素数分布中蒋函数

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1777 年最伟大数学家 Euler 说：“数学家还没有发现素数序列中的一些规则。我们有理由相信它是一个人类智慧尚未洞悉的奥秘。”20 世纪最伟大数学家 Erdös 说：“至少还需要 100 万年，我们才能真正理解素数。”说明素数研究多么困难！多么复杂！但是我的兴趣就是要研究没有人研究的问题。用我的思路、我的方法进行研究。不管这个问题多么困难。这篇论文是把我过去对素数研究作一个总结，使人们更容易理解。我发现只有素数定理才算一个真正定理：即欧几里德证明了有无限多个素数，并找到计算低于 N 素数个数公式 N/logN，其它素数定理都是猜想，因为只获得上限和下限公式并没有证明它有无限多素数解。

用蒋函数我证明了素数分布中几乎所有问题。素数问题是有规则的，不是随机的。本文只有一个定理。孪生素数和哥德巴赫猜想只能作为两个特例。这个定理包括素数分布所有问题。素数研究最近最大成果是格林和陶哲轩证明存在任意长的素数等差数列。陶哲轩因此获得 2006 年国际数学家大会菲尔茨奖。王元对陶哲轩评价：“我不敢想象天下会有这样伟大的成就。”陶哲轩文章内容就是本文 Example8。格林—陶哲轩论文是错误的：1、他们没有证明<素数等差数列>无限多素数解。2、他们没有找计算素数个数的公式，他们是抄前人的公式，计算个数概念不清。请看文献[20]，公式应该是 $\nu_k N(\log N)^{-k}$，不是 $\nu_k N^2(\log N)^{-k}$。他们 66 页论文没有直接讨论素数等差数列，他们根本不懂素数，普林斯顿《数学年刊》发表这样一篇错误的论文，说明这杂志编委不懂素数，因为全世界数学家都不懂素数，把这篇错误论文评为 2006 年国际数学家的菲尔茨奖，这就是当代素数研究水平。本文 Example 8 已给素数等差数列彻底解决。

Jiang's function $J_{n+1}(\omega)$ in prime distribution

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Abstract

We define that prime equations

$$f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots P_n) \tag{5}$$

are polynomials (with integer coefficients) irreducible over integers, where $P_1,\cdots,P_n$ are all prime. If Jiang's function $J_{n+1}(\omega)=0$ then (5) has finite prime solutions. If $J_{n+1}(\omega)\neq 0$ then there are infinitely many primes $P_1,\cdots,P_n$ such that $f_1,\cdots f_k$ are all prime. We obtain a unite prime formula in prime distribution

$$\pi_{k+1}(N,n+1)=\left|\left\{P_1,\cdots,P_n\le N: f_1,\cdots,f_k \text{ are all prime}\right\}\right|$$

$$\sim=\prod_{i=1}^{k}(\deg f_i)^{-1}\times\frac{J_{n+1}(\omega)\omega^k}{n!\phi^{k+n}(\omega)}\frac{N^n}{\log^{k+n}N}(1+o(1)). \tag{8}$$

Jiang's function is accurate sieve function. Using Jiang's function we prove about 600 prime theorems[6]. Jiang's function provids proofs of the prime theorems which are simple enough to understand and accurate enough to be useful.

Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate.

Leonhard Euler

It will be another million years, at least, before we understand the primes.

Paul Erdös

Support that Euler totient function

$$\phi(\omega)=\prod_{2\le P}(P-1)=\infty \quad \text{as} \quad \omega\to\infty, \qquad (1)$$

where $\omega=\prod_{2\le P} P$ is called primorial.

Support that $(\omega,h_i)=1$, where $i=1,\cdots,\phi(\omega)$. We have prime equations

$$P_1=\omega n+1,\cdots,P_{\phi(\omega)}=\omega n+h_{\phi(\omega)} \qquad (2)$$

where $n=0,1,2,\cdots$.

(2) is called infinitely many prime equations (IMPE). Every equation has infinitely many prime solutions. We have

$$\pi_{h_i}=\sum_{\substack{P_i\le N\\ P_i\equiv h_i \pmod{\omega}}}1=\frac{\pi(N)}{\phi(\omega)}(1+o(1)).,\qquad (3)$$

where π_{h_i} denotes the number of primes $P_i\le N$ in $P_i=\omega n+h_i$ $n=0,1,2,\cdots$, $\pi(N)$ the number of primes less than or equal to N.

We replace set of prime numbers by IMPE. (2) is the fundamental tool for proving the prime theorems in prime distribution.

Let $\omega=30$ and $\phi(30)=8$. From (2) we have eight prime equations

$$P_1=30n+1,\ P_2=30n+7,\ P_3=30n+11,\ P_4=30n+13,\ P_5=30n+17,$$
$$P_6=30n+19,\ P_7=30n+23,\ P_8=30n+29,\ n=0,1,2,\cdots \qquad (4)$$

Every equation has infinitely many prime solutions.

THEOREM. We define that prime equations

$$f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots,P_n) \qquad (5)$$

are polynomials (with integer coefficients) irreducible over integers, where $P_1,\cdots,P_n$ are all prime. There exist infinitely many $n-$tuplets of $P_1,\cdots,P_n$ such that each f_k is prime.

PROOF. Firstly, we have Jiang's function [1-11]

$$J_{n+1}(\omega)=\prod_{3\le P}[(P-1)^n-\chi(P)], \tag{6}$$

where $\chi(P)$ is called sieve constant and denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k} f_i(q_1,\cdots,q_n)\equiv 0 \pmod P, \tag{7}$$

where $q_1=1,\cdots,P-1,\cdots,q_n=1,\cdots,P-1$.

$J_{n+1}(\omega)$ denotes the number of $n-$tuplets of $P_1,\cdots,P_n$ prime equations for which $f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots,P_n)$ are prime equation. If $J_{n+1}(\omega)=0$ then (5) has finite prime solutions. If $J_{n+1}(\omega)\ne 0$ using $\chi(P)$ we sift out from (2) prime equations that can not be represented $P_1,\cdots,P_n$, then residual prime equations of (2) are $P_1,\cdots,P_n$ prime equations for which $f_1(P_1,\cdots,P_n),\cdots,$ $f_k(P_1,\cdots,P_n)$ are all prime equation, hence we prove that there exist infinitely many $n-$ tuplets of primes $P_1,\cdots,P_n$ for which $f_1(P_1,\cdots,P_n),\cdots,$ $f_k(P_1,\cdots,P_n)$ are all prime.

Secondly, we have the best asymptotic formula [2,3,4,6]

$$\pi_{k+1}(N,n+1)=\left|\{P_1,\cdots,P_n\le N: f_1,\cdots,f_k \text{ are all prime}\}\right|$$

$$=\prod_{i=1}^{k}(\deg f_i)^{-1}\times\frac{J_{n+1}(\omega)\omega^k}{n!\phi^{k+n}(\omega)}\frac{N^n}{\log^{k+n}N}(1+o(1)). \tag{8}$$

（8）is called a unite prime formula in prime distribution. Let $n=1,k=0$, $J_2(\omega)=\phi(\omega)$. From (8) we have prime number theorem

$$\pi_1(N,2)=\left|\{P_1\le N: P_1 \text{ is prime}\}\right|=\frac{N}{\log N}(1+o(1)).. \tag{9}$$

Number theorists believe that there are infinitely many twin primes, but they do not have rigorous proof of this old conjecture by any method. All prime theorems are conjectures except the prime number theorm, because they do not prove that prime equations have infinitely many prime solutions. We prove the following conjectures by this theorem.

Example 1. Twin primes $P, P+2$ (300BC)**.**

From (6) and (7) we have Jiang's function

$$J_2(\omega) = \prod_{3\le P}(P-2) \neq 0$$

Since $J_2(\omega) \neq 0$ in (2) exist infinitely many P prime equations for which $P+2$ is prime equation, hence we prove that there are infinitely many primes P for which $P+2$ is prome.

Let $\omega = 30$ and $J_2(30) = 3$. From (4) we have three P prime equatins

$$P_3 = 30n + 11, \quad P_5 = 30n + 17, \quad P_8 = 30n + 29.$$

From (8) we have the best asymptotic formula

$$\pi_2(N,2) = \left|\{P \le N : P+2 \text{ prime}\}\right| = \frac{J_2(\omega)\omega}{\phi^2(\omega)} \frac{N}{\log^2 N}(1+o(1))$$

$$= 2\prod_{3\le P}\left(1 - \frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)).$$

In 1996 we proved twin primes conjecture [1]

Remark. $J_2(\omega)$ denotes the number of P prime equations,

$\dfrac{\omega}{\phi^2(\omega)} \dfrac{N}{\log^2 N}(1+o(1))$ the number of solutions of primes for every P prime equation.

Example 2. Even Goldbach's conjecture $N = P_1 + P_2$**.** Every even number $N \ge 6$ is the sum of two primes**.**

From (6) and (7) we have Jiang's function

$$J_2(\omega)=\prod_{3\le P}(P-2)\prod_{P|N}\frac{P-1}{P-2}\ne 0\,.$$

Since $J_2(\omega)\ne 0$ as $N\to\infty$ in (2) exist infinitely many P_1 prime equations for which $N-P_1$ is prime equation, hence we prove that every even number $N\ge 6$ is the sum of two primes.

From (8) we have the best asymptotic formula

$$\pi_2(N,2)=\left|\{P_1\le N, N-P_1 \text{ prime}\}\right|=\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+o(1)).$$

$$=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\prod_{P|N}\frac{P-1}{P-2}\frac{N}{\log^2 N}(1+o(1))\,.$$

In 1996 we proved even Goldbach's conjecture [1]

Example 3. Prime equations $P, P+2, P+6$ **.**

From (6) and (7) we have Jiang's function

$$J_2(\omega)=\prod_{5\le P}(P-3)\ne 0\,,$$

$J_2(\omega)$ is denotes the number of P prime equations for which $P+2$ and $P+6$ are all prime equation. Since $J_2(\omega)\ne 0$ in (2) exist infinitely many P prime equations such that $P+2$ and $P+6$ are all prime equation, hence we prove that there are infinitely many primes P such that $P+2$ and $P+6$ are all prime.

Let $\omega=30,\ J_2(30)=2$. From (4) we have two P prime equations

$$P_3=30n+11,\quad P_5=30n+17\,.$$

From (8) we have the best asymptotic formula

$$\pi_3(N,2)=\left|\{P\le N: P+2, P+6 \text{ prime}\}\right|=\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+o(1))\,.$$

Example 4. Odd Goldbach's conjecture $N=P_1+P_2+P_3$. Every odd number $N\ge 9$ is the sum of three primes.

From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left(P^2-3P+3)\right)\prod_{P|N}\left(1-\frac{1}{P^2-3P+3}\right)\ne 0 .$$

Since $J_3(\omega)\ne 0$ as $N\to\infty$ in (2) exist infinitely many pairs of P_1 and P_2 prime equations for which $N-P_1-P_2$ is prime equation, hence we prove that every odd number $N\ge 9$ is the sum of three primes.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\left\{P_1,P_2\le N: N-P_1-P_2 \text{ prime}\right\}\right|=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)) .$$

$$=\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\prod_{P|N}\left(1-\frac{1}{P^3-3P+3}\right)\frac{N^2}{\log^3 N}(1+o(1)) .$$

Example 5. Prime equation $P_3=P_1P_2+2$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left(P^2-3P+2\right)\ne 0$$

$J_3(\omega)$ denotes the number of pairs of P_1 and P_2 prime equations for which P_3 is prime equation. Since $J_3(\omega)\ne 0$ in (2) exist infinitely many pairs of P_1 and P_2 prime equations for which P_3 is prime equation, hence we prove that there are infinitely many pairs of primes P_1 and P_2 for which P_3 is prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\left\{P_1,P_2\le N: P_1P_2+2 \text{ prime}\right\}\right|=\frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Note. $\deg(P_1P_2)=2$.

Example 6 [12]. Prime equation $P_3=P_1^3+2P_2^3$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left[(P-1)^2-\chi(P)\right]\ne 0$$

where $\chi(P)=3(P-1)$ if $2^{\frac{P-1}{3}}\equiv 1(\operatorname{mod} P)$; $\chi(P)=0$ if $2^{\frac{P-1}{3}}\not\equiv 1(\operatorname{mod} P)$;

$\chi(P) = P-1$ otherwise.

Since $J_3(\omega) \neq 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations for which P_3 is prime equation, hence we prove that there are infinitely many pairs of primes P_1 and P_2 for which P_3 is prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3) = \left|\left\{P_1, P_2 \leq N : P_1^3 + 2P_2^3 \text{ prime}\right\}\right| \sim \frac{J_3(\omega)\omega}{6\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+o(1)).$$

Example 7 [13]. Prime equation $P_3 = P_1^4 + (P_2+1)^2$.
From (6) and (7) we have Jiang's function

$$J_3(\omega) = \prod_{3 \leq P}\left[(P-1)^2 - \chi(P)\right] \neq 0$$

where $\chi(P) = 2(P-1)$ if $P \equiv 1 (\bmod 4)$; $\chi(P) = 2(P-3)$ if $P \equiv 1 (\bmod 8)$; $\chi(P) = 0$ otherwise.
Since $J_3(\omega) \neq 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations for which P_3 is prime equation, hence we prove that there are infinitely many pairs of primes P_1 and P_2 for which P_3 is prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3) = \left|\left\{P_1, P_2 \leq N : P_3 \text{ prime}\right\}\right| = \frac{J_3(\omega)\omega}{8\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+o(1)).$$

Example 8 [14-19]. Arithmetic progressions consisting only of primes. We define the arithmetic progressions of length k.

$$P_1, P_2 = P_1 + d, P_3 = P_1 + 2d, \cdots, P_k = P_1 + (k-1)d, (P_1, d) = 1. \qquad (10)$$

From (8) we have the best asymptotic formula

$$\pi_k(N,2) = \left|\left\{P_1 \leq N : P_1, P_1 + d, \cdots, P_1 + (k-1)d \text{ are all prime}\right\}\right|$$

$$= \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}(1+o(1))..$$

If $J_2(\omega) = 0$ then (10) has finite prime solutoins. If $J_2(\omega) \neq 0$ then there are infinitely many primes P_1 for which $P_2,\cdots,P_k$ are all prime. In [20] Conjecture 1.2 (Hardy-Littlewood conjecture on $k-$term APs) and Theorem 1.3(G.-Tao) are false.

To eliminate d from (10) we have

$$P_3 = 2P_2 - P_1, \quad P_j = (j-1)P_2 - (j-2)P_1, 3 \le j \le k .$$

From (6) and (7) we have Jiang's function

$$J_3(\omega) = \prod_{3\le P<k}(P-1)\prod_{k\le P}(P-1)(P-k+1) \ne 0$$

Since $J_3(\omega)\ne 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations for which $P_3,\cdots,P_k$ are all prime equation, hence we prove that there are infinitely many pairs of primes P_1 and P_2 for which $P_3,\cdots,P_k$ are all prime.

From (8) we have the best asymptotic formula

$$\pi_{k-1}(N,3) = \left|\left\{P_1,P_2 \le N : (j-1)P_2 - (j-2)P_1 \text{ prime}, 3\le j\le k\right\}\right|$$

$$= \frac{J_3(\omega)\omega^{k-2}}{2\phi^k(\omega)}\frac{N^2}{\log^k N}(1+o(1))$$

$$= \frac{1}{2}\prod_{2\le P<k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k\le P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)) .$$

Example 9. It is a well-known conjecture that one of $P, P+2, P+2^2$ is always divisible by 3. To generalize above to the $k-$primes, we prove the following conjectures. Let n be a square-free even number.

1. $P, P+n, P+n^2$,

where $3\mid(n+1)$.

Frome (6) and (7) we have $J_2(3)=0$, hence one of $P, P+n, P+n^2$ is always divisible by 3.

2. $P, P+n, P+n^2,\cdots,P+n^4$,

where $5\mid(n+b), b=2,3$.

From (6) and (7) we have $J_2(5)=0$, hence one of $P, P+n, P+n^2,\cdots,P+n^4$

is always divisible by 5.

3. $P, P+n, P+n^2, \cdots, P+n^6$,

where $7 \mid (n+b), b=2,4.$

From (6) and (7) we have $J_2(7)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^6$ is always divisible by 7.

4. $P, P+n, P+n^2, \cdots, P+n^{10}$,

where $11 \mid (n+b), b=3,4,5,9.$

From (6) and (7) we have $J_2(11)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{10}$ is always divisible by 11.

5. $P, P+n, P+n^2, \cdots, P+n^{12}$,

where $13 \mid (n+b), b=2,6,7,11.$

From (6) and (7) we have $J_2(13)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{12}$ is always divisible by 13.

6. $P, P+n, P+n^2, \cdots, P+n^{16}$,

where $17 \mid (n+b), b=3,5,6,7,10,11,12,14,15.$

From (6) and (7) we have $J_2(17)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{16}$ is always divisible by 17.

7. $P, P+n, P+n^2, \cdots, P+n^{18}$,

where $19 \mid (n+b), b=4,5,6,9,16.17.$

From (6) and (7) we have $J_2(19)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{18}$ is always divisible by 19.

Example 10. Let n be an even number.

1. $P, P+n^i, i=1,3,5,\cdots,2k+1$,

From (6) and (7) we have $J_2(\omega) \neq 0$, hence we prove that there exist infinitely many primes P such that $P, P+n^i$ are all prime for any k.

2. $P, P+n^i, i=2,4,6,\cdots,2k$.

From (6) and (7) we have $J_2(\omega) \neq 0$, hence we prove that there exist infinitely many primes P such that $P, P+n^i$ are all prime for any k.

Example 11. prime equation $2P_2 = P_1 + P_3$

Frome (6) and (7) we have Jiang's function

$$J_3(\omega) = \prod_{3 \le P}(P^2 - 3P + 2) \ne 0$$

Since $J_3(\omega) \ne 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations for which P_3 is prime equation, hence we prove that there are infinitely many pairs of primes P_1 and P_2 for which P_3 is prime.

From (8) we have the best asymptotic formula

$$\pi_2(N,3) = \left|\{P_1, P_2 \le N : P_3 \text{ prime}\}\right| = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+o(1)).$$

In the same way we can prove $2P_2^2 = P_3 + P_1$ which has the same Jiang's function.

Jiang's funciton is accurate sieve function. Using it we can prove any irreducible prime equations in prime distribution. There are infinitely many twin primes but we do not have rigorous proof of this old conjecture by any method [20]. As strong as the numerical evidence may be, we still do not even know whether there are infinitely many pairs of twin primes [21]. All the prime theorems are conjectures except the prime number theorem, because they do not prove the simplest twin primes. They conjecture that the prime distribution is randomness [12-25],because they do not understand theory of prime numbers.

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References

[1] Chun-Xuan Jiang, On the Yu-Goldbach prime theorem, Guangxi Sciences (Chinese) 3(1996), 91-2.

[2] Chun-Xuan Jiang, Foundations of Santilli's isonumber theory, Part I, Algebras Groups and Geometries, 15(1998), 351-393.

[3] ChunXuan Jiang, Foundations of Santilli's isonumber theory, Part II, Algebras Groups and Geometries, 15(1998), 509-544.

[4] Chun-Xuan Jiang, Foundations Santilli's isonumber theory, In: Foundamental open problems in sciences at the end of the millennium, T.

Gill, K. Liu and E. Trell (Eds) Hadronic Press, USA, (1999), 105-139.
[5] Chun-Xuan Jiang, Proof of Schinzel's hypothesis, Algebras Groups and Geometries, 18(2001), 411-420.
[6] Chun-Xuan Jiang, Foundations of Santilli's isonmuber theory with applications to new cryptograms, Fermat's theorem and Goldbach's conjecture, Inter. Acad. Press, 2002, MR2004c: 11001, http://www.i-b-r.org/jiang.pdf
[7] Chun-Xuan Jiang,Prime theorem in Santilli's isonumber theory, 19(2002), 475-494.
[8] Chun-Xuan Jiang, Prime theorem in Santilli's isonumber theory (II), Algebras Groups and Geometries, 20(2003), 149-170.
[9] Chun-Xuan Jiang, Disproofs of Riemann's hypothesis, Algebras Groups and Geometries, 22(2005), 123-136. http://www.i-b-r.org/docs/Jiang Riemann.pdf
[10] Chun-Xuan Jiang, Fifteen consecutive integers with exactly k prime factors, Algebras Groups and Geometries, 23(2006), 229-234.
[11] Chun-Xuan Jiang, The simplest proofs of both arbitrarily long arithmetic progression of primes, preprint, 2006.
[12] D. R. Heath-Brown, Primes represented by x^3+2y^3, Acta Math., 186 (2001), 1-84.
[13] J. Friedlander and H. Iwaniec, The polynomial x^2+y^4 captures its primes, Ann. Math., 148(1998), 945-1040.
[14] E. Szemerédi, On sets of integers containing no k elements in arithmetic progressions, Acta Arith., 27(1975), 299-345.
[15] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, J. Analyse Math., 31(1997), 204-256.
[16] W. T. Gowers, A new proof of Szemerédi's theorem, GAFA, 11(2001), 465-588.
[17] B. Kra, The Green-Tao theorem on arithmetic progressions in the primes: An ergodic point of view, Bull. Amer. Math. Soc., 43(2006), 3-23.
[18] B. Green and T. Tao, The primes contain arbitrarily long arithmetic progressions, Ann. Math., 167(208), 481-547.
[19] T. Tao, The dichotomy between structure and randomness, arithmetic

progressions, and the primes, In: proceedings of the international congress of mathematicians (Madrid. 2006), Europ. Math. Soc. Vol. 581-608, 2007.
[20] B. Green, Long arithmetic progressions of primes, Clay Mathematics Proceedings Vol. 7, 2007,149-159.
[21] H. Iwanice and E. Kowalski, Analytic number theory, Amer. Math. Soc., Providence, RI, 2004
[22] R. Crandall and C. Pomerance, Prime numbers a computational perpective, Spring-Verlag, New York, 2005.
[23] B. Green, Generalising the Hardy-Littlewood method for primes, In: Proceedings of the international congress of mathematicians (Madrid. 2006), Europ. Math. Soc., Vol. II, 373-399, 2007.
[24] K. Soundararajan, Small gaps between prime numbers: The work of Goldston-Pintz-Yildirim, Bull. Amer. Math. Soc., 44(2007), 1-18.
[25] A. Granville, Harald Cramér and distribution of prime numbers, Scand. Actuar. J, 1995(1)(1995), 12-28.