The Chinese Remainder Theorem • Goldbach's Conjecture (A) • Hardy-Littewood's Conjecture (A)

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Abstract: $N = p_i + (N-p_i) = p + (N-p)$. ① If p is congruent to N modulo p_i, Then (N-p) is a composite integer, When i=1, 2, …, r, if p and N are incongruent modulo p_i, Then p and (N-p) are solutions of Goldbach's Conjecture (A); ② By Chinese Remainder Theorem we can calculate the primes and solutions of Goldbach's Conjecture (A) with different system of congruence; ③The (N-**p**) must have solution of Goldbach's Conjecture (A), The number of solutions of Goldbach's Conjecture (A) is increasing as $N\to\infty$, and finding unknown particulars for Hardy-Littewood's Conjecture (A).

Key words: congruent, Chinese Remainder Theorem, Goldbach's Conjecture (A), Hardy-Littewood's Conjecture (A).

"*Every even positive integer greater than 2 can be written as the sum of two primes.*" This conjecture was stated by Christion Goldbach in a letter to Leonhard Euler in 1742.

Let $p_i < \sqrt{N}$, $\sqrt{N} < p < N - \sqrt{N}$. We have $N = p_i + (N-p_i) = p + (N-p)$. ①If p is congruent to N modulo p_i, then (N-p) is a composite integer, (See **Theorem 1**.) When i=1, 2, …, r, if p and N are incongruent modulo p_i, Then p and (N-p) are solutions of Goldbach's Conjecture (A); (See **Theorem 2**.) ②By Chinese Remainder Theorem, we can calculate the primes and solutions of Goldbach's Conjecture (A) with different system of congruence (3), (5). (See **Theorem 3, 4.**) ③The (N-**p**) must have solution of Goldbach's Conjecture (A), (See **Theorem 5**.) The number of solutions of Goldbach's Conjecture (A) is increasing as $N\to\infty$, (See **Theorem 6**.) and finding unknown particulars for Hardy-Littewood's Conjecture (A),(See **Theorem 6**.)

1, Term, Terminology, Symbol.

N — Even positive integers. Let $2 \leqslant p_i \leqslant p_r < \sqrt{N} < p_{r+1} < N < p^2_{r+1} < p_1 p_2 \ldots p_r$.

p_i, p_r, p_{r+1} — Prime number. i=1, 2, 3, …, r. $r = \pi(\sqrt{N})$.

(1) $w_r = p_1 p_2 \cdots p_r = \prod_{2 \leqslant p \leqslant \sqrt{N}} p$.

p — Prime number. $p_{r+1} \leqslant p \leqslant (N - p_r - 1)$. We have $(N-p) > p_r$. Every p can be written as $p = p(a_i) + np_i$. $1 \leqslant p(a_i) \leqslant p_i - 1$.

$p(a_i)$ — Remainder that divided p by p_i. $1 \leqslant p(a_i) \leqslant p_i - 1$.

Let $f_1(a_i) = 1, 2, \ldots, (p_i - 1)$.

p — All p. We have $\mathbf{p} = f_1(a_i) + np_i$, $f_1(a_i) = 1, 2, \ldots, (p_i - 1)$.

Let $N = p_i + (N-p_i) = p + (N-p)$, When N=98, 126, 128, … The $(N-p_i)$=composite integers. We prove that (N-**p**) must have prime.

Lemma 1. If $r \geqslant 4$, Then $N < p^2_{r+1} < p_1 p_2 \ldots p_r = w_r$.

When $r < 4$, we can finding $p_1 p_2 \ldots p_r < N$, Therefore, This paper studies $r \geqslant 4$, $N \geqslant 50$.

$N(a_i)$ — Remainder that divided N by p_i. We have $N = N(a_i) + np_i$. $0 \leqslant N(a_i) \leqslant p_i - 1$.

$N(a_i)_r$ — A group of systematic remainders that divided N by $p_1, p_2, \ldots, p_r$. $N(a_i)_r = N(a_1), N(a_2), \ldots, N(a_r)$. For example, N=90, r=4, $90(a_i)_4 = 90(a_1), 90(a_2), \ldots, 90(a_4) = 0,0,0,6$.

$f_2(a_i)$ — Take $N(a_i)$ out of $f_1(a_i)$, we can obtain $f_2(a_i)$. ① When $p_i \mid N$, $N(a_i)=0 \neq f_2(a_i)$, the number of element of $f_2(a_i)$ is (p_i-1); ② When $(p_i, N)=1$, $0 < N(a_i) \leqslant p_i-1$. $f_1(a_i) = 1,2,\ldots,(p_i-1)$. The $N(a_i)$ is one element of $f_1(a_i)$, The number element of $f_2(a_i)$ is (p_i-2).

$N(1,1)_i$ — The number of solutions of Goldbach's Conjecture (A) lying in the interval $(0, p_r+1)$ and $(N- p_r -1, N)$.

$N(1,1)_r$ — The number of solutions of Goldbach's Conjecture (A) lying in the interval $(p_r+1, N- p_r -1)$[1].

$N(1,1)$ $(=r_2(N))$ — The number of solutions of Goldbach's Conjecture (A) lying in the interval $(0,N)$. $N(1,1)= N(1,1)_r +2 N(1,1)_i$.

2. Distinguish of Solutions of Goldbach's Conjecture (A).

Theorem 1. If N is congruent to p modulo p_i, Then the (N-p) is a composite integer.
Proof. $N \equiv p \pmod{p_i}$, $p_i \mid (N-p)$, We have $(N-p)=kp_i$. $(k \geqslant 1.)$ As before, $(N-p) > p_r$, $k > 1$, The (N-p) is composite integer. Theorem 1 is proved.
Theorem 2. If i=1, 2, 3, ..., r. N and p are incongruent modulo $p_1, p_2, \ldots, p_r$. Then p and (N-p) are solutions of Goldbach's Conjecture (A).
Proof. i=1, 2, 3, ..., r. N and p are incongruent modulo $p_1, p_2, \ldots, p_r$. In other words, the (N-p) is not divisible by any prime not exceeding $\sqrt{N}$. The (N-p) is a prime. The p and (N-p) are solutions of Goldbach's Conjecture (A). Theorem 2 is proved.

3. Finding Primes and Solutions of Goldbach's Conjecture (A).

Lemma 2. ***The Chinese Remainder Theorem .*** Let $\mathbf{m_1, m_2, \ldots, m_r}$ be pairwise relatively prime positive integers. Then the system of congruences

$$(2)\quad \begin{aligned} x &\equiv \mathbf{a_1} \pmod{\mathbf{m_1}}, \\ x &\equiv \mathbf{a_2} \pmod{\mathbf{m_2}}, \\ &\;\;\vdots \\ x &\equiv \mathbf{a_r} \pmod{\mathbf{m_r}}, \end{aligned}$$

has a unique solution modulo $\mathbf{M=m_1m_2\ldots m_r}$.
Theorem 3. The u is number of solutions of system of congruences (3). When y<N, the y is a prime.

$$(3)\quad \begin{aligned} y &\equiv f_1(a_1) \pmod{p_1} \\ y &\equiv f_1(a_2) \pmod{p_2} \\ &\;\;\vdots \\ y &\equiv f_1(a_r) \pmod{p_r} \end{aligned}$$

$$(4)\quad u=(p_1-1)(p_2-1)\ldots(p_r-1)=\prod_{2 \leqslant p \leqslant \sqrt{N}} (p-1)$$

Proof. The $f_1(a_1)=1$, The number of elements of $f_1(a_1)$ is (p_1-1);

The $f_1(a_2)=1,2$. The number of elements of $f_1(a_2)$ is (p_2-1); ...

The $f_1(a_r)=1,2,\ldots,(p_r-1)$. The number of elements of $f_1(a_r)$ is (p_r-1).

When i=1,2,..., r. we taking one element of the $f_1(a_i)$, We can obtain different system of congruences (3), The number of the different system of congruences (3) is $(p_1-1)(p_2-1)\ldots(p_r-1)=u$.

By (3), if $y<N$, the y is not divisible by any prime not exceeding $\sqrt{N}$. The y is a prime. Theorem 3 is proved.

Theorem 4. The v is number of solutions of system of congruences (5). When $(p_r+1)<y<(N-p_r-1)$, the y and (N-y) are solutions of Goldbach's Conjecture (A).

$$
\begin{aligned}
(5)\quad & y\equiv f_2(a_1) \pmod{p_1}\\
& y\equiv f_2(a_2) \pmod{p_2}\\
& \quad\vdots\\
& y\equiv f_2(a_r) \pmod{p_r}
\end{aligned}
$$

$$(6)\quad v=\prod_{\substack{(p,N)=1\\ 3\leqslant p\leqslant\sqrt{N}}}(p-2)\prod_{\substack{(p,N)=p\\ 2\leqslant p\leqslant\sqrt{N}}}(p-1)=\prod_{3\leqslant p\leqslant\sqrt{N}}(p-2)\prod_{\substack{p\mid N\\ 3\leqslant p\leqslant\sqrt{N}}}\frac{p-1}{p-2}\quad \text{(get rid of } p_1-1=1\text{。)}$$

Proof. When $(N, p_i)=1$, The number of elements of $f_2(a_i)$ is (p_i-2); When $(N, p_i)= p_i$, The number of elements of $f_2(a_i)$ is (p_i-1). (When $p_i>2$, We have $(p_i-1)=(p_i-2)\dfrac{p_i-1}{p_i-2}$.)

When i=1,2,...,r. we taking one element of the $f_2(a_i)$, We can obtain different system of congruences (5), The number of the different system of congruences (5) is multiply $\prod(p_i-2)\prod(p_j-1)$. (See 6.)

By (5), if $(p_r+1)<y<(N-p_r-1)$, ① Because $f_2(a_i)\neq 0$, by Theorem 3, the y is a prime; ② Because $f_2(a_i)\neq N(a_i)$, N and y are incongruent modulo $p_1, p_2,\ldots, p_r$. By Theorem 2, the (N-y) is a prime. The y and (N-y) are solutions of Goldbach's Conjecture (A). Theorem 4 is proved.

4. The Proof of Goldbach's Conjecture (A).

Theorem 5. (N-**p**) must have prime.

Proof. $(N-\mathbf{p})>p_r$, Suppose $(N-\mathbf{p})$=composite integer$=h_ip_i$. ($h_i>1$.) We have $\mathbf{p}=N-h_ip_i=N(p_i)+np_i-h_ip_i=N(p_i)+(n-h_i)p_i$. The $\mathbf{p}=N(p_i)+(n-h_i)p_i$ are in contradiction with $\mathbf{p}=f_1(a_i)+np_i$. (The $N(p_i)$ is one of the $f_1(a_i)$.) The contradiction shows, that there are some primes in (N-**p**). Theorem 5 is proved.

Lemma 3. $$\pi(N)=\varepsilon N\prod_{2\leqslant p\leqslant\sqrt{N}}\frac{p-1}{p}$$

Proof. The $\pi(N)$ and u are number of positive integer that are not divisible by any prime not exceeding $\sqrt{N}$.

Noticing $\dfrac{\pi(N)}{N}\neq\dfrac{u}{w_r}$, we have $\dfrac{\pi(N)}{N}=\varepsilon\dfrac{u}{w_r}$, and $\pi(N)=\varepsilon N\dfrac{u}{w_r}$. Lemma 3 is proved.

Theorem 6. The number of solutions of Goldbach's Conjecture (A) is increasing as $N\to\infty$.

Proof. By Lemma 3, we have

$$1 = \prod_{2\leqslant p\leqslant\sqrt{N}} \frac{p}{p-1}\frac{p}{p-1} \times \prod_{2\leqslant p\leqslant\sqrt{N}} \frac{p-1}{p}\frac{p-1}{p}$$

$$= \prod_{2\leqslant p\leqslant\sqrt{N}} \frac{p}{p-1}\frac{p}{p-1} \times \frac{1}{\varepsilon}\frac{\pi(N)}{N}\frac{1}{\varepsilon}\frac{\pi(N)}{N}$$

$$= 4 \prod_{3\leqslant p\leqslant\sqrt{N}} \frac{p}{p-1}\frac{p}{p-1} \times \frac{1}{\varepsilon}\frac{\pi(N)}{N}\frac{1}{\varepsilon}\frac{\pi(N)}{N}$$

The N(1,1) and v are number of positive integer that are not divisible by any prime not exceeding $\sqrt{N}$. Added to this, these positive integer and N are incongruent modulo p_i.

Noticing $\frac{N(1,1)}{N} \neq \frac{v}{w_r}$. We have $\frac{N(1,1)}{N} = \psi\frac{v}{w_r}$, and have (7).

(7) $$N(1,1) = \psi \frac{v}{w_r} N$$

$$= \psi \frac{N}{2} \prod_{3\leqslant p\leqslant\sqrt{N}} \frac{p-2}{p} \prod_{\substack{p|N \\ 3\leqslant p\leqslant\sqrt{N}}} \frac{p-1}{p-2} \times 1$$

$$= \frac{\psi}{\varepsilon\varepsilon}\frac{2\pi(N)\pi(N)}{N} \prod_{3\leqslant p<\sqrt{N}} \frac{p-2}{p-1}\frac{p}{p-1} \prod_{\substack{p|N \\ 3\leqslant p<\sqrt{N}}} \frac{p-1}{p-2}$$

$$= \frac{\psi}{\varepsilon\varepsilon}\frac{2\pi(N)\pi(N)}{N} \prod_{3\leqslant p<\sqrt{N}} \left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ 3\leqslant p<\sqrt{N}}} \frac{p-1}{p-2}$$

The $\frac{2\pi(N)\pi(N)}{N} \prod (1-\frac{1}{(p-1)^2}) \prod \frac{p-1}{p-2} \sim \frac{2N}{\ln^2 N} \prod (1-\frac{1}{(p-1)^2}) \prod \frac{p-1}{p-2}$ (= $r_2(N)$. It is Hardy-Littewood's Conjecture (A).) The (7) is increasing as $N\to\infty$. Theorem 6 is proved.

The ε and ψ are some unknown particulars for Hardy-Littewood's Conjecture (A).

5. Discussion.

This Goldbach's Conjecture (A) the proof.

If $N\to\infty$, proof $\frac{\psi}{\varepsilon\varepsilon}\to 1$, Then Hardy-Littewood's Conjecture (A) is proved.

The others particulars of Hardy-Littewood's Conjecture (A) is still under discussion.

Reference material:

[1] Tong Xin Ping , Even formulae when calculating Goldbach Problem, Journal of youjiang teachers' college for nationalities, 1997, 3, 10-12.