On the Foundamental Theorem in Arithmetic Progession of Primes

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Abstract

Using Jiang function we prove the foundamental theorem in arithmetic progression of primes [1-3]. The primes contain only $k < P_{g+1}$ long arithmetic progressions, but the primes have no $k > P_{g+1}$ long arithmetic progressions.Terence Tao is recipient of 2006 Fields medal.Green and Tao proved that the primes contain arbitrarily long arithmetic progressions which is absolutely false[4-9].They do not understand the arithmetic progression of primes [4-15].

Theorem. The fundamental theorem in arithmetic progression of primes.

We define the arithmetic progression of primes [1-3].

$$P_{i+1} = P_1 + \omega_g i, i = 0,1,2,\cdots,k-1, \qquad (1)$$

where $\omega_g = \prod_{2\le P\le P_g}$ is called a common difference, P_g is called g -th prime.

We have Jiang function [1-3]

$$J_2(\omega) = \prod_{3\le P}(P-1-X(P)), \qquad (2)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k-1}(q+\omega_g i) \equiv 0 (\bmod P), \qquad (3)$$

where $q = 1,2,\cdots,P-1$.

If $P|\omega_g$, then $X(P)=0$; $X(P)=k-1$ otherwise. From (3) we have

$$J_2(\omega) = \prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-k). \qquad (4)$$

If $k = P_{g+1}$ then $J_2(P_{g+1}) = 0$, $J_2(\omega)=0$, there exist finite primes P_1 such that $P_2,\cdots,P_k$ are primes. If $k < P_{g+1}$ then $J_2(\omega)\ne 0$, there exist infinitely many primes P_1 such that $P_2,\cdots,P_k$ are primes. The primes contain only $k < P_{g+1}$ long arithmetic progressions, but the primes have no $k > P_{g+1}$ long arithmetio progressions. We have the best asymptotic formula [1-3]

$$\pi_k(N,2) = \left|\{P_1+\omega_g i = \text{prime}, 0\le i\le k-1, P_1\le N\}\right|$$

$$= \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}(1+o(1)), \qquad (5)$$

where $\omega = \prod_{2\le P} P, \phi(\omega) = \prod_{2\le P}(P-1)$, ω IS called primorial, $\phi(\omega)$ Euler function.

Suppose $k = P_{g+1}-1$. From (1) we have

$$P_{i+1} = P_1 + \omega_g i, i = 0,1,2,\cdots, P_{g+1} - 2 . \qquad (6)$$

From (4) we have [1-2]

$$J_2(\omega) = \prod_{3 \le P \le P_g} (P-1) \prod_{P_{g+1} \le P} (P - P_{g+1} + 1) \to \infty \text{ as } \omega \to \infty \qquad (7)$$

We prove that there exist infinitely many primes P_1 such that $P_2, \cdots, P_{P_{g+1}-1}$ are primes for all P_{g+1}.

From (5) we have

$$\pi_{P_{g+1}-1}(N,2) =$$

$$\prod_{2 \le P \le P_g} \left(\frac{P}{P-1} \right)^{P_{g+1}-2} \prod_{P_{g+1} \le P} = \frac{P^{P_{g+1}-2}(P - P_{g+1} + 1)}{(P-1)^{P_{g+1}-1}} \frac{N}{(\log N)^{P_{g+1}-1}} (1 + o(1)). \qquad (8)$$

From (8) we are able to find the smallest solutions $\pi_{P_{g+1}-1}(N,2) > 1$ for large P_{g+1}.
Theorem is foundations for arithmetic progression of primes

Example 1. Suppose $P_1 = 2, \omega_1 = 2, P_2 = 3$. From (6) we have the twin primes theorem

$$P_2 = P_1 + 2. \qquad (9)$$

From (7) we have

$$J_2(\omega) = \prod_{3 \le P} (P-2) \to \infty \text{ as } \omega \to \infty, \qquad (10)$$

We prove that there exist infinitely many primes P_1 such that P_2 are primes. From (8) we have the best asymptotic formula

$$\pi_2(N,2) = 2 \prod_{3 \le P} \left(1 - \frac{1}{(P-1)^2} \right) \frac{N}{\log^2 N} (1 + o(1)). \qquad (11)$$

Twin prime theorem is the first theorem in arithmetic progression of primes. Green and Tao do not prove the twin prime theorem. Therefore Green – Tao theorem is absolutely false [4-9]. The prime distribution is order rather than randomness. The arithmetic progreessions of primes are not directly related to ergodic theory, harmonic analysis, discrete geometry and additive combinatorics. Conjectures and theorems on arithmetic progressions of primes are absolutely false [4-15], because they do not understand the arithmetic progressions of primes.

Example 2. Suppose $P_2 = 3, \omega_2 = 6, P_3 = 5$. From (6) we have

$$P_{i+1} = P_1 + 6i, i = 0,1,2,3. \quad (12)$$

From (7) we have

$$J_2(\omega) = 2 \prod_{5 \le P} (P-4) \to \infty \quad \text{as} \quad \omega \to \infty, \quad (13)$$

We prove that there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are primes. From (8) we have the best asymptotic formula

$$\pi_4(N,2) = 27 \prod_{5 \le P} \frac{P^3(P-4)}{(P-1)^4} \frac{N}{\log^4 N}(1+o(1)). \quad (14)$$

Example 3. Suppose $P_9 = 23, \omega_9 = 223092870, P_{10} = 29$. From (6) we have

$$P_{i+1} = P_1 + 223092870i, i = 0,1,2,\cdots,27. \quad (15)$$

From (7) we have

$$J_2(\omega) = 36495360 \prod_{29 \le P} (P-28) \to \infty \quad \text{as} \quad \omega \to \infty, \quad (16)$$

We prove that there exist infinitely many primes P_1 such that $P_2, \cdots, P_{28}$ are primes. From (8) we have the best asymptotic formula

$$\pi_{28}(N,2) = \prod_{2 \le P \le 23} \left(\frac{P}{P-1}\right)^{27} \prod_{29 \le P} \frac{P^{27}(P-28)}{(P-1)^{28}} \frac{N}{\log^{28} N}(1+o(1)). \quad (17)$$

From (17) we are able to find the smallest solutions $\pi_{28}(N_0,2) > 1$.

On May 17, 2008, Wroblewski and Raanan Chermoni found the first known case of 25 primes:

$$6171054912832631 + 366384 \times \omega_{23} \times n, \text{ for } n = 0 \text{ to } 24.$$

Theorem can help in finding for 26, 27, 28, …, primes in arithmetic progressions of primes.

Corollary 1. Arithmetics progression with two prime variables

Suppose $\omega_g = d$. From (1) we have

$$P_1, P_2 = P_1 + d, P_3 = P_1 + 2d, \cdots, P_k = P_1 + (k-1)d, (P_1, d) = 1. \quad (18)$$

From (18) we obtain the arithmetic progression with two prime variables: P_1 and P_2,

$$P_3 = 2P_2 - P_1, \quad P_j = (j-1)P_2 - (j-2)P_1, \quad 3 \le j \le k < P_{g+1} \quad . \quad (19)$$

We have Jiang function [3]

$$J_3(\omega) = \prod_{3 \le P} [(P-1)^2 - X(P)], \quad (20)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=3}^{k} [(j-1)q_2 - (j-2)q_1] \equiv 0 (\bmod P), \quad (21)$$

where $q_1 = 1,2,\cdots, P-1; q_2 = 1,2,\cdots, P-1.$

From (21) we have

$$J_3(\omega) = \prod_{3 \le P \le k} (P-1) \prod_{k<P} (P-1)(P-k+1) \to \infty \quad \text{as} \quad \omega \to \infty. \quad (22)$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3, \cdots, P_k$ are primes for $3 \le k < P_{g+1}$

we have the best asymptotic formula

$$\pi_{k-1}(N,3) = \left| \{ (j-1)P_2 - (j-2)P_1 = \text{prime}, 3 \le j \le k, P_1, P_2 \le N \} \right|$$

$$= \frac{J_3(\omega)\omega^{k-2}}{\phi^k(\omega)} \frac{N^2}{\log^k N} (1 + o(1)), \quad (23)$$

From (23) we have the best asymptotic formula

$$\pi_{k-1}(N,3)=\prod_{2\le P\le k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k<P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \quad (24)$$

From (24) we are able to find the smallest solution $\pi_{k-1}(N_0,3)>1$ for large $k<P_{g+1}$.

Example 4. Suppose $k=3$ and $P_{g+1}>3$. From (19) we have

$$P_3=2P_2-P_1. \quad (25)$$

From (22) we have

$$J_3(\omega)=\prod_{3\le P}(P-1)(P-2)\to\infty \text{ as } \omega\to\infty, \quad (26)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 are primes. From (24) we have the best asymptotic formula

$$\pi_2(N,3)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N^2}{\log^3 N}(1+o(1))=1.32032\frac{N^2}{\log^3 N}(1+o(1)). \quad (27)$$

Example 5. Suppose $k=4$ and $P_{g+1}>4$. From (19) we have

$$P_3=2P_2-P_1, \qquad P_4=3P_2-2P_1. \quad (28)$$

From (22) we have

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-3)\to\infty \text{ as } \omega\to\infty, \quad (29)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 and P_4 are primes. From (24) we have the best asymptotic formula

$$\pi_3(N,3)=\frac{9}{2}\prod_{5\le P}\frac{P^2(P-3)}{(P-1)^3}\frac{N^2}{\log^4 N}(1+o(1)). \quad (30)$$

Example 6. Suppose $k=5$ and $P_{g+1}>5$. From (19) we have

$$P_3 = 2P_2 - P_1, \qquad P_4 = 3P_2 - 2P_1, \qquad P_5 = 4P_2 - 3P_1 \quad . \tag{31}$$

From (22) we have

$$J_3(\omega) = 2\prod_{5\le P}(P-1)(P-4) \to \infty \text{ as } \omega \to \infty, \tag{32}$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3, P_4 and P_5 are primes. From (24) we have the best asymptotic formula

$$\pi_4(N,3) = \frac{27}{2}\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N^2}{\log^5 N}(1+o(1)). \tag{33}$$

Green and Tao study only **corollary 1**, which is not the theorem [4-9].

Corollary 2. Arithmetic progression with three prime variables

From (18) we obtain the arithmetic progression with three prime variables: P_1, P_2 and P_3

$$P_4 = P_3 + P_2 - P_1, \quad P_j = P_3 + (j-3)P_2 - (j-3)P_1, \; 4 \le j \le k < P_{g+1} \tag{34}$$

We have Jiang function

$$J_4(\omega) = \prod_{3\le P}((P-1)^3 - X(P)), \tag{35}$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=4}^{k}(q_3 + (j-3)q_2 - (j-3)q_1) \equiv 0 (\text{mod } P), \tag{36}$$

where $q_i = 1,2,\cdots,P-1, i = 1,2,3$.

Example 7. Suppose $k = 4$ and $P_{g+1} > 4$. From (34) we have

$$P_4 = P_3 + P_2 - P_1. \tag{37}$$

From (35) and (36) we have

$$J_4(\omega) = \prod_{3\le P}(P-1)(P^2 - 3P + 3) \to \infty \text{ as } \omega \to \infty, \tag{38}$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that P_4 are primes. we have the best asymptotic formula

$$\pi_2(N,4)=2\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\frac{N^3}{\log^4 N}(1+o(1)). \qquad (39)$$

For $k\ge 5$ from (35) and (36) We have Jiang function

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2$$
$$\times\prod_{(k-1)\le P}(P-1)[(P-1)^2-(P-2)(k-3)]\to\infty$$

as $\omega\to\infty$. (40)

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that $P_4,\cdots,P_k$ are primes for $5\le k<P_{g+1}$.
we have the best asymptotic formula

$$\pi_{k-2}(N,4)=\left|\{P_3+(j-3)P_2-(j-3)P_1=\text{prime},4\le j\le k,P_1,P_2,P_3\le N\}\right|$$

$$=\frac{J_4(\omega)\omega^{k-3}}{\phi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)). \qquad (41)$$

From (41) we have

$$\pi_{k-2}(N,4)$$

$$=\prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N}(1+o(1)).$$

(42)

From (42) we are able to find the smallest solution $\pi_{k-2}(N_0,4)>1$ for large

$k<P_{g+1}$.

Corollary 3. Arithmetic progression with four prime variables

From (18) we obtain the arithmetic progression with four prime variables: P_1,P_2,P_3 and P_4

$$P_5=P_4+2P_3-3P_2+P_1, \qquad P_j=P_4+(j-3)P_3-(j-2)P_2+P_1,$$
$$5\le j\le k<P_{g+1} \qquad (43)$$

We have Jiang function

$$J_5(\omega)=\prod_{3\le P}\left[(P-1)^4-X(P)\right], \tag{44}$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=5}^{k}[q_4+(j-3)q_3-(j-2)q_2+q_1]\equiv 0 \pmod P, \tag{45}$$

where

$$q_i=1,\cdots,P-1, i=1,2,3,4$$

Example 8. Suppose $k=5$ and $P_{g+1}>5$. From (43) we have

$$P_5=P_4+2P_3-3P_2+P_1. \tag{46}$$

From (44) and (45) we have

$$J_5(\omega)=12\prod_{5\le P}(P-1)(P^3-4P^2+6P-4)\to\infty \quad \text{as} \quad \omega\to\infty. \tag{47}$$

We prove there exist infinitely many primes P_1,P_2,P_3 and P_4 such that P_5 are primes.

We have the best asymptotic formula

$$\pi_2(N,5)=\frac{J_5(\omega)\omega}{\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+o(1)). \tag{48}$$

Example 9. Suppose $k=6$ and $P_{g+1}>6$. From (43) we have

$$P_5=P_4+2P_3-3P_2+P_1, \qquad P_6=P_4+3P_3-4P_2+P_1. \tag{49}$$

From (44) and (45) we have

$$J_5(\omega)=10\prod_{5\le P}(P-1)(P^3-5P^2+10P-9)\to\infty \quad \text{as} \quad \omega\to\infty. \tag{50}$$

We prove there exist infinitely many primes P_1,P_2,P_3 and P_4 such that P_5 and P_6 are primes.

We have the best asymptotic formula

$$\pi_3(N,5)=\frac{J_5(\omega)\omega^2}{\phi^6(\omega)}\frac{N^4}{\log^6 N}(1+o(1)). \tag{50}$$

For $k \ge 7$ from (44) and (45) we have Jiang function

$$J_5(\omega) = 6 \prod_{5 \le P \le (k-4)} (P-1)(P^2 - 3P + 3)$$

$$\times \prod_{(k-4)<P} \left\{ (P-1)^4 - (P-1)^2 \left[(P-3)(k-4)+1 \right] - (P-1)(2k-9) \right\} \to \infty$$

as $\omega \to \infty$ (51)

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that $P_5, \cdots, P_k$ are primes.

We have best asymptotic formula

$$\pi_{k-3}(N,5) = \left| \left\{ P_4 + (j-3)P_3 - (j-2)P_2 + P_1 = \text{prime}, 5 \le j \le k, P_1, \cdots, P_4 \le N \right\} \right|$$

$$= \frac{J_5(\omega)\omega^{h-4}}{\phi^k(\omega)} \frac{N^4}{\log^k N}(1 + o(1)). \quad (52)$$

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