The Non-Equilibrium Thermodynamic Environment and Prigogine's Dissipative Structures

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Abstract

This essay is based on the fundmental assumption that any physical system of synergetic parts is a thermodynamic system. The universality of thermodynamics is due to the fact that thermodynamic homogeneous properties, such as pressure, temperature and their analogs, do not depend upon size or shape. That is, thermodynamics is a topological (not a geometrical) theory. By use of Cartan's methods of exterior differential forms and their topological properties of closure, it is possible to define and construct examples for the universal concepts of:

[1] Continuous Topological Evolution of topological properties - which in effect is a dynamical version of the First Law.

[2] Topological Torsion and Pfaff Topological Dimension - which distinguishes equilibrium (PTD < 3, TT $= 0$) and non-equilibrium systems (PTD > 2, TT $\neq 0$).

[3] A Topological Thermodynamic Environment - of PTD $= 4$.

[4] Thermodynamic irreversible processes, which cause self-similar evolution in the environment, and emergence of self-organized states of PTD $= 3$ as topological defects in the PTD $= 4$ environment. These results clarify and give credence to Prigogine's conjectures about dissipative structures.

[5] A universal thermodynamic phase function, Θ, which can have a singular cubic factor equivalent to a deformed, universal, van der Waals gas. This van der Waals gas admits negative pressure and dark matter properties, which are current themes in Astronomy and GR.

Contents

1 How do you encode the thermodynamic environment?

It has been said that the "Aether is the empty space upon which the universe sits. [3]" This assessment of the early primitive idea of the Aether leads to the notion of an empty void vacuum, which does not exist as "matter", but supposedly acts as a background for physical phenomena. In topological terms, the void is the empty set. In a thermodynamic sense, the empty set has a Pfaff Topological Dimension of PTD = 0. It is difficult to rationalize how an empty void can interact with anything. Therefor, this concept of an empty void, the null set, is rejected as a definition of the thermodynamic environment of interacting systems.

However, topological non-equilibrium thermodynamics [42] suggests there may be a more modern substitute for the Aether. In earlier works I chose to describe this idea of a thermodynamic environment as the "Physical Vacuum" [arXiv:gr-qc/0602118], and then later as the "Thermodynamic Physical Vacuum" [46]. In order to get rid of the erroneous implications of the word "vacuum", a better choice of words would have been: "the Thermodynamic Physical Environment".

An objective herein is to utilize a formal definition of an ecological environment based on topological thermodynamics, a definition that has universal application, and includes a rational description of those non-equilibrium interactions that exchange radiation and matter with the environment. The evolutionary processes considered must be capable of describing self (including fractal) similarity, and emergent, self organization, as those processes which are thermodynamic irreversible.

As explained in the text that follows, the topological candidate for a definition of the Thermodynamic Physical Environment is

Definition 1 *The universal Thermodynamic Physical Environment is an Open system of PTD = 4, with a disconnected Cartan topology. It is not a void vacuum of PTD = 0.*

The methods used are based on topological properties that are not constrained by geometrical concepts of size and length or shape. The continuous topological evolution of these topological (not geometric) properties turns out to be the cohomological statement of the First Law of Thermodynamics. It is remarkable that the topological methods lead to the local emergence of compact connected sets of PTD = 3 as the result of thermodynamically irreversible processes in the Thermodynamic Physical Environment of PTD = 4. The analysis presented below gives clarification of, and formal justification to, Prigogine's conjectures. Could this definition of the Thermodynamic Physical Environment be the equivalent to a modernized Aether?

1.1 Thermodynamic Systems and Processes

Topological thermodynamics is based on two fundamental components [42]:

1. The first fundamental component is that of a thermodynamic system, which is encoded in terms of an exterior differential 1-form of Action, A. The coefficients of the 1-form are homogeneous of degree zero, and behave as the components of a covariant field intensities (think, Pressure, Temperature, **E**, **B**) with respect to diffeomorphisms. It is not always possible to find integrating factors, λ, such that A/λ is closed; in general, $d(A/\lambda) \neq 0$. It is this last property that delineates the differences between [equilibrium, isolated, closed and open] thermodynamic systems, in terms of the Pfaff Topological Dimension = [1,2,3,4] of the 1-form of Action, A.

2. The second fundamental component is that of a thermodynamic process direction field encoded in terms of an exterior differential M-1-form density. The M-1-form density consists of a vector direction field, V, multiplied by a density function, ρ, such that $J = \rho V$ is closed, $dJ = 0$. The coefficients of the Vector direction field behave as the components of a contra-variant field excitations (think Volume, Entropy, **D**, **H**) with respect to diffeomorphisms. In contrast to 1-forms, it is always possible to find density distributions ρ (integrating factors) such that the M-1-form, ρV, is closed (has no limit points).

1.2 Universality

As the topological method used in the definition of thermodynamic physical environment is based upon the number of deformation invariants and their continuous topological evolution, the results do not depend upon scales. The concept of a thermodynamic physical environment can be applied to problems in aqueous chemistry, as well as plasmas, and other non-equilibrium thermodynamic phenomena. As the topological concepts depend upon synergetically interacting parts, the methods - in principle - could apply to non-equilibrium economic and non-equilibrium political systems.

One question to be answered is: "What is the ecological environment (an Aether?) from which Universe has emerged?" Topological thermodynamics suggests that the void vacuum, of PTD = 0, should be replaced with the topological non-equilibrium Open thermodynamic state, as a background of PTD = 4. This PTD = 4 state is a thermodynamic field with a *disconnected* topology, which can contain emergent matter as topological singularities or defects of PTD = 3 embedded in the PTD = 4 background.

1.3 The point of Departure

The point of departure starts with a topological (not statistical) formulation of Thermodynamics, which furnishes a universal foundation for the Partial Differential Equations of classical hydrodynamics and electrodynamics [42]. The topology that is of significance is defined in terms of Cartan's topological structure [36], which can be constructed from an exterior differential 1-form, A, defined on a pre-geometric domain of base variables, often assumed to be $\{x, y, z, t\}$. Cartan's topological structure is a connected topology on domains of Pfaff Topological Dimension PTD = 2, or less, but Cartan's topological structure is a disconnected topology on domains of PTD = 3 or more. The connected topology can be put into correspondence with equilibrium-isolated thermodynamic systems, while the disconnected topology is to be associated with non-equilibrium thermodynamic systems. In particular, the Cartan methods of exterior differential forms lead to a deeper understanding of non-equilibrium thermodynamic systems, for the exterior differential can be shown to be a limit point generator for the Cartan topology. Topological closure of a differential form is the union of the p-form and its exterior differential. This result leads straight-away to cohomology theory. For equilibrium systems where the Cartan topology is a connected Topology, there is no concept of torsion. When non-equilibrium systems are considered, the Disconnected Cartan Topology automatically includes the concept of Topological Torsion; the 3-form $A\hat{}dA$ is not zero.

. The theory of Topological Thermodynamics, based upon Continuous Topological Evolution [37] of Cartan's topological structure, can explain why topologically coherent, compact structures, far from equilibrium, will emerge as long-lived locally connected subspaces, or topological defects, in the non-equilibrium disconnected topology. The processes, in domains of Pfaff Topological Dimension 4, that produce such defects are thermodynamically irreversible, but topologically continuous. Continuous Topological Evolution in terms of exterior differential forms is to be recognized as the dynamical equivalent of the First Law of Thermodynamics.

I want to present the idea (with examples) that:

Theorem 2 *Irreversible processes in non-equilibrium thermodynamic systems of Pfaff Topological dimension 4 (a symplectic variety) can cause the evolution of compact connected topological defects in the otherwise disconnected topology. The defect structures may or may not form the parts of a disconnected topology of Pfaff Topological dimension 3, a Contact variety. The contact structure admits a Hamiltonian dynamics, and therefore the defect structures can have relatively long lifetimes, modulo topological perturbations introduced by macroscopic spinor dynamics.*

There exist C2 smooth thermodynamically irreversible processes that can describe the topological evolution from an Open non-equilibrium turbulent domain of Pfaff Topological Dimension 4 to Closed, but non-equilibrium, domains of Pfaff Topological Dimension 3, and ultimately to equilibrium domains of Pfaff dimension 2 or less. In fluid dynamics, one would say that Topological domains of Pfaff Topological Dimension 3 emerge via thermodynamically irreversible, dissipative processes as topologically coherent, deformable defects, embedded in the *turbulent* environment of Pfaff Topological Dimension 4.

Now I am well aware of the fact that Thermodynamics (much less Topological Thermodynamics) is a topic often treated with apprehension. In addition, I must confess, that as undergraduates at MIT we used to call the required physics course in Thermodynamics, The Hour of Mystery! Let me present a few quotations (taken from Uffink, [41]) that describe the apprehensive views of several very famous scientists:

> Any mathematician knows it is impossible to understand an elementary course in thermodynamics .... V. Arnold 1990.
>
> It is always emphasized that thermodynamics is concerned with reversible processes and equilibrium states, and that it can have nothing to do with irreversible processes or systems out of equilibrium ......Bridgman 1941
>
> No one knows what entropy really is, so in a debate (if you use the term entropy) you will always have an advantage ...... Von Neumann (1971)

On the other hand Uffink states:

> Einstein, ..., remained convinced throughout his life that thermodynamics is the only universal physical theory that will never be overthrown.

I wish to demonstrate that from the point of view of Continuous Topological Evolution (which is based upon Cartan's theory of exterior differential forms) many of the mysteries of non-equilibrium thermodynamics, irreversible processes, and turbulent flows, can be resolved. In addition, the non-equilibrium methods can lead to many new processes and patentable devices and concepts.

Non-equilibrium thermodynamics can be constructed in terms of disconnected Cartan topology of Pfaff Topological Dimension of 3 or more. As irreversibility requires a change in topology, the point of departure for this article will be to use the thermodynamic theory of continuous topological evolution in 4D space-time. It can be demonstrated, by example, that the non-equilibrium component of the Cartan topology can support topological change, thermodynamic irreversible processes and turbulent solutions to the Navier-Stokes equations, while the equilibrium topological component cannot [29]. In addition, it will be demonstrated that complex isotropic *macroscopic* Spinors are the source of topological fluctuations and irreversible processes in the topological dynamics of non-equilibrium systems. This, perhaps surprising, fact has been ignored by almost all researchers in classical hydrodynamics who use classic real vector analysis and symmetries to produce conservation laws, which do not require Spinor components. The flaw in such symmetrical based theories is that they describe evolutionary processes that are time reversible. Time irreversibility requires topological change.

1.4 Exterior Differential Forms

During the period 1965-1992 it became apparent that new theoretical foundations were needed to describe non-equilibrium systems and continuous irreversible processes. Irreversible processes require topological (not geometrical) evolution. Exterior differential forms overcomes the limitations of real vector (and tensor) analysis. I selected Cartan's methods of exterior differential topology to encode Continuous Topological Evolution. The reason for this choice is that many years of teaching experience indicated that such methods were rapidly learned by all research scientists and engineers. In short:

1. Vector and Tensor analysis is not adequate to study the evolution of topology. The tensor constraint of diffeomorphic equivalences implies that the topology of the initial state must be equal to the topology of the final state. Turbulence, for example, is a thermodynamic, irreversible process which can not be described by tensor fields alone.

2. However, Cartan's methods of exterior differential systems and the topological perspective of Continuous Topological Evolution (not geometrical evolution) can be used to construct viable descriptions of non-equilibrium thermodynamic systems and can distinguish between reversible and irreversible processes. That is, Cartan's methods can be used determine the arrow of time.

3. The exterior differential can be shown to be a limit point generator, such that $\Sigma \cup d\Sigma$ defines the concept of Topological Closure. The exterior differential forms, in effect, define the topological theory of Cohomology, which is not same as theory of Homology. Cartan's Magic formula [11] is a homotopy formula that will describe the topological evolution of the exterior differential forms. In fact, the Magic formula is the dynamical equivalent of the First Law of Thermodynamics. No equivalent concept is associated with Homology.

4. Processes acting on thermodynamic systems can be represented (to within a distribution parameter, ρ) by a vector direction field, V, with the composite defined as a current density, $J = \rho V$. There are two types of current densities, those current densities which are impair, and are sensitive to orientation, and those current densities which are pair, and do no depend on density. The Impair processes lead to the concept of charge which is chiral and quantized, and the Pair processes lead to the concept of mass which is not chiral but quantized to boson numbers.

5. Cartan's theory of exterior differential forms is built over completely antisymmetric structures, and therefore is the method of choice for studying topological evolution. In topological thermodynamics, processes are defined in terms of directionfields which may or may not be tensors. The ubiquitous concepts of 1-1 diffeomorphic equivalence, and non-zero congruences, for the eigen directionfields of symmetric matrices do not apply to the eigen directionfields of antisymmetric matrices. The eigen direction fields of antisymmetric matrices (which are equivalent to Cartan's isotropic Spinors) may be used to define components of a thermodynamic process, but such Spinors have a null congruence (zero valued quadratic form), admit chirality, and are not 1-1. Where classic geometric evolution is described in terms of symmetries and conservation laws, topological evolution is described in terms of antisymmetries. The concept of Spinors slips through the net when physical systems are described through symmetries.

The Cartan theory of extended differential forms can be used to study topological change. The word *extended* is used to emphasize the fact that differential forms are functionally well defined with respect a larger class of transformations than those used to define tensors. Extended differential forms behave as scalars with respect to C1 maps which do not have an inverse, much less an inverse Jacobian. Both the inverse map and the inverse Jacobian are required by a diffeomorphism. The exterior differential form on the final state of such C1 non-invertible maps permits the functional form of the differential form on the initial state to be functionally well defined in a retrodictive, pullback sense - not just at a point, but over a neighborhood.

Theorem 3 *Tensor fields can be neither retrodicted nor predicted in functional form by maps that are not diffeomorphisms [15].*

1.5 The Universal Topological Thermodynamic Environment.

Herein, the thermodynamic environment is defined as the non-equilibrium thermodynamic system of Pfaff Topological Dimension 4. The concept implies that the underlying Cartan topology is a disconnected topology. Subsets in the thermodynamic environment can exchange both mass (mole number - where moles mean numbers of atoms, molecules, baryons, stars, galaxies...) and radiation. The environment can support thermodynamically irreversible, dissipative processes, which can lead to self similar structures and the emergence of topological defects of PTD = 3 in the PTD = 4 environment. Cartan's methods coupled with continuous topological evolution permit examples and solutions of these topics to be found. Several examples will be described in detail below.

2 Continuous Topological Evolution

2.1 Objectives of CTE

The objectives of the theory of Continuous Topological Evolution are to:

1. Establish the long sought for connection between irreversible thermodynamic processes and dynamical systems – without statistics!

2. Demonstrate the connection between thermodynamic irreversibility and Pfaff Topological Dimension equal to 4. The result suggests that "2-D Turbulence is a myth" for it is a thermodynamic system of Pfaff Topological Dimension equal to 3 [22].

3. Demonstrate that topological thermodynamics leads to universal topological equivalences between Electromagnetism, Hydrodynamics, Cosmology, and Topological Quantum Mechanics.

4. Demonstrate that Cartan's methods of exterior differential forms permits important topological concepts to be displayed in a useful, engineering format.

2.2 New Concepts deduced from CTE

The theory of Continuous Topological Evolution introduces several new important concepts that are not apparent in a geometric equilibrium analysis.

1. Continuous Topological Evolution is the dynamical equivalent of the FIRST LAW OF THERMODYNAMICS.

2. The Pfaff Topological Dimension, PTD, is a topological property associated with any Cartan exterior differential 1-form, A. The PTD can change via topologically continuous processes.

3. Topological Torsion is a 3-form (in any 4D geometrical domain) that can be used to describe irreversible processes which produce self-similar thermodynamic systems. As a 4D non-equilibrium direction field it is completely determined by the coefficient functions that encode the thermodynamic system. Other process directionfields are determined by the system topology based upon the 1-form of Action, A, and the refinement based on the topology of the 1-form of work, W.

4. Closed thermodynamic topological defects of Pfaff Topological Dimension 3 can emerge from Open thermodynamic systems of Pfaff Topological Dimension 4 by means of irreversible dissipative processes that represent topological evolution and change. When the topologically coherent defect structures emerge, their evolution can be dominated by a Hamiltonian component (modulo topological fluctuations), which maintains the topological deformation invariance, and yields hydrodynamic wakes [21] and other Soliton structures. These objects are of Pfaff Topological Dimension 3 and are far from equilibrium. They behave as if they were "stationary excited" states above the equilibrium ground state. Falaco Solitons are an easily reproduced hydrodynamic example that came to my attention in 1986 [43] [34]. A movie is available online [35].

2.3 Overall Mathematical Synopsis

Irreversible processes, of continuous topological (not geometrical) evolution, in Open Symplectic non-equilibrium systems of Pfaff Topological (not geometrical) Dimension PFD = 4 can cause the local emergence of Closed Contact non-equilibrium systems (of PTD = 3). These locally connected, deformable, topologically coherent, PTD = 3, states appear as topological defects in the 4D Symplectic structure; hydrodynamic wakes are an example. The Symplectic structure with a disconnected topology of PTD = 4, is defined as the non-equilibrium thermodynamic environment, for it admits the interaction and exchange of mass and radiation among its subsets. The topological defects of PTD = 3 are far from equilibrium as they can exchange radiation, but not mass with the environment. The vector direction fields, that encode thermodynamic processes within the defect structures admit a basis of three eigen solutions, one of which has an extremal Hamiltonian realization which preserves topological properties. The other two basis vectors are complex Spinor solutions that describe topological fluctuations and topological change. If a process in the PTD = 3 domain is dominated by the Hamiltonian component, the emergent topological defects will maintain a relatively long-lived Soliton-like structure. Experimental examples (Falaco Solitons) can be created in a swimming pool. These defect structures represent "stationary excited states" far from equilibrium. The existence of continuous topological thermodynamic evolution gives credence to a general theory of self-organized states far from equilibrium, as conjectured by I. Prigogine. Universal engineering design criteria can be extracted from the topological methods to improve process efficiency. For example, $E \circ B \Rightarrow 0$ will minimize irreversible dissipation in a plasma. Ecological applications of topological thermodynamics apply universally to all synergetic topological systems, be they mechanical, biological, economical or political.

3 Topological Thermodynamics

3.1 Synopsis

The concepts of Topological Thermodynamics in a space-time variety are reviewed (briefly) in terms of Cartan's method of exterior differential forms. A thermodynamic system is encoded in terms of a 1-form of Action, A. Thermodynamic processes are encoded in terms of the Lie differential with respect to a directionfield, V, acting on the 1-form, A, to produce a 1-form, Q. The process directionfield can have Vector and Spinor components.

The definition of the Lie differential is a statement of cohomology and defines Q as the composite of a 1-form, W, and a perfect differential, dU. The formula abstractly represents a dynamical version of the First Law of Thermodynamics. It is a statement about cohomology theory, where the difference between the inexact 1-form of Heat, Q, and the inexact 1-form of Work, W, is a perfect differential, $dU = Q - W$.

The existence of a 1-form on a 4D space-time variety generates a Cartan topology. If the Pfaff Topological (not geometrical) Dimension of the 1-form of Action, A, is 2 or less, then the thermodynamic system is an isolated or equilibrium system on the 4D variety. If the Pfaff Topological Dimension of A is greater than 3, then the system is a non-equilibrium system on the 4D variety. Examples of systems of Pfaff Topological Dimension 4 which admit processes which are thermodynamically irreversible are given in the reference monographs (see footnote page 1)

3.2 The Axioms of Topological Thermodynamics

The topological methods used herein are based upon Cartan's theory of exterior differential forms. The thermodynamic view assumes that the physical systems to be studied can be encoded in terms of a 1-form of Action Potentials (per unit source, or, per mole), A, on a four-dimensional variety of ordered independent variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$. The variety supports a differential volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. This statement implies that the differentials of the $\mu = 4$ base variables are functionally independent. No metric, no connection, no constraint of gauge symmetry is imposed upon the four-dimensional pre-geometric variety. Topological constraints can be expressed in terms of exterior differential systems placed upon this set of base variables [1].

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables will be changed to the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety, at this, thermodynamic, level. For instance, it is NOT assumed that the variety is spatially Euclidean.

With this notation, the Axioms of Topological Thermodynamics can be summarized as:

Axiom 1. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials,* $A_\mu(x, y, z, t...)$*, on a four-dimensional abstract variety of ordered independent variables,* $\{x, y, z, t\}$*. The variety supports differential volume element* $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$*.*

Axiom 2. *Thermodynamic processes are assumed to be encoded, to within a factor,* $\rho(x, y, z, t...)$*, in terms of a contravariant Vector and/or complex Spinor directionfields, symbolized as* $V_4(x, y, z, t)$*.*

Axiom 3. *Continuous Topological Evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [11]). The Lie differential with respect to the process,* ρV_4*, when applied to an exterior differential 1-form of Action,* $A = A_\mu dx^\mu$*, is equivalent, abstractly, to the first law of thermodynamics.*

$$\begin{aligned}
\text{Cartan's Magic Formula } L_{(\rho \mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), && (1)\\
\text{First Law } &: W + dU = Q, && (2)\\
\text{Inexact Heat 1-form } Q &= W + dU = L_{(\rho\mathbf{V}_4)}A, && (3)\\
\text{Inexact Work 1-form } W &= i(\rho\mathbf{V}_4)dA, && (4)\\
\text{Internal Energy } U &= i(\rho\mathbf{V}_4)A. && (5)
\end{aligned}$$

Axiom 4. *Equivalence classes of systems and continuous processes can be defined in terms of the Pfaff Topological Dimension and topological structure generated by of the 1-forms of Action,* A, Work, W, and Heat, Q.

Axiom 5. If $Q\hat{}dQ \neq 0$, *then the thermodynamic process is irreversible.*

3.3 Cartan's Magic Formula ≈ First Law of Thermodynamics

The Lie differential (not Lie derivative) is the fundamental generator of Continuous Topological Evolution. When acting on an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, Cartan's magic (algebraic) formula is equivalent *abstractly* to the first law of thermodynamics:

$$\begin{aligned}
L_{(\rho\mathbf{V}_4)}A &= i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), && (6)\\
&= W + dU = Q. && (7)
\end{aligned}$$

In effect, Cartan's magic formula leads to a topological basis of thermodynamics, where the thermodynamic Work, W, thermodynamic Heat, Q, and the thermodynamic internal energy, U, are defined *dynamically* in terms of Continuous Topological Evolution. In effect, the First Law is a statement of Continuous Topological Evolution in terms of deRham cohomology theory; the difference between two non-exact differential forms is equal to an exact differential, $Q - W = dU$.

My recognition (some 30 years ago) of this correspondence between the Lie *differential* and the First Law of thermodynamics has been the corner stone of my research efforts in applied topology.

It is important to realize that the Cartan formula is to be interpreted algebraically. Many textbook presentations of the Cartan-Lie differential formula presume a dynamic constraint, such that the vector field $\mathbf{V}_4(x, y, z, t)$ be the generator of a single parameter group. If true, then the topological constraint of Kinematic Perfection can be established as an exterior differential system of the format:

$$\textbf{Kinematic Perfection :} \quad dx^k - \mathbf{V}^k dt \Rightarrow 0. \tag{8}$$

The topological constraint of Kinematic Perfection, in effect, defines (or presumes) a limit process. This constraint leads to the concept of the Lie *derivative*[1] of the 1-form A. The evolution then is represented by the infinitesimal propagation of the 1-form, A, down the flow lines generated by the 1-parameter group. Cartan called this set of flow lines "the tube of trajectories".

However, such a topological, kinematic constraint is *not* imposed in the presentation found in this essay; the directionfield, $\mathbf{V}_4$, may have multiple parameters. This observation leads to the important concept of topological fluctuations (about Kinematic Perfection), such as given by the expressions:

$$\begin{aligned}
\textbf{Topological} \quad &: \quad \textbf{Fluctuations} \\
(dx^k - \mathbf{V}^k dt) &= \Delta \mathbf{x}^k \neq 0, \qquad (\sim \text{Pressure}) \qquad (9)\\
(dV^k - \mathbf{A}^k dt) &= (\Delta \mathbf{V}^k) \neq 0, \quad (\sim \text{Temperature}) \qquad (10)\\
d(\Delta \mathbf{x}^k) &= -(d\mathbf{V}^k - \mathbf{A}^k dt)\hat{}\, dt = -(\Delta \mathbf{V}^k)\hat{}\, dt, \qquad (11)
\end{aligned}$$

In this context it is interesting to note that in Felix Klein's discussions [5] of the development of calculus, he says

> "The primary thing for him (Leibniz) was not the differential quotient (the derivative) thought of as a limit. The differential, dx, of the variable x had for him (Leibniz) actual existence..."

[1] Professor Zbigniew Oziewicz told me that Slebodzinsky was the first to formulate the idea of the Lie derivative in his thesis (in Polish).

The Leibniz concept is followed throughout this presentation. It is important for the reader to remember that the concept of a differential form is different from the concept of a derivative, where a (topological) limit has been defined, thereby constraining the topological evolution.

The topological methods to be described below go beyond the notion of processes which are confined to equilibrium systems of kinematic perfection. Non-equilibrium systems and processes which are thermodynamically irreversible, as well as many other classical thermodynamic ideas, can be formulated in precise mathematical terms using the topological structure and refinements generated by the three thermodynamic 1-forms, A, W, and Q.

3.4 The Pfaff Sequence and the Pfaff Topological Dimension

3.4.1 The Pfaff Topological Dimension of the System 1-form, A

It is important to realize that the Pfaff Topological Dimension of the system 1-form of Action, A, determines whether the thermodynamic system is Open, Closed, Isolated or Equilibrium. Also, it is important to realize that the Pfaff Topological Dimension of the thermodynamic Work 1-form, W, determines a specific category of reversible and/or irreversible processes. It is therefore of some importance to understand the meaning of the Pfaff Topological Dimension of a 1-form. Given the functional format of a general 1-form, A, on a 4D variety it is an easy step to compute the Pfaff Sequence, using one exterior differential operation, and several algebraic exterior products. For a differential 1-form, A, defined on a geometric domain of 4 base variables, the Pfaff Sequence is defined as:

$$\textbf{Pfaff Sequence} \quad \{A, dA, A\hat{}dA, dA\hat{}dA...\} \tag{12}$$

It is possible that over some domains, as the elements of the sequence are computed, one of the elements (and subsequent elements) of the Pfaff Sequence will vanish. The number of non-zero elements in the Pfaff Sequence (PS) defines the Pfaff Topological Dimension (PTD) of the specified 1-form[2]. Modulo singularities, the Pfaff Topological Dimension determines the minimum number M of N functions of base variables ($N \geq M$) required to define the topological properties of the connected component of the 1-form A.

The Pfaff Topological Dimension of the 1-form of Action, A, can be put into correspondence with the four classic topological structures of thermodynamics. Equilibrium, Isolated, Closed, and Open systems. The classic thermodynamic interpretation is that the first two structures do not exchange mass (mole numbers) or radiation with their environment. The

[2] The Pfaff Topological dimension has been called the "class" of a 1-form in the old literature. I prefer the more suggestive name.

Closed structure can exchange radiation with its environment but not mass (mole numbers). The Open structure can exchange both mass and radiation with its environment. The following table summarizes these properties. For reference purposes, I have given the various elements of the Pfaff sequence specific names:

Topological p-form name	**PS** element	**Nulls**	**PTD**	Thermodynamic system
Action	A	$dA = 0$	1	**Equilibrium**
Vorticity	dA	$A\hat{}dA = 0$	2	**Isolated**
Torsion	$A\hat{}dA$	$dA\hat{}dA = 0$	3	**Closed**
Parity	$dA\hat{}dA$	$-$	4	**Open**

Table 1 Applications of the Pfaff Topological Dimension.

The four thermodynamic systems can be placed into two disconnected topological categories. If the Pfaff Topological Dimension of A is 2 or less, the first category is determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$. This Cartan topology is a connected topology. In the case that the Pfaff Topological Dimension is greater than 2, the Cartan topology is based on the union of two closures, $\{A \cup dA \ \cup \ A\hat{}dA \cup dA\hat{}dA\}$, and is a disconnected topology.

It is a topological fact that there exists a (topologically) continuous C2 process from a disconnected topology to a connected topology, but there does not exist a C2 continuous process from a connected topology to a disconnected topology. This fact implies that topological change can occur continuously by a "pasting" processes representing the *decay* of turbulence by "condensations" from non-equilibrium to equilibrium systems. On the other hand, the *creation* of Turbulence involves a discontinuous (non C2) process of "cutting" into parts. This warning was given long ago [20] to prove that computer analyses that smoothly match value and slope will not replicate the *creation* of turbulence, but can faithfully replicate the *decay* of turbulence.

3.4.2 The Pfaff Topological Dimension of the Thermodynamic Work 1-form, W

The topological structure of the thermodynamic Work 1-form, W, can be used to refine the topology of the physical system; recall that the physical system is encoded by the Action 1-form, A.

Claim 4 *The PDE's that represent the system dynamics are determined by the Pfaff Topological Dimension of the 1-form of Work, W, and the 1-form of Action, A, that encodes the physical system.*

The Pfaff Topological Dimension of the thermodynamic Work 1-form depends upon both the physical system, A, and the process, $\mathbf{V}_4$. In particular if the Pfaff Dimension of the thermodynamic Work 1-form is zero, $(W = 0)$, then system dynamics is generated by an extremal vector field which admits a Hamiltonian realization. However, such extremal direction fields can occur only when the Pfaff Topological Dimension of the system encoded by A is odd, and equal or less than the geometric dimension of the base variables.

For example, if the geometric dimension is 3, and the Pfaff Topological Dimension of A is 3, then there exists a unique extremal field on the Contact manifold defined by dA. This unique directionfield is the unique eigen directionfield of the 3x3 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 3, then there exists a two extremal fields on the geometric manifold. These directionfields are those generated as the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 4, then there do not exist extremal fields on the Symplectic manifold defined by dA. All of the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) are complex isotropic spinors with pure imaginary eigenvalues not equal to zero.

4 Topological Torsion and other Continuous Processes.

4.0.3 Reversible Processes

Physical Processes are determined by directionfields[3] with the symbol, $\mathbf{V}_4$, to within an arbitrary function, ρ. There are several classes of direction fields that are defined as follows [6]:

$$\begin{aligned}
\textbf{Associated Class} &: \quad i(\rho\mathbf{V}_4)A = 0, && (13)\\
\textbf{Extremal Class} &: \quad i(\rho\mathbf{V}_4)dA = 0, && (14)\\
\textbf{Characteristic Class} &: \quad i(\rho\mathbf{V}_4)A = 0, && (15)\\
\text{and} &: \quad i(\rho\mathbf{V}_4)dA = 0, && (16)\\
\textbf{Helmholtz Class} &: \quad d(i(\rho\mathbf{V}_4)dA) = 0, && (17)
\end{aligned}$$

Extremal Vectors (relative to the 1-form of Action, A) produce zero thermodynamic work, $W = i(\rho\mathbf{V}_4)dA = 0$, and admit a Hamiltonian representation. Associated Vectors (relative

[3]Which include both vector and spinor fields.

to the 1-form of Action, A) can be adiabatic if the process remains orthogonal to the 1-form, A. Helmholtz processes (which include Hamiltonian processes, Bernoulli processes and Stokes flow) conserve the 2-form of Topological vorticity, dA. All such processes are thermodynamically reversible. Many examples of these systems are detailed in the reference monographs (see footnote on page 1).

4.0.4 Irreversible Processes

There is one directionfield that is uniquely defined by the coefficient functions of the 1-form, A, that encodes the thermodynamic system on a 4D geometric variety. This vector exists only in non-equilibrium systems, for which the Pfaff Topological Dimension of A is 3 or 4. This 4 vector is defined herein as the topological Torsion vector, $\mathbf{T}_4$. To within a factor, this directionfield[4] has the four coefficients of the 3-form $A\hat{}dA$, with the following properties:

$$\begin{aligned}
\textbf{Properties of} &\quad : \quad \textbf{Topological Torsion } \mathbf{T}_4 \text{ on } \Omega_4 && (18)\\
i(\mathbf{T}_4)\Omega_4 &= i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA, && (19)\\
W &= i(\mathbf{T}_4)dA = \sigma\ A, && (20)\\
dW &= d\sigma\hat{}A + \sigma dA = dQ && (21)\\
U &= i(\mathbf{T}_4)A = 0, \quad \mathbf{T}_4 \text{ is associative} && (22)\\
i(\mathbf{T}_4)dU &= 0 && (23)\\
i(\mathbf{T}_4)Q &= 0 \qquad \mathbf{T}_4 \text{ is adiabatic} && (24)\\
L_{(\mathbf{T}_4)}A &= \sigma\ A, \quad \mathbf{T}_4 \text{ is homogeneous} && (25)\\
L_{(\mathbf{T}_4)}dA &= d\sigma\hat{}A + \sigma dA = dQ, && (26)\\
Q\hat{}dQ &= L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0, && (27)\\
dA\hat{}dA &= d(A\hat{}dA) = d\{(i(\mathbf{T}_4)\Omega_4\} = (div_4\mathbf{T}_4)\Omega_4, && (28)\\
L_{(\mathbf{T}_4)}\Omega_4 &= d\{(i(\mathbf{T}_4)\Omega_4\} = (2\sigma)\Omega_4, && (29)
\end{aligned}$$

If the Pfaff Topological Dimension of A is 4 (an Open thermodynamic system), then $\mathbf{T}_4$ has a non-zero 4 divergence, (2σ), representing an expansion or a contraction of the 4D volume element Ω_4. The Heat 1-form, Q, generated by the process, $\mathbf{T}_4$, is NOT integrable. Q is of Pfaff Topological Dimension greater that 2, when $\sigma \neq 0$. Furthermore the $\mathbf{T}_4$ process

[4] A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

is locally adiabatic as the change of internal energy in the direction of the process path is zero. Therefore, in the Pfaff Topological Dimension 4 case, where $dA\hat{}dA \neq 0$, the $\mathbf{T}_4$ direction field represents an *irreversible, adiabatic process.*

When σ is zero *and* $d\sigma = 0$, but $A\hat{}dA \neq 0$, the Pfaff Topological Dimension of the system is 3 (a Closed thermodynamic system). In this case, the $\mathbf{T}_4$ direction field becomes a characteristic vector field which is both extremal and associative, and induces a Hamilton-Jacobi representation (the ground state of the system for which $dQ = 0$).

For any process and any system, equation (27) can be used as a test for irreversibility.

It seems a pity, that the concept of the Topological Torsion vector and its association with non-equilibrium systems, where it can be used to establish design criteria to minimize energy dissipation, has been ignored by the engineering community.

4.1 Self-Similarity, Topological Torsion and Dissipative processes.

In general, if a process $\mathbf{V}_4$ acting on a p-form ω^p satisfies the equation,

$$L_{(\mathbf{V}_4)}\omega^p = i(\mathbf{V}_4)d\omega^p + d(i(\mathbf{V}_4)\omega^p) = \sigma\ \omega^p, \tag{30}$$

the p-form is said to be homogeneous of degree σ [9]. When σ is a constant, the evolutionary process $\mathbf{V}_4$ generates integer and fractal replicas of the p-form. The formula of continuous topological evolution then gives a precise definition of the concept of evolutionary self similarity. If, in addition, the p-form is closed and σ is a constant, it follows that the p-form must be exact

$$\omega^p = d(i(\mathbf{V}_4)\omega^p)/\sigma. \tag{31}$$

For process with the direction field in the direction of the 4 component Topological Torsion vector, the fundamental equation of continuous topological evolution indicates that the topological process is not only thermodynamically irreversible and dissipative, but also is homogenous of degree σ, an indication of self similarity. The 1-form A evolves into a multiple of itself determined by the value of σ:

$$L_{(\mathbf{T}_4)}A = \sigma\ A, \tag{32}$$

The numeric value of σ need not be an integer, indicating that the self similarity property induced by the irreversible dissipative process could be fractal.

However, of all processes, those with a direction field in the direction of the Topological Torsion vector are special. First, such processes are adiabatic; the change in internal energy is zero. Second, the Work done is proportional to the Action.

$$W = i(\rho\mathbf{T}_4)dA = \sigma\ A, \tag{33}$$

$$U = i(\rho\mathbf{T}_4)A = 0, \tag{34}$$

$$L_{(\mathbf{T}_4)}A = W + dU = i(\rho\mathbf{T}_4)dA = \sigma\ A. \tag{35}$$

The interesting cases occur when the dissipation coefficient, σ, is not an evolutionary constant. If the irreversible evolution produces values of σ that are zero, then in those local domains the PTD of the 1-form of Action, A, is reduced to PTD = 3 from PTD = 4. These PTD = 3 states appear to emerge as topological defect structures in the PTD = 4 thermodynamic environment. Subsequent evolution of the PTD = 3 states have an extremal, hence Hamiltonian, component (which is not dissipative). Hence the PTD=3 state does not have a dissipative structure, but it is produced by a dissipative (irreversible) process. These mathematical results clarify and modify the conjectures of Progogine: the emergent states need not be dissipative, but they are produced by dissipative irreversible processes (not structures). Progogine's emphasis on non-equilibrium thermodynamics remains as a golden rule.

Examples are given below.

4.2 The Spinor class

It is rather remarkable (and only fully appreciated by me in February, 2005) that there is a large class of direction fields useful to the topological dynamics of thermodynamic systems (given herein the symbol $\rho\mathbf{S}_4$) that do not behave as vectors (with respect to rotations). They are isotropic complex vectors of zero length, defined as Spinors by E. Cartan [2], but which are most easily recognized as the eigen directionfields relative to the antisymmetric matrix, $[F]$, generated by the component of the 2-form $F = dA$:

$$\textbf{The Spinor Class} \quad [F] \circ |\rho\mathbf{S}_4\rangle = \lambda\, |\rho\mathbf{S}_4\rangle \neq 0, \tag{36}$$

$$\langle\rho\mathbf{S}_4| \circ |\rho\mathbf{S}_4\rangle = 0, \quad \lambda \neq 0 \tag{37}$$

In the language of exterior differential forms, if the Work 1-form is not zero, the process must contain Spinor components:

$$W = i(\rho\mathbf{S}_4)dA \neq 0 \tag{38}$$

As mentioned above, Spinors have metric properties, behave as vectors with respect to transitive maps, but do not behave as vectors with respect to rotations (see p. 3, [2]).

Spinors generate harmonic forms and are related to conjugate pairs of minimal surfaces. The notation that a Spinor is a complex isotropic *directionfield* is preferred over the names "complex isotropic *vector*", or "null *vector*" that appear in the literature. As shown below, the familiar formats of Hamiltonian mechanical systems exclude the concept of Spinor process directionfields, for the processes permitted are restricted to be represented by direction fields of the extremal class, which have zero eigenvalues.

Remark 5 *Spinors are normal consequences of antisymmetric matrices, and, as topological artifacts, they are not restricted to physical microscopic or quantum constraints. According to the topological thermodynamic arguments, Spinors are implicitly involved in all processes for which the 1-form of thermodynamic Work is not zero. Spinors play a role in topological fluctuations and irreversible processes.*

The thermodynamic Work 1-form, W, is generated by a completely antisymmetric 2-form, F, and therefore, if not zero, must have Spinor components. In the odd dimensional Contact manifold case there is one eigen Vector, with eigenvalue zero, which generates the extremal processes that can be associated with a Hamiltonian representation. The other two eigendirection fields are Spinors. In the even dimensional Symplectic manifold case, any non-zero component of work requires that the evolutionary directionfields must contain Spinor components. All eigen directionfields on symplectic spaces are Spinors.

The fundamental problem of Spinor components is that there can be more than one Spinor direction field that generates the same geometric path. For example, there can be Spinors of left or right handed polarizations and Spinors of expansion or contraction that produce the same optical (null congruence) path. This result does not fit with the classic arguments of mechanics, which require unique initial data to yield unique paths. Furthermore, the concept of Spinor processes can annihilate the concept of time reversal symmetry, inherent in classical hydrodynamics. The requirement of uniqueness is not a requirement of non-equilibrium thermodynamics, where Spinor "entanglement" has to be taken into account.

5 An EM Application

5.1 Synopsis

The electromagnetic format was chosen because my teaching experience demonstrated that the ideas of non-equilibrium phenomena are more readily recognized in terms of electromagnetic concepts. It will be demonstrated how the PDE's representing the Hamiltonian version of the hydrodynamic Lagrange-Euler equations arise from the constraint that the work 1-form, W, should vanish (Pfaff Topological Dimension of $W = 0$). Such processes are defined as extremal processes in the theory of the calculus of variations.

The Bernoulli flow can be obtained by constraining the thermodynamic Work 1-form to be exact, $W = d\Theta$ (Pfaff Topological Dimension 1), and the Helmholtz flow will follow from the constraint that the thermodynamic Work 1-form be closed, but not necessarily exact, $dW = 0$. Such reversible dynamical processes belong to the connected component of the Work 1-form.

Irreversible processes belong to the disconnected topological component of the Work 1-form. An important example is the process defined in terms of the Topological Torsion direction field on a Symplectic manifold. Such processes are self-similar relative to the 1-form of Action, A, and are thermodynamically irreversible. The irreversible dissipation coefficient will be found to be proportional to $\sigma = (\mathbf{E} \circ \mathbf{B})$.

Other Applications are detailed in my several monographs [42], [43], [**?**], [45], [46]. Of immediate interest is the application to problems of Turbulence [arXiv:physics/0102003]

5.2 An Electromagnetic format

The thermodynamic identification of the terms in Cartan's magic formula are not whimsical. To establish an initial level of credence in the terminology, consider the 1-form of Action, A, where the component functions are the symbols representing the familiar vector and scalar potentials in electromagnetic theory. The coefficient functions have arguments over the four independent variables $\{x, y, z, t\}$,

$$A = A_\mu(x, y, z, t)dx^\mu = \mathbf{A} \circ d\mathbf{r} - \phi\ dt. \tag{39}$$

Construct the 2-form of field intensities as the exterior differential of the 1-form of Action,

$$\begin{aligned} F &= dA = (\partial A_k/\partial x^j - \partial A_j/\partial x^k)dx^j \hat{}\, dx^k & (40)\\ &= F_{jk}dx^j \hat{}\, dx^k = +\mathbf{B}_z dx \hat{}\, dy ... + \mathbf{E}_x dx \hat{}\, dt ... \quad . & (41) \end{aligned}$$

The engineering variables are defined as electric and magnetic field intensities:

$$\mathbf{E} = -\partial\mathbf{A}/\partial t - grad\ \phi, \qquad \mathbf{B} = curl\ \mathbf{A}. \tag{42}$$

Relative to the ordered set of base variables, $\{x, y, z, t\}$, define a process directionfield, $\rho\mathbf{V}_4$, as a 4-vector with components, $[\mathbf{V}, 1]$, with a scaling factor, ρ.

$$\rho[\mathbf{V}_4] = \rho[\mathbf{V}, 1]. \tag{43}$$

Note that this direction field can be used to construct a useful 3-form of (matter) current, C, in terms of the 4-volume element, $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$:

$$C = i(\rho\mathbf{V}_4)dx\hat{}dy\hat{}dz\hat{}dt = i(\mathbf{C}_4)\Omega_4. \tag{44}$$

The process 3-form, C, is not necessarily the same as electromagnetic charge current density 3-form of electromagnetic theory, J. The 4-divergence of C, need not be zero: $dC \neq 0$.

Using the above expressions, the evaluation of the thermodynamic work 1-form in terms of 3-vector engineering components becomes:

$$\begin{aligned}
&\textbf{The thermodynamic Work 1-form:} & & W = i(\rho\mathbf{V}_4)dA = i(\rho\mathbf{V}_4)F, & (45)\\
& & \Rightarrow\ & -\rho\{\mathbf{E}+\mathbf{V}\times\mathbf{B}\}\circ d\mathbf{r}+\ \rho\{\mathbf{V}\circ\mathbf{E}\}dt & (46)\\
& & =\ & -\rho\{\mathbf{f}_{Lorentz}\}\circ d\mathbf{r}+\ \rho\{\mathbf{V}\circ\mathbf{E}\}dt. & (47)\\
&\textbf{The Lorentz force} & =\ & -\{\mathbf{f}_{Lorentz}\}\circ d\mathbf{r}\ \text{(spatial component)} & (48)\\
&\textbf{The dissipative power} & =\ & +\{\mathbf{V}\circ\mathbf{E}\}dt\ \text{(time component)}. & (49)
\end{aligned}$$

For those with experience in electromagnetism, note that the construction yields the format, automatically and naturally, for the "Lorentz force" as a *derivation consequence*, without further ad hoc assumptions. The dot product of a 3 component force, $\mathbf{f}_{Lorentz}$, and a differential spatial displacement, $d\mathbf{r}$, defines the elementary classic concept of "differential work". The 4-component thermodynamic Work 1-form, W, includes the spatial component and a differential time component, Pdt, with a coefficient which is recognized to be the "dissipative power", $P = \{\mathbf{V}\circ\mathbf{E}\}$. The thermodynamic Work 1-form, W, is not necessarily a perfect differential, and therefore can be path dependent. Closed cycles of Work need not be zero.

Next compute the Internal Energy term, relative to the process defined as $\rho\mathbf{V}_4$:

$$\textbf{Internal Energy: } U = i(\rho\mathbf{V}_4)A = \rho(\mathbf{V}\circ\mathbf{A}-\phi). \tag{50}$$

The result is to be recognized as the "interaction" energy density in electromagnetic plasma systems. It is apparent that the internal energy, U, corresponds to the interaction energy of the physical system; that is, U is the internal stress energy of system deformation. Therefore, the electromagnetic terminology can be used to demonstrate the premise that Cartan's magic formula is not just another way to state that the first law of thermodynamics makes practical sense. The topological methods permit the long sought for integration of mechanical and thermodynamic concepts, without the constraints of equilibrium systems, and/or statistical analysis.

It is remarkable that although the symbols are different, the same basic constructions and conclusions apply to many classical physical systems. The correspondence so established between the Cartan magic formula acting on a 1-form of Action, and the first law of thermodynamics is taken both literally and seriously in this essay. The methods yield explicit constructions for testing when a process acting on a physical system is irreversible. The methods permit irreversible adiabatic processes to be distinguished from reversible adiabatic processes, analytically. Adiabatic processes need not be "slow" or quasi-static.

Given any 1-form, A, W, and/or Q, the concept of Pfaff Topological Dimension (for each of the three 1-forms, A, W, Q) permits separation of processes and systems into equivalence classes. For example, dynamical processes can be classified in terms of the topological Pfaff dimension of the thermodynamic Work 1-form, W. All extremal Hamiltonian systems have a thermodynamic Work 1-form, W, of topological Pfaff dimension of 1, ($dW = 0$). Hamiltonian systems can describe reversible processes in non-equilibrium systems for which the topological Pfaff dimension is 3. Such systems are topological defects whose topology is preserved by the Hamiltonian dynamics, but all processes which preserve topology are reversible. In non-equilibrium systems, topological fluctuations can be associated with Spinors of the 2-form, $F = dA$. Even if the dominant component of the process is Hamiltonian, Spinor fluctuations can cause the system (ultimately) to decay.

5.3 Topological 3-forms and 4-forms in EM format

Construct the elements of the Pfaff Sequence for the EM notation,

$$\{A, F = dA, A\hat{}F, F\hat{}F\}, \tag{51}$$

and note that the algebraic expressions of Topological Torsion, $A\hat{}F$, can be evaluated in terms of 4-component engineering variables $\mathbf{T}_4$ as:

$$\begin{aligned} & \textbf{Topological Torsion vector} && (52)\\ A\hat{}F &= i(\mathbf{T}_4)\Omega_4 = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt && (53)\\ \mathbf{T}_4 &= [\mathbf{T}, h] = -[\mathbf{E}\times\mathbf{A} + \mathbf{B}\phi, \mathbf{A}\circ\mathbf{B}]. && (54)\end{aligned}$$

The exterior 3-form, $A\hat{}F$, with physical units of $(\hbar/\text{unit\_mole})^2$, is not found (usually) in classical discussions of electromagnetism[5].

[5] The unit mole number is charge, e, in EM theory.

If $\mathbf{T}_4$ is used as to define the direction field of a process, then

$$L_{(\mathbf{T}_4)}A = \sigma A, \qquad i(\mathbf{T}_4)A = 0. \tag{55}$$

$$\text{where } 2\sigma = \{div_4(\mathbf{T}_4)\} = 2(\mathbf{E} \circ \mathbf{B}). \tag{56}$$

The important (universal) result is that if the acceleration associated with the direction field, **E**, is parallel to the vorticity associated with the direction field, **B,** then according to the equations starting with eq. (18) et. seq. the process is dissipative and irreversible. This result establishes the design criteria for engineering applications to minimize dissipation from turbulent processes.

The Topological Torsion vector has had almost no utilization in applications of classical electromagnetic theory.

5.4 Topological Torsion quanta

The 4-form of Topological Parity, $F\hat{}F$, can be evaluated in terms of 4-component engineering variables as:

Topological Parity

$$d(A\hat{}F) = F\hat{}F = \{div_4(\mathbf{T}_4)\}\Omega_4 = \{2\mathbf{E} \circ \mathbf{B}\}\ \Omega_4. \tag{57}$$

This 4-form is also known as the second Poincare Invariant of Electromagnetic Theory.

The fact that $F\hat{}F$ need not be zero implies that the Pfaff Topological Dimension of the 1-form of Action, A, must be 4, and therefore A represents a non-equilibrium Open thermodynamic system. Similarly, if $F\hat{}F = 0$, but $A\hat{}F \neq 0$, then the Pfaff Topological Dimension of the 1-form of Action, A, must be 3, and the physical system is a non-equilibrium Closed thermodynamic system. When $F\hat{}F = 0$, the corresponding three-dimensional integral of the closed 3-form, $A\hat{}F$, when integrated over a closed 3D-cycle, becomes a deRham period integral, defined as the Torsion quantum. In other words, the closed integral of the (closed) 3-form of Topological Torsion becomes a deformation (Hopf) invariant with integral values proportional to the integers.

$$\textbf{Torsion quantum} = \iiint_{3D\_cycle} A\hat{}F. \tag{58}$$

On the other hand, topological evolution and transitions between "quantized" states of Torsion require that the respective Parity 4-form is are not zero. As,

$$L_{(\mathbf{T}_4)}\Omega_4 = d\{i(\mathbf{T}_4)\Omega_4\} = (2\sigma)\ \Omega_4 = 2(\mathbf{E} \circ \mathbf{B})\ \Omega_4 \neq 0, \tag{59}$$

it is apparent that the evolution of the differential volume element, Ω_4, depends upon the existence and colinearity of both the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$. It is here that contact is made with the phenomenological concept of "4D bulk" viscosity $= 2\sigma$. It is tempting to identify σ^2 with the concept of entropy production. Note that the Topological Torsion directionfield appears only in non-equilibrium systems. These results are universal and can be used in hydrodynamic systems, as discussed in [44].

5.5 Hydrodynamics

In many treatments of fluid mechanics the (geometrical) continuum hypothesis is invoked from the start. The idea is "matter" occupies all points of the space of interest, and that properties of the fluid can be represented by piecewise continuous functions of space and time, as long as length and time scales are not too small. The problem is that at very small scales, one has been led to believe the molecular or atomic structure of particles will become evident, and the "macroscopic" theory will breakdown. However, these problems of scale, size and length are geometric issues, important to many applications, but not pertinent to a topological perspective, where shape and size are unimportant. Suppose that the dynamics can be formulated in terms of topological concepts, such as those found in Homology or Cohomology theories, which the sets of interest are independent from sizes and shapes. Then such a theory of a Topological Continuum would be valid at all scales, but would be valid only for equilibrium systems. built on a connected topology. The continuum is not admissible in a disconnected topology.

The "breakdown" of the continuum model is not relevant. The topological system may consist of many disconnected parts when the system is not in thermodynamic equilibrium or isolation, and the parts can have topological obstructions or defects, some of which can be used to construct period integrals that are topologically "quantized". Hence the "quantization" of the micro-scaled geometric systems can have it genesis in the non-equilibrium theory of thermodynamics. However, from the topological perspective, the rational topological quantum values can also occur at all scales.

By 1969 it had become evident to me that electromagnetism (without geometric constraints), when written in terms of differential forms, was a topological theory, and that the concept of dissipation and irreversible processes required more than that offered by Hamiltonian mechanics. At that time I was interested in possible interactions of the gravitational field and the polarizations of an electromagnetic signal. One of the first ideas discovered about topological electrodynamics was that there existed an intrinsic transport theorem [12] that introduced the concept of what is now called Topological Spin, $A\hat{}G$, into electromagnetic theory [45]. As a transport theorem not recognized by classical electromagnetism, the first publication was as a letter to Physics of Fluids. That started my interest in a topological formulation of fluids.

It was not until 1974 that the Lie differential acting on exterior differential forms was established as the key to the problem of intrinsically describing dissipation and the production of topological defects in physical systems; but methods of visualization of such topological defects in classical electrodynamics were not known [13]. It was hoped that something in the more visible fluid mechanics arena would lend credence to the concepts of topological defects. The first formulations of the PDE's of fluid dynamics in terms of differential forms and Cartan's Magic formula followed quickly [14].

In 1976 it was argued that topological evolution was at the cause of turbulence in fluid dynamics, and the notion of what is now called Topological Torsion, $A\hat{}F$, became recognized as an important concept. It was apparent that streamline flow imposed the constraint that $A\hat{}F = 0$ on the equations of hydrodynamics. Turbulent flow, being the antithesis of streamline flow, must admit $A\hat{}F \neq 0$. In 1977 it was recognized that topological defect structures could become "quantized" in terms of deRham period integrals [16], forming a possible link between topology and both macroscopic and microscopic quantum physics. The research effort then turned back to a study of topological electrodynamics in terms of the dual polarized ring laser, where it was experimentally determined that the speed of an electromagnetic signal outbound could be different from the speed of an electromagnetic signal inbound: a topological result not within the realm of classical theory.

Then in 1986 the long sought for creation and visualization of topological defects in fluids [17] became evident. The creation of Falaco Solitons in a swimming pool was the experiment that established credence in the ideas of what had, by that time, become a theory of continuous topological evolution. It was at the Cambridge conference in 1989 [18] that the notions of topological evolution, hydrodynamics and thermodynamics were put together in a rudimentary form, but it was a year later at the Permb conference in 1990 [19] that the ideas were well established. The Permb presentation also suggested that the ambiguous (at that time) notion of coherent structures in fluids could be made precise in terms of topological coherence. A number of conference presentations followed in which the ideas of continuous thermodynamic irreversible topological evolution in hydrodynamics were described [20], but the idea that the topological methods of thermodynamics could be used to distinguish non-equilibrium processes and non-equilibrium systems and irreversible processes with out the use of statistics slowly came into being in the period 1985-2005 [23]. These efforts have been summarized in [42]

6 Emergence and dissipative processes

6.1 Synopsis

The problem of Emergence of topologically coherent compact defects in an open thermodynamic system will be attacked from the point of continuous topological evolution. First, the properties of the different species of topological defects will be discussed. These defects are non-equilibrium closed domains (of $PTD = 3$) which can emerge by C2 smooth irreversible process in open domains (of $PTD = 4$), as excited states far from equilibrium, yet with long relative lifetimes. Falaco Solitons are an easily reproduced *experimental* example of such topological defects, and are discussed in detail in [43] [35].

The analytic properties of two different species of $PTD = 3$ defect domains will be given in detail. In addition, an analytic solution of a thermodynamically irreversible process that causes the defect domain to emerge will be demonstrated. An example of a process that creates the topological defect in finite time will be given.

Finally, an example will be given where by combinations of Spinor solutions (fluctuations) produce piecewise linear processes. These piecewise linear processes are thermodynamically reversible, while the Spinor solutions of which they are composed, are not.

6.2 Emergent Defects and the Arrow of Time

Suppose an evolutionary process starts in a domain of Pfaff Topological Dimension 4, for which a process in the direction of the Topological Torsion vector, $\mathbf{T}_4$, is known to represent an irreversible process. Examples can demonstrate that the irreversible process can proceed to a domain of the geometric variety for which the dissipation coefficient, σ, becomes zero. Physical examples [43] such as the skidding bowling ball proceed with irreversible dissipation ($PTD = 6$) until the "no-slip" condition is reached ($PTD = 5$). In fluid systems the topological defects can emerge as long lived states far from equilibrium. The process is most simply visualized as a "condensation" from a turbulent gas, such as the creation of a star in the model which presumes the universe is a very dilute, turbulent van der Waals gas near its critical point. The red spot of Jupiter, a hurricane, the ionized plasma ring in a nuclear explosion, Falaco Solitons, the wake behind an aircraft are all exhibitions of the emergence process to long lived topological structures far from equilibrium. It is most remarkable that the emergence of these experimental defect structures occurs in finite time.

The idea is that a subdomain of the original system of Pfaff Topological Dimension 4 can evolve continuously with a change of topology to a region of Pfaff Topological Dimension 3. The emergent subdomain of Pfaff Topological Dimension 3 is a topological defect, with topological coherence, and often with an extended lifetime (as a soliton structure with a dominant Hamiltonian evolutionary path), embedded in the Pfaff dimension 4 turbulent background.

The Topological Torsion vector in a region of Pfaff Topological Dimension 3 is an extremal vector direction field in systems of Pfaff Topological Dimension 3; it then has a zero 4D divergence, and leaves the volume element invariant. Moreover the existence of an extremal direction field implies that the 1-form of Action can be given a Hamiltonian representation, $P_k dq^k + H(P,q,t)dt$. In the domain of Pfaff dimension 3 for the Action, A, the subsequent continuous evolution of the system, A, relative to the process $\mathbf{T}_4$, can proceed in an energy conserving, Hamiltonian manner, representing a "stationary" or "excited" state far from equilibrium (the ground state). This argument is based on the assumption that the Hamiltonian component of the direction field is dominant, and any Spinor components in the $PTD = 3$ domain, representing topological fluctuations, can be ignored. These excited states, far from equilibrium, can be interpreted as the evolutionary topological defects that emerge and self-organize due to irreversible processes in the turbulent dissipative system of Pfaff dimension 4.

The descriptive words of self-organized states far from equilibrium have been abstracted from the intuition and conjectures of I. Prigogine [10]. The methods of Continuous Topological Evolution correct the Prigogine conjecture that "dissipative structures" can be caused by dissipative processes and fluctuations. The long-lived excited state structures created by irreversible processes are non-equilibrium, deformable topological defects almost void of irreversible dissipation. The topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Using Cartan's methods of exterior differential systems, thermodynamic irreversibility and the arrow of time can be well defined in a topological sense, a technique that goes beyond (and without) statistical analysis [24]. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

The problem of C2 smoothness will be attacked from the point of view of topological thermodynamics. First, two distinct examples will be given demonstrating two different emergent $PTD = 3$ states, that emerge from different 4D rotations (see p. 108, [40]). Then, an example demonstrating the decay of a $PTD = 4$ state into a $PTD = 3$ state will be given in detail. The electromagnetic notation will be used, but the results can be converted into hydrodynamic format using the techniques found in [44].

6.3 Examples of PTD = 3 domains and their Emergence

The properties of those $PTD = 3$ domains which emerge by C2 irreversible solutions from domains of $PTD = 4$ are of particular interest. From Section 7, it is apparent that the key feature of $PTD = 3$ domains is that the electric $\mathbf{E}$ field (acceleration field $\mathbf{a}$ in hydrodynamics) must be orthogonal to the magnetic $\mathbf{B}$ field (vorticity field $\boldsymbol{\omega}$ in hydrodynamics). There are 8 cases to consider (including chirality),

Pfaff Topological Dimension 3

$$\mathbf{E} = \mathbf{0}, \qquad \pm\mathbf{B} \neq \mathbf{0}, \tag{60}$$

$$\mathbf{B} = \mathbf{0}, \qquad \pm\mathbf{E} \neq \mathbf{0}, \tag{61}$$

$$\mathbf{E} \circ \mathbf{B} = \mathbf{0}, \qquad \text{with chirality choices, } \pm\mathbf{E} = \pm\mathbf{B} \neq \mathbf{0}, \tag{62}$$

of which two will be discussed in detail.

6.3.1 The Finite Helicity case (both E and B finite) PTD = 3

Start with the 4D thermodynamic domain, and first consider the 1-form of Action, A, with the format[6]:

$$A = A_x(z)dx + A_y(z)dy - \phi(z)dt, \tag{63}$$

and its induced 2-form, $F = dA$,

$$\begin{aligned} F &= dA = (\partial A_x(z)/\partial z)dz\hat{\ }dx + (\partial A_y(z)/\partial z)dz\hat{\ }dy - (\partial\phi(z)/\partial z)dz\hat{\ }dt, && (64)\\ &= B_x(z)dz\hat{\ }dx - B_y(z)dz\hat{\ }dy + E_z(z)dz\hat{\ }dt. && (65)\end{aligned}$$

The 3-form of Topological Torsion 3-form becomes

$$\begin{aligned} i(\mathbf{T}_4)\Omega_4 &= A\hat{\ }F \text{ where} && (66)\\ \mathbf{T}_4(z) &= [E_zA_y + \phi B_x,\ -E_zA_x + \phi B_y,\ 0,\ A_xB_x + A_yB_y] && (67)\\ \text{with } div_4(\mathbf{T}_4(z)) &= 2(\mathbf{E} \circ \mathbf{B}) = 0, \qquad \mathbf{A} \circ \mathbf{B} \neq 0. && (68)\end{aligned}$$

6.3.2 The Zero Helicity case (both E and B finite) PTD = 3

Start with the 4D thermodynamic domain, and consider the 1-form of Action, A, with the format:

$$A = A_x(x, y)dx + A_y(x, y)dy - \phi(x, y)dt, \tag{69}$$

[6] The +E, +B chirality has been selected.

and its induced 2-form, $F = dA$,

$$
\begin{aligned}
F &= dA = \{(\partial A_y(x,y)/\partial x) - (\partial A_x(x,y)/\partial x)dx\hat{}dy\} \\
&\quad -(\partial \phi(x,y)/\partial x)dx\hat{}dt - (\partial \phi(x,y)/\partial y)dy\hat{}dt, && (70)\\
&= B_z(x,y)dx\hat{}dy + E_x(x,y)dx\hat{}dt + E_y(x,y)dy\hat{}dt. && (71)
\end{aligned}
$$

The 3-form of Topological Torsion 3-form becomes,

$$
\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= A\hat{}F \qquad \text{where} && (72)\\
\mathbf{T}_4(x,y) &= [0, 0, (E_x A_y - E_y A_x) + \phi B_z, 0] && (73)\\
\text{with } div_4(\mathbf{T}_4(x,y)) &= 2(\mathbf{E}\circ\mathbf{B}) = 0, \qquad \mathbf{A}\circ\mathbf{B} = 0. && (74)
\end{aligned}
$$

This case of zero helicity ($\mathbf{A}\circ\mathbf{B} = \mathbf{0}$), has the Topological Torsion vector, $\mathbf{T}_4(x,y)$, colinear with the $\mathbf{B}$ field.

6.3.3 Zero Helicity case: PTD = 4 decays to PTD = 3 $(\mathbf{E}\circ\mathbf{B}) \Rightarrow 0$

The two distinct cases, modulo chirality, are suggestive of the idea (see p. 108 [40]) that the rotation group of a 4D domain is not simple. The example,immediately above, is particularly useful because the algebra of the decay from Pfaff dimension 4 to 3 is transparent.

Start with the 4D thermodynamic domain, and consider the 1-form of Action, A, with the format:

$$
A = A_x(x,y)dx + A_y(x,y)dy - \phi(x,y,z,t)dt, \tag{75}
$$

and its induced 2-form, $F = dA$,

$$
\begin{aligned}
F &= dA = \{(\partial A_y(x,y)/\partial x) - (\partial A_x(x,y)/\partial x)dx\hat{}dy\} \\
&\quad -(\partial \phi(x,y,z)/\partial x)dx\hat{}dt - (\partial \phi(x,y,z)/\partial y)dy\hat{}dt - (\partial \phi(x,y,z)/\partial z)dz\hat{}dt, && (76)\\
&= B_z(x,y)dx\hat{}dy && (77)\\
&\quad +E_x(x,y,z,t)dx\hat{}dt + E_y(x,y,z,t)dy\hat{}dt + E_z(x,y,z,t)dy\hat{}dt. && (78)
\end{aligned}
$$

The 3-form of Topological Torsion 3-form becomes,

$$
\begin{aligned}
i(\mathbf{T}_4)\Omega_4 &= A\hat{}F \text{ with PTD}(A) = 4 && (79)\\
\mathbf{T}_4(x,y,z,t) &= [-E_z A_y, +E_z A_x, (E_x A_y - E_y A_x) + \phi B_z, 0] && (80)\\
\text{with } div_4(\mathbf{T}_4(x,y,z,t)) &= 2\{E_z(x,y,z,t)B_z(x,y)\} \neq 0. && (81)
\end{aligned}
$$

In this case, the helicity ($\mathbf{A}\circ\mathbf{B} = \mathbf{0}$) is still zero, but now the Topological Torsion vector, $\mathbf{T}_4(x,y,z,t)$, has three spatial components. Moreover, the Process generated by $\mathbf{T}_4(x,y,z)$ is thermodynamically irreversible, as $(\mathbf{E}\circ\mathbf{B}) \neq 0$. The example 1-form is of PTD = 4.

To demonstrate the emergence of the $PTD = 3$ state, suppose the potential function in this example has the format,

$$\phi = \psi(x, y) + \varphi(z)e^{-\alpha t} \tag{82}$$

$$E_z(z, t) = -(\partial\varphi(z)/\partial z)e^{-\alpha t} = E_z(z)e^{-\alpha t}. \tag{83}$$

Then the irreversible dissipation function decays as $\{E_z(z)B_z\}e^{-\alpha t}$. By addition of Spinor fluctuation terms to represent the very small components of irreversible dissipation at late times, the $PTD = 3$ solution,

$$\mathbf{T}_4(x, y) = [0, 0, (E_x A_y - E_y A_x) + \phi B_z, 0] \tag{84}$$

becomes dominant, and represents a long lived "stationary" state far from equilibrium, modulo the small Spinor decay terms[7].

To demonstrate the emergence of the $PTD = 3$ state in finite time, suppose the potential function in this example has the format,

$$\phi = \psi(x, y) \pm \varphi(z)\sqrt{(-t - t_c)^3} \tag{85}$$

$$E_z(z, t) = \mp(\partial\varphi(z)/\partial z)\sqrt{(-t - t_c)^3} = \pm E_z(z)\sqrt{(-t - t_c)^3}. \tag{86}$$

Then the irreversible dissipation function decays in a cuspoidal way (typical of the approach to an edge of regression of an envelope function) according to the formula, $\{E_z(z)B_z\}\sqrt{(-t - t_c)^3}$. The PTD of the system is 4 for $t < t_c$, and becomes equal to 3 for $t = t_c$.

6.4 Piecewise Linear Vector Processes vs. C2 Spinor processes

It will be demonstrated on thermodynamic spaces of Pfaff Topological Dimension 3, that there exist piecewise continuous processes (solutions to the Navier-Stokes equations) which are thermodynamically reversible. These Vector processes can be fabricated by combinations of Spinor processes, each of which is irreversible. This topological result demonstrates, by example, the difference between piecewise linear 3-manifolds and smooth complex manifolds. It appears that the key feature of the irreversible processes is that they have a fixed point of "rotation or expansion".

Consider those abstract physical systems that are represented by 1-forms, A, of Pfaff Topological Dimension 3. The concept implies that the topological features can be described

[7] The experimental fact that the defect structures emerge in finite time is still an open *topological* problem, although some *geomtric* success has been achieved through Ricci flows.

in terms of 3 functions (of perhaps many geometrical coordinates and parameters) and their differentials. For example, if one presumes the fundamental independent base variables are the set $\{x, y, z\}$, with an exterior differential oriented volume element consisting of a product[8] of exact 1-forms $\Omega_3 = +dx\hat{}dy\hat{}dz$, (then a local) Darboux representation for a physical system could have the appearance,

$$A = xdy + dz. \tag{87}$$

The objective is to use the features of Cartan's magic formula to compute the possible evolutionary features of such a system. The evolutionary dynamics is essentially the first law of thermodynamics:

$$L_{\rho\mathbf{V}}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A) = W + dU = Q. \tag{88}$$

The elements of the Pfaff sequence for this Action become,

$$\begin{aligned} A &= xdy + dz., & (89)\\ dA &= dx\hat{}dy, & (90)\\ A\hat{}dA &= dx\hat{}dy\hat{}dz, & (91)\\ dA\hat{}dA &= 0. & (92) \end{aligned}$$

Note that for this example the coefficient of the 3-form of Topological Torsion is not zero, and depends upon the Enstrophy (square of the Vorticity) of the fluid flow.

6.5 The Vector Processes

Relative to the position vector $\mathbf{R} = [x, y, z]$ of ordered topological coordinates $\{x, y, z\}$, consider the 3 abstract, linearly independent, orthogonal (supposedly) vector direction fields:

$$\mathbf{V}_x = \left| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle, \tag{93}$$

$$\mathbf{V}_y = \left| \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\rangle, \tag{94}$$

$$\mathbf{E} = \left| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle. \tag{95}$$

[8]More abstract systems could be constructed from differential forms which are not exact.

These direction fields can be used to define a class of (real) Vector processes, but these real vectors do not exhibit the complex Spinor class of eigendirection fields for the 2-form, dA. The Spinor eigendirection fields are missing from this basis frame. The important fact is that thermodynamic processes defined in terms of a real basis frame (and its connection) are incomplete, as such processes ignore the complex spinor direction fields.

For each of the real direction fields, deform the (assumed) process by an arbitrary function, ρ. Then construct the terms that make up the First Law of topological thermodynamics. First construct the contractions to form the internal energy for each process,

$$U_{\mathbf{V}_x} = i(\rho\mathbf{V}_x)A = 0, \qquad dU_{\mathbf{V}_x} = 0, \tag{96}$$

$$U_{\mathbf{V}_y} = i(\rho\mathbf{V}_y)A = \rho x, \quad dU_{\mathbf{V}_y} = d(\rho x), \tag{97}$$

$$U_{\mathbf{E}} = i(\rho\mathbf{E})A = \rho, \qquad dU_{\mathbf{E}} = d\rho. \tag{98}$$

The *extremal* vector $\mathbf{E}$ is the unique eigenvector with eigenvalue zero relative to the maximal rank antisymmetric matrix generated by the 2-form, dA. The *associated* vector $\mathbf{V}_x$ (relative to the 1-form of Action, A, is orthogonal to the y, z plane. Recall that any associated vector represents a local adiabatic process, as the Heat flow is transverse to the process. The linearly independent thermodynamic Work 1-forms for evolution in the direction of the 3 basis vectors are determined to be,

$$W_{\mathbf{V}_x} = i(\rho\mathbf{V}_x)dA = +\rho dy, \tag{99}$$

$$W_{\mathbf{V}_y} = i(\rho\mathbf{V}_y)dA = -\rho dx, \tag{100}$$

$$W_{\mathbf{E}} = i(\rho\mathbf{E})dA = 0. \tag{101}$$

From Cartan's Magic Formula representing the First Law as a description of topological evolution,

$$L_{(\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V})A) \equiv Q, \tag{102}$$

it becomes apparent that,

$$Q_{\mathbf{V}_x} = +\rho dy, \qquad dQ_{\mathbf{V}_x} = +d\rho\hat{}dy, \tag{103}$$

$$Q_{\mathbf{V}_y} = +x d\rho, \qquad dQ_{\mathbf{V}_y} = -d\rho\hat{}dx, \tag{104}$$

$$Q_{\mathbf{E}} = d\rho \qquad dQ_{\mathbf{E}} = 0, \tag{105}$$

All processes in the extremal direction satisfy the conditions that $Q_{\mathbf{E}}\hat{}dQ_{\mathbf{E}} = 0$. Hence, all extremal processes are reversible. It is also true that evolutionary processes in the direction of the other basis vectors, separately, are reversible, as the 3-form $Q\hat{}dQ$ vanishes for $\mathbf{V}_x$, $\mathbf{V}_y$, or $\mathbf{E}$. Hence all such *piecewise* continuous, *transitive*, processes are thermodynamically reversible.

Note further that the "rotation" induced by the antisymmetric matrix $[dA]$ acting on $\mathbf{V}_x$ yields $\mathbf{V}_y$ and the 4th power of the matrix yields the identity rotation,

$$\begin{aligned}
[dA] \circ |\mathbf{V}_x\rangle &= |\mathbf{V}_y\rangle\,, & (106)\\
[dA]^2 \circ |\mathbf{V}_x\rangle &= -\,|\mathbf{V}_x\rangle\,, & (107)\\
[dA]^4 \circ |\mathbf{V}_x\rangle &= +\,|\mathbf{V}_x\rangle\,. & (108)
\end{aligned}$$

This concept is a signature of Spinor phenomena.

6.6 The Spinor Processes

Now consider processes defined in terms of the Spinors. The eigendirection fields of the antisymmetric matrix representation of $F = dA$,

$$[F] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{109}$$

are given by the equations:

$$\begin{aligned}
\text{EigenSpinor1}\quad |Sp1\rangle &= \left| \begin{matrix} 1 \\ \sqrt{\text{-}1} \\ 0 \end{matrix} \right\rangle \quad \text{Eigenvalue} = +\sqrt{\text{-}1}, & (110)\\
\text{EigenSpinor2}\quad |Sp2\rangle &= \left| \begin{matrix} 1 \\ -\sqrt{\text{-}1} \\ 0 \end{matrix} \right\rangle \quad \text{Eigenvalue} = \text{-}\sqrt{\text{-}1} & (111)\\
\text{EigenVector1}\,|E\rangle &= \left| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle \quad \text{Eigenvalue} = 0 & (112)
\end{aligned}$$

Now consider the processes defined by ρ times the Spinor eigendirection fields. Compute the change in internal energy, dU, the Work, W and the Heat, Q, for each Spinor eigendirection field:

$$\begin{aligned}
U_{\rho\mathbf{Sp_1}} &= i(\rho\mathbf{Sp_1})A = \sqrt{-1}\rho x & d(U_{\rho\mathbf{Sp_1}}) &= \sqrt{-1}d(\rho x), & (113)\\
U_{\rho\mathbf{Sp_2}} &= i(\rho\mathbf{Sp_2})A = -\sqrt{-1}\rho x & d(U_{\rho\mathbf{Sp_2}}) &= -\sqrt{-1}d(\rho x), & (114)\\
U_{\rho\mathbf{E}} &= i(\rho\mathbf{E})A = \rho, & d(U_{\rho\mathbf{E}}) &= d\rho,. & (115)
\end{aligned}$$

$$W_{\rho\mathbf{Sp_1}} = i(\rho\mathbf{Sp_1})dA = \rho(dy - \sqrt{\text{-}1}dx), \tag{116}$$
$$W_{\rho\mathbf{Sp_2}} = i(\rho\mathbf{Sp_2})dA = +\rho(dy + \sqrt{\text{-}1}dx) \tag{117}$$
$$W_{\rho\mathbf{E}} = i(\rho\mathbf{V_1})dA = 0,. \tag{118}$$

$$Q_{\rho\mathbf{Sp_1}} = L_{i(\rho\mathbf{Sp_1})}A = \rho(dy - \sqrt{\text{-}1}dx) + \sqrt{\text{-}1}d(\rho x), \tag{119}$$
$$Q_{\rho\mathbf{Sp_2}} = L_{i(\rho\mathbf{Sp_2})}A = \rho(dy + \sqrt{\text{-}1}dx) - \sqrt{\text{-}1}d(\rho x), \tag{120}$$
$$Q_{\rho\mathbf{E}} = L_{i(\rho\mathbf{E})}A = d\rho. \tag{121}$$

6.7 Irreversible Spinor processes

Next compute the 3-forms of $Q\hat{}dQ$ for each direction field, including the spinors:

$$Q_{\rho\mathbf{E}}\hat{}dQ_{\rho\mathbf{E}} = 0, \tag{122}$$
$$Q_{\rho\mathbf{Sp_1}}\hat{}dQ_{\rho\mathbf{Sp_1}} = -\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy, \tag{123}$$
$$Q_{\rho\mathbf{Sp_2}}\hat{}dQ_{\rho\mathbf{Sp_2}} = +\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy. \tag{124}$$

It is apparent that evolution in the direction of the Spinor fields can be irreversible in a thermodynamic sense, if $d\rho\hat{}dx\hat{}dy$ is not zero. This is not true for the "piecewise linear" combinations of the complex Spinors that produce the real vectors, $\mathbf{V}_x$ and $\mathbf{V}_y$.

Evolution in the direction of "smooth" combinations of the base vectors may not satisfy the reversibility conditions, $Q\hat{}dQ = 0$, when the combination involves a fixed point in the x, y plane. For example, it is possible to consider smooth rotations (polarization chirality) in the x, y plane:

$$V_{\text{rotation right}} = \mathbf{V}_x + \sqrt{\text{-}1}\mathbf{V}_y = Sp1, \tag{125}$$
$$Q\hat{}dQ = -\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy. \tag{126}$$

$$V_{\text{rotation left}} = \mathbf{V}_x - \sqrt{\text{-}1}\mathbf{V}_y = Sp2, \tag{127}$$
$$Q\hat{}dQ = +\sqrt{\text{-}1}\rho d\rho\hat{}dx\hat{}dy. \tag{128}$$

The non-zero value of $Q\hat{}dQ$ for the continuous rotations are related to the non-zero Godbillon-Vey class [8]. A key feature of the rotations is that they have a fixed point in the plane;

the motions are not transitive. If the physical system admits an equation of state of the form, $\theta = \theta(x, y, \rho) = 0$, then the rotation or expansion processes are not irreversible.

Note that the (supposedly) Vector processes of the preceding subsection are combinations of the Spinor processes,

$$\mathbf{V}_x = (a \cdot Sp1 + b \cdot Sp2)/2 \tag{129}$$

$$\mathbf{V}_y = -\sqrt{\text{-}1}(a \cdot Sp1 - b \cdot Sp2)/2. \tag{130}$$

Almost always, a process defined in terms a linear combinations of the Spinor direction fields will generate a Heat 1-form, Q, that does not satisfy the Frobenius integrability theorem, and therefore all such processes are thermodynamically irreversible: $Q\hat{}dQ \neq 0$. However, with the requirement that a^2 is precisely the same as b^2, then either piecewise linear process is reversible, for $Q\hat{}dQ = 0$.

If the coefficients, and therefore the Spinor contributions, have slight fluctuations, the cancellation of the complex terms is not precise. Then either of the (now approximately) piecewise continuous process will NOT be reversible due to Spinor fluctuations.

Remark 6 *The facts that piecewise (sequential) C1 transitive evolution along a set of direction fields in odd (3) dimensions can be thermodynamically reversible, $Q\hat{}dQ = 0$, while (smooth) C2 evolution processes composed from complex Spinors can be thermodynamically irreversible, $Q\hat{}dQ \neq 0$, is a remarkable result which appears to have a relationship to Nash's theorem on C1 embedding. Physically, the results are related to tangential discontinuities such as hydrodynamic wakes.*

For systems of Pfaff dimension 4, all of the eigendirection fields are Spinors. The Spinors occur as two conjugate pairs. If the conjugate variables are taken to be x,y and z,t then the z,t spinor pair can be interpreted in terms of a chirality of expansion or contraction, where the x,y pair can be interpreted as a chirality of polarization. In this sense it may be said that thermodynamic time irreversibility is an artifact of dimension 4.

It is remarkable that a rotation and an expansion can be combined (eliminating the fixed point) to produce a thermodynamically *reversible* process.

Ian Stewart points out that there are three types of manifold structure: piecewise linear, smooth, topological. Theorems on piecewise-linear manifolds may not be true on smooth manifolds. The work above seems to describe such an effect. Piecewise continuous processes are reversible, where smooth continuous processes are not (see page 106, [40])!

7 Epilogue:

7.1 The Cartan-Hilbert Action 1-form

To demonstrate by example how the topological methods can be extended to higher dimensional systems and fiber bundles, consider those physical systems that can be described by a function $\mathsf{L}(\mathbf{q},\mathbf{v},t)$ and a 1-form of Action (per unit mole) given by Cartan-Hilbert format,

$$A=\mathsf{L}(\mathbf{q}^k,\mathbf{v}^k,t)dt+\mathbf{p}_k\cdot(d\mathbf{q}^k-\mathbf{v}^k dt). \tag{131}$$

The classic Lagrange function, $\mathsf{L}(\mathbf{q}^k,\mathbf{v}^k,t)dt$, is extended to include fluctuations in the kinematic variables, $(d\mathbf{q}^k-\mathbf{v}^k dt)\neq 0$. It is no longer assumed that the equation of Kinematic Perfection is satisfied. Fluctuations of the topological constraint of Kinematic Perfection are permitted;

$$\textbf{Topological Fluctuations in position: } \Delta\mathbf{q}=(d\mathbf{q}^k-\mathbf{v}^k dt)\neq 0. \tag{132}$$

As the fluctuations are 1-forms, it is some interest to compute their Pfaff Topological Dimension. The first step in the construction of the Pfaff Sequence is to compute the exterior differential of the fluctuation 1-form:

$$\begin{aligned}
\textbf{Fluctuation 2-form}: \quad & d(\Delta\mathbf{q})=-(d\mathbf{v}^k-\mathbf{a}^k dt)\hat{}dt && (133)\\
= \quad & -\Delta\mathbf{v}\hat{}dt, && (134)\\
\textbf{Topological Fluctuations in velocity} \quad : \quad & \Delta\mathbf{v}=(d\mathbf{v}^k-\mathbf{a}^k dt)\neq 0. && (135)
\end{aligned}$$

It is apparent that the Pfaff Topological Dimension of the fluctuations is at most 3, as $\Delta\mathbf{q}\hat{}\Delta\mathbf{v}\hat{}dt\neq 0$, and has a Heisenberg component,

When dealing with fluctuations in this prologue, the geometric dimension of independent base variables will not be constrained to the 4 independent base variables of the Thermodynamic model. At first glance it appears that the domain of definition is a (3n+1)-dimensional variety of independent base variables, $\{\mathbf{p}_k,\mathbf{q}^k,\mathbf{v}^k,t\}$. Do not make the assumption that the $\mathbf{p}_k$ are constrained to be canonically defined. Instead, consider $\mathbf{p}_k$ to be a (set of) Lagrange multiplier(s) to be determined later. Also, do not assume at this stage that $\mathbf{v}$ is a kinematic velocity function, such that $(d\mathbf{q}^k-\mathbf{v}^k dt)\Rightarrow 0$. The classical idea is to assert that topological fluctuations in position are related to pressure, and topological fluctuations in velocity are related to temperature.

For the given Action, construct the Pfaff Sequence (12) in order to determine the Pfaff dimension or class [9] of the Cartan-Hilbert 1-form of Action. The top Pfaffian is defined as the non-zero p-form of largest degree p in the sequence. The top Pfaffian for the Cartan-Hilbert Action is given by the formula,

Top Pfaffian is 2n+2

$$(dA)^{n+1} = (n+1)!\{\Sigma_{k=1}^{n}(\partial \mathsf{L}/\partial v^k - p_k)dv^k\}\hat{}\Omega_{2n+1}, \tag{136}$$

$$\Omega_{2n+1} = dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{137}$$

The formula is a bit surprising in that it indicates that the Pfaff Topological Dimension of the Cartan-Hilbert 1-form is 2n+2, and not the geometrical dimension $3n+1$. For n = 3 "degrees of freedom", the top Pfaffian indicates that the Pfaff Topological Dimension of the 2-form, dA is $2n+2=8$. The value $3n+1=10$ might be expected as the 1-form was defined initially on a space of $3n+1$ "independent" base variables. The implication is that there exists an irreducible number of independent variables equal to $2n+2=8$ which completely characterize the differential topology of the first order system described by the Cartan-Hilbert Action. It follows that the exact 2-form, dA, satisfies the equations

$$(dA)^{n+1} \neq 0, \; but \quad A\hat{}(dA)^{n+1} = 0. \tag{138}$$

Remark 7 *The idea that the 2-form, dA, is a symplectic generator of even maximal rank, 2n+2, implies that ALL eigendirection fields of the 2-form, $F = dA$, are complex isotropic Spinors, and all processes on such domains have Spinor components.*

The format of the top Pfaffian requires that the bracketed factor in the expression above, $\{\Sigma_{k=1}^{n}(\partial \mathsf{L}/\partial v^k - p_k)dv^k\}$, can be represented (to within a factor) by a perfect differential, dS.

$$dS = (n+1)!\{\Sigma_{k=1}^{n}(\partial \mathsf{L}/\partial v^k - p_k)dv^k\}. \tag{139}$$

The result is also true for any closed addition γ added to A; e.g., the result is "gauge invariant". Addition of a closed 1-form does not change the Pfaff dimension from even to odd. On the other hand the result is not renormalizable, for multiplication of the Action 1-form by a function can change the algebraic Pfaff dimension from even to odd.

On the 2n+2 domain, the components of (2n+1)-form $T = A\hat{}(dA)^n$ generate what has been defined herein as the Topological Torsion vector, to within a factor equal to the Torsion Current. The coefficients of the (2n+1)-form are components of a contravariant vector density $\mathbf{T}^m$ defined as the Topological Torsion vector, the same concept as defined previously on a 4D thermodynamic domain, but now extended to (2n+2)-dimensions. This vector is orthogonal (transversal) to the 2n+2 components of the covariant vector, $\mathbf{A}_m$. In other words,

$$A\hat{}T = A\hat{}(A\hat{}(dA)^n) = 0 \Rightarrow i(\mathbf{T})(A) = \sum \mathbf{T}^m \mathbf{A}_m = 0. \tag{140}$$

This result demonstrates that the extended Topological Torsion vector represents an adiabatic process. This topological result does not depend upon geometric ideas such as metric. It was demonstrated above that, on a space of 4 independent variables, evolution in the direction of the Topological Torsion vector is irreversible in a thermodynamic sense, subject to the symplectic condition of non-zero divergence, $d(A\hat{}dA) \neq 0$. The same concept holds on dimension 2n+2.

The 2n+2 symplectic domain so constructed can not be compact without boundary for it has a volume element which is exact. By Stokes theorem, if the boundary is empty, then the surface integral is zero, which would require that the volume element vanishes; but that is in contradiction to the assumption that the volume element is finite. For the 2n+2 domain to be symplectic, the top Pfaffian can never vanish. The domain is therefore orientable, but has two components, of opposite orientation. Examination of the constraint that the symplectic space be of dimension 2n+2 implies that the Lagrange multipliers, $\mathbf{p}_k$, cannot be used to define momenta in the classical "conjugate or canonical" manner.

Define the non-canonical components of the momentum, $\hbar k_j$, as,

$$\textbf{non-canonical momentum: } \hbar k_j = (p_j - \partial \mathsf{L}/\partial v^j), \tag{141}$$

such that the top Pfaffian can be written as,

$$(dA)^{n+1} = (n+1)!\{\Sigma_{j=1}^{n} \hbar k_j dv^j\}\hat{}\Omega_{2n+1}, \tag{142}$$

$$\Omega_{2n+1} = dp_1\hat{}...dp_n\hat{}dq^1\hat{}..dq^n\hat{}dt. \tag{143}$$

For the Cartan-Hilbert Action to be of Pfaff Topological Dimension 2n+2, the factor $\{\Sigma_{j=1}^{n} \hbar k_j dv^j\} \neq 0$. It is important to note, however, that as $(dA)^{n+1}$ is a volume element of geometric dimension 2n+2, the 1-form $\Sigma_{j=1}^{n} \hbar k_j dv^j$ is exact (to within a factor, say $T(\mathbf{q}^k, t,\ \mathbf{p}_k, S_\mathbf{v})$); hence,

$$\Sigma_{j=1}^{n} \hbar k_j dv^j = T dS_\mathbf{v}. \tag{144}$$

Tentatively, this 1-form, $dS_\mathbf{v}$, will be defined as the Topological Entropy production relative to topological fluctuations of momentum, kinematic differential position and velocity. If $\hbar k_j$ is defined as the deviation about the canonical definition of momentum, $\hbar k_j = \Delta p_j$, and noting the the expression for the top Pfaffian can be written as $(n+1)!\{\Sigma_{j=1}^{n} \hbar k_j \Delta v^j\}\hat{}\Omega_{2n+1}$, leads to an expression for the entropy production rate in the suggestive "Heisenberg" format:

$$T dS_\mathbf{v} = \Delta p_j \Delta v^j. \tag{145}$$

7.2 The Cosmological Thermodynamic Environment.

The coefficient functions of the 1-form of Action, A, that defines the PTD $= 4$ thermodynamic environment, can be used to construct a 4x4 Jacobian matrix of functions. This matrix always has a 4th order characteristic polynomial, Θ, that vanishes. The characteristic function Θ can be used to define a thermodynamic phase function, which is invariant with respect to all similarity transformations. In this sense, the topological idea of universality and independence from the observers choice of coordinates becomes evident.

The universal phase function admits an envelope and an edge of regression, which have classic thermodynamic interpretations. It is also true, that if the matrix is singular, then there can exist a cubic factor to the quartic polynomial, and the image of this hypersurface has the features of a deformed van der Waals gas.

The deformed van der Waals gas admits (non-equilibrium states) of negative pressure (as every steam engineer knows) and attraction, due to van der Waals condensation. These are tenable non-exotic concepts that direct attention to more mundane explanations for the dark matter (van der Waals condensation), dark energy (van der Waals negative pressure) properties that have been observed by astronomers. The details of this topological thermodynamic theory of the Cosmos appears in [43].

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