

2. This argument makes sense also for discrete groups $G \subset SU(2)$ since the representations of G realized in terms of WCW spinor s extend to the representations of SU(2) naturally.

3. Arbitrarily high tensor powers of $\mathcal{M}$ are possible and one can wonder why only finite-dimensional Lie algebra results. The fact that $\mathcal{N}$ has finite depth as a sub-factor means that the tensor products in tensor powers of $\mathcal{N}$ are representable by a finite Dynkin diagram. Finite depth could thus mean that there is a periodicity involved: the kn tensor powers decomposes to representations of a Lie algebra with $3n$ Cartan algebra generators. Thus the additional requirement would be that the number of tensor powers of $\mathcal{M}$ is multiple of n.

3. Space-time correlate for the tensor powers $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$

By quantum classical correspondence there should exist space-time correlate for the formation of tensor powers of $\mathcal{M}$ regarded as $\mathcal{N}$ module. A concrete space-time realization for this kind of situation in TGD would be based on n-fold cyclic covering of H implied by the $H \rightarrow H/G_a \times G_b$ bundle structure in the case of say G_b. The sheets of the cyclic covering would correspond to various factors in the n-fold tensor power of $SU(2)$ and one would obtain a Lie algebra, affine algebra or its quantum counterpart with n Cartan algebra generators in the process naturally. The number n for space-time sheets would be also a space-time correlate for the finite depth of $\mathcal{N}$ as a factor.

WCW spinors could provide fermionic representations of $G \subset SU(2)$. The Dynkin diagram characterizing tensor products of representations of $G \subset SU(2)$ with doublet representation suggests that tensor products of doublet representations associated with n sheets of the covering could realize the Dynkin diagram.

Singlet representation in the Dynkin diagram associated with irreps of G would not give rise to an SU(2) sub-algebra in ADE Lie algebra and would correspond to the scaling generator. For ordinary Dynkin diagram representing gauge group algebra scaling operator would be absent and therefore also the exceptional node. Thus the difference between $(\mathcal{M} : \mathcal{N} = 4)$ and $(\mathcal{M} : \mathcal{N} < 4)$ cases would be that in the Kac-Moody group would reduce to gauge group $\mathcal{M} : \mathcal{N} < 4$ because Kac-Moody central charge k and therefore also Virasoro central charge resulting in Sugawara construction would vanish.

4. Do finite subgroups of $SU(2)$ play some role also in $\mathcal{M} : \mathcal{N} = 4$ case?

One can ask wonder the possible interpretation for the appearance of extended Dynkin diagrams in $(\mathcal{M} : \mathcal{N} = 4)$ case. Do finite subgroups $G \subset SU(2)$ associated with extended Dynkin diagrams appear also in this case. The formal analog for $H \rightarrow G_a \times G_b$ bundle structure would be $H \rightarrow H/G_a \times SU(2)$. This would mean that the geodesic sphere of CP_2 would define the fiber. The notion of number theoretic braid meaning a selection of a discrete subset of algebraic points of the geodesic sphere of CP_2 suggests that $SU(2)$ actually reduces to its subgroup G also in this case.

5. Why Kac-Moody central charge can be non-vanishing only for $\mathcal{M} : \mathcal{N} = 4$?

From the physical point of view the vanishing of Kac-Moody central charge for $\mathcal{M} : \mathcal{N} < 4$ is easy to understand. If parton corresponds to a homologically non-trivial geodesic sphere, space-time surface typically represents a string like object so that the generation of Kac-Moody central extension would relate directly to the homological non-triviality of partons. For instance, cosmic strings are string like objects of form $X^2 \times Y^2$, where X^2 is minimal surface of M^2 and Y^2 is a holomorphic sub-manifold of CP_2 reducing to a homologically non-trivial geodesic sphere in the simplest situation. A conjecture that deserves to be shown wrong is that central charge k is proportional/equal to the absolute value of the homology (Kähler magnetic) charge h.

6. More general situation

McKay correspondence generalizes also to the case of subgroups of higher-dimensional Lie groups q [A172]. The argument above makes sense also for discrete subgroups of more general compact Lie groups H since also they define unique sub-factors. In this case, algebras having Cartan algebra with nk generators, where n is the dimension of Cartan algebra of H, would emerge in the process. Thus there are reasons to believe that TGD could emulate practically any dynamics having gauge group or Kac-Moody type symmetry. An interesting question concerns the interpretation of non-ADE type principal graphs associated with subgroups of SU(2).

7. Flavor groups of hadron physics as a support for HFF?

The deformation assigning to an n-fold tensor power of representations of Lie group G with k-dimensional Cartan algebra a representation of a Lie group with nk-dimensional Cartan algebra could be also seen as a dynamically generated symmetry. If quantum measurement is characterized by the choice of Lie group G defining measured quantum numbers and defining Jones inclusion characterizing the measurement resolution, the measurement process itself would generate these dynamical symmetries. Interestingly, the flavor symmetry groups of hadron physics cannot be justified from the structure of the standard model having only electro-weak and color group as fundamental symmetries. In TGD framework flavor group $SU(n)$ could emerge naturally as a fusion of n quark doublets to form a representation of $SU(n)$.

Conformal representations of braid group and a possible further generalization of McKay correspondence

Physically especially interesting representations of braid group and associated Temperley-Lieb-Jones algebras (TLJ) are representations provided by the n-point functions of conformal field theories studied in [A178] . The action of the generator of braid group on n-point function corresponds to a duality transformation of old-fashioned string model (or crossing) represented as a monodromy relating corresponding conformal blocks. This effect can be calculated. Since the index $r = \mathcal{M} : \mathcal{N}$ appears as a parameter in TLJ algebra, the formulas expressing the behavior of n-point functions under the duality transformation reveal also the value of index which might not be easy to calculate otherwise.

Note that in TGD framework the arguments of n-point function would correspond to the strands of the number theoretic braid and thus to the points of the geodesic sphere S^2 associated with the light-cone boundary $\delta M^4_\pm$. The projection to the geodesic sphere of CP_2 projection would be same for all these strands.

WZW model for group G and Kac-Moody central charge k quantum phase is discussed in [A178] . The non-triviality of braiding boils to the fact that quantum group G_q defines the effect of braiding operation. Quantum phase is given as $q = exp(i\pi/(k + C(G))$, where $C(G)$ is the value of Casimir operator in adjoint representation. The action of the braid group generator reduces to the unitary matrix relating the basis defined by the tensor product of representations of G_q to the basis obtained by application of a generator of the braid group. For n-point functions of primary fields belonging to a representation D of G, index is the square of the quantum dimension $d_q(D)$ of the corresponding representation of G_q. Hence each primary field correspond to its own inclusion of HFF, which corresponds to $n \to \infty$-point function.

The result could have been guessed as the dimension of quantum Clifford algebra emerging naturally in inclusion when HFF is represented as an infinite tensor power of $M(d(D), C)$. For $j = 1/2$ representation of $SU(2)$ standard Jones inclusions with $r < 4$ are obtained. The resulting inclusion is irreducible ($\mathcal{N}' \cap \mathcal{M} = C$, where $\mathcal{N}'$ is the commutator of $\mathcal{N}'$). Using the representation of HFF as infinite tensor power of $M(2, C)$ the result would not be so easy to understand.

The mathematical challenge would be to understand how the representations HFF as an infinite tensor power of $M(n, C)$ relate to each other for different values of n. It might be possible to understand the relationship between different infinite tensor power representations of HFF by representing $M(n_1, C)$ as a sub-algebra of a tensor power of a finite tensor power of $M(n_2, C)$. Perhaps a detailed construction of the maps between representations of HFF as infinite tensor power of $M(n, C)$ for various values of n could reveal further generalizations of McKay correspondence.

13.3.7 Technical questions related to Hecke algebra and Frobenius element

Frobenius elements

Frobenius element Fr_p is mapped to a conjugacy class of Galois group using the decomposition of prime p to prime ideals in the algebraic extension K/F.

1. At the level of braid group Frobenius element Fr_p corresponds to some conjugacy class of Galois group acting imbedded to S_n (only the conjugacy equivalence class is fixed) and thus can be mapped to an element of the braid group. Hence it seems possible to assign to Fr_p an element of infinitely cyclic subgroup of the braid group.

2. One can always reduce in given representation the element of given conjugacy class to a diagonal matrix so that it is possible to chose the representatives of Fr_p to be commuting operators. These operators would act as a spinor rotation on quantum Clifford algebra elements defined by Jones inclusion and identifiable as element of some cyclic group of the group G defining the sub-factor via the diagonal embedding.

3. Fr_p for a given finite Galois group G should have representation as an element of braid group to which G is imbedded as a subgroup. It is possible to chose the representatives of Fr_p so that they commute. Could one chose them in such a manner that they belong to the commuting subgroup defined by even (odd) generators e_i? The choice of representatives for Fr_p for various Galois groups must be also consistent with the hierarchies of intermediate extensions of rationals associated with given extension and characterized by subgroups of Galois group for the extension.

How the action of commutative Hecke algebra is realized in hyper-finite factor and braid group?

One can also ask how to imbed Hecke algebra to the braid algebra. Hecke algebra for a given value of prime p and group $GL(n, R)$ is a polynomial algebra in Hecke algebra generators. There is a fundamental difference between Hecke algebra and Frobenius element Fr_p in the sense that Fr_p has finite order as an element of finite Galois group whereas Hecke algebra elements do not except possibly for representations. This means that Hecke algebra cannot have a representation in a finite Galois groups.

Situation is different for braid algebra generators since they do not satisfy the condition $e_i^2 = 1$ and odd and even generators of braid algebra commute. The powers of Hecke algebra generators would correspond to the powers of basic braiding operation identified as a π twist of neighboring strands. For unitary representations eigenvalues of e_i are phase factors. Therefore Hecke algebra might be realized using odd or even commuting sub-algebra of braid algebra and this could allow to deduce the Frobenius-Hecke correspondence directly from the representations of braid group. The basic questions are following.

1. Is it possible to represent Hecke algebra as a subalgebra of braid group algebra in some natural manner? Could the infinite cyclic group generated by braid group image of Fr_p belong represent element of Hecke algebra fixed by the Langlands correspondence? If this were the case then the eigenvalues of Frobenius element Fr_p of Galois group would correspond to the eigen values of Hecke algebra generators in the manner dictated by Langlands correspondence.

2. Hecke operators $H_{p,i}$, $i = 1, .., n$ commute and expressible as two-side cosets in group $GL(n, Q_p)$. This group acts in $\mathcal{M}$ and the action could be made rather explicit by using a proper representations of $\mathcal{M}$ (note however that physical situation can quite well distinguish between various representations). Does the action of the Hecke sub-algebra fixed by

Hecke-Frobenius correspondence co-incide with the action of Frobenius element Fr_p identified as an element of braid sub-group associated with some cyclic subgroup of the Galois group identified as a group defining the sub-factor?

13.4 Langlands conjectures and the most recent view about TGD

Langlands program [A56, A150, A148] relies on very general conjectures about a connection between number theory and harmonic analysis relating the representations of Galois groups with the representations of certain kinds of Lie groups to each other. Langlands conjecture has many forms and it is indeed a conjecture and many of them are imprecise since the notions involved are not sharply defined.

Peter Woit noticed that Edward Frenkel had given a talk with rather interesting title "What do Fermat's Last Theorem and Electro-magnetic Duality Have in Common?" [A105]? I listened the talk and found it very inspiring. The talk provides bird's eye of view about some basic aspects of Langlands program using the language understood by physicist. Also the ideas concerning the connection between Langlands duality and electric-magnetic duality generalized to S-duality in the context of non-Abelian gauge theories and string theory context and developed by Witten and Kapustin [A174] and followers are summarized. In this context $D = 4$ and twisted version of $\mathcal{N} = 4$ SYM familiar from twistor program and defining a topological QFT appears.

For some years ago I made my first attempt to understand what Langlands program is about and tried to relate it to TGD framework [K35]. At that time I did not really understand the motivations for many of the mathematical structures introduced. In particular, I did not really understand the motivations for introducing the gigantic Galois group of algebraic numbers regarded as algebraic extension of rationals.

1. Why not restrict the consideration to finite Galois groups [A38] or their braided counterparts (as I indeed effectively did [K35])? At that time I concentrated on the question what the enormous Galois group of algebraic numbers regarded as algebraic extension of rationals could mean, and proposed that it could be identified as a symmetric group consisting of permutations of infinitely many objects. The definition of this group is however far from trivial. Should one allow as generators of the group only the permutations affecting only finite number of objects or permutations of even infinite number of objects?

   The analogous situation for the sequences of binary digits would lead to a countable set of sequence of binary digits forming a discrete set of finite integers in real sense or to 2-adic integers forming a 2-adic continuum. Something similar could be expected now. The physical constraints coming the condition that the elements of symmetric group allow lifting to braidings suggested that the permutations permuting infinitely many objects should be periodic meaning that the infinite braid decomposes to an infinite number of identical N-braids and braiding is same for all of them. The p-adic analog would be p-adic integers, which correspond to rationals having periodical expansion in powers of p. Braids would be therefore like pinary digits. I regarded this choice as the most realistic one at that time. I failed to realize the possibility of having analogs of p-adic integers by general permutations. In any case, this observation makes clear that the unrestricted Galois group is analogous to a Lie group in topology analogous to p-adic topology rather than to discrete group. Neither did I realize that the Galois groups could be finite and be associated with some other field than rationals, say a Galois group associated with the field of polynomials of n-variable with rational coefficients and with its completion with coefficients replaced by algebraic numbers.

2. The ring of adeles [A4] can be seen as a Cartesian product of non-vanishing real numbers $R_\times$ with the infinite Cartesian product $\prod Z_p$ having as factors p-adic integers Z_p for all values of prime p. Rational adeles are obtained by replacing R with rationals Q and requiring that multiplication of rational by integers is equivalent with multiplication of any Z_p with rational. Finite number of factors in Z_p can correspond to Q_p: this is required in to have finite adelic norm defined as the product of p-adic norms. This definition implicitly regards rationals as

common to all number fields involved. At the first encounter with adeles I did not realize that this definition is in spirit with the basic vision of TGD.

The motivation for the introduction of adele is that one can elegantly combine the algebraic groups assignable to rationals (or their extensions) and all p-adic number fields or even more general function fields such as polynomials with some number of argument at the same time as a Cartesian product of these groups as well as to finite fields. This is indeed needed if one wants to realize number theoretic universality which is basic vision behind physics as generalized number theory vision. This approach obviously means enormous economy of thought irrespective of whether one takes adeles seriously as a physicist.

In the following I will discuss Taniyama-Shimura-Weil theorem and Langlands program from TGD point view.

13.4.1 Taniyama-Shimura-Weil conjecture from the perspective of TGD

Taniyama-Shimura-Weil theorem

It is good to consider first the Taniyama -Shimura-Weil conjecture [A92] from the perspective provided by TGD since this shows that number theoretic Langlands conjecture could be extremely useful for practical calculations in TGD framework.

1. Number theoretical universality requires that physics in real number field and various p-adic number fields should be unified to a coherent hole by a generalization of the notion of number: different number fields would be like pages of book intersecting along common rationals. This would hold true also for space-time surfaces and imbedding space but would require some preferred coordinates for which rational points would determined the intersection of real and p-adic worlds. There are good reasons for the hypothesis that life resides in the intersection of real and p-adic worlds.

   The intersection would correspond at the level of partonic 2-surfaces rational points of these surfaces in some preferred coordinates, for which a finite-dimensional family can be identified on basis of the fundamental symmetries of the theory. Allowing algebraic extensions one can also consider also some algebraic as common points. In any case the first question is to count the number of rational points for a partonic 2-surface.

   2-dimensional Riemann surfaces serve also as a starting point of number theoretic Langlands problem and the same is true for the geometric Langlands program concentrating on Riemann surfaces and function fields defined by holomorphic functions.

2. The number theoretic side of Taniyama-Shimura-Weil (TSW briefly) theorem for elliptic surfaces, which is essential for the proof of Fermat's last theorem, is about counting the integer (or equivalently rational) points of the elliptic surfaces

$$y^2 = x^3 + ax + b \ , \ \ a,b \in Z \ .$$

   The theorem relates number theoretical problem to a problem of harmonic analysis, which is about group representations. What one does is to consider the above Diophantine equation modulo p for all primes p. Any solution with finite integers smaller than p defines a solution in real sense if $mod \ p$ operation does not affect the equations. Therefore the existence of a finite number of solutions involving finite integers in real sense means that for large enough p the number a_p of solutions becomes constant.

3. On harmonic analysis one studies so called modular forms $f(\tau)$, where τ is a complex coordinate for upper half plane defining moduli space for the conformal structures on torus. Modular forms have well defined transformation properties under group $Gl_2(R)$: the action is defined by the formula $\tau \rightarrow (a\tau + b)/(c\tau + d)$. The action of $Gl_2(Z)$ or its appropriate subgroup is such that the modular form experiences a mere multiplication by a phase factor: $D(hk) = c(h,k)D(h)D(k)$. The phase factors obey cocycle conditions $D(h,k)D(g,hk) = D(gh,k)D(g,h)$ guaranteeing the associativity of the projective representation.

Modular transformations are clearly symmetries represented projectively as quantum theory indeed allows to do. The geometric interpretation is that one has projective representations in the fundamental domain of upper plane defined by the identification of the points differing by modular transformations. In conformally symmetric theories this symmetry is essential. Fundamental domain is analogous to lattice cell. One often speaks of cusp forms: cusp forms vanish at the boundary of the fundamental domain defined as the quotient of the upper half plane by a subgroup -call it Γ of the modular group $Sl_2(Z)$. The boundary corresponds to $Im(\tau) \rightarrow \infty$ or equivalently $q = exp(i2\pi\tau) \rightarrow 0$.

Remark: In TGD framework modular symmetry says that elementary particle vacuum functionals are modular invariants. For torus one has the above symmetry but for Riemann surface with higher genus modular symmetries correspond to a subgroup of $Sl_{2g}(Z)$.

4. One can expand the modular form as Fourier expansion using the variable $q = exp(i2\pi\tau)$ as

$$f(\tau) = \sum_{n>0} b_n q^n \quad .$$

$b_1 = 1$ fixes the normalization. $n > 0$ in the sum means that the form vanishes at the boundary of the fundamental domain associated with the group Γ. The TSW theorem says that for prime values $n = p$ one has $b_p = a_p$, where a_p is the number of mod p integer solutions to the equations defining the elliptic curve. At the limit $p \rightarrow \infty$ one obtains the number of real actual rational points of the curve if this number is finite. This number can be also infinite. The other coefficients b_n can be deduced from their values for primes since b_n defines what is known as a multiplicative character in the ring of integers implying $b_{mn} = b_m b_n$ meaning that b_n obeys a decomposition analogous to the decomposition of integer into a product of primes.

The definition of the multiplicative character is extremely general: for instance it is possible to define quantum counterparts of multiplicative characters and of various modular forms by replacing integers with quantum integers defined as products of quantum primes for all primes except one -call it p_0, which is replaced with its inverse: this definition of quantum integer appears in the deformation of distributions of integer valued random variable characterized by rational valued parameters and is motivated by strange findings of Shnoll [K5]. The interpretation could be in terms of TGD based view about finite measurement resolution bringing in quantum groups and also preferred p-adic prime naturally.

5. TSW theorem allows to prove Fermat's last theorem: if the latter theorem were wrong also TSW theorem would be wrong. What also makes TSW theorem so wonderful is that it would allow to count the number of rational points of elliptic surfaces just by looking the properties of the automorphic forms in $Gl_2(R)$ or more general group. A horrible looking problem of number theory is transformed to a problem of complex analysis which can be handled by using the magic power of symmetry arguments. This kind of virtue does not matter much in standard physics but in quantum TGD relying heavily on number theoretic universality situation is totally different. If TGD is applied some day the counting of rational points of partonic surfaces is everyday practice of theoretician.

How to generalize TSW conjecture?

The physical picture of TGD encourages to imagine a generalization of the Tanyama-Shimura-Weil conjecture.

1. The natural expectation is that the conjecture should make sense for Riemann surfaces of arbitrary genus g instead of $g = 1$ only (elliptic surfaces are tori). This suggests that one should one replace the upper half plane representing the moduli space of conformal equivalence classes of toric geometries with the $2g$-dimensional (in the real sense) moduli space of genus g conformal geometries identifiable as Teichmüller space.

   This moduli space has symplectic structure analogous to that of $g + g$-dimensional phase space and this structure relates closely to the cohomology defined in terms of integrals of holomorphic forms over the $g+g$ cycles which each handle carrying two cycles. The moduli are defined by the values of the holomorphic one-forms over the cycles and define a symmetric matrix Ω_{ij} (modular parameters), which is modular invariant [K17]. The modular parameters related $Sp_2g(Z)$ transformation correspond to same conformal equivalence class.

If Galois group and effective symmetry group G are representable as symplectic flows at the light-like boundary of $CD(\times CP_2)$, their action automatically defines an action in the moduli space. The action can be realized also as a symplectic flow defining a braiding for space-like braids assignable to the ends of the space-time surface at boundaries of CD or for time-like braids assignable to light-like 3-surfaces at which the signature of the induced metric changes and identified as orbits of partonic 2-surfaces analogous to black hole horizons.

2. It is possible to define modular forms also in this case. Most naturally they correspond to theta functions used in the construction of elementary particle functionals in this space [K17]. Siegel modular forms transform naturally under the symplectic group $Sp_{2g}(R)$ and are projectively invariant $Sp_{2g}(Z)$. More general moduli spaces are obtained by allowing also punctures having interpretation as the ends of braid strands and very naturally identified as the rational points of the partonic 2-surface. The modular forms defined in this extended moduli space could carry also information about the number of rational points in the same manner as the automorphic representations of $Gl_2(R)$ carry information about the number of rational points of elliptic curves.

3. How Tanyama-Shimura-Weil conjecture should be generalized? Also now one can consider power series of modular forms with coefficients b_n defining multiplicative characters for the integers of field in question. Also now the coefficients a_p could give the number of integer/rational points of the partonic 2-surface in mod p approximation and at the limit $p \to \infty$ the number of points a_p would approach to a constant if the number of points is finite.

4. The only sensical interpretation is that the analogs of elementary particle vacuum functionals [K17] identified as modular forms must be always restricted to partonic 2-surfaces having the same number of marked points identifiable as the end points of braid strands rational points. It also seems necessary to assume that the modular forms factorize to a products of two parts depending on Teichmüller parameters and positions of punctures. The assignment of fermionic and bosonic quantum numbers with these points conforms with this interpretation. As a special case these points would be rational. The surface with given number or marked points would have varying moduli defined by the conformal moduli plus the positions of the marked points. This kind of restriction would be physically very natural since it would mean that only braids with a given number of braid strands ending at fixed number of marked points at partonic 2-surfaces are considered in given quantum state. Of course, superpositions of these basis states with varying braid number would be allowed.

13.4.2 Unified treatment of number theoretic and geometric Langlands conjectures in TGD framework

One can already now wonder what the relationship of the TGD view about number theoretic Langlands conjecture to the geometric Langlands conjecture could be?

1. The generalization of Taniyama-Shinamure-Weil theorem to arbitrary genus would allow to deduce the number of rational points already for finite but large enough values of p from the Taylor coefficients of an appropriate modular form. Is this enough for the needs of TGD? The answer is "No". One must be able to count also numbers of "rational 2-surfaces" in the space of 2-surfaces and the mere generalization of TSW conjecture does not allow this. Geometric Langlands replacing rational points with "rational" surfaces is needed.

   If the geometric Langlands conjecture holds true in the spirit with TGD, it must allow to deduce the number of rational variants of of partonic 2-surfaces assignable to given quantum state defined to be a state with fixed number of braid strands for each partonic 2-surface of the collection. What is new is that collections of partonic 2-surfaces regarded as sub-manifolds of $M^4 \times CP_2$ are considered.

2. Finite measurement resolution conjectured to be definable in terms of effective symmetry group G defined by the inclusion of hyper-finite factors of type II_1 [K80](HFFs in the sequel) effectively replaces partonic 2-surfaces with collections of braid ends and the natural idea is that the orbits of these collections under finite algebraic subgroup of symmetry group defining finite measurement resolution gives rise to orbit with finite number of points (point understood now as collection of rational points). The TGD variant of the geometric Langlands conjecture would allow to deduce the number of different collections of rational braid ends for the quantum state considered (one particular WCW spinor field) from the properties of automorphic form.

3. Quantum group structure is associated with the inclusions of HFFs, with braid group representations, integrable QFTs, and also with the quantum Yangian symmetry [A184, A159] suggested strongly by twistor approach to TGD. In zero energy ontology physical states define Lie-algebra and the multi-locality of the scattering amplitudes with respect to the partonic 2-surfaces (that is at level of WCW) suggests also quantum Yangian symmetry. Therefore the Yangian of the Kac-Moody type algebra defining measurement resolution is a natural candidate for the symmetry considered. What is important is that the group structure is associated with a finite-dimensional Lie group.

This picture motivates the question whether number theoretic and geometric Langlands conjecture could be realized in the same framework? Could electric-magnetic duality generalized to S-duality imply these dualities and bring in the TGD counterpart of effective symmetry group G in some manner. This framework would be considerably more general than the 4-D QFT framework suggested by Witten and Kapustin [A174] and having very close analogies with TGD view about space-time.

The following arguments support the view that in TGD Universe number theoretic and geometric Langlands conjectures could be understood very naturally. The basic notions are following.

1. Zero energy ontology and the related notion of causal diamond CD (CD is short hand for the cartesian product of causal diamond of M^4 and of CP_2). This notion leads to the notion of partonic 2-surfaces at the light-like boundaries of CD and to the notion of string world sheet.

2. Electric-magnetic duality realized in terms of string world sheets and partonic 2-surfaces. The group G and its Langlands dual ${}^LG$ would correspond to the time-like and space-like braidings. Duality predicts that the moduli space of string world sheets is very closely related to that for the partonic 2-surfaces. The strong form of 4-D general coordinate invariance implying electric-magnetic duality and S-duality as well as strong form of holography indeed predicts that the collection of string world sheets is fixed once the collection of partonic 2-surfaces at light-like boundaries of CD and its sub-CDs is known.

3. The proposal is that finite measurement resolution is realized in terms of inclusions of hyperfinite factors of type II_1 at quantum level and represented in terms of confining effective gauge group [K80]. This effective gauge group could be some associate of G: gauge group, Kac-Moody group or its quantum counterpart, or so called twisted quantum Yangian strongly suggested by twistor considerations ("symmetry group" hitherto). At space-time level the finite measurement resolution would be represented in terms of braids at space-time level which come in two varieties correspond to braids assignable to space-like surfaces at the two light-like boundaries of CD and with light-like 3-surfaces at which the signature of the induced metric changes and which are identified as orbits of partonic 2-surfaces connecting the future and past boundaries of CDs.

   There are several steps leading from G to its twisted quantum Yangian. The first step replaces point like particles with partonic 2-surfaces: this brings in Kac-Moody character. The second step brings in finite measurement resolution meaning that Kac-Moody type algebra is replaced with its quantum version. The third step brings in zero energy ontology: one cannot treat single partonic surface or string world sheet as independent unit: always the collection of partonic 2-surfaces and corresponding string worlds sheets defines the geometric structure so that multilocality and therefore quantum Yangian algebra with multilocal generators is unavoidable.

In finite measurement resolution geometric Langlands duality and number theoretic Langlands duality are very closely related since partonic 2-surface is effectively replaced with the punctures representing the ends of braid strands and the orbit of this set under a discrete subgroup of G defines effectively a collection of "rational" 2-surfaces. The number of the "rational" surfaces in geometric Langlands conjecture replaces the number of rational points of partonic 2-surface in its number theoretic variant. The ability to compute both these numbers is very relevant for quantum TGD.

4. The natural identification of the associate of G is as quantum Yangian of Kac-Moody type group associated with Minkowskian open string model assignable to string world sheet representing a string moving in the moduli space of partonic 2-surface. The dual group corresponds to Euclidian string model with partonic 2-surface representing string orbit in the moduli space of the string world sheets. The Kac-Moody algebra assigned with simply laced G is obtained using the standard tachyonic free field representation obtained as ordered exponentials of Cartan algebra generators identified as transversal parts of M^4 coordinates for the braid strands. The importance of the free field representation generalizing to the case of non-simply laced groups in the realization of finite measurement resolution in terms of Kac-Moody algebra cannot be over-emphasized.

5. Langlands duality involves besides harmonic analysis side also the number theoretic side. Galois groups (collections of them) defined by infinite primes and integers having representation as symplectic flows defining braidings. I have earlier proposed that the hierarchy of these Galois groups define what might be regarded as a non-commutative homology and cohomology. Also G has this kind of representation which explains why the representations of these two kinds of groups are so intimately related. This relationship could be seen as a generalization of the MacKay correspondence between finite subgroups of $SU(2)$ and simply laced Lie groups.

6. Symplectic group of the light-cone boundary acting as isometries of the WCW geometry [K16] allowing to represent projectively both Galois groups and symmetry groups as symplectic flows so that the non-commutative cohomology would have braided representation. This leads to braided counterparts for both Galois group and effective symmetry group.

7. The moduli space for Higgs bundle playing central role in the approach of Witten and Kapustin to geometric Landlands program [A174] is in TGD framework replaced with the conformal moduli space for partonic 2-surfaces. It is not however possible to speak about Higgs field although moduli defined the analog of Higgs vacuum expectation value. Note that in TGD Universe the most natural assumption is that all Higgs like states are "eaten" by gauge bosons so that also photon and gluons become massive. This mechanism would be very general and mean that massless representations of Poincare group organize to massive ones via the formation of bound states. It might be however possible to see the contribution of p-adic thermodynamics depending on genus as analogous to Higgs contribution since the conformal moduli are analogous to vacuum expectation of Higgs field.

Number theoretic Langlands conjecture in TGD framework

Number theoretic Langlands conjecture generalizes TSW conjecture to a duality between two kinds of groups.

1. At the number theoretic side of the duality one has an n-dimensional representation of Galois group for the algebraic numbers regarded as algebraic extension of rationals. In the more general case one can consider arbitrary number field identified as algebraic extension of rationals. One can assign to the number field its rational adele. In the case of rationals this brings in both real numbers and p-adic numbers so that huge amount of information can be packed to the formulas. For anyone who has not really worked concretely with number theory it is difficult to get grasp of the enormous generality of the resulting theory.

2. At the harmonic analysis side of the conjecture one has n-dimensional representation of possibly non-compact Lie group G and its Langlands dual ${}^LG$ appearing also in the non-Abelian form of electric-magnetic duality. The idea that electric-magnetic duality generalized

to S-duality could provide a physical interpretation of Langlands duality is suggestive. $U(n)$ is self dual in Langlands sense but already for $G = SU(3)$ one has ${}^LG = SU(3)/Z_3$. For most Lie groups the Lie algebras of G and ${}^LG$ are identical but even the Lie algebras can be different. $Gl_2(R)$ is replaced with any reductive algebraic group and in the matrix representation of the group the elements of the group are replaced by adeles of the discrete number field considered.

3. Langlands duality relates the representations of the Galois group in question to the automorphic representations of G. The action of the Lie group is on the argument of the modular form so that one obtains infinite-dimensional representation of G for non-compact G analogous to a unitary representation of Lorentz group. The automorphic forms are eigenstates of the Casimir operator of G. Automorphy means that a subgroup Γ of the modular group leaves the automorphic form invariant modulo phase factor.

4. The action of the modular transformation $\tau \rightarrow -1/\tau$ in the case of $Gl_2(R)$ replaces G with ${}^LG$. In the more general case (for the moduli space of Riemann surfaces of genus g possessing n punctures) the definition of the modular transformation induce the change $G \rightarrow^L G$ does not look obvious. Even the idea that one has only two groups related by modular transformation is not obvious. For electromagnetic duality with τ interpreted in terms of complexified gauge coupling strength this interpretational problem is not encountered.

Geometric Langlands conjecture in TGD framework

Consider next the geometric Langlands conjecture from TGD view point.

1. The geometric variant of Langlands conjecture replaces the discrete number field F (rationals and their algebraic extensions say) with function number field- say rational function with rational coefficients- for which algebraic completion defines the gigantic Galois group. Witten and Kapustin [A174] proposed a concrete vision about how electric-magnetic duality generalized to S-duality could allow to understand geometric Langlands conjecture.

2. By strong form of general coordinate invariance implying holography the partonic 2-surfaces and their 4-D tangent space data (not completely free probably) define the basic objects so that WCW reduces to that for partonic 2-surfaces so that the formulation of geometric Langlands conjecture for the local field defined by holomorphic rational functions with rational coefficients at partonic 2-surface might make sense.

3. What geometric Langlands conjecture could mean in TGD framework? The transition from space-time level to the level of world of classical worlds suggests that polynomials with rational functions with rational coefficients define the analog of rational numbers which can be regarded to be in the intersection real and p-adic WCWs. Instead of counting rational points of partonic 2-surface one might think of counting the numbers of points in the intersection of real and p-adic WCWs in which life is suggested to reside. One might well consider the possibility that a kind of volume like measure for the number of these point is needed. Therefore the conjecture would be of extreme importance in quantum TGD. Especially so if the intersection of real and p-adic worlds is dense subset of WCW just as rationals form a dense subset of reals and p-adic numbers.

Electric-magnetic duality in TGD framework

Consider first the ideas of Witten and Kapustin in TGD framework.

1. Witten and Kapustin suggest that electric-magnetic duality and its generalization to S-duality in non-abelian is the physical counterpart of $G \leftrightarrow^L G$ duality in geometric Langlands. The model is essentially a modification $\mathcal{N} = 4$ SUSY to $\mathcal{N} = 2$ SUSY allowing this duality with Minkowski space replaced with a Cartesian product of two Riemann surfaces. In TGD framework M^4 would correspond naturally to space-time sheet allowing a slicing to string world sheets and partonic 2-surfaces. Witten and Kapustin call these 2-dimensional surfaces branes of type A and B with motivation coming from M-theory. The generalization of the basic dimensional formulas of S-duality to TGD framework implies that light-like 3-surfaces at

which the signature of the induced metric changes and space-like 3-surfaces at the boundaries of CDs are analogs of brane orbits. Branes in turn would be partonic 2-surfaces. S-duality would be nothing but strong form of general coordinate invariance.

2. Witten and Kapustin introduce the notions of electric and magnetic eigen branes and formulate the duality as a transformation permuting these branes with each other. In TGD framework the obvious identification of the electric eigen branes are as string world sheets and these can be indeed identified essentially uniquely. Magnetic eigen branes would correspond to partonic 2-surfaces.

3. Witten and Kapustin introduce gauge theory with given gauge group. In TGD framework there is no need to introduce gauge theory description since the symmetry group emerges as the effective symmetry group defining measurement resolution. Gauge theory is expected to be only an approximation to TGD itself. In fact, it seems that the interpretation of G as Lie-group associated with Kac-Moody symmetry is more appropriate in TGD framework. This would mean generalization of 2-D sigma model to string model in moduli space. The action of G would not be visible in the resolution used.

4. Edward Frenkel represents the conjecture that there is mysterious 6-dimensional theory behind the geometric Langlands duality. In TGD framework this theory might correspond to twistorial formulation of quantum TGD using instead of $M^4 \times CP_2$ the space $CP_3 \times CP_3$ with space-time surfaces replaced by 6-D sphere bundles.

Finite measurement resolution realized group theoretically

The notion of finite measurement resolution allows to identify the effective symmetry groups G and ${}^LG$ in TGD framework. The most plausible interpretation of G is as Lie group giving rise to Kac-Moody type symmetry and assignable to a string model defined in moduli space of partonic 2-surfaces. By electric magnetic duality the roles of the string world sheet and partonic 2-surface can be exchanged provided the replacement $G \rightarrow G_L$ is performed. The duality means a duality of closed Euclidian strings and Minkowskian open strings.

1. The vision is that finite measurement resolution realized in terms of inclusions of HFFs corresponds to effective which is gauge or Kac-Moody type local invariance extended to quantum Yangian symmetry. A given finite measurement resolution would correspond to effective symmetry G giving rise to confinement so that the effective symmetry indeed remains invisible as finite measurement resolution requires. The finite measurement resolution should allow to emulate almost any gauge theory or string model type theory. This theory might allow super-symmetrization reducing to broken super-symmetries of quantum TGD generated by the fermionic oscillator operators at partonic 2-surfaces and string world sheets.

2. Finite measurement resolution implies that the orbit of the partonic 2-surface reduces effectively to a braid. There are two kinds of braids. Time-like braids have their ends at the boundaries of CD consisting of rational points in the intersection of real and p-adic worlds. Space-like braids are assignable to the space-like 3-surfaces at the boundaries of CD and their ends co-incide with the ends of time-like braids. The electric-magnetic duality says that the descriptions based using either kind of braids is all that is needed and that the descriptions are equivalent.

   The counterpart of $\tau \rightarrow -1/\tau$ should relate these descriptions. This need not involve transformation of effective complex Kähler coupling strength although this option cannot be excluded. If this view is correct the descriptions in terms of string world sheets and partonic 2-surfaces would correspond to electric and magnetic descriptions, which is indeed a very natural interpretation. This geometric transformation should replace G with ${}^LG$.

3. Finite measurement resolution effectively replaces partonic 2-surface with a discrete set of points and space-time surface with string world sheets or partonic 2-surfaces. The natural question is whether finite measurement resolution also replaces geometric Langlands and the "rational" intersection of real and p-adic worlds with number theoretic Langlands and rational points of the partonic 2-surface. Notice that the rational points would be common

to the string world sheets and partonic 2-surfaces so that the duality of stringy and partonic descriptions would be very natural for fintie measurement resolution.

The basic question is how the symmetry group G emerges from finite measurement resolution. Are all Lie groups possible? Here the theory of Witten and Kapustin suggests guidelines.

1. What Witten and Kapustin achieve is a transformation of a twisted $\mathcal{N}=4$ SUSY in $M^4 = \Sigma \times C$, where Σ is "large" as compared to Riemann surface C SUSY to a sigma model in Σ with values of fields in the moduli space of Higgs bundle defined in C. If one accepts the basic conjecture that at least regions of space-time sheets allow a slicing by string world sheets and partonic 2-surfaces one indeed obtains $M^4 = \Sigma \times C$ type structure such that Σ corresponds to string world sheet and C to partonic 2-surface.

   The sigma model -or more generally string theory- would have as a natural target space the moduli space of the partonic 2-surfaces. This moduli space woudl have as coordinates its conformal moduli and the positions of the punctures expressible in terms of the imbedding space coordinates. For M^4 coordinates only the part transversal to Σ would represent physical degree of freedom and define complex coordinate. Each puncture would give rise to two complex E^2 coordinates and 2 pairs of complex CP_2 coordinates. If one identifies the string world sheets as an inverse image of a homologically non-trivial geodesic sphere as suggested in [K34]. This would eliminate CP_2 coordinates as dynamical variables and one would have just n complex valued coordinates.

2. How to construct the Lie algebra of the effective symmetry group G defining the measurement resolution? If G is gauge group there is no obvious guess for the recipe. If G defines Kac-Moody algebra the situation is much better. There exists an extremely general construction allowing a stringy construction of Kac-Moody algebra using only the elements of its Cartan algebra with central extension defined by integer valued central extension parameter k. The vertex operators defining the elements of the complement of the Cartan algebra of complexified Kac-Moody algebra are ordered exponentials of linear combinations of the Cartan algebra generators with coefficient given by the weights of the generators, which are essentially the quantum numbers assignable to them as eigenvalues of Cartan algebra generators acting in adjoint representations.

   The explicit expression for the Kac-Moody generator as function of complex coordinate of Riemann sphere S^2 is

   $$J_\alpha(z) =: exp(\alpha \cdot \phi(z))) : \quad .$$

   $J_\alpha(z)$ represents a generator in the complement of Cartan algebra in standard Cartan basis having quantum numbers α and $\phi(z)$ represents the Cartan algebra generator allowing decomposition into positive and negative frequency parts. The weights α must have the same length $((\alpha,\alpha)=2)$ meaning that the Lie group is simply laced. This representation corresponds to central extension parameter $k=1$. In bosonic string models these operators are problematic since they represent tachyons but in the recent context this not a problem. The central extension parameter c for the associated Virasoro representation is also non-vanishing but this should not be a problem now.

3. What is remarkable that depending on the choice of the weights α one obtains a large number of Lie algebras with same dimension of Cartan algebra. This gives excellent hopes of realizing in finite measurement resolution in terms of Kac-Moody type algebras obtained as ordered exponentials of the operators representing quantized complex E^2 coordinates. Any complexified simply laced Lie group would define a Kac-Moody group as a chacterizer of finite measurement resolution. Simply laced groups correspond by MacKay correspondence finite subgroups of $SU(2)$, which suggests that only Galois groups representable as subgroups of $SU(2)$ can be realized using this representation. It however seems that free field representations can be defined for an arbitrary affine algebra: these representations are discussed by Edward Frenkel [A146].

4. The conformal moduli of the partonic 2-surface define part of the target space. Also they could play the role of conformal fields on string world sheet. The strong form of holography poses heavy constraints on these fields and the evolution of the conformal moduli could be completely fixed once their values at the ends of string world sheets at partonic 2-suraces are known. Are also the orbits of punctures fixed completely by holography from initial values for "velocities" at partonic 2-surfaces corresponding to wormhole throats at which the signature of the metric changes? If this were the case, stringy dynamics would reduce to that for point like particles defined by the punctures. This cannot be true and the natural expectation is that just the finite spatial measurement resolution allows a non-trivial stringy dynamics as quantum fluctuations below the measurement resolution.

   The assumption that electromagnetic charge is well-defined for the modes of the induced spinor field implies in the generic case that the modes are localized to 2-D surfaces carrying vanishing induced W fields and above weak scale also vanishing induced Z^0 field. This makes sense inside the Minkowskian regions at least. The boundaries of the string world sheets carrying fundamental fermions would define uniquely braids and their intersections with partonic 2-surfaces would define the braid points. The imbedding space coordinates of these points in preferred coordinates should be rational in the intersection of realities and p-adicities.

   Finite measurement resolution would pose upper limit of the number of the string world sheets and thus to the fermion number of wormhole throat.

5. One can assign to the lightlike parton orbits at which the signature of the induced metric changes a Chern-Simons term compensating corresponding term coming from Kähler action so that only the Chern-Simons contributions associated with the ends of the space-time surface remain. Supersymmetry requires that Kähler-Dirac action contains Chern-Simons Dirac term at the partonic orbits. Since spinor modes are localized at string world sheets - at least in Minkowskian space-time regions - this term is actually localized at their 1-D boundaries. The assumption is that spinor modes are generalized eigenstates of C-S-D operator with eigenvalues $p^k\gamma_k$, where p_k is four-momentum of the virtual fermion line. Finite measurement resolution would mean IR and UV cutoffs to the spectrum of p^k. IR cutoff would be due to the finite size of causal diamond (CD) and UV cutoff to the lower bound for the size of sub-CDS involved.

   Note that Kähler action contains also measurement interaction terms at the space-like ends of the space-time surface. They fix the values of some classical conserved quantities to be equal to their quantum counterparts for the space-time surfaces allowed in quantum superpositions [K26]. Also here finite measurement resolution is expected.

6. The electric-magnetic duality induces S-duality permuting G and ${}^LG$ and the roles of string world sheet as 2-D space-time and partonic 2-surface defining defining the target manifold of string model. The moduli spaces of string world sheets and partonic 2-surfaces are in very close correspondence as implied by the strong form of holography.

How Langlands duality relates to quantum Yangian symmetry of twistor approach?

The are obvious objections against the heuristic considerations represented above.

1. One cannot restrict the attention on single partonic 2-surface or string world sheet. It is the collection of partonic 2-surfaces at the two light-like boundaries of CD and the string world sheets which define the geometric structure to which one should assign both the representations of the Galois group and the collection of world sheets as well as the groups G and ${}^LG$. Therefore also the group G defining the measurement resolution should be assigned to the entire structure and this leaves only single option: G defines the quantum Yangian defining the symmetry of the theory. If this were not complicated enough, note that one should be also able to take into account the possibility that there are CDs within CDs.

2. The finite measurement resolution should correspond to the replacement of ordinary Lie group with something analogous to quantum group. In the simplest situation the components of quantum spinors cease to commute: as a consequence the components correlate and

the dimension of the system is reduced to quantum dimension smaller than the algebraic dimension $d = 2$. Ordinary (p, q) wave mechanics is a good example about this: now the dimension of the system is reduced by a factor two from the dimension of phase space to that of configuration space.

3. Quantum Yangian algebra is indeed an algebra analogous to quantum group and according to MacKay did not receive the attention that it received as a symmetry of integrable systems because quantum groups became the industry [A184]. What can one conclude about the quantum Yangian in finite measurement resolution. One can make only guesses and which can be defended only by their internal consistency.

   (a) Since the basic objects are 2-dimensional, the group G should be actually span Kac-Moody type symplectic algebra and Kac-Moody algebra associated with the isometries of the imbedding space: this conforms with the proposed picture. Frenkel has discussed the relations between affine algebras, Langlands duality, and Bethe ansatz already at previous millenium [A147].

   (b) Finite measurement resolution reduces the partonic 2-surfaces to collections of braid ends. Does this mean that Lie group defining quantum Yangian group effectively reduces to something finite-dimensional? Or does the quantum Yangian property already characterize the measurement resolution as one might conclude from the previous argument? The simplest guess is that one obtains quantum Yangian containing as a factor the quantum Yangian associated with a Kac-Moody group defined by a finite-D Lie group with a Cartan algebra for which dimension equals to the total number of ends of braid strands involved. Zero energy states would be singlets for this group. This identification conforms with the general picture.

   (c) There is however an objection against the proposal. Yangian algebra contains a formal complex deformation parameter h but all deformations are equivalent to $h = 1$ deformation by a simple re-scaling of the generators labelled by non-negative integers trivial for $n = 0$ generators. Is Yangian after all unable to describe the finite measurement resolution. This problem could be circumvented by replacing Yangian with so called (twisted) quantum Yangian characterized by a complex quantum deformation parameter q. The representations of twisted quantum Yangians are discussed in [A159].

   (d) The quantum Yangian group should have also as a factor the quantum Yangian assigned to the symplectic group and Kac-Moody group for isometries of H with M^4 isometries extended to the conformal group of M^4. Finite measurement resolution would be realized as a q-deformation also in these degrees of freedom.

   (e) The proposed identification looks consistent with the general picture but one can also consider a reduction of continuous Kac-Moody type algebra to its discrete version obtained by replacing partonic 2-surfaces with the ends of braid strands as an alternative.

4. The appearance of quantum deformation is not new in the context of Langlands conjecture. Frenkel has proposed Langlands correspondence for both quantum groups [A151], and finite-dimensional representations of quantum affine algebras [A152].

The representation of Galois group and effective symmetry group as symplectic flow

Langlands duality involves both the Galois group and effective gauge or Kac-Moody groups G and ${}^LG$ extended to quantum Yangian and defining the automorphic forms and one should understand how these groups emerge in TGD framework.

1. What is the counterpart of Galois group in TGD? It need not be the gigantic Galois group of algebraic numbers regarded as an extension of rationals or algebraic extension of rationals. Here the proposal that infinite primes, integers and rationals are accompanied by collections of partonic 2-surfaces is very natural. Infinite primes can be mapped to irreducible polynomials of n variables and one can construct a procedure which assigns to infinite primes a collection of Galois groups. This collection of Galois groups characterizes a collection of partonic 2-surfaces.

2. How the Galois group is realized and how the symmetry group G realization finite measurement resolution is realized. How the finite-dimensional representations of Galois group lift to the finite-dimensional representations of G. The proposal is that Galois group is lifted to its braided counterpart just like braid group generalizes the symmetric group. One can speak about space-like and time-like braidings so that one would have two different kind of braidings corresponding to stringy and partonic pictures and it might be possible to understand the emergence of G and ^LG. The symplectic group for the boundary of CD define the isometries of WCW and by its infinite-dimensionality it is unique candidate for realizing representation of any group as its subgroup. he braidings are induced by symplectic flows.

3. Obviously also the symmetry groups G and ^LG should be realized as symplectic flows in appropriate moduli spaces. There are two different symplectic flows corresponding to space-like and time-like braids so that G and ^LG can be different and might differ even at the level of Lie algebra. The common realization of Galois group and symmetry group defining measurement resolution would imply Langlands duality automatically. The electric magnetic duality would in turn correspond to the possibility of two kinds of braidings. It must be emphasized that Langlands duality would be something independent of electric-magnetic duality and basically due to the realization of group representations as projective representations realized in terms of braidings. Note that also the automorphic forms define projective representations of G.

Why should the finite Galois group (possibly so!) correspond to Lie group G as it does in number theoretic Langlands correspondence?

1. The dimension of the representation of Galois group is finite and this dimension would correspond to the finite dimension for the representation of G defined by the finite-dimensional space in which G acts. This space is very naturally the moduli space of partonic 2-surfaces with n punctures corresponding to the n braid ends. A possible additional restriction is that the end points of braids are only permuted under the action of G. If the representations of the Galois group indeed automatically lift to the representations of the group defining finite measurement resolution, then Langlands duality would follow automatically.

2. The group G would correspond to the Galois group in very much the same manner as finite subgroups of $SU(2)$ correspond to simply laced Lie groups in MacKay correspondence [A59]. This would generalize Mc Kay correspondence to much more general theorem holding true for the inclusions of HFFs.

An interesting open question is whether one should consider representations of the collection of Galois groups assignable to the construction of zeros for polynomials associated with infinite prime or the gigantic Galois group assignable to algebraic numbers. The latter group could allow naturally p-adic topology. The notion of finite measurement resolution would strongly suggest that one should consider the braided counterpart of the finite Galois group. This would give also a direct connection with the physics in TGD Universe. Langlands correspondence would be basic physics of TGD Universe.

The practical meaning of the geometric Langlands conjecture

This picture seems to lead naturally to number theoretic Langlands conjecture. What geometric Langlands conjecture means in TGD Universe?

1. What it means to replace the braids with entire partonic 2-surfaces. Should one keep the number of braid strands constant and allow also non-rational braid ends? What does the number of rational points correspond at WCW level? How the automorphic forms code the information about the number of rational surfaces in the intersection?

2. Quantum classical correspondence suggests that this information is represented at space-time level. Braid ends characterize partonic 2-surfaces in finite measurement resolution. The quantum state involves a quantum super position of partonic 2-surfaces with the same number of rational braid strands. Different collections of rational points are of course possible. These

collections of braid ends should be transformed to each other by a discrete algebraic subgroup of the effective symmetry group G. Suppose that the orbit for a collection of n braid end points contains N different collections of braid points.

One can construct irreps of a discrete subgroup of the symmetry group G at the orbit. Could the number N of points at the orbit define the number which could be identified as the number of rational surfaces in the intersection in the domain of definition of a given WCW spinor field defined in terms of finite measurement resolution. This would look rather natural definition and would nicely integrate number theoretic and geometric Langlands conjectures together. For infinite primes which correspond to polynomials also the Galois groups of local number fields would also entire the picture naturally.

3. One can of course consider the possibility of replacing them with light-like 3-D surfaces or space-like 3- surfaces at the ends of causal diamonds but this is not perhaps not essential since holography implies the equivalence of these identifications. The possible motivation would come from the observations that vanishing of two holomoprhic functions at the boundary of CD defines a 3-D surface.

How TGD approach differs from Witten-Kapustin approach?

The basic difference as compared to Witten-Kapustin approach [A174] is that the moduli space for partonic 2-surfaces replaces in TGD framework the moduli space for Higgs field configurations. Higgs bundle defined as a holomorphic bundle together with Higgs field is the basic concept. In the simplest situations Higgs field is not a scalar but holomorphic 1-form at Riemann surface Y (analog of partonic 2-surface) related closely to the gauge potential of $M^4 = C \times Y$ whose components become scalars in spontaneous compactification to C. This is in complete analogy with the fact that the values of 1-forms defining the basis of cohomology group for partonic 1-surface for cycles defining the basis of 1-homology define conformal moduli.

A possible interpretation is in terms of geometrization of all gauge fields and Higgs field in TGD framework. Color and electroweak gauge fields are geometrized in terms of projections of color Killing vectors and induced spinor connection. Conformal moduli space for the partonic 2-surface would define the geometrization for the vacuum expectation value of the Higgs field.

One can even argue that dynamical Higgs is not consistent with the notion that the modulus characterizes entire 2-surfaces. Maybe the introducing of the quantum fluctuating part of Higgs field is not appropriate. Also the fact, that for Higgs bundle Higgs is actually 1-form suggests that something might be wrong with the notion of Higgs field. Concerning Higgs the recent experimental situation at LHC is critical: it might well turn out that Higgs boson does not exist. In TGD framework the most natural option is that Higgs like particles exist but all of them are "eaten" by gauge bosons meaning that also photon, gluons possess a small mass. Something analogous to the space of Higgs vacuum expectation values might be however needed and this something could correspond to the conformal moduli space. In TGD framework the particle massivation is described in terms of p-adic thermodynamics and the dominant contribution to the mass squared comes from conformal moduli. It might be possible to interpret this contribution as an average of the contribution coming from geometrized Higgs field.

One challenge is to understand whether the moduli spaces assignable to partonic 2-surfaces and with string world sheets are so closely related that they allow the analog of mirror symmetry of the super-string models relating 6-dimensional Calabi-Yau manifolds. For Calabi-Yau:s the mirror symmetry exchanges complex and Kähler structures. Could also now something analogous make sense.

1. Strong form of general coordinate invariance and the notion of preferred extremal implies that the collection of partonic 2-surfaces fixes the collection of string world sheets (these might define single connected sheet as a connected sum). This alone suggests that there is a close correspondence between moduli spaces of the string world sheets and of partonic 2-surfaces.

2. One problem is that space-time sheets in the Minkowskian regions have hyper-complex rather than complex structure. The analog of Kähler form must represent hypercomplex imaginary

unit and must be an antisymmetric form multiplied by the complex imaginary unit so that its square equals to the induced metric representing real unit.

3. How the moduli defined by integrals of complex 1-forms over cycles generalize? What one means with cycles now? How the handle numbers g_i of handles for partonic 2-surfaces reveal themselves in the homology and cohomology of the string world sheet? Do the ends of the string world sheets at the orbits of a given partonic 2-surface define curves which rotate around the handles and is the string would sheet a connected structure obtained as topological sum of this kind of string world sheets. Does the dynamics for preferred extremals of Kähler dictate this?

In the simplest situation (abelian gauge theory) the Higgs bundle corresponds to the upper half plane defined by the possible values of the inverse of the complexified coupling strength

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \quad .$$

Does the transformation for τ defined in this manner make sense?

1. The vacuum functional is the product of exponent of imaginary Kähler action from Minkowskian regions and exponent of real Kähler action from Euclidian regions appears as an exponent proportional to this kind of parameter. The weak form of electric-magnetic duality reduces Kähler action to 3-D Chern-Simons terms at light-like wormhole throats plus possible contributions not assignable to wormhole throats. This realizes the almost topological QFT property of quantum TGD and also holography and means an enormous calculational simplification. The complexified Kähler coupling strength emerges naturally as the multiplier of Chern-Simons term if the latter contributions are not present.

2. There is however no good reason to believe that string world sheets and partonic two-surface should correspond to the values of τ and $-1/\tau$ for a moduli space somehow obtained by gluing the moduli spaces of string worlds sheets and partonic 2-surfaces. More general modular symmetries for τ seem also implausible in TGD framework. The weak form of electric magnetic duality leads to the effective complexification of gauge coupling but there is no reason to give up the idea about the quantum criticality implying quantization of Kähler coupling strength.

3. From the foregoing it is clear that the identification of G as a Kac-Moody type group extended to quantum Yangian and assignable to string model in conformal moduli space is strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.

4. The notion of finite measurement resolution suggesting strongly quantum group like structure is what distinguishes TGD approach from Witten's approach and from the foregoing it is clear that the identification of G as a group defining Kac-Moody type group assignable to string model in conformal moduli space and further extended to quantum Yangian is the strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.

5. In his lecture Edward Frenkel explains that the recent vision about the conformal moduli is as parameters analogous to gauge coupling constants. It might well be that the moduli could take the role of gauge couplings. This might allow to have a fresh view to the conjecture that the lowest three genera are in special role physically because all these Riemann surfaces are hyper-elliptic (this means global Z_2 conformal symmetry) and because for higher genera elementary particle vacuum functionals vanish for hyper-elliptic Riemann surfaces [K17].

To sum up, the basic differences seem to be due to zero energy ontology, finite measurement resolution, and the identification of space-time as a 4-surface implying strong form of general coordinate invariance implying electric-magnetic and S-dualities implying also the replacement of Higgs bundle with the conformal moduli space.

13.4.3 About the structure of the Yangian algebra

The attempt to understand Langlands conjecture in TGD framework led to a completely unexpected progress in the understanding of the Yangian symmetry expected to be the basic symmetry of quantum TGD and the following vision suggesting how conformal field theory could be generalized to four-dimensional context is a fruit of this work.

The structure of the Yangian algebra is quite intricate and in order to minimize confusion easily caused by my own restricted mathematical skills it is best to try to build a physical interpretation for what Yangian really is and leave the details for the mathematicians.

1. The first thing to notice is that Yangian and quantum affine algebra are two different quantum deformations of a given Lie algebra. Both rely on the notion of R-matrix inducing a swap of braid strands. R-matrix represents the projective representations of the permutation group for braid strands and possible in 2-dimensional case due to the non-commutativity of the first homotopy group for 2-dimensional spaces with punctures. The R-matrix $R_q(u,v)$ depends on complex parameter q and two complex coordinates u,v. In integrable quantum field theories in M^2 the coordinates u,v are real numbers having identification as exponentials representing Lorenz boosts. In 2-D integrable conformal field theory the coordinates u,v have interpretation as complex phases representing points of a circle. The assumption that the coordinate parameters are complex numbers is the safest one.

2. For Yangian the R-matrix is rational whereas for quantum affine algebra it is trigonometric. For the Yangian of a linear group quantum deformation parameter can be taken to be equal to one by a suitable rescaling of the generators labelled by integer by a power of the complex quantum deformation parameter q. I do not know whether this true in the general case. For the quantum affine algebra this is not possible and in TGD framework the most interesting values of the deformation parameter correspond to roots of unity.

Slicing of space-time sheets to partonic 2-surfaces and string world sheets

The proposal is that the preferred extremals of Kähler action are involved in an essential manner the slicing of the space-time sheets by partonic 2-surfaces and string world sheets. Also an analogous slicing of Minkowski space is assumed and there are infinite number of this kind of slicings defining what I have called Hamilton-Jaboci coordinates [K10]. What is really involved is far from clear. For instance, I do not really understand whether the slicings of the space-time surfaces are purely dynamical or induced by special coordinatizations of the space-time sheets using projections to special kind of sub-manifolds of the imbedding space, or are these two type of slicings equivalent by the very property of being a preferred extremal. Therefore I can represent only what I think I understand about the situation.

1. What is needed is the slicing of space-time sheets by partonic 2-surfaces and string world sheets. The existence of this slicing is assumed for the preferred extremals of Kähler action [K10]. Physically the slicing corresponds to an integrable decomposition of the tangent space of space-time surface to 2-D space representing non-physical polarizations and 2-D space representing physical polarizations and has also number theoretical meaning.

2. In zero energy ontology the complex coordinate parameters appearing in the generalized conformal fields should correspond to coordinates of the imbedding space serving also as local coordinates of the space-time surface. Problems seem to be caused by the fact that for string world sheets hyper-complex coordinate is more natural than complex coordinate. Pair of hyper-complex and complex coordinate emerge naturally as Hamilton-Jacobi coordinates for Minkowski space encountered in the attempts to understand the construction of the preferred extremals of Kähler action.

Also the condition that the flow lines of conserved isometry currents define global coordinates lead to the to the analog of Hamilton-Jacobi coordinates for space-time sheets [K10]. The physical interpretation is in terms of local polarization plane and momentum plane defined by local light-like direction. What is so nice that these coordinates are highly unique and determined dynamically.

3. Is it really necessary to use two complex coordinates in the definition of Yangian-affine conformal fields? Why not to use hyper-complex coordinate for string world sheets? Since the inverse of hyper-complex number does not exist when the hyper-complex number is light-like, hyper-complex coordinate should appear in the expansions for the Yangian generalization of conformal field as positive powers only. Intriguingly, the Yangian algebra is "one half" of the affine algebra so that only positive powers appear in the expansion. Maybe the hyper-complex expansion works and forces Yangian-affine instead of doubly affine structure. The appearance of only positive conformal weights in Yangian sector could also relate to the fact that also in conformal theories this restriction must be made.

4. It seems indeed essential that the space-time coordinates used can be regarded as imbedding space coordinates which can be fixed to a high degree by symmetries: otherwise problems with general coordinate invariance and with number theoretical universality would be encountered.

5. The slicing by partonic 2-surfaces could (but need not) be induced by the slicing of CD by parallel translates of either upper or lower boundary of CD in time direction in the rest frame of CD (time coordinate varying in the direction of the line connecting the tips of CD). These slicings are not global. Upper and lower boundaries of CD would definitely define analogs of different coordinate patches.

Physical interpretation of the Yangian of quantum affine algebra

What the Yangian of quantum affine algebra or more generally, its super counterpart could mean in TGD framework? The key idea is that this algebra would define a generalization of super conformal algebras of super conformal field theories as well as the generalization of super Virasoro algebra. Optimist could hope that the constructions associated with conformal algebras generalize: this includes the representation theory of super conformal and super Virasoro algebras, coset construction, and vertex operator construction in terms of free fields. One could also hope that the classification of extended conformal theories defined in this manner might be possible.

1. The Yangian of a quantum affine algebra is in question. The heuristic idea is that the two R-matrices - trigonometric and rational- are assignable to the swaps defined by space-like braidings associated with the braids at 3-D space-like ends of space-time sheets at light-like boundaries of CD and time like braidings associated with the braids at 3-D light-like surfaces connecting partonic 2-surfaces at opposite light-like boundaries of CD. Electric-magnetic duality and S-duality implied by the strong form of General Coordinate Invariance should be closely related to the presence of two R-matrices. The first guess is that rational R-matrix is assignable with the time-like braidings and trigonometric R-matrix with the space-like braidings. Here one must or course be very cautious.

2. The representation of the collection of Galois groups associated with infinite primes in terms of braided symplectic flows for braid of braids of .... braids implies that there is a hierarchy of swaps: swaps can also exchange braids of ...braids. This would suggest that at the lowest level of the braiding hierarchy the R-matrix associated with a Kac-Moody algebra permutes two braid strands which decompose to braids. There would be two different braided variants of Galois groups.

3. The Yangian of the affine Kac-Moody algebra could be seen as a 4-D generalization of the 2-D Kac-Moody algebra- that is a local algebra having representation as a power series of complex coordinates defined by the projections of the point of the space-time sheet to geodesic spheres of light-cone boundary and geodesic sphere of CP_2.

4. For the Yangian the generators would correspond to polynomials of the complex coordinate of string world sheet and for quantum affine algebra to Laurent series for the complex coordinate

of partonic 2-surface. What the restriction to polynomials means is not quite clear. Witten sees Yangian as one half of Kac-Moody algebra containing only the generators having $n \geq 0$. This might mean that the positivity of conformal weight for physical states essential for the construction of the representations of Virasoro algebra would be replaced with automatic positivity of the conformal weight assignable to the Yangian coordinate.

5. Also Virasoro algebra should be replaced with the Yangian of Virasoro algebra or its quantum counterpart. This construction should generalize also to Super Virasoro algebra. A generalization of conformal field theory to a theory defined at 4-D space-time surfaces using two preferred complex coordinates made possible by surface property is highly suggestive. The generalization of conformal field theory in question would have two complex coordinates and conformal invariance associated with both of them. This would therefore reduce the situation to effectively 2-dimensional one rather than 3-dimensional: this would be nothing but the effective 2-dimensionality of quantum TGD implied by the strong form of General Coordinate Invariance.

6. This picture conforms with what the generalization of $D = 4$ $\mathcal{N} = 4$ SYM by replacing point like particles with partonic 2-surfaces would suggest: Yangian is replaced with Yangian of quantum affine algebra rather than quantum group. Note that it is the finite measurement resolution alone which brings in the quantum parameters q_1 and q_2. The finite measurement resolution might be relevant for the elimination of IR divergences.

How to construct the Yangian of quantum affine algebra?

The next step is to try to understand the construction of the Yangian of quantum affine algebra.

1. One starts with a given Lie group G. It could be the group of isometries of the imbedding space or subgroup of it or even the symplectic group of the light-like boundary of $CD \times CP_2$ and thus infinite-dimensional. It could be also the Lie group defining finite measurement resolution with the dimension of Cartan algebra determined by the number of braid strands.

2. The next step is to construct the affine algebra (Kac-Moody type algebra with central extension). For the group defining the measurement resolution the scalar fields assigned with the ends of braid strands could define the Cartan algebra of Kac-Moody type algebra of this group. The ordered exponentials of these generators would define the charged generators of the affine algebra.

   For the imbedding space isometries and symplectic transformations the algebra would be obtained by localizing with respect to the internal coordinates of the partonic 2-surface. Note that also a localization with respect to the light-like coordinate of light-cone boundary or light-like orbit of partonic 2-surface is possible and is strongly suggested by the effective 2-dimensionality of light-like 3-surfaces allowing extension of conformal algebra by the dependence on second real coordinate. This second coordinate should obviously correspond to the restriction of second complex coordinate to light-like 3-surface. If the space-time sheets allow slicing by partonic 2-surfaces and string world sheets this localization is possible for all 2-D partonic slices of space-time surface.

3. The next step is quantum deformation to quantum affine algebra with trigonometric R-matrix $R_{q_1}(u,v)$ associated with space-like braidings along space-like 3-surfaces along the ends of CD. u and v could correspond to the values of a preferred complex coordinate of the geodesic sphere of light-cone boundary defined by rotational symmetry. It choice would fix a preferred quantization axes for spin.

4. The last step is the construction of Yangian using rational R-matrix $R_{q_2}(u,v)$. In this case the braiding is along the light-like orbit between ends of CD. u and v would correspond to the complex coordinates of the geodesic sphere of CP_2. Now the preferred complex coordinate would fix the quantization axis of color isospin.

These arguments are of course heuristic and do not satisfy any criteria of mathematical rigor and the details could of course change under closer scrutinity. The whole point is in the attempt to understand the situation physically in all its generality.

How 4-D generalization of conformal invariance relates to strong form of general coordinate invariance?

The basic objections that one can rise to the extension of conformal field theory to 4-D context come from the successes of p-adic mass calculations. p-Adic thermodynamics relies heavily on the properties of partition functions for super-conformal representations. What happens when one replaces affine algebra with (quantum) Yangian of affine algebra? Ordinary Yangian involves the original algebra and its dual and from these higher multi-local generators are constructed. In the recent case the obvious interpretation for this would be that one has Kac-Moody type algebra with expansion with respect to complex coordinate w for partonic 2-surfaces and its dual algebra with expansion with respect to hyper-complex coordinate of string world sheet.

p-Adic mass calculations suggest that the use of either algebra is enough to construct single particle states. Or more precisely, local generators are enough. I have indeed proposed that the multi-local generators are relevant for the construction of bound states. Also the strong form of general coordinate invariance implying strong form of holography, effective 2-dimensionality, electric-magnetic duality and S-duality suggests the same. If one could construct the states representing elementary particles solely in terms of either algebra, there would be no danger that the results of p-adic mass calculations are lost. Note that also the necessity to restrict the conformal weights of conformal representations to be non-negative would have nice interpretation in terms of the duality.

13.4.4 Summary and outlook

It is good to try to see the relationship between Langlands program and TGD from a wider perspective and relate it to other TGD inspired views about problems of what I would call recent day physical mathematics. I try also to become (and remain!) conscious about possible sources of inconsistencies to see what might go wrong.

I see the attempt to understand the relation between Langlands program and TGD as a part of a bigger project the goal of which is to relate TGD to physical mathematics. The basic motivations come from the mathematical challenges of TGD and from the almost-belief that the beautiful mathematical structures of the contemporary physical mathematics must be realized in Nature somehow.

The notion of infinite prime is becoming more and more important concept of quantum TGD and also a common denominator. The infinite-dimensional symplectic group acting as the isometry group of WCW geometry and symplectic flows seems to be another common denominator. Zero energy ontology together with the notion of causal diamond is also a central concept. A further common denominator seems to be the notion of finite measurement resolution allowing discretization. Strings and super-symmetry so beautiful notions that it is difficult to imagine physics without them although super string theory has turned out to be a disappointment in this respect. In the following I mention just some examples of problems that I have discussed during this year.

Infinite primes are certainly something genuinely TGD inspired and it is reasonable to consider their possible role in physical mathematics.

1. The set theoretic view about the fundamentals of mathematics is inspired by classical physics. Cantor's view about infinite ordinals relies on set theoretic representation of ordinals and is plagued by difficulties (say Russel's paradox) [K70]. Infinite primes provide an alternative to Cantor's view about infinity based on divisility alone and allowing to avoid these problems. Infinite primes are obtained by a repeated second quantization of an arithmetic quantum field theory and can be seen as a notion inspired by quantum physics. The conjecture is that quantum states in TGD Universe can be labelled by infinite primes and that standard model symmetries can be understood in terms of octonionic infinite primes defined in appropriate manner.

   The replacement of ordinals with infinite primes would mean a modification of the fundamentals of physical mathematics. The physicists's view about the notion set is also much more restricted than the set theoretic view. Subsets are typically manifolds or even algebraic varieties and they allow description in terms of partial differential equations or algebraic equations.

Boolean algebra is the quintessence of mathematical logic and TGD suggests that quantum Boolean algebra should replace Boolean algebra [K70]. The representation would be in terms of fermionic Fock states and in zero energy ontology fermionic parts of the state would define Boolean states of form $A \rightarrow B$. This notion might be useful for understanding the physical correlates of Boolean cognition and might also provide insights about fundamentals of physical mathematics itself. Boolean cognition must have space-time correlates and this leads to a space-time description of logical OR *resp.* AND as a generalization of trouser diagram of string models *resp.* fusion along ends of partonic 2-surfaces generalizing the 3-vertex of Feyman diagrammatics. These diagrams would give rise to fundamental logic gates.

2. Infinite primes can be represented using polynomials of several variables with rational coefficients [K70]. One can solve the zeros of these polynomials iteratively. At each step one can identify a finite Galois group permuting the roots of the polynomial (algebraic function in general). The resulting Galois groups can be arranged into a hierarchy of Galois groups and the natural idea is that the Galois groups at the upper levels act as homomorphisms of Galois groups at lower levels. A generalization of homology and cohomology theories to their non-Abelian counterparts emerges [K82]: the square of the boundary operation yields unit element in normal homology but now an element in commutator group so that abelianization yields ordinary homology. The proposal is that the roots are represented as punctures of the partonic 2-surfaces and that braids represent symplectic flows representing the braided counterpars of the Galois groups. Braids of braids of.... braids structrure of braids is inherited from the hierarchical structure of infinite primes.

   That braided Galois groups would have a representation as symplectic flows is exactly what physics as generalized number theory vision suggests and is applied also to understand Langlands conjectures. Langlands program would be modified in TGD framework to the study of the complexes of Galois groups associated with infinite primes and integers and have direct physical meaning.

The notion of finite measurement resolution realized at quantum level as inclusions of hyperfinite factors and at space-time level in terms of braids replacing the orbits of partonic 2-surfaces - is also a purely TGD inspired notion and gives good hopes about calculable theory.

1. The notion of finite measurement resolution leads to a rational discretization needed by both the number theoretic and geometric Langlands conjecture. The simplest manner to understand the discretization is in terms of extrema of Chern-Simons action if they correspond to "rational" surfaces. The guess that the rational surfaces are dense in the WCW just as rationals are dense in various number fields is probably quite too optimistic physically. Algebraic partonic 2-surfaces containing typically finite number of rational points having interpretation in terms of finite measurement resolution. Same might apply to algebraic surfaces as points of WCW in given quantum state.

2. The charged generators of the Kac-Moody algebra associated with the Lie group G defining measurement resolution correspond to tachyonic momenta in free field representation using ordered exponentials. This raises unpleasant question. One should have also a realization for the coset construction in which Kac-Moody variant of the symplectic group of $\delta M^4_\pm$ and Kac-Moody algebra of isometry group of H assignable to the light-like 3-surfaces (isometries at the level of WCW *resp.* H) define a coset representation. The actions of corresponding super Virasoro algebras are identical. Now the momenta are however non-tachyonic.

   How these Kac-Moody type algebras relate? From p-adic mass calculations it is clear that the ground states of super-conformal representations have tachyonic conformal weights. Does this mean that the ground states can be organized into representations of the Kac-Moody algebra representing finite measurement resolution? Or are the two Kac-Moody algebra like structures completely independent. This would mean that the positions of punctures cannot correspond to the H-coordinates appearing as arguments of symplectic and Kac-Moody algebra. The fact that the groups associated with algebras are different would allow this.

TGD is a generalization of string models obtained by replacing strings with 3-surfaces. Therefore it is not surprising that stringy structures should appear also in TGD Universe and the strong form of general coordinate invariance indeed implies this. As a matter fact, string like objects appear also in various applications of TGD: consider only the notions of cosmic string [K20] and nuclear string [L5]. Magnetic flux tubes central in TGD inspired quantum biology making possible topological quantum computation [K24] represent a further example.

1. What distinguishes TGD approach from Witten's approach is that twisted SUSY is replaced by string model like theory with strings moving in the moduli space for partonic 2-surfaces or string world sheets related by electric-magnetic duality. Higgs bundle is replaced with the moduli space for punctured partonic 2-surfaces and its electric dual for string world sheets. The new element is the possibility of trouser vertices and generalization of 3-vertex if Feynman diagrams having interpretation in terms of quantum Boolean algebra.

2. Stringy view means that all topologies of partonic 2-surfaces are allowed and that also quantum superpositions of different topologies are allowed. The restriction to single topology and fixed moduli would mean sigma model. Stringy picture requires quantum superposition of different moduli and genera and this is what one expects on physical grounds. The model for CKM mixing indeed assumes that CKM mixing results from different topological mixings for U and D type quarks [K49] and leads to the notion of elementary particle vacuum functional identifiable as a particular automorphic form [K17].

3. The twisted variant of $\mathcal{N} = 4$ SUSY appears as TQFT in many mathematical applications proposed by Witten and is replaced in TGD framework by the stringy picture. Supersymmetry would naturally correspond to the fermionic oscillator operator algebra assignable to the partonic 2-surfaces or string world sheet and SUSY would be broken.

When I look what I have written about various topics during this year I find that symplectic invariance and symplectic flows appear repeatedly.

1. Khovanov homology provides very general knot invariants. In [**?**] rephrased Witten's formulation about Khovanov homology as TQFT in TGD framework. Witten's TQFT is obtained by twisting a 4-dimensional $\mathcal{N} = 4$ SYM. This approach generalizes the original 3-D Chern-Simons approach of Witten. Witten applies twisted 4-D $\mathcal{N} = 4$ SYM also to geometric Langlands program and to Floer homology.

   TGD is an almost topological QFT so that the natural expectation is that it yields as a side product knot invariants, invariants for braiding of knots, and perhaps even invariants for 2-knots: here the dimension $D = 4$ for space-time surface is crucial. One outcome is a generalization of the notion of Wilson loop to its 2-D variant defined by string world sheet and a unique identification of string world sheet for a given space-time surface. The duality between the descriptions based on string world sheets and partonic 2-surfaces is central. I have not yet discussed the implications of the conjectures inspired by Langlands program for the TGD inspired view about knots.

2. Floer homology generalizes the usual Morse theory and is one of the applications of topological QFTs discussed by Witten using twisted SYM. One studies symplectic flows and the basic objects are what might regarded as string world sheets referred to as pseudo-holomorphic surfaces. It is now wonder that also here TGD as almost topological QFT view leads to a generalization of the QFT vision about Floer homology [K82]. The new result from TGD point of view was the realization that the naivest possible interpretation for Kähler action for a preferred extremal is correct. The contribution to Kähler action from Minkowskian regions of space-time surface is imaginary and has identification as Morse function whereas Euclidian regions give the real contribution having interpretation as Kähler function. Both contributions reduce to 3-D Chern-Simons terms and under certain additional assumptions only the wormhole throats at which the signature of the induced metric changes from Minkowskian to Euclidian contribute besides the space-like regions at the ends of the space-time surface at the light-like boundaries of CD.

3. Gromov- Witten invariants are closely related to Floer homology and their definition involves quantum cohomology in which the notion of intersection for two varieties is more general taking into account "quantum fuzziness". The stringy trouser vertex represent the basic diagram: the incoming string world sheets intersect because they can fuse to single string world sheet. Amazingly, this is just that OR in quantum Boolean algebra suggested by TGD. Another diagram would be *and* responsible for genuine particle reactions in TGD framework. There would be a direct connection with quantum Boolean algebra.

Number theoretical universality is one of the corner stones of the vision about physics as generalized number theory. One might perhaps say that a similar vision has guided Grothendieck and his followers.

1. The realization of this vision involves several challenges. One of them is definition of p-adic integration. At least integration in the sense of cohomology is needed and one might also hope that numerical approach to integration exists. It came as a surprise to me that something very similar to number theoretical universality has inspired also mathematicians and that there exist refined theories inspired by the notion of motive introduced by Groethendieck to to define universal cohomology applying in all number fields. One application and also motivation for taking motives very seriously is motivic integration which has found applications in in physics as a manner to calculate twistor space integrals defining scattering amplitudes in twistor approach to $\mathcal{N} = 4$ SUSY. The essence of motivic integral is that integral is an algebraic operation rather than defined by a measure. One ends up with notions like scissor group and integration as processing of symbols. This is of course in spirit with number theoretical approach where integral as measure is replaced with algebraic operation. The problem is that numerics made possible by measure seems to be lost.

2. The TGD inspired proposal for the definition of p-adic integral relies on number theoretical universality reducing the integral essentially to integral in the rational intersection of real and p-adic worlds. An essential role is played at the level of WCW by the decomposition of WCW to a union of symmetric spaces allowing to define what the p-adic variant of WCW is. Also this would conform with the vision that infinite-dimensional geometric existence is unique just from the requirement that it exists. One can consider also the possibility of having p-adic variant of numerical integration [K82].

Twistor approach has led to the emergence of motives to physics and twistor approach is also what gives hopes that some day quantum TGD could be formulated in terms of explicit Feynman rules or their twistorial generalization [K79, K81].

1. The Yangian symmetry and its quantum counterpart were discovered first in integrable quantum theories is responsible for the success fo the twistorial approach. What distinguishes Yangian symmetry from standard symmetries is that the generators of Lie algebra are multilocal. Yangian symmetry is generalized in TGD framework since point like particles are replaced by partonic 2-surfaces meaning that Lie group is replaced with Kac-Moody group or its generalization. Finite measurement resolution however replaces them with discrete set of points defining braid strands so that a close connection with twistor approach and ordinary Yangian symmetry is suggestive in finite measurement resolution. Also the fact that Yangian symmetry relates closely to topological string models supports the expectation that the proposed stringy view about quantum TGD could allow to formulate twistorial approach to TGD.

2. The vision about finite measurement resolution represented in terms of effective Kac-Moody algebra defined by a group with dimension of Cartan algebra given by the number of braid strands must be consistent with the twistorial picture based on Yangians and this requires extension to Yangian algebra- as a matter to quantum Yangian. In this picture one cannot speak about single partonic 2-surface alone and the same is true about the TGD based generalization of Langlands program. Collections of two-surfaces and possibly also string world sheets are always involved. Multi-locality is also required by the basic properties of quantum states in zero energy ontology.

3. The Kac-Moody group extended to quantum Yangian and defining finite measurement resolution would naturally correspond to the gauge group of $\mathcal{N} = \triangle$ SUSY and braid points to the arguments of N-point functions. The new element would be representation of massive particles as bound states of massless particles giving hopes about cancellation of IR divergences and about exact Yangian symmetry. Second new element would be that virtual particles correspond to wormholes for which throats are massless but can have different momenta and opposite signs of energies. This implies that absence of UV divergences and gives hopes that the number of Feynman diagrams is effectively finite and that there is simple expression of twistorial diagrams in terms of Feynman diagrams [K81].

13.5 Appendix

13.5.1 Hecke algebra and Temperley-Lieb algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_ne_{n+1} &= e_ne_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{13.5.0}$$

The algebra reduces to that for symmetric group for $t = 1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and imbedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

Temperley-Lieb algebra assignable with Jones inclusions of hyper-finite factors of type II_1 with $\mathcal{M} : \mathcal{N} < 4$ is given by the relations

$$\begin{aligned} e_{n+1}e_nen+1 &= e_{n+1} \\ e_ne_{n+1}e_n &= e_n \ , \\ e_n^2 &= te_n \ , \ \ , t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ , n = 3,4,... \end{aligned} \tag{13.5.-1}$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$\begin{aligned} e_{n+1}e_nen+1 &= te_{n+1} \\ e_ne_{n+1}e_n &= te_n \ , \\ e_n^2 &= e_n = e_n^* \ , \ \ , t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ , n = 3,4,... \end{aligned} \tag{13.5.-2}$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

13.5.2 Some examples of bi-algebras and quantum groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

Simplest bi-algebras

Let $k(x_1, .., x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1, ..., x_n), A)$ of algebra homomorphisms $k(x_1, ..., x_n) \to A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1, ...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x], A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a, b, c, d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a, b, c, d)$ to $M_2^{\otimes 2} = k(a', a'', b', b'', c', c'', d', d'')$ to $k(a, b, c, d)$ in matrix form as

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a, b, c, d)$ spanned by the generators a, b, c, d by the ideal $det - 1 = ad - bc - 1 = 0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

$$\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a, b, c, d of the algebra. For instance, one has $\Delta(a) = a \to a \otimes a + b \otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a, b, c, d by the relations

$$\begin{array}{cc} ba = qab \ , & db = qbd \ , \\ ca = qac \ , & dc = qcd \ , \\ bc = cb \ , & ad - da = (q^{-1} - 1)bc \ , \end{array}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

$\mu, \eta, \Delta, \epsilon$ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A, B, C, D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+, X_-] = H \quad , \quad [H, X_\pm] = \pm 2X_\pm \quad . \tag{13.5.-1}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{aligned}
&\Delta(J) = J\otimes 1 + 1\otimes J \quad , \quad S(J) = -J \quad , \quad \epsilon(J) = 0 \quad , \quad J = X_\pm, H \quad , \\
&S(1) = 1 \quad , \qquad\qquad\qquad \epsilon(1) = 1 \quad .
\end{aligned} \tag{13.5.0}$$

The enveloping algebras of Borel algebras $U(B_\pm)$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \to H^\star$ reads as

$$\begin{array}{l}
a \to a^\star \quad , \quad ab = (ab)^\star = b^\star a^\star \quad , \\
1 \to 1 \quad , \qquad H \to H^\star = hH \quad , \qquad X_+ \to (X_+)^\star = hX_- \quad .
\end{array} \tag{13.5.1}$$

The commutation relations of $U_q(sl(2)$ read as

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \quad , \quad [H, X_\pm] = \pm 2X_\pm \quad . \tag{13.5.2}$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_\pm$:

$$\begin{aligned}
&\Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2}\otimes X_\pm \quad , \\
&S(X_\pm) = -q^{\pm 1} X_\pm \quad .
\end{aligned} \tag{13.5.3}$$

When q is not a root of unity, the universal R-matrix is given by

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \quad . \tag{13.5.4}$$

When q is m:th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [A125].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \to \Psi(x)$ $M_q(2) \to U_q^\star$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\langle uv, x\rangle = \langle u\otimes v, \Delta(x)\rangle \ , \quad \langle u, xy\rangle = \langle \Delta(u), x\otimes y\rangle \ ,$$

$$\langle 1, x\rangle = \epsilon(x) \ , \qquad\qquad \langle u, 1\rangle = \epsilon(u)$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \rho(K = q^H) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ ,$$

extended to a representation

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \to U_q^\star$.

General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [A125]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:
i) A is indecomposable, that is does not reduce to a direct sum of matrices.
ii) $a_{ij} \leq 0$ holds true for $i < j$.
iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.
A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.

The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{ll} k_ik_j = k_jk_i \ , & k_ie_j = q_i^{a_{ij}}e_jk_i \ , \\ k_if_j = q_i^{-a_{ij}}e_jk_i \ , & e_if_j - f_je_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \ , \end{array} \tag{13.5.5}$$

and so called Serre relations

$$\begin{array}{l} \sum_{l=0}^{1-a_{ij}}(-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} e_i^{1-a_{ij}-l}e_je_i^l = 0, \ \ i \neq j \ , \\ \sum_{l=0}^{1-a_{ij}}(-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} f_i^{1-a_{ij}-l}f_jf_i^l = 0 \ , \ \ i \neq j \ . \end{array} \tag{13.5.6}$$

Here $q_i = q^{D_i}$ where one has $D_ia_{ij} = a_{ij}D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) = k_i \otimes k_i \ , \tag{13.5.7}$$

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \ , \tag{13.5.8}$$

$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes 1 \ . \tag{13.5.9}$$

(13.5.10)

The action of antipode S is defined as

$$S(e_i) = -e_ik_i^{-1} \ , \quad S(f_i) = -k_if_i \ , \quad S(k_i) = -k_i^{-1} \ . \tag{13.5.11}$$

Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [A125].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l+1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1) \times (l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0:th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i = 0, 1, .., l)$ satisfying the relations of Eq. 13.5.6 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0} e_i \ , \quad [d, f_i] = \delta_{i0} f_i, \quad [d, k_i] = 0 \ . \tag{13.5.12}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \ , \tag{13.5.13}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \ . \tag{13.5.14}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \ , \tag{13.5.15}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\begin{aligned} \Psi(a, b) &= -\Psi(b, a) \ , \\ \Psi([a, b], c) + \Psi([b, c], a) + \Psi([c, a], b) &= 0 \ . \end{aligned} \tag{13.5.15}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l) \otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,]+\Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned} J_n^x &= x \otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \ . \end{aligned} \tag{13.5.15}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c = 0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n > 0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{13.5.15}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}} e_i \ , & D_t(f_i) = t^{\delta_{i0}} f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{13.5.16}$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \ , \tag{13.5.17}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{13.5.18}$$

and satisfies the equations

$$\begin{aligned} &\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\ &(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\ &(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\ &\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ . \end{aligned} \tag{13.5.19}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations e_i, f_i, k_i, $i > 0$.

Chapter 14

Quantum Arithmetics and the Relationship between Real and p-Adic Physics

14.1 Introduction

The construction of quantum counterparts for various mathematical structures of theoretical physics have been a fashion for decades. Quantum counterparts for groups, Lie algebras, coset spaces, etc... have been proposed often on purely formal grounds. In TGD framework quantum group like structures emerge via the hyper-finite factors of type II_1 (HFFs) about which WCW spinors represent a canonical example [K80]. The inclusions of HFFs provide a very attractive manner to realize mathematically the notion of finite measurement resolution.

In the following a proposal for what might be called quantum integers and quantum matrix groups is discussed. One can imagine two basic definitions of quantum integers n_q: option I and II. For option I the map $n \rightarrow n_q$ respects prime decomposition so that one obtains quantum variant of primeness. For option II ordinary primeness in the ordinary sense of word is lost as it is lost also for p-adic numbers (only p is prime for Q_p).

Also quantum rationals belonging to algebraic extension of rationals can be defined as well as their algebraic extensions. Quantum arithmetics differs from the usual one in that quantum sum is defined in such a manner that the map $n \rightarrow n_q$ commutes also with sum besides the product: $m_q +_q n_q = (m+n)_q$. Quantum matrix groups differ from their standard counterparts in that the matrix elements are not non-commutative. The matrix multiplication involving summation over products is however replaced with quantum summation.

The hope is that these new mathematical structures could allow a better understanding of the relationship between real and p-adic physics for various values of p-adic prime p, to be called l in the sequel because of its preferred physical nature resembling that of l-adic prime in l-adic cohomology. The correspondence with the ordinary quantum groups [A76] is also considered and suggested to correspond to a discretization following as a correlate of finite measurement resolution.

14.1.1 Overall view about variants of quantum integers

The starting point of quantum arithmetics is the map $n \rightarrow n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition

of n by dividing n to its prime factors and performing the mapping $p \to p_q$. The map takes prime first to finite field $G(p,1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p,1)$ there are no primes.

Clearly, many variants of quantum integers can be found and it is difficult to decide which of them - if any - has interesting from TGD point of view.

1. If one wants to really model something using quantum integers, the second options is perhaps the realistic one: the reason is that the decomposition into prime factors requires a lot of computation time.

2. A second fictive criterion would be whether the definition is maximally general. Does the definition makes sense for infinite primes? The simplest infinite primes at the first level of hierarchy have physical interpretation as many-particle states consisting of bosons and fermions, whose momentum values correspond to finite primes. The interpretation generalizes to higher levels of the hierarchy. A simple argument show that the option keeping information about prime factorization of the p-adic number allowing also infinite primes as factors makes sense only if prime factors are not expanded in series with respect to the prime p and if p does not correspond to a fermionic mode. The quantum map using prime root of unity therefore makes sense for all but fermionic primes. The presence of exceptional primes in number theory is basic phenomenon: typically they correspond to primes for which factorization is not unique in algebraic extension.

14.1.2 Motivations for quantum arithmetics

Quantum arithmetics has several motivations in TGD framework.

Model for Shnoll effect

The model for Shnoll effect [K5] suggests that this effect could be understand in terms of a deformation of probability distribution $f(n)$ (n non-negative integer) for random fluctuations. The deformation would replace the rational parameters characterizing the distribution with new ones obtained by mapping the parameters to new ones by using the analog of canonical identification respecting symmetries.

The idea of the model of Shnoll effect was to modify the quantum map $n \to n_q$ in such a manner that it is consistent with the prime decomposition of ordinary integers. This deformation would involve two parameters: quantum phase $q = exp(i\pi/m)$ and preferred prime l, which need not be independent however: $m = l$ is a highly suggestive restriction.

What could be the deeper mathematics behind dualities?

Dualities certainly represent one of the great ideas of theoretical physics of the last century. On could say that electric-magnetic duality due to Montonen and Olive [B7] is the mother of all dualities. Later a proliferation, one might say even inflation, of dualities has taken place. AdS/CFT correspondence [B29] is one example relating to each other perturbative QFT working in short scales and string theory working in long scales.

Also in TGD framework several dualities suggests itself. All of them seem to relate to dichotomies such as weak–strong, perturbative–non-perturbative, point like particle–string. Also number theory seems to be involved in an essential manner.

1. If $M^8 - -M^4 \times CP_2$ duality is true it is possible to regard space-times as surfaces in M^8 or $M^4 \times CP_2$ [K72]. The proper treatment of Minkowskian signature requires complexified version M^8_c of M^8 allowing identification as complexified octonions. One manner to interpret the duality would as the analog of q-p duality in wave mechanics. Surfaces in M^8 (or M^8_c) would be analogous to momentum space representation of the physical states: space-time surfaces in M^8 would represent in some sense the points for the tangent space of the "world of classical worlds" (WCW) just like tangent for a curve gives the first approximation for the curve near a given point.

   The argument supporting $M^8 - -M^4 \times CP_2$ duality involves the basic facts about classical number fields - in particular octonions and their complexification - and one can understand $M^4 \times CP_2$ in terms of number theory. The analog of the color group in M^8 picture would be the isometry group $SO(4)$ of E^4 which happens to be the symmetry group of the old fashioned hadron physics. Does this mean that $M^4 \times CP_2$ corresponds to short length scales and perturbative QCD whereas M^8 would correspond to long length scales and non-perturbative approach?

2. Second duality would relate partonic 2-surfaces and string world sheets playing a key role in the recent view about preferred extremals of Kähler action [L17]. Partonic 2-surfaces are magnetic monopoles and TGD counterparts of elementary particles, which in QFT approach are regarded as point like objects. The description in terms of partonic 2-surfaces forgetting that they are parts of bigger magnetically neutral structures would correspond to perturbative QFT. The description in terms of string like objects with vanishing magnetic charge is needed in longer length scales. Electroweak symmetry breaking and color confinement would be the natural applications. The essential point is that stringy description corresponds to long length scales (strong coupling) and partonic description to short length scales (weak coupling).

   Number theory seems to be involved also now: string world sheets could be seen as commutative (hyper-complex) 2-surfaces of space-time surface with hyper-quaternionic tangent space structure and partonic 2-surfaces as co-commutative (co-hyper-complex) 2-surfaces. To avoid inflation of clumsy "hyper-"s, the terms "associative"/"co-associative" and "commutative"/"co-commutative" will be used in the sequel.

   The localization of the modes of induced spinor fields to string world sheets and partonic 2-surfaces could be seen as a physical realization this and is implied by the requirement that spinor modes are eigenstates of em charge operator [K89].

3. Space-time surface itself would decompose to associative and co-associative regions and a duality also at this level is suggestive [L15], [K10]. The most natural candidates for dual space-time regions are regions with Minkowskian and Euclidian signatures of the induced metric with latter representing the generalized Feynman graphs. Minkowskian regions would correspond to non-perturbative long length scale description and Euclidian regions to perturbative short length scale description. This duality should relate closely to quantum measurement theory and realize the assumption that the outcomes of quantum measurements are always macroscopic long length scale effects. Again number theory is in a key role.

Real and p-adic physics and their unification to a coherent whole represent the basic pieces of physics as generalized number theory program.

1. p-Adic physics in minimal sense would mean a discretization of real physics relying on effective p-adic topology. p-Adic physics could also mean genuine p-adic physics at p-adic space-time sheets identified as space-time correlates of cognition and intentions.

Real continuity and smoothness is a powerful constraint on short distance physics. p-Adic continuity and smoothness pose similar constraints in short scales an therefore on real physics in long length scales if one accepts that real and space-time surfaces (partonic 2-surfaces for minimal option) intersect along rational points and possible common algebraics in preferred coordinates. p-Adic fractality implying short range chaos and long range correlations is the outcome. Therefore p-adic physics could allow to avoid the landscape problem of M-theory due to the fact that the IR limit is unpredictable although UV behavior is highly unique.

2. The recent argument [L17] suggesting that the areas for partonic 2-surfaces and string world sheets could characterize Kähler action leads to the proposal that the large N_c expansion [B1] in terms of the number of colors defining non-perturbative stringy approach to strong coupling phase of gauge theories could have interpretation in terms of the expansion in powers of $1/\sqrt{p}$, p the p-adic prime. This expansion would converge extremely rapidly since N_c would be of the order of the ratio of the secondary and primary p-adic length scales and therefore of the order of $\sqrt{p}$: for electron one has $p = M_{127} = 2^{127} - 1$.

3. Could there exist a duality between genuinely p-adic physics and real physics? Could the mathematics used in p-adic mass calculations - in particular canonical identification $\sum_n x_n p^n \rightarrow \sum x_n p^{-n}$ - be extended to apply to quantum TGD itself and allow to understand the non-perturbative long length scale effects in terms of short distance physics dictated by continuity and smoothness but in different number field? Could a proper generalization of the canonical identification map allow to realize concretely the real–p-adic duality?

Could quantum arithmetics allow a variant of canonical identification respecting both symmetries and continuity?

One could argue that a generalization of the canonical identification [K47] and its variants is needed in order to solve the tension between algebra (symmetries) and topology: the correspondence via common rationals respects algebra and symmetries but is discontinuous. Canonical identification is continuous but does not respect algebra.

Concerning the correspondence between p-adics and reals the notion of p-adic manifolds seems to represent a real step of progress. The notion of p-adic manifold [K95] is based on simple idea. The chart maps of p-adic manifolds (now space-time surfaces) are to real manifolds (space-time surfaces) rather than p-adic counterpart of Euclidian space and realized in terms of some variant of canonical identification restricted to a discrete subset of rational points of manifold- now space-time surface- and preferred extremal property allows to find a space-time surface which contains these points. In accordance with finite measurement resolution, the correspondence is not unique.

The real image is interpreted as realization of intention represented as p-adic space-time surface. The reverse maps providing p-adic charges about real space-time surface are interpreted as cognitive representations. Building of cognitive representation and realization of intention as action could be time reversals of each other in the sense that quantum jump could lead from p-adic sector to real and vice versa: this requires zero energy ontology (ZEO) in order to make sense.

All forms of canonical identification break to some extent symmetries and continuity (this forces the restriction to a discrete subset of space-time points). One could accept this or ask whether a generalization of canonical identification resolving the tension between symmetries and continuity could exist.

It seems that this is not the case. The tension seems to be unresolvable and have interpretation in terms of finite measurement resolution. At best a given continuous symmetry group would be replaced by some of its discrete subgroups. Of course, both real and p-adic variants of symmetries are realized but the problem is that they are very different and canonical identification in its basic form does not give close connection between them.

This chapter was written before the emergence of the notion of p-adic manifold and in the hope that the symmetry respecting generalization of canonical identification might exist. In the new situation quantum variant of canonical identification provides a new variant of the map taking discretization of the p-adic space-time surfaces to its real counterpart.

Quantum integers and preferred extremals of Kähler action

One might hope that quantum integers have some deep function. Somehow the fact that the images of primes $1 < p_i < p$ are algebraic numbers might relate to this. Maybe their function might relate to the notion of p-adic manifold [K95]. The basic challenge is to continue the discrete canonical image of the p-adic space-time points to continuous and differentiable preferred extremal of Kähler action. O_c-real analytic functions (O_c denotes complexified octonions) [K97] defining four-surfaces in M_c^8 mappable to space-time surface in H by $M^8 - H$ correspondence might allow to code preferred extremals by real-valued analytic functions. A hierarchy of polynomials with rational or even algebraic arguments suggests itself.

Quantum integers might define discretization of real space-time surface by mapping p-adic integers (continuum) representing preferred imbedding space coordinates to a set of quantum integers n_q, $0 \leq n < p$.

The notion of deformation has played central role in attempts to generalize physics and one can see quantum physics as a deformation of classical physics. Suppose that p-adic preferred extremal is characterized by functions which are polynomials/ rational functions. Suppose that one can interpret these functions as functions in the ring of quantum integers. Since differentiability makes sense for the quantum ring one could hope that these functions could define preferred extremal in the ring of quantum integers and perhaps also in real imbedding space.

14.1.3 Correspondence along common rationals and canonical identification: two manners to relate real and p-adic physics

The relationship between real and p-adic physics deserves a separate discussion.

Identification along common rationals

The first correspondence between reals and p-adics is based on the idea that rationals are common to all number fields implying that rational points are common to both real and p-adic worlds. This requires preferred coordinates. It also leads to a fusion of different number fields along rationals and common algebraics to a larger structure having a book like structure [K71, K47].

1. Quite generally, preferred space-time coordinates would correspond to a subset of preferred imbedding space coordinates, and the isometries of the imbedding space give rise to this kind of coordinates which are however not completely unique. This would give rise to a moduli space corresponding to different symmetry related coordinates interpreted in terms of different choices of causal diamonds (CDs: recall that CD is the intersection of future and past directed light-cones.

2. Cognitive representation in the rational (partly algebraic) intersection of real and p-adic worlds would necessarily select certain preferred coordinates and this would affect the physics in a delicate manner. The selection of quantization axis would be basic example of this symmetry breaking. Finite measurement resolution would in turn reduce continuous symmetries to discrete ones. It deserves to be mentioned that for color color symmetries $SU(3)$ the space for the choices of quantization axes is flag-manifold $SU(3)/U(1) \times U(1)$ having interpretation as twistor space of CP_2: CP_2 is the only compact 4-manifold allowing twistor space with complex structure. M^4 twistors are assigned with light-like vectors defining plane $M^2 \subset M^4$ in turn defining quantization axis for spin.

3. Typically real and p-adic variants of given partonic 2-surface would have discrete and possibly finite set of rational points plus possible common algebraic points. The intersection of real and p-adic worlds would consist of discrete points. At more abstract level rational functions with rational coefficients used to define partonic 2-surfaces would correspond to common 2-surfaces in the intersection of real and p-adic WCW:s. As a matter fact, the quantum arithmetics would make most points algebraic numbers.

4. The correspondence along common rationals respects symmetries but not continuity: the graph for the p-adic norm of rational point is totally discontinuous. Most non-algebraic reals

and p-adics do not correspond to each other. In particular, transcendental at both sides belong to different worlds with some exceptions like e^p which exists p-adically.

Canonical identification and its variants

There is however a totally different view about real–p-adic correspondence.

The predictions of p-adic mass calculations are mapped to real numbers via the canonical identification applied to the p-adic value of mass squared [K47, K46]. One can imagine several forms of canonical identification but this affects very little the predictions since the convergence in powers of p for the mass squared thermal expectation is extremely fast.

As a matter fact, I proposed for more that 15 years ago that canonical identification could be essential element of cognition mapping external world to p-adic cognitive representations realized in short length scales and vice versa.

If so, then real–p-adic duality would be a cornerstone of cognition [K50]. Common rational points would relate to the intentionality which is second aspect of the p-adic real correspondence: the transformation of real to p-adic surfaces in quantum jump would be the correlate for the transformation of intention to action. The realization of intention would correspond to the correspondence along rationals and common algebraics (the more common points real and p-adic surface have, the more faithful the realization of intentional action) and the generation of cognitive representations to the canonical identification.

The already mentioned, notion of p-adic manifolds [K95] relies on this notion and provides a very promising approach to the description of space-time correlates of cognition. Various forms of canonical identification would define cognitive representations and their reverses.

Canonical identification is continuous but does does not respect symmetries: the action of the p-adic symmetry followed by a canonical identification to reals is not equal to the canonical identification map followed by the real symmetry.

Can one fuse the two views about real-p-adic correspondence

Could the two views about real-p-adic correspondence be fused if appropriately generalized canonical identification is interpreted as a concrete duality mapping short length scale physics and long length scale physics to each other? There are however hard technical problems involved.

1. Canonical identification is not consistent with general coordinate invariance unless one can identify some physicallyt preferred coordinate system. For imbedding spaces the isometries guarantee the existence of rather limited space of this kind of coordinate systems: linear coordinates for M^4 and complex coordinate systems relatd by color isometries for CP_2. This suggests taht canonical identification should be realized at the level of imbedding space.

2. Canonical identification would be locally continuous in both directions. Note that for the points with finite pinary expansion (ordinary integers) the map is two-valued. Note also that rationals can be expanded in infinite powers series with respect to p and one can ask whether one should do this or map $q = m/n$ to $I(m)/I(n)$ (the representation of rational is unique if m and n have no common factors). Symmetries represented by matrix groups with rational matrix elements require the latter option.

   One can map rationals by $m/n \rightarrow I(m)/I(n)$. One can also express m and n as power series of p^k as $x = \sum x_n p^{nk}$ and perform the map as $x \rightarrow \sum x_n p^{-nk}$. This allows to preserve symmetries in arbitrary good measurement resolution characterized by the power p^{-k} on real side. The reason would be that rationals m/n with $m < pk$ and $n < p^k$ would be mapped to themselves: algebra wins. If m or n or both are larger than p^k the behavior associated with canonical identification sets in: topology wins.

3. This compromize between algebra and topology looks nice but an additional problem emerges when one brings in more TGD. If one wants to map differentiable p-adic space-time surfaces (preferred extremals of Kähler action) to differentiable real surfaces (preferred extremals of Kähler action), canonical identification cannot work since it is not differentiable. Second pinary cutoff above which one simply throws out the pinary digits, is needed. p-Adic space-time sheets are discretized and mapped to a discrete subsets of the real space-time sheet.

Completion to a preferred extremal is needed and assigning a preferred extremal to a discrete point set becomes the challenge. The p-adic manifold concept relies essentially on this idea about p-adic-real correspondence.

This chapter was originally written few years before the idea of p-adic manifold. The question was whether one could circumvent the tension between symmetries and continuity without approximations? After few years the answer is definitely "No!".

Despite this I have decided to keep this chapter since the quantum variant of canonical identification could also be involved with the definition of p-adic manifold. In particular, the fact that it maps p-adic numbers to algebraic numbers in the algebraic extension defined by p:th root of unity might have some deep meaning and relate to the connection between Galois group of maximal Abelian extension of rationals and adeles consisting of the Cartesian product of real and various p-adic number fields.

Could the canonical identification based on quantum integers provide a generalization of the notion of symmetry itself in order to circumvent ugly constructions? This is the question to be addressed in this chapter.

14.1.4 Brief summary of the general vision

Some of the basic questions of the p-adicization program are following.

1. Is there a duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p \to 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

   Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant. The physical interpretation could be in terms of inclusions of hyper-finite factors of type II_1 parametrized by quantum phases and allowing to interpret the action of the included algebra as having no effects on the state in the measurement resolution used [K80]. When quantum phase approaches unity one would obtained ordinary canonical identification.

2. What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this [K39])?

   This chapter studies some ideas but does not provide a clearcut answer to these questions.

Two options for quantum integers

In the sequel two options for definining quantum arithmetics are discussed: Options I and II. These are not the only one imaginable but represent kind of diametrical opposites. The two options are defined in the following manner.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to $l \ modp$ (to guarantee positivity of the quantum integer) decomposed into primes $l < p$ and these in turn to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of the product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n) p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

4. Non-negativity of quantum primes is important in the modelling of Shnoll effect by a deformation of probability distribution $P(n)$ by replacing the argument n by quantum integers and the parameters of the distribution by quantum rationals [K5]. One could also replace quantum prime by its square without losing the map of products to products.

5. At the limit when the quantum phase approaches to unit, ordinary quantum integers with p-adic norm 1 approach to ordinary integers in real sense and ordinary arithmetics results. Ordinary integers in real sense are obtained for option II when the coefficients of the pinary expansion of n are much smaller than p and p approaches infinity. Same is true for option I if the prime factors of the integer are much smaller than p.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

Quantum counterparts of classical groups

Quantum arithmetics inspires the notion of quantum matrix group as a counterpart of quantum group for which matrix elements are non-commuting numbers. Now the elements would be ordinary numbers. Quantum classical correspondence and the notion of finite measurement resolution realized at classical level in terms of discretization suggest that these two views about quantum groups are closely related. The preferred prime p defining the quantum matrix group is identified as p-adic prime or its power and the inversion $p \to 1/p$ is group homomorphism so that symmetries are respected.

Option I gives p-adic counterparts of classical groups. p-Adic numbers are replaced with the ring generated by the quantum images of p-adic numbers, which each correspond to some power of p: this extension gives powers series in p. By requiring the group conditions for a subgroup of special linear group to be satisfied in order $O(p) = 0$ one obtains classical groups for finite fields $G(p, 1)$ by simply requiring that group conditions are satisfied in order $O(p) = 0$. One can also have also classical groups associated with finite fields $G(p, n)$ having p^n elements.

Option II is more interesting and quantum counterparts could be seen as counterparts of classical groups obtained by replacing group elements with the elements of ring defined by Kac-Moody type algebra. The difference to Option I and its variants is that one does not map p-adic integer to $G(p, 1)$ by $n \to n \mod p$ before quantum map but applies it to the entire p-adic integer.

1. The quantum counterparts of special linear groups $SL(n, F)$ exists always. For the covering group $SL(2, C)$ of $SO(3, 1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if quantum arithmetics is characterized by a prime rather than general integer and when the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.

2. For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n = 2^{2r}(8k + 7))$. The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n \to 2^{2r} n$. This means scaling by 2 for the integers appearing in the square sum representation.

3. $r_3(n)$ is proportional to the so called class number function $h(-n)$ telling how many non-equivalent decompositions algebraic integers have in the quadratic algebraic extension generated by $\sqrt{-n}$.

The findings about quantum $SO(3)$ encourages to consider a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

1. The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p) = 0$ approximation) do not support this idea in the case of Mersenne primes.

2. An alternative idea discussed in [K86] is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- Physics as generalized number theory [L29]

14.2 Various options for quantum arithmetics

In this section the notion of quantum arithmetics as a deformation of p-adic number field to a ring is discussed. One can imagine several options for quantum arithmetics. Both for Option I and II p-adic integers are mapped to a subset of a ring of quantum integers and the sum operation for the ring has nothing to with that for p-adic numbers. In both cases the elements of ring makes sense as real numbers.

14.2.1 Comparing options I and II

The two options for defining quantum arithmetics are represented in the introduction so that it is no point writing the formulas again. It is interesting to compare these options.

Consider first what is common to these options.

1. For option I all integers are decomposed into products of primes mapped to their quantum counterparts by $p_1 \rightarrow p_1 \ mod \ p \rightarrow \prod p_i p_i^{k_i}$ followed by the mapping of p_i to its quantum counterpart. The modding operator for op_1 guarantees positivity of the outcome. Hence the information about prime decomposition is not lost completely. Also the information about p-adic norm is preserved if p is mapped to itself or $1/p$ (this depending on whether one speaks about p-adic or real variant of quantum integer). This option clearly respects algebra.

   For option II the information about prime decomposition is lost and there is no point of decomposing the coefficients of powers of p to prime factors. The information about pinary expansion is not lost. This option in turn respects continuity.

2. Quantum map $n \rightarrow n_q$ precedes canonical identification so that it could be interpreted as a modification for the chart map defined by canonical identification in the proposed definition for p-adic manifold already mentioned [K95](see the appendix of the book) .

3. For both options the quantum image belongs to a ring which is larger than the image since for neither options the sum of two quantum integers need not be image of p-adic number. This makes possible to assign classical groups to this ring.

4. p-Adic–real duality can be identified as the analog of canonical identification induced by the map $p \to 1/p$ in the pinary expansion of quantum rational. This maps maps p-adic and real physics to each other and real long distances to short ones and vice versa. This map is especially interesting as a map for defining cognitive representations. The map $p^n \to p^{-n}$ is generalization of this map an maps p-adic integers $k < p^n$ to itself. Note that subgroups of $Gl(m, R)$ consisting of matrices with integer valued elements p^n are especially interesting p-adically since one avoid p-adic rationals for which canonical identification map allows several variants.

The differences between options I and II relate to how one treats integers $n > p$.

1. For option I one decomposes given integer to a product of primes and all primes are mapped to their quantum counterparts so that products go to products. Sums are not however mapped to sums. Quantum primes can be also negative. For $q = exp(i\pi/p)$ integers vanishing modulo p go to zero if one defines n_q by using the general formula for quantum integer. Also the extension of the map to rationals m/n meets with difficulties if n_q can vanish. It seems that p must be mapped to $1/p$ to avoid these problems and this is done in the proposal developed in the model for Shnoll effect [?]. With this modification the image of integer is always product of quantum primes by some power of p and one does not obtain series in powers of p typical for p-adic numbers and canonical identification.

   If quantum map would respect both product and sum, the quantum counterparts of subgroups of classical matrix groups with elements elements smaller than p^n would exist. This condition cannot be satisfied. It is not clear whether subgroups of matrix groups exist for which their quantum counterparts defined by matrices with matrix elements smaller than p^n are groups too.

   This suggests that one must extend the image of p-adic integers (and its extension to that of p-adic rationals) to a ring defined by quantum sums and assign matrix group acting as symmetries to this ring. Matrix groups for which symmetries preserve volume the determinant of the matrix equals to unity so that the inverse exists always even when number field is replaced wit ring so that the existence of generalized matrix groups does not seem to be a problem.

2. For Option II one expands integer in powers p^k and maps the coefficients $n_k < p$ by quantum map just as for the first option. The quantum counterparts of p-adic integers generate a larger ring via products and sums.

   One obtains the analog of Kac-Moody algebra with coefficients for a given power of p defining an algebra analogy to polynomial algebra. One can define also rationals and obtains a structure analogous to a function field. This field allows projection to p-adic numbers but is much larger than p-adic numbers. The construction works also for more general quantum phases q than those defined by primes and $q = exp(i\pi/p^n)$ is an especially interesting case. For this option the symmetries of quantum p-adics would be preserved in the canonical identification.

14.2.2 Quantum arithmetics

The starting point idea was that quantum arithmetics maps products to products and sums to sums. It has turned out that this need not be the case for the sum and even in the case of product one can ask whether the assumption is necessary. For Option I both sum and product are respected but this option is more or less equivalent with p-adic numbers. For Option II the images of primes generate Kac-Moody type algebra, sums are not mapped to sums, and the number of elements of quantum algebra is larger than that of p-adic number field. Also in this case one can consider the option giving up the condition that products are mapped to products.

Are products mapped to products?

The first question is whether products are mapped to products. For Option I this is true by construction but for option II it does not hold anymore.

1. The multiplicative structure of ordinary integers should be respected in the map taking ordinary integers to quantum integers:

$$n = kl \rightarrow n_q = k_q l_q \quad . \tag{14.2.1}$$

   This is guaranteed if the map is induced by the map of ordinary primes to quantum primes. This means that one decomposes n to a product of primes l and maps $l \rightarrow l_q$. For primes $l < p$ the map reads as $l \rightarrow l_q = (q^l - \overline{q}^{-l})/(q - \overline{q})$, $q = exp(i\pi/p)$ and gives a positive number.

   For option I all primes are mapped by quantum map and products are mapped to products. For $l > p$ this is not the case for option II: one expands primes $l > p$ as $l = \sum l_m p^m$, $l_m < p$, and maps the coefficients l_m to quantum integers without decomposition to primes $l_i < p$. Products are not mapped to products now. One can of course modify option two by applying option I to the coefficients of pinary expansion.

Are sums mapped to sums?

Second question is about whether quantum map commutes with sum. There are two options.

1. For Option I also the sum of quantum integers is well-defined and also induces sum of the quantum rationals. If the sum $+_q$ for quantum integers reflects the summation of ordinary integers, one has

$$(k + l)_q = k_q +_q l_q \quad . \tag{14.2.2}$$

   This is not the case in general: consider only the situation in which the sum of pinary digits for some power of p is p as an example. Sums cannot go to sums for option I. One however form sums of the quantum images of p-adics and the generate a ring whose elements can be projected to p-adic numbers by projecting the summands separately. Hence one obtains a ring and this is enough to talk about classical matrix groups for which matrix has unit determinant.

2. Also for option II it is impossible to map sums of elements to sums of their images. p-Adic numbers are mapped to a sub-space of a ring of quantum p-adics generated by the the images l_q of primes $l < m$, where m defines the quantum phase. In other words, one forms all possible products and sums of the these generators and also their negatives. The sum is defined as the complete analog of sum for Kac-Moody algebras: $a+b = \sum a_n m^n + \sum b_n m^n = \sum (a_n + b_n) m^n$ and obviously differs from m-adic sum. The general element of algebra is $x = \sum x_n m^n$, where one has

$$x_n = \sum_{\{n_i\}} N(\{n_i\}) \prod_i x_i^{n_i} \quad , \quad x_i = p_{i,q}, \quad p_i < m \quad , \quad q = exp(i\pi/m) \quad .$$

   Here $N(\{n_i\})$ is integer. $m = p$ gives what might be called quantum p-adic numbers. Note that also zeroth order term giving rise to integers as constant term of polynomials is also present. The map would produce integers from zeroth order terms so that skeptic could see the construction too complex.

   One has what could be regarded as analog of polynomial algebra with coefficients of polynomials given by integers. Note that the coefficients can be also negative since quantum map combined with canonical identification maps -1 to -1: canonical identification mapping -1

to $(p-1)_q(1+p+p^2...)$ would give only non-negative real numbers. If one wants that also the images under canonical identification form a field (so that $-x$ for given x belongs to the system) one must assume that -1 is mapped to -1. Also the condition that one obtains classical groups requires this. One can form also rationals of this algebra as ratios of this kind of polynomials and a subset of them projects naturally to p-adic rationals.

3. One can project quantum integers for Option II to p-adic numbers by mapping the products of powers of generators l_q, $l<m$ to products of ordinary p-adic primes $l<m$ in the sums defining the coefficients in the expansion in powers of m to ordinary p-adic integers. This projection defines a structure analogous to a covering space for p-adic numbers. The covering contains infinite number of elements since also the negatives of generators are allowed in the construction. The covering by elements with positive coefficients of m^n is finite.

4. Quantum p-adics for option II form a ring but do they form a field? One might hope this since quantum p-adics are very much analogous to a function field for which the argument of function is defined by integer characterizing the powers of p in quantum pinary expansion. One would have the analogy of function field in the set of integers. This means that one can indeed speak of quantum rationals M/N which can be mapped to reals by $I(M/N)=I(M)/I(N)$.

What could be the interpretation for 1-to-many character of the quantum map of p-adics to reals? One possibility is that it could reflect the non-uniqueness due to finite measurement resolution. One can ask whether it might be possible to extend the canonical identification with finite measurement resolution reflected as pinary cutoffs so that all p-adic points would be mapped to reals in such a manner that the real images would be differentiable. This is probably not possible and by accepting this one ends up with the notion of p-adic manifold [K95]. p-Adic manifolds could be constructed also by using quantum variant of canonical identification and this might have some physical relevance and relate to finite resolution and inclusions of von Neuman algebras representing it.

The identification of rationals as points common to all number fields suggests that one should define differentiability as a weaker property consistent with finite measurement resolution: function is differentiable if it is differentiable at rational points represented as ratios of integers.

About the choice of the quantum parameter q

Some comments about the quantum parameter q are in order.

1. The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n-\overline{q}^n}{q-\overline{q}} \ . \tag{14.2.3}$$

Here q is *any* complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$\begin{aligned} p_q &= \frac{q^p-\overline{q}^p}{q-\overline{q}} \\ n_q &= \prod_p p_q^{n_p} \ \text{ for } \ n=\prod_p p^{n_p} \ . \end{aligned} \tag{14.2.3}$$

2. In the general case quantum phase is complex number with magnitude different from unity:

$$q = exp(\eta)exp(i\pi/m) \ . \tag{14.2.4}$$

The quantum map is 1-1 for a non-vanishing value of η and the limit $m \rightarrow \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteeing a unique definition of quantum sum. In the p-adic context with $m = p$ the number $exp(\eta)$ exists as an ordinary p-adic number only for $\eta = np$. One can of course introduce a finite-dimensional extension of p-adic numbers generated by $e^{1/k}$.

3. The root of unity must correspond to an element of algebraic extension of p-adic numbers. Here Fermat's theorem $a^{p-1} \ mod \ p = 1$ poses constraints since $p-1$:th root of unity exists as ordinary p-adic number. Hence $m = p-1$:th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers.

4. If q reduces to quantum phase, the points $n = 0, 1, -1$ are fixed points of $n \rightarrow n_q$ for ordinary integers so that one could say that all these numbers are common to integers and quantum integers for all values of $q = exp(i\pi/m)$. For p-adic integers $-1 = (p-1)(1 + p + p^2 + ..$ is problematic. Should one use direct formula mapping it to -1 or should one map the expansion to $(p-1)_q(1 + p + p^2 + ...)$? This option looks more plausible.

   (a) For the first option the images under canonical can have both signs and can form a field. For the latter option would obtain only non-negative quantum p-adics for ordinary p-adic numbers. They do not form a field. For algebraic extensions of p-adics by roots of unity one can obtain more general complex numbers as quantum images. For the latter option also the quantum p-adic numbers projecting to a given prime l regarded as p-adic integer form a finite set and correspond to all expansions $l = \sum l_k p^k$ where l_k is product of powers of primes $p_i < p$ but one can have also $l_k > p$.

   (b) Quantum integers containing only the $O(p^0)$ term in the binary expansion for a sub-ring. Corresponding quantum rationals could form a field defining a kind of covering for finite field $G(p, 1)$.

   (c) The image $I(m/n)$ of the pinary expansion of p-adic rational is different from $I(m)/I(n)$. The formula $m/n \rightarrow I(m)/I(n)$ is the correct manner to define canonical identification map. In this case the real counterparts of p-adic quantum integers do not form the analog of function fields since the numbers in question are always non-negative.

5. For p-adic rationals the quantum map reads as $m/n \rightarrow m_q/n_q$ by definition. But what about p-adic transcendentals such as e^p? There is no manner to decompose these numbers to finite primes and it seems that the only reasonable map is via the mapping of the coefficients x_n in $x = \sum x_n p^n$ to their quantum adic counterparts. It seems that one must expand all quantum transcendentals having as a signature non-periodic pinary expansion to quantum p-adics to achieve uniqueness. Second possibility is to restrict the consideration to rational p-adics. If one gives up the condition that products are mapped to products, one can map $n = n_k p^k$ to $n_q = \sum n_{kq} p^k$. Only the products of p-adic integers $n < p$ smaller than p would be mapped to products.

6. The index characterizing Jones inclusion [A172] [K25] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

To sum up, one can imagine several options and it is not clear which option is the correct one (if any). Certainly Option I for which the quantum map is only part of canonical identification is the simpler one- perhaps quite too simple. The model for Shnoll effect requires only Option I. The notion of quantum integer as defined for Opion II imbeds p-adic numbers to a much larger structure imbeddable to reals and therefore much more general than that proposed in the model of Shnoll effect [K5] but gives identical predictions when the parameters characterizing the probability distribution $f(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $f(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

14.2.3 Canonical identification for quantum rationals and symmetries

The fate of symmetries in canonical identification map is different for options I and II. Before continuing, one can of course ask why canonical identification should map p-adic symmetries to real symmetries. There is no obvious answer to the question.

1. For option II the prime p in the expansion $\sum x_n l^n$ is interpreted as a symbolic coordinate variable and the product of two quantum integers is analogous to the product of polynomials reducing to a convolution of the coefficient using quantum sum. The coefficient of a given power of p in the product would be just the convolution of the coefficients for factors using quantum sum. In the sum coefficients would be just the quantum sums of coefficients of summands.

2. Option I maps p-adic integers to their quantum counterparts by mapping the prime factors to their quantum counterparts defined by $q = exp(i\pi/p)$. The sums of the resulting quantum integers define a linear space consisting of sums $\sum k_n q^n$ of quantum phases with integer coefficients k_n subject to the condition that the sum $\sum_{0\leq n<p} q^n$ vanishes. Given p-adic integer is mapped to single phase q^n. The map of all p-adic integers to p quantum phases means loss of information and generation or ring creates information not related to the p-adic numbers themselves.

   (a) One can also define quantum rationals by writing a given rational in unique manner as $r = p^k m/n$, expanding m and n as finite power series in p, and by replacing the coefficients with their quantum counterparts. The mapping of quantum rationals to their real counterparts would be by canonical identification $p \rightarrow 1/p$ in m_q/n_q. Also the completion of quantum rationals obtained by allowing infinite powers series for m and n makes sense and defines by canonical identification what might be called quantum reals.

   (b) Quantum arithmetics defined in this manner does not reflect faithfully the ordinary p-adic arithmetics and also leads to a problem with symmetries. In the product of ordinary p-adic integers the convolution for given power of p can lead to overflow and this leads to the emergence of modulo arithmetics. As a consequence, the canonical identification $\sum x_n l^n \rightarrow \sum x_n l^{-n}$ does not respect product and sum in general (simple example: $I((xl)^2) = x^2 l^{-2} \neq (I(xl))^2 = (x^2 modl)l^{-2} + (x^2 - x^2 modl)l^{-3}$ for $x > l/2$). Therefore canonical identification induced by $l \rightarrow 1/l$ does not respect symmetries represented affinely (as linear transformations and translations) although it is continuous.

   (c) For quantum rationals defined as ratios m_q/n_q of quantum integers and mapped to $I(m_q)/I(n_q)$ the situation improves dramatically but is not cured completely. The breaking of symmetries could have a natural interpretation in terms of finite measurement resolution. For instance, one could argue that p-adic space-time sheets are extrema of Kähler action in algebraic sense and their real counterparts obtained by canonical identification are kind of smoothed out quantum average space-time surfaces, which do not satisfy real field equations and are not even differentiable. In this framework p-adicization would defined quantum average space-time as a p-adically smooth object which nice geometric properties.

Consider next Option II for quantum p-adics.

1. The original motivation for quantum rationals was to obtain correspondence between p-adics and reals respecting symmetries. For option II this dream can be achieved if the symmetries are defined for quantum rationals rather than p-adic numbers. Whether this means that quantum rationals are somehow deeper notion that p-adic number field is an interesting question. Since quantum rationals are obtained from quantum integers defining a

Kac-Moody type algebra in powers of p^n symmetry conditions for quantum rational matrices reduce to conditions in terms of quantum integers and hold separately for each power of p. Therefore the value of p does not actually matter, and the replacement $p \to 1/p$ respects the symmetries.

For instance, for the quantum counterpart of group $SL(2,Z)$ assuming that p^N is the largest power in the matrix elements the condition $det(A) = 1$ gives $2N+1$ conditions for $4(N+1)$ parameters leaving $2N+3$ parameters. The matrix elements are integers so that actual conditions are more stringent.

2. Quantum integers generate a space in which the space of coefficients of p^n is the module generated by the sums $\sum k_n q^n$ of quantum phases with integer coefficients k_n subject to the condition that the sum $\sum_{0\leq n<p} q^n$ vanishes. The huge extension of the original space is an obvious problem.

3. For this option non-uniqueness is a potential problem. One can have several quantum integers projecting to the same finite integer in powers of p. The number would be actually infinite when the coefficients of powers of p can occur with both signs. Does the non-uniqueness mean that quantum p-adics are more fundamental than p-adics?

4. The non-uniqueness inspires questions about the relationship between quantum field theory and number theory. Could the sum over different quantum representatives for p-adic integers define the analog of the functional integral in the ideal measurement resolution? Could loop corrections correspond number theoretically to the sum over all the alternatives allowed in a given measurement resolution defined by maximal number of powers of p in expansions of m and n in $r = m/n$? This would extend the vision about physics as generalized number theory considerably.

Note that quantum p-adic numbers are algebraic numbers so that quantum integers are algebraic numbers with prime p remaining ordinary integer.

14.2.4 More about the non-uniqueness of the correspondence between p-adic integers and their quantum counterparts

For both options the projection from quantum integers to p-adic numbers is many-to-one.

For option I p-adic integer is mapped to an integer proportional to a quantum integers proportional to power of p expressing its p-adic norm. Since the primes p_i in the decomposition of n are effectively replaced with $p_i \mod p$, a large number of integers with same p-adic norm is mapped to same quantum integer. A lot of information is lost.

For Option II p-adic number is mapped to a series in powers of p so that information is not lost. It is interesting to have some idea about how many quantum counterparts given p-adic integer has in this case and what might be their physical interpretation. If -1 is mapped to -1 rather than $(p-1)_q(1+p+p^2+...)$ in quantum map and therefore also in canonical identification quantum p-adics form an analog of a function field. The number of quantum p-adics projected to same integer is infinite.

The number of quantum p-adics for which the coefficients of the polynomials of quantum primes $p_1 < p$ regarded as variables are positive is finite. These kind of quantum integers could be called strictly positive. It is easy to count the number of different strictly positive quantum counterparts of p-adic integer $n = n_0 + n_1 p + n_2 p^2 + ... + n_k p^k$ - that is elements of the ring of quantum integers projected to a given p-adic integer n.

1. For both options the number of quantum integers projected to a given integer n is simply the number of all partitions of to a sum of integers, whose number can vary from 1 to n and thus expressible as the sum $D(n) = \sum_{k=1}^{n} d(n,k)$ of numbers of partitions to k integers. Interestingly, the number of states with total conformal weight n constructible using at most k Virasoro generators equals to $d(n,k)$ and the total number of states with conformal weight n is just $D(n)$. This result follows if one does not assumes that different quantum representatives are really different. One cannot exclude the possibility that the condition $\sum_{n=0}^{p-1} q^n = 0$ for quantum phases implies this kind of dependencies.

Similar situation occurs in the construction of tensor powers of group representations for any additive quantum number for which the basic unit is fixed. Could quantum classical correspondence be realized as a mapping of different states of a tensor product as different quantum p-adic space-time sheets?

2. The partition of n in all possible manners resembles combinatorially the insertion of loop corrections in all possible manners to a Feynman diagram containing corresponds up to p^{k-1}. Maybe the sum over quantum corrections could be reduced to the summation of amplitudes in which p-adic integer is mapped to its quantum counterpart in all possible manners. In zero energy ontology quantum corrections to generalized Feynman diagrams in a new p-adic length scaled defined by p^k indeed more or less reduces to the addition of zero energy states as a new tensor factor in all possible manners so that structurally the process would be like adding tensor factor.

To number of geometric objects to which one can assign quantum counterparts is rather limited. For the points of imbedding space with rational coordinates the number of quantum rational counterparts would be finite. If either of the integers appearing in the p-adic rational become infinite as a real integer, the number of quantum rationals becomes infinite and one obtains continuum in p-adic sense since p-adic integers form a continuum.

An infinite number of points of a $D > 0$-dimensional quantum counterpart of p-adic surface project to the same p-adic point. The restriction to a finite number of pinary digits makes sense only at the ends of braid strands at partonic 2-surfaces. This provides additional support for the effective 2-dimensionality and the braid representation for the finite measurement resolution. The selection of braid ends is strongly constrained by the condition that the number of pinary digits for the imbedding space coordinates is finite.

The interesting question is whether the summation over the infinite number of quantum copies of the p-adic partonic 2-surface could correspond to the functional integral over partonic 2-surfaces with braid ends fixed and thus having only one term in their pinary expansion. This kind of functional integral is indeed encountered in quantum TGD.

1. The summations in which the quantum positions of braid ends form a finite set would correspond to finite pinary cutoff. Second question is what the quantum summation for partonic 2-surfaces means: certainly there must be correlations between very nearby points if the summation is to make sense. The notion of finite measurement resolution suggests that summation reduces to that over the quantum positions of the braid ends.

2. Indeed, the reduction of the functional integral to a summation over quantum copies makes sense only if it can be carried out as a limit of a discrete sum analogous to Riemann sum and giving as a result what might be called quantum p-adic integral. This limit would mean inclusion of an increasing number of points of the partonic 2-surface to the quantum sum defined by the increasing pinary cutoff. One would also sum over the number of braid strands. This approach could make sense physically if the collection of p-adic partonic 2-surfaces together with their tangent space data corresponds to a maximum of Kähler function. Quantum summation would correspond to a functional integral over small deformations with weight coming from the p-adic counterpart of vacuum functional mapped to its quantum counterpart. Canonical identification would give the real or complex counterpart of the integral.

14.2.5 The three basic options for quantum arithmetics

I have proposed two alternative definitions for quantum integers. In [K86] a third option is discussed.

1. For option I quantum counterparts of p-adic integers are identified as products of quantum counterparts for the primes dividing them. Powers of p are mapped to their inverses (straightforward quantum map would take them to zero). The quantum integers can be extended to ring (and algebra) by allowing sum operation. Field property is in general lost.

2. The approach adopted in the sequel is based on Option II based on the identification of quantum p-adics as an analog of Kac-Moody algebra with powers p^n in the same role as the powers z^n for Kac-Moody algebra. The two algebras have identical rules for sum and multiplication, and one does not require the arithmetics to be induced from the p-adic arithmetics (as assumed originally) since this would lead to a loss of associativity in the case of sum. Therefore the quantum counterparts of primes $l \neq p$ generate the algebra. One can also make the limitation $l < p^N$ to the generators. The counterparts of fixed integers in the map of integers to quantum integers are $0, 1, -1$ are $, 0, 1, -1$ as is easy to see. The number of quantum integers projecting to same p-adic integer is infinite.

3. One can consider also quantum m-adic option with expansion $l = \sum l_k m^k$ in powers of integer m with coefficients decomposable to products of primes $l < m$. This option is consistent with p-adic topology for primes p divisible by m and is suggested by the inclusion of hyper-finite factors [K25] characterized by quantum phases $q = exp(i\pi/m)$. Giving up the assumption that coefficients are smaller than m gives what could be called quantum covering of m-adic numbers. For this option all quantum primes l_q are non-vanishing. Phases $q = exp(i\pi/m)$ characterize Jones inclusions of hyper-finite factors of type II_1 assumed to characterize finite measurement resolution.

4. The definition of quantum p-adics discussed in [K86] replaces integers with Hilbert spaces of same dimension and $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$. Also co-product and co-sum must be introduced and assign to the arithmetics quantum dynamics, which leads to proposal that sequences of arithmetic operations can be interpreted arithmetic Feynman diagrams having direct TGD counterparts. This procedure leads to what might be called quantum mathematics or Hilbert mathematics since the replacement can be made for any structure such as rationals, algebraic numbers, reals, p-adic numbers, even quaternions and octonions. Even set theory has this kind of generalization. The replacement can be made also repeatedly so that one obtains a hierarchy of structures very similar to that obtained in the construction of infinite primes by a procedure analogous to repeated second quantization. One possible interpretation is in terms of a hierarchy of logics of various orders. Needless to say this definition is the really deep one and actually inspired by quantum TGD itself. In this picture the quantum p-adics as they are defined here would relate to the canonical identification map to reals and this map would apply also to Hilbert p-adics.

14.3 Do commutative quantum counterparts of Lie groups exist?

The proposed definition of quantum rationals involves exceptional prime p expected to define what might be called p-adic prime. In p-adic mass calculations canonical identification is based on the map $p \rightarrow 1/p$ and has several variants but quite generally these variants fail to respect symmetries. Canonical identification for space-time coordinates fails also to be general coordinate invariant unless one has preferred coordinates. A possible interpretation could be that cognition affects physics: the choice of coordinate system to describe physics affects the physics.

The natural question is whether the proposed definition of quantum integers as series of powers of p-adic prime p with coefficients which are arbitrary quantum rationals not divisible by p with product defined in terms of convolution for the coefficients of the series in powers of p using quantum sum for the summands in the convolution could change (should one say "save"?) the situation.

To see whether this is the case one must find whether the quantum analogues of classical matrix groups exist. To avoid confusion it should be emphasized that these quantum counterparts are distinct from the usual quantum groups having non-commutative matrix elements. Later a possible connection between these notions is discussed. In the recent case matrix elements commute but sum is replaced with quantum sum and the matrix element is interpreted as a powers series or polynomial in symbolic variable $x = p$ or $x = 1/p$, p prime such that coefficients are rationals not divisible by p.

The crucial points are the following ones.

1. All classical groups [A18] are subgroups of the special linear groups [A87] $SL_n(F)$, $F = R, C$, consisting of matrices with unit determinant. One can also replace F with the integers of the field F to get groups like $SL(2, Z)$. Classical groups are obtained by posing additional conditions on $SL_n(F)$ such as the orthonormality of the rows with respect to real, complex or quaternionic inner product. Determinant defines a homomorphism mapping the product of matrices to the product of determinants in the field F.

   Could one generalize rational special linear group (matrices with determinant 1) and its algebraic extensions by replacing the group elements by ratios of polynomials of a formal variable x, which has as its value the preferred prime p such that the coefficients of the polynomials are quantum integers not divisible by p? For Option I the situation one has just ratios of p-adic integers finite as real integers and for Option II the integers are polynomials $x = \sum x_n p^n$, where one has

   $$x_n = \sum_{\{n_i\}} N(\{n_i\}) \prod_i x_i^{n_i} \quad , \quad x_i = p_{i,q}, \quad p_i < p \quad , \quad q = exp(i\pi/p) \quad .$$

   Here $N(\{n_i\})$ is integer. Could one perform this generalization in such a manner that the canonical identification $p \to 1/p$ maps this group to an isomorphic group? If quantum p-adic counterpart of the group is non-trivial, this seems to be the case since p plays the role of an argument of a polynomial with a specific values.

2. The identity $det(AB) = det(A)det(B)$ and the fact that the condition $det(A) = 1$ involves at the right hand side only the unit element common to all quantum integers suggests that this generalization could exist. If one has found a set of elements satisfying the condition $det_q(A) = 1$ all quantum products satisfy the same condition and subgroup of rational special linear group is generated.

14.3.1 Quantum counterparts of special linear groups

Special linear groups [A87] defined by matrices with determinant equal to 1 contain classical groups as subgroups and the conditions for their quantum counterparts are therefore the weakest possible. Special linear group makes sense also when one restricts the matrix elements to be integers of the field so that one has for instance $SL_n(Z)$. Opiton I reduces to that for ordinary p-adics. For Option II each power of p can be treated independently so that the situation is easier. The treatment of conditions in two cases differs only in that overflows in p are possible for Option I. The numbers of conditions are same.

Let us consider $SL_n(Z)$ first.

1. To see that the generalization exists in the case of special linear groups one just just writes the matrix elements a_{ij} in series in powers of p

   $$a_{ij} = \sum_n a_{ij}(n) p^n \quad . \tag{14.3.1}$$

   This expansion is very much analogous to that for the Kac-Moody algebra element and also the product and sum obey similar algebraic structure. p is treated as a symbolic variable in the conditions stating $det_q(A) = 1$. It is essential that $det_q(A) = 1$ holds true when p is treated as a formal symbol so that each power of p gives rise to separate conditions.

2. For SL_n the definition of determinant involves sum over products of n elements. Quantum sums of these elements are in question.

3. Consider now the number of conditions involved. The number of matrix elements is in real case $N^2(k+1)$, where k is the highest power of p involved. $det(A) = 1$ condition involves powers of p up to l^{Nk} and the total number of conditions is $kN + 1$ - one for each power. For higher powers of p the conditions state the vanishing of the coefficients of p^m. This is achieved

elegantly in the sense of modulo arithmetics if the quantum sum involved is proportional to l_q.

The number of free parameters is

$$\# \quad = (k+1)N^2 - kN - 1 = kN(N-1) + N^2 - 1 \ . \tag{14.3.2}$$

For $N = 2, k = 0$ one obtains $\# = 3$ as expected for SL(2,R). For $N = 2, k = 1$ one obtains $\# = 5$. This can be verified by a direct calculation. Writing $a_{ij} = b_{ij} + c_{ij}p$ one obtains three conditions

$$det_q(B) = 1 \ , \ Tr_q(BC) = 0 \ , \ det_q(C) = 0 \ . \tag{14.3.3}$$

for the 8 parameters leaving 5 integer parameters.

Integer values of the parameters are indeed possible. Using the notation

$$b_{ij} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \ , \quad c_{ij} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \tag{14.3.4}$$

one can write the solutions as

$$\begin{array}{l} (a_1, b_1) = k(c_1, d_1) \ , \quad (c_1, d_1) = l(a_0 - kc_0, b_0 - kd_0) \ , \\ a_0 d_0 - b_0 c_0 = 1 \ . \end{array} \tag{14.3.5}$$

Therefore 6 integers characterize the solution.

4. Complex case can be treated in similar manner. In this case the number of three parameters is $2(k+1)N^2$, the number of conditions is $2(kN+1)$ and the number of parameters is

$$\# = 2(k+1)N^2 - 2(kN+1) \ . \tag{14.3.6}$$

5. Since the conditions hold separately for each power of p, the formulate $det_q(AB) = det_q(A)det_q(B)$ implies that the matrices satisfying the conditions generate a subgroup of SL_n.

One can generalize the argument to rational values of matrix elements in a simple manner. The matrix elements can be written in the form $A_{ij} = Z_{ij}/K$ and the only modification of the equations is that the zeroth order term in p gives $det(Z) = K^n$ for SL_n. One can expand K^n in powers of p and it gives inhomogenous term to in each power of p. For instance, if K is zeroth order in p, solutions to the conditions certainly exist.

The result means that rational subgroups of special linear groups $SL_n(R)$ and $SL(n,C)$ and also the real and complex counterparts of $SL(n,Z)$ quantum matrix groups characterized by prime p exist in both real and p-adic context and can be related by the map $p \rightarrow 1/p$ mapping short and length scales to each other.

It is remarkable that only the Lorentz groups $SO(2,1)$ and $SO(3,1)$ have covering groups are isomorphic to $SL(2,R)$ and $SL(2,C)$ allow these subgroups. All classical Lie groups involve additional conditions besides the condition that the determinant of the matrix equals to one and all these groups except symplectic groups fail to allow the generalization of this kind for arbitrary values of k. Therefore four-dimensional Minkowski space is in completely exceptional position.

14.3.2 Do classical Lie groups allow quantum counterparts?

In the case of classical groups one has additional conditions stating orthonormality of the rows of the matrix in real, complex, or quaternionic number field. It is quite possible that the conditions might not be satisfied always and it turns out that for G_2 and probably also for other exceptional groups this is the case.

1. Non-exceptional classical groups

It is easy to see that all non-exceptional classical groups quantum counterparts in the proposed sense for sufficiently small values of k and in the case of symplectic groups quite generally. In this case one must assume rational values of group elements and one can transform the conditions to those involving integers by writing $A_{ij} = Z_{ij}/K$. The expansion of K gives for orthogonal groups the condition that the lengths of the integer rows defining Z_{ij} have length K^2 plus orthogonality conditions. $det(A) = 1$ condition holds true also now since a subgroup of special linear group is in question.

1. Consider first orthogonal groups $SO(N)$.

   (a) For $q = 1$ there are N^2 parameters. There are N conditions stating that the rows are unit vectors and $N(N-1)/2$ conditions stating that they are orthogonal. The total number of free parameters is $\# = N(N-1)/2$.

   (b) If the highest power of p is k there are $(k+1)N^2$ parameters and $(2k+1)[N+N(N-1)/2] = (2k+1)(N+1)/2$ conditions. The number of parameters is

   $$\# = N^2(k+1) - \frac{N(N+1)(2k+1)}{2} = \frac{N(N-2k+1)}{2} \quad . \tag{14.3.7}$$

   This is negative for $k > (N+1)/2$. It is quite not clear how to interpret this result. Does it mean that when one forms products of group elements satisfying the conditions the powers higher than $k_{max} = [(N+1)/2]$ vanish by quantum modulo arithmetics. Or do the conditions separate to separate conditions for factors in AB: this indeed occurs in the unitarity conditions as is easy to verify. For $SO(3)$ and $SO(2,1)$ this would give $k_{max} = 2$. For $SO(3,1)$ one would have $k_{max} = 2$ too. Note that for the covering groups $SL(2,R)$ and $SL(2,C)$ there is no restrictions of this kind.

   (c) The normalization conditions for the coefficients of the highest power of a given row imply that the vector in question has vanishing length squared in quantum inner product. For $q = 1$ this implies that the coefficients vanish. The repeated application of this condition one would obtain that $k = 0$ is the only possible solution. For $q \neq 1$ the conditions can be satisfied if the quantum length squared is proportional to $l_q = 0$. It seems that this condition is absolutely essential and serves as a refined manner to realize p-adic cutoff and quantum group structure and p-adicity are extremely closely related to each other. This conclusion applies also in the case of unitary groups and symplectic groups.

   (d) Complex forms of rotation groups can be treated similarly. Both the number of parameters and the number of conditions is doubled so that one obtains $\# = N^2(k+1) - N(N+1)(2k+1) = N(N-2k+1)$ which is negative for $k > (N+1)/2$.

2. Consider next the unitary groups $U(N)$. Similar argument leads to the expression

   $$\# = 2N^2(k+1) - (2k+1)N^2 = N^2 \tag{14.3.8}$$

   so that the number of three parameters would be N^2- same as for $U(N)$. The determinant has modulus one and the additional conditions requires that this phase is trivial. This is expected to give $k+1$ conditions since the fixed phase has l-adic expansion with $k+1$ powers. Hence the number of parameters for $SU(N)$ is

$$\# = N^2 - k + 1 \tag{14.3.9}$$

giving the condition $k_{max} < N^2 - 1$ which is the dimension of $SU(N)$.

3. Symplectic group can be regarded as a quaternionic unitary group. The number of parameters is $4N^2(k+1)$ and the number of conditions is $(2k+1)(N+2N(N-1)) = N(2N-1)(2k+1)$ so that the number of three parameters is $\# = 4N^2(k+1)-(2k+1)N(N-1) = (2k+3)N^2 + N(2k+1)$. Fixing single quaternionic phase gives 3(k+1) conditions so that the number of parameters reduces to

$$\# = (2k+3)N^2 + (2k+1)N - 3(k+1) = (k+1)(2N^2+2N-3) + N(N-1) \tag{14.3.10}$$

which is positive for all values of N and k so that also symplectic groups are in preferred position. This is rather interesting, since the infinite-dimensional variant of symplectic group associated with the $\delta M^4 \times CP_2$ is in the key role in quantum TGD and one expects that in finite measurement resolution its finite-dimensional counterparts should appear naturally.

2. Exceptional groups are exceptional

Also exceptional groups [A31] [A31] related closely to octonions allow an analogous treatment once the nature of the conditions on matrix elements is known explicitly. The number of conditions can be deduced from the dimension of the ordinary variant of exceptional group in the defining matrix representation to deduce the number of conditions. The following argument allows to expect that exceptional groups are indeed exceptional in the sense that they do not allow non-trivial quantum counterparts.

The general reason for this is that exceptional groups are very low dimensional subgroups of matrix groups so that for the quantum counterparts of these groups the number N_{cond} of group conditions is too large since the number of parameters is $(k+1)N^2$ in the defining matrix representation (if such exists) and the number of conditions is at least $(2k+1)N_{class}$, where N_{class} is the number of condition for the classical counterpart of the exceptional group. Note that r-linear conditions the number of conditions is proportional to $rk+1$.

One can study the automorphism group G_2 [A36] of octonions as an example to demonstrate that the truth of the conjecture is plausible.

1. G_2 is a subgroup of $SO(7)$. One can consider 7-D real spinor representation so that a representation consists of real 7×7matrices so that one has $7^2 = 49$ parameters. One has $N(N+1)/2$ orthonormality conditions giving for $N = 7$ orthonormality conditions 28 conditions. This leaves 21 parameters. Besides this one has conditions stating that the 7-dimensional analogs of the 3-dimensional scalar-3-products $A\cdot(B\times C)$ for the rows are equal 1, -1, or 0. The number of these conditions is $N(N-1)(N-2)/3!$. For $N = 7$ this gives 35 conditions meaning that these conditions cannot be independent of orthonormalization conditions The number of parameters is $\# = 49 - 35 = 14$ - the dimension of G_2 - so that these conditions must imply orthonormality conditions.

2. Consider now the quantum counterpart of G_2. There are $(k+1)N^2 = 49(k+1)$ parameters altogether. The number of cross product conditions is $(3k+1)\times 35$ since the highest power of p in the scalar-3-product is l^{3k}. This would give

$$\# = -56k + 14 \ . \tag{14.3.11}$$

This number is negative for $k > 0$. Hence G_2 would not allow quantum variant. Could this be interpreted by saying that the breaking of G_2 to $SU(3)$ must take place and indeed occurs in quantum TGD as a consequence of associativity conditions for space-time surfaces.

3. The conjecture is that the situation is same for all exceptional groups.

The general results suggest that both the covering group of the Lorenz group of 4-D Minkowski space and the hierarchy symplectic groups have very special mathematical role and that the notions of finite measurement resolution and p-adic physics have tight connections to classical number fields, in particular to the non-associativity of octonions.

14.3.3 Questions

In the following some questions are introduced and discussed.

How to realize p-adic-real duality at the space-time level?

The concrete realization of p-adic–real duality would require a map from p-adic realm to real realm and vice-versa induced by the map $p \rightarrow 1/p$ leading from p-adic number field to real number field or vice versa.

If possible, the realization of p-adic real duality at the space-time level should not pose additional conditions on the preferred extremals themselves. Together with effective 2-dimensionality this suggests that the map from p-adic realm to real realm maps partonic 2-surfaces to partonic 2-surfaces defining at least partially the boundary data for holography.

The situation might not be so simple as this.

1. One must however also consider the possibility that its is 3-D space-like surfaces at the ends of CDs which are mapped by the duality from p-adic realm to real realm or vice versa. A possible reason is that this kind of surfaces can be easily defined as intersections $F_i(z, r\xi^2, \xi^2) = 0$, $i = 1, 2$ of two complex valued functions F_i of complex coordinate z and radial light-like coordinate for $\delta M^4_\pm = S^2 \times T_+$ and two complex coordinates ξ^i, $i = 1, 2$ of CP_2: the number of conditions is 4 and this gives D= 7-4=3-dimensional space-like surface as a solution. These surfaces - that is functions F_i cannot be completely free but solutions of field equations in the direction of radial coordinate, and this might pose a difficulty.

2. It is also possible that some local 4-D tangent space data at partonic 2-surfaces are needed to characterize the space-time surface. An alternative possibility is that the failure of standard form of determinism for Kähler action forces to introduce partonic 2-surfaces in various scales and the breaking of strict 2-dimensionality does not occur locally. This option would correspond at quantum level radiative corrections in shorter scales down to CP_2 scale and might be seen as aesthetically more attractive option.

3. The realization of p-adic real duality by applying the proposed form of canonical identification to quantum rational points requires preferred coordinates. For the minimum option defined by the map of partonic 2-surfaces (no 4-D tangent space data) this would mean that one must have preferred coordinates for partonic 2-surfaces. It is easy to imagine how to identify this kind of preferred complex coordinate. The complex coordinate could correspond to a preferred complex coordinate for $S^2 \subset \delta M^4_\pm$ or for a homologically non-trivial geodesic sphere of CP_2. The complex coordinates would transform linearly under the maximal compact subgroup of SO(3) *resp.* SU(3).

How commutative quantum groups could relate to the ordinary quantum groups?

The interesting question is whether and how the commutative quantum groups relate to ordinary quantum groups.

This kind of question is also encountered when considers what finite measurement resolution means for second quantized induced spinor fields [K26]. Finite measurement resolution implies a cutoff on the number of the modes of the induced spinor fields on partonic 2-surfaces. As a consequence, the induced spinor fields at different points cannot ant-commute anymore. One can however require anti-commutativity at a discrete set of points with the number of points "more or less equal" to the number of modes. Discretization would follow naturally from finite measurement resolution in its quantum formulation.

The same line of thinking might apply to to quantum groups. The matrix elements of quantum group might be seen as quantum fields in the field of real or complex numbers or possibly p-adic number field or of its extension. Finite measurement resolution means a cutoff in the number of modes and commutativity of the matrix elements in a discrete set of points of the number field rather than for all points. Finite measurement resolution would apply already at the level of symmetry groups themselves. The condition that the commutative set of points defines a group would lead to the notion of commutative quantum group and imply p-adicity as an additional and completely universal outcome and select quantum phases $exp(i\pi/p)$ in a preferred position. Also the generalization of canonical identification so central for quantum TGD would emerge naturally.

One must of course remember that the above considerations probably generalize so that one should not take the details of the discussion too seriously.

How to define quantum counterparts of coset spaces?

The notion of commutative quantum group implies also a generalization of the notion of coset space G/H of two groups G and $H \subset G$. This allows to define the quantum counterparts of the proper time constant hyperboloid and $CP_2 = SU(3)/U(2)$ as discrete spaces consisting of quantum points identifiable as representatives of cosets of the coset space of discrete quantum groups. This approach is very similar but more precise than the earlier approach in which the points in discretization had angle coordinates corresponding to roots of unity and radial coordinates with discretization defined by p-adic prime.

The infinite-dimensional "world of classical worlds" (WCW) can be seen as a union of infinite-dimensional symmetric spaces (coset spaces) [K16] and the definition as a quantum coset group could make sense also now in finite measurement resolution. This kind of approach has been already suggested and might be made rigorous by constructing quantum counterparts for the coset spaces associated with the infinite-dimensional symplectic group associated with the boundary of causal diamond. The problem is that matrix group is not in question. There are however good hopes that the symplectic group could reduces to a finite-dimensional matrix group in finite measurement resolution. Maybe it is enough to achieve this reduction for matrix representations of the symplectic group.

14.3.4 Quantum p-adic deformations of space-time surfaces as a representation of finite measurement resolution?

A mathematically fascinating question is whether one could use quantum arithmetics as a tool to build quantum deformations of partonic 2-surfaces or even of space-time surfaces and how could one achieve this. These quantum space-times would be commutative and therefore not like non-commutative geometries assigned with quantum groups. Perhaps one could see them as commutative semiclassical counterparts of non-commutative quantum geometries just as the commutative quantum groups discussed in [K83] could be seen commutative counterparts of quantum groups.

As one tries to develop a new mathematical notion and interpret it, one tends to forget the motivations for the notion. It is however extremely important to remember why the new notion is needed.

1. In the case of quantum arithmetics Shnoll effect is one excellent experimental motivation. The understanding of canonical identification and realization of number theoretical universality are also good motivations coming already from p-adic mass calculations. A further motivation comes from a need to solve a mathematical problem: canonical identification for ordinary p-adic numbers does not commute with symmetries.

2. There are also good motivations for p-adic numbers. p-Adic numbers and quantum phases can be assigned to finite measurement resolution in length measurement and in angle measurement. This with a good reason since finite measurement resolution means the loss of ordering of points of real axis in short scales and this is certainly one outcome of a finite measurement resolution. This is also assumed to relate to the fact that cognition organizes the world to objects defined by clumps of matter and with the lumps ordering of points does not matter.

3. Why quantum deformations of partonic 2-surfaces (or more ambitiously: space-time surfaces) would be needed? Could they represent convenient representatives for partonic 2-surfaces (space-time surfaces) within finite measurement resolution?

    (a) If this is accepted, there is no compelling need to assume that this kind of space-time surfaces are preferred extremals of Kähler action.

    (b) The notion of quantum arithmetics and the interpretation of p-adic topology in terms of finite measurement resolution however suggest that they might obey field equations in preferred coordinates but not in the real differentiable structure but in what might be called quantum p-adic differentiable structure associated with prime p.

    (c) Canonical identification would map these quantum p-adic partonic (space-time surfaces) to their real counterparts in a unique continuous manner and the image would be real space-time surface in finite measurement resolution. It would be continuous but not differentiable and would not of course satisfy field equations for Kähler action anymore. What is nice is that the inverse of the canonical identification which is two-valued for finite number of pinary digits would not be needed in the correspondence.

    (d) This description might be relevant also to quantum field theories (QFTs). One usually assumes that minima obey partial differential equations although the local interactions in QFTs are highly singular so that the quantum average field configuration might not even possess differentiable structure in the ordinary sense! Therefore quantum p-adicity might be more appropriate for the minima of effective action.

    The cautious conclusion would be that commutative quantum deformations of space-time surfaces could have a useful function in TGD Universe.

Consider now in more detail the identification of the quantum deformations of space-time surfaces.

1. Rationals are in the intersection of real and p-adic number fields and the representation of numbers as rationals $r = m/n$ is the essence of quantum arithmetics. This means that m and n are expanded to series in powers of p and coefficients of the powers of p which are smaller than p are replaced by the quantum counterparts. They are quantum quantum counterparts of integers smaller than p. This restriction is essential for the uniqueness of the map assigning to a give rational quantum rationals.

2. One must get also quantum p-adics and the idea is simple: if the pinary expansions of m and n in positive powers of p are allowed o become infinite, one obtains a continuum very much analogous to that of ordinary p-adic integers with exactly the same arithmetics. This continuum can be mapped to reals by canonical identification. The possibility to work with numbers which are formally rationals is utmost importance for achieving the correct map to reals. It is possible to use the counterparts of ordinary pinary expansions in p-adic arithmetics.

3. One can defined quantum p-adic derivatives and the rules are familiar to anyone. Quantum p-adic variants of field equations for Kähler action make sense.

    (a) One can take a solution of p-adic field equations and by the commutativity of the map $r = m/n \rightarrow r_q = m_q/n_q$ and of arithmetic operations replace p-adic rationals with their quantum counterparts in the expressions of quantum p-adic imbedding space coordinates h^k in terms of space-time coordinates x^α.

    (b) After this one can map the quantum p-adic surface to a continuous real surface by using the replacement $p \rightarrow 1/p$ for every quantum rational. This space-time surface does not anymore satisfy the field equations since canonical identification is not even differentiable. This surface - or rather its quantum p-adic pre-image - would represent a space-time surface within measurement resolution. One can however map the induced metric and induced gauge fields to their real counterparts using canonical identification to get something which is continuous but non-differentiable.

4. This construction works nicely if in the preferred coordinates for imbedding space and partonic (space-time) surface itself the imbedding space coordinates are rational functions of space-time coordinates with rational coefficients of polynomials (also Taylor and Laurent series with rational coefficients could be considered as limits). This kind of assumption is very restrictive but in accordance with the fact that the measurement resolution is finite and that the representative for the space-time surface in finite measurement resolution is to some extent a convention. The use of rational coefficients for the polynomials involved implies that for polynomials of finite degree WCW reduces to a discrete set so that finite measurement resolution has been indeed realized quite concretely!

Consider now how the notion of finite measurement resolution allows to circumvent the objections against the construction.

1. Manifest GCI is lost because the expression for space-time coordinates as quantum rationals is not general coordinate invariant notion unless one restricts the consideration to rational maps and because the real counterpart of the quantum p-adic space-time surface depends on the choice of coordinates. The condition that the space-time surface is represented in terms of rational functions is a strong constraint but not enough to fix the choice of coordinates. Rational maps of both imbedding space and space-time produce new coordinates similar to these provided the coefficients are rational.

2. Different choices for imbedding space and space-time surface lead to different quantum p-adic space-time surface and its real counterpart. This is an outcome of finite measurement resolution. Since one cannot order the space-time points below the measurement resolution, one cannot fix uniquely the space-time surface nor uniquely fix the coordinates used. This implies the loss of manifest general coordinate invariance and also the non-uniqueness of quantum real space-time surface. The choice of coordinates is analogous to gauge choice and quantum real space-time surface preserves the information about the gauge.

14.4 Could one understand p-adic length scale hypothesis number theoretically?

p-Adic length scale hypothesis states that primes near powers of two are physically interesting. In particular, both real and Gaussian Mersenne primes seem to be fundamental and can be tentatively assigned to charged leptons and living matter in the length scales between cell membrane thickness and size of the cell nucleus. They can be also assigned to various scaled up variants of hadron physics and with lepto-hadron physics suggested by TGD.

14.4.1 Number theoretical evolution as a selector of the fittest p-adic primes?

How could one understand p-adic length scale hypothesis? The general explanation would be in terms of number theoretic evolution by quantum jumps selecting the primes that are the fittest. The vision discussed in [K86] d leads to the proposal that the fittest p-adic primes are those which do not split in the physically preferred algebraic extensions of rationals. Algebraic extensions are naturally characterized by infinite primes characterizing also stable bound states of particles. Therefore these stable infinite primes or equivalently stable bound states would characterize also the p-adic primes which are fit. This explanation looks rather attractive.

p-Adic evolution would mean also a selection of preferred scales for CDs, instead of integer multiples of CP_2 scale only prime multiples or possibly prime power multiples would be favored and primes near powers of two were especially fit. A possible "biological" explanation is that for the preferred primes the number of quantum states is especially large making possible to build complex sensory and cognitive representations about external world.

The proposed vision about commutative quantum groups encourages to consider a number theoretic explanation for the p-adic length scale hypothesis consistent with the evolutionary explanation is that the quantum counterpart of symmetry groups are especially large for preferred primes. Large symmetries indeed imply large numbers of states related by symmetry transformations and high representational capacity provided by the p-adic–real duality. It is easy to make a rough test of the proposal for $G = SO(3)$, $SU(2)$ or $SU(3)$ associated with p-adic integers modulo p reducing to the counterpart of G for finite field might be especially large for physically preferred primes. Mersenne primes do not however seem to be special in this sense so that the following considerations can be taken as an exercise in the use of number theoretic functions and the reader can quite well skip the section.

14.4.2 Only Option I is considered

One considers *only the Option I*, which reduces to ordinary p-adic numbers effectively since quantum map induced by $p_i \rightarrow p_{iq}$ for $p_i < p$ can be combined with canonical identification. The arguments developed say nothing about option II. For option I the group transformations for which the conditions hold true modulo p make sense if matrix elements are integers satisfying $a_{ij} < p$. This makes sense for large values of p associated with elementary particles. This implies a reduction to finite field $G(p,1)$. The original argument was more general and used same condition but involved an error.

1. For $SL(2,C)$ - the covering group of Lorentz group - one obtains no constraints and all quantum phases $exp(i\pi/n)$ are allowed: this would mean that all CDs are in the same position. The rational $SL(2,C)$ matrices whose determinant is zero modulo p form a group assignable to finite field and and it might be that for some values of p this group is exceptionally large. $SL(2,C)$ defines also the covering group of conformal symmetries of sphere.

2. For orthogonal, unitary, and symplectic groups only $n = p$, p prime allows $k > 0$ and genuine p-adicity. Since $SO(3,1)$, $SO(3)$, $SU(2)$ and $SU(3)$ should alow p-adicization this selects CDs with size scale characterized by prime p.

3. For orthogonal, unitary, and symplectic groups one obtains non-trivial solutions to the unitarity conditions only if the highest power of p corresponds quantum image of a vector with zero norm modulo p as follows from the basic properties of quantum arithmetics.

   (a) In the case of $SO(3)$ one has the condition

   $$\sum_{i=1}^{3} x_i^2 = 1 + k \times p \tag{14.4.1}$$

   Note that this condition can degenerate to a condition stating that a sum of two squares is multiple of prime. As noticed the prime must be large and $x_i^2 < p$ holds true.

   (b) For the covering group $SU(2)$ of $SO(3)$ one has the condition

   $$\sum_{i=1}^{4} x_i^2 = 1 + k \times p \tag{14.4.2}$$

   since two complex numbers for the row of SU(2) matrix correspond to four real numbers.

   (c) For $SU(3)$ one has the condition

   $$\sum_{i=1}^{6} x_i^2 = 1 + k \times p \tag{14.4.3}$$

   corresponding to 3 complex numbers defining the row of SU(3) matrix.

What can one say about these conditions? The first thing to look is whether the conditions can be satisfied at all. Second thing to look is the number of solutions to the conditions.

14.4.3 Orthogonality conditions for $SO(3)$

The conditions for $SO(3)$ are certainly the strongest ones so that it is reasonable to study this case first.

1. One must remember that there are also integers -in particular primes- allowing representation as a sum of two squares. For instance, Fermat primes whose number is very small, allow representation $F_n = 2^+1$. More generally, Fermat's theorem on sums of two squares states that and odd prime is expressible as sum of two squares only if it satisfies $p \ mod \ 4 = 1$. The second possibility is $p \ mod \ 4 = 3$ so that roughly one half of primes satisfy the $p \ mod \ 4 = 1$ condition: Mersenne primes do not satisfy it.

   The more general condition giving sum proportional to prime is satisfied for all $n = k^2 l$, $k = 1, 2, ...$

2. For the sums of three non-vanishing squares one can use the well-known classical theorem stating that integer n can be represented as a sum of three squares only if it is it not of the form [A51]

$$n = 2^{2r}(8k+7) \tag{14.4.4}$$

   For instance, squares of odd integers are of form $8k+1$ and multiplied by any power of two satisfy the condition of being expressible as a sum of three squares.

   If n satisfies (does not satisfy) this condition then nm^2 satisfies (doe not satisfy) it for any m this since m^2 gives some power of 2 multiplied by a $8k+1$ type factor so that one can say that square free odd integers for which the condition $n \neq 7 \pmod 8$ generate this set of integers. Note that the integers representable as sums of three non-vanishing squares do not allow a representation using two squares. The product of odd primes $p_1 = 8m_1 + k_1$ and $p_1 = 8m_2 + k_2$ fails to satisfy the condition only if one has $k_1 = 3$ and $k_2 = 5$. The product of n primes $p_i = 8m_i + k_i$ must satisfy the condition $\prod k_i \neq 7 \pmod 8$ in order to serve as a generating square free prime.

   In the recent case one must have $n \ mod \ p = 1$. For Mersenne primes $m = 1 + kM_n$ allows representation as a sum of three squares for most values of k. In particular, for $k = 1$ one obtains $m = 2^n$ allowing at least the representation $m = 2^{n-1} + 2^{n-1}$. One must also remember that all that is needed is that sufficiently small multiples of Mersenne primes correspond to large value of $r_3(n)$ if the proposed idea has any sense.

14.4.4 Number theoretic functions $r_k(n)$ for $k = 2, 4, 6$

The number theoretical functions $r_k(n)$ telling the number of vectors with length squared equal to a given integer n are well-known for $k = 2, 3, 4, 6$ and can be used to gain information about the constraints posed by the existence of quantum groups $SO(2)$, $SO(3)$, $SU(2)$ and $SU(3)$. In the following the easy cases corresponding to $k = 2, 4, 6$ are treated first and after than the more difficult case $k = 3$ is discussed. For the auxiliary function the reader can consult to the Appendix.

The behavior of $r_2(n)$

$r_2(n)$ gives information not only about quantum $SO(2)$ but also about $SO(3)$ since 2-D vectors define 3-D vectors in an obvious manner. The expression for $r_2(n)$ is given by

$$r_2(n) = \sum_{d|n} \chi(d) \ , \ \chi(d) = \left(\frac{-4}{d}\right) \ . \tag{14.4.5}$$

$\chi(d)$ is so called principal character defined in appendix. For $n = 1 + M_k = 2^k$ only powers of 2 and 1 divide n and for even numbers principal character vanishes so that one obtains $r_2(1+M_k) = \chi(1) = 1$. This corresponds to the representation $2^k = 2^{k-1} + 2^{k-1}$.

The behavior of $r_4(n)$

The expression for $r_4(n)$ reads as

$$r_4(n) = \begin{cases} 8\sigma(n) & \text{if } n \text{ is odd }, \\ 24\sigma(m) & \text{if } n = 2^\nu m,\ m \text{ odd }. \end{cases} \quad . \tag{14.4.6}$$

For $n = M_k + 1 = 2^k$ one has $r_4(n) = 24\sigma(1) = 24$.

The asymptotic behavior of σ function is known so that it is relatively easy to estimate the behavior of $r_4(n)$. The behavior involves random looking local fluctuation which can be understood as reflective the multiplicative character implying correlation between the values associated with multiples of a given prime.

The behavior of $r_6(n)$

The analytic expression for $r_6(n)$ is given by

$$\begin{aligned} r_6(n) &= \sum_{d|n} \left[16\chi(\frac{n}{d}) - 4\chi(d)\right] d^2 \ , \\ \chi(n) &= \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{cases} \end{aligned} \tag{14.4.6}$$

For $n = M_k + 1 = 2^k$ this gives $r_6(n) = 12 \times 2^{2k} - 4$ so that the number of representation is very large for large Mersenne primes.

14.4.5 What can one say about the behavior of r_3?

The proportionality of $r_3(D)$ to the order of $h(-D)$ [A8] of the ideal class group [A49] [A49] for quadratic extensions of rationals [A8] inspires some conjectures.

1. The conjecture that preferred primes p correspond to large commutative quantum groups translates to a conjecture that the order of ideal class group is large for the algebraic extension generated by $\sqrt{-p-1}$ or more generally $\sqrt{-kp-1}$ - at least for some values of k. Could suitable integer multiples primes near power of 2 - in particular Mersenne primes - be such primes? Note that only integer multiple is required by the basic argument.

2. Also some kind of approximate fractal behavior $r_k(sp) \simeq r_k(p)f_k(s)$ for some values of s analogous to that encountered for $r_4(D)$ for all values of s might hold true since $k = 3$ is a critical transition dimension between $k = 2$ and $k = 3$. In particular, an approximate periodicity in octaves of primes might hold true: $r_k(2^s p) \simeq r_k(p)$: this would support p-adic length scale hypothesis and make the commutative quantum group large.

Expression of r_3 in terms of class number function

To proceed one must have an explicit expression for the class number function $h(D)$ and the expression of r_3 in terms of $h(D)$.

1. The expression for $h(D)$ discussed in the Appendix reads as gives

$$h(-D) = -\frac{1}{D}\sum_1^D r \times \left(\frac{-D}{r}\right) \quad . \tag{14.4.7}$$

The symbols($\left(\frac{-D}{r}\right)$ are Dirichlet and Kronecker symbols defined in the Appendix. Note that for $D = M_k + 1 = 2^k$ the algebraic expansion in question reduces to that generated by $\sqrt{-2}$ so that the algebraic extension is definitely special.

2. One can express $r_3(|D|)$ in terms of $h(D)$ as

$$r_3(|D|) = 12(1-(\frac{D}{2}))h(D) \ . \tag{14.4.8}$$

Note that $(\frac{p}{2})$ refers to Kronecker symbol.

3. From Wolfram one finds the following expressions of $r_3(n)$ for square free integers

$$\begin{array}{ll} r_3(n) = 24h(-n) & n = 3 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 12h(-4n) & n = 1,2,5,6 \ (\mathrm{mod}\ 8) \ , \\ r_3(n) = 0 & n = 7 \ (\mathrm{mod}\ 8) \ . \end{array} \tag{14.4.9}$$

4. The generating function for r_3 [A90] is third power of theta function θ_3.

$$\sum_{n\geq 0} r_3(n)x^n = \theta_3^3(n) = 1+6x+12x^2+8x^3+6x^4+24x^5+24x^6+12x^8+30x^9+... \tag{14.4.10}$$

This representation follows trivially from the definition of θ function as sum $\sum_{n=-\infty}^{\infty} x^{n^2}$.

The behavior of $h(-D)$ for large arguments is not easy to deduce without numerical calculations which probably get too heavy for primes of order M_{127}. The definition involves sum of p terms labeled by $r = 1,...,p$, and each term is a product is product of terms expressible as a product over the prime factors of of r with over all term being a sign factor. "Interference " effects between terms of different sign are obviously possible in this kind of situation and one might hope that for large primes these effects imply wild fluctuations of $r_3(p)$.

Simplified formula for $r_3(D)$

Recall that the proportionality of $r_3(|D|)$ to the ideal class number $h(D)$ is for $D < -4$ given by

$$r_3(|D|) = 12[1-\left(\frac{D}{2}\right)]h(D) \ . \tag{14.4.11}$$

The expression for the Kronecker symbol appears in the formula as well as formulas to be discussed below and reads as

$$\left(\frac{D}{2}\right) = \left\{ \begin{array}{ll} 0 & \text{if } D \text{ is even} \ , \\ 1 & \text{if } D = -1 \ (\mathrm{mod}\ 8) \ , \\ -1 & \text{if } D = \pm 3 \ (\mathrm{mod}\ 8) \ . \end{array} \right. \tag{14.4.12}$$

The proportionality factor vanishes for $D = 2^{2r}(8m+7)$ and equals to 12 for even values of D and to 24 for $D = \pm 3$ (mod 8).

To get more detailed information about r_3 one can begin from class number formula [A17] for $D < -4$ reading as

$$h(D) = \frac{1}{|D|}\sum_{r=1}^{|D|} r\left(\frac{D}{r}\right) \ . \tag{14.4.13}$$

Each Jacobi symbol $(\frac{D}{r})$ decomposes to a product of Legendre and Kronecker symbols $\left(\frac{D}{p_i}\right)$ in the decomposition of odd integer r to a product of primes p_i.

For $\left(\frac{D}{p_i}\right) = 1$ p_i splits into a product of primes in quadratic extension generated by $\sqrt{D}$. If it vanishes p_i is square of prime in the quadratic extension. In the recent case neither of these options are possible for the primes involved as is easy to see by using the definition of algebraic integers. Hence one has $\left(\frac{D}{p_i}\right) = -1$ for all odd primes to transform the formula for $D < -4$ to the form

$$\begin{aligned} h(D) &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r)-\nu_2(r)} \\ &= \frac{1}{|D|}\sum_{r=1}^{|D|} r[-\left(\frac{D}{2}\right)]^{\nu_2(r)}(-1)^{\Omega(r))} \ . \end{aligned} \tag{14.4.12}$$

Here $\nu_2(r)$ characterizes the power of 2 appearing in r and $\Omega(r)$ is the number of prime divisors of r with same divisor counted so many times as it appears. Hence the sign factor is same for all integers r which are obtained from the same square free integer by multiplying it by a product of even powers of primes.

Consider next various special cases.

1. For even values $D < -4$ (say $D = -1 - M_n$) only odd integers r contribute to the sum since the Kronecker symbols vanish for even values of r.

$$h(D = 2d) = \frac{1}{|D|}\sum_{1\leq r<|D| \ \ odd} r(-1)^{\Omega(r)} \tag{14.4.11}$$

2. For $D = \pm 1 \pmod 8$, the factors $\left(\frac{D}{2}\right) = -1$ implies that one can forget the factors of 2 altogether in this case (note that for $D = -1 \pmod 8$ $r_3(|D|)$ vanishes unlike $h(D)$).

$$h(D = \pm 1 (mod \ \ 8)) = \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r))} \tag{14.4.10}$$

3. For $D = \pm 3 \pmod 8$, the factors $\left(\frac{D}{2}\right) = 1$ implies that one has

$$h(D = \pm 3 (mod \ \ 8)) = \frac{1}{|D|}\sum_{r=1}^{|D|} r(-1)^{\Omega(r)-\nu_2(r)} \tag{14.4.10}$$

The magnitudes of the terms in the sum increase linearly but the sign factor fluctuates wildly so that the value of $h(-D)$ varies chaotically but must be divisible by p and negative since $r_3(p)$ must be a positive integer.

Could thermodynamical analogy help?

For $D < -4$ $h(D)$ is expressible in terms of sign factors determined by the number of prime factors or odd prime factors modulo two for integers or odd integers $r < D$. This raises hopes that $h(D)$ could be calculated for even large values of D.

1. Consider first the case $D = \pm 1 \pmod{8}$). The function $\lambda(r) = (-1)^{\Omega(r)}$ is known as Liouville function [A57]. From the product expansion of zeta function in terms of "prime factors" it is easy to see that the generating function for $\lambda(r)$

$$\begin{aligned} \sum_n \lambda(n) n^{-s} &= \frac{\zeta(2s)}{\zeta(s)} = \frac{1}{\zeta_F(s)} \ , \\ \zeta_(s) &= \prod_p (1-p^{-s})^{-1} \ , \ \zeta_F(s) = \prod_p (1+p^{-s}) \ . \end{aligned} \tag{14.4.10}$$

   Recall that $\zeta(s)$ *resp.* $\zeta_F(s)$ has a formal interpretation as partition functions for the thermodynamics of bosonic *resp.* fermionic system. This representation applies to $h(D = \pm 1(mod 8))$.

2. For $D = 2d$ the representation is obtained just by dropping away the contribution of all even integers from Liouville function and this means division of $(1+2^{-s})$ from the fermionic partition function $\zeta_F(s)$. The generating function is therefore

$$\sum_{n \ odd} \lambda(n) n^{-s} = \prod_{p \ odd} (1+p^{-s})^{-1} = (1+2^{-s}) \frac{1}{\zeta_F(s)} \ . \tag{14.4.11}$$

3. For $h(D = \pm 3(\ mod \ 8))$. One most modify the Liouville function by replacing $\Omega(r)$ by the number of odd prime factors but allow also even integers r. The generating function is now

$$\sum_n \lambda(n)(-1)^{\nu_2(n)} n^{-s} = \frac{1}{1-2^{-s}} \prod_{p \ odd} (1+p^{-s})^{-1} = \frac{1}{1-2^{-s}} \frac{1}{\zeta_F(s)} \ . \tag{14.4.12}$$

The generating functions raise the hope that it might be possible to estimate the values of the $h(D)$ numerically for large values of D using a thermodynamical analogy.

1. $h(D)$ is obtained as a kind of thermodynamical average $\langle r(-1)^{\Omega(r)} \rangle$ for particle number r weighted by a sign factor telling the number of divisors interpreted as particle number. s plays the role of the inverse of the temperature and infinite temperature limit $s = 0$ is considered. One can also interpret this number as difference of average particle number for states restricted to contain even *resp.* odd particle number identified as the number of prime divisors with 2 and even particle numbers possibly excluded.

2. The average is obtained at temperature corresponding to $s = 0$ so that $n^{-s} = 1$ holds true identically. The upper bound $r < D$ means cutoff in the partition sum and has interpretation as an upper bound on the energy $log(r)$ of many particle states defined by the prime decomposition. This means that one must replace Riemann zeta and its analogs with their cutoffs with $n \leq |D|$. Physically this is natural.

3. One must consider bosonic system all the cases considered. To get the required sign factor one must associated to the bosonic partition functions assigned with individual primes in $\zeta(s)$ the analog of chemical potential term $exp(-\mu/T)$ as the sign factor $exp(i\pi) = -1$ transforming ζ to $1/\zeta_F$ in the simplest case.

One might hope that one could calculate the partition function without explicitly constructing all the needed prime factorizations since only the number of prime factors modulo two is needed for $r \leq |D|$.

Expression of r_3 in terms of Dirichlet L-function

It is known [A65] that the function $r_3(D)$ is proportional to Dirichlet L-function $L(1,\chi(D))$ [A25]:

$$\begin{aligned} r_3(|D|) &= \frac{12\sqrt{D}}{\pi}L(1,\chi(D))) \ , \\ L(s,\chi) &= \sum_{n>0}\frac{\chi(n,D)}{n^s} \ , \end{aligned} \tag{14.4.11}$$

$\chi(n,D)$ is Dirichlet character [A24] which is periodic and multiplicative function - essentially a phase factor- satisfying the conditions

$$\begin{array}{ll} \chi(n,D)\neq 0 & \text{if n and D have no common divisors} > 1 \ , \\ \chi(n,D)=0 & \text{if n and D have a common divisor} > 1 \ , \\ \chi(mn,D)=\chi(m,D)\chi(n,D) \ , & \chi(m+D,D)=\chi(m,D) \ , \\ \chi(1,D)=1 \ . & \end{array} \tag{14.4.12}$$

1. $L(1,\chi(D))$ varies in average sense slowly but fluctuates wildly between certain bounds. One can say that there is local chaos.

   The following estimates for the bounds are given in [A124]:

   $$c_1(D) \equiv k_1log(log(D) < L_1(1,\chi(D)) < c_2(D) \equiv k_2log(log(D)) \ . \tag{14.4.13}$$

   Also other bounds are represented in the article.

Could preferred integers correspond to the maxima of Dirichlet L-function?

The maxima of Dirichlet L-function are excellent candidates for the local maxima of $r_3(D)$ since $\sqrt{D}$ is slowly varying function.

1. As already found, integers $n = 1 + M_k = 2^k$ cannot represent pronounced maxima of $r_3(n)$ since there are no representation as a sum of three squares and the proportionality constant vanishes. Note that in this case the representation reduces to a representation in terms of two integers. In this special case it does not matter whether L-function has a maximum or not.

   (a) Could just the fact that the representation for $n = 1 + M_k = 2^k$ in terms of three primes is not possible, select Mersenne primes $M_n > 3$ as preferred ones? For $SU(2)$, which is covering group of $SO(3)$ the representation as a sum of four squares is possible. Could it be that the spin 1/2 character of the fermionic building blocks of elementary particles means that a representation as sum of four squares is what matters. But why the non-existence of representation of n as a sum of three squares might make Mersenne primes so special?

2. Could also primes near power of 2 define maxima? Unfortunately, the calculations of [A124] involve averaging, minimum, and maximum over 10^6 integers in the ranges $n \times 10^6 < D < (n+1) \times 10^6$, so that they give very slowly varying maximum and minimum.

3. Could Dirichlet function have some kind of fractal structure such that for any prime one would have approximate factorization? The naivest guesses would be $L(1,\chi_{kl}) \simeq f_1(k)L(1,\chi_l)$ with $k = 2^s$. This would mean that the primes for which $D(1,\chi_p)$ is maximum would be of special importance.

4. p-Adic fractality and effective p-adic topology inspire the question whether L-function is p-adic fractal in the regions above certain primes defining effective p-adic topology $D(1,\chi_{p^k}) \simeq f_1(k)DK(1,\chi_p)$ for preferred primes.

Interference as a helpful physical analogy?

Could one use physical analog such as interference for the terms of varying sign appearing in L-function to gain some intuition about the situation?

1. One could interpret L-function as a number theoretic Fourier transform with D interpreted as a wave vector and one has an interference of infinite number of terms in position space whose points are labelled by positive integers defining a half -lattice with unit lattice length. The magnitude of n:th summand $1/n$ and its phase is periodic with period $D = kp$. The value of the Fourier component is finite except for $D = 0$ which corresponds to Riemann Zeta at $s = 1$. Could this means that the Fourier component behaves roughly like $1/D$ apart from an oscillating multiplicative factor.

2. The number theoretic counterparts of plane waves are special in that besides D-periodicity they are multiplicative making thema lso analogs of logarithmic waves. For ordinary Fourier components one additivity in the sense that $\Psi(k_1 + k_2) = \Psi(k_1)\Psi(k_2)$. Now one has $\Psi(k_1k_2) = \Psi(k_1)\Psi(k_2)$ so that $log(D)$ corresponds to ordinary wave vector. p-Adic fractality is an analog for periodicity in the sense of logarithmic waves so that powers rather than integer multiples of the basic scale define periodicity. Could the multiplicative nature of Dirichlet characters imply p-adic - or at least 2-adic - fractality, which also means logarithmic periodicity?

3. Could one say that for these special primes a constructive interference takes place in the sum defining the L-function. Certainly each prime represents the analog of fundamental wavelength whose multiples characterize the summands. In frequency space this would mean fundamental frequency and its sub-harmonics.

Period doubling as physical analogy?

1. For $k = 4$ all scales are present because of the multiplicative nature of σ function. Now only the Dirichlet characters are multiplicative which suggests that only few integers define preferred scales? Prime power multiples of the basic scale are certainly good candidates for preferred scales but amongst them must be some very special prime powers. $p = 2$ is the only even prime so that it is the first guess.

2. Could the system be chaotic or nearly chaotic in the sense of period doubling so that octaves of preferred primes interfere constructively? Why constructively? Could complete chaos -interpreted as randomness- correspond to a destructive interference and minimum of the L-function?

3. What about scalings by squares of a given prime? It seems that these scalings cannot be excluded by any simple argument. The point is that $r_3(n)$ contains also the factor $\sqrt{n}$ which must transform by integer in the scaling $n \to kn$. Therefore k must be power of square.

This leaves two extreme options. Both options are certainly testable by simple numerical calculations for small primes. For instance one can use generating function $\theta_3^3(x) = \sum r_3(n)x^n$ to kill the conjectures.

1. The first option corresponds to scalings by all integers that are squares. This option is also consistent with the condition $n \neq 2^k(8m+7)$ since both the scaling by a square of odd prime and by a square of 2 preserve this condition since one has $n^2 = 1 \pmod 8$ for odd integers. This is also consistent with the finding that $r_3(n) = 1$ holds true only for a finite number of integers. A simple numerical calculation for the sums of 3 squares of 16 first integers demonstrates that the conjecture is wrong.

2. The second option corresponds only to the scaling by even powers of two and is clearly the minimal option. This period quadrupling for n corresponds to period doubling for the components of 3-vector. A calculation of the sums of squares of the 16 first integers demonstrates that for $n = 3, 6, 9, 11, ..$ the conjecture the value of $r_3(n)$ is same so that the conjecture might hold true! If it holds true then Dirichlet L-function should suffer scaling by 2^{-r} in the scaling $n \rightarrow 2^{2r}n$. The integer solutions for n scaled by 2^r are certainly solutions for $2^{2r}n$. Quite generally, one has $r_3(m^2n) \geq r_3(n)$ for any integer m. The non-trivial question is whether some new solutions are possible when the scaling is by 2^{2r}.

   A simple argument demonstrates that there cannot be any other solutions to $\sum_{n_i=1}^{3} m_i^2 = 2^{2r}n$ than the the scaled up solutions $m_i = 2n_i$ obtained from $\sum_{n_i=1}^{3} n_i^2 = n$. This is seen by noticing that non-scaled up solutions must contain $1, 2, or 3$ integers m_i, which are odd. For this kind of integers one has $m^2 = 1$ (mod 4) so that the sum ($\sum_i m_i^2$)= 1,2, or 3 (mod 4) whereas the the right hand side vanishes mod 4.

3. If D is interpreted as wave vector, period quadrupling could be interpreted as a presence of logarithmic wave in wave-vector space with period $2log(2)$.

Does 2-adic quantum arithmetics prefer CD scales coming as powers of two?

For $p = 2$ quantum arithmetics looks singular at the first glance. This is actually not the case since odd quantum integers are equal to their ordinary counterparts in this case. This applies also to powers of two interpreted as 2-adic integers. The real counterparts of these are mapped to their inverses in canonical identification.

Clearly, odd 2-adic quantum quantum rationals are very special mathematically since they correspond to ordinary rationals. It is fair to call them "classical" rationals. This special role might relate to the fact that primes near powers of 2 are physically preferred. CDs with $n = 2^k$ would be in a unique position number theoretically. This would conform with the original - and as such wrong - hypothesis that only these time scales are possible for CDs. The preferred role of powers of two supports also p-adic length scale hypothesis.

The discussion of the role of quantum arithmetics in the construction of generalized Feynman diagrams in [K31] allows to understand how for a quantum arithmetics based on particular prime p particle mass squared - equal to conformal weight in suitable mass units- divisible by p appears as an effective propagator pole for large values of p. In p-adic mass calculations real mass squared is obtained by canonical identification from the p-adic one. The construction of generalized Feynman diagrams allows to understand this strange sounding rule as a direct implication of the number theoretical universality realized in terms of quantum arithmetics.

14.5 How quantum arithmetics affects basic TGD and TGD inspired view about life and consciousness?

The vision about real and p-adic physics as completions of rational physics or physics associated with extensions of rational numbers is central element of number theoretical universality. The physics in the extensions of rationals are assigned with the interaction of real and p-adic worlds.

1. At the level of the world of classical worlds (WCW) the points in the intersection of real and p-adic worlds are 2-surfaces defined by equations making sense both in real and p-adic sense. Rational functions with polynomials having rational (or algebraic coefficients in some extension of rationals) would define the partonic 2-surface. One can of course consider more stringent formulations obtained by replacing 2-surface with certain 3-surfaces or even by 4-surfaces.

2. At the space-time level the intersection of real and p-adic worlds corresponds to rational points common to real partonic 2-surface obeying same equations (the simplest assumption). This conforms with the vision that finite measurement resolution implies discretization at the level of partonic 2-surfaces and replaces light-like 3-surfaces and space-like 3-surfaces at the ends of causal diamonds with braids so that almost topological QFT is the outcome.

How does the replacement of rationals with quantum rationals modify quantum TGD and the TGD inspired vision about quantum biology and consciousness?

14.5.1 What happens to p-adic mass calculations and quantum TGD?

The basic assumption behind the p-adic mass calculations and all applications is that one can assign to a given partonic 2-surface (or even light-like 3-surface) a preferred p-adic prime (or possibly several primes).

The replacement of rationals with quantum rationals in p-adic mass calculations implies effects, which are extremely small since the difference between rationals and quantum rationals is extremely small due to the fact that the primes assignable to elementary particles are so large ($M_{127} = 2^{127} - 1$ for electron). The predictions of p-adic mass calculations remains almost as such in excellent accuracy. The bonus is the uniqueness of the canonical identification making the theory unique.

The problem of the original p-adic mass calculations is that the number of common rationals (plus possible algebraics in some extension of rationals) is same for all primes p. What is the additional criterion selecting the preferred prime assigned to the elementary particle?

Could the preferred prime correspond to the maximization of number theoretic negentropy for a quantum state involved and therefore for the partonic 2-surface by quantum classical correspondence? The solution ansatz for the modified Dirac equation indeed allows this assignment [K26]: could this provide the first principle selecting the preferred p-adic prime? Here the replacement of rationals with quantum rationals improves the situation dramatically.

1. Quantum rationals are characterized by a quantum phase $q = exp(i\pi/p)$ and thus by prime p (in the most general but not so plausible case by an integer n). The set of points shared by real and p-adic partonic 2-surfaces would be discrete also now but consist of points in the algebraic extension defined by the quantum phase $q = exp(i\pi/p)$.

2. What is of crucial importance is that the number of common quantum rational points of partonic 2-surface and its p-adic counterpart would depend on the p-adic prime p. For some primes p would be large and in accordance with the original intuition this suggests that the interaction between p-adic and real partonic 2-surface is stronger. This kind of prime is the natural candidate for the p-adic prime defining effective p-adic topology assignable to the partonic 2-surface and elementary particle. Quantum rationals would thus bring in the preferred prime and perhaps at the deepest possible level that one can imagine.

14.5.2 What happens to TGD inspired theory of consciousness and quantum biology?

The vision about rationals as common to reals and p-adics is central for TGD inspired theory of consciousness and the applications of TGD in biology.

1. One can say that life resides in the intersection of real and p-adic worlds. The basic motivation comes from the observation that number theoretical entanglement entropy can have negative values and has minimum for a unique prime [K42]. Negative entanglement entropy has a natural interpretation as a genuine information and this leads to a modification of Negentropy Maximization Principle (NMP) allowing quantum jumps generating negentropic entanglement. This tendency is something completely new: NMP for ordinary entanglement entropy would force always a state function reduction leading to unentangled states and the increase of ensemble entropy.

   What happens at the level of ensemble in TGD Universe is an interesting question. The pessimistic view [K42], [L16] is that the generation of negentropic entanglement (see fig. `http://www.tgdtheory.fi/appfigures/cat.jpg` or fig. 21 in the appendix of this book) is accompanied by entropic entanglement somewhere else guaranteeing that second law still holds true. Living matter would be bound to pollute its environment if the pessimistic view is correct. I cannot decide whether this is so: this seems like deciding whether Riemann hypothesis is true or not or perhaps unprovable.

2. Replacing rationals with quantum rationals however modifies somewhat the overall vision about what life is. It would be quantum rationals which would be common to real and p-adic variants of the partonic 2-surface. Also now an algebraic extension of rationals would be in question so that the proposal would be only more specific. The notion of number theoretic entropy still makes sense so that the basic vision about quantum biology survives the modification.

3. The large number of common points for some prime would mean that the quantum jump transforming p-adic partonic 2-surface to its real counterpart would take place with a large probability. Using the language of TGD inspired theory of consciousness one would say that the intentional powers are strong for the conscious entity involved. This applies also to the reverse transition generating a cognitive representation if p-adic-real duality induced by the canonical identification is true. This conclusion seems to apply even in the case of elementary particles. Could even elementary particles cognize and intend in some primitive sense? Intriguingly, the secondary p-adic time scale associated with electron defining the size of corresponding CD is .1 seconds defining the fundamental 10 Hz bio-rhythm. Just an accident or something very deep: a direct connection between elementary particle level and biology perhaps?

14.6 Appendix: Some number theoretical functions

Explicit formulas for the number $r_k(n)$ of the solutions to the conditions $\sum_1^k x_k^2 = n$ are known and define standard number theoretical functions closely related to the quadratic algebraic extensions of rationals. The formulas for $r_k(n)$ require some knowledge about the basic number theoretical functions to be discussed first. Wikipedia contains a good overall summary about basic arithmetic functions [A8] including the most important multiplicative and additive arithmetic functions.

Included are character functions which are periodic and multiplicative: examples are symbols (m/n) assigned with the names of Legendre, Jacobi, and Kronecker as well as Dirichlet character.

14.6.1 Characters and symbols

Principal character

Principal character [A8] $\chi(n)$ distinguishes between three situations: n is even, $n = 1 \pmod 4$, and $n = 3 \pmod 4$ and is defined as

$$\chi(n) = \left(\frac{-4}{n}\right) = \begin{cases} 0 & \text{if n=0 (mod 2)} \\ +1 & \text{if } n = 1 \pmod 4 \\ -1 & \text{if } n = 3 \pmod 4 \end{cases} \quad (14.6.1)$$

Principal character is multiplicative and periodic with period $k = 4$.

Legendre and Kronecker symbols

Legendre symbol $\left(\frac{n}{p}\right)$ characterizes what happens to ordinary primes in the quadratic extensions of rationals. Legendre symbol is defined for odd integers n and odd primes p as

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } n = 0 \text{ (mod p)} \ , \\ +1 & \text{if } n \neq 0 \text{ (mod p) and } n = x^2 \text{ (mod p)} \ , \\ -1 & \text{if there is no such } x \ . \end{cases} \quad (14.6.2)$$

When D is so called fundamental discriminant- that is discriminant $D = b^2 - 4c$ for the equation $x^2 - bx + c = 0$ with integer coefficients b, c, Legendre symbols tells what happens to ordinary primes in the extension:

1. $\left(\frac{D}{p}\right) = 0$ tells that the prime in question divides D and that p is expressible as a square in the quadratic extension of rationals defined by $\sqrt{D}$.

2. $\left(\frac{D}{p}\right) = 1$ tells that p splits into a product of two different primes in the quadratic extension.

3. For $\left(\frac{D}{p}\right) = -1$ the splitting of p does not occur.

This explains why Legendre symbols appear in the ideal class number $h(D)$ characterizing the number of different splittings of primes in quadratic extension.

Legendre symbol can be generalized to Kronecker symbol well-defined for also for even integers D. The multiplicative nature requires only the definition of $\left(\frac{n}{2}\right)$ for arbitrary n:

$$\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \ , \\ (-1)^{\frac{n^2-1}{8}} & \text{if n is odd} \ . \end{cases} \tag{14.6.3}$$

Kronecker symbol for $p = 2$ tells whether the integer is even, and if odd whether $n = \pm 1$ (mod 8) or $a = \pm 3$ (mod 8) holds true. Note that principal character $\chi(n)$ can be regarded as Dirichlet character $\left(\frac{-4}{n}\right)$.

For $D = p$ quadratic resiprocity [A74] allows to transform the formula

$$\chi_p(n) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\left(\frac{p}{n}\right) = (-1)^{(p-1)/2}(-1)^{(n-1)/2}\prod_{p_i|n}\left(\frac{p}{p_i}\right) \ . \tag{14.6.4}$$

Dirichlet character

Dirichlet character [A24] $\left(\frac{a}{n}\right)$ is also a multiplicative function. Dirichlet character is defined for all values of a and odd values of n and is fixed completely by the conditions

$$\begin{aligned} &\chi_D(k) = \chi_D(k+D) \ , \qquad \chi_D(kl) = \chi_D(k)\chi_D(l) \ , \\ &\text{If } D|n \text{ then } \chi_D(n) = 0 \ , \quad \text{otherwise } \chi_D(n) \neq 0 \ . \end{aligned} \tag{14.6.5}$$

Dirichlet character associated with quadratic residues is real and can be expressed as

$$\chi_D(n) = \left(\frac{n}{D}\right) = \prod_{p_i|D}\left(\frac{n}{p_i}\right) \ . \tag{14.6.6}$$

Here $\left(\frac{n}{p_i}\right)$ is Legendre symbol described above. Note that the primes p_i are odd. $\left(\frac{n}{1}\right) = 1$ holds true by definition.

For prime values of D Dirichet character reduces to Legendre symbol. For odd integers Dirichlet character reduces to Jacobi symbol defined as a product of the Legendre symbols associated with the prime factors. For $n = p^k$ Dirichlet character reduces to $(\left(\frac{p}{n}\right))^k$ and is non-vanishing only for odd integers not divisible by p and containing only odd prime factors larger than p besides power of 2 factor.

14.6.2 Divisor functions

Divisor functions [A26] $\sigma_k(n)$ are defined in terms of the divisors d of integer n with $d = 1$ and $d = n$ included and are also multiplicative functions. $\sigma_k(n)$ is defined as

$$\sigma_k(n) = \sum_{d|n} d^k \ , \tag{14.6.7}$$

and can be expressed in terms of prime factors of n as

$$\sigma_k(n) = \sum_i (p_i^k + p_i^{2k} + ... + p_i^{a_i k}) \ . \tag{14.6.8}$$

$\sigma_1 \equiv \sigma$ appears in the formula for $r_4(n)$.

The figures in Wikipedia [A40] give an idea about the locally chaotic behavior of the sigma function.

14.6.3 Class number function and Dirichlet L-function

In the most interesting $k = 3$ case the situation is more complicated and more refined number theoretic notions are needed. The function $r_3(D)$ is expressible in terms of so called class number function $h(n)$ characterizing the order of the ideal class group for a quadratic extension of rationals associated with D, which can be negative. In the recent case $D = -p$ is of special interest as also $D = -kp$, especially so for $k = 2^r$. $h(n)$ in turn is expressible in terms of Dirichlet L-function so that both functions are needed.

1. Dirichlet L-function [A25] can be regarded as a generalization of Riemann zeta and is also conjectured to satisfy Riemann hypothesis. Dirichlet L-function can be assigned to any Dirichlet character χ_D appearing in it as a function valued parameter and is defined as

$$L(s, \chi_D) = \sum_n \frac{\chi_D(n)}{n^s} \ . \tag{14.6.9}$$

   For $\chi_1 = 1$ one obtains Riemann Zeta. Also L-function has expression as product of terms associated with primes converging for $Re(s) > 1$, and must be analytically continued to get an analytic function in the entire complex plane. The value of L-function at $s = 1$ is needed and for Riemann zeta this corresponds to pole. For Dirichlet zeta the value is finite and $L(1, \chi_{-n})$ indeed appears in the formula for $r_3(n)$.

2. Consider next what class number function h means.

   (a) Class number function [A17] characterizes quadratic extensions defined by $\sqrt{D}$ for both positive and negative values of D. For these algebraic extensions the prime factorization in the ring of algebraic integers need not be unique. Algebraic integers are complex algebraic numbers which are not solutions of a polynomial with coefficients in Z and with leading term with unit coefficient. What is important is that they are closed under addition and multiplication. One can also defined algebraic primes. For instance, for the quadratic extension generated by $\sqrt{\pm 5}$ algebraic integers are of form $m + n\sqrt{\pm 5}$ since $\sqrt{\pm 5}$ satisfies the polynomial equation $x^2 = \pm 5$.

   Given algebraic integer n can have several prime decompositions: $n = p_1 p_2 = p_3 p_4$, where p_i algebraic primes. In a more advance treatment primes correspond to ideals of the algebra involved: obviously algebra of algebraic integers multiplied by a prime is closed with respect to multiplication with any algebraic integer.

   A good example about non-unique prime decomposition is $6 = 2\times 3 = (1+\sqrt{-5})(\sqrt{1-\sqrt{-5}}$ in the quadratic extension generated by $\sqrt{-5}$.

   (b) Non-uniqueness means that one has what might be called fractional ideals: two ideals I and J are equivalent if one can write $(a)J = (b)I$ where (n) is the integer ideal consisting of algebraic integers divisible by algebraic integer n. This is the counterpart for the non-uniqueness of prime decomposition. These ideals form an Abelian group known as ideal class group [A49]. For algebraic fields the ideal class group is always finite.

   (c) The order of elements of the ideal class group for the quadratic extension determined by integer D can be written as

$$h(D) = \frac{1}{D}\sum_{1}^{|D|} r \times \left(\frac{D}{r}\right) \quad , \quad D < -4 \quad . \tag{14.6.10}$$

Here $\left(\frac{D}{r}\right)$ denotes the value of Dirichlet character. In the recent case D is negative.

3. It is perhaps not completely surprising that one can express $r_3(|D|)$ characterizing quadratic form in terms of $h(D)$ charactering quadratic algebraic extensions as

$$r_3(|D|) = 12(1 - \left(\frac{D}{2}\right))h(D) \quad , \quad D < -4 \quad . \tag{14.6.11}$$

Here $\left(\frac{D}{2}\right)$ denotes Kronecker symbol.

Chapter 15

Quantum Adeles

15.1 Introduction

Quantum arithmetics [K83] is a notion which emerged as a possible resolution of long-lasting challenge of finding mathematical justification for the canonical identification mapping p-adics to reals.

15.1.1 What quantum p-adics could be?

The basic idea is that p-adic numbers could have quantum counterparts. This idea has developed through several twists and turns and involved moments of almost despair.

The first attempts

The first attempts where based on the replacement of p-adic numbers with quantum p-adics in the hope that the arithmetics could be lifted to quantum level.

1. The earlier work with quantum arithmetics [K83] suggests a modification of p-adic numbers by replacing the coefficients a_n p-adic pinary expansions with their quantum counterparts $(a_n)_q$ allowing the coefficients a_n of prime powers to be integers not divisible by p and involving only primes $l < p$ in the prime decomposition (for $l > p$ the quantum counterpart can be negative). $a_n > p$ is allowed for the "interesting but risky" and $a_n < p$ is required for "less-interesting but safe" option.

2. For the "interesting" option the assignment of quantum integer to a given p-adic integer is not unique. A natural looking but not absolutely necessary constraint is that the assignment respects the decomposition of the p-adic integer to powers of prime. With this assumption the construction of quantum integers would reduce to that for primes l. The quantum counterpart of $l > p$ is not unique if the coefficients of powers of p can be larger than p. There exists preferred quantum counterpart obtained by assuming that $a_n < p$. Restricting the consideration to these quantum integers gives just p-adic integers if one regards quantum map $n \rightarrow n_q$ and canonical identification as unrelated notions.

3. Quantum p-adic integers for the "interesting option" could be in some sense to p-adic integers what the integers in the extension of number field are for the number field and attempts to identify quantum Galois group for given prime were made. The attempt to define basic arithmetic operations for quantum p-adics led however to difficulties and motivated to assign to the conjecture quantum Galois group wave functions so that the quantum sum and product would be defined for the wave functions assigned for the quantum p-adic integers. This option looked also too complex to be fundamental. Also the question whether this option gives rise to a generalization of number field, remained open, and no natural identification of quantum Galois group was found.

Eventually I was forced to ask whether it would be wiser to be conservative and concentrate on the "less-interesting" option and try to make it more interesting. Could the emergence 1-to-many correspondence between ordinary and quantum p-adics be something totally unrelated to the construction of quantum p-adics? Could it emerge in the quantum map $n \rightarrow n_q$ taking into account the effects of finite measurement resolution and meaning symmetry breaking: the different p-adic expansions of n allowing the coefficient a_n of p^n to be integers divisible only by primes $l < p$ but having also values $a_n > p$ would be mapped to different quantum p-adic numbers. If this were the case, quantum p-adics must mean something else than was thought first.

The replacement of numbers with sequences of arithmetic operations and integers with Hilbert spaces

The first attempt to solve the problems related to the definition of $+_q$ and $\times_q$ was inspired by zero energy ontology and led to a replacement of numbers with sequences of arithmetic operations describable by analogs of Feyman diagrams. The comparison with generalized Feynman diagrams allowed to realize how "less-interesting" option could become "interesting": numbers could be replaced with Hilbert spaces and all the conditions would be trivially satisfied!

1. The notion of generalized Feynman diagram suggests that of arithmetic Feynman diagram describing a sequence of arithmetic operations performed for a set of incoming integers and producing a set of outgoing integers. The basic 3-vertices of the arithmetic Feynman diagram would be $\times_q$ and $+_q$ and their co-operations. The moves of Feynman diagrams leaving the amplitude invariant would code for associativity and distributivity. All loops could be eliminated by these moves and diagram transformed to a canonical tree diagram in which incoming *resp.* outgoing lines could be permuted. This kind of reduction to tree diagrams is an old proposal that I gave up as too "romantic" [K9] but which re-emerged from zero energy ontology where the assumption that also internal lines (wormhole throats) are massless and on shell although the sign of energy can be negative, poses extremely powerful kinematical constraints reducing the number of Feynman diagrams. Incoming lines would correspond to integers decomposing into products of primes and an attractive interpretation is that these primes correspond to braid strands.

2. The basic vertices in quantum TGD correspond to the stringy 3-vertex and 3-vertex for Feynman graphs. They correspond at Hilbert space level naturally to tensor product and direct sum. Could $\times_q$ and $+_q$ correspond to $\otimes$ and $\oplus$ obeying also associativity and distributivity and could quantum arithmetics for Hilbert spaces apply to quantum TGD? If so, the integers characterizing the lines of arithmetic Feynman diagrams would correspond to Hilbert space dimensions - or rather, Hilbert spaces and quantum states - and in the vertices the incoming states fuse to a direct sum $\oplus$ or tensor product $\otimes$!

3. One could assign to integer n a multiple covering defined by the state basis of n-dimensional Hilbert space. This is just what one wants! The quantum Galois group would be subgroup of the permutation group permuting the elements of this basis. The analogy with covering spaces suggests cyclic group Z_n. The non-trivial quantum Galois group would thus emerge also for the "less-interesting" but non-risky option so that the conservative approach might work after all!

4. The Hilbert spaces in question could represent physical states - in p-adic context one could speak about cognitive representations. It also turns out possible to relate these Hilbert spaces directly to the singular coverings of imbedding space associated with the hierarchy of Planck constants assigned with dark matter in TGD Universe. This gives a concrete content for the quantum Galois group as cyclic permutations of the sheets of the covering of the imbedding space. Hilbert spaces can be identified as function spaces associated with the discrete point sets of the covering projected to the same point. Also a beautiful connection with infinite primes defining algebraic extensions of rationals emerges and infinite primes would characterize physical states by characterizing their dimensions of Hilbert spaces assignable to the incoming and outgoing lines.

5. Quantum arithmetics would be arithmetics of Hilbert spaces and of states assigned to them. This arithmetics allows also extension to rationals and algebraic numbers, and even the Hilbert space variants of algebraic complex numbers, quaternions and octonions can be considered. Also quantum adeles can be defined in terms of Hilbert spaces. These generalization are expected to be crucial for the understanding of generalized Feynman diagrams.

15.1.2 Quantum TGD and Hilbert adeles

Irrespective of whether the isomorphism holds true quantum adeles - if they exist - could provide a very powerful tool also for the formulation of quantum TGD and realize the old intuition that AGG is a symmetry group of quantum TGD [K35] .

1. The innocent TGD inspired question posed already earlier is whether the fusion of real and various p-adic physics together could be realized in terms of adeles providing - if not anything else - an ingenious book keeping device allowing to do real physics and all p-adic physics simultaneously by replacing the whole stuff by single letter A! Now however replaced with A_q.

2. The function spaces associated with quantum adeles decompose to tensor products of function spaces associated with the completions of rationals and one can speak about rational entanglement between different number fields. Rational entanglement can be generalized to algebraic entanglement when one replaces rationals with their algebraic extension and primes with corresponding primes. Could it be that this rational/algebraic entanglement is the rational/algebraic suggested to characterize living matter and to which one can assign negative entanglement entropy having interpretation as a measure for genuine information?

3. The basic vision of TGD inspired quantum bio-physics is that life resides in the intersection of real and p-adic worlds in which rational/algebraic entanglement is natural. One can argue that rational and algebraic entanglement are unstable and that it cannot be realized in any system - living or not. The objection is that Negentropy Maximization Principle (NMP [K42]) favors the generation of negentropic entanglement and once formed between two material systems described by real numbers is stable. Could it be that the mechanism producing this kind of entanglement is the necessary rational/algebraic entanglement between different number fields - between matter and mind one might say - and that quantum jumps transforming p-adic space-time sheets to real ones generates rational/algebraic entanglement between systems consisting of matter. Intention transforming to action would be the interpretation for this process.

4. The construction of generalized Feynman diagrams leads to a picture in which propagator lines give rise to expressions in various p-adic number fields and vertices naturally to multi-p-adic expressions involving p-adic primes of incoming lines. This picture has also natural generalization to quantum variants of p-adic numbers and the expressions are eventually mapped to real numbers by canonical identification induced by $p \rightarrow 1/p$ for quantum rationals appearing in various lines and in vertices of the generalized Feynman diagram. This construct would naturally to a tensor product of state spaces assignable to different p-adic primes and also reals so that M-matrix elements would be naturally in this tensor product. Note that the function space associated with (quantum) adeles is naturally tensor product of functions spaces associated with Cartesian factors of the adele ring with rationals defining the entanglement coefficients. All this of course generalizes by replacing rationals by their algebraic extensions.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- Physics as generalized number theory [L29]

15.2 Earlier attempts to construct quantum arithmetics

Quantum arithmetics [K83] provides a possible resolution of a long-lasting challenge of finding a mathematical justification for the canonical identification mapping p-adics to reals playing a key role in TGD - in particular in p-adic mass calculations [K46].

In [K83] two basic options for quantum arithmetics were discussed. For option I products of integers are mapped to products of quantum integers achieved by mapping primes l to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$. For option II this is not the case.

In this chapter a third and much more general option is discussed. In order to give the needed context, the options discussed in [K83] are however briefly discussed first.

15.2.1 Overall view about variants of quantum integers

The starting point of quantum arithmetics is the map $n \to n_q$ taking integers to quantum integers: $n_q = (q^n - q^{-n})/(q - q^{-1})$. Here $q = exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q = exp(i\pi/p)$ are of special interest. Also prime prime power roots $q = exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of "real and p-adic worlds" finite integers can be regarded both p-adic and real.

1. If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p \to p_q$. The map takes prime first to finite field $G(p, 1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

2. For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1 < p$. This map can be seen as a modification of canonical identification.

3. If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p, 1)$ there are no primes.

Clearly, many variants of quantum integers can be found and it is difficult to decide which of them - if any - has interesting from TGD point of view.

1. If one wants to really model something using quantum integers, the second options is perhaps the realistic one: the reason is that the decomposition into prime factors requires a lot of computation time.

2. A second fictive criterion would be whether the definition is maximally general. Does the definition makes sense for infinite primes? The simplest infinite primes at the first level of hierarchy have physical interpretation as many-particle states consisting of bosons and fermions, whose momentum values correspond to finite primes. The interpretation generalizes to higher levels of the hierarchy. A simple argument show that the option keeping information about prime factorization of the p-adic number allowing also infinite primes as factors makes sense only if prime factors are not expanded in series with respect to the prime p and if p does not correspond to a fermionic mode. The quantum map using prime root of unity therefore makes sense for all but fermionic primes. The presence of exceptional primes in number

theory is basic phenomenon: typically they correspond to primes for which factorization is not unique in algebraic extension.

Two options for quantum integers

Two options for definining quantum arithmetics are discussed on [K83]: Options I and II. These are not the only one imaginable but represent kind of diametrical opposites. The two options are defined in the following manner.

1. For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to $l\ modp$ (to guarantee positivity of the quantum integer) decomposed into primes $l < p$ and these in turn to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$ so that image of the product is product of images. Sums are *not* mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

2. Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

   The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n)p^n$. For p-adic numbers this is not the case.

3. For both options it is natural to consider the variant for which one has expansion $n = \sum_k n_k p^{kr}$, $n_k < p^r$, $r = 1, 2....$ p^k would serve as cutoff.

4. Non-negativity of quantum primes is important in the modelling of Shnoll effect by a deformation of probability distribution $P(n)$ by replacing the argument n by quantum integers and the parameters of the distribution by quantum rationals [K5]. One could also replace quantum prime by its square without losing the map of products to products.

5. At the limit when the quantum phase approaches to unit, ordinary quantum integers with p-adic norm 1 approach to ordinary integers in real sense and ordinary arithmetics results. Ordinary integers in real sense are obtained for option II when the coefficients of the pinary expansion of n are much smaller than p and p approaches infinity. Same is true for option I if the prime factors of the integer are much smaller than p.

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

About the choice of the quantum parameter q

Some comments about the quantum parameter q are in order.

1. The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n - \overline{q}^n}{q - \overline{q}} . \quad (15.2.1)$$

   Here q is *any* complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$
\begin{aligned}
p_q &= \frac{q^p - \overline{q}^p}{q - \overline{q}} \\
n_q &= \prod_p p_q^{n_p} \quad \text{for} \quad n = \prod_p p^{n_p} \quad .
\end{aligned}
\tag{15.2.1}
$$

2. In the general case quantum phase is complex number with magnitude different from unity:

$$
q = exp(\eta)exp(i\pi/m) \quad . \tag{15.2.2}
$$

The quantum map is 1-1 for a non-vanishing value of η and the limit $m \to \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteeing a unique definition of quantum sum. In the p-adic context with $m = p$ the number $exp(\eta)$ exists as an ordinary p-adic number only for $\eta = np$. One can of course introduce a finite-dimensional extension of p-adic numbers generated by $e^{1/k}$.

3. The root of unity must correspond to an element of algebraic extension of p-adic numbers. Here Fermat's theorem $a^{p-1}\ mod\ p = 1$ poses constraints since $p-1$:th root of unity exists as ordinary p-adic number. Hence $m = p - 1$:th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers.

4. If q reduces to quantum phase, the $n = 0, 1, -1$ are fixed points of $n \to n_q$ for ordinary integers so that one could say that all these numbers are common to integers and quantum integers for all values of $q = exp(i\pi/m)$. For p-adic integers $-1 = (p-1)(1 + p + p^2 + ..$ is problematic. Should one use direct formula mapping it to -1 or should one map the expansion to $(p-1)_q(1 + p + p^2 + ...)$? This option looks more plausible.

   (a) For the first option the images under canonical can have both signs and can form a field. For the latter option would obtain only non-negative quantum p-adics for ordinary p-adic numbers. They do not form a field. For algebraic extensions of p-adics by roots of unity one can obtain more general complex numbers as quantum images. For the latter option also the quantum p-adic numbers projecting to a given prime l regarded as p-adic integer form a finite set and correspond to all expansions $l = \sum l_k p^k$ where l_k is product of powers of primes $p_i < p$ but one can have also $l_k > p$.

   (b) Quantum integers containing only the $O(p^0)$ term in the binary expansion for a sub-ring. Corresponding quantum rationals could form a field defining a kind of covering for finite field $G(p, 1)$.

   (c) The image $I(m/n)$ of the pinary expansion of p-adic rational is different from $I(m)/I(n)$. The formula $m/n \to I(m)/I(n)$ is the correct manner to define canonical identification map. In this case the real counterparts of p-adic quantum integers do not form the analog of function fields since the numbers in question are always non-negative.

5. For p-adic rationals the quantum map reads as $m/n \to m_q/n_q$ by definition. But what about p-adic transcendentals such as e^p? There is no manner to decompose these numbers to finite primes and it seems that the only reasonable map is via the mapping of the coefficients x_n in $x = \sum x_n p^n$ to their quantum adic counterparts. It seems that one must expand all quantum transcendentals having as a signature non-periodic pinary expansion to quantum p-adics to achieve uniqueness. Second possibility is to restrict the consideration to rational p-adics. If one gives up the condition that products are mapped to products, one can map $n = n_k p^k$ to $n_q = \sum n_{kq} p^k$. Only the products of p-adic integers $n < p$ smaller than p would be mapped to products.

6. The index characterizing Jones inclusion [A172] [K25] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

To sum up, one can imagine several options and it is not clear which option is the correct one. Certainly Option I for which the quantum map is only part of canonical identification is the simpler one but for this option canonical identification respects discrete symmetries only approximately. The model for Shnoll effect requires only Option I. The notion of quantum integer as defined for Opion II imbeds p-adic numbers to a much larger structure and therefore much more general than that proposed in the model of Shnoll effect [K5] but gives identical predictions when the parameters characterizing the probability distribution $f(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $f(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

15.2.2 The third option for quantum p-adics

The definition of quantum p-adics discussed in this chapter replaces integers with Hilbert spaces of same dimension and + and × with direct sum ⊕ and tensor product ⊗. Also co-product and co-sum must be introduced and assign to the arithmetics quantum dynamics, which leads to proposal that sequences of arithmetic operations can be interpreted arithmetic Feynman diagrams having direct TGD counterparts. This procedure leads to what might be called quantum mathematics or Hilbert mathematics since the replacement can be made for any structure such as rationals, algebraic numbers, reals, p-adic numbers, even quaternions and octonions.

Even set theory has this kind of generalization. The replacement can be made also repeatedly so that one obtains a hierarchy of structures very similar to that obtained in the construction of infinite primes by a procedure analogous to repeated second quantization. One possible interpretation is in terms of a hierarchy of logics of various orders. Needless to say this definition is the really deep one and actually inspired by quantum TGD itself. In this picture the quantum p-adics as they are defined here would relate to the canonical identification map to reals and this map would apply also to Hilbert p-adics.

15.3 Hilbert p-adics, Hilbert adeles, and TGD

One can imagine also a third generalization of the number concept. One can replace integer n with n-dimensional Hilbert space and sum and product with direct sum and tensor product and introduced their co-operations, the definition of which is non-trivial. This procedure yields also Hilbert space variants of rationals, algebraic numbers, p-adic number fields, and even complex, quaternionic and octonionic algebraics. Also adeles can be replaced with their Hilbert space counterparts. Even more, one can replace the points of Hilbert spaces with Hilbert spaces and repeat this process, which is very similar to the construction of infinite primes having interpretation in terms of repeated second quantization. This process could be the counterpart for construction of n^{th} order logics and one might speak of Hilbert or quantum mathematics. It would also generalize the notion of algebraic holography.

This vision emerged from the connections with generalized Feynman diagrams, braids, and with the hierarchy of Planck constants realized in terms of coverings of the imbedding space. Hilbert space generalization of number concept seems to be extremely well suited for the purposes of TGD. For instance, generalized Feynman diagrams could be identifiable as arithmetic Feynman diagrams describing sequences of arithmetic operations and their co-operations. The definition of co-operations would define quantum dynamics. Physical states would correspond to the Hilbert space states assignable to numbers.

15.3.1 Could the notion of Hilbert mathematics make sense?

After having worked one month with the iea I found myself in a garden of branching paths and realized that something must be wrong. Is the idea about quantum p-adics a disgusting fix idee or is it something deeper?

The successful manner to make progress in this this kind of situation has been the combination of existing firmly established ideas with the newcomer. Could the attempt to relate quantum p-adics to generalized Feynman graphs, infinite primes, and hierarchy of Planck constants help?

Second good strategy is maximal simplification. In the recent case this encourages sticking to the most conservative option for which quantum p-adics are obtained from ordinary p-adics by mapping the coefficients of powers of p to quantum integers. This option has also a variant for which one has expansion in powers of p^N defining pinary cutoff. At the level of p-adic numbers different values of N make no difference but at the level of finite measurement resolution situation is different. Also quantum m-adicity would have natural interpretation in terms of measurement resolution rather than fundamental algebra.

Replacing integers with Hilbert spaces

Consider now the argument leading to the interpretation of p-adic integers as Hilbert space dimensions and the formulation of quantum p-adics as p-adic Hilbert spaces whose state basis defines a multiple covering of integer defining the dimension of the Hilbert space.

1. The notion of generalized Feynman diagram and zero energy ontology suggest suggests that of arithmetic Feynman diagram describing a sequence of arithmetic operations performed for a set of incoming integers and producing a set of outgoing integers. This approach indeed led to the discovery that integers could be replaced by Hilbert spaces.

2. The basic 3-vertices of the arithmetic Feynman diagram would be $\times_q$ and $+_q$ and their co-operations. The moves of Feynman diagrams leaving the amplitude invariant would code for associativity and distributivity. All loops could be eliminated by these moves and diagram transformed to a canonical tree diagram in which incoming *resp.* outgoing lines could be permuted.

3. Incoming lines would correspond to integers decomposing into products of primes and an attractive interpretation is that these primes correspond to braid strands for generalized Feynman diagrams.

4. The basic vertices in quantum TGD correspond to the stringy 3-vertex and 3-vertex for Feynman graphs. They correspond at Hilbert space level naturally to tensor product and direct sum. Could $\times_q$ and $+_q$ correspond to tensor product and direct sum obeying also associativity and distributivity?! If so, the integers characterizing the lines of arithmetic Feynman diagrams would correspond to Hilbert space dimensions - or rather, Hilbert spaces - and in vertices the incoming states fuse to direct sum of tensor product!

5. What this would mean is that one could assign to each p-adic integer a multiple covering defined by the state basis of the corresponding Hilbert space. This is just what one wants! The quantum Galois group would be subgroup of the permutation group permuting the elements of this basis. The analogy with covering spaces suggests just cyclic group. The non-trivial quantum Galois group would emerge also for the "less-interesting" but non-risky option so that the conservative approach might work!

6. The Hilbert spaces in question could represent physical states - maybe cognitively in the p-adic context. It also turns out possible to relate these Hilbert spaces directly to the singular coverings of imbedding space associated with the hierarchy of Planck constants assigned with dark matter in TGD Universe. This gives a concrete content for the quantum Galois group as cyclic permutations of the sheets of the covering of the imbedding space and Hilbert spaces can be identified as function spaces associated with the discrete point sets of covering projected to the same point. Also a beautiful connection with infinite primes defining algebraic extensions of rationals emerges and infinite primes would characterize

physical states by characterizing their dimensions of Hilbert spaces assignable to the incoming and outgoing lines.

This approach works for the ordinary p-adic integers. There is no need to allow coefficients $a_n > p$ ("interesting" option) in the expansion $\sum a_n p^n$ of p-adic numbers but still consisting of primes $l < p$. "Interesting" option would emerge as one takes finite measurement resolution into account by mapping the Hilbert spaces defining coefficients of Hilbert space pinary expansion with their quantum counterparts. More precisely.

1. At Hilbert space level pinary expansion of p-adic Hilbert space becomes direct sum $\oplus_n a_n \otimes p^n$. $a_n = \otimes_i p_i$, $p_i < p$, denotes tensor product of prime Hilbert spaces for which I use the same label as for p-adic numbers. p^n denotes Hilbert space with dimension p^n. In real context it is very natural to decompose real Hilbert spaces to tensor products of prime Hilbert spaces.

2. Quantum p-adic numbers would be obtained by mapping the Hilbert space valued coefficients a_n of the to their quantum counterparts $(a_n)_q$, which are conjectured to allow precise definition in terms of inclusions of hyper-finite factors with Jones inclusions associated with the quantum counterpart of 2-D Hilbert space. The quantum map would reduce to the mapping of the tensor factors p_1 of a_n to $(p_1)_q$. Same would apply to quantum states. The map would be defined as $\oplus a_n \otimes p^n \to \oplus (a_n)_q \otimes p^n$, $(a_n)_q = \otimes_{p_1} (p_1)_q$. The map $p_1 \to (p_1)_q$ would take into account finite measurement resolution.

3. "Interesting" option would be obtained as follows. It is possible to express given p-adic number in many manners if one only requires that the coefficients a_n in the direct sum are tensor products of prime Hilbert spaces with dimension $p_1 < p$ but does not assume $a_n < p$. For instance, for $p = 3$ and $n = 8$ one has $8 = 2 \oplus 2 \otimes$ or $8 = 2 \otimes 2 \otimes 2$. These representations are p-adically equivalent. Quantum map however spoils this equivalence. $2 \oplus 2 \otimes 3 \to 2_q \oplus 2_q \otimes 3$ and $8 = 2 \otimes 2 \otimes 2 \to 2_q \otimes 2_q \otimes 2_q$ are not same quantum Hilbert spaces. The "interesting" option would thus emerge as one takes into account the finite measurement resolution.

4. One could say that the quantum Hilbert spaces associated with a given p-adic Hilbert space form a covering space like structure. Quantum Galois group identified as a subgroup of permutations of these quantum Hilbert spaces need not make sense however.

After this lengthy motivating introduction I want to describe some details of the arithmetics of p-adic Hilbert spaces. This arithmetics is formally identical with the ordinary integer arithmetics. What is however interesting is that one can generalize it so that one obtains something that one could call Hilbert spaces of dimension which is negative, rational, algebraic, or even complex, and even quaternionic or octonionic. It might be necessary to have these generalizations if one wants full generality.

1. Consider first what might be called p-adic Hilbert spaces. For brevity I will denote Hilbert spaces in the same manner as p-adic numbers: reader can replace "n" with "H_n" if this looks more appropriate. p-Adic Hilbert spaces have direct sum expansions of form

$$n = \oplus_k a_k \otimes p^k \quad .$$

All integers appearing in the formula can be also interpreted as Hilbert space dimensions. In the real context it is very natural to decompose real Hilbert spaces to tensor products of prime Hilbert spaces.

2. How to define Hilbert spaces with negative dimension? In p-Adic context this is not a problem. Hilbert space with dimension -1 is given by Hilbert spaces with dimension $(p-1)/(1-p) = (p-1)(1+p+p^2+...)$ converging p-adically and given by

$$-1 = \oplus_k (p-1) \otimes p^k \quad .$$

In real context one must consider pairs of Hilbert spaces (m, n) and define equivalence $(m, n) = (m + k, n + k)$. In canonical representation Hilbert space with positive dimension m corresponds to $(m, 0)$ and Hilbert spaces with negative dimension $-m$ to $(0, m)$. This procedure is familiar from the theory of vector bundles where one subtracts vector bundles and defines their negatives.

3. In p-adic context one can also define p-adic Hilbert spaces with rational dimension if the p-adic norm of the rational (m/n) is smaller than 1. This is achieved simply by the expansion

$$\frac{m}{n} = \oplus_k a_k \otimes p^k \quad .$$

In real context tone can define Hilbert spaces with rational valued dimension just as one defines rational numbers - that is as pairs of Hilbert spaces (m,n) with equivalence $(m,n) \equiv (km,kn)$.

4. One can even define Hilbert spaces with dimensions in algebraic extensions of rationals.

   (a) Consider first the real case and the extension defined by Gaussian integers for which integers are of form $m+in \equiv (m,n)$. What is needed is just the product rule: $(m,n) \otimes (r,s) = (m \otimes r - \oplus(-n \otimes s), m \otimes s \oplus r \otimes n)$. This expression is completely well-defined in the p-adic context and also in real context if one accepts the proposed defined of integer Hilbert spaces as pairs of ordinary Hilbert spaces. For $Q(\sqrt{5})$ one would have $(m,n) \times (r,s) = (m \otimes r \oplus 5 \otimes n \otimes s, m \otimes s \oplus r \otimes n)$. In n-dimensional case one just replaces Hilbert spaces with n-multiple of ordinary Hilbert spaces and uses the multiplication rules.

   (b) In p-adic context similar approach works when the algebraic extension requires also extension of p-adic numbers. In p-adic context however many algebraic numbers can exist as ordinary p-adic numbers. For instance, for $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as well as its Hilbert space counterpart. For quadratic extensions of $p >$ 2-adic numbers the 4-D extension involving the addition of two square roots all square roots except that of p exist -adically.

Quantum Hilbert spaces and generalization to extensions of rationals

The map of p-adic integers to their quantum counterparts generalizes so that it applies to Hilbert spaces. This means that prime Hilbert spaces are mapped to the quantum counterparts. What this means is not quite obvious. Quantum groups appearing in the context of Jones inclusions lead to the emergence of quantum spinors that is quantum counterparts of 2-D Hilbert spaces. This suggest that more general inclusions lead to prime-dimensional quantum Hilbert spaces. The idea is simple: quantum matrix algebra M/N with quantum dimension $(2_q)^2$ is defined as a coset space of hyper-finite factor M and included factor $N \subset M$. This quantum matrix algebra acts in quantum spinor space of dimension 2_q. The generalization would introduce p_q-dimensional quantum Hilbert spaces.

A good test for the proposal is whether it generalizes naturally to algebraic extensions of rationals.

1. For algebraic extensions some ordinary primes split into products of primes associated with the extension. The problem is that for these algebraic primes the factors $exp(i\pi/P)$ fail to be algebraic numbers and finite roots of unity and its is not at all clear whether the naive generalization of the notion of quantum p-adic makes sense. This suggests that only the ordinary primes which do not split into products of primes of extension remain and one can define quantum p-adics only for these whereas the other primes correspond to ordinary algebraic extension of p-adic numbers. This would make algebraic extension of rationals the coefficient group of adele consisting of p-adic numbers fields associated with non-split primes only. Note that rationals or their extension would naturally appear as tensor factor of adeles meaning that their action can be thought to affect any of the factors of the adele.

2. For split primes the p-adic Hilbert spaces must be defined for their algebraic prime factors. The proposed procedure of defining Hilbert space counterparts for algebraic extensions of rationals provides a recipe for how to achieve this. These Hilbert spaces the quantum map would be trivial.

3. Hilbert space counterpart for the algebraic extension of rationals and of p-adics makes also sense. The Hilbert space assigned with integer which splits into primes of extension splits also to a tensor product of prime Hilbert spaces assignable with the extension. The splitting of integers and primes is highly analogous to the decomposition of hadron to quarks and gluons. This decomposition is not seen at the level of rationals representing observed.

What about Hilbert spaces with real number valued dimension?

What Hilbert space variant of a real number could mean? What Hilbert space with dimension equal to arbitrary real number could mean? One can imagine two approaches.

1. The first approach is based on the map of Hilbert p-adics to real p-adics by a map used to map p-adic numbers to reals. The formula would be $\oplus_n a_n \otimes p^n \rightarrow \oplus (a_n)_q \otimes p^{-n}$. $(a_n)_q = \otimes_l l_q^{e_l}$, were l_q is quantum Hilbert space of prime dimension. Also the Hilbert space p^{-n} would be well-defined as a Hilbert rational defined as a pair of Hilbert spaces.

   For hyper-finite factors of type II_1 Hilbert spaces with continuous dimension emerge naturally. The reason is that the dimension of the Hilbert space is defined as quantum trace of identity operator characterized by quantum phase this dimension is finite and continuous. This allows a spectrum of sub-Hilbert spaces with continuously varying real dimension. The appearance of quantum Hilbert spaces in the canonical identification map conforms with this and even for dimension $0 < n < p$ gives rise to quantum Hilbert space with algebraic quantum dimension given as $n = \prod l_q^{e_l}$ for $n = \prod_l l^{e_l}$.

2. Second approach relies on the mimicry of the completion of ordinary rationals to real numbers. One can define Hilbert space analogs of rationals and algebraics by defining positive and negative rationals as pairs of Hilbert spaces with equivalence relation $(m,n) \equiv (m \oplus r, n \oplus r)$. Taking pairs of these pairs with equivalence relation $(M,N) \equiv (K \otimes M, K \otimes N)$ one obtains Hilbert spaces corresponding to rational numbers. Algebraic extensions are obtained similarly. By taking limits just in the same manner as for real numbers one would obtain Hilbert reals with transcendental dimensions. For instance, e could be defined as the limit of tensor power $(1 \oplus 1/n)^n$, $n \rightarrow \infty$.

Again one must remember that the co-vertices define the hard part of the problem and their definition means postulate of quantum dynamics. This would be the genuinely new element and transform mathematics to quantum physics. Every sequences of algebraic operations having a realization as Feynman diagram involving arithmetic operations as positive energy part of Feynman diagrams and co-operations as the negative energy part of diagram connected by single line.

It should not go un-noticed that the direct sum and tensor product decompositions of possibly infinite-dimensional Hilbert spaces are very essential for the interpretation. For infinite-dimensional Hilbert spaces these decompositions would be regarded as equivalent for an abstract definition of Hilbert space. In physical applications tensor product and direct sum representations have also very concrete physical content.

Hilbert calculus?

What this approach suggests is a generalization of calculus in both real and and p-adic context. The first thing to do is to define Hilbert functions as Hilbert space valued functions as $x \rightarrow f(x)$. This could be done formally by assigning to Hilbert space associated with point x Hilbert space associated with the point $f(x)$ for all values of x. Function could have representation as Taylor series or Laurent series with sum replaced with direct sum and products with tensor products. The correspondence $x \rightarrow f(x)$ would have as a counterpart the analog of Feynman diagram describing the Taylor series with final line defining the value $f(x)$. Also derivatives and integrals would be at least formally defined. This would requite separate diagram for every point x. One can consider also the possibility of more abstract definition of $f(x)$. For instance the set of coefficients $\{f_n\}$ in the Taylor series of f would defined a collection of Hilbert spaces.

One should be able to define also co-functions in terms of co-vertices. The value of co-function at point y would give all the values of x for which one has $f(x) = y$. Co-function would correspond to a quantum superposition of values of inverse function and to time reversed zero energy states.

The breaking of time reversal would be inherent in the very definition of function as an arrow from one Hilbert set to another Hilbert set and typically the functions involved would be many-valued form beginning. Perhaps it would be better to speak from the beginning about relations between two sets rather than functions. The physical realization of Hilbert calculus would be obtained by assigning to incoming arguments represented as Hilbert space quantum states.

Quantum mathematics?

Could one transform entire mathematics to quantum mathematics - or would it be better to say Hilbert mathematics? Reader can decide. Consider first Hilbert set theory. The idea wold be to replace numbers with Hilbert spaces. This would give Hilbert structure. By replacing Hilbert spaces with their quantum counterparts characterized by quantum dimensions n_q one would obtain which might be called quantum Hilbert structure.

1. At the level of set theory this would mean replacement of sets with Hilbert sets. A set with n elements would correspond intuitively to n-dimensional Hilbert space. Therefore Hilbert sets would provide much more specific realization of set theory than abstract set theory in which the elements of set can be anything. For n-dimensional Hilbert space however the ordering of the elements of basis induces automatically the ordering of the elements of the set. Does the process of counting the elements of set corresponds to this ordering. Direct sum would be the counterpart of set theoretic union. One could construct natural numbers inductively as direct sums $(n+1) = n \oplus 1$. To be subset would correspond to sub-Hilbert space property. Intersection of two Hilbert sets would correspond to the direct sum of common direct summands. Also set difference and symmetric difference could be defined.

2. The set theoretic realization of Boolean logic would have Hilbert variant. This would mean that logical statements could be formulated using Hilbert variants of basic logical functions.

3. Cartesian product of sets would correspond to a tensor product of Hilbert spaces. This would bring in the notion of prime since Hilbert integers would have decomposition into tensor products of Hilbert primes. Note that here one can consider the symmetrization of tensor product modulo phase factor and this could give rise to bosonic and fermionic statistics and perhaps also to anyonic statistics when the situation is 2-dimensional as it indeed is for partonic 2-surfaces.

4. What about sets of sets?

   (a) The elements of n-dimensional Hilbert space consist of numbers in some number field. By replacing these numbers with corresponding Hilbert spaces one would obtain Hilbert space of Hilbert spaces as a counterpart for sets of sets. One would have Hilbert space whose points are Hilbert spaces: Hilbert-Hilbert space!. This process could be continued indefinitely and would give rise to a hierarchy formed by Hilbert^n-spaces. This would be obviously something new and mean self-referential property. For Hilbert^n-spaces one would the points at n:th level of hierarchy with points of the number field involved and obtain a concrete realization. The construction of infinite primes involves formations of sets of rationals and sets of these sets, etc.... and would have also interpretation as formation of a hierarchy of Hilbert sets of sets of.....

   (b) Power set as set of subsets of set would be obtained from direct sum of Hilbert spaces, by replacing the points of each Hilbert space with corresponding Hilbert spaces.

   (c) One could define the analog of set theoretic intersection also for tensor products as the set of common prime Hilbert factors for two Hilbert sets. For ordinary integers defined as sets the intersection in this sense would correspond to the common prime factors. In Cartesian product the intersection would correspond to common Cartesian factors.

5. The completely new and non-trivial element bringing in the quantum dynamics is brought in by co-operations for union and intersection. The solution to the equation $f(x) = y$ could be represented as a number theoretic Feynman diagram in zero energy ontology. Positive energy part would correspond to y and diagram beginning from y would represent co-function of $f(x)$

identifiable as its inverse. Negative energy state would represent a quantum superposition of the values of x representing the solutions.

6. One can ask whether a Feynman diagrammatic representation for the statements like $\exists x \in A$ such that $f(x) = g(x)$ and $\forall x \in A f(x) = g(x)$ exists. One should be able to construct quantum state which is superposition of solutions to the condition $f(x) = g(x)$. If this state is non-vanishing the solution exists.

   This kind of statements are statements of first order logic involving existential quantifiers whereas the statements of predicate logic would correspond simply to a calculation of a value of function at given point. The hierarchy of Hilbert^n spaces brings in mind strongly the hierarchy of infinite primes assigned already earlier to a hierarchy of logics. Could the statements of n:th order logic require the use of Hilbert^n- spaces. The replacement of numbers with Hilbert spaces could correspond to formation of statements of first order logic. The individual quantum states satisfying the statement would represent the statements of predicate logic.

   The construction of infinite primes can be regarded as repeated second quantization in which the many particle states of the previous level become single particle states of the new level. Maybe also the hierarchy of Hilbert^n-spaces could be seen in terms of a hierarchy of second quantizations.

   Infinite primes lead to the notion of algebraic holography meaning that real point has infinitely rich number theoretical anatomy due to the existence of real units expressible as ratios of infinite integers reducing to real unit in real topology. The possibility to replace the points of space-time with Hilbert spaces and to continue this process indefinitely would realize the same idea.

Number theoretic Feynman diagrams

Could one imagine a number theoretical quantum dynamics in which integers are replaced with sequences of arithmetic operations? If numbers are replaced with Hilbert spaces and if one can assigns to each number a state of the Hilbert space accompanying it, this seems to be possible.

1. All algebraic functions would be replaced with their algebraic expressions, which would be interpreted as analogs of zero energy states in which incoming arguments would represent positive energy part and the result of operation outgoing state. This would also unify algebra and co-algebra thinking and the information about the values of the arguments of the function would not be forgotten in the operations.

2. The natural constraints on the dynamics would be trivial. In $+_q$ vertex a direct sum of incoming states and in $\times_q$ gives rise to tensor product. This also at the level of Hilbert spaces involved. The associativity and commutativity of direct sum and tensor product guarantee automatically the these properties for the vertices. The associativity and commutativity conditions are analogous to associativity conditions for 3-point functions of conformal field theories. Distributivity condition is something new. Co-product and co-sum obey completely analogous constraints as product and sum.

3. For product the total numbers of prime factors is conserved for each prime appearing in the product meaning that the total momenta $n_i log(p_i)$ are conserved separately for each prime in the process involving only products. This kind of conservation law is natural also for infinite primes and one can indeed map the simplest infinite primes at the lowest level analogous to free Fock states of bosons and fermions to ordinary rationals so that the addition of Galois degrees of freedom tentatively identified as cyclic permutations of the state basis for Hilbert space associated with given prime would give for a particle labelled by prime p additional internal degrees of freedom. In fact, one can illustrate infinite prime as in terms of two braids corresponding to the numerator and denominator of corresponding rational and the primes appearing in rationals take the role of braid strands. For $\times_q$ the conservation of quantum numbers would correspond to conservation of representations. This guarantees commutativity and associativity of product. One can also allow co-product and co-sum and

they obey completely analogous constraints as product and sum and they have counterparts at the level of Hilbert spaces two studied in the theory of quantum groups.

One can represent algebraic operations using the analogs of Feynman diagrams and there is an intriguing analogy with generalized Feynman diagrams which forces to ask whether the generalized Feynman diagrams of quantum TGD could be interpreted in terms of quantum counterparts algebraic equations transformed if one extends the number field to quaternions and their possibly existing p-adic counterparts.

1. Multiplicative and additive inverses - in the case that they exist - can be seen as kind of conjugation operations analogous to C and P which commute with each other. Their product $n \to -1/n$ could be seen as the analog of T if $CPT = 1$ is taken as identity. Co-product and co-sum would would be obtained from product and sum by CP or T.

2. One can represent the integer $X = X(\{n_k\})$ resulting from a sequence of algebraic operations $+_q$ and $\times_q$ performed for integers n_k appearing as inputs of a Feynman diagram having the value of X as outgoing line. $n_{+,k}$ represent incoming external lines and intermediate products of algebraic operations appear as internal "off-mass-shell" lines. $+_q$ and $\times_q$ represent the basic vertices. This gives only tree diagrams with single outgoing line representing the (quantum value) of X.

   Associativity and commutativity for $+_q$ resp. $\times_q$ would mean that the lines of diagram with 3 incoming particles and two vertices can be modified by permuting the incoming lines in all possible manners. Distributivity $a \times_q (b +_q c) = a \times_q b +_q a \times_q c$ does not correspond anything familiar from conformal field theories since the line representing a appears twice on the right hand side of the identity and there are 3 vertices whereas left hand side involves 2 vertices. In TGD framework the interpretation of the analogs of stringy decay vertices in terms of propagation along two different paths allows however to interpret these vertices as counterparts of $+_q$ whereas the TGD counterparts of vertices of Feynman diagrams would correspond to $\times_q$. $+q$ would correspond at state space level to direct sum and $\times_q$ to tensor product.

3. The lines of Feynman diagrams are naturally replaced with braids - just as in quantum TGD. The decomposition of the incoming quantum rational $q = m/n$ to primes defines a braid with two colors of braid strands corresponding to the primes appearing in m and n so that a close connection with braid diagrams emerges. This of course raises the question whether one could allow non-trivial braiding operation for two braid strands represented by primes. Non-triviality would mean that $p_1p_2 = p_2p_1$ would not hold true only in projective sense so that the exchange would induce a phase factor. This would suggest that the commutativity of the basic operations - or at least multiplication - might hold true only apart from quantum phase factor. This would not be too surprising since quantum phases are the essence of what it is to be quantum integer.

4. The diagrammatical counterparts of co-operations are obtained by time reversal transforming incoming to outgoing lines and vice versa. If one adds co-products and sums to the algebraic operations producing X one obtains diagrams with loops. If ordinary algebraic rules generalizes the diagrams with loops must be transformable to diagrams without them by algebraic "moves". The simplification of arithmetic formulas that we learn in elementary school would correspond to a sequence of "moves" leading to a tree diagram with single internal line at the middle and representing $X = Y$. One can form also diagrams of form $X = Y = Z = ...$ just as in zero energy ontology.

5. In zero energy ontology a convenient manner to represent a identity $X = Y$ - call it a "quantum correlate for mathematical thought" - involving only sums and products and therefore no loops is as a tree diagram involving only two kinds of 3-vertices, namely $+_q$ and $\times_q$ and their co-algebra vertices representing time reversed processes. In zero energy ontology this kind of representation would correspond to either the condition $X/Y = 1$ or as $X - Y = 0$. In both cases one can say that the total quantum numbers would be conserved - that is net quantum numbers assignable to prime factors of X vanish for zero energy state. The diagram involves always single integral line representing the identical values of X and Y. Line

representing X would be preceded by a tree diagram involving only product and sum vertices and Y would involve only co-product and co-sum. For ordinary arithmetics every algebraic operation is representable in this kind of diagram, which suggests that infinite number of different diagrams involving loops are equivalent to this diagram with single internal line.

6. The resulting braid Feynman diagrammatics would obey extremely powerful rules due to the possibility of the "moves". All possible independent equations $X = Y$ would define the basis of zero energy states. In quantum TGD the breaking of time reversal invariance is unavoidable and means that only the positive or negative energy parts of the diagram can have well defined quantum numbers. The direct translation would be that the zero energy states correspond to sums over all diagrams for which either positive/negative energy part corresponds to given rationals and the negative/positive energy part of the state is superposition of states consisting of rationals. This would mean non-trivial U-matrix dictated by the coefficients of the superpositions and genuine arithmetic quantum dynamics.

15.3.2 Hilbert p-adics, hierarchy of Planck constants, and finite measurement resolution

The hierarchy of Planck constants assigns to the N-fold coverings of the imbedding space points N-dimensional Hilbert spaces. The natural identification of these Hilbert spaces would be as Hilbert spaces assignable to space-time points or with points of partonic 2-surfaces. There is however an objection against this identification.

1. The dimension of the local covering of imbedding space for the hierarchy of Planck constants is constant for a given region of the space-time surface. The dimensions of the Hilbert space assignable to the coordinate values of a given point of the imbedding space are defined by the points themselves. The values of the 8 coordinates define the algebraic Hilbert space dimensions for the factors of an 8-fold Cartesian product, which can be integer, rational, algebraic numbers or even transcendentals and therefore they vary as one moves along space-time surface.

2. This dimension can correspond to the locally constant dimension for the hierarchy of Planck constants only if one brings in finite measurement resolution as a pinary cutoff to the pinary expansion of the coordinate so that one obtains ordinary integer-dimensional Hilbert space. Space-time surface decomposes into regions for which the points have same pinary digits up to p^N in the p-adic case and down to p^{-N} in the real context. The points for which the cutoff is equal to the point itself would naturally define the ends of braid strands at partonic 2-surfaces at the boundaries of CD:s.

3. At the level of quantum states pinary cutoff means that quantum states have vanishing projections to the direct summands of the Hilbert spaces assigned with pinary digits p^n, $n > N$. For this interpretation the hierarchy of Planck constants would realize physically pinary digit representations for number with pinary cutoff and would relate to the physics of cognition.

One of the basic challenges of quantum TGD is to find an elegant realization for the notion of finite measurement resolution. The notion of resolution involves observer in an essential manner and this suggests that cognition is involved. If p-adic physics is indeed physics of cognition, the natural guess is that p-adic physics should provide the primary realization of this notion.

The simplest realization of finite measurement resolution would be just what one would expect it to be except that this realization is most natural in the p-adic context. One can however define this notion also in real context by using canonical identification to map p-adic geometric objets to real ones.

Does discretization define an analog of homology theory?

Discretization in dimension D in terms of pinary cutoff means division of the manifold to cube-like objects. What suggests itself is homology theory defined by the measurement resolution and by the fluxes assigned to the induced Kähler form.

1. One can introduce the decomposition of n-D sub-manifold of the imbedding space to n-cubes by $n-1$-planes for which one of the coordinates equals to its pinary cutoff. The construction works in both real and p-adic context. The hyperplanes in turn can be decomposed to $n-1$-cubes by $n-2$-planes assuming that an additional coordinate equals to its pinary cutoff. One can continue this decomposition until one obtains only points as those points for which all coordinates are their own pinary cutoffs. In the case of partonic 2-surfaces these points define in a natural manner the ends of braid strands. Braid strands themselves could correspond to the curves for which two coordinates of a light-like 3-surface are their own pinary cutoffs.

2. The analogy of homology theory defined by the decomposition of the space-time surface to cells of various dimensions is suggestive. In the p-adic context the identification of the boundaries of the regions corresponding to given pinary digits is not possible in purely topological sense since p-adic numbers do not allow well-ordering. One could however identify the boundaries sub-manifolds for which some number of coordinates are equal to their pinary cutoffs or as inverse images of real boundaries. This might allow to formulate homology theory to the p-adic context.

3. The construction is especially interesting for the partonic 2-surfaces. There is hierarchy in the sense that a square like region with given first values of pinary digits decompose to p square like regions labelled by the value $0,...,p-1$ of the next pinary digit. The lines defining the boundaries of the 2-D square like regions with fixed pinary digits in a given resolution correspond to the situation in which either coordinate equals to its pinary cutoff. These lines define naturally edges of a graph having as its nodes the points for which pinary cutoff for both coordinates equals to the actual point.

4. I have proposed earlier [K14] what I have called symplectic QFT involving a triangulation of the partonic 2-surface. The fluxes of the induced Kähler form over the triangles of the triangulation and the areas of these triangles define symplectic invariants, which are zero modes in the sense that they do not contribute to the line element of WCW although the WCW metric depends on these zero modes as parameters. The physical interpretation is as non-quantum fluctuating classical variables. The triangulation generalizes in an obvious manner to quadrangulation defined by the pinary digits. This quadrangulation is fixed once internal coordinates and measurement accuracy are fixed. If one can identify physically preferred coordinates - say by requiring that coordinates transform in simple manner under isometries - the quadrangulation is highly unique.

5. For 3-surfaces one obtains a decomposition to cube like regions bounded by regions consisting of square like regions and Kähler magnetic fluxes over the squares define symplectic invariants. Also Kähler Chern-Simons invariant for the 3-cube defines an interesting almost symplectic invariant. 4-surface decomposes in a similar manner to 4-cube like regions and now instanton density for the 4-cube reducing to Chern-Simons term at the boundaries of the 4-cube defines symplectic invariant. For 4-surfaces symplectic invariants reduce to Chern-Simons terms over 3-cubes so that in this sense one would have holography. The resulting structure brings in mind lattice gauge theory and effective 2-dimensionality suggests that partonic 2-surfaces are enough.

The simplest realization of this homology theory in p-adic context could be induced by canonical identification from real homology. The homology of p-adic object would the homology of its canonical image.

1. Ordering of the points is essential in homology theory. In p-adic context canonical identification $x = \sum x_n p^n \to \sum x_n p^{-n}$ map to reals induces this ordering and also boundary operation for p-adic homology can be induced. The points of p-adic space would be represented by n-tuples of sequences of pinary digits for n coordinates. p-Adic numbers decompose to disconnected sets characterized by the norm p^{-n} of points in given set. Canonical identification allows to glue these sets together by inducing real topology. The points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ having p-adic norms p^{-n} and p^{-n-1} are mapped to the same real point p^{-n} under canonical identification and therefore the points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ can be said to define the endpoints of a continuous interval

in the induced topology although they have different p-adic norms. Canonical identification induces real homology to the p-adic realm. This suggests that one should include canonical identification to the boundary operation so that boundary operation would be map from p-adicity to reality.

2. Interior points of p-adic simplices would be p-adic points not equal to their pinary cutoffs defined by the dropping of the pinary digits corresponding p^n, $n > N$. At the boundaries of simplices at least one coordinate would have vanishing pinary digits for p^n, $n > N$. The analogs of $n-1$ simplices would be the p-adic points sets for which one of the coordinates would have vanishing pinary digits for p^n, $n > N$. $n-k$-simplices would correspond to points sets for which k coordinates satisfy this condition. The formal sums and differences of these sets are assumed to make sense and there is natural grading.

3. Could one identify the end points of braid strands in some natural manner in this cohomology? Points with $n \leq N$ pinary digits are closed elements of the cohomology and homologically equivalent with each other if the canonical image of the p-adic geometric object is connected so that there is no manner to identify the ends of braid strands as some special points unless the zeroth homology is non-trivial. In [K84] it was proposed that strand ends correspond to singular points for a covering of sphere or more general Riemann surface. At the singular point the branches of the covering would co-incide.

   The obvious guess is that the singular points are associated with the covering characterized by the value of Planck constant. As a matter fact, the original assumption was that *all* points of the partonic 2-surface are singular in this sense. It would be however enough to make this assumption for the ends of braid strands only. The orbits of braid strands and string world sheet having braid strands as its boundaries would be the singular loci of the covering.

Does the notion of manifold in finite measurement resolution make sense?

A modification of the notion of manifold taking into account finite measurement resolution might be useful for the purposes of TGD.

1. The chart pages of the manifold would be characterized by a finite measurement resolution and effectively reduce to discrete point sets. Discretization using a finite pinary cutoff would be the basic notion. Notions like topology, differential structure, complex structure, and metric should be defined only modulo finite measurement resolution. The precise realization of this notion is not quite obvious.

2. Should one assume metric and introduce geodesic coordinates as preferred local coordinates in order to achieve general coordinate invariance? Pinary cutoff would be posed for the geodesic coordinates. Or could one use a subset of geodesic coordinates for $\delta CD \times CP_2$ as preferred coordinates for partonic 2-surfaces? Should one require that isometries leave distances invariant only in the resolution used?

3. A rather natural approach to the notion of manifold is suggested by the p-adic variants of symplectic spaces based on the discretization of angle variables by phases in an algebraic extension of p-adic numbers containing n^{th} root of unity and its powers. One can also assign p-adic continuum to each root of unity [K26]. This approach is natural for compact symmetric Kähler manifolds such as S^2 and CP_2. For instance, CP_2 allows a coordinatization in terms of two pairs (P^k, Q^k) of Darboux coordinates or using two pairs $(\xi^k, \overline{\xi}^k)$, $k = 1, 2$, of complex coordinates. The magnitudes of complex coordinates would be treated in the manner already described and their phases would be described as roots of unity. In the natural quadrangulation defined by the pinary cutoff for $|\xi^k|$ and by roots of unity assigned with their phases, Kähler fluxes would be well-defined within measurement resolution. For light-cone boundary metrically equivalent with S^2 similar coordinatization using complex coordinates $(z, \overline{z})$ is possible. Light-like radial coordinate r would appear only as a parameter in the induced metric and pinary cutoff would apply to it.

Hierachy of finite measurement resolutions and hierarchy of p-adic normal Lie groups

The formulation of quantum TGD is almost completely in terms of various symmetry group and it would be highly desirable to formulate the notion of finite measurement resolution in terms of symmetries.

1. In p-adic context any Lie-algebra gG with p-adic integers as coefficients has a natural grading based on the p-adic norm of the coefficient just like p-adic numbers have grading in terms of their norm. The sub-algebra g_N with the norm of coefficients not larger than p^{-N} is an ideal of the algebra since one has $[g_M, g_N] \subset g_{M+N}$: this has of course direct counterpart at the level of p-adic integers. g_N is a normal sub-algebra in the sense that one has $[g, g_N] \subset g_N$. The standard expansion of the adjoint action $g g_N g^{-1}$ in terms of exponentials and commutators gives that the p-adic Lie group $G_N = exp(tpg_N)$, where t is p-adic integer, is a normal subgroup of $G = exp(tpg)$. If indeed so then also G/G_N is group, and could perhaps be interpreted as a Lie group of symmetries in finite measurement resolution. G_N in turn would represent the degrees of freedom not visible in the measurement resolution used and would have the role of a gauge group.

2. The notion of finite measurement resolution would have rather elegant and universal representation in terms of various symmetries such as isometries of imbedding space, Kac-Moody symmetries assignable to light-like wormhole throats, symplectic symmetries of $\delta CD \times CP_2$, the non-local Yangian symmetry, and also general coordinate transformations. This representation would have a counterpart in real context via canonical identification I in the sense that $A \rightarrow B$ for p-adic geometric objects would correspond to $I(A) \rightarrow I(B)$ for their images under canonical identification. It is rather remarkable that in purely real context this kind of hierarchy of symmetries modulo finite measurement resolution does not exist. The interpretation would be that finite measurement resolution relates to cognition and therefore to p-adic physics.

3. Matrix group G contains only elements of form $g = 1 + O(p^m)$, $m \geq 1$ and does not therefore involve matrices with elements expressible in terms roots of unity. These can be included by writing the elements of the p-adic Lie-group as products of elements of above mentioned G with the elements of a discrete group for which the elements are expressible in terms of roots of unity in an algebraic extension of p-adic numbers. For p-adic prime p p:th roots of unity are natural and suggested strongly by quantum arithmetics [K83].

15.3.3 Quantum adeles

Before saying anything about Hilbert space adeles it is better to consider ordinary adeles.

1. Fusing reals and quantum p-adic integers for various values of prime p to Cartesian product $A_Z = R \times (\prod_p Z_p)$ gives the ring of integer adeles. The tensor product $Q \otimes_Z A_Z$ gives rise to rational adeles. $_Z$ means the equivalence $(nq, a) \equiv (q, na)$. This definition generalization to any number field including algebraic extensions of rationals. It is not quite clear to me how essential the presence of R as Cartesian factor is. One can define ideles as invertible adeles by inverting individual p-adic numbers and real number in the product. If the component in the Cartesian product vanishes, the component of inverse also vanishes.

2. The definition of a norm of adele is not quite straightforward.

   (a) The norm of quantum adeles defined as product of real and p-adic norms is motivated by the formula for the norm of rational numbers as the product of its p-adic norms. This definition of norm however looks non-physical and non-mathematical. For instance, it requires that all p-adic components of quantum adele are non-vanishing and most of them have norm equal to one and are therefore p-adic integers of norm one. This condition would also break general coordinate invariance at the level of quantum adelic imbedding space very strongly. Also for adelic spinors and adelic Hilbert space this condition is definitely non-sensical.

(b) The physically acceptable norm for adeles should reflect the basic properties of p-adic norm for a given p-adic field in the product but should also have the characteristic property of Hilbert space norm that the norm squared is sum of the norms squared for the factors of the adele. The solution to these demands seems to be simple: map the p-adic number to its quantum counterpart in each factor and map this number to real number by canonical identification. After this form the real Hilbert space norm of the resulting element of infinite-dimensional real Hilbert space. This norm generalizes in a natural manner to linear spaces possessing adeles as components. Most importantly, for this norm the elements of adele having finite number of components have a non-vanishing norm and field property is possible.

Consider now what happens when one replaces p-adic integers with p-adic Hilbert spaces and p-adic numbers as components of the vectors of the Hilbert space.

1. As far as arithmetics is considered, the definition of Hilbert space adeles for p-adic number fields is formally the same as that of ordinary adeles. It of course takes time to get accustomed to think that rationals correspond to a pair of Hilbert spaces and their product is formulated for this pair.

2. p-Adic Hilbert spaces would be linear spaces with p-adic coefficients that is vectors with p-adic valued components. Inner product and norm would be defined by mapping the components of vectors to real/complex numbers by mapping them first to quantum p-adics and them to reals by canonical identification. Note that the attempts to define p-adic Hilbert spaces using p-adic norm or formal p-adic valued norm mapped to real number by canonical identification lead to difficulties since already in 2-D case the equation $x^2 + y^2 = 0$ has solution $y = \sqrt{-1}x$ for $p \ mod \ 4 = 1$ since in this case $\sqrt{-1}$ exists p-adically.

3. A possible problem relates to the fact that all p-adic numbers are mapped to non-negative real numbers under canonical identification if the coefficients a_n in the expansion $\sum_n a_n p^n$ consists of primes $l < p$ for which quantum counterpart is non-negative. For ordinary p-adic numbers orthogonal vectors in a given basis would be simply vectors with no common non-vanishing components. Does this mean the existence of a preferred basis with elements $(0, .., 0, 1, 0...)$ so that any other unitarily related basis would be impossible. Or should one introduce cyclic algebraic extension of p-adic numbers with n-elements $exp(i2\pi k/n)$ for which one obtains linear superposition and can form new unitarily related basis taking into account the restrictions posed by p-adicity. This option is suggested also by the identification of the Hilbert space as wave functions in the local singular covering of imbedding space. The phases form also in a natural manner cyclic group Z_n identifiable as quantum Galois group assignable to integer n and decomposing to a product of cyclic groups Z_{p_i}, $p_i|n$.

Also real numbers form a Cartesian factor of adeles. The question what Hilbert spaces with dimension equal to arbitrary real number could mean has been already discussed and there are two approaches to the problem: one based on canonical identification and quantum counterparts of p-adic numbers and one to a completion of Hilbert rationals.

15.4 Generalized Feynman diagrams as quantum arithmetic Feynman diagrams?

The idea that the generalized Feynman diagrams of TGD could have interpretation in terms of arithmetic QFT is not new but the quantum arithmetic Feynman diagrams give much more precise content to this idea.

1. The possibility to eliminate all loops is by "moves" is an old idea (briefly discussed in [K9]), which I introduced as a generalization of the old fashioned s-t duality of string models. One motivation was of course the resulting cancellation of diverges. I however gave up this idea as too romantic [K9]. The properties of the counterparts of twistor diagrams in zero energy ontology re-inspires this idea.

2. The basic question concerns the possible physical interpretation of the two kinds of 3-vertices and their co-vertices, which are also included and mean decomposition of incoming particle characterized by integer m to quantum superposition of two particle states characterized by integers n, p satisfying $m = n + p$ for the co-sum and $m = n \times p$ for co-product. The amplitudes of different state pairs n, p in fact determine the quantum dynamics and typically the irreversible dynamics leading from state with well-defined quantum number characterized by integers would be due to the presence of co-vertices meaning de-localization.

3. If quantum p-adic integers correspond to Hilbert spaces then the identifications $+_q = \oplus$ and $\times_q = \otimes$ become possible. The challenge is to fix uniquely their co-vertices and this procedure fixes completely number theoretic Feynman amplitudes. Quantum dynamics would reduce to co-arithmetics. Or should one say that mathematics could reduce to quantum dynamics?

4. $\times_q$ and $+_q$ alone look very quantal and the generalization of string model duality means that besides cyclic permutations arbitrary permutations of incoming *resp.* outgoing lines act as symmetries. The natural question is whether this symmetry generalizes to permutations of all lines. This of course if commutativity in strict sense holds true also for quantum arithmetics: it could be that it holds true only in projective sense. Distributivity has however no obvious interpretation in terms of standard quantum field theory. The arithmetics for integers would naturally reflect the arithmetics of Hilbert spaces dimensions induced by direct sum and tensor product

15.4.1 Quantum TGD predicts counterparts for $\times_q$ and $+_q$ vertices

Also quantum TGD allows two kinds of vertices identifiable in terms of the arithmetic vertices and this gives strong physical constraints on $+_q$ vertices.

1. First kind of vertices are the direct topological analogs of vertices of ordinary Feynman diagrams and there are good arguments suggesting that only 3-vertices are possible and would mean joining of 3 light-like 3-surfaces representing lines of generalized Feyman diagram along their 2-dimensional ends. At the these vertices space-time fails to be a manifold but 3-surface and partonic 2-surface are manifolds. These vertices correspond naturally to $\times_q$ or equivalently $\otimes$.

2. The vertices of second kind correspond to the stringy vertices, in particular the analog of stringy trouser vertices. The TGD based interpretation - different from stringy interpretation- is that no decay takes place for a particle: rather the same particles travels along different routes. These vertices correspond to four-surfaces, which are manifolds but 3-surfaces and partonic 2-surface fail to be manifolds at the vertex. There is a strong temptation to interpret $+_q$ - or equivalently $\oplus$ - as the counterpart of stringy vertices so that the two lines entering to $+_q$ would represent same incoming particle and should have in some sense same quantum numbers in the situation when the particle is an eigenstate of the quantum numbers in question? This would allow to understand the strange looking quantum distributivity and also to deduce what can happen in $+_q$ vertex.

3. What does the conservation of quantum numbers mean for quantum Galois quantum numbers identified in the proposed manner as quantum number associated with the cyclic groups assignable to the integers appearing in the vertex? For $\times_q$ vertex the answer is simple since tensor product is formed. This means that the number theoretic momentum is conserved. For direct sum one obtains direct sum of the incoming states and one cannot speak about conservation of quantum numbers since the final state does not possess well-defined quantum numbers.

15.4.2 How could quantum numbers of physical states relate to the number theoretic quantum numbers?

Quite generally, the above proposal would allow to represent all n-plets of rationals as zero energy states with either positive or negative arrow of time and one could assign to these states M-matrices as entanglement coefficients and define quantum jump as a sequence of two state function reductions occurring to states with opposite arrow of time. This kind of strong structural similarities with quantum TGD are hardly not a accident when one takes into account the connection with infinite primes and one could hope that zero energy states and generalized Feynman graphs could represent the arithmetics of Hilbert adeles with very dramatic consequences due to the arithmetic moves allowing to eliminate loops and permuted incoming lines without affecting the diagram except by a phase factor. The hierarchy of infinite primes suggests strongly the generalization of this picture since the resulting states would correspond only to the infinite integers at the lowest level of the hierarchy and identifiable in terms of free Fock states of super-symmetric arithmetic QFT.

The possible reduction of generalized Feynman diagrams to Hilbert adelic arithmetics raises several questions and one can try to proceed by requiring consistency with the earlier speculations.

1. How the quantum numbers like momentum, spin and various internal quantum numbers relate to the number theoretic quantum numbers $k = n2\pi/p$ defined only modulo p? The natural idea is that they find a representation in the number theoretical anatomy of the state so that these quantum numbers corresponds to waves with these momenta at the orbits of quantum Galois group. Momentum UV cutoff would have interpretation in terms of finite measurement resolution completely analogous to that encountered in condensed matter physics for lattice like systems. This would realize self-reference in the sense that cognitive part of the quantum state would represent quantum numbers characterizing the real part of the quantum state.

2. What about the quantum p-adics themselves characterizing incoming and out-going states in number theoretic vertices? There would be a conservation of number theoretical "momentum" characterized by logarithm of a rational in $\times_q$ vertex. Does this momentum have any concrete physical counterpart? Perhaps not since it would be associated with quantum p-adic degrees of freedom serving as correlates for cognition. In fact, the following argument suggest interpretation in terms of a finite dimension (finite by finite measurement resolution) of a Hilbert space associated with the orbit of a partonic 2-surface.

   (a) The prime factors of integer characterizing the orbit of a partonic 2-surface correspond naturally to braid strands for generalized Feynman diagrams. This suggests that the primes in question can be assigned with braid strands and would be indeed something new. The product of the primes associated with the particles entering $\times_q$ vertex would be same as the product of primes leaving this vertex. In the case of $+_q$ vertex the integer associated with each line would be same. One cannot identify these primes as p-adic primes since entire orbit of partonic 2-surface and therefore all braid strands are characterized by single common p-adic prime p.

   (b) Hilbert spaces with prime dimension are in a well-defined sense primes for tensor product, and any finite-dimensional Hilbert space decomposes into a product of prime Hilbert spaces. Hence the integer n associated with the line of a generalized Feynman diagram could characterize the dimension of the finite-dimensional Hilbert space (by finite measurement resolution) associated with it. The decomposition of n to prime factors would correspond to a decomposition of this Hilbert space to a tensor product of prime factors assignable to braid strands. This would define a direct Hilbert space counterpart for the decomposition of braid into braid strands and would be very natural physically and actually define the notion of elementarity. The basic selection rule for $\times_q$ vertex would be that the prime factors of incoming Hilbert spaces recombining to form Hilbert spaces of outgoing particles. For the $+_q$ incoming Hilbert spaces of dimensions n_1 and n_b would fuse to $n_1 + n_2$ dimensional direct sum. $a(b + c) = ab + ac$ would state that the tensor product with direct sum is sum of tensor products with direct summands. Therefore the two kind of vertices as well as corresponding vertices of quantum TGD would correspond to basic algebraic operations for finite-dimensional Hilbert spaces very natural for finite measurement resolution.

   (c) Could the different quantum versions of p-adic prime $l > p$ correspond to different direct sum decompositions of a Hilbert space with prime dimension to Hilbert spaces with prime dimensions appearing in the quantum pinary expansion in powers of p?

The coefficients of powers of p defined as products of quantum primes $l < p$ would be quantum dimensions and reflect effects caused by finite measurement resolution whereas the powers of p would correspond to ordinary dimensions. This decomposition would correspond to a natural decomposition to a direct sum by some natural criterion related to finite measurement resolution. For instance, power p^n could correspond to n-ary p-adic length scale. The decomposition would take place for every strand of braid.

The objection is that for algebraic extensions of rationals the primes of the extension can be algebraic number so that the corresponding Hilbert space dimension would be complex algebraic number. It seems that only the primes $l > p$ which do not split for the algebraic extension used (and thus label quantum p-adic number fields in the adele) can be considered as prime dimensions for the Hilbert spaces associated with braid strands. The latter option is more natural and would mean that the number theoretic evolution generating increasingly higher-dimensional algebraic extensions implies selection of both preferred p-adic primes and preferred prime dimensions for state spaces. One implication would be that the quantum Galois group assignable to given p-adic integer would in general be smaller for an algebraic extension of rationals than for rationals since only the non-splittable primes in its factorization would contribute to the quantum Galois group.

(d) As already discussed, the most plausible interpretation is that the pair of co-prime integers defining the quantum rational defines a pair of Hilbert space dimensions possibly assignable to fermions and bosons respectively. Interestingly, for the simplest infinite primes representing Fock states and mappable to rationals m/n the integers m and n could be formally associated with many-boson and many-fermion states.

(e) Because of multiplicative conservation law in $\times_q$ vertex quantum p-adic numbers does not have a natural interpretation as ordinary quantum numbers - say momentum components. The problem is that the momentum defined as logarithm of multiplicatively conserved number theoretic momentum would not be p-adic number without the introduction of an infinite-dimensional transcendental extension to guarantee the existence of logarithms of primes.

(f) If this vision is correct, the representation of ordinary quantum numbers as quantum Galois quantum numbers would be a representation in a state space formed by (quantum) state spaces of various quantum dimensions and thus rather abstract but quite possible in TGD framework. This is of course a huge generalization from the simple wave mechanical picture based on single Hilbert space but in spirit with abstract category-theoretical thinking about what integers are category-theoretically. The integers appearing as integers in the Cartesian factors of adeles would represent Hilbert space dimensions in the case of generalized Feynman diagrams. The arithmetic Feynman rules would be only a part of story: as such very abstract but made concrete by braid representation.

3. Note that the interpretation of $+$ and $\times$ vertices in terms of Hilbert space dimensions makes sense also in the real context whereas the further decomposition into direct sum in powers of p^n does not make sense anymore.

15.4.3 Number theoretical quantum numbers and hierarchy of Planck constants

What could be the TGD inspired physical interpretation of these mysterious looking Hilbert spaces possessing prime dimensions and having no obvious identification in standard physics context?

How the Hilbert space dimension relates to the value of Planck constant?

The first question is how the Hilbert space dimension assigned to a given line of a generalized Feynman diagram relates to the the value of Planck constant.

1. As already noticed, the decomposition of integer to primes would naturally correspond to its decomposition to braid strands to which one can assign Hilbert spaces of prime-valued

dimension $D = l$ appearing as factors of integer n. This suggests a Hilbert space is defined by wave functions in a set B_n with n points,. This Hilbert space naturally decomposes into a tensor product of Hilbert spaces with Hilbert spaces associated with point sets B_l containing l of points with $l|n$.

2. The only space of this kind that comes in mind relates to the proposed hierarchy of (effective) Planck constants coming as integer multiples of ordinary Planck constant. For the simplest option Planck constant $\hbar_n = n\hbar_0$ would correspond to a local (singular) covering of the imbedding space due to the n-valuedness of the time derivatives of the imbedding space coordinates as function of canonical momentum densities which is due to the huge vacuum degeneracy of Kähler action.

3. The discrete group Z_n would act as a natural symmetry of the covering and would decomposes a $Z_n = \prod_{l|n} Z_l^{e_l}$ and the orbits of Z_l in the covering would define naturally the sets B_l. Given prime l in the decomposition would correspond to an l-fold covering of a braid strand and to a wave function in this space.

4. The proposal for the hierarchy of Planck constants assumes that different sheets of this singular covering degenerate to single sheet at partonic 2-surfaces at the ends of CD. Furthermore, the integers n would decompose to to products $n = n_1n_2$ corresponding to directions of time-like braids along wormhole throat and along the space-like 3-surface at the end of CD defining by effective 2-dimensionality (strong form of holography) two space-time coordinates playing the role of time coordinate in the field equations for preferred extremals. Note that the information about the presence of covering would be carried at partonic 2-surfaces by the tangent space data characterized by the n_i-valued normal derivatives.

5. The simplest option is that Hilbert space dimension corresponds to Planck constant for a given line of generalized Feynman diagram. This would predict that in the multiplicative vertex also the values of Planck constants characterizing the numbers of sheets for many-sheeted coverings would satisfy the condition $n_3 = n_1n_2$. The assumption that the multiplicative vertex corresponds to the gluing of incoming lines of generalized Feynman diagram together along their ends seems however to require $n_1 = n_2 = n_3$. Furthermore, the identification of Hilbert space dimension as Planck constant is also inconsistent with the vision about book like structure of the imbedding space explaining the darkness as relative darkness due to the fact that only particles with the same value of Planck constant can appear in the same vertex [K25].

   The way out of the difficulty is to assume that the value of Planck constant $\hbar = n\hbar_0$ corresponds to $n = n_3 = n_1n_2$ or has n_3 as a factor. For $n = n_3$ the states with Hilbert space dimensions n_1 and n_2 are invariant under cyclic groups Z_{n_2} and Z_{n_1} respectively. For n containing n_3 as a genuine divisor analogous conditions would hold true.

6. p-Adic prime p would make itself manifest in the further decomposition of the l-dimensional Hilbert spaces to a direct sum of sub-Hilbert spaces with dimensions given by the terms $l_{n,q}p^n$ in the expression of l as quantum integer. The fact that the only prime ideal for p-adic integers is pQ_p should relate to this. It is quite possible that this decomposition occurs only for the p-adic sectors of the Hilbert adelic imbedding space.

   What suggests itself is symmetry breaking implying the decomposition of the covering A_n of braid strand to subsets $A_{n,m}$ with numbers of elements given by $\#_{n,m} = l_mp^m$ with l_m divisible only by primes $p_1 < p$. Wave functions would be localized to the sets $A_{n,m}$, and inside $A_{n,m}$ one would have tensor product of wave functions localized into the sets A_l with $l < p$ and $l|l_m$.

   Hilbert space dimensions would be now quantum dimensions associated with the quantum phase $exp(i\pi/l)$: this should be due to the finite measurement resolution and relate to the fact that one has divided away the hyper-finite factor N from the factor $M \supset N$.

   The index characterizing Jones inclusion [A172] [K25] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. Note that $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

If the above identification is correct, the conservation laws in $\times_q$ and $+_q$ vertices would give rather precise information about what can happen for the values of Planck constants in thes vertices. In $\times_q$ co-vertices Hilbert space-timensions would combine multiplicatively to give the common value of Planck constant and in $\oplus_q$ co-vertices additively. The phase transitions changing Planck constant, for instance for photons, are central for quantum TGD and the selection rules would not allow them only if they correspond to a formation of a Bose-Einstein condensate like state or its decay by $\times_q$- or $+_q$-vertex.

Could one identify the Hilbert space dimension as value of Planck constant?

It has been already seen that the identification of Hilbert space dimension with Planci constant it is not consistent with the idea that product vertex means that the lines of generalized Feynman graph are glued along their 2-D ends together. I did not however realize this when I wrote the first version of this section and I decided to keep the earlier discussion about the option for which Planck constants correspond to Hilbert space dimensions so that $n_3 = n_1 n_2$ holds true for Planck constants. The question was whether it could be consistent with the idea of dark matter as matter with non-standard value of Planck constant. By replacing "Planck constant" with "Hilbert space dimension" below one obtains a discussion giving information about the selection rules for Hilbert space dimensions.

1. In $\times_q$-vertex the Planck constants for the outgoing particles would be smaller and factors of incoming Planck constant. In $\times_q$ co-vertex Planck constant would increase. I have considered analogous selection rules already earlier. $\times_q$ vertex does not allow the fusion of photons with the ordinary value of Planck constant to fuse to photons with larger value of Planck constant.

   By conservation of energy the frequency of a photon like state resulting in the fusion is given by $f = \sum n_k f_k / N_{out} \prod_k n_k$ for $\hbar_k = n_k \hbar_0$, where N_{in} and N_{out} are the numbers of quanta in the initial and final state. For a common incoming frequency $f_k = f_0$ this gives $f/f_0 = \sum_k n_k/(N_{out} \prod_k n_k)$. If one assumes that spin unit for photon increases to $\prod_k n_k \hbar_0$ and spins are parallel one obtains from angular momentum conservation $N_{out} \prod_k n_k = N_{in} \sum n_k$ giving $N_{out} = \prod_k n_k N_{in} / \sum n_k = n^{N_{in}}/N_{in} n$, which in turn gives $f/f_0 = 1/N_{in}$. This looks rather natural.

   In the presence of a feed of $r = \hbar/\hbar_0$¿1 particles $\times_q$ vertex could lead to a phase transition generating particles with large values of Planck constant. Large values of Planck constant are in a key role in TGD based model of living matter since Compton lengths and other quantum scales are proportional to $\hbar$ so that large values of $\hbar$ make possible macroscopic quantum phases. The phase transition leading to living matter could be this kind of phase transition in presence of feed of $r > 1$ particles.

2. For $+_q$ co-vertex $r = \hbar/\hbar_0$ could be additive and for incoming photons with same frequency and Planck constants $\hbar_k$ the outgoing state with Planck constant $\sum_k \hbar_k$ energy conservation is guaranteed if the frequency stays same. This vertex would allow the transformation of ordinary photons to photons with large Planck constant, and one could say that effectively the photons fuse to form single photon. This is consistent with the quantization of spin since the unit of spin increases. For this option the presence of particles with ordinary value of Planck constant would be enough to generate particles with $r > 1$ and this in turn could lead to a the phase transition generation living matter.

3. One can of course ask whether it should be $r - 1 = \hbar/\hbar_0 + 1$, which corresponds to the integer n. For this option the third particle of $+_q$ vertex with two incoming particles with ordinary Planck constant would have ordinary Planck constant. For $\times_q$ vertex containing two incoming particles with $r = n$, $n = 1$ ($n = 2$), also the third particle would have $n = 1$ ($n = 2$). $\times_q$ and $+_q$ vertices could not generate $n > 1$ particles from particles with ordinary Planck constant. The phase transition leading from inanimate to living matter would require

$n > 1$ states as a seed (one has $2 + 2 \rightarrow 3$ for $+_q$ vertex). A quantum jump generating a CD containing this kind of particles could lead to this kind of situation.

4. These selection rules would mean a deviation of the earlier proposal that only particles with same values of Planck constant can appear in a given vertex [K25]. This assumption explains nicely why dark matter identified as phases with non-standard value of Planck constant decouples from ordinary matter at vertices. Now this explanation would be modified. If $\times_q$ vertex contains two particles with $r = n + 1$ for $r = n$ option ($r = 1$ or 2 for $r = n + 1$ option), also the third particle has ordinary value of Planck constant so that ordinary matter effectively decouples from dark matter. For $+_q$ vertex the decoupling of the ordinary from dark matter occurs for $r = n + 1$ option but not for $r = n$ option. Hence $r = n + 1$ could explain the virtual decoupling of dark and ordinary matter from each other. The assumption that Planck constant is same for all incoming lines and corresponds to $n_3 = n_1 n_2$ defines however much more plausible option.

What happens in phase transitions changing the value of Planck constant?

The phase transitions changing the value of Planck constant are in a central role in TGD inspired quantum biology. The typical phase transition of this kind would change the Planck constant of photon. This phase transition would formally correspond to a 2-vertex changing the value of Planck constant. Can one pose selection rules to the change of Planck constant? By the above assumptions both the incoming and outgoing line correspond to Hilbert space dimension which is a factor of the integer defining Planck constant. If the value of the Hilbert space dimension is not changed in the process, the incoming and outgoing Planck constants must have this dimension as a common factor.

15.4.4 What is the relation to infinite primes?

Already quantum p-adics would mean a dramatic generalization of number concept by assigning to rationals ane even algebraic numbers Hilbert spaces and their states. Quantum adeles would mean a further generalization of number concept by gluing together reals and Hilbert space variants of p-adic number fields.

TGD leads also to another generalization of number concept based on the hierarchy of infinite primes [K70]. This generalization also leads to a generalization of real number in the sense that one can construct infinite number of real units as infinite rationals which reduce to units in real sense. This would mean that space-time point has infinitely complex number theoretic anatomy not visible at the level of real physics [K72].

The possibly existing relationship between these generalizations is of course interesting. Infinite primes can be mapped to polynomial primes and this means that one can assign to them algebraic extensions of rationals and corresponding Galois groups and in [K82] I discussed a conjecture that the elements of these Galois groups could be represented as symplectic flows assignable to braids which emerge naturally as counterparts of partonic 2-surfaces in finite measurement resolution. This would suggest a possible relationship.

The construction of infinite primes relies on the product $X = \prod_p p$ of finite primes interpreted physical as analog of Dirac vacuum with all negative energy states filled. Simplest infinite primes are constructed by kicking away fermions from this vacuum and by adding also bosons labeled by primes. One obtains also the analogs of bound states as infinite primes which can be mapped to irreducible polynomials. The roots of the polynomial code for the infinite prime and the algebraic extension. The infinite primes corresponding to nth order polynomials decompose to products of n simplest infinite primes of algebraic extension so that the corresponding Galois group emerges naturally.

The construction can be repeated endlessly by taking the infinite primes of the existing highest level and forming the product X of them and repeating the process. What these means that the many-particle states of the previous level define single particle states of the new level. One can map these infinite primes to polynomial primes for polynomials of several variables. Also this hierarchy might allow generalization obtained by assigning to infinite primes the orbits of their Galois groups. The earlier considerations [K44] suggest strongly a reduction of the description to the lowest level and involving only algebraic numbers.

What do we understand about infinite primes?

Let us first try to summarize what we understand about infinite primes. What seems very natural is the postulate that arithmetic QFT associated with infinite primes conserves multiplicative number theoretic momenta defined by ordinary primes with separate conservation law for each prime. This law would hold for $\times_q$ vertices very naturally whereas for $+_q$ vertices it would be broken. Recall that these two vertices correspond to the TGD counterparts of 3-vertices for Feynman diagrams and stringy diagrams respectively and also to tensor product and direct sum.

1. What seems clear is that infinite prime characterizes an algebraic extension of rationals (or of its extension) in the case that infinite primes is defined in terms of finite primes of extension. Infinite prime dictates also the p-adic primes which are possible and appear in the quantum adele assignable to infinite prime.

2. The integer exponents of ordinary primes appearing in the infinite and finite part of the simplest lowest level infinite prime could define infinite number of conserved number theoretic momenta, one for each prime p and having $log(p)$, p prime, as a unit. Separate conservation follows from the algebraic independence. These number theoretic momenta do not make sense p-adically, which means that in p-adic context the multiplicative form of the conservation law is the appropriate one. Therefore it is appropriate to speak of multiplicative momenta. Therefore the relationship with ordinary additively conserved momenta does not lok plausible.

   Arithmetic QFT interpretation allows also to interpret the numbers n_p in p^{n_p} as particle numbers assignable to bosonic quanta and fermionic quanta in the case of the simplest infinite primes with "small part" representing fermions kicked out from the Dirac sea possibly accompanied by bosonic quanta. The conservation law at $\times_q$ vertices would mean conservation of total particle numbers assignable to primes p.

3. For the simplest primes at the lowest level identifiable as linear polynomials with integer coefficients there are two separate integers defining number theoretic momenta. The first integer corresponds to the finite part of infinite prime and the second one to the finite part of the infinite prime to which one assigns number theoretic fermions. These two parts are separately conserved. Since the integers have no common prime factors, one can also speak about rational valued multiplicative number theoretic momentum. The physical interpretation for the absence of common factors would be that given mode cannot simultaneously containing and not contain fermionic excitation. For higher irreducible polynomials of order n interpreted in terms of bound states there are $n+1$ integers defining a collection of number theoretic momenta. For the representation as a monic polynomial one has a collection of n rational valued number theoretic momenta.

4. The notion of multiplicative number theoretic momentum generalizes.

   (a) At the second level of the hierarchy ordinary primes are replaced with prime polynomials $P_n(x)$ of single variable. At the n^{th} level they are prime polynomials $P_n(x_1, ..., x_{n-1})$ of $n-1$ variables. The value of the number theoretic momentum at n^{th} s level can be said to be a polynomial $P_n(x_1, ..., x_{n-1})$ rather than integer.

   (b) This looks very abstract but can be concretized. For instance, each coefficient of $P_n(x, y)$ at second level as polynomial of y defines a polynomial $P_k(x)$ at the first level and $P_k(x)$ is characterized by a collection of number theoretic momenta defined by its integer coefficients in the representation as a polynomial with integer coefficients. Therefore $P_k(x)$ can be identified as the collection of $k+1$ integer coefficients or k rational coefficients in the monic representation identified as number theoretic momenta for a k-particle state. $P_n(x, y)$ in turn corresponds to a collection of n many-particles states with i^{th} one containing k_i particles, $i = 1, ...n$. The interpretation in terms of n-braid with braid strands decomposing to k_i braid strands is natural and conforms with the fractality of TGD Universe.

   (c) This example allows to deduce the number theoretic interpretation of the polynomial at the n^{th} level and one can continue this abstraction hierarchy ad infinitum. Eventually each prime at a given level of hierarchy reduces to a collection of number theoretic momenta defined by ordinary integers grouped in a manner characterized by the infinite prime. Physically this would characterize how these number theoretic elementary particles group to particles at the first level, these to particles at second level, and so on.

(d) The possibility to express the irreducible polynomial as a product of first order polynomials with zeros which algebraic numbers gives for the bound state a representation as free many-particle state but with number theoretic momenta which are algebraic rationals in algebraic extension of rationals. These number theoretic momenta can be also complex and therefore do not allow interpretation as Hilbert space dimensions. This decomposition is analogous to a decomposition of hadron to quarks. The rational coefficients expressible in terms of the roots of the polynomial code for Galois invariants analogous to the observables assignable to hadrons and accessible to the experimenter.

5. The basic conservation law of arithmetic QFT and of TGD would be that the multiplicative number theoretic momenta labelled by finite primes are separately conserved in $\times_q$ vertices but not in $+_q$ vertices. The conservation number theoretic quantum numbers allows the interpretation of Hilbert space dimensions in terms of the hierarchy of Planck constants, and this leads to a proposal that infinite primes code the pairs of finite integers with no common factors assignable to the pairs of time-like and space-like braid strands.

   If one takes seriously the notion of number theoretic fermion, one could assign to space-like braid strands only bosonic excitations and to time-like braid strands fermion and possibly also bosonic excitations. The interpretation could be in terms of the super-conformal algebras containing both fermionic and bosonic generators. The hierarchy of infinite primes would correspond to a hierarchy of braids containing lower level braids as their strands as suggested already earlier [K82]. What would be new would be a concrete assignment of primes to braid strands and detailed identification in terms of time-like and space-like braids.

   This kind of assignment would mean a rather dramatic step of progress in the understanding of the complexities of generalized Feynman diagrams. One not completely settled old question is what selects the p-adic prime assignable to given partonic 2-surface.

This is the stable looking part of the vision about infinite primes, and any attempt to relate it to quantum p-adics and quantum adeles should respect this picture.

Hyper-octonionic primes correspond to p-adic primes in extension of rationals

The earlier interpretation hyper-complex and appropriately defined quaternionic and octonionic generalizations is in terms of standard model quantum numbers [K26]. It seems that also this identification survives under the selective pressures by new ideas but that one cannot replace hyper-complex primes with their infinite counterparts. Rather, hyper-complex prime generalizes p-adic prime as a preferred prime by replacing ordinary integers with hyper-complex integers. The definition of infinite primes in quaternionic and octonionic context is plagued by the problems caused by non-commutativity and associativity so that the conclusion is well-come.

1. The solutions of modified Dirac equation suggest the interpretation of the M^2 projections of four-momenta as "hyper-complex" primes or perhaps more realistically. their integer multiples. These momenta are conserved additively rather than multiplicatively at vertices to which $\times_q$ is assigned and only their exponents - naturally phase factors - would be conserved multiplicatively.

2. Could this identification generalize from hyper-octonionic primes to hyper-octonionic infinite primes? This does not seem to be the case. The multiplicative conservation in $\times_q$ vertices for number theoretic momenta is in conflict with additive conservation for ordinary quantum numbers. Additive conservation is also in conflict with interpretation in terms Hilbert space dimensions allowing concretization in terms of the hierarchy of Planck constants. Of course, hyper-complex Hilbert space dimension does not make sense either.

3. One must remember that there are many kinds of primes involved and a little list helps to see what the correct interpretation for hyper-complex primes could be.

(a) There are the primes l appearing in the decomposition of infinite primes and having interpretation in terms of Hilbert space dimensions. The conservation of multiplicative number theoretical momenta is natural at $\times_q$ vertices.

(b) There are the p-adic primes p, and on basis of p-adic mass calculations it is this prime to which it is natural to assign additively conserved momenta. p characterizes the "active" sector of adeles and therefore also the various quantum variants of the prime l in which quantum primes $p_1 < p$ appear as factors. p characterizes partonic 2-surface.

(c) The Abelizanization of the quantum Galois group assignable to prime l decomposes into prime factors Z_{p_2} and the phases $exp(i2\pi/p_1)$ might provide cognitive representations in finite measurement resolution for various standard model quantum numbers.

4. The only reasonable interpretation seems to be that the hyper-complex momenta and possible other quantum numbers assignable to them correspond to p-adic prime p for rationals or for an algebraic extension of rationals to the ring hyper-complex rationals. The failure of field property implies that the inverse of hyper-complex number fails to exist when it defines a light-like vector of M^2. This has however a concrete physical interpretation and light-like hyper-complex momentum for a massless state is massless only when the momentum of the state transverse to M^2 vanishes so that also propagator defined by M^2 momentum diverges.

What the identification of M^2 momenta as hyper-complex integers really means, deserves some comments.

1. Suppose that particle's p-adic mass squared is of form $m^2 = np$ as predicted by p-adic mass calculations. Assume that m^2 corresponds to M^2 momentum squared with preferred M^2 characterizing given causal diamond CD. Assume also that total M^4 mass squared vanishes in accordance with the idea that all states - even those representing virtual particles - carried by wormhole throats are massless. In accordance with the adelic vision, assume that the prime p does not split in the algebraic extension of rationals used (simplest extension would be $Q[\sqrt{-1}]$). This requires $p \ mod \ 4 = 3$ in accordance with Mersenne prime hypothesis. The idea is that p does not split for ordinary algebraic extension but splits in the ring of hyper-complex numbers.

2. The preferred plane $M^2 \subset M^4$ corresponds to a preferred hyper-complex plane of complexified (by commuting imaginary unit i) hyper-octonionic space M^8. M^2-momentum has therefore purely number theoretic interpretation being due to the slitting of $M^2 = np$ to a product of hypercomplex integer $N = N_0 + eN_z$ and its conjugate $N_0 - eN_z$). The hyper-complex imaginary unit $e = iI$ satisfying $e^2 = 1$ and $I^2 = -i^2 = -1$ would correspond to z-axes of M^2. Here is I is the preferred octonionic imaginary unit and i an imaginary unit commuting with it. One could say that 2-D particle momentum emerges via the emergence of hyper-complex extension of rationals of their extension. This would also generalize to quaternions and one could say that M^4 momentum emerges via extension of rationals to hyper-quaternions.

3. M^2 momentum squared would satisfy $P_0^2 - P_z^2 = (P_0 - eP_z)(P_0 + eP_z) = np$. The prime p does not split in the algebraic extension of rationals used but splits in the ring of hyper-complex numbers. Assume first $n = 1$. In this case the splitting of $p \ mod \ = 3$ ($p \ mod \ = 1$) to $p = (p_0 + ep_z)(p_0 - ep_z)$ implies p_0 is even (odd) and p_z is odd (even). For $n > 1$ one must have $(n_0 - en_1)(n_0 + en_1) = n$ and similar conditions apply to n so that one would have for M^2 momentum $P_0 + eP_z = (n_0 \pm en_z)(p_0 \pm ep_z)$.

4. Momentum components are hyper-complex integer multiples of hyper-complex prime so that that the allowed momenta would form an ideal of hyper-complex numbers. This is mathematically very nice but might be quite too strong a condition physically although it is typically encountered in systems in which particle is enclosed in box. Now the box would correspond to CD with periodic boundary conditions at the ends of CD for the modified Dirac equation. One could consider also a weaker condition for with the integer n is replaced with a rational (m/n) such that neither m nor n contains p as a prime factor.

5. The peculiar looking prediction would be that M^2 momentum cannot be purely time-like. In other words, the particle cannot be at rest M^2. Observer for which CD defines the rest system could not perform a state function reduction leading to a situation in which the particle is at rest with respect to the observer! In fact, this kind of situation is encountered also for particle in box since boundary conditions do not allow constant mode. If one recalls that all particles would be massless in M^4 sense, this condition does not look so strange.

Infinite primes and Hilbert space dimensions

Arithmetic QFT picture would strongly suggests that the number theoretic momenta at the lowest level are conserved in $\times_q$ vertices at least. For $+_q$ vertices the conservation cannot hold true. The conservation could mean that the total number of powers of given prime in state is same for positive and negative energy states.

Of course, much richer spectrum of conservation laws can be imagined since one could require similar conservation laws also at the higher levels of hierarchy, where various number theoretic momenta correspond to numbers prime polynomials at lower level present in the state. The physical interpretation would be that the numbers of bound states particles are conserved meaning that these particles can be regarded as stable. On physical grounds this kind of conservation laws can be only approximate.

1. Could infinite primes label infinite-dimensional prime Hilbert spaces as finite primes do? Could the interpretation for the object $X = \prod_p p$ be in terms of a tensor product of all prime-dimensional Hilbert spaces. Infinite primes with positive finite part would have interpretation as direct sums of this space and finite integer-dimensional Hilbert space. When the finite part of the infinite prime is negative the interpretation would not be so straightforward, and this option does not look attractive.

2. A much more plausible option is that infinite prime at the first level defines an algebraic extension of rationals (or of its extension) and that this gives rise to a collection of norm for algebraic extension induced by complex norm. As a matter fact, these points at which this norm vanishes might have interpretation as complex coordinates for a corresponding braid strand in n-strand bound state braid in preferred complex coordinates for the partonic 2-surface. A possible geometric interpretation for these points inspired by the notion of dessins d'enfant is that the partonic 2-surface as an abstract Riemann surface representable as a covering of sphere becomes singular at these points as several sheets of covering co-incide.

3. The infinite primes of the lowest level of the hierarchy formally representing Fock states of free bosons and fermions can be mapped to rationals. These rationals could define pairs of Hilbert space dimensions assignable to bosonic and fermionic parts of the state and could this allow identification as quantum p-adic integer in each sector of the adele and the identification in terms of integer dimension in the real sector of quantum adeles. The fact that the two integers have no common factors would only mean that given mode cannot both contain and not contain fermionic excitation.

   One could even consider the possibility of concrete assignment of the first dimension in terms of fermionic braid strands with bosonic excitations and second dimension in terms of purely bosonic braid strands. This interpretation is very natural since the super-conformal algebras creating states have both purely bosonic and purely fermionic generators. These braids could correspond to space-like and time-like (actually light-like) braids having their ends at partonic 2-surfaces.

   The Galois groups associated with primes appearing as factors of the primes would correspond naturally to additional internal degrees of freedom. This identification makes sense also for the infinite primes represented by irreducible polynomials since the coefficients of the polynomial representable in terms of the roots of polynomials define rationals having interpretation as number theoretic momenta. Therefore the interpretation in terms of Hilbert space dimensions makes sense when rationals are interpreted as pairs of dimensions for Hilbert spaces.

4. What about the infinite primes representing bound states and mappable to irreducible polynomials with rational coefficients and defining polynomial primes characterized by a collection of roots [K44]. These roots define an algebraic extension of rationals and this suggests that the quantum adele associated with the infinite prime in question is defined accordingly. The infinite primes mappable to n^{th} order monic polynomials would have interpretation as many particle states consisting of single particle states which correspond to algebraic number rather than rational. The rational coefficients of the monic polynomial would define the rationals defining pairs of Hilbert space dimensions.

5. The natural identification for the Hilbert spaces in question would be in terms of the singular local coverings of imbedding space associated with the hierarchy of Planck constants suggested to emerge from the vacuum degeneracy of Kähler action. The integer n decomposing to primes would correspond to sub-braids labeled by prime factors l of n and consisting of l strands in the l-fold sub-covering.

The consistency with the quantum adeles would force the following highly speculative picture. Main justification comes from its internal consistency and consistency with generalize Feynman graphs.

1. Infinite prime (integer, rational) defines the algebraic extension used and the allowed quantum p-adic number fields contributing as factors to the corresponding quantum adele. p-Adic primes, which can be also algebraic primes if one starts from extension of rationals, by definition do not split in the algebraic extension. Infinite primes assignable to particle states obey the conservation of multiplicative number theoretic momenta and define naturally collections of pairs if Hilbert space dimensions assignable to the particles and decomposing to primes l assignable to braid strands. The integers characterizing the rational defining number theoretic momentum correspond to time-like and space-like braid strands and only the time- or space-like strand carries fermionic quantum numbers.

2. These Hilbert spaces have a natural interpretation in terms of the hierarchy of Planck constants realizable in terms of local singular coverings of the imbedding space forced by the enormous vacuum degeneracy of Kähler action.

3. Hyper-complex primes are identifiable as generalizations of p-adic primes and have nothing to do with infinite primes. They could code for standard model quantum number.

4. The quantum Galois quantum numbers assignable to primes l for given p-adic prime p and appearing in the infinite prime characterizing the state would provide a cognitive representation of the standard model quantum numbers.

5. Mersenne primes and primes near powers of 2 and $p = 2$ also should be selected as a p-adic prime in this manner.

6. The basic uncertain aspect of the scenario is whether the notion of quantum p-adic with coefficients in quantum pinary expansion satisfying only the condition $x_n < p^N$ for $N > 1$, with N dictated by the pinary cutoff, makes sense. Physically $N > 1$ is very natural generalization. Most of the preceding considerations remain intact even if $N = 1$ is the only internally consistent option. What is lost is the representation of quantum numbers using quantum Galois group and the crazy proposal that quantum Galois group could be isomorphic to AGG.

This is only the simplest possibility that I can imagine now and reader is encouraged to imagine something better!

The relationship between the infinite primes of TGD and of algebraic number theory

While preparing this chapter I experienced quite a surprise as I learned that something called infinite primes emerges in algebraic number theory [A5]. Infinite primes in this sense looked first to me like a heuristic concept characterizing norms for algebraic extensions of rationals induced by the complex norm for the imbeddings of the extension to complex plane. The nomenclature is motivated by the analogy with p-adic norms defined by algebraic primes. It however turns out that there is a close connection with infinite primes at the first level of the hierarchy.

1. The embeddings (ring homomorphisms) of Galois extension to complex plane induce a collection of norms induced by the complex norm. The analogy with p-adic norms labelled by primes serves as a partial motivation for calling these norms infinite primes. The imbeddings are induced by the imbeddings of the roots of an irreducible monic polynomials $P_n(x) = x^n + ...$ with rational coefficients, which defines a polynomial prime so that infinite primes in the sense of algebraic number theory correspond to a polynomial primes.

2. The imbeddings (ring homomorphisms) of the extension of K in C an be defined to those reducing to imbeddings in R and those not. The imbeddings to R correspond in one-one manner to real roots and complex imbeddings come in pairs corresponding to complex root and its conjugate. The norm is defined as $|z - z_k|$, where z_k is the root. The number of imbeddings and therefore of norms is $r = r_1 + 2r_2$, where r is the the degree of the extension K/Q and also the degree of its Galois group for Galois extensions (defined by polynomials with rational coefficients).

3. Also in TGD framework the infinite primes at the lowest level of hierarchy can be mapped to irreducible monic polynomials of single variable: at n^{th} level polynomials of n variables is required. Now however also polynomials $P_1(x)$, whose roots are rationals and have interpretation in terms of free Fock states, are included. Note that the replacement of the variable z with $z - m/n$ shifts the roots of a monic polynomial by m/n so that the corresponding algebraic extension is not modified. For the simplest infinite primes the norm would correspond to $|z - m/n|$. Therefore infinite prime indeed characterizes the algebraic extension and its imbeddings and the "real" factor of quantum adeles is identifiable with this algebraic extension endowed with any of these norms.

15.4.5 What selects preferred primes in number theoretical evolution?

Preferred p-adic length scales seem to correspond to primes near powers of two, in particular Mersenne primes. The proposed explanation is that number theoretic evolution as emergence of higher-dimensional extensions of rationals and also of p-adics somehow selects Mersenne primes as fittest. But what fitness could mean? This is the question. The answer to the question might be banally simple. The fittest primes could be stable in the process of generation of algebraic extensions! Stability means very concretely that primes do not split into products of primes of the extension and therefore can define p-adic primes for quantum adeles! Number theoretic evolution by algebraic extensions would gradually kill p-adic primes.

The splitting to primes need not be unique (if it is one speaks of principal ideal domain). For instance, in $Q[\sqrt{-5}]$ for which factorization to algebraic primes is not unique (but is unique to prime ideals): $6 = 2\times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. In this kind of situation it is better to speak about prime ideals since this makes the splitting unique for what is known as Dedekind domains. The ideal class group characterizes the non-uniqueness of splitting to primes and consists of equivalence classes of fractional ideals (essentially integers defined by some fixed integer) under equivalence defined by multiplication by a rational of extension. The non-uniqueness of the factorization is characterized by so called ideal class group [A49].

Are Mersenne primes especially stable against splitting to algebraic primes? Generally, for an especially large set of algebraic extensions, or for some special but physically important extensions? The cautious guess is that Mersenne primes could be special in the sense that the set of (physically relevant) algebraic extensions for which they do not split is especially large. This in turn would raise the infinite primes defining these special algebraic extensions in a special physical role. A possible physical interpretation for these infinite primes would be in terms of bound states. Therefore the stability of Mersenne primes could be translate to the stability of the bound states for which Mersenne primes are stable.

quadratic fields $Q[\sqrt{d}]$ are the simplest algebraic extensions of rationals since they correspond to second order prime polynomials and are also relatively well-studied so that one can look them at first. For $Q[\sqrt{d}]$ there are general results about the splitting of primes.

1. Quite generally, given prime p can be inert, split to a product of two distinct prime ideals, or can be ramified. The so called discriminant D characterizes the situation: for $d \ mod \ 4 = 1$ equals to $D = d$ and otherwise to $D = 4d$.

2. If p - say M_k - is an odd prime not dividing d, p splits only if one has

$$D \ mod \ p = x^2$$

   In this case one has $(D/p) = 1$, where (D/p) is Legendre symbol having values in the set $\{0, 1, -1\}$. $(D/p) = -1$ means stability of p against slitting.

   Legendre symbol is a multiplicative function in the set of integers D meaning that if p splits under D_1 and D_2 it splits also under D_1D_2 , and if p does not split under D_1 nor under D_2 it splits under D_1D_2. The multiplicative property implies $(4p_1/p) = (2/p)^2 \times (p_1/p) = (p_1/p)$. It is obviously enough to check whether the splitting occurs for primes p_1. Non-splitting prime p_1 gives rise to a set of non-splitting integers obtained by multiplying p_1 with any splitting prime. Also odd powers of non-splitting p_1 define this kind of sets.

3. Also the following properties of Legendre symbol are useful. One has $(D/p) = (p/D)$ if either $D \ mod \ 4 = 1$ or $p \ mod \ 4 = 1$ holds true. $D \ mod \ 4 = 3$ and $p \ mod \ 4 = 3$ one has $(D/p) = -(p/D)$. One has also $(-1/p) = (-1)^{(p-1)/2}$ and $(2/p) = (-1)^{(p^2-1)/8}$.

4. If the p-adic number fields, which do not allow $\sqrt{-1}$ as ordinary p-adic number are in special role then there might be hopes about the understanding of the special role of Mersenne primes. Mersenne primes are also stable for Gaussian integers and quadratic extensions $Q[\sqrt{\pm d}]$ of rationals defined by positive integers d, which are products $d = d_1d_2$ of two integers. d_1 factorizes to a product of primes $p_1 \ mod \ 4 = 1$ splitting M_k, and d_2 is a product of an odd number of primes $p_1 \ mod \ 4 = 1$ not splitting M_k.

5. One must also distinguish between the algebraic extensions of rationals and finite dimensional extensions of p-adic numbers (also powers e^k, $k < p$ define finite-dimensional extension). For instance, one can consider a quadratic extension $Q[\sqrt{-1}]$ for rationals defining similar extension for the allowed p-adic primes $p \ mod \ 4 = 3$ and fuse it with a quadratic extension $Q[\sqrt{d}]$ for which $d \ mod \ 4 = 1$ holds true. For adeles the extension of rationals and the extensions of p-adic numbers can be said to separate.

Some special examples are in order to make the situation more concrete.

1. A good example about physically very relevant quadratic extension is provided by Gaussian integers, which correspond to Galois extension $Q[\sqrt{-1}]$ [A37]. $p = 2$ splits as $2 = (1+i)(1-i) = -i(1+i)^2 = i(1-i)^2$ and the splitting to primes is non-unique. The splitting to prime ideals is however unique so that $p = 2$ is not ramified.

   The primes $p \ mod \ 4 = 1$ split also as stated by Fermat's theorem of two squares. Mersenne primes satisfy $p \ mod \ 4 = 3$ but some additional criterion is needed to select them. Primes $p \ mod \ 4 = 3$ do not and cannot define p-adic primes appearing in quantum adele for Gaussian rationals. Note that for $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as p-adic number, which might cause problems in the p-adic formulation of quantum mechanics. These observations suggest that p-adic primes $p \ mod 4 = 1$ suffer extinction when $\sqrt{-1}$ emerges in the number theoretic evolution and only the primes $p \ mod 4 = 3$ remain. One could also start from the extension $Q[\sqrt{-1}]$ rather than rationals as the role of $\sqrt{-1}$ in quantum theory suggests so that the primes $p \ mod 4 = 3$ would be the only allowed quantum p-adic primes.

2. for $Q[\sqrt{2}]$ for which 2-adicity would not be possible. What happens for Mersenne primes? One can write $M_3 = 7 = (\sqrt{2}+3)(-\sqrt{2}+3)$ where $3 \pm \sqrt{2}$ is an algebraic integer as a root of a monic polynomial $P(x) = x^2 - 6x + 7$ so that the splitting of M_3 occurs in $Q[\sqrt{2}]$. Therefore it seems that the absence of $\sqrt{2}$ and allowance of 2-adicity is necessary for Mersenne-adicity. This conforms with the naive physical picture that the p-adic scales defined by Mersennes are in excellent approximation n-ary 2-adic length scales.

One should check whether the extension defined by $\sqrt{2}$ is somehow special as compared to the extensions defined by odd primes. Certainly the fact that this prime is the only even prime makes it rather special. It allows extension with $\sqrt{-1}$ and p-adic extension allowing all square roots except those of 2 is spanned by four square roots unlike similar extensions for other p-adic numbers fields which require only two square roots.

3. Suppose $D = p_1$ with $p_1 \ mod \ 4 = 1$. For $p = M_k$ quadratic resiprocity implies that the condition is equivalent with $M_k \ mod \ p_1 = x^2$. Neither the extensions $Q[\sqrt{p_1}]$ nor $Q[\sqrt{-p_1}]$ induce splitting of M_k for $p_1 \ mod \ 4 = 1$. For $M_3 = 7$ and $p_1 \in \{5, 13, 17\}$ no splitting of M_3 takes place but for $p_1 = 29$ splitting occurs. This suggests that there is no general rule guaranteeing the stability of Mersenne primes in this case.

4. Suppose $D = p_1 \ mod \ 4 = 3$. One has $(4p_1, M_k) = (p_1, M_k)$ by the multiplicative character of the Legendre symbol. Quadratic resiprocity gives now $(p_1, M_k) = -(M_k, p_1)$ so that splitting occurs for M_k only if it does not occur for p_1. If splitting occurs for p_1 it does not occur for $-p_1$ and vice versa. $p_1 = 7$ and $M_2 = 3$ serve as a testing sample. One has $(3, 7) = 1$ so that the splitting of $M_2 = 3$ takes place for $Q[\sqrt{7}]$ but not for $Q(\sqrt{-7})$ and the splitting of $M_3 = 7$ takes place for $Q[\sqrt{-3}]$ but not for $Q(\sqrt{3})$.No obvious general rule seems to hold.

15.4.6 Generalized Feynman diagrams and adeles

The notion of Hilbert adeles seems to fit nicely with the recent view about generalized Feynman diagrams. The basic heuristic idea is the idea about fusion of physics in various number fields. p-Adic mass calculations lead to the conclusion that elementary particles are characterize by p-adic primes and inside hadron quarks obeying different effective or real p-adic topologies are present. One can speak about real and p-adic space-time sheets and real and p-adic spinors and also WCW has real and p-adic sectors. There is a hierarchy of algebraic extensions of rationals and presumably of also p-adic numbers. Even more general finite-dimensional extensions containing for instance Neper number e and its roots are also possible and involve extensions of p-adic numbers.

At the level of Feynman graphs this means that different lines correspond to different p-adic topologies and I have already proposed how this could give rise p-adic length scale hypothesis when the Feynman amplitudes in the tensor product of quantum variants p-adic number fields are mapped to reals by canonical identification [K31]. Rational or even more general entanglement between different number fields would be essential.

The vertices of generalized Feynman diagrams for different incoming p-adic number fields could be multi-p p-adic objects in quantum sense involving powers expansions in powers of integer n decomposed to product of powers of quantum primes associated with its factors with coefficients not divisible by the factors. An alternative option is that vertices are rational numbers common to all number fields serving as entanglement coefficients. A third option is that they are real numbers in corresponding tensor factor. One should also formulate symmetries in p-adic sectors and the simplest option is that symmetries represented as affine transformations simply reduce to products of the symmetries in various p-adic sectors of the imbedding space.

The challenge is to formulate all this in a concise and elegant manner. It seems that adeles generalized to Hilbert adeles might indeed provide this formulation. The naive basic recipe would be extremely simple: whenever you have a real number, replace it with Hilbert adele. You can even replace the points of Hilbert spaces involved with corresponding Hilbert spaces! One could replace imbedding space, space-time surfaces, and WCW as well as imbedding space spinors and spinor fields and WCW spinors and spinor fields with the hierarchy of their Hilbert adelic counterparts obtaining in this manner what might be interpreted as cognitive representations.

15.5 Quantum Mathematics and Quantum Mechanics

Quantum Mathematics (QM) suggests that the basic structures of Quantum Mechanics (QM) might reduce to fundamental mathematical and metamathematical structures, and that one even consider the possibility that Quantum Mechanics reduces to Quantum Mathematics with mathematician included or expressing it in a s manner: QM=QM!

The notes below were stimulated by an observation raising a question about a possible connection between multiverse interpretation of quantum mechanics and quantum mathematics. The heuristic idea of multiverse interpretation is that quantum state repeatedly branches to quantum states which in turn branch again. The possible outcomes of the state function reduction would correspond to different branches of the multiverse so that one could save keep quantum mechanics deterministic if one can give a well-defined mathematical meaning to the branching. Could quantum mathematics allow to somehow realize the idea about repeated branching of the quantum universe? Or at least to identify some analog for it? The second question concerns the identification of the preferred state basis in which the branching occurs.

Quantum Mathematics replaces numbers with Hilbert spaces and arithmetic operations + and $\times$ with direct sum $\oplus$ and tensor product $\otimes$.

1. The original motivation comes from quantum TGD where direct sum and tensor product are naturally assigned with the two basic vertices analogous to stringy 3-vertex and 3-vertex of Feynman graph. This suggests that generalized Feynman graphs could be analogous to sequences of arithmetic operations allowing also co-operations of $\oplus$ and $\otimes$.

2. One can assign to natural numbers, integers, rationals, algebraic numbers, transcendentals and their p-adic counterparts for various prime p Hilbert spaces with formal dimension given by the number in question. Typically the dimension of these Hilbert spaces in the ordinary sense is infinite. Von Neuman algebras known as hyper-finite factors of type II_1 assume as a convention that the dimension of basic Hilbert space is one although it is infinite in the standard sense of the word. Therefore this Hilbert space has sub-spaces with dimension which can be any number in the unit interval. Now however also negative and even complex, quaternionic and octonionic values of Hilbert space dimension become possible.

3. The decomposition to a direct sum matters unlike for abstract Hilbert space as it does also in the case of physical systems where the decomposition to a direct sum of representations of symmetries is standard procedure with deep physical significance. Therefore abstract Hilbert space is replaced with a more structured objects. For instance, the expansion $\sum_n x_n p^n$ of a p-adic number in powers of p defines decomposition of infinite-dimensional Hilbert space to a direct sum $\oplus_n x_n \otimes p^n$ of the tensor products $x_n \otimes p^n$. It seems that one must modify the notion of General Coordinate Invariance since number theoretic anatomy distinguishes between the representations of space-time point in various coordinates. The interpretation would be in terms of cognition. For instance, the representation of Neper number requires infinite number of pinary digits whereas finite integer requires onlya finite number of them so that at the level of cognitive representations general coordinate invariance is broken.

   Note that the number of elements of the state basis in p^n factor is p^n and $m \in \{0, ..., p-1\}$ in the factor x_n. Therefore the Hilbert space with dimension $p^n > x_n$ is analogous to the Hilbert space of a large effectively classical system entangled with the microscopic system characterized by x_n. p-Adicity of this Hilbert space in this example is for the purpose of simplicity but raises the question whether the state function reduction is directly related to cognition.

4. On can generalize the concept of real numbers, the notions of manifold, matrix group, etc... by replacing points with Hilbert spaces. For instance, the point $(x_1, .., x_n)$ of E^n is replaced with Cartesian product of corresponding Hilbert spaces. What is of utmost importance for the idea about possible connection with the multiverse idea is that also this process can be also repeated indefinitely. This process is analogous to a repeated second quantization since intuitively the replacement means replacing Hilbert space with Hilbert space of wave functions in Hilbert space. The finite dimension and its continuity as function of space-time point must mean that there are strong constraints on these wave functions. What does this decomposition to a direct sum mean at the level of states? Does one have super-selection rules stating that quantum interference is possible only inside the direct summands?

5. Could one find a number theoretical counterpart for state function reduction and preparation and unitary time evolution? Could zero energy ontology have a formulation at the level of the number theory as earlier experience with infinite primes suggest? The proposal was that

zero energy states correspond to ratios of infinite integers which as real numbers reduce to real unit. Could zero energy states correspond to states in the tensor product of Hilbert spaces for which formal dimensions are inverses of each other so that the total space has dimension 1?

15.5.1 Unitary process and state function reduction in ZEO

The minimal view about unitary process and state function reduction is provided by ZEO [K6].

1. Zero energy states correspond to a superposition of pairs of positive and negative energy states. The M-matrix defining the entanglement coefficients is product of Hermitian square root of density matrix and unitary S-matrix, and various M-matrices are orthogonal and form rows of a unitary U-matrix. Quantum theory is square root of thermodynamics. This is true even at single particle level. The square root of the density matrix could be also interpreted in terms of finite measurement resolution.

2. It is natural to assume that zero energy states have well-defined single particle quantum numbers at the either end of CD as in particle physics experiment. This means that state preparation has taken place and the prepared end represents the initial state of a physical event. Since either end of CD can be in question, both arrows of geometric time identifiable as the Minkowski time defined by the tips of CD are possible.

3. The simplest identification of the U-matrix is as the unitary U-matrix relating to each other the state basis for which M-matrices correspond to prepared states at two opposite ends of CD. Let us assume that the preparation has taken place at the "lower" end, the initial state. State function reduction for the final state means that one measures the single particle observables for the "upper" end of CD. This necessarily induces the loss of this property at the "lower" end. Next preparation in turn induces localization in the "lower" end. One has a kind of time flip-flop and the breaking of time reversal invariance would be absolutely essential for the non-triviality of the process.

The basic idea of Quantum Mathematics is that M-matrix is characterized by Feynman diagrams representing sequences of arithmetic operations and their co-arithmetic counterparts. The latter ones give rise to a superposition of pairs of direct summands (factors of tensor product) giving rise to same direct sum (tensor product). This vision would reduce quantum physics to generalized number theory. Universe would be calculating and the consciousness of the mathematician would be in the quantum jumps performing the state function reductions to which preparations reduce.

Note that direct sum, tensor product, and the counterpart of second quantization for Hilbert spaces in the proposed sense would be quantum mathematics counterpart for set theoretic operations, Cartesian product and formation of the power set in set theory.

15.5.2 ZEO, state function reduction, unitary process, and quantum mathematics

State function reduction acts in a tensor product of Hilbert spaces. In the p-adic context to be discussed n the following $x_n \otimes p^n$ is the natural candidate for this tensor product. One can assign a density matrix to a given entangled state of this system and calculate the Shannon entropy. One can also assign to it a number theoretical entropy if entanglement coefficients are rationals or even algebraic numbers, and this entropy can be negative. One can apply Negentropy Maximization Principle to identify the preferred states basis as eigenstates of the density matrix. For negentropic entanglement the quantum jump does not destroy the entanglement.

Could the state function reduction take place separately for each subspace $x_n \otimes p^n$ in the direct sum $\oplus_n x_n \otimes p^n$ so that one would have quantum parallel state function reductions? This is an old proposal motivated by the many-sheeted space-time. The direct summands in this case would correspond to the contributions to the states localizable at various space-time sheets assigned to different powers of p defining a scale hierarchy. The powers p^n would be associated with zero modes by the previous argument so that the assumption about independent reduction would reflect the super-selection rule for zero modes. Also different values of p-adic prime are present and tensor product between them is possible if the entanglement coefficients are rationals or even algebraics. In the formulation using adeles the needed generalization could be formulated in a straightforward manner.

How can one select the entangled states in the summands $x_n \otimes p^n$? Is there some unique choice? How do unitary process and state function reduction relate to this choice? Could the dynamics of Quantum Mathematics be a structural analog for a sequence of state function reductions taking place at the opposite ends of CD with unitary matrix U relating the state basis for which single particle states have well defined quantum numbers either at the upper or lower end of CD? Could the unitary process and state function reduction be identified solely from the requirement that zero energy states correspond to tensor products Hilbert spaces, which correspond to inverses of each other as numbers? Could the extension of arithmetics to include co-arithmetics make the dynamics in question unique?

15.5.3 What multiverse branching could mean?

Could QM allow to identify a mathematical counterpart for the branching of quantum states to quantum states corresponding to preferred basis? Could one can imagine that a superposition of states $\sum c_n \Psi_n$ in a direct summand $x_n \otimes p^n$ is replaced by a state for which Ψ_n belong to different direct summands and that branching to non-interfering sub-universes is induced by the proposed super-selection rule or perhaps even induces state function reduction? These two options seem to be equivalent experimentally. Could this de-coherence process perhaps correspond to the replacement of the original Hilbert space characterized by number x with a new Hilbert space corresponding to number y inducing the splitting of $x_n \otimes p^n$? Could the interpretation of finite integers x_n and p^n as p-adic numbers $p_1 \neq p$ induce the de-coherence?

This kind of situation is encountered also in symmetry breaking. The irreducible representation of a symmetry group reduces to a direct sum of representations of a sub-group and one has in practice super-selection rule: one does not talk about superpositions of photon and Z^0. In quantum measurement the classical external fields indeed induce symmetry breaking by giving different energies for the components of the state. In the case of the factor $x_n \otimes p^n$ the entanglement coefficients define the density matrix characterizing the preferred state basis. It would seem that the process of branching decomposes this state space to a direct sum 1-D state spaces associated with the eigenstates of the density matrix. In symmetry breaking superposition principle holds true and instead of quantum superposition for different orientations of "Higgs field" or magnetic field a localization selecting single orientation of the "Higgs field" takes place. Could state function reduction be analogous process? Could non-quantum fluctuating zero modes of WCW metric appear as analogs of "Higgs fields". In this picture quantum superposition of states with different values of zero modes would not be possible, and state function reduction might take place only for entanglement between zero modes and non-zero modes.

15.5.4 The replacement of a point of Hilbert space with Hilbert space as a second quantization

The fractal character of the Quantum Mathematics is what makes it a good candidate for understanding the self-referentiality of consciousness. The replacement of the Hilbert space with the direct sum of Hilbert spaces defined by its points would be the basic step and could be repeated endlessly corresponding to a hierarchy of statements about statements or hierarchy of n^{th} order logics. The construction of infinite primes leads to a similar structure.

What about the step leading to a deeper level in hierarchy and involving the replacement of each point of Hilbert space with Hilbert space characterizing it number theoretically? What could it correspond at the level of states?

1. Suppose that state function reduction selects one point for each Hilbert space $x_n \otimes p^n$. The key step is to replace this direct sum of points of these Hilbert spaces with direct sum of Hilbert spaces defined by the points of these Hilbert spaces. After this one would select point from this very big Hilbert space. Could this point be in some sense the image of the Hilbert space state at previous level? Should one imbed Hilbert space $x_n \otimes p^n$ isometrically to the

Hilbert space defined by the preferred state $x_n \otimes p^n$ so that one would have a realization of holography: part would represent the whole at the new level. It seems that there is a canonical manner to achieve this. The interpretation as the analog of second quantization suggest the identification of the imbedding map as the identification of the many particle states of previous level as single particle states of the new level.

2. Could topological condensation be the counterpart of this process in many-sheeted space-time of TGD? The states of previous level would be assigned to the space-time sheets topologically condensed to a larger space-time sheet representing the new level and the many-particle states of previous level would be the elementary particles of the new level.

3. If this vision is correct, second quantization performed by theoreticians would not be a mere theoretical operation but a fundamental physical process necessary for cognition! The above proposed unitary imbedding would imbed the states of the previous level as single particle states to the new level. It would seem that the process of second quantization, which is indeed very much like self-reference, is completely independent from state function reduction and unitary process. This picture would conform with the fact that in TGD Universe the theory about the Universe is the Universe and mathematician is in the quantum jumps between different solutions of this theory.

Returning to the motivating question: it seems that the endless branching of the states in multiverse interpretation cannot correspond to a repeated second quantization but could have interpretation as a de-coherence identifiable as de-localization in zero modes. If state function is allowed, it corresponds to a localization in zero modes analogous to Higgs mechanism. The Quantum Mathematics realization for a repeated second quantization would represent a genuinely new kind of process which does not reduce to anything already known.

15.6 Speculations related to Hilbert adelization

This section contains further speculations related to realization of number theoretical universality in terms of Hilbert adeles and to the notion of number theoretic emergence. One can construct infinite hierarchy of Hilbert adeles by replacing the points of Hilbert spaces with Hilbert spaces repeatedly: this generalizes the repeated second quantization used to construct infinite primes and realizes also algebraic holography since the points of space have infinitely complex structure. There are strong restrictions on the values of coordinates of Hilbert space for the p-adic sectors of the adele and the number of state basis satisfying orthonormality conditions is very restricted: a good guess is that unitary transformations reduce to a permutation group and that its cyclic subgroup defines quantum Galois group. Also the Hilbert counterpart of real factor of adeles is present and in this case there are no such restrictions.

A logical use of terms is achieved if one refers by term "quantum Hilbert adele" to the adele obtained by replacing the Hilbert space coefficients $a_n < p$ of pinary expansions with their quantum Hilbert spaces. On the other hand the hierarchy of Hilbert adeles is very quantal since it is analogous to a hierarchy of second quantizations so that Hilbert adeles could be also called quantum adeles. Reader can decide.

15.6.1 Hilbert adelization as a manner to realize number theoretical universality

Hilbert adelization is highly suggestive realization of the number theoretical universality. The very construction of adeles and their Hilbert counterparts is consistent with the idea that rational numbers are common to all completions of rationals. This suggests a generalization of the formalism of physics allowing to realize number theoretical universality in terms of adeles and their Hilbert counterparts. What this would mean the replacement of real numbers everywhere by adeles containing real numbers as one Cartesian factor. Field equations make sense for the adeles separately in each Cartesian factor.

If one can define differential calculus for the Hilbert reals and p-adics as seems to be the case, this abstraction might make sense. There seems to be no obvious objection for field property and the entire hierarchy of n-Hilbert spaces could be seen as a cognitive self-referential representation of the mathematical structure allowing perhaps also physical realization if the structure is consistent with the general axioms.

Field equations would thus make sense also for an infinite hierarchy formed by Hilbert^n adeles. The fascinating conjecture is that quantum physics reduces to quantum mathematics and one might hope that TGD provides a realization for this physics because of its very strong ties with number theory.

Hilbert adelication at imbedding space level

The Hilbert adelization at the level of imbedding space makes senses if adelization works so that one can consider only adelization.

1. Could imbedding space coordinates regarded as adeles? In the p-adic sectors general coordinate invariance would require some preferred coordinate choices maybe unique enough by symmetry considerations. One can also consider a spontaneous breaking of GCI by cognitive representations. Adelization would code field equations in various p-adic number fields to single field equation for adeles and would not bring anything new.

2. What could field equations mean for Hilbert adeles? One could imagine that ordinary field equations as local algebraic statements are expressed separately at each point of space-time surface giving infinite number of equations of form $F^k(x) = 0$, where k labels imbedding space coordinates. Moving to the first level of hierarchy would mean that one replaces the points of Hilbert spaces involved with Hilbert spaces. The connection with the first order logic would suggest that the points of the Hilbert spaces representing points of imbedding space and space-time - in general infinite-dimensional for real and p-adic numbers - represent points of imbedding space and of space-time. This second quantization would transform infinite number of statements of predicate logic to a statement of first order logic.

   This certainly sounds hopelessly abstract and no-one would seriously consider solving field equations in this manner. But maybe mathematical thinking relying on quantum physics could indeed do it like this? At the next level of hierarchy one might dream of combining field equations for entire families of solutions of field equations to single equation and so on. Maybe these families could correspond to supports of WCW spinor fields in WCW. At the next level statements would be about families of WCW spinors fields and so on - ad infinitum. In fact, WCW spinors can be seen as quantum superpositions of logical statements in fermionic Fock space and WCW spinor fields would assign to WCW a direct sum of this kind of statements, one to each point of WCW. This sounds infinitely infinite but one must remember that the sub-WCW consisting of surfaces expressible in terms of rational functions is discrete.

3. The conjecture that field equations reduce to octonion real-analyticity requires that octonions and quaternions make sense also p-adically. The problem is that the p-adic variants of octonions and quaternions do not form a field: the reason is that even the equation $x^2 + y^2 = 0$ can have solutions in p-adic number fields so that the inverses of quaternions and octonions, and even p-adic complex numbers need not make sense. The p-adic counterparts of quaternions and octonions however exist as a ring so that one could speak about polynomials and Taylor series whereas the definition of rationals and therefore rational functions would involve problems. Octonion real-analyticity and quaternion real-analyticity and therefore also space-time surfaces defined by polynomials or even by infinite Taylor series could make sense also for the p-adic variants of octonions and quaternions.

Could imbedding space spinors be regarded as adelic and even Hilbert adelic spinors? Again the problems reduce to the adelic level.

1. Adelization could be perhaps seen as a convenient book keeping device allowing to encapsulate the infinite number of physics in various quantum p-adic number fields to single physics. Hilbert adelic structures could however provide much deeper realization of physics as generalized number theory. One can indeed ask whether the action of the p-adic quantum counterparts of various symmetries could representable in the quantum quantum Galois groups

for Hilbert adeles: these groups might reduce to cyclic groups and might relate to cyclic coverings of imbedding space at the level of physics.

The minimal interpretation would be as a cognitive representation of quantum numbers of physical states at the first "material" level of hierarchy using the number theoretic Hilbert space anatomy of the point to achieve the representation. The representative capacity would be infinite for transcendental numbers with infinite number of pinary digits and finite for rational numbers. For real unit if would be miminal and zero could not represent anything. Quantum entanglement would be possible for tensor product coefficients and quantum superposition would be possible due to direct sum of pinary digits.

2. Imbedding space spinor fields could be regarded as Cartesian products (direct sums) of spinor fields in real and various p-adic imbedding spaces having values in the same number field. Also the induced metric and spinor connection would correspond to Cartesian product rather than tensor product. The isometries of the imbedding space would have matrix representation in terms of adeles on the adelic components of spinors and imbedding space coordinates.

Hilbert adelication at the level of WCW

What about quantum TGD at the level of WCW? Could Hilbert adelication apply also at this level? Could one use the same general recipes to adelize? The step from adele to the hierarchy of Hilbert adeles does not seem to be a conceptual problem and the basic problem is to understand what adele means.

1. Could WCW described in terms of generalized number theory? Could adelic WCW be defined as the Cartesian product of real WCW and p-adic WCWs? The observations about dessins d'enfant [A23] [K84] suggest that the description of WCW could be reduced to the description in terms of orbits of algebraic 2-surfaces identified as partonic 2-surfaces at the boundaries of CDs (also the 4-D tangent space data at them codes for physics).

2. For a Cartesian product of finite-dimensional spaces spinors are formed as tensor products associated with with the Cartesian factors. Adelic WCW is Cartesian sum of real and p-adic variants. Could Hilbert adelic WCW spinors be identified as a tensor product of WCW spinors defined in the Hilbert adelic variant of WCW. This would conform with the physical vision that real and p-adic physics (matter and cognition) correspond to tensor factors of a larger state space. Furthermore, spinos generalizes scalar functions and the function space for adele valued functions with adelic argument forms in a natural manner tensor product of function spaces for various completions of reals. Note that one can speak about rational quantum entanglement since rational numbers are common to all the Cartesian factors.

3. Could also the moduli space of conformal equivalence classes of partonic 2-surfaces be regarded as adele in the sense that Teichmueller parameters from adele. This requires that the Teichmueller space of conformal equivalence classes of Riemann surfaces corresponds to the p-adic version of real Teichmueller space: this has been actually assumed in p-adic mass calculations [K17, K39].

   One could start from the observation that algebraic Riemann surfaces are dense in the space of all Riemann surfaces. This means that the algebraic variant of Teichmueller space is able to characterize the conformal equivalence classes. What happens when one adds the Riemann surfaces for which the coefficients of the Belyi function and rational functions defining are allowed to be in real or p-adic completion of rationals. A natural guess is that completion of the algebraic variant of Teichmueller space results in this manner. If this is argument makes sense then adelic moduli space makes sense too.

   There are however technical delicacies involved. Teichmueller parameters are defined as values of 1-forms for the homology generators of Riemann surface. What does one mean with the values of these forms when one has a surface containing only algebraic points and ordinary integral is not well-defined? Also in the p-adic context the definition of the integral is problematic and I have devoted a lot of time and energy to this problem (see for instance [K82]). Could the holomorphy of these forms help to define them in terms of residue calculus? This option looks the most plausible one.

What about the partial well-ordering of p-adic numbers induced by the map $n \rightarrow n_q$ combined with canonical identification: could this allow an elegant notion of integration by using the partial well-ordering. Note that one cannot say which of the numbers 1 and $-(p-1)\sum n = 1^{\infty} p^n$ is bigger in this ordering, and this induces similar problem for all p-adic integers which have finite number of pinary digits.

Problems to solutions and new questions

Usually one becomes fully conscious of a problem only after one has found the solution of the problem. The vision about Hilbert adeles - as a matter fact, already adeles- solves several nasty nuisances of this kind and I have worked hardly to prevent these problems from running off under the rug.

1. What one means with integer -1 is not a problem for p-adic mathematics. It becomes a problem for physical interpretation when one must relate real and p-adic physics to each other since canonical identification maps p-adic numbers to non-negative reals. This leads to problems with Hilbert space inner product but algebraic extensions of p-adic numbers by roots of unity allow to define p-adic Hilbert spaces but it seems that the allowed state basis are very restricted since the number of unitary isometries of Hilbert space is restricted dramatically by number theoretical existence requirement. The optimistic interpretation would that full quantum superposition is highly restricted in cognitive sectors by the condition of number theoretic existence.

2. What one means with complex p-adics is second problem. $\sqrt{-1}$ exists p-adically for $p \ mod \ 4 = 1$ so that one cannot introduce it via algebraic extension of p-adics in this case. This is a problem of p-adic quantum mechanics. Allowance of only p-adic primes p which do not split for the extension containing imaginary unit seems to be a general solution of the problem.

3. p-Adic counterparts of quaternions, and octonions do not exist for the simple reason that the p-adic norm can be vanishing even for p-adic complex number for p-adic fields allowing $\sqrt{-1}$. This problem can be circumvented by giving up the requirement that one has number field.

4. The norm for adeles exist as a product of real and norm and p-adic norms but is not physical. Also the assignment of Hilbert space structure to adeles is problematic. Canonical identification combined with $n \rightarrow n_q$ allows the mapping p-adic components of adele to real numbers and this allows to define natural inner product and norm analogous to Hilbert space norm for adeles and their Hilbert counterparts.

5. p-Adic numbers are not well ordered. This implies that difficulties with the definition of integral since definite integral relies heavily on well-orderedness of reals. Canonical identification suggests that quantum p-adics are well ordered: $a < b$ holds true if it holds true for the images under canonical identification. This gives hopes about defining also definite integral. For integrable functions the natural definition of quantum p-adic valued integral would be by using substitution for integral function. One - and rather ugly - option is to define the integral as ordinary real integral for the canonical image of the quantum p-adic valued function. This because this image is not expected to be smooth in real sense even if p-adic function is smooth.

6. p-Adic integration is plagued also by the problem that already for rational integrals one obtains numbers like $log(n)$ and π and is forced to introduce infinite-dimensional extension of p-adic numbers. For $log(n)$ one could restrict the consideration to p-adic primes p satisfying $n \ mod \ p = 1$ but this looks like a trick. Could this difficulty be circumvented somehow for p-adic numbers? The only possibility that one can imagine would be canonical identification map combined with $n \rightarrow n_q$ and the interpretation of integral as a real number.

   This could provide also the trick to interpret the integrals involving powers of π possible emerging from Feynman diagrams in sensible manner. All integrals can be reduced with the use of Laurent series to integrals of powers of x so that integral calculus would exist in analytic sense for analytic functions of quantum p-adic numbers.

7. What does one mean with the p-adic counterpart of CP_2 or more generally, with the p-adic counterpart of any non-linear manifold? What does one mean with the complex structure of p-adic CP_2 for $p \ mod \ 4 = 1$? Should one restrict the consideration to $p \ mod \ 4 = 3$? What does one mean with groups and coset spaces? One can inceed have a satisfactory looking definition based on algebraic extensions and effective discretization by introducing roots of unity replacing complex phases as continuous variables [K82].

   One could consider two options.

   (a) Could the p-adic counterpart of real $M^4 \times CP_2$ be M^8? The objection is that algebraic groups are however fundamental for mathematics and typically non-linear manifolds. Therefore there are excellent motivations for their (Hilbert) adelic existence. Projective spaces are in turn central in algebraic geometry and in this spirit one might hope that CP_2 could have non-trivial p-adic counterpart defined as quantum p-adic projective space.

   (b) Another option accepts that adeles contain only those p-adic number fields as Cartesian factors for which the prime does not split. This excludes automatically $p \ mod \ 4 = 1$ if $\sqrt{-1}$ is present from the beginning in the algebraic extension of rationals defining the adeles. What happens if one does not assume this. Does CP_2 degenerate to real projective space RP_2? What happens to M^4 if regarded as a Cartesian product of hyper-complex numbers and complex numbers. Does it reduce to M^2. Could the not completely well understood role of M^2 in quantum TGD relate to this kind of reduction?

The new view raises also questions challenging previous basic assumptions.

1. Could adeles and their octonionic counterpart allow to understand the origin of commutative complexification for quaternions and octonions in number theoretic vision about TGD? How could the commutative imaginary unit emerge number theoretically?

2. One must also reconsider $M^8 - M^4 \times CP_2$ duality. For instance, could M^8 be the natural choice in p-adic sectors and $M^4 \times CP_2$ in the real sector?

3. The preferred extremals of Kähler action are conjectured to be quaternionic in some sense. There are two proposals for what this means. Could it be that the sense in which the space-time surfaces are quaternionic depends on whether the surface is real or quantum p-adic?

4. The idea that rationals are in the intersection of reals and p-adics is central in the applications of TGD. How does this vision change? For $p = 2$ quantum rationals in the sense that pinary coefficients are quantum integer, are ordinary rational numbers. For $p > 2$ the pinary coefficients are in general mapped to algebraic numbers involving l_q, $0 < l < p$. The common points with reals would in general algebraic numbers.

Do basic notions require updating in the Hilbert adelic context?

In the adelic context one must take a fresh look to what one means with phrases like "imbedding space" and "space-time surfaces". The phrase "space-time surface as a preferred extremal of Kähler action" might be quite too strong a statement in adelic context and could actually make sense only in the real sector of the quantum adelic imbedding space. Also the phrase "p-adic variant of $M^4 \times CP_2$" might involve un-necessarily strong implicit assumptions since for p-adic integers one has automatically the counterparts of compactness even for M^8. The proposed identification of the quantum p-adic numbers as Hilbert p-adic quantum numbers reduces the question to whether p-adic counterparts of various structures exist or are needed as such.

1. We "know" that the real imbedding space must be $M^4 \times CP_2$. What about p-adic counterpart of the imbedding space? Is it really possible to have a p-adic counterpart of CP_2 or could non-linearity destroy this kind of hopes? Are there any strong reasons for having the counterpart of $M^4 \times CP_2$ in p-adic sectors? Could one have $M^4 \times CP_2$ only in real sector and M^8 in p-adic sectors. Complex structure of CP_2 requires $p \ mod \ 4 = 3$. This is not a problem if one assumes that adeles contain only the p-adic primes which do not split in the extension

of rationals containing imaginary unit. Definition as coset space $CP_2 = SU(3)/U(2)$ is one possible manner to proceed and seems to work also.

One can also wonder whether octonion real-analyticity really makes sense for $M^4 \times CP_2$ and its p-adic variants. The fact that real analyticity makes sense for S^2 suggests that it does. In any case, octonion real-analyticity would make life very easy for p-adic sectors if regarded as octonionic counterpart of M^8 rather than $M^4 \times CP_2$.

2. If the p-adic factors are identified as linear spaces with M^8 regarded as sub-space of the ring of complexified p-adic octonions, octonion real-analyticity for polynomial functions with rational coefficients could replace field equations in the ring formed by Z_p. Note however that octonion real-analyticity requires the Wick rotation mapping to ordinary octonions, the identification of the 4-surface from the vanishing of the imaginary part of the octonion real-analytic function, and map back to Minkowski space by Wick rotation. This is well-defined procedure used routinely in quantum field theories but could be criticized as mathematically somewhat questionable. One could consider also the definition of Minkowski space inner product as real part of z_1z_2 for quaternions and use similar formula for octonions. This would give Minkowski norm squared for $z_1 = z_2$.

   Linear space would also allow to realize the idea that partonic 2-surfaces are in some sense trivial in most sectors reducing to points represented most naturally by the tips of causal diamonds (CDs). For p-adic sectors CP_2 would be replaced with E^4 and for most factors M^8_p the partonic 2-surfaces would reduce to the point $s = 0$ of E^4 representing the origin of coordinates in which E^4 rotations act linearly.

3. The conjecture is that preferred extremals correspond to loci for the zeros of the imaginary or real part of octonion real-analytic function. Is this identification really necessary? Could it be that in the *real* sector the extremals correspond to quaternionic 4-surfaces in the sense that they have quaternionic tangent spaces? And could the identification as loci for the zeros of the imaginary or real part of octonion real-analytic function be the sensible option in the p-adic sectors of the adelic imbedding space: in particular if these sectors correspond to octonionic M^8. If this were the case, $M^8 - M^4 \times CP_2$ duality would have a meaning differing from the original one and would relate the real sector of adelic imbedding space to its p-adic sectors in manner analogous to the expression of real rational as a Cartesian product of powers of p-adic primes in various sectors of adele.

My cautious conclusion is that the earlier vision is correct: $M^4 \times CP_2$ makes sense in all sectors.

15.6.2 Could number theoretic emergence make sense?

The observations made in this and previous sections encourage to ask whether some kind of number theoretic emergence could make sense. One would end up step by step from rationals to octonions by performing algebraic extensions and completions. At some step also the attribute "Hilbert" would lead to a further abstraction and relate closely to the evolution of cognition. This would mean something like follows.

Rationals → algebraic extensions → algebraic numbers → completions of rationals to reals and p-adics → completions of algebraic 2-surfaces to real and p-adic ones in algebraic extensions reals and classical number fields→ hierarchy of Hilbert variants of these structures as their cognitive representations.

The Maximal Abelian Galois group (MAGG) for rationals is isomorphic to the multiplicative group of ideles and involves reals and various p-adic number fields. How could one interpret the Hilbert variant of this structure. Could some kind of physical and cognitive evolution lead from rationals to octonions and eventually to Universe according to TGD? Could it be that the gradual emergence of algebraic numbers and AGG (Absolute Galois Group defined as Galois group of algebraic numbers as extension of rationals) brings in various completions of rationals and further extensions to quaternions and octonions and symmetry groups like SU(2) acting as automorphisms of quaternions as extension of reals and $SU(3) \subset G_2$ where G_2 acts as Galois for the extension of octonions as extension of reals?

Objections against emergence

The best manner to develop a new idea is by inventing objections against it. This applies also to the notion of algebraic emergence. The objections actually allow to see the basic conjectures about preferred extremals of Kähler action in new light.

1. Algebraic numbers emerge via extensions of rationals and complex numbers via completion of algebraic numbers. But can higher dimensions really emerge? This is possible but only when they correspond to those of classical number fields: reals, quaternions, and octonions. This is enough in TGD framework. Adelization could lead to the emergence of real space-time and its p-adic variants. Completion of solutions of algebraic equations to p-adic and real number fields is natural. Also the extensions of reals and complex numbers to quaternions and octonions are natural and could be seen as emergence.

2. All algebraic Riemann surfaces are compact but the reverse of this does not hold true. Partonic 2-surfaces are fundamental in TGD framework. Once the induced metric of the compact partonic 2-surface is known, one can regard it as a Riemann surface. Only if it is algebraic surface, the action of Galois group on it is well-defined as an action on the algebraic coefficients appearing in rational functions defining the surface. This is consistent with the basic vision about life as something in the intersection of real and p-adic worlds and therefore having as correlates algebraic partonic 2-surfaces. The non-algebraic partonic 2-surfaces are naturally present and if they emerge they must do so via completion to reals occurring also at adelic level.

   All partonic 2-surfaces allow a representation as projective varieties in CP_3 which forces again the question about possible connection with twistors.

   Representation as algebraic projective varieties in say CP_3 does not imply this kind of representation in $\delta CD \times CP_2$. This kind of representation can make sense for 3-surfaces consisting of light like geodesics emanating from the tip of the CD. If one wants to obtain 2-surfaces one must restrict light-like radial coordinate r to be a real function of complex variables so that the 2-surface cannot be algebraic surface defined as a null locus of holomorphic functions unless r is taken to be a constant equal to algebraic number. Note that the light rays of 3-D light-cone are parametrized by S^2, which corresponds to $CP_1 \subset CP_3$. This kind of partonic 2-surfaces might correspond to maxima for Kähler function.

3. Could one do without the non-algebraic partonic 2-surfaces? This is not the case if one believes on the notion of number theoretic entanglement entropy which can be negative for rational or even algebraic entanglement and presumably also for its quantum variant. Non-algebraic partonic 2-surfaces would naturally correspond to reals as a Cartesian factor of adeles. All partonic 2-surfaces which do not allow a representation as algebraic surfaces would belong to this factor of adelic imbedding space. The ordinary real number based physics would prevail in this sector and entanglement in this sector would be in generic case real so that ordinary definition of entropy would work. In quantum p-adic sectors entanglement probabilities would be quantum rational (in the sense of $n \rightarrow n_q$) and the generalization of number theoretic entanglement entropy should make sense. Completion must be taken as would be part of the emergence.

Could imbedding space spinors really emerge? The dimension of the space of imbedding space spinors is dictated by the dimension of the imbedding space. Therefore it is difficult to image how 8+8-complex-dimensional spinors could emerge from spinors in the set of algebraic numbers since these spinors are naturally 2-dimensional for algebraic numbers which are geometrically 2-dimensional. Does this mean that one must introduce algebraic octonions and their complexifications from the very beginning? Not necessarily.

1. The idea that also the imbedding space spinors emerge algebraically suggests that imbedding space spinors in p-adic sectors are octonionic (p-adic octonions form a ring but this might be enough). In real sector both interpretations might make sense and have been considered [K79]. For octonionic spinors ordinary gamma matrices are replaced with the analogs

of gamma matrices obtained as tensor products of sigma matrices having quaternionic interpretation and of octonionic units. For these gamma matrices SO(1,7) as vielbein group is replaced with G_2. Physically this corresponds to the presence of a preferred time direction defined by the line connecting the tips of CD. It would seem that $SO(1,7)$ must be assigned with the ordinary imbedding space spinors assignable to the reals as a factor of quantum adeles. The relationship between the ordinary and octonionic imbedding space spinors is unclear. One can however ask whether the p-adic spinors in various factors of adelic spinors could correspond to the octonionic modification of gamma matrices so that these spinors would be 1-D spinors algebraically extended to octonionic spinors.

2. Also quaternionic spinors make sense and could emerge in a well-defined sense. The basic conjecture is that the preferred extremals of Kähler action are quaternionic surfaces in some sense. This could mean that the octonionic tangent space reduces to quaternionic one at each point of the space-time surface. This condition involves partial derivatives and these make sense for p-adic number fields . The "real" gamma matrices would be ordinary gamma matrices. In p-adic sectors at least octonion real-analyticity would be the natural condition allowing to identify quaternionic 4-surfaces [K72] if one allows only Taylor series expansions.

Emergence of reals and p-adics via quantum adeles?

MAGG (Maximal Abelian Galois Group) brings in reals and various p-adic number fields although one starts from algebraic numbers as maximal abelian extension of rationals. Does this mean emergence?

1. Could one formulate the theory by starting from algebraic numbers? The proposal that octonion real-analytic functions can be used to define what quaternionicity looks sensible for quantum p-adic space-time surfaces. For real space-time surfaces octonion real-analyticity might be an unrealistic condition and quaternionicity as the condition that octonionic gamma matrices generate quaternionic algebra in the tangent space looks more plausible alternative. Quantum p-adic space-time surfaces would be naturally algebraic but in real context also non-algebraic space-time surfaces and partonic 2-surfaces are possibe. In real sector partial differential equations would prevail and in quantum p-adic sectors algebraic equations would dictate the dynamics.

2. The p-adic variants of quaternions and octonions do not exist as fields. The vanishing of the sum of Euclidian norm for quaternions and octonions for p-adic octonions and quaternions makes it impossible to define p-adic quaternion and octonionic fields. There are also problems due to the fact that $\sqrt{-1}$ exists as p-adic number for $p \ mod \ 4 = 1$.

3. The notion of quaternionic space-time surface requires complexified octonions with additional imaginary unit i commuting with octonionic imaginary units I_k. Space-time surfaces are identified as surfaces in the sub-space of complexified octonions of form $o_0 + i \sum o_k I^k$. Could i relate to the algebraic extensions of rationals and could complexified quantum p-adic imbedding spaces have complex coordinates $x + iy$?

4. Polynomial equations with real algebraic coefficients make sense even if adeles where not a field and one can assign to the roots of polynomials with quaternionic and octonionic argument Galois group if one restricts to solution which reduce to complex solutions in some complex plane defined by preferred imaginary unit. For quaternions Galois group consist of rotations in SO(3) acting via adjoint action combined with AAG. For octonions Galois group consists of G_2 elements combined with AAG. $SU(3)$ leaves the preferred imaginary unit invariant and $U(2)$ the choice of quaternionic plane. Are there any other solutions of polynomial equations than those reducing to complex plane?

Is it really necessary to introduce p-adic space-time sheets?

The (Hilbert) adelization of imbedding space, space-time, and WCW as well as spinors fields of imbedding space and WCW would be extremely elegant manner to realize number theoretic universality. One must however keep the skeptic attitude. The definition of p-adic imbedding space and space-time surfaces is not free of technical problems. The replacement of $M^4 \times CP_2$ with M^8 in p-adic sectors could help solve these problems. The conservative approach would be based on giving up p-adicization in imbedding space degrees of freedom. It is certainly not an imaginative option but must be considered as a manner to gain additional insights.

1. p-Adic mass calculations do not mention anything about the p-adicization of space-time sheets unless one wants to answer the question what is the concrete realizations of various conformal algebras. Only p-adic and adelic interpretation of conformal weights would be needed. Adelic interpretation of conformal weights makes sense. The replacement $n \rightarrow n_q$ (interpreted originally as quantum p-adicization) brings in only $O(p^2)$ corrections which are typically extremely small in elementary particle scales.

2. Is the notion of p-adic or Hilbert p-adic (Hilbert adelic) spinor field in imbedding space absolutely necessary? If one has p-adic spinors one must have also p-adic spinor connection. This does not require p-adic imbedding space and space-time surface if one restricts the consideration to algebraic points and if the components of connection are algebraic numbers or even rational numbers and allow p-adic interpretation. This assumption is however in conflict with the universality of adelization.

3. What about Hilbert adelic WCW spinor fields. They are needed to give both p-adic and real quantum states. These fields should have adelic values. Their arguments could be algebraic partonic surfaces. There would be no absolute need to perform completions of algebraic partonic 2-surfaces although this would be very natural on basis of number theoretical universality.

4. p-Adic space-time sheets are identified as correlates of intention and cognition. Transformation of intention to action as leakage from p-adic to real sector of imbedding space. This idea provides strong support for p-adic space-time. But could one assume only that the quantum states are p-adic or quantum p-adic but that space-time is real? Does it mean only that the WCW spinor field or zero energy state assignable to light-like 3-surface or partonic 2-surface is Hilbert adelic. Quantum transitions between states for which initially WCW spinor field is p_i-adic and in the final state p_f-adic. Only the number field for WCW spinors would change in the transition. One could say that partonic 2-surface is p-adic if the value of WCW spinor field assigned with it is p-adic. This idea does not look attractive and is in complete conflict with the adelization idea.

5. The vision about life in the intersection of real and p-adic worlds is very attractive. The p-adicization of algebraic surfaces is very natural as completion meaning that one just solves the algebraic equations using series in powers of p. Imaginary unit is key number of quantum theory and the fact that $\sqrt{-1}$ exists for $p \mod 4 = 1$ is potential problem for p-adic quantum mechanics. For these primes also splitting occurs in the ring of Gaussian integers. For quantum adeles this problem disappears if one allows only the p-adic number fields for which p does not slit in algebraic extension (now Gaussian rationals).

15.7 Appendix: Some possibly motivating considerations

The path to the idea that quantum adeles could represent algebraic numbers originated from a question having no obvious relation to quantum p-adics or quantum adeles and I will proceed in the following by starting from this question.

Function fields are much simpler objects to handle than rationals and their algebraic extensions. In particular, the objects of function fields have inverses and inverse is well defined also for sum of elements. This is not true in the ring of adeles. This is the reason why geometric Langlands is easier than the number theoretic one. Also the basic idea of Langlands correspondence is that it is possible to translate problems of classical number theory (rationals and their extensions) to those involving functions fields. Could it be possible to represent the field of rationals as a function field in some sense? Quantum arithmetics gives a slight hope that this might be possible.

15.7.1 Analogies between number theoretic and function field theoretic ramification

Consider first the analogies between number theoretic and geometric ramification (probably trivialities for professionals but not for a physicist like me!). The relationship between number theoretic and geometric ramification is interesting and mathematician could of course tell a lot about it. My comments are just wonderings of a novice.

1. The number theoretic ramification takes place for the primes of number field when it is extended. If one knows the roots of the polynomials involved with the rational function $f(z)$ defining Belyi function one knows the coefficient field F of polynomial and its algebraic extension K and can deduce the representations of ordinary primes as products of those of F and of the primes of the coefficient field F as products of those of K. In particular, one can find the ramified primes of ordinary integers and of integers of F.

2. The ramification however occurs also for ordinary integers and means that their decomposition to primes involves higher powers of some primes: $n = \prod_l l^{e_l}$ with $e_l > 1$ for some primes l dividing n. Could one introduce an extension of some ring structure in which ordinary primes would be analogous to the primes in the extension of rationals?

3. Geometric ramification takes place for polynomials decomposing to products of first order monomials $P(z) = z - z_k$ with roots which are in algebraic extension of coefficients. The polynomials can however fail to be irreducible meaning that they have multiple roots. For multiple roots one obtains a ramified zero of a root and for Belyi functions these critical points correspond to zeros which are ramified when the degree is larger than zero. The number theoretic ramification implies that the polynomials involved have several algebraic roots and when they coincide, a geometric ramification takes place. Degeneration of roots of polynomial implies ramification.

4. Ordinary integers clearly correspond to the space of polynomials and the integers, which are not square free are analogous to polynomials with multiple roots. The ramification of prime in the extension of rationals and also the appearance of higher powers of p in non-square free integer is analogous to the degeneration of roots of polynomial.

15.7.2 Could one assign analog of function field to integers and analogs prime polynomials to primes?

Could one assign to integer (prime) a map analogous to (prime) polynomial? Prime polynomial can be labeled by its zero and polynomial by its zeros. What kind of maps could represent ordinary primes and integers. What could be the argument of this kind of maps and do zeros of these map label them? What could be the ring in which the counterparts of polynomials are defined?

Could quantum arithmetics [K83] help to answer these questions?

1. Quantum arithmetics involves the map $f_q : n = \prod_{l|n} l^{e_l} \rightarrow n_q = \prod_{l|n} l_q^{e_l}$, where l are primes in the prime decomposition of n and quantum primes s $l_q = (q^l - q^{-ln})/(q - q^{-1})$ are defined by the phase $q = exp(i\pi/p)$, where p is the preferred prime. Note that one has $p_q = 0$ and $(p+1)_q = -1$. Note also that one has $q = exp(i\pi/p)$ rather than $q = exp(i2\pi/p)$ (as in the earlier version of article). This is necessary to get the denominator correctly also for $p = 2$ and to make quantum primes l_q non-negative for $l < p$. Under $n \rightarrow n_q$ all integers n divisible by p are mapped to zero. This would suggest that the counterparts of prime polynomials are the maps f_q, $q = q_p$ and that the analogs of polynomials are products $\prod_p f_{q_p}$ defined in some sense.

2. The more conventional view about quantum integers defines analogous map as $n \rightarrow n_q = (q^n - q^{-n})/(q - q^{-1})$. Choosing $q = exp(i\pi/p)$ one finds also now that integers divisible by p are mapped to zero. By finding the primes for which n is mapped to zero one finds the prime decomposition of n. Now one does not however have a decomposition to a product of quantum primes as above. Similar statement is of course true also for the above definition

of quantum decomposition: the maps $n \rightarrow n_q$ are analogous to polynomials and primes are analogous to the zeros of these polynomials.

3. One can also consider $q = exp(i\pi/m)$ and used decomposition primes which are smaller than m. This would give non-vanishing quantum integers. They would correspond to quantum q-adicity with $q = m$ integer: q-adic numbers do not form a field. q could be even rational. As a special case these numbers give rise to multi-p p-adicity. The Jones inclusions of hyperfinite factors of type II_1 [K25] suggests that also these quantum phases should be considered. The index $[M : N] = 4cos^2(2\pi/n)$ of the inclusion would correspond to quantum matrix dimension 2_q^2, for $q = exp(i\pi/n)$ corresponding to quantum 2-spinors so that quantum dimension p_q could be interpreted as dimension of p-dimensional quantum Hilbert space.

Chapter 16

About Absolute Galois Group

16.1 Introduction

Langlands correspondence represents extremely abstract mathematics - perhaps too abstract for a simple minded physicist with rather mundane thinking habits. It takes years to get just a grasp about the basic motivations and notions, to say nothing about technicalities. Therefore I hope that my own prattlings about Langlands correspondence could be taken with a merciful understanding attitude. I cannot do anything for it: I just want desperately to understand what drives these mathematical physicists and somehow I am convinced that this exotic mathematics could be extremely useful for my attempts to develop the TGD view about Universe and everything. Writing is for me the only manner to possibly achieve understanding - or at least a momentary illusion of understanding - and I can only apologize if the reader has feeling of having wasted time by trying to understand these scribblings.

Ed Frenkel lectured again about geometric Langlands correspondence and quantum field theories and this inspired a fresh attempt to understand what the underlying notions could mean in TGD framework. Frenkel has also article about the relationship between geometric Langlands program and conformal field theories [A150]. My own attempt might be regarded as hopeless but to my view it is worth of reporting.

The challenge of all challenges for a number theorist is to understand the Galois group of algebraic numbers regarded as extension of rationals - by its fundamental importance this group deserves to be called Absolute Galois Group (AGG, [A2]). This group is monstrously big since it is in some sense union of all finite-D Galois groups. Another fundamental Galois group is the Maximal Abelian Galois Group (MAGG) associated with maximal Abelian extension of rationals [A60]. This group is isomorphic with a subgroup assignable to the ring of adeles [A4].

16.1.1 Could AGG act as permutation group for infinite number of objects?

My own naive proposal for years ago is that AGG could be identified as infinite-dimensional permutation group S_∞ [K35]. What the subscript ∞ means is of course on non-trivial question. The set of all finite permutations for infinite sequence of objects at integer positions (to make this more concrete) or also of permutations which involve infinite number of objects? Do these object reside along integer points of half-line or the entire real line? In the latter case permutations acting as integer shifts along the real line are possible and bring in discrete translation group.

A good example is provided by 2-adic numbers. If only sequences consisting of a finite number of non-vanishing bits are allowed, one obtains ordinary integers - a discrete structure. If sequences having strictly infinite number of non-vanishing bits are allowed, one obtains 2-adic integers forming a continuum in 2-adic topology, and one can speak about differential calculus. Something very similar could take place in the case of AGG and already the example of maximal Abelian Galois group which has been shown to be essentially Cartesian product of real numbers and all p-adic number fields Q_p divided by rationals suggests that Cartesian product of all p-adic continuums is involved.

What made this proposal so interesting from TGD point of view is that the group algebra of S_∞ defined in proper manner is hyper-finite factor of II_1 (HFF) [K35]. HFFs are fundamental in TGD: WCW spinors form as a fermionic Fock spaces HFF. This would bring in the inclusions of HFFs, which could provide new kind understanding of AGG. Also the connection with physics might become more concrete. The basic problem is to identify how AGG acts on quantum states and the obvious guess is that they act on algebraic surfaces by affecting the algebraic number valued coefficients of the polynomials involved. How to formulate this with general coordinate invariant (GCI) manner is of course a challenge: one should be able to identify preferred coordinates or at least class of them related by linear algebraic transformations if possible. Symmetries make possible to consider candidates for this kind of coordinates but it is far from obvious that p-adic CP_2 makes sense - or is even needed!

In [K35] I proposed a realization of AGG or rather- its covering replacing elements of permutation group with flows - in terms of braids. Later I considered the possibility to interpret the mapping of the Galois groups assignable to infinite primes to symplectic flows on braids [K82]. This group is covering group of AGG with permutations being replaced with flows which in TGD framework could be realized as symplectic flows. Again GCI is the challenge. I have discussed the symplectic flow representation of generalized Galois groups assigned with infinite primes (allowing mapping to polynomial primes) in [K82] speculating in the framework provided by the TGD inspired physical picture. Here the notion of finite measurement resolution leading to finite Galois groups played a key role.

16.1.2 Dessins d'enfant

Any algebraic surface defined as a common zero locus of rational (in special case polynomial) functions with algebraic coefficients defines a geometric representation of AGG. The action on algebraic coefficients is induced the action of AGG on algebraic numbers appearing as coefficients and in the roots of the polynomials involved. One can study many things: the subgroups of AGG leaving given algebraic surface invariant, the orbits of given algebraic surface under AGG, the subgroups leaving the elements at the orbit invariant, etc... . This looks simple but is extremely difficult to realize in practice.

One working geometric approach of this kind to AGG relies on so called dessins d'enfant [A23] to be discussed later. These combinatorial objects provide an amazingly simple diagrammatic approach allowing to understand concretely what the action of AGG means geometrically at the level of algebraic Riemann surfaces. What is remarkable that every algebraic Riemann surface (with polynomials involved having algebraic coefficients) is compact by Belyi's theorem [A11] and bi-holomorphisms generate non-algebraic ones from these.

In TGD partonic 2-surfaces are the basic objects and necessarily compact. This puts bells ringing and suggests that the old idea about AGG as symmetry group of WCW might make sense in the algebraic intersection of real and p-adic worlds at the level of WCW identifies as the seat of life in TGD inspired quantum biology. Could this mean that AGG acts naturally on partonic 2-surfaces and its representations assign number theoretical quantum numbers to living systems? An intriguing additional result is that all compact Riemann surfaces can be representation as projective varities in CP_3 assigned to twistors. Could there be some connection?

16.1.3 Langlands program

Another approach to AGG is algebraic and relies on finite-dimensional representations of AGG. If one manages to construct a matrix representation of AGG, one can identify AGG invariants as eigenvalues of the matrices characterizing their AGG conjugacy class. Langlands correspondence [A150, A148] is a conjecture stating that the representations of adelic variants of algebraic matrix groups [A3] .

Adelic representations are obtained by replacing the matrix elements with elements in the ring of rational adeles which is tensor product of rationals with Cartesian product of real numbers and all p-adic number fields with and they provide representations of AGG. Ideles represent elements of abelianization of AGG. Various completions of rationals are simply collected to form single super structure.

Number theoretic invariants - such as numbers for points of certain elliptic curves (polynomials with integer coefficients) - correspond to invariants for the representations of algebraic groups assignable to the automorphic functions defined in the upper plane $H = SL(2,R)/O(2)$ and invariant under certain subgroup Γ of modular group acting as modular symmetries in this space and defining in this manner an algebraic Riemann surface as a coset space H/Γ with finite number of cusps in which the automorphic function vanishes. The vanishing conditions coded by Γ code also for number theoretic information.

The conjecture is that number theoretic questions could allow translation to questions of harmonic analysis and algebraic equations would be replaced by differential equations much simpler to handle. Also a direct connection with subgroups of modular group Γ of $SL(2,Z)$ emerges and number theoretic functions like zeta and η functions emerge naturally in the complex analysis.

The notion of adeles generalizes. Instead of rationals one can consider any extension of rationals and the MAGG and AGG associated with it. p-Adic number fields of the adele are replaced with their extensions and algebraic extension of rationals appears as entanglement coefficients. This also conforms with the TGD based vision about evolution and quantum biology based on a hierarchy of algebraic extensions of rationals. For these reasons it seems that adeles or something akin to them is tailor-made for the goals and purposes of TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [L18]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [L19]. The topics relevant to this chapter are given by the following list.

- Physics as generalized number theory [L29]
- Quantum physics as generalized number theory [L30]

16.2 Langlands program

Langlands programs starts from the idea that finite-dimensional representations of AGG provide information about AGG. If one manages to construct a matrix representation of AGG, one can identify AGG invariants as eigenvalues of the matrices characterizing their AGG conjugacy class. Langlands correspondence [A150, A148] is a conjecture stating that the representations of adelic variants of algebraic matrix groups [A3] .

Adelic representations are obtained by replacing the matrix elements with elements in the ring of adeles and they provide representations of AGG. Number theoretic invariants - such as numbers for points of certain elliptic curves (polynomials with integer coefficients) - correspond to invariants for the representations of algebraic groups assignable to the automorphic functions defined in the upper plane $H = SL(2,R)/O(2)$ and invariant under certain subgroup Γ of modular group acting as modular symmetries in this space and defining in this manner an algebraic Riemann surface as a coset space H/Γ with finite number of cusps in which the automorphic function vanishes. The vanishing conditions coded by Γ code also for number theoretic information.

Langlands conjecture states that number theoretic questions could allow translation to questions of harmonic analysis and algebraic equations would be replaced by differential equations much simpler to handle. Also a direct connection with subgroups of modular group Γ of $SL(2,Z)$ emerges and number theoretic functions like zeta and η functions emerge naturally in the analysis. I hasten to admit that I have failed to understand intuitively the deeper motivations for this conjecture but there is support for it.

16.2.1 Adeles

This approach leads to adeles [A4].

1. AGG is extremely complex and the natural approach is to try something less ambitious first and construct representations of the Maximal Abelian Galois Group of rationals (MAGG) [A60] assigned to an extension containing all possible roots of unity. One can show that

MAGG is isomorphic to the group of invertible adeles divided by rationals. This is something concrete as compared to AGG albeit still something extremely complex.

2. The ring of rational adeles [A4] discovered by Chevalley is formed by the Cartesian product of all p-adic number fields and of reals and its non-vanishing elements have the property that only finite number of p-adic numbers in $(...., a_{p_n}, ...) \times a$ are not p-adic integers (that is possess norm > 1). Algebraic operations are purely local: multiplications in every completion of rationals involved. One can also understand this space as a tensor product of rationals with integer adeles defined by the cartesian product of reals and various p-adic integers. One can say that adeles organize reals and all p-adic number fields to infinite-dimensional Cartesian product and that identified rational numbers as common to all of them so that multiplication by rational acts just as it act in a finite dimensional Cartesian product. The idea that rationals are common to all completions of rationals is fundamental for quantum TGD so that adeles are expected to be important.

3. The ring property of adeles makes possible to talk about polynomials of adele valued argument having rational coefficients and one can extend algebraic geometry to adeles as long as one talks about varieties defined by polynomials. Existence of polynomials makes it possible to talk about matrices with adele valued elements. The notion of determinant is well-defined and one can also define the inverse of adele matrix so that classical algebraic groups have also adele counterpart. This is of utmost significance in Langlands program and means a breathtaking achievement in book keeping: all the p-adic number fields would be caught under single symbol "A"!

4. Ideles are rational adeles with inverse. Ideles form a group but sum of two ideles is not always idele so that ideles do not form a number field and one cannot dream of constructing genuine differential calculus of ideles or talking about rational functions of ideles. Also rational functions fail to make sense. This means quite a strong constraint: if one wants adelic generalization of physics the solutions of field equations must be representable in terms of polynomials or infinite Taylor series.

The conjecture of Langlands is that the algebraic groups with matrix elements replaced with adeles provide finite-dimensional representations of adeles in what can be loosely called group algebra of adelic algebraic group.

The construction of representation uses complex valued functions defined in the ring of adeles. This function algebra decomposes naturally to a tensor product of function algebras associated with reals and various p-adic number fields and one can speak about rational entanglement between these functions. From the TGD point of view this is very interesting since rational entanglement plays a key role in TGD inspired quantum biology.

16.2.2 Construction of representations of adelic Gl_2

I have explained some details about the construction of the representation of adelic Gl_2 in the Appendix and earlier in [K35].

1. The basic idea is to start from the tensor product of representations in various completions of rationals using the corresponding group algebras. It is natural to require that the functions are invariant under the *left* multiplication by $Gl_2(Q)$ and eigenstates of $Gl_2(R)$ Casimir operator C under the *right* multiplication. The functions are smooth in the sense that they are smooth in $Gl_2(R)$ and locally constant in $Gl_2(Q_p)$.

2. The diagonal subgroup $Z(A)$ consists of products of diagonal matrices in $Gl_2(A)$. Characters are defined in $Z(A)$ as group homomorphisms to complex numbers. The maximal compact subgroup $K \subset Gl_2(A)$ is the Cartesian product of $Gl_2(Z_p)$ and $O_2(R)$ and finite-dimensionality under the action of these groups is also a natural condition.

3. The representations functions satisfy various constraints described in detail in the appendix and in the article of Frenkel [A150]. I just try to explain what I see as the basic ideas.

(a) Functions f form a finite-dimensional vector space under the action of elements of the maximal compact subgroup K. Multiplication from left by diagonal elements reduces to a multiplication with character. The functions are eigenstates of the Casimir operator of $Gl_2(R)$ acting from left with a discrete spectrum of eigen values. they are bounded in $Gl_2(A)$. These conditions are rather obvious.

(b) Besides this the functions satisfy also the so called cuspidality conditions, the content of which is not obvious for a novice like me. These conditions imply that the functions are invariant under the action for $Gl_2(Z_p)$ apart from finite number of primes called ramified. For these primes invariance holds true only under subgroup $\Gamma_0(p^{n_k})$ of $Sl_2(Z_p)$ consisting of 2×2-matrices for which the elements $a_{21} \equiv c$ vanish modulo p^n.

(c) What is non-trivial and looks like a miracle to a physicist is that one can reduce everything to the study of so called automorphic functions [A10] defined in $\Gamma_0(N)/Sl(2,R)$, $N = \prod p^{n_k}$. Intuitively one might try to understand this from the idea that adeles for which elements in Z_p are powers of p represent rational numbers. That various p-adic physics somehow factorize the real physics would be the misty idea which in TGD inspired theory of consciousness translates to the idea that various p-adic physics make possible cognitive representations of real physics. Somehow the whole adele effectively reduces to a real number. Automorphic functions have a number theoretic interpretation and this is certainly one of the key motivations between Langlands program.

4. Automorphic functions reduce to complex analytic functions in the upper half plane $H = SL_2(R)/O(2)$ transforming in a simple manner under $\Gamma_0(N)$ (modular form of weight k). What one is left with are modular forms of weight k and level N in upper half plane.

(a) The overall important cuspidality conditions characterized by integer N imply that the automorphic functions vanish at the cusp points of the algebraic Riemann surface defined as $H/\Gamma_0(N)$. The modular form can be expanded in Fourier series $f = \sum a_n q^n$ in powers of $q = exp(i2\pi\tau)$, where τ parameterizes upper half plane.

(b) The Fourier coefficients a_n satisfy the condition $a_{mn} = a_m a_n$ and one ends up with the conclusion that for each elliptic curve [A28] $y^2 = x^3 + ax + b$ (a and b are rational numbers satisfying $4a^3 + 27b^2 \neq 0$ and reduce to integer is the recent case) there should exist a modular form with the property that a_p codes for the numbers of points of this elliptic curve in finite field F_p for all but finite number of primes! This is really amazing and mysterious looking result.

(c) τ can be interpreted as a complex coordinate parametrizing the conformal moduli of tori. Is this a pure accident or could this relate to the fact that the coefficients turn out to give numbers of roots for algebraic elliptic surfaces, which are indeed tori? Could cuspidality conditions have interpretation as vanishing of the modular forms for tori with moduli corresponding to cusps: could these be are somehow singular as elliptic surfaces? The objection is that the elliptic surfaces as sub-manifolds of C^2 have a unique induced metric and therefore correspond to a unique conformal modulus τ. But what about other Kähler metrics than the standard metric for C^2 and imbeddings to other complex spaces as algebraic surfaces? Could adelic Gl_2 representations generalize to adelic representations of Gl_{2g} acting on Teichmueller parameters of Riemann surface with genus g?

The notion of adeles generalizes. Instead of rationals one can consider any extension of rationals and the MAGG and AGG associated with it. p-Adic number fields of the adele are replaced with their extensions and algebraic extension of rationals appears as entanglement coefficients. This also conforms with the TGD based vision about evolution and quantum biology based on a hierarchy of algebraic extensions of rationals. For these reasons it seems that adeles or something akin to them is tailor-made for the goals and purposes of TGD.

16.3 Compactness is guaranteed by algebraicity: dessins d'enfant

"*This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focussed. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any childs drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.*"

This piece of text was written by Grothendieck. He described here the profound impact of the notion of dessins d'enfant [A23] on him. The translation of the notion to english is "child's drawings". These drawings are graphical representations of Riemann surfaces understood as pairs formed by an algebraic Riemann surface and its universal covering space from which Riemann surface is obtained as a projection which can be many-to-one one map. This diagram allows to construct the Riemann surface modulo bi-holomorphism. Algebraic Riemann surface means that the equations defining it involve only rational functions with coefficients which are algebraic numbers. This implies that the action of AGG on the algebraic Riemann surface is well defined as action on the coefficients. One can assign to the dessin d'enfant combinatorial invariants for the action of AGG.

16.3.1 Dessins d'enfant

1. Dessin d'enfant is a bipartite graph (`http://en.wikipedia.org/wiki/Bipartite_graph`) [A173] meaning that it is possible to label the nodes of the graphs by black and white points in such a manner that the black and white points alternate along edge paths. One can identify black and white nodes as sets U and V and every edge of the graph connects points of U and V. For instance, bipartite graph does not posses any odd edge cycles. Every tree is bipartite and every planar graphs with even number of edges is bipartite. The vertices of the bipartite graph are topologically characterized by the number of lines emerging to the vertex and also 2-vertices are possible. The surface and the embedding can be described combinatorially using rotation system assigned with each vertex of the graph and telling the order in which the edges would be crossed by a path that travels clockwise on the surface around the vertex.

2. The notions of dessin d'enfant and counterpart for Belyi function [A11] defining the projection from the covering of sphere to sphere dates back to the work of Felix Klein. A very deep and very surprising theorem by Belyi (`http://en.wikipedia.org/wiki/Belyi's_theorem`) states that all algebraic curves represent compact Riemann surfaces. These surfaces are ramified coverings of the Riemann sphere ramified at three points only which in suitable complex coordinates can be taken to be the rational points 0,1, ∞ of real axis. Ramification means that the rational function f with algebraic number coefficients - known as Belyi's function - projecting the Riemann surface as covering of sphere to sphere has critical points which are pre-images of these three points. In the neighborhood of the critical points the projection map known as Belyi's function is characterized by degree telling how many points are mapped to single point of sphere. At the critical point itself these points coincide. A simplified example of criticality is z^n at origin.

   The Riemann surface in question can be taken to be H/Γ compactified by finite number of cusp points. Here H is upper half plane Γ a subgroup of modular group having finite index

3. Dessin d'enfant allows to code combinatorially the data about the Belyi function so that one can construct both the surface and its Belyi function from this data apart from bi-holomorhism. The interpretation as projection from covering allows to get grasp about the geometric meaning of dessin d'enfant. Physicist reader is probably familiar with the graphical representation of cusp catastrophe. The projection of the critical points and curves of cusp

catastrophe as function of the two control parameters to the control parameter plane replaced in the recent case by complex plane is highly analogous to dessin d'enfant. The boundary of cusp catastrophe in which cusp projection is three-to-one has V-shape and at the sides of V the covering of plane is 2-to-1 and and at the vertex and outside cusp region 1-to-1. The edges of V correspond to the edges of the dessin d'enfant and the vertex of V to a node of dessin d'enfant.

The number of edges entering given critical point tells the degree of the Belyi function at that critical point. Dessin d'enfant is imbedded on an oriented surface - plane in the simplest situation but also sphere and half plane can be considered. The lines of the graph correspond to curves at which two branches of the covering coincide.

The Wikipedia article [A23] about dessin d'enfant discusses a nice example about the construction of dessin d'enfant and is recommended for the reader.

4. The Belyi function could be any holomorphic function from X to Riemann sphere having only 0,1, and ∞ as critical values and the function f is determined only up to bi-holomorphism. If X is algebraic surface, f is rational function with algebraic coefficients.

5. What makes the dessin d'enfant so remarkable is that AGG has natural action on the algebraic coefficients of the rational functions defining algebraic Riemann surfaces and therefore on dessin d'enfant. For instance, the sequence of integers form by the degrees of the projection map at the critical points is geometric Galois invariant. One can identify the stabilize of dessin as the sub-group of AGG leaving dessin d'enfant invariant. One can identify the orbit of dessin d'enfant under AGG and the subgroup of AGG leaving the points of orbit invariant.

16.3.2 Could one combine quantum adelic representations with dessin d'enfant representations?

As already noticed, dessin d'enfant representation of AGG allows to have representations of AGG at the orbits of dessins d'enfant. If the orbit consists of a finite number n of points, one obtains representations of AGG in the finite-dimensional discrete Hilbert space spanned by the points, and representation matrices are $n \times n$ matrices.

Suppose that the Galois group of quantum adeles is indeed isomorphic with the commutator group of AGG. If this is the case then quantum adele valued amplitudes defined in the discrete space formed by the orbits of dessins d'enfant would provide a representation of AGG with commutator group acting on the fiber analogous to spin degrees of freedom and AGG on the base space having role analogous to that of Minkowski space.

One can imagine an approach mimicking the construction of induced representations [A50] of Mackey inspired by the representations of Poincare group. In this approach one identifies orbit of group G as a space carrying the fields with spin. The subgroup H of G leaving a given point of representation space invariant is same at all points of orbit apart from conjugation. The field would have values in H or group algebra of H or in space in which H acts linearly. In the recent case H could adelic Galois group of quantum adeles identified as AGG or the subgroup G_I of AGG leaving the dessins d'enfant invariant.

What can one say about G_I. How large it is? Can one identify it or its abelization A_{G_I} and assign it to the points of orbits to construct analogs of induced representations?

1. If the orbit of dessin d'enfant is finite as the fact that the number of its points is invariant under the action of AGG suggests, G_I must be infinite. This would suggests that also A_{G_I} is infinite. Does A_{G_I} possess adele representation? Is this adele representation identifiable as a sub-adele of A_{AGG} in some sense? Could it be obtained by dropping some quantum variants of Z_p:s from the decomposition of adele? What the interpretation of these lacking primes could be? Could these primes correspond to the primes which split in the extensions. If this is the case one could consider the representations in which A_{G_I} forms the fiber space at each point of dessin d'enfant.

2. One can consider also weaker option for which only so called ramified primes are dropped from the adele for rationals to obtain the adele for algebraic extension. In adele construction there

are problematic primes p. For rational primes (or corresponding ideals) the representation of p is as a product of primes of extension as $p = \prod P_i^{e_i}$ e_i are called degrees of ramification. For some $e_i > 1$ one has ramification analogous to the dependence of form $(z - z_0)^n$, $n > 1$ of holomorphic function around critical point have interpretation as ramified primes and corresponding factors Z_p are dropped from the adele. To eliminate the problems cause by number theoretic ramification one can t drop ramified primes from the adele in the extensions of algebraic numbers associated with the roots of the polynomials appearing in the Belyi map. Could the resulting adele be the counterpart for the reduced MGGA?

16.3.3 Dessins d'enfant and TGD

What might be the relevance of Belyi's theorem and dessins d'enfant for TGD?

1. In TGD framework effective 2-dimensionality implies that basic objects are partonic 2-surfaces together with their data related to the 4-D tangent space a them. I have already earlier proposed that Absolute Galois group could have a natural action in the world of the classical worlds (WCW). The horrible looking problem is how to achieve General Coordinate Invariance (GCI) for this action.

   Partonic 2-surfaces are compact so that they allow a representation as algebraic surfaces. The notion of dessin d'enfant suggests that partonic 2-surfaces could be described as simple combinatorial objects defined by dessin d'enfant as far as the action of Galois group is considered. This representation would be manifestly general coordinate invariant and would allow to construct representations as Galois group in terms of discrete wave functions at the orbits of dessin d'enfant. One can also expect that the representation reduces to those of finite Galois groups.

2. Second central problem is the notion of braid which is proposed to provide a realization for the notion of finite measurement resolution.The recent view is that time-like braids on light like surfaces and space-like braids at the 3-surfaces defining the ends of space-time surfaces contain braid strands as Legendrian knots for which the projection of Kähler gauge potential has vanishing inner product with the tangent vector of the braid strand. For light-like 3-surfaces this does not imply that the tangent vector of strand is orthogonal to the strand: if the tangent vector is light-like the condition is automatically satisfied and light-like braid strands define a good but - as it seems - not a unique guess for what the braid strands are. Note however that the condition that braid strands correspond to boundaries of string world sheets gives additional conditions. At space-like 3-surfaces orthogonality to induced Kähler gauge potential fixes the direction of the tangent vector field only partially.

   Suppose one manages to fix completely the equations for braid strands - say by the identification as light-like strands. What about the end points of strands? How uniquely their positions are determined? Number theoretical universality suggests that the end points are rational or algebraic points as points of imbedding space but again GCI poses a problem. Symmetry arguments suggest that one could use group theoretically preferred coordinates for M^4 and CP_2 and identify also the coordinates of partonic 2-surface as imbedding space coordinates for their projections to geodesic spheres of $\delta M^4_\pm$ and geodesic sphere of CP_2.

   A possible resolution of this problem comes from the fact that partonic 2-surface allows an interpretation as algebraic surface. Braid ends could correspond to the critical points of the Belyi function defining the projection from the covering so that they would be algebraic points in the complex coordinates of partonic 2-surfaces fixed apart from algebraic bi-holomorphism. One would a concrete topological interpretation for why the braid ends are so special. I have already earlier proposed that braid ends could correspond to singularities associated with coordinate patches.

3. Is it possible to have compact Riemann which cannot be represented as algebraic surfaces? Belyi's theorem does not deny this. For instance rational functions with real coefficients for polynomials are possible and must give rise to compact surfaces. Inherently non-algebraic partonic 2-surfaces are possible and for them one cannot define representations of AGG at the orbits of dessin d'denfant since the action of AGG on f is not well defined now.

This relates in an interesting manner to the conjecture [K42] that life resides in the in the intersection of real and p-adic worlds. At WCW level this would mean that the equations for the partonic 2-surfaces makes sense in any completion of rationals. For algebraic partonic 2-surfaces this is indeed the case if arbitrary high-dimensional algebraic extensions of p-adic numbers are allowed. Taking this seriously one can ask whether the existence of the representations of Galois group at the level of WCW is an essential aspect of what it is to be living. Could one assign Galois quantum numbers to the quantum states of living system? These would be realized in the discrete space provided by different quantum counterparts of a given integer and one would have discrete wave functions in these discrete spaces.

4. One also learns from Wikipedia that any compact Riemann surface is a projective variety and thus representable using polynomial equations in projective space. It also allows an imbedding as as a surface n 3-dimensional complex projective space CP_3. Wikipedia states that if compactness condition is added the Riemann the surface is necessarily algebraic: here however algebraic means rational functions with arbitrary real or complex coefficients. Above it means algebraic coefficients. Whether this CP_3 could have anything to do with the twistor space appearing in Witten's twistor string model [B39] and also in the speculated twistorial formulation of TGD [K79] remains an open question.

5. Modular invariance plays central role in TGD [K17], and a natural additional condition on the representations of AGG would be that the quantum states in WCW are modular invariant. The action of AGG induces a well-defined action on the conformal moduli of the partonic 2-surfaces and therefore on Teichmueller parameters. This discrete action need not be simple - say linear- but it would be action in n-dimensional space. Modular invariance requires that the action of AGG transformation induces a conformal scaling of the induced metric and changes the conformal moduli by an action of modular group $Sl(2g, Z)$. For torus topology this group is Sl(2,Z) appearing in modular invariant functions assigned to the representations of AGG in the group algebra of adelic algebraic groups.

6. Could the combination of dessins d'enfant as a geometric representation and adelic matrix representations for the abelianizer of the isotropy group G_I of dessin d'enfant provide additional insights in to Langlands conjecture? The problem is that AGG elements do not leave MGGA invariant.

7. Bi-partite graphs appear also in the construction of inclusions of hyper-finite factors of type II_1 (HFF). The TGD inspired proposal that AGG allows identification as S_∞ and the group algebra of permutation group S_∞ is HFF. In optimistic mood one might see dessins d'enfant as a piece of evidence for this identification of AGG and adele formed from the Galois group of quantum p-adic integers as its commutator group.

16.4 Appendix: Basic concepts and ideas related to the number theoretic Langlands program

The following representation of the basic ideas of Langlands program reflects my very limited understanding of the extremely refined conceptual framework involved. This pieces of text can be found almost as such also in [K35] and Ed Frenkel provides more detailed discussion in his article [A150, A148].

16.4.1 Langlands correspondence and AGG

The representations of group carry information about the group and the natural question is how to represent the AGG and deduce invariants of AGG in this manner. Eigenvalues for the representation matrices are invariants characterizing conjugacy classes of the group. The generators of MAGG abelled by primes define so called Frobenius elements and the eigenvalues and traces for their representation matrics defined invariants of this kind. The big question is how to construct representations of the AGG. Langlands program is an attempt to answer this question.

1. 1-D representations of AGG corresponds those of maximal Abelian Galois group which is the factor group of AGG by its commutator group. The natural intuitive guess is that the n-dimensional representations of AGG in the group algebra of adelic algebraic group $Gl(n)$ could provide higher-dimensional representations of AGG. $Gl(n)$ would give rise to a kind of AGG spin. The action of AGG commutator group would be mapped to$GL_n(A)$ action. Does this mean that AGG is mapped homomorphically to adelic matrices in $Gl_n(A)$ as one might first think? I am not able to answer the question. From Wikipedia one learns that so called Langlands dual [A55] extends AGG by the algebraic Lie group G_L so that one obtains semi-direct product of complex G_L with the AGG which acts on the algebraic root data of G_L. The adelic representations of G_L are said to control those of G. In this form the correspondence gives information about group representations rather than number theory.

   Remark: One naive guess would be that one could realize the representations of AGG by adjoint action $x \rightarrow gxg^{-1}$ in the commutator subgroup of AGG, which is maximal normal subgroup and closed with respect to this action. Also the adjoint action of the factor group defined my maximal Abelian group in this group could define representation? The guess of the outsider is that the practical problem is that the commutator group is not known.

2. Number theoretic Langlands program is however more than study of the relationships between representations of $Gl(F)$ and its adelic variant $Gl(A_F)$. The basic conjecture is the existence of duality between number theory and harmonic analysis. On number theoretical side one typically studies algebraic curves. Typical question concerns the number of rational points in modulo p approximation to the equations determining the algebraic curve. The conjecture about number theoretic Langlands correspondence was inspired by the observation that Fourier series expansions of automorphic forms code via their coefficients this kind of data and the proof of Fermat's theorem can be seen as application of this correspondence.

   There is support for the conjecture that adelic representations carry purely number theoretic information in the case of $Gl(n)$. The number theoretical invariants defined by the trace for the representation matrix for the Frobenius element generating the Abelian Galois group would corresponds to the trace of so called Hecke operator at the side of the harmonic analysis.

3. Intuitive motivations for the Langlands duality come from the fact the notion of algebraic surface defined by a polynomials with integer coefficients is number theoretically universal: the argument can belong to finite field, rational numbers or their extension, real numbers, or any p-adic number field and can represent even element of function field. Function fields defined algebraic functions at algebraic curves in finite fields are somehow between classical number fields and function fields associated with Riemann surfaces to which one can apply the tools of harmonic analysis.

16.4.2 Abelian class field theory and TGD

The context leading to the discovery of adeles (`http://en.wikipedia.org/wiki/Adele_ring`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p \ mod \ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p \ mos \ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (`http://en.wikipedia.org/wiki/Class_field_theory`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

Adeles and ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://en.wikipedia.org/wiki/Class_field_theory`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [A56, A150, A148]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where imbedding space is replaced with Cartesian product of real imbedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://en.wikipedia.org/wiki/Adele_ring`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod_p' Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (`http://en.wikipedia.org/wiki/Idele_class_group`Idele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

16.4.3 Langlands correspondence and modular invariance

A strong motivation for Langlands correspondence is modular invariance - or rather its restricted form - which emerges in both number theory and in the automorphic representations of Gl_2 and relates directly to the ramification of primes for Galois extensions- now maximal Abelian extension. In TGD framework the restricted modular invariance could have interpretation in terms of concrete representations of AGG involving the action of AGG on the adelic variants of Teichmueller parameters characterizing the algebraic surfaces its variants in various number fields.

It is not necessary to know the explicit action of AGG to modular parameters. What is however needed is modular invariance in some sense. The first - and hard-to-realize - option is that allowed subgroup of AGG leaves the conformal equivalence class of Riemann surface invariant. Second option is that the action of both AGG and modular group $Sl(2g, Z)$ or its subgroup leave the states of representation invariant. This is the case if AGG induces Gl_{2g} transformations in each Cartesian factor of the adele and the states defined in the group algebra of Gl_{2g} are invariant. For ramified primes however modular invariance can break down to subgroup of Sl_{2g}. These conditions lead to automorphic modular forms.

These arguments are very heuristic and following arguments due to Frenkel give better view about the situation.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL_e(2, A)$ and more generally $GL_e(n, A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [A62] , which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL_e(n, A)$.

2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL_e(n, A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n, R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k = 1, .., n$ acting in group algebra of $Gl(n, R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

3. The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_e^L(F)$ of $G(F)$.

16.4.4 Correspondence between n-dimensional representations of $Gal(\overline{F}/F)$ and representations of $GL_e(n, A_F)$ in the space of functions in $GL_e(n, F)\backslash GL_e(n, A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^\times$, where $Z_p^\times$ consists of invertible p-adic integers [A150] .

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p\in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^\times \backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^\times \backslash A_F^\times$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL_e(1, A_F)$ in the space of functions defined in $GL_e(1, F)\backslash GL_e(1, A_F)$.*

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL_e(n, A_F)$ in the space of functions defined in $GL_e(n, F)\backslash GL_e(n, A_F)$.*

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

2. The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \to \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \to \infty$ from a diagonal imbedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \to \infty$ the imbedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

3. These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [A81] . In the case of elliptic curves [A150] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL_e(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL_e(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL_e(n, E)$.

   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative manners to state the same thing.

Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \rightarrow Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [A150] . The representations can be also l-adic being defined in $GL_e(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [A150] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p, 1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^\times$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^\times|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p \ mod \ n$.

Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [A150] for the route from automorphic adelic representations of $GL_e(2, R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL_e(2, Q)$ are constructed in the space of smooth bounded functions $GL_e(2, Q)\backslash GL_e(2, A) \rightarrow C$ or equivalently in the space of $GL_e(2, Q)$ left-invariant functions in $GL_e(2, A)$. A denotes adeles and $GL_e(2, A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL_e(2, A)$. The motivating idea is the central role of double coset decompositions $G = K_1 A K_2$, where K_i are compact subgroups and A denotes the space of double cosets $K_1 g K_2$ in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.

   To my best understanding $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL_e(2, Z_p)$. For ramified primes K_p consists of $SL_e(2, Z_p)$ matrices with $c \in p^{n_p} Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2, R)$ with eigenvalue ρ so that irreducible representations of $gl(2, R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+ X - + X_- X_+ \ ,$$

where one has

$$X_0 \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array} \right) .$$

3. The center $A^{\times}$ of $GL_e(2,A)$ consists of $A^{\times}$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^{\times} \rightarrow C$ is a character providing a multiplicative representation of $A^{\times}$.

4. Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g) du = 0$$

is satisfied [A150] . Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

The groups $gl(2,R)$, $O(2)$ and $GL_e(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL_e(2,A_F) \times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL_e(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL_e(2,Q)\backslash GL_e(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is conductor [A150]. This means enormous simplification since one gets ride of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Carteisian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL_e(2,Z)$ consists of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \quad , \quad c \ mod \ N = 0.$$

$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL_e(2,Z)$ so that also $SL_e(2,Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \rightarrow \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL_e(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL_e(2,R)$ to upper half-plane $H_u = SL_e(2,R)/SO(2)$

The representations of $(gl(2,C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A150] . For the discrete series representation π giving square integrable representation in $SL_e(2,R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ \ , \ \ X_+ v_\infty = 0 \ \ .$$

The latter module is in turn generated by the lowest weight vector

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_\infty \ \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL_e(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL_e(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL_e(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) \ = \ \sum_{n=0}^{\infty} a_n q^n \ \ . \tag{16.4.1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL_e(2,Z_p)$ bi-invariant functions on $GL_e(2,Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A150] .

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmueller parameters of algebraic Riemann surface are affected by absolute Galois group. This induces Sl(2g,Z) transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroup of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

Chapter 1

Appendix

Originally this appendix was meant to be a purely technical summary of basic facts but in its recent form it tries to briefly summarize those basic visions about TGD which I dare to regarded stabilized. I have added illustrations making it easier to build mental images about what is involved and represented briefly the key arguments. This chapter is hoped to help the reader to get fast grasp about the concepts of TGD.

The basic properties of imbedding space and related spaces are discussed and the relationship of CP_2 to standard model is summarized. The notions of induction of metric and spinor connection, and of spinor structure are discussed. Many-sheeted space-time and related notions such as topological field quantization and the relationship many-sheeted space-time to that of GRT space-time are discussed as well as the recent view about induced spinor fields and the emergence of fermionic strings. Various topics related to p-adic numbers are summarized with a brief definition of p-adic manifold and the idea about generalization of the number concept by gluing real and p-adic number fields to a larger book like structure. Hierarchy of Planck constants can be now understood in terms of the non-determinism of Kähler action and the recent vision about connections to other key ideas is summarized.

A-1 Imbedding space $M^4 \times CP_2$ and related notions

Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$ the Cartesian product of empty Minkowski space - the space-time of special relativity - and compact 4-D space CP_2 with size scale of order 10^4 Planck lengths. One can say that imbedding space is obtained by replacing each point m of empty Minkowski space with 4-D tiny CP_2. The space-time of general relativity is replaced by a 4-D surface in H which has very complex topology. The notion of many-sheeted space-time gives an idea about what is involved.

Fig. 1. Imbedding space $H = M^4 \times CP_2$ as Cartesian product of Minkowski space M^4 and complex projective space CP_2. `http://www.tgdtheory.fi/appfigures/Hoo.jpg`

Denote by M^4_+ and M^4_- the future and past directed lightcones of M^4. Denote their intersection, which is not unique, by CD. In zero energy ontology (ZEO) causal diamond (CD) is defined as cartesian product $CD \times CP_2$. Often I use CD to refer just to $CD \times CP_2$ since CP_2 factor is relevant from the point of view of ZEO.

Fig. 2. Future and past light-cones M^4_+ and M^4_-. Causal diamonds (CD) are defined as their intersections. `http://www.tgdtheory.fi/appfigures/futurepast.jpg`

Fig. 3. Causal diamond (CD) is highly analogous to Penrose diagram but simpler. `http://www.tgdtheory.fi/appfigures/penrose.jpg`

A rather recent discovery was that CP_2 is the only compact 4-manifold with Euclidian signature of metric allowing twistor space with Kähler structure. M^4 is in turn is the only 4-D space with Minkowskian signature of metric allowing twistor space with Kähler structure so that $H = M^4 \times CP_2$ is twistorially unique.

One can loosely say that quantum states in a given sector of "world of classical worlds" (WCW) are superpositions of space-time surfaces inside CDs and that positive and negative energy parts of zero energy states are localized and past and future boundaries of CDs. CDs form a hierarchy. One can have CDs within CDs and CDs can also overlap. The size of CD is characterized by the proper time distance between its two tips. One can perform both translations and also Lorentz boosts of CD leaving either boundary invariant. Therefore one can assign to CDs a moduli space and speak about wave function in this moduli space.

In number theoretic approach it is natural to restrict the allowed Lorentz boosts to some discrete subgroup of Lorentz group and also the distances between the tips of CDs to multiples of CP_2 radius defined by the length of its geodesic. Therefore the moduli space of CDs discretizes. The quantization of cosmic recession velocities for which there are indications, could relate to this quantization.

A-2 Basic facts about CP_2

CP_2 as a four-manifold is very special. The following arguments demonstrates that it codes for the symmetries of standard models via its isometries and holonomies.

A-2.1 CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-2.1}$$

Here λ is any non-zero complex number. Note that CP_2 can be also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three coordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [A210] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-2.1}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi + \Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi - \Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-2.1}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

Fig. 4. CP_2 as manifold. `http://www.tgdtheory.fi/appfigures/cp2.jpg`

A-2.2 Metric and Kähler structure of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i\bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}}d\xi^a d\bar{\xi}^b \ , \tag{A-2.2}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2\partial_a\partial_{\bar{b}}K \ , \tag{A-2.3}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-2.3}$$

The Kähler function for S^2 has the same form. It gives the S^2 metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r,\phi) = (tan(\theta/2),\phi)$.

The representation of the CP_2 metric is deducible from S^5 metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2+\sigma_2^2)}{F} \ , \tag{A-2.4}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-2.3}$$

R denotes the radius of the geodesic circle of CP_2. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2\sum_A e_k^A e_l^A \ , \tag{A-2.4}$$

are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , & e^1 &= \frac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \frac{r\sigma_2}{\sqrt{F}} \ , & e^3 &= \frac{r\sigma_3}{F} \ . \end{aligned} \tag{A-2.5}$$

The explicit representations of vierbein vectors are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , & e^1 &= \frac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\ e^2 &= \frac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , & e^3 &= \frac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \end{aligned} \tag{A-2.5}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = \frac{dr^2}{F^2} + \frac{r^2}{4F^2}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \tag{A-2.5}$$

The vierbein connection satisfying the defining relation

$$de^A = -V^A_B \wedge e^B \ , \tag{A-2.6}$$

is given by

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ . \end{array} \tag{A-2.7}$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{lllllll} R_{01} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , & R_{23} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} & = & e^0 \wedge e^2 - e^3 \wedge e^1 \ , & R_{31} & = & -e^0 \wedge e^2 + e^3 \wedge e^1 \ , \\ R_{03} & = & 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , & R_{12} & = & 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \tag{A-2.8}$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$J = -ig_{a\bar{b}}d\xi^a d\bar{\xi}^b \ , \tag{A-2.9}$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$J^k{}_r J^{rl} = -s^{kl} \ . \tag{A-2.10}$$

The form J is integer valued and by its covariant constancy satisfies free Maxwell equations. Hence it can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling). Locally one has therefore

$$J = dB \ , \tag{A-2.11}$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

It should be noticed that the magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$\begin{array}{lll} B & = & 2re^3 \ , \\ J & = & 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2}dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F}sin\Theta d\Theta d\Phi \ . \end{array} \tag{A-2.10}$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1,1).

Useful coordinates for CP_2 are the so called canonical coordinates in which Kähler potential and Kähler form have very simple expressions

$$
\begin{aligned}
B &= \sum_{k=1,2} P_k dQ_k \ , \\
J &= \sum_{k=1,2} dP_k \wedge dQ_k \ .
\end{aligned}
\tag{A-2.10}
$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$
\begin{aligned}
P_1 &= -\frac{1}{1+r^2} \ , \\
P_2 &= \frac{r^2 cos\Theta}{2(1+r^2)} \ , \\
Q_1 &= \Psi \ , \\
Q_2 &= \Phi \ .
\end{aligned}
\tag{A-2.8}
$$

A-2.3 Spinors in CP_2

CP_2 doesn't allow spinor structure in the conventional sense [A191] . However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1- factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential $exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

A-2.4 Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the imbedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [A165] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]] \in t \ \text{ for } \ X,Y,Z \in t \ . \tag{A-2.9}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S^2_I \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S^2_{II} \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S^2_I. S^2_{II} is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-3 CP_2 geometry and standard model symmetries

A-3.1 Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B26] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \tag{A-3.0}$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \times \gamma_5$, $1 \times \gamma_5$ and $\gamma_5 \times 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4) = SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \tag{A-3.1}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r-\frac{1}{r})e^3 \ , & V_{12} & = & (2r+\frac{1}{r})e^3 \ , \end{array} \tag{A-3.2}$$

and

$$B = 2re^3 \ , \tag{A-3.3}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \tag{A-3.4}$$

where one have defined

$$\begin{array}{lll} I_L^1 & = & \frac{(\Sigma_{01}-\Sigma_{23})}{2} \ , \\ I_L^2 & = & \frac{(\Sigma_{02}-\Sigma_{13})}{2} \ . \end{array} \tag{A-3.4}$$

A_{ch} is clearly left handed so that one can perform the identification

$$W^\pm = \frac{2(e^1 \pm ie^2)}{r} \ , \tag{A-3.5}$$

where $W^\pm$ denotes the charged intermediate vector boson.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$\begin{array}{lll} X & = & re^3 \ , \\ Y & = & \frac{e^3}{r} \ , \end{array} \tag{A-3.5}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\begin{array}{lll} \bar{\gamma} & = & aX+bY \ , \\ \bar{Z}^0 & = & cX+dY \ , \end{array} \tag{A-3.5}$$

where the normalization condition

$$ad-bc=1 \ ,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$\begin{aligned} A_{nc} &= [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\ &+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ . \end{aligned} \tag{A-3.4}$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5 \Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$c = -d \ . \tag{A-3.5}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-3.6}$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$\begin{aligned} Q_{em} &= \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ , \\ I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ . \end{aligned} \tag{A-3.6}$$

The fields γ and Z^0 are defined via the relations

$$\begin{aligned} \gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ , \\ Z^0 &= 4(a+b)\bar{Z}^0 = 4(X - Y) \ . \end{aligned} \tag{A-3.6}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} \ , \tag{A-3.7}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of Weinberg angle is a dynamical problem. The angle is completely fixed once the YM action is fixed by requiring that action contains no cross term of type γZ^0. Pure symmetry non-broken electro-weak YM action leads to a definite value for the Weinberg angle. One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) \ , \tag{A-3.8}$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \end{aligned} \tag{A-3.7}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-3.8}$$

Evaluating the expressions above one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{03} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{A-3.8}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \tag{A-3.9}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ , \end{aligned} \tag{A-3.9}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$\begin{aligned} X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\ K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ , \end{aligned} \tag{A-3.9}$$

In the general case the value of the coefficient K is given by

$$K = \sum_i \left[-\frac{(18+2n_i^2)sin^2\theta_W}{9}\right] \ , \tag{A-3.10}$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18+n_i^2))} \ . \tag{A-3.11}$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \tag{A-3.12}$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is quite close to the typical value 9/24 of GUTs [B40] .

A-3.2 Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

1. Symmetries must be realized as purely geometric transformations.
2. Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B13] .

The action of the reflection P on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-3.13}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-3.12}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \tag{A-3.12}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-4 The relationship of TGD to QFT and string models

TGD could be seen as a generalization of quantum field theory (string models) obtained by replacing pointlike particles (strings) as fundamental objects with 3-surfaces.

Fig. 5. TGD replaces point-like particles with 3-surfaces. `http://www.tgdtheory.fi/appfigures/particletgd.jpg`

The fact that light-like 3-surfaces are effectively metrically 2-dimensional and thus possess generalization of 2-dimensional conformal symmetries with light-like radial coordinate defining the analog of second complex coordinate suggests that this generalization could work and extend the super-conformal symmetries to their 4-D analogs.

The boundary $\delta M^4_+ = S^2 \times R_+$- of 4-D light-cone M^4_+ is also metrically 2-dimensional and allows extended conformal invariance. Also the group of isometries of light-cone boundary and of light-like 3-surfaces is infinite-dimensional since the conformal scalings of S^2 can be compensated by S^2-local scaling of the light-like radial coordinate of R_+. These simple facts mean that 4-dimensional Minkowski space and 4-dimensional space-time surfaces are in completely unique position as far as symmetries are considered.

String like objects obtained as deformations of cosmic strings $X^2 \times Y^2$, where X^2 is minimal surface in M^4 and Y^2 a holomorphic surface of CP_2 are fundamental extremals of Kähler action having string world sheet as M^4 projections. Cosmic strings dominate the primordial cosmology of TGD Universe and inflationary period corresponds to the transition to radiation dominated cosmology for which space-time sheets with 4-D M^4 projection dominate.

Also genuine string like objects emerge from TGD. The conditions that the em charge of modes of induces spinor fields is well-defined requires in the generic case the localization of the modes at 2-D surfaces -string world sheets and possibly also partonic 2-surfaces. This in Minkowskian space-time regions.

Fig. 6. Well-definedness of em charge forces the localization of induced spinor modes to 2-D surfaces in generic situtation in Minkowskian regions of space-time surface. http://www.tgdtheory.fi/appfigures/fermistring.jpg

TGD based view about elementary particles has two aspects.

1. The space-time correlates of elementary particles are identified as pairs of wormhole contacts with Euclidian signature of metric and having 4-D CP_2 projection. Their throats behave effectively as Kähler magnetic monopoles so that wormhole throats must be connected by Kähler magnetic flux tubes with monopole flux so that closed flux tubes are obtained.

2. Fermion number is carried by the modes of the induced spinor field. In Minkowskian space-time regions the modes are localized at string world sheets connecting the wormhole contacts.

Fig. 7. TGD view about elementary particles. a) Particle corresponds 4-D generalization of world line or b) with its light-like 3-D boundary (holography). c) Particle world lines have Euclidian signature of the induced metric. d) They can be identified as wormhole contacts. e) The throats of wormhole contacs carry effective Kähler magnetic charges so that wormhole contacts must appear as pairs in order to obtain closed flux tubes. f) Wormhole contacts are accompnied by fermionic strings connecting the throats at same sheet: the strings do not extend inside the wormhole contacts. http://www.tgdtheory.fi/appfigures/elparticletgd.jpg

Particle interactions involve both stringy and QFT aspects.

1. The boundaries of string world sheets correspond to fundamental fermions. This gives rise to massless propagator lines in generalized Feynman diagrammatics. One can speak of "long" string connecting wormhole contacts and having hadronic string as physical counterpart. Long strings should be distinguished from wormhole contacts which due to their super-conformal invariance behave like "short" strings with length scale given by CP_2 size, which is 10^4 times longer than Planck scale characterizing strings in string models.

2. Wormhole contact defines basic stringy interaction vertex for fermion-fermion scattering. The propagator is essentially the inverse of the superconformal scaling generator L_0. Wormhole contacts containing fermion and antifermion at its opposite throats beheave like virtual bosons so that one has BFF type vertices typically.

3. In topological sense one has 3-vertices serving as generalizations of 3-vertices of Feynman diagrams. In these vertices 4-D "lines" of generalized Feynman diagrams meet along their 3-D ends. One obtains also the analogs of stringy diagrams but stringy vertices do not have the usual interpretation in terms of particle decays but in terms of propagation of particle along two different routes.

Fig. 8. a) TGD analogs of Feynman and string diagrammatics at the level of space-time topology. b) The 4-D analogs of both string diagrams and QFT diagrams appear but the interpretation of the analogs stringy diagrams is different. http://www.tgdtheory.fi/appfigures/tgdgraphs.jpg

A-5 Induction procedure and many-sheeted space-time

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

Induction procedure for gauge fields

Induction procedure for gauge potentials and spinor structure is a standard procedure of bundle theory. If one has imbedding of some manifold to the base space of a bundle, the bundle structure can be induced so that it has as base space the imbedded manifold. In the recent case the imbedding of space-time surface to imbedding space defines the induction procedure. The induce gauge potentials and gauge fields are projections of the spinor connection of the imbedding space to the space-time surface. Induction procedure makes sense also for the spinor fields of imbedding space and one obtains geometrization of both electroweak gauge potentials and of spinors.

Fig. 9. Induction of spinor connection and metric as projection to the space-time surface. `http://www.tgdtheory.fi/appfigures/induct.jpg`

Induced gauge fields for space-times for which CP_2 projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP_2 projection, only vacuum extremals and space-time surfaces for which CP_2 projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP_2 projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \quad .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP_2 projection color rotations and weak symmetries commute.

A-5.1 Many-sheeted space-time

TGD space-time is many-sheeted: in other words, there are in general several space-sheets which have projection to the same M^4 region. Second manner to say this is that CP_2 coordinates are many-valued functions of M^4 coordinates. The original physical interpretation of many-sheeted space-time time was not correct: it was assumed that single sheet corresponds to GRT space-time and this obviously leads to difficulties since the induced gauge fields are expressible in terms of only four imbedding space coordinates.

Fig. 10. Illustration of many-sheeted space-time of TGD. `http://www.tgdtheory.fi/appfigures/manysheeted.jpg`

Superposition of effects instead of superposition of fields

The first objection against TGD is that superposition is not possible for induced gauge fields and induced metric. The resolution of the problem is that it is effects which need to superpose, not the fields.

Test particle topologically condenses simultaneously to all space-time sheets having a projection to same region of M^4 (that is touches them). The superposition of effects of fields at various space-time sheets replaces the superposition of fields.This is crucial for the understanding also how GRT space-time relates to TGD space-time, which is also in the appendix of this book).

Wormhole contacts

Wormhole contacts are key element of many-sheeted space-time. One does not expect them to be stable unless there is non-trivial Kähler magnetic flux flowing through then so that the throats look like Kähler magnetic monopoles.

Fig. 11. Wormhole contact. `http://www.tgdtheory.fi/appfigures/wormholecontact.jpg`

Since the flow lines of Kähler magnetic field must be closed this requires the presence of another wormhole contact so that one obtains closed monopole flux tube decomposing to two Minkowskian pieces at the two space-time sheets involved and two wormhole contacts with Euclidian signature of the induced metric. These objects are identified as space-time correlates of elementary particles and are clearly analogous to string like objects.

The relationship between the many-sheeted space-time of TGD and of GRT space-time

The space-time of general relativity is single-sheeted and there is no need to regard it as surface in H although the assumption about representability as vacuum extremal gives very powerful constraints in cosmology and astrophysics and might make sense in simple situations.

The space-time of GRT can be regarded as a long length scale approximation obtained by lumping together the sheets of the many-sheeted space-time to a region of M^4 and providing it with an effective metric obtained as sum of M^4 metric and deviations of the induced metrics of various space-time sheets from M^4 metric. Also induced gauge potentials sum up in the similar manner so that also the gauge fields of gauge theories would not be fundamental fields.

Fig. 12. The superposition of fields is replaced with the superposition of their effects in many-sheeted space-time. `http://www.tgdtheory.fi/appfigures/fieldsuperpose.jpg`

Space-time surfaces of TGD are considerably simpler objects that the space-times of general relativity and relate to GRT space-time like elementary particles to systems of condensed matter physics. Same can be said about fields since all fields are expressible in terms of imbedding space coordinates and their gradients, and general coordinate invariance means that the number of bosonic field degrees is reduced locally to 4. TGD space-time can be said to be a microscopic description whereas GRT space-time a macroscopic description. In TGD complexity of space-time topology replaces the complexity due to large number of fields in quantum field theory.

Topological field quantization and the notion of magnetic body

Topological field quantization also TGD from Maxwell's theory. TGD predicts topological light rays ("massless extremals (MEs)) as space-time sheets carrying waves or arbitrary shape propagating with maximal signal velocity in single direction only and analogous to laser beams and carrying light-like gauge currents in the generi case. There are also magnetic flux quanta and electric flux quanta. The deformations of cosmic strings with 2-D string orbit as M^4 projection gives rise to magnetic flux tubes carrying monopole flux made possible by CP_2 topology allowing homological Kähler magnetic monopoles.

Fig. 13. Topological quantization for magnetic fields replaces magnetic fields with bundles of them defining flux tubes as topological field quanta. `http://www.tgdtheory.fi/appfigures/field.jpg`

The imbeddability condition for say magnetic field means that the region containing constant magnetic field splits into flux quanta, say tubes and sheets carrying constant magnetic field. Unless one assumes a separate boundary term in Kähler action, boundaries in the usual sense are forbidden except as ends of space-time surfaces at the boundaries of causal diamonds. One obtains typically pairs of sheets glued together along their boundaries giving rise to flux tubes with closed cross section possibly carrying monopole flux.

These kind of flux tubes might make possible magnetic fields in cosmic scales already during primordial period of cosmology since no currents are needed to generate these magnetic fields: cosmic string would be indeed this kind of objects and would dominated during the primordial period. Even superconductors and maybe even ferromagnets could involve this kind of monopole flux tubes.

A-5.2 Imbedding space spinors and induced spinors

One can geometrize also fermionic degrees of freedom by inducing the spinor structure of $M^4 \times CP_2$.

CP_2 does not allow spinor structure in the ordinary sense but one can couple the opposite H-chiralities of H-spinors to an $n = 1$ ($n = 3$) integer multiple of Kähler gauge potential to obtain a respectable modified spinor structure. The em charges of resulting spinors are fractional (integer valued) and the interpretation as quarks (leptons) makes sense since the couplings to the induced spinor connection having interpretation in terms electro-weak gauge potential are identical to those assumed in standard model.

The notion of quark color differs from that of standard model.

1. Spinors do not couple to color gauge potential although the identification of color gauge potential as projection of $SU(3)$ Killing vector fields is possible. This coupling must emerge only at the effective gauge theory limit of TGD.

2. Spinor harmonics of imbedding space correspond to triality $t = 1$ ($t = 0$) partial waves. The detailed correspondence between color and electroweak quantum numbers is however not correct as such and the interpretation of spinor harmonics of imbedding space is as representations for ground states of super-conformal representations. The wormhole pairs associated with physical quarks and leptons must carry also neutrino pair to neutralize weak quantum numbers above the length scale of flux tube (weak scale or Compton length). The total color quantum numbers or these states must be those of standard model. For instance, the color quantum numbers of fundamental left-hand neutrino and lepton can compensate each other for the physical lepton. For fundamental quark-lepton pair they could sum up to those of physical quark.

The well-definedness of em charge is crucial condition.

1. Although the imbedding space spinor connection carries W gauge potentials one can say that the imbedding space spinor modes have well-defined em charge. One expects that this is true for induced spinor fields inside wormhole contacts with 4-D CP_2 projection and Euclidian signature of the induced metric.

2. The situation is not the same for the modes of induced spinor fields inside Minkowskian region and one must require that the CP_2 projection of the regions carrying induced spinor field is such that the induced W fields and above weak scale also the induced Z^0 fields vanish in order to avoid large parity breaking effects. This condition forces the CP_2 projection to be 2-dimensional. For a generic Minkowskian space-time region this is achieved only if the spinor modes are localized at 2-D surfaces of space-time surface - string world sheets and possibly also partonic 2-surfaces.

3. Also the Kähler-Dirac gamma matrices appearing in the modified Dirac equation must vanish in the directions normal to the 2-D surface in order that Kähler-Dirac equation can be satisfied. This does not seem plausible for space-time regions with 4-D CP_2 projection.

4. One can thus say that strings emerge from TGD in Minkowskian space-time regions. In particular, elementary particles are accompanied by a pair of fermionic strings at the opposite space-time sheets and connecting wormhole contacts. Quite generally, fundamental fermions would propagate at the boundaries of string world sheets as massless particles and wormhole contacts would define the stringy vertices of generalized Feynman diagrams. One obtains geometrized diagrammatics, which brings looks like a combination of stringy and Feynman diagrammatics.

5. This is what happens in the the generic situation. Cosmic strings could serve as examples about surfaces with 2-D CP_2 projection and carrying only em fields and allowing delocalization of spinor modes to the entire space-time surfaces.

A-5.3 Space-time surfaces with vanishing em, Z^0, or Kähler fields

In the following the induced gauge fields are studied for general space-time surface without assuming the extremal property. In fact, extremal property reduces the study to the study of vacuum extremals and surfaces having geodesic sphere as a CP_2 projection and in this sense the following arguments are somewhat obsolete in their generality.

Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r,Θ,Ψ,Φ) for CP_2, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2}dr\wedge(d\Psi+cos(\Theta)d\Phi)+\frac{r^2}{2F}sin(\Theta)d\Theta\wedge d\Phi \ , \\ F &= 1+r^2 \ . \end{aligned} \tag{A-5.0}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2}dr\wedge(d\Psi+cos(\Theta)d\Phi)+(3+p)\frac{r^2}{2F}sin(\Theta)d\Theta\wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-5.0}$$

where Θ_W denotes Weinberg angle.

1. The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2F}(d(r^2)/d\Theta)(k+cos(\Theta)) &+ (3+p)sin(\Theta)=0 \ , \end{aligned} \tag{A-5.0}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned} r &= \sqrt{\frac{X}{1-X}} \ , \\ X &= D\left[|\frac{(k+u}{C}|\right]^{\epsilon} \ , \\ u &\equiv cos(\Theta) \ , \ C=k+cos(\Theta_0) \ , \ D=\frac{r_0^2}{1+r_0^2} \ , \ \epsilon=\frac{3+p}{3+2p} \ , \end{aligned} \tag{A-5.-1}$$

where C and D are integration constants. $0\leq X\leq 1$ is required by the reality of r. $r=0$ would correspond to $X=0$ giving $u=-k$ achieved only for $|k|\leq 1$ and $r=\infty$ to $X=1$ giving $|u+k|=[(1+r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k)\times[\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}}\leq k+1 \ ,$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p} J \ . \end{aligned} \quad \text{(A-5.-1)}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

2. The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$ is useful.

3. The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$\begin{aligned} Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\ r &= \sqrt{\frac{X}{1-X}} \ , \ X = D|k+u| \ , \\ \gamma &= -\frac{p}{2} Z^0 \ . \end{aligned} \quad \text{(A-5.-2)}$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em,Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$\begin{aligned} ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta} d\Theta^2 + s^{eff}_{\Phi\Phi} d\Phi^2] \ , \\ s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\ s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ , \end{aligned} \quad \text{(A-5.-3)}$$

and is useful in the construction of vacuum imbedding of, say Schwartchild metric.

Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$\begin{aligned} \Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\ \Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ . \end{aligned} \quad \text{(A-5.-3)}$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global imbedding for, say a constant magnetic field. Although global imbedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteeing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \quad \text{(A-5.-2)}$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

A-6 p-Adic numbers and TGD

A-6.1 p-Adic number fields

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A121] . p-Adic numbers are representable as power expansion of the prime number p of form

$$x = \sum_{k\geq k_0} x(k)p^k, \ x(k) = 0, ..., p-1 \ . \quad \text{(A-6.1)}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \ . \quad \text{(A-6.2)}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \ , \quad \text{(A-6.3)}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x,y) = |x-y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} \ . \tag{A-6.4}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D \ . \tag{A-6.5}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.
2. Distances of points x and y inside single class are smaller than distances between different classes.
3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B32] . The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

A-6.2 Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

Basic form of canonical identification

There exists a natural continuous map $I: R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0,1,..,p-1\} \ . \end{aligned} \tag{A-6.5}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{A-6.4}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^k \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \quad \text{(A-6.3)}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see Fig. 5.4.2) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

Fig. 14. The real norm induced by canonical identification from 2-adic norm. `http://www.tgdtheory.fi/appfigures/norm.png`

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p |y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \quad \text{(A-6.3)}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p |y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \quad \text{(A-6.3)}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 \quad = \quad (\sum_n x_n^2)_R \quad . \tag{A-6.4}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-6.5}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

Generalization of number concept and notion of imbedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic imbedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R_p^n of R^n each of which projects to a copy of R_+^n (four quadrants in the case of plane). The common points of p-adic and real imbedding spaces are rational points and most p-adic points are at real infinity.

Real numbers and various algebraic extensions of p-adic number fields are thus glued together along common rationals and also numbers in algebraic extension of rationals whose number belong to the algebraic extension of p-adic numbers. This gives rise to a book like structure with rationals and various algebraic extensions of rationals taking the role of the back of the book. Note that Neper number is exceptional in the sense that it is algebraic number in p-adic number field Q_p satisfying $e^p \ mod \ p = 1$.

Fig. 15. Various number fields combine to form a book like structure. `http://www.tgdtheory.fi/appfigures/book.jpg`

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real imbedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

A-6.3 The notion of p-adic manifold

The notion of p-adic manifold is needed in order to fuse real physics and various p-adic physics to a larger structure which suggests that real and p-adic number fields should be glued together along common rationals bringing in mind adeles. The notion is problematic because p-adic topology is totally disconnected implying that p-adic balls are either disjoint or nested so that ordinary definition of manifold using p-adic chart maps fails. A cure is suggested to be based on chart maps from p-adics to reals rather than to p-adics (see the appendix of the book)

The chart maps are interpreted as cognitive maps, "thought bubbles" with reverse map interpreted as a transformation of intention to action and would be realized in terms of canonical identification or some of its variants.

Fig. 16. The basic idea between p-adic manifold. `http://www.tgdtheory.fi/appfigures/padmanifold.jpg`

There are some problems.

1. Canonical identification does not respect symmetries since it does not commute with second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map arithmetic operations which requires pinary cutoff below which chart map takes rationals to rationals so that commutativity with arithmetics and symmetries is achieved in finite resolution: above the cutoff canonical identification is used

2. Canonical identification is continuous but does not map smooth p-adic surfaces to smooth real surfaces requiring second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map requiring completion of the image to smooth preferred extremal of Kähler action so that chart map is not unique in accordance with finite measurement resolution

3. Canonical identification vreaks general coordinate invariance of chart map: (cognition-induced symmetry breaking) minimized if p-adic manifold structure is induced from that for p-adic imbedding space with chart maps to real imbedding space and assuming preferred coordinates made possible by isometries of imbedding space: one however obtains several inequivalent p-adic manifold structures depending on the choice of coordinates: these cognitive representations are not equivalent.

A-7 Hierarchy of Planck constants and dark matter hierarchy

Hierarchy of Planck constants was motivated by the "impossible" quantal effects of ELF em fields on vertebrate cyclotron energies $E = hf = \hbar \times eB/m$ are above thermal energy is possible only if $\hbar$ has value much larger than its standard value. Also Nottale's finding that planetary orbits migh be understood as Bohr orbits for a gigantic gravitational Planck constant.

Hierachy of Planck constant would mean that the values of Planck constant come as integer multiples of ordinary Planck constant: $h_{eff} = n \times h$. The particles at magnetic flux tubes characterized by h_{eff} would correspond to dark matter which would be invisible in the sense that only particle with same value of h_{eff} appear in the same vertex of Feynman diagram.

Hierarchy of Planck constants would be due to the non-determism of the Kähler action predicting huge vacuum degeneracy allowing all space-time surfaces which are sub-manfolds of any $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2. For agiven Y^2 one obtains new manifolds Y^2 by applying symplectic transformations of CP_2.

Non-determinism would mean that the 3-surface at the ends of causal diamond (CD) can be connected by several space-time surfaces carrying same conserved Kähler charges and having same values of Kähler action. Conformal symmetries defined by Kac-Moody algebra associated with the imbedding space isometries could act as gauge transformations and respect the light-likeness property of partonic orbits at which the signature of the induced metric changes from Minkowskian to Euclidian (Minkowskianb space-time region transforms to wormhole contact say). The number of conformal equivalence classes of these surfaces could be finite number n and define discrete physical degree of freedom and one would have $h_{eff} = n \times h$. This degeneracy would mean "second quantization" for the sheets of n-furcation: not only one but several sheets can be realized.

This relates also to quantum criticality postulated to be the basic characteristics of the dynamics of quantum TGD. Quantum criticalities would correspond to an infinite fractal hierarchy of broken conformal symmetries defined by sub-algebras of conformal algebra with conformal weights coming as integer multiples of n. This leads also to connections with quantum criticality and hierarchy of broken conformal symmetries, p-adicity, and negentropic entanglement which by consistency with standard quantum measurement theory would be described in terms of density matrix proportional $n \times n$ identity matrix and being due to unitary entanglement coefficients (typical for quantum computing systems).

Formally the situation could be described by regarding space-time surfaces as surfaces in singular n-fold singular coverings of imbedding space. A stronger assumption would be that they are expressible as as products of n_1 -fold covering of M^4 and n_2-fold covering of CP_2 meaning analogy with multi-sheeted Riemann surfaces and that M^4 coordinates are n_1-valued functions and CP_2 coordinates n_2 -valued functions of space-time coordinates for $n = n_1 \times n_2$. These singular coverings of imbedding space form a book like structure with singularities of the coverings localizable at the boundaries of causal diamonds defining the back of the the book like structure.

Fig. 17. Hierarchy of Planck constants. `http://www.tgdtheory.fi/appfigures/planckhierarchy.jpg`

A-8 Some notions relevant to TGD inspired consciousness and quantum biology

Below some notions relevant to TGD inspired theory of consciousness and quantum biology.

A-8.1 The notion of magnetic body

Topological field quantization inspires the notion of field body about which magnetic body is especially important example and plays key role in TGD inspired quantum biology and consciousness theory. This is a crucial departure fromt the Maxwellian view. Magnetic body brings in third level to the description of living system as a system interacting strongly with environment. Magnetic body would serve as an intentional agent using biological body as a motor instrument and sensory receptor. EEG would communicated the information from biological body to magnetic body and Libet's findings from time delays of consciousness support this view.

The following pictures illustrate the notion of magnetic body and its dynamics relevant for quantum biology in TGD Universe.

Fig. 18. Magnetic body associated with dipole field. `http://www.tgdtheory.fi/appfigures/fluxquant.jpg`

Fig. 19. Illustration of the reconnection by magnetic flux loops. `http://www.tgdtheory.fi/appfigures/reconnect1.jpg`

Fig. 20. Illustration of the reconnection by flux tubes connecting pairs of molecules. `http://www.tgdtheory.fi/appfigures/reconect2.jpg`

Fig. 21. Flux tube dynamics. a) Reconnection making possible magnetic body to "recognize" the presence of another magnetic body, b) braiding, knotting and linking of flux tubes making possible topological quantum computation, c) contraction of flux tube in phase transition reducing the value of h_{eff} allowing two molecules to find each other in dense molecular soup. `http://www.tgdtheory.fi/appfigures/fluxtubedynamics.jpg`

A-8.2 Number theoretic entropy and negentropic entanglement

TGD inspired theory of consciousness relies heavily p-Adic norm allows an to define the notion of Shannon entropy for rational probabilities (and even those in algebraic extension of rationals) by replacing the argument of logarithm of probability with its p-adic norm. The resulting entropy can be negative and the interpretation is that number theoretic entanglement entropy defined by this formula for the p-adic prime minimizing its value serves as a measure for conscious information. This negentropy characterizes two-particle system and has nothing to do with the formal negative negentropy assignable to thermodynamic entropy characterizing single particle. Negentropy Maximization Principle (NMP) implies that number theoretic negentropy increases during evolution by quantum jumps. The condition that NMP is consistent with the standard quantum measurement theory requires that negentropic entanglement has a density matrix proportional to unit matrix so that in 2-particle case the entanglement matrix is unitary.

Fig. 22. Schrödinger cat is neither dead or alive. For negentropic entanglement this state would be stable. `http://www.tgdtheory.fi/appfigures/cat.jpg`

A-8.3 Life as something residing in the intersection of reality and p-adicities

In TGD inspired theory of consciousness p-adic space-time sheets correspond to space-time correlates for thoughts and intentions. The intersections of real and p-adic preferred extremals consist of points whose coordinates are rational or belong to some extension of rational numbers in preferred imbedding space coordinates. They would correspond to the intersection of reality and various p-adicities representing the "mind stuff" of Descartes. There is temptation to assign life to the intersection of realities and p-adicities. The discretization of the chart map assigning to real space-time surface its p-adic counterpart would reflect finite cognitive resolution.

At the level of "world of classical worlds" (WCW) the intersection of reality and various p-adicities would correspond to space-time surfaces (or possibly partonic 2-surfaces) representable in terms of rational functions with polynomial coefficients with are rational or belong to algebraic extension of rationals.

The quantum jump replacing real space-time sheet with p-adic one (vice versa) would correspond to a buildup of cognitive representation (realization of intentional action).

Fig. 23. The quantum jump replacing real space-time surface with corresponding p-adic manifold can be interpreted as formation of though, cognitive representation. Its reversal would correspond to a transformation of intention to action. `http://www.tgdtheory.fi/appfigures/padictoreal.jpg`

A-8.4 Sharing of mental images

The 3-surfaces serving as correlates for sub-selves can topologically condense to disjoint large space-time sheets representing selves. These 3-surfaces can also have flux tube connections and this makes possible entanglement of sub-selves, which unentangled in the resolution defined by the size of sub-selves. The interpretation for this negentropic entanglement would be in terms of sharing of mental images. This would mean that contents of consciousness are not completely private as assumed in neuroscience.

Fig. 24. Sharing of mental images by entanglement of subselves made possible by flux tube connections between topologically condensed space-time sheets associated with mental images. `http://www.tgdtheory.fi/appfigures/sharing.jpg`

A-8.5 Time mirror mechanism

Zero energy ontology (ZEO) is crucial part of both TGD and TGD inspired consciousness and leads to the understanding of the relationship between geometric time and experience time and how the arrow of psychological time emerges. One of the basic predictions is the possibiity of negative energy signals propagating backwards in geometric time and having the property that entropy basically associated with subjective time grows in reversed direction of geometric time. Negative energy signals inspire time mirror mechanism (see fig. http://www.tgdtheory.fi/appfigures/timemirror.jpg or fig. 24 in the appendix of this book) providing mechanisms of both memory recall, realization of intentational action initiating action already in geometric past, and remote metabolism. What happens that negative energy signal travels to past and is reflected as positive energy signal and returns to the sender. This process works also in the reverse time direction.

Fig. 25. Zero energy ontology allows time mirror mechanism as a mechanism of memory recall. Essentially "seeing" in time direction is in question. http://www.tgdtheory.fi/appfigures/timemirror.jpg

REFERENCES

Mathematics

[A1] A directory of all known zeta functions. http://secamlocal.ex.ac.uk/~mwatkins/zeta/directoryofzetafunctions.htm.

[A2] Absolute Galois Group. http://en.wikipedia.org/wiki/Absolute_Galois_group.

[A3] Adele group. http://en.wikipedia.org/wiki/Adele_group.

[A4] Adelic ring. http://en.wikipedia.org/wiki/Adelic.

[A5] Algebraic number theory. http://en.wikipedia.org/wiki/Algebraic_number_theory.

[A6] Amenable Group. http://en.wikipedia.org/wiki/Amenable_group.

[A7] An Introduction to Analytic Number Theory. http://algo.inria.fr/banderier/Seminar/Vardi/index.htm.

[A8] Arithmetic functions. http://en.wikipedia.org/wiki/Arithmetic_function.

[A9] Atyiah-Singer index-theorem. http://en.wikipedia.org/wiki/Atiyah-Singer_index_theorem.

[A10] Automorphic functions. http://en.wikipedia.org/wiki/Automorphic_functions.

[A11] Belyi's theorem. http://en.wikipedia.org/wiki/Belyi's_theorem.

[A12] Betti cohomology. http://en.wikipedia.org/wiki/Betti_cohomology).

[A13] Birational geometry. http://en.wikipedia.org/wiki/Birational_geometry.

[A14] Braid group. http://en.wikipedia.org/wiki/Braid_group).

[A15] Category theory. http://en.wikipedia.org/wiki/Category_theory.

[A16] Category theory. http://en.wikipedia.org/wiki/Category_theory.

[A17] Class number formula. http://en.wikipedia.org/wiki/Class_number_formula.

[A18] Classical groups. http://en.wikipedia.org/wiki/Classical_groups.

[A19] Coalgebra. http://en.wikipedia.org/wiki/Coalgebra.

[A20] Cohomology. http://en.wikipedia.org/wiki/Cohomology.

[A21] Crystalline cohomology. http://en.wikipedia.org/wiki/Crystaline_cohomology.

[A22] De Rham cohomology. http://en.wikipedia.org/wiki/De_Rham_cohomology).

[A23] Dessin d'enfant. http://en.wikipedia.org/wiki/Dessin_d'enfant.

[A24] Dirichlet character. http://en.wikipedia.org/wiki/Dirichlet_character.

[A25] Dirichlet L-function. http://en.wikipedia.org/wiki/Dirichlet_L-function.

[A26] Divisor function. http://en.wikipedia.org/wiki/Divisor_function.

[A27] Dolbeault cohomology. http://en.wikipedia.org/wiki/Dolbeault_cohomology).

[A28] Elliptic surface. http://en.wikipedia.org/wiki/Elliptic_surface.

[A29] Etale cohomology. http://en.wikipedia.org/wiki/Etale_cohomology.

[A30] Exact sequences. http://en.wikipedia.org/wiki/Exact_sequences.

[A31] Exceptional groups. http://en.wikipedia.org/wiki/Exceptional_groups.

[A32] Farey sequence. http://en.wikipedia.org/wiki/Farey_sequence.

[A33] Floer homology. http://en.wikipedia.org/wiki/Floer_homology.

[A34] Frobenius theorem (differential topology). http://en.wikipedia.org/wiki/Frobenius_theorem_(differential_topology).

[A35] Functor. http://en.wikipedia.org/wiki/Functor.

[A36] G_2 (mathematics). http://en.wikipedia.org/wiki/G2_(mathematics).

[A37] Galois extension. http://en.wikipedia.org/wiki/Galois_extension.

[A38] Galois group. http://www.mathpages.com/home/kmath290/kmath290.htm.

[A39] Geometrization conjecture. http://en.wikipedia.org/wiki/Geometrization_conjecture.

[A40] Graph of sigma function. http://en.wikipedia.org/wiki/File:Sigma_function.svg.

[A41] Gromov-Witten invariant. http://en.wikipedia.org/wiki/GromovWitten_invariant.

[A42] Homology. http://en.wikipedia.org/wiki/Homology_(mathematics).

[A43] Homotopy. http://en.wikipedia.org/wiki/Homotopy).

[A44] Hurwitz zeta function. http://en.wikipedia.org/wiki/Hurwitz_zeta_function.

[A45] Hyperbolic 3-manifold. http://en.wikipedia.org/wiki/Hyperbolic_3-manifold.

[A46] Hyperbolic manifold. http://en.wikipedia.org/wiki/Hyperbolic_manifold.

[A47] Hyperfinite type II factor. http://en.wikipedia.org/wiki/Hyperfinite_type_II-1_factor.

[A48] Icosahedron. http://en.wikipedia.org/wiki/icosahedron.

[A49] Ideal class groip. http://en.wikipedia.org/wiki/Ideal_class_group.

[A50] Induced representations. http://en.wikipedia.org/wiki/Induced_representations.

[A51] Integer sum of three squares. http://www.proofwiki.org/wiki/Integer_as_Sum_of_Three_Squares.

[A52] Isomorphism theorem. http://en.wikipedia.org/wiki/Isomorphism_theorem.

[A53] Knot theory. http://en.wikipedia.org/wiki/Knot_theory.

[A54] Künneth theorem. http://en.wikipedia.org/wiki/Kunneth_theorem.

[A55] Langlands dual. http://en.wikipedia.org/wiki/Langlands_dual.

[A56] Langlands program. http://en.wikipedia.org/wiki/Langlands_program.

[A57] Liouville function. http://en.wikipedia.org/wiki/Liouville_function.

[A58] Loop space. http://en.wikipedia.org/wiki/Loop_space.

[A59] MacKay correspondence. http://en.wikipedia.org/wiki/ADE_classification.

[A60] Maximal Abelian Extension. http://en.wikipedia.org/wiki/Maximal_abelian_extension.

[A61] Mellin transform. http://en.wikipedia.org/wiki/Mellin_transform.

[A62] Modular forms. http://en.wikipedia.org/wiki/Modular_forms.

[A63] Motivic cohomology. http://en.wikipedia.org/wiki/Motivic_cohomology.

[A64] Motivic Galois group. http://en.wikipedia.org/wiki/Motivic_Galois_groupmotivic Galois group.

[A65] Number theory. http://en.wikipedia.org/wiki/Number_theory.

[A66] Octonions. http://en.wikipedia.org/wiki/Octonions.

[A67] Operad theory. http://en.wikipedia.org/wiki/Operad.

[A68] p-Adic Numbers and Bruhat-Tits Tree. http://www.math.mcgill.ca/goren/667.2010/Atefeh.pdf.

[A69] Platonic solid. http://en.wikipedia.org/wiki/Platonics_solid.

[A70] Poincare duality. http://en.wikipedia.org/wiki/Poincare_duality.

[A71] Profinite group. http://en.wikipedia.org/wiki/Profinite_group.

[A72] Pythagorean triangles. http://en.wikipedia.org/wiki/Pythagorean_triangle.

[A73] Q-theta function. http://en.wikipedia.org/wiki/Q-theta_function.

[A74] Quadratic resiprocity theorem. http://en.wikipedia.org/wiki/Quadratic_reciprocity.

[A75] Quantum cohomology. http://en.wikipedia.org/wiki/Quantum_cohomology).

[A76] Quantum group. http://en.wikipedia.org/wiki/Quantum_group.

[A77] Quaternions. http://en.wikipedia.org/wiki/Quaternion.

[A78] Ramification. http://en.wikipedia.org/wiki/Ramification.

[A79] Ramification. http://en.wikipedia.org/wiki/Unramified.

[A80] Relative cohomology. http://en.wikipedia.org/wiki/Relative_cohomology.

[A81] Representation theory of the symmetric group. http://en.wikipedia.org/wiki/Representation_theory_of_the_symmetric_group.

[A82] Ricci flow. http://en.wikipedia.org/wiki/Ricci_flow.

[A83] Riemann-Hilbert problem. http://en.wikipedia.org/wiki/Riemann-Hilbert_problem.

[A84] Riemann Zeta function. http://en.wikipedia.org/wiki/Riemann_zeta.

[A85] Scheme. http://en.wikipedia.org/wiki/Scheme).

[A86] Set theory. http://en.wikipedia.org/wiki/Set_theory.

[A87] Special linear group. http://en.wikipedia.org/wiki/Special_linear_group.

[A88] Steenrod-Eilenberg axioms. http://en.wikipedia.org/wiki/Steenrod-Eilenberg_axioms.

[A89] Stiefel-Whitney class. http://en.wikipedia.org/wiki/StiefelWhitney_class.

[A90] Sum of squares function. http://mathworld.wolfram.com/SumofSquaresFunction.html.

[A91] Symplecitification of science. http://www.pims.math.ca/~gotay/Symplectization(E).pdf.

[A92] Taniyama-Shimura Weil conjecture. http://en.wikipedia.org/wiki/Taniyama-Shimura_conjecture.

[A93] Theta function. http://en.wikipedia.org/wiki/Theta_function.

[A94] This Week's Finds in Mathematical Physics: Week 230. http://math.ucr.edu/home/baez/week230.html.

[A95] Topological sigma models. http://ccdb4fs.kek.jp/cgi-bin/img/allpdf?198804521.

[A96] Topological string theory. http://en.wikipedia.org/wiki/Topological_string_theory.

[A97] Torus knot. http://en.wikipedia.org/wiki/Torus_knot.

[A98] Twisted K-theory. http://en.wikipedia.org/wiki/Twisted_K-theory.

[A99] Uniformization theorem. http://en.wikipedia.org/wiki/Uniformization_theorem.

[A100] Video about Edward Witten's talk relating to his work with knot invariants. http://video.ias.edu/webfm_send/1787.

[A101] Videos on Arizona Winter School 2007 on non-Archimedean geometry. http://swc.math.arizona.edu/aws/2007/notes.html.

[A102] Weekly Finds (Week 102). http://math.ucr.edu/home/baez/week202.html.

[A103] Weil cohomology. http://en.wikipedia.org/wiki/Weil_cohomology.

[A104] Weil conjectures. http://en.wikipedia.org/wiki/Weil_conjectures.

[A105] What do Fermat's Last Theorem and Electro-magnetic Duality Have in Common? http://online.kitp.ucsb.edu/online/bblunch/frenkel/.

[A106] Witt vector. http://en.wikipedia.org/wiki/Witt_vector.

[A107] Yang tableau. http://en.wikipedia.org/wiki/Young_tableau.

[A108] Yangian symmetry. http://en.wikipedia.org/wiki/Yangian.

[A109] Zariski topology. http://en.wikipedia.org/wiki/Zariski_topology).

[A110] Zeta function. http://en.wikipedia.org/wiki/Zeta_function.

[A111] *Objects of categories as complex numbers*, 2007.

[A112] S. S. Abhyankar. Algebraic Geometry for Scientists and Engineers. *Mathematical Surveys and Monographs*, 35, 1980.

[A113] R. B. J. T. Allenby and E. J. Redfern. *Introduction to Number Theory with Computing.* Edward Arnold, 1989.

[A114] J. Baez. Categorification. `http://arxiv.org/pdf/math/9802029v1`, 1998.

[A115] John Baez. Quantum Quandaries. `http://math.ucr.edu/home/baez/quantum/node1.html`, 2007.

[A116] John Baez. Question about representations of finite groups. `http://golem.ph.utexas.edu/category/2007/07/question_about_representations.html#more`, 2007.

[A117] John C. Baez. The Octonions. *Bull. Amer. Math. Soc.*. *http://math.ucr.edu/home/baez/Octonions/octonions.html*, 39(2002), 2001.

[A118] M. Barnsley. *Fractals Everywhere.* Academic Press, 1988.

[A119] M. V. Berry and J. P. Keating. *H=xp and the Riemann Zeros*, pages 355–367. Kluwer, New York, 1999.

[A120] A. L. Besse. *Einstein Manifolds.* Springer Verlag, 1987.

[A121] Z. I. Borevich and I. R. Shafarevich. *Number Theory.* Academic Press, 1966.

[A122] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics.* Springer Verlag, New York, 1979.

[A123] L. Brekke and P. G. O. Freund. p-Adic Numbers in Physics. *Phys. Rep.*, 233(1), 1993.

[A124] D. A. Buell. The last exhaustive computation of class groups. In *Number theory: Fifth Conference of the Canadian Number Theory Association, August 17-22, 1996, Carleton University, Ottawa, Ontario, Canada*, pages 35–53, 1996.

[A125] M. Ruiz-Altaba C. Gomez and G. Sierra. *Quantum Groups and Two-Dimensional Physics.* Cambridge University Press, Cambridge, 1996.

[A126] H. Saleur C. Itzykson and J-B. Zuber (Editors). *Conformal Invariance and Applications to Statistical Mechanics.* World Scientific, 1988.

[A127] P. Cartier. A Mad Day's Work: From Grothendienck to Connes and Kontsevich: the Evolution of Concepts of Space and Symmetry. *American Mathematical Society*, 38(4):389–408, 2001.

[A128] A. Connes and D. Kreimer. *Hofp algebras, renormalization, and non-commutative geometry*, volume 1999. Kluwer, 1998.

[A129] G. Cornelissen and F. Kato. The p-Adic Icosahedron. `http://www.ams.org/notices/200507/fea-cornelissen.pdf`, 2006.

[A130] P. Das D. Bisch and S. K. Gosh. Planar algebra of group-type subfactors. `http://arxiv4.library.cornell.edu/abs/0807.4134`, 2008.

[A131] J. Daboul and R. Delborough. Matrix Representations of Octonions and Generalizations. `http://arxiv.org/abs/hep-th/9906065`, 1999.

[A132] J. Dixmier. *Von Neumann Algebras.* North-Holland, Amsterdam, 1981.

[A133] J. J. Duistermaat and G. J. Heckmann. *Inv. Math.*, 69, 1982.

[A134] T. Durt. A new expression for mutually unbiased bases in prime power dimensions. `http://arxiv.org/abs/quant-ph/0409090`, 2004.

[A135] F. Dyson. Birds and Frogs. *Notices of AMS. http://www.ams.org/notices/200902/rtx090200212p.pdf*, 56(2), 2009.

[A136] H. M. Edwards. *Riemann's Zeta Function.* Academic Press, London, 1974.

[A137] M. Eichler. *Introduction to the theory of algebraic numbers and functions.* Academic Press, New York, 1966.

[A138] Eisenhart. *Riemannian Geometry.* Princeton University Press, 1964.

[A139] R. Elwes. The Ultimate logic: to infinity and beyond. http://www.newscientist.com/article/mg21128231.400-ultimate-logic-to-infinity-and-beyond.html, 2011.

[A140] J. Esmonde and M. Ram Murty. *Problems in Algebraic Number Theory.* Springer Verlag, New York, 1991.

[A141] J. Brillhart et al. Factorizations of $b^m \pm 1$. *American Mathematical Society*, 1990.

[A142] K. E. Smith et al. *Invitation to Algebraic Geometry.* Springer Verlag, New York, 2000.

[A143] L. Chekhov et al. Multiloop Calculations in P-adic String Theory and Bruhat-Tits Trees. *Comm. Math. Phys., http://projecteuclid.org/DPubS/Repository/1.0/Disseminate?view=body&id=pdf_1&handle=euclid.cmp/1104179635*, 125:675–711, 1989.

[A144] D. Evans and Y. Kawahigashi. *Quantum symmetries on operator algebras.* Oxford University Press, New York, 1998.

[A145] D. S. Freed. The Geometry of Loop Groups, 1985.

[A146] E. Frenkel. Free Field Realizations in Representation Theory and Conformai Field Theory. http://www.mathunion.org/ICM/ICM1994.2/Main/icm1994.2.1256.1269.ocr.pdf.

[A147] E. Frenkel. Affine Algebras, Langlands duality, and Bethe Ansatz. http://arxiv.org/abs/q-alg/9506003, 1999.

[A148] E. Frenkel. Recent Advances in Langlands program. *American Mathematical Society*, 41(2):151–184, 2004.

[A149] E. Frenkel. Representation Theory: Its rise and Its Role in Number Theory. http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/gibbs-ps.pdf, 2004.

[A150] E. Frenkel. Lectures on Langlands program and conformal field theory. http://arxiv.org/abs/hep-th/0512172, 2005.

[A151] E. Frenkel and D. Hernandez. Langlands duality for representations of quantum groups. http://arxiv.org/abs/0809.4453, 2008.

[A152] E. Frenkel and D. Hernandez. Langlands duality for finite-dimensional representations of quantum affine algebras. http://arxiv.org/abs/0902.0447, 2009.

[A153] D. Gaiotto and E. Witten. Knot Invariants from Four-Dimensional Gauge Theory. http://arxiv.org/abs/1106.4789v1, 2011.

[A154] G. W. Gibbons and C. N. Pope. CP_2 as gravitational instanton. *Comm. Math. Phys.*, 55, 1977.

[A155] A. Givental. A tutorial in Quantum Cohomology. http://math.berkeley.edu/~giventh/papers/lqc.pdf.

[A156] R. Goldblatt. *The Categorial Analysis of Logic.* North-Holland, Amsterdam, 1984.

[A157] A. Goncharov. Volumes of hyperbolic manifolds and mixed Tate motives. http://arxiv.org/abs/alg-geom/9601021, 1996.

[A158] F. Q. Gouvêa. *p-Adic Numbers: An Introduction.* Springer Verlag, 1997.

[A159] L. Gow and A. Molev. Representations of twisted q-Yangians. `http://arxiv.org/abs/0909.4905`, 2009.

[A160] R. Guy. *Unsolved Problems in Number Theory.* Springer Verlag, 1994.

[A161] P. M. Hajac. On and around the Drinfeld double. `http://www.impan.pl/~burgunde/WSBC09/Ddouble_Hajac.pdf`.

[A162] T. C. Hales. What is Motivic Measure? *Bulletin of the American Mathematical Society*, 42(2):119–135, 2005.

[A163] R. Harvey. *Spinors and Calibrations.* Academic Press, New York, 1990.

[A164] S. W. Hawking and C. N. Pope. Generalized Spin Structures in Quantum Gravity. *Phys. Lett.*, (1), 1978.

[A165] S. Helgason. *Differential Geometry and Symmetric Spaces.* Academic Press, New York, 1962.

[A166] D. R. Hofstadter. *Gödel, Escher, Bach: an Eternal Braid.* Penguin Books, 1980.

[A167] J. Huertal. A Short History of the Interaction Between QFT and Topology. `http://math.ucr.edu/~alex/cobordism_lecture5.pdf`.

[A168] R. A. Minklos I. M. Gelfand and Z. Ya. Shapiro. *Representations of the rotation and Lorentz groups and their applications.* Pergamon Press, 1963.

[A169] C. J. Isham and J. Butterfield. Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity. `http://arxiv.org/abs/gr-gc/9910005`, 1999.

[A170] C. Iztykson and J-B. Zuber. *Quantum Field Theory*, volume 549. Mc Graw-Hill, New York, 1980.

[A171] F. R. Jones. *Braid groups, Hecke algebras and type II_1 factors.* 1983.

[A172] V. Jones. In and around the origin of quantum groups. `http://arxiv.org/abs/math/0309199`, 2003.

[A173] V. F. R. Jones. *The planar algebra of a bipartite graph*, pages 94–117. World Science Publishing, 2000.

[A174] A. Kapustin and E. Witten. Electric-Magnetic Duality And The Geometric Langlands Program. `http://arxiv.org/abs/hep-th/0604151`, 2006.

[A175] G. Kato. Sheaf Cohomology of Conscious Entity. *Int. J. of Comp. Antic. Systems*, 2002.

[A176] G. Kato and D. C. Struppa. Category Theory and Consciousness. 2001.

[A177] L. H. Kauffman and S. Lambropoulou. Hard Unknots and Collapsing Tangles. `http://arxiv.org/abs/math/0601525`, 2006.

[A178] R. K. Kaul. The representations of Temperley-Lieb-Jones algebras. `http://arxiv.org/abs/hep-th/9312214`, 1993.

[A179] A. Khrennikov. *p-Adic Valued Distributions in Mathematical Physics.* Kluwer, Dordrecht, 1994.

[A180] A. Yu. Khrennikov. p-Adic Probability and Statistics. *Dokl. Akad Nauk*, (6), 1992.

[A181] A. Yu. Khrennikov. p-Adic Probability and Statistics. *Dokl. Akad Nauk*, (6), 1992.

[A182] M. Kontsevich. Operads and Motives in Deformation Quantization. `http://arxiv.org/abs/math/9904055`, 1999.

[A183] W. T. Lu and S. Sridhar. Correlations Among the Riemann Zeros: Invariance, Resurgence, Prophecy and Self-Duality. http://arxiv.org/pdf/nlin/0405058, 2004.

[A184] N. J. MacKay. Introduction to Yangian symmetry in integrable field theory. http://wenku.baidu.com/view/01e4ebdbad51f01dc281f180.html, 2009.

[A185] S. Majid. *Foundations of Quantum Group Theory.* 2005.

[A186] J. Mickelson. Gerbes, (Twisted) K-Theory, and the Supersymmetric WZW Model. http://arxiv.org/abs/hep-th/0206139, 2002.

[A187] J. S. Milne. Motives - Grothendieck's Dream. http://www.jmilne.org/math/xnotes/MOT.pdf, 2009.

[A188] J. Milnor. *Topology form Differential Point of View.* The University Press of Virginia, Virginia, 1965.

[A189] B. Mitchell. *Theory of Categories.* Academic Press, 1965.

[A190] B. I. Plotkin. Wolters-Noordhoff, 1972.

[A191] C. N. Pope. Eigenfunctions and $Spin^c$ Structures on CP_2, 1980.

[A192] M. Reid. McKay correspondence. http://arxiv.org/abs/alg-geom/9702016.

[A193] A. Robinson. *Nonstandard Analysis.* North-Holland, Amsterdam, 1974.

[A194] E. E. Rosinger. How Far Should the Principle of Relativity Go? http://arxiv.org/abs/0710.0226, 2007.

[A195] E. E. Rosinger. Quantum Foundations: Is Probability Ontological? http://arxiv.org/abs/physics/0703019, 2007.

[A196] E. E. Rosinger. Group Invariant Entanglements in Generalized Tensor Products. http://arxiv.org/abs/0808.0095, 2008.

[A197] E. E. Rosinger. Heisenberg Uncertainty in Reduced Power Algebras. http://arxiv.org/abs/0901.4825, 2009.

[A198] D. Ruberman. Comment in discussion about unitary cobordisms. http://math.ucr.edu/home/baez/quantum/ruberman.html.

[A199] G. Sorkin S. Solla and S. White. *Configuration space analysis for optimization problems.* Springer Verlag, Berlin, 1986.

[A200] D. Salamon. Lectures on Floer homology. http://www.math.ethz.ch/~salamon/PREPRINTS/floer.pdf, 1997.

[A201] S. Sawin. Links, Quantum Groups, and TQFT's. http://arxiv.org/abs/q-alg/9506002, 1995.

[A202] R. Schaflitzel. II_1-sub-factors associated with the C^*-Tensor Category of a Finite Group. http://arxiv.org/abs/funct-an/9701008, 1997.

[A203] J. Schray and C. A. Manogue. Octonionic representations of Clifford algebras and triality. http://arxiv.org/abs/hep-th/9407179, 1994.

[A204] M. R. Schroeder. *Number Theory in Science and Communication.* Springer Verlag, 1990.

[A205] H. Schubert. *Categories.* Springer Verlag, New York, 1972.

[A206] T. Smith. D4-D5-E6 Physics. http://galaxy.cau.edu/tsmith/d4d5e6hist.html, 1997.

[A207] M. Spivak. *Differential Geometry I,II,III,IV.* Publish or Perish, Boston, 1970.

[A208] Richard Stefanik. *Structuralism, Category Theory and Philosophy of Mathematics*. MSG Press, Washington, 1994.

[A209] H. Sugawara. A field theory of currents. *Phys. Rev.*, 176, 1968.

[A210] B. Gilkey T. Eguchi and J. Hanson. *Phys. Rep.*, 66:1980, 1980.

[A211] M. Temkin. Introduction to Berkovich analytic spaces. `http://www.math.huji.ac.il/~temkin/papers/Lecture_Notes_Berkovich_Analytic_Spaces.pdf`.

[A212] N. H. V. Temperley and E. H. Lieb. Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices:some exact results for the percolation problem. *Proc. Roy. Soc. London*, 322(1971), 1971.

[A213] R. Thom. *Commentarii Math. Helvet.*, 28, 1954.

[A214] E. C. Titchmarch. *The Theory of Riemann Zeta Function*. Oxford Univ. Press, Oxford, 1986.

[A215] Wallace. *Differential Topology*. W. A. Benjamin, New York, 1968.

[A216] A. Werner. Buildings and Berkovich spaces (lecture slides). `http://www.uni-frankfurt.de/fb/fb12/mathematik/ag/personen/werner/talks/dmvmuench10.pdf`.

[A217] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121:351–399, 1989.

[A218] E. Witten. Fivebranes and Knots. `http://arxiv.org/pdf/1101.3216v1`, 2010.

[A219] E. C. Zeeman. *Catastrophe Theory*. Addison-Wessley Publishing Company, 1977.

Theoretical Physics

[B1] $1/N$ expansion. `http://en.wikipedia.org/wiki/1/N_expansion`.

[B2] Chern-Simons theory. `http://en.wikipedia.org/wiki/ChernSimons_theory`.

[B3] Global Scaling. `http://www.dr-nawrocki.de/globalscalingengl2.html`.

[B4] K-theory (physics). `http://en.wikipedia.org/wiki/K-theory_(physics)`.

[B5] Korteweg-de Vries equation. `http://en.wikipedia.org/wiki/Kortewegde_Vries_equation`.

[B6] Lax pair. `http://en.wikipedia.org/wiki/Lax_pair`.

[B7] Montonen Olive Duality. `http://en.wikipedia.org/wiki/Montonen-Olive_duality`.

[B8] Non-linear Schrdinger equation. `http://en.wikipedia.org/wiki/Non-linear_Schrdinger_equation`.

[B9] Sine-Gordon equation. `http://en.wikipedia.org/wiki/Sine-Gordon`.

[B10] SUSY Summary. `http://bolvan.ph.utexas.edu/~vadim/Classes/01f/396T/table.pdf`.

[B11] Israel Gelfand. `http://terrytao.wordpress.com/2009/10/07/israel-gelfand/#more-2860`, 2009.

[B12] John Baez. This Weeks Finds (Week 259). http://math.ucr.edu/home/baez/week259.html, 2007.

[B13] J. Björken and S. Drell. *Relativistic Quantum Fields*. Mc Graw-Hill, New York, 1965.

[B14] D. Chowdbury. *Spin Glasses and other Frustrated Systems*. World Scientific, 1986.

[B15] R. E. Cutkosky. *J. Math. Phys.*, (1):429–433, 1960.

[B16] G. Moore D.-E. Diaconescu and E. Witten. E8 Gauge Theory, and a Derivation of K-Theory from M-Theory). http://arXiv.org/abs/hep-th/0005090v3, 2005.

[B17] T. Ericson and J. Rafelski. The tale of the Hagedorn temperature. *Cern Courier*. *http://www.cerncourier.com/main/toc/43/7*, 43(7), 2002.

[B18] A. B. Goncharov et al. Classical Polylogarithms for Amplitudes and Wilson Loops. http://arxiv.org/abs/1006.5703, 2010.

[B19] Nima Arkani-Hamed et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM. http://arxiv.org/find/hep-th/1/au:+Bourjaily_J/0/1/0/all/0/1, 2010.

[B20] S. Ansoldi et al. p-Branes ElectricMagnetic Duality and Stueckelberg/Higgs Mechanism: a PathIntegral Approach. http://arxiv.org/abs/hep-th/0004044v2, 2000.

[B21] H. Evslin. What doesn't K-theory classify? http://arxiv.org/abs/hep-th/0610328, 2006.

[B22] D. S. Freed and E. Witten. Anomalies in string theory with D-branes. http://arXiv.org/abs/hep-th/9907189v2, 1999.

[B23] J. Maldacena G. Horowitz and A. Strominger. http://arxiv.org/abs/hep-th/9603019, 1996.

[B24] A. B. Goncharov. A simple construction of Grassmannian polylogarithms. http://arxiv.org/abs/0908.2238, 2011.

[B25] H. Haken. *Synergetics*. Springer Verlag, 1978.

[B26] K. Huang. *Quarks,Leptons* & *Gauge Fields*. World Scientific, 1982.

[B27] J. Henn J. Drummond and J. Plefka. Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory. http://cdsweb.cern.ch/record/1162372/files/jhep052009046.pdf, 2009.

[B28] G. Moore J. Maldacena and N. Seiberg. D-Brane Instantons and K-Theory Charges). http://arxiv.org/abs/hep-th/010810, 2001.

[B29] I. R. Klebanov. TASI Lectures: Introduction to the AdS/CFT Correspondence. http://arxiv.org/abs/hep-th/0009139, 2000.

[B30] A. Lakthakia. *Beltrami Fields in Chiral Media*, volume 2. World Scientific, Singapore, 1994.

[B31] R. Lister. Simulated Annealing: Quasi-fractals and Quasi-failures. http://www.csu.edu.au/ci/vol2/lister/node1.html, 1997.

[B32] G. Parisi. *Field Theory, Disorder and Simulations*. World Scientific, 1992.

[B33] R. Penrose. The Central Programme of Twistor Theory. *Chaos, Solitons and Fractals*, 10, 1999.

[B34] M. Rocek and M. Disconzi. Elementary realization of BRST symmetry and gauge fixing. http://www.math.sunysb.edu/~wdlinch3/NewPages-Images/BRSTandGaugefixing-Martin%20Rocek.pdf.

[B35] N. H. March S. Lundqvist and M. P. Tosi. *Order and Chaos in Nonlinear Physical Systems*, volume 295. Plenum, New York, 1988.

[B36] A. Sen. Tachyon Condensation on the Brane Antibrane system). `http://arxiv.org/abs/hep-th/9805170`, 1998.

[B37] P. Townsend. p-Brane democracy). `http://xxx.lanl.gov/abs/hep-th/9507048`, 1995.

[B38] E. Verlinde. Global Aspects of Electric-Magnetic Duality). `http://arxiv.org/abs/hep-th/9506011v3`, 1995.

[B39] E. Witten. Perturbative Gauge Theory As a String Theory In Twistor Space. `http://arxiv.org/abs/hep-th/0312171`, 2003.

[B40] A. Zee. *The Unity of Forces in the Universe*. World Science Press, Singapore, 1982.

Particle and Nuclear Physics

[C1] A. E. Nelson D. B. Kaplan and N. Weiner. Neutrino Oscillations as a Probe of Dark Energy. `http://arxiv.org/abs/hep-ph/0401099`, 2004.

[C2] U. Egede. A theoretical limit on Higgs mass. `http://www.hep.lu.se/atlas//thesis/egede/thesis-node20.html`, 1998.

[C3] S. E. Shnoll et al. Realization of discrete states during fluctuations in macroscopic processes. *Uspekhi Fisicheskikh Nauk*, 41(10):1025–1035, 1998.

Condensed Matter Physics

[D1] A Bibliography of $1/f$ noise. `http://linkage.rockefeller.edu/wli/1fnoise`.

[D2] D. J. P. Morris et al. Dirac Strings and Magnetic Monopoles in Spin Ice Dy2Ti2O7. *Physics World*, 326(5951):411–414, 2009.

[D3] R. Mackenzie and F. Wilczek. *Rev. Mod. Phys. A*, 3:2827, 1988.

Cosmology and Astro-Physics

[E1] S. E. Shnoll et al. Realization of discrete fluctuations in macroscopic processes. *Physics-Uspekhi.* *http://home.t01.itscom.net/allais/blackprior/shnoll/shnoll-1.pdf*, 41(10):1025–1035, 1998.

[E2] L. Z. Fang and H. Sato. Is the Periodicity in the Distribution of Quasar Red Shifts an Evidence of Multiple Connectedness of the Universe? *Gen. Rel. and Grav.*, (11), 1985.

[E3] D. Da Roacha and L. Nottale. Gravitational Structure Formation in Scale Relativity. http://arxiv.org/abs/astro-ph/0310036, 2003.

[E4] S. Weinberg. *Gravitation and Cosmology.* Wiley, New York, 1967.

Fringe Physics

[H1] V. V. Roshchin and S. M. Godin. An Experimental Investigation of the Physical Effects in a Dynamic Magnetic System. *New Energy Technologies*, 1, 2001.

[H2] V. V. Roshchin and S. M. Godin. An Experimental Investigation of the Physical Effects in a Dynamic Magnetic System. *New Energy Technologies*, 1, 2001.

Biology

[I1] R. Sheldrake. *A New Science of Life: The Hypothesis of Formative Causation.* Inner Traditions Intl Ltd., 1995.

Neuroscience and Consciousness

[J1] S. J. Blackmore. Near death experiences: in or out of the body? *Skeptical Inquirerhttp://www.uwe.ac.uk/fas/staff/sb/si91nde.html*, 1991:34–45, 1991.

[J2] O. Sacks. *The man who mistook his wife for a hat.* Touchstone books, 1998.

[J3] S. Suzuki. *Zen Mind, Beginner's Mind.* Waterhill,, New York, 1988.

Books related to TGD

[K1] M. Pitkänen. 1983.

[K2] M. Pitkänen. Construction of S-matrix. In *Towards S-matrix.* Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#smatrix, 2000.

[K3] M. Pitkänen. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. In *Towards M-matrix.* Onlinebook. http://tgdtheory.fi/public_html/pdfpool/append.pdf, 2006.

[K4] M. Pitkänen. Is it Possible to Understand Coupling Constant Evolution at Space-Time Level? In *Towards M-Matrix.* Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#rgflow, 2006.

[K5] M. Pitkänen. A Possible Explanation of Shnoll Effect. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy.* Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html#ShnollTGD, 2006.

[K6] M. Pitkänen. About Nature of Time. In *TGD Inspired Theory of Consciousness.* Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html#timenature, 2006.

[K7] M. Pitkänen. An Overview about Quantum TGD: Part I. In *Topological Geometrodynamics: Overview.* Onlinebook. http://tgdtheory.fi/public_html/tgdview/tgdview.html#evoII, 2006.

[K8] M. Pitkänen. An Overview about the Evolution of Quantum TGD. In *Topological Geometrodynamics: Overview.* Onlinebook. http://tgdtheory.fi/public_html/tgdview/tgdview.html#evoI, 2006.

[K9] M. Pitkänen. Appendix A: Quantum Groups and Related Structures. In *Hyper-finite Factors and Dark Matter Hierarchy.* Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#bialgebra, 2006.

[K10] M. Pitkänen. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#class, 2006.

[K11] M. Pitkänen. Basic Properties of CP_2. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/pdfpool/chappendc.pdf, 2006.

[K12] M. Pitkänen. *Bio-Systems as Conscious Holograms*. Onlinebook. http://tgdtheory.fi/public_html/hologram/hologram.html, 2006.

[K13] M. Pitkänen. *Bio-Systems as Self-Organizing Quantum Systems*. Onlinebook. http://tgdtheory.fi/public_html/bioselforg/bioselforg.html, 2006.

[K14] M. Pitkänen. Category Theory and Quantum TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#categorynew, 2006.

[K15] M. Pitkänen. Configuration Space Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#cspin, 2006.

[K16] M. Pitkänen. Construction of Configuration Space Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#compl1, 2006.

[K17] M. Pitkänen. Construction of elementary particle vacuum functionals. In *p-Adic length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html#elvafu, 2006.

[K18] M. Pitkänen. Construction of Quantum Theory: M-matrix. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#towards, 2006.

[K19] M. Pitkänen. Construction of Quantum Theory: Symmetries. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#quthe, 2006.

[K20] M. Pitkänen. Cosmic Strings. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#cstrings, 2006.

[K21] M. Pitkänen. Dark Forces and Living Matter. In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#darkforces, 2006.

[K22] M. Pitkänen. Dark Matter Hierarchy and Hierarchy of EEGs. In *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html/tgdeeg/tgdeeg.html#eegdark, 2006.

[K23] M. Pitkänen. Dark Nuclear Physics and Condensed Matter. In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#exonuclear, 2006.

[K24] M. Pitkänen. DNA as Topological Quantum Computer. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#dnatqc, 2006.

[K25] M. Pitkänen. Does TGD Predict the Spectrum of Planck Constants? In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#Planck, 2006.

[K26] M. Pitkänen. Does the Modified Dirac Equation Define the Fundamental Action Principle? In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#Dirac, 2006.

[K27] M. Pitkänen. Does the QFT Limit of TGD Have Space-Time Super-Symmetry? In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#susy, 2006.

[K28] M. Pitkänen. Evolution in Many-Sheeted Space-Time. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#prebio, 2006.

[K29] M. Pitkänen. Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#mblocks, 2006.

[K30] M. Pitkänen. General Ideas about Many-Sheeted Space-Time: Part I. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#topcond, 2006.

[K31] M. Pitkänen. Generalized Feynman Diagrams as Generalized Braids. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#braidfeynman, 2006.

[K32] M. Pitkänen. *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html, 2006.

[K33] M. Pitkänen. Identification of the Configuration Space Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#kahler, 2006.

[K34] M. Pitkänen. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#knotstgd, 2006.

[K35] M. Pitkänen. Langlands Program and TGD. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdeeg/tgdnumber.html#Langlandia, 2006.

[K36] M. Pitkänen. Macroscopic Quantum Phenomena and CP_2 Geometry. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#super, 2006.

[K37] M. Pitkänen. *Magnetospheric Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/magnconsc/magnconsc.html, 2006.

[K38] M. Pitkänen. Many-Sheeted DNA. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#genecodec, 2006.

[K39] M. Pitkänen. Massless states and particle massivation. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html#mless, 2006.

[K40] M. Pitkänen. *Mathematical Aspects of Consciousness Theory*. Onlinebook. http://tgdtheory.fi/public_html/mathconsc/mathconsc.html, 2006.

[K41] M. Pitkänen. Matter, Mind, Quantum. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html#conscic, 2006.

[K42] M. Pitkänen. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html#nmpc, 2006.

[K43] M. Pitkänen. New Particle Physics Predicted by TGD: Part I. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html#mass4, 2006.

[K44] M. Pitkänen. Non-standard Numbers and TGD. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#infsur, 2006.

[K45] M. Pitkänen. Nuclear String Hypothesis. In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#nuclstring, 2006.

[K46] M. Pitkänen. *p-Adic length Scale Hypothesis and Dark Matter Hierarchy.* Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html, 2006.

[K47] M. Pitkänen. p-Adic Numbers and Generalization of Number Concept. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#padmat, 2006.

[K48] M. Pitkänen. p-Adic Particle Massivation: Elementary Particle Masses. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html#mass2, 2006.

[K49] M. Pitkänen. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Length Scale Hypothesis and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/paddark/paddark.html#mass3, 2006.

[K50] M. Pitkänen. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html#cognic, 2006.

[K51] M. Pitkänen. p-Adic Physics: Physical Ideas. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#phblocks, 2006.

[K52] M. Pitkänen. *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html, 2006.

[K53] M. Pitkänen. Quantum Field Theory Limit of TGD from Bosonic Emergence. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#emergence, 2006.

[K54] M. Pitkänen. Quantum Hall effect and Hierarchy of Planck Constants. In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.#anyontgd, 2006.

[K55] M. Pitkänen. *Quantum Hardware of Living Matter*. Onlinebook. http://tgdtheory.fi/public_html/bioware/bioware.html, 2006.

[K56] M. Pitkänen. Quantum Model for Bio-Superconductivity: I. In *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html//tgdeeg/tgdeeg/tgdeeg.html#biosupercondI, 2006.

[K57] M. Pitkänen. Quantum Model for Bio-Superconductivity: II. In *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html//tgdeeg/tgdeeg/tgdeeg.html#biosupercondII, 2006.

[K58] M. Pitkänen. Quantum Model for Nerve Pulse. In *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html//tgdeeg/tgdeeg/tgdeeg.html#pulse, 2006.

[K59] M. Pitkänen. *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook.http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html, 2006.

[K60] M. Pitkänen. *Quantum TGD*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquant.html, 2006.

[K61] M. Pitkänen. Riemann Hypothesis and Physics. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#riema, 2006.

[K62] M. Pitkänen. Self and Binding. In *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html#selfbindc, 2006.

[K63] M. Pitkänen. Some Fresh Ideas about Twistorialization of TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#twistornotices, 2006.

[K64] M. Pitkänen. TGD and Astrophysics. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#astro, 2006.

[K65] M. Pitkänen. TGD and Cosmology. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#cosmo, 2006.

[K66] M. Pitkänen. *TGD and EEG*. Onlinebook. http://tgdtheory.fi/public_html/tgdeeg/tgdeeg.html, 2006.

[K67] M. Pitkänen. *TGD and Fringe Physics*. Onlinebook. http://tgdtheory.fi/public_html/freenergy/freenergy.html, 2006.

[K68] M. Pitkänen. TGD and Nuclear Physics. In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#padnucl, 2006.

[K69] M. Pitkänen. *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html, 2006.

[K70] M. Pitkänen. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#visionc, 2006.

[K71] M. Pitkänen. TGD as a Generalized Number Theory: p-Adicization Program. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#visiona, 2006.

[K72] M. Pitkänen. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#visionb, 2006.

[K73] M. Pitkänen. *TGD Inspired Theory of Consciousness*. Onlinebook. http://tgdtheory.fi/public_html/tgdconsc/tgdconsc.html, 2006.

[K74] M. Pitkänen. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#tgdgrt, 2006.

[K75] M. Pitkänen. Three new physics realizations of the genetic code and the role of dark matter in bio-systems. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#dnatqccodes, 2006.

[K76] M. Pitkänen. Time, Spacetime and Consciousness. In *Bio-Systems as Conscious Holograms*. Onlinebook. http://tgdtheory.fi/public_html/hologram/hologram.html#time, 2006.

[K77] M. Pitkänen. *Topological Geometrodynamics: an Overview*. Onlinebook.http://tgdtheory.fi/public_html/tgdview/tgdview.html, 2006.

[K78] M. Pitkänen. Topological Quantum Computation in TGD Universe. In *Genes and Memes*. Onlinebook. http://tgdtheory.fi/public_html/genememe/genememe.html#tqc, 2006.

[K79] M. Pitkänen. Twistors, N=4 Super-Conformal Symmetry, and Quantum TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#twistor, 2006.

[K80] M. Pitkänen. Was von Neumann Right After All. In *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html#vNeumann, 2006.

[K81] M. Pitkänen. Yangian Symmetry, Twistors, and TGD. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#Yangian, 2006.

[K82] M. Pitkänen. Motives and Infinite Primes. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#infmotives, 2011.

[K83] M. Pitkänen. Quantum Arithmetics and the Relationship between Real and p-Adic Physics. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#qarithmetics, 2011.

[K84] M. Pitkänen. About Absolute Galois Group. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#agg, 2012.

[K85] M. Pitkänen. Construction of Quantum Theory: More about Matrices. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#UandM, 2012.

[K86] M. Pitkänen. Quantum Adeles. In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#galois, 2012.

[K87] M. Pitkänen. Quantum Mind and Neuroscience. In *TGD based view about living matter and remote mental interactions*. Onlinebook. http://tgdtheory.fi/public_html/pdfpool/lianPN.pdf, 2012.

[K88] M. Pitkänen. *TGD Based View About Living Matter and Remote Mental Interactions*. Onlinebook. http://tgdtheory.fi/public_html/tgdlian/tgdlian.html, 2012.

[K89] M. Pitkänen. The Recent Vision About Preferred Extremals and Solutions of the Modified Dirac Equation. In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#dirasvira, 2012.

[K90] M. Pitkänen. *Hyper-finite Factors and Dark Matter Hierarchy*. Onlinebook. http://tgdtheory.fi/public_html/neuplanck/neuplanck.html, 2013.

[K91] M. Pitkänen. *p-Adic length Scale Hypothesis*. Onlinebook. http://tgdtheory.fi/public_html/padphys/padphys.html, 2013.

[K92] M. Pitkänen. *Quantum TGD*. Onlinebook. http://tgdtheory.fi/public_html/tgdquantum/tgdquantum.html, 2013.

[K93] M. Pitkänen. Various General Ideas Related to TGD. In *Quantum TGD as Infinite-dimensional Spinor Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#tgdmisc, 2013.

[K94] M. Pitkänen. What are the counterparts of Einstein's equations in TGD? In *Physics in Many-Sheeted Space-Time*. Onlinebook. http://tgdtheory.fi/public_html/tgdclass/tgdclass.html#EinsteinTGD, 2013.

[K95] M. Pitkänen. What p-Adic Icosahedron Could Mean? And What about p-Adic Manifold? In *TGD as a Generalized Number Theory*. Onlinebook. http://tgdtheory.fi/public_html/tgdnumber/tgdnumber.html#picosahedron, 2013.

[K96] M. Pitkänen. Recent View about Kähler Geometry and Spin Structure of "World of Classical Worlds" . In *Quantum Physics as Infinite-Dimensional Geometry*. Onlinebook. http://tgdtheory.fi/public_html/tgdgeom/tgdgeom.html#wcwnew, 2014.

[K97] M. Pitkänen. Unified Number Theoretical Vision. In *Towards M-Matrix*. Onlinebook. http://tgdtheory.fi/public_html/tgdquant/tgdquantum.html#numbervision, 2014.

Articles about TGD

[L1] M. Pitkänen. A Strategy for Proving Riemann Hypothesis. http://arxiv.org/abs/math/0111262, 2002.

[L2] M. Pitkänen. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. http://tgdtheory.fi/pdfpool/append.pdf, 2006.

[L3] M. Pitkänen. About Correspondence Between Infinite Primes, Space-time Surfaces, and Configuration Space Spinor Fields. http://tgdtheory.fi/articles/brahma.pdf, 2007.

[L4] M. Pitkänen. First edge of the cube. http://matpitka.blogspot.com/2007/05/first-edge-of-cube.html, 2007.

[L5] M. Pitkänen. Further Progress in Nuclear String Hypothesis. http://tgdtheory.fi/articles/nuclstring.pdf, 2007.

[L6] M. Pitkänen. TGD briefly. http://www.helsinki.fi/~matpitka/TGDbrief.pdf, 2007.

[L7] M. Pitkänen. TGD inspired theory of consciousness. http://www.helsinki.fi/~matpitka/tgdconsc.pdf, 2007.

[L8] M. Pitkänen. TGD Inspired Quantum Model of Living Matter. http://tgdtheory.fi/quantumbio.pdf, 2008.

[L9] M. Pitkänen. TGD Inspired Theory of Consciousness. http://tgdtheory.fi/tgdconsc.pdf, 2008.

[L10] M. Pitkänen. Topological Geometrodynamics: What Might Be The Basic Principles. http://tgdtheory.fi/articles/tgd2008.pdf, 2008.

[L11] M. Pitkänen. Topological Geometrodynamics: What Might Be the Basic Principles. http://tgdtheory.fi/articles/tgd2008.pdf, 2008.

[L12] M. Pitkänen. Physics as Generalized Number Theory II: Classical Number Fields. https://www.createspace.com/3569411, July 2010.

[L13] M. Pitkänen. Physics as Generalized Number Theory I: p-Adic Physics and Number Theoretic Universality. https://www.createspace.com/3569411, July 2010.

[L14] M. Pitkänen. Physics as Generalized Number Theory III: Infinite Primes. https://www.createspace.com/3569411, July 2010.

[L15] M. Pitkänen. An attempt to understand preferred extremals of K ahler action. http://tgdtheory.fi/articles/prefextremals.pdf, 2011.

[L16] M. Pitkänen. Generalization of thermodynamics allowing negentropic entanglement and a model for conscious information processing. http://tgdtheory.fi/articles/thermolife.pdf, 2011.

[L17] M. Pitkänen. Is Kähler action expressible in terms of areas of minimal surfaces? http://tgdtheory.fi/articles/minimalsurface.pdf, 2011.

[L18] M. Pitkänen. CMAP representations about TGD. http://www.tgdtheory.fi/cmaphtml.html, 2014.

[L19] M. Pitkänen. CMAP representations about TGD, and TGD inspired theory of consciousness and quantum biology. http://www.tgdtheory.fi/tgdglossary.pdf, 2014.

[L20] Matti Pitkänen. Basic notions behind $M^8 - Hduality$.http://www.tgdtheory.fi/webCMAPs/BasicnotionsbehindM^8-Hduality.html. 2014.

Matti Pitkänen. Infinite primes.http://www.tgdtheory.fi/webCMAPs/Infiniteprimes.html. 2014.

Matti Pitkänen. $M^8{-}Hduality$.http://www.tgdtheory.fi/webCMAPs/M^8-Hduality.html. 2014.

Matti Pitkänen. p-Adic manifold.http://www.tgdtheory.fi/webCMAPs/p-Adicmanifold.html. 2014.

Matti Pitkänen. p-Adic mass calculations.http://www.tgdtheory.fi/webCMAPs/p-Adicmasscalculations.html. 2014.

Matti Pitkänen. p-Adic number fields and cognition and intention.http://www.tgdtheory.fi/webCMAPs/p-Adicnumberfieldsandcognitionandintention.html. 2014.

Matti Pitkänen. p-Adic number fields.http://www.tgdtheory.fi/webCMAPs/p-Adicnumberfields.html. 2014.

Matti Pitkänen. p-Adic physics.http://www.tgdtheory.fi/webCMAPs/p-Adicphysics.html. 2014.

Matti Pitkänen. p-Adice length scale hypothesis.http://www.tgdtheory.fi/webCMAPs/p-Adicelengthscalehyp
html. 2014.

Matti Pitkänen. Physics as generalized number theory.http://www.tgdtheory.fi/webCMAPs/Physicsasgeneralizednumbertheory.html. 2014.

Matti Pitkänen. Quantum physics as generalized number theory.http://www.tgdtheory.fi/webCMAPs/Quantumphysicsasgeneralizednumbertheory.html. 2014.

Matti Pitkänen. Quaternionic planes of octonions.http://www.tgdtheory.fi/webCMAPs/Quaternionicplanesof
html. 2014.

Matti Pitkänen. TGD and classical number fields.http://www.tgdtheory.fi/webCMAPs/TGDandclassicalnumbe
html. 2014.

Index

TGD AS A GENERALIZED NUMBER THEORY

Topological Geometrodynamics (TGD) is a modification of general relativity inspired by the problems related to the definition of inertial and gravitational energies in general relativity. TGD is also a generalization of super string models. Physical space-times are seen as four-dimensional surfaces in certain 8-dimensional space H. The choice of H is fixed by symmetries of standard model and leads to a geometrization of known classical fields and elementary particle numbers. In fermionic sector strings indeed emerge.

Many-sheeted space-time replaces Einsteinian space-time, which follows as a long length scale approximation in which sheets of the many-sheeted space-time are lumped together. The extension of number concept based on the fusion of real numbers and p-adic number fields implies a further generalisation of the space-time concept allowing to identify space-time correlates of cognition and intentionality.

Zero energy ontology forces an extension of quantum measurement theory to a theory of consciousness and a hierarchy of phases identified as dark matter is predicted with far reaching implications for the understanding of consciousness and living systems. This all implies an elegant theoretical projection of our reality honoring the work by renowned scientists (such as Wheeler, Feynman, Penrose, Einstein, Josephson to name a few) and creating a solid foundation for modeling our Universe in terms of geometry.

Ph.D. Matti Pitkänen • *Hanko, Finland*

Matti Pitkänen started to work with the basic idea of TGD at 1977, published his thesis work about TGD at 1982, and has since then worked to transform the basic vision to a consistent predictive mathematical framework, to solve various interpretational issues, and understand the relationship of TGD with existing theories.

TGD Web Pages: http://www.tgdtheory.com
TGD Diary and Blog: http://matpitka.blogspot.com