Appendix 3.D

Quantum and Hadronic Mechanics, the Diffusion and Iso-Heisenberg Representations

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1 Introduction

It is appropiate to start by quoting Prof. Santilli: "a first meaning of the novel hadronic mechanics is that of providing the first known methods for quantitative studies of the interplay between matter and the underlying substratum. The understanding is that space is the final frontier of human knowledge, with potential outcomes beyond the most vivid science fiction of today". In this almost prophetic observation, Prof. Santilli has pointed out to the essential role of the substratum, its geometrical structure and the link with consciousness. In the present article, which we owe to the kind invitation of Prof. Santilli, we shall present similar views, specifically in presenting both quantum and hadronic mechanics as space-time fluctuations, and we shall discuss the role of the substratum. As for the problem of human knowledge, we shall very briefly indicate on how the present approach may be related to the fundamental problem of consciousness, which is that of self-reference.

A central problem of contemporary physics is the distinct world views provided by QM and GR(short for quantum mechanics and general relativity, respectively), and more generally of gravitation. In a series of articles [1-4,25] and references therein, we have presented an unification between space-time structures, Brownian motions, fluid dynamics and QM. The starting point is the unification of space-time geometry and classical statistical theory, which has been possible due to a complementarity of the objects characterizing the Brownian motion, i.e. the noise tensor which produces a metric, and the drift vector field which describes the average velocity of the Brownian, in jointly describing both the space-time geometry and the stochastic processes. These space-time structures can be defined starting from flat Euclidean or Minkowski space-time, and they have in addition to a metric a torsion tensor which is formed from the metric conjugate of the drift vector field. The key to this unification lies in that the laplacian operator defined by this geometrical structure is the differential generator of the Brownian motions; stochastic analysis which deals with the transformation rules of classical observables on diffusion paths ensures that this unification is valid in both directions [26]. Thus , in this equivalence, one can choose the Brownian motions as the original structures determining a space-time structure, or conversely, the space-time structures produce a Brownian motion process. Space-time geometries with torsion have lead to an extension of the theory of gravitation which was first explored in joint work by Einstein with Cartan [5], so that the foundations for the gravitational field, for the special case in which the torsion reduces to its trace, can be found in these Brownian motions. Furthermore, in [2] we have shown that the relativistic quantum potential coincides , up to a conformal factor, with the metric scalar curvature. In this setting we are lead to conceive that there is no actual propagation of disturbances but instead an holistic modification of the whole space-time structure due to an initial perturbation which provides for the Brownian process modification of the original configuration. Furthermore, the present theory which has a kinetic Brownian motion generation of the geometries, is related to Le Sage's proposal of a Universe filled with all pervading tiny particles moving in all directions as a pushing (in contrast with Newton's pulling force) source for the gravitational field [39]. Le Sage's perspective was found to be compatible with cosmological observations by H. Arp [40]. This analysis stems from the assumption of a non-constant mass in GR which goes back to Hoyle and Narlikar, which in another perspective developed by Wu and Lin generates rotational forces [41]. These rotational forces can be ascribed to the drift trace-torsion vector field of the Brownian processes through the Hodge duality transformation [3], or still to the vorticity generated by this vector field. In our present theory, motions in space *and* time are fractal, they generate the gravitational field, and furthermore they generate rotational fields, in contrast with the pulling force of Newton's theory and the pushing force of Le Sage, or in the realm of the neutron, the Coulomb force. Furthermore, in our construction the drift has built-in terms given by the conjugate of electromagnetic-like potential 1-forms, whose associated intensity two-form generate vorticity, i.e. angular momentum; these terms include the Hertz potential which is the basis for the construction of superluminal solutions of Maxwell's equations; see [2] and references therein. So the present geometries are very different from the metric geometries of general relativity and are not in conflict with present cosmological observations.

The space-time geometrical structures of this theory can be introduced by the Einstein λ transformations on the tetrad fields [5,2], from which the usual Weyl scale transformations on the metric can be derived, but contrarily to Weyl geometries, these structures have torsion and they are integrable in contrast with Weyl's theory; we have called these connections as RCW structures (short for Riemann-Cartan-Weyl) [1-4]. This construction is a special case of the construction of Riemannian or Lorentzian metrics presented in Section 3.5.3, in which the generalized isotopic unit takes a diagonal form with equal elements given by (the square of) a scale function, while the number field, the differential and integral calculus are the usual ones of practice in differential geometry; these restrictions will be lifted to work with a full isotopic theory for HM in extending the theory developed for QM; in distinction with HM, the usual scale transformations do not depend on anything but the space-time coordinates, thus excluding the more general non-linear non-hamiltonian case contemplated by HM. In distinction with GR which due to the lack of a source leads to inconsistencies discussed in Section 1.4, a theory based on torsion and in particular in the case of a so-called absolute parallelism in which the torsion is derived from the differential of the cotetrad field (the so-called Weitzenbock spaces), has a geometrically defined energy-momentum tensor which is built from the torsion tensor [23,44]. Furthermore, the trace-torsion has built-in electromagnetic potential terms. We must recall that in Section 1.4 it was proved that gravitational mass has partially an electromagnetic origin. So our original setup in terms of torsion fields which can be non-null in flat Minkowski or Euclidean spaces (while in these spaces curvature is null), does not lead in principle to the inconsistencies observed before. There are other differences between the present approach and GR which we would like to discuss. In the latter theory, the space-time structure is absolute in the sense that it is defined without going through a self-referential characterization. With the introduction of torsion, and especially in the case of the trivial metric with null associated curvature tensor, we are introducing a self-referential characterization of the geometry since the definition of the manifold by the torsion, is through the concept of locus of a point (be that temporal or spatial). Indeed, space and time can only be distinguished if we can distinguish inhomogenities, and this is the intent of torsion, to measure the dislocation (in space and time) in the manifold [52]. Thus all these theories stem from a geometrical operation which has a logical background related to the concept of distinction (and more fundamentally, the concept of identity, which is prior to that of distinction) and its implementation through the operation of comparison by parallel transport with the affine connection with non-vanishing torsion. [1] In comparison, in GR there is also an operation of distinction carried out by the parallel transport of pair of vector fields with the Levi-Civita metric connection yielding a trivial difference, i.e. the torsion is null and infinitesimal parallelograms trivially close, so that it does not lead to the appearence of inhomogenities as resulting from this primitive operation of distinction; these are realized through the curvature derived from the metric. But to close this discussion, we refer again to the inconsistencies that an approach based on the curvature viz a viz the present approach which places the appearence of space-time in terms of deformations of the vacuum, and as such, has the same genesis as Isorelativity developed by Prof. Santilli and presented in Section 3.5.5.

We have shown that this approach leads to non-relativistic QM both in configuration space [3] and in the projective Hilbert state-space through the stochastic Schroedinger equation [5] (in the latter case, it was proved shown that

[1] This can be further related with multivalued logics and the appearence of time waves related to paradoxes, which in a cognitive systems approach yields the Schroedinger representation; furthermore this conception leads to the notion of reentrance of a space-time domain into itself, as a self-referential cybernetic system, and ultimately to multidimensional time; this may ultimately be linked to semiotics and its role in biology [44].

this geometry is related to the reduction of the wave function can be described by decoherence through noise [3,5]) , and further to Maxwell's equation and its equivalence with the Dirac-Hestenes equation of relativistic QM [2,21]. The fact that non-relativistic QM can be linked to torsion fields was unveiled recently [3]. In fact, torsion fields have been considered to be as providing deviations of GR outside the reach of present precision measurements [22]. It turns out that quantum wave-functions verifying linear or non-linear Schroedinger equations are another universal, or if wished, mundane examples of torsion fields. We shall show in the present article, that this approach extends to the strong interactions as described by HM and thus that the isotopic lift of the Schroedinger wave function is also a source for torsion, albeit one which incorporates the full non-linearity and non-hamiltonian character of the strong interactions. The quantum random ensembles which generate the quantum geometries, or which dually can be seen as generated by them, in the case of the Schroedinger equation can be associated with harmonic oscillators with disordered random phase and amplitude first proposed by Planck, which have the same energy spectrum as the one derived originally by Schroedinger [56]. The probabilities of these ensembles are classical since they are associated with classical Brownian motions in the configuration and projective Hilbert-state manifolds, in sharp contrast with the Copenhagen interpretation of QM which is constructed in terms of single system description, and they are related to the scalar amplitude of the spinor field in the case of the Dirac field, and in terms of the modulus of the complex wave function in the non-relativistic case [2,3,21]. We would like to recall at this stage that Khrennikov has proved that Kolmogorov's axiomatics of classical probability theory, in a contextual approach which means an a-priori consideration of a complex of physical conditions, permits the reconstruction of quantum theory [27]. Thus, Khrennikov 's theory places the validity of quantum theory in ensembles, in distinction with the Copenhagen interpretation, and is known as the Vaxho interpretation of quantum mechanics. In the present approach we obtain both a geometrical characterization of the quantum domain through random ensembles performing Brownian motions which generate the space and time geometries, and additionally a characterization for single systems through the topological Bohr-Sommerfeld invariants associated with the trace-torsion by introducing the concept of Pfaffian system developed by Kiehn in his geometro-topological theory of processes [42], specifically applied to the trace-torsion one-form [44]. Most remarkably, in our setting another relevant example of these space-time geometries is provided by viscous fluids obeying the invariant Navier-Stokes equations of fluid-dynamics, or alternatively the kinematical dynamo equation for the passive transport of magnetic fields on fluids [1,4]. This is of importance with respect to cosmology, since cosmological observations have registered turbulent large-scale structures which are described in terms of the Navier-Stokes equations [45].

There have been numerous attempts to relate non-relativistic QM to diffusion equations; the most notable of them is Stochastic Mechanics due to Nelson

[9]. Already Schroedinger proposed in 1930-32 that his equation should be related to the theory of Brownian motions (most probably as a late reaction to his previous acceptance of the single system probabilistic Copenhagen interpretation), and further proposed a scheme he was not able to achieve, the so-called interpolation problem which requires to describe the Brownian motion and the wave functions in terms of interpolating the initial and final densities in a given time-interval [9]. More recently Nagasawa presented a solution to this interpolation problem and further elucidated that the Schroedinger equation is in fact a Boltzmann equation [14], and thus the generation of the space and time structures produced by the Brownian motions has a statistical origin. [2] Neither Nagasawa nor Nelson presented these Brownian motions as space-time structures, but rather as matter fields *on* the vacuum.[3]. Furthermore, Kiehn has proved that the Schroedinger equation in spatial 2D can be exactly transformed into the Navier-Stokes equation for a compressible fluid, if we further take the kinematical viscosity ν to be $\frac{\hbar}{m}$ with m the mass of the electron [12]. We have argued in [3] that the Navier-Stokes equations share with the Schroedinger equation, that both have a RCW geometry at their basis: While in the Navier-Stokes equations the trace-torsion is $\frac{-1}{2\nu}u$ with u the time-dependent velocity one-form of the viscous fluid, in the Schroedinger equation, the trace-torsion one-form incorporates the logarithmic differential of the wave function -just like in Nottale's theory [11]- and further incorporates electromagnetic potential terms in the trace-torsion one-form. This correspondence between trace-torsion one-forms is what lies at the base of Kiehn's correspondance, with an important addendum: While in the approach of the Schroedinger equation the probability density is related to the Schroedinger scale factor (in incorporating the complex phase) and the Born formula turns out to be a formula and not an hypothesis, under the transformation to the Navier-Stokes equations it turns out that the probability density of non-relativistic quantum mechanics, is the enstrophy density of the fluid, i.e. the square of the vorticity, which thus plays a *geometrical* role that substitutes the probability density. Thus, in this approach, while there

[2] We have discussed in [3] that the solution of the interpolation problem leads to consider time to be more than a classical parameter, but an active operational variable, as recent experiments have shown [46] which have elicited theoretical studies in [55]; other experiments that suggest an active role of time are further discussed in [3].

[3] Another developments following Nelson's approach, in terms of an initial fractal structure of space-time and the introduction of Nelson's forward and backward stochastic derivatives, was developed by Nottale in his Scale Theory of Relativity [11]. Remarkably, his approach has promoted the Schroedinger equation to be valid for large scale structures, and predicted the existence of exo-solar planets which were observationally verified to exist [13]. This may further support the idea that the RCW structures introduced in the vacuum by scale transformations, are valid independently of the scale in which the associated Brownian motions and equations of QM are posited. Nottale's covariant derivative operator turns to be a particular case of our RCW laplacian [3]. We would like to mention also the important developments of a theory of space-time with a Cantorian structure being elaborated in numerous articles by M. El Naschie [47] and a theory of fractals and stochastic processes of QM which has been elaborated by G. Ord [48].

exist virtual paths sustaining the random behaviour of particles (as is the case also of the Navier-Stokes equations) and interference such as in the two-slit experiments can be interpreted as a superposition of Brownian paths [14], the probability density has a purely geometrical fluid-dynamical meaning. This is of great relevance with regards to the fundamental role that the vorticity, i.e. the fluid's particles angular-momentum has as an organizing structure of the geometry of space and time. In spite that the torsion tensor in this theory is naturally restricted to its trace and thus generates a differential one-form, in the non-propagating torsion theories it is interpreted that the vanishing of the completely skew-symmetric torsion implies the absence of spin and angular momentum densities [22], it is precisely the role of the vorticity to introduce angular momentum into the present theory.

To explain the fundamental kinematical role of torsion in QM and classical mechanics of systems with Lie group symmetries, we note that if we consider as configuration space a Lie group, there is a canonical connection whose torsion tensor coefficients are non other than the coefficients of the Lie-algebra under the Lie bracket operation [38]. Thus a Lie group symmetry is characterized by the torsion tensor for the canonical connection. Thus the Lie-Santilli isotopic theory implies a deformation of the torsion tensor of the canonical connection by the generalized unit [15-20].[4] With regards to another role of torsion in classical mechanics, it appears as describing friction, or more generally, non-anholonomic terms which produce additional terms in the equations of motion, which were obliterated by contemporary physics with the exception of Birkhoffian mechanics and discussed in Sections 1.2.4, 3.1, 3.3 and 4.1.2 by Prof. Santilli, which originated in the monographs [60]. In fact the attention of this author to HM at an early stage, stemmed from his work (jointly with S.Sternberg) in classical mechanical systems with angular momentum, which could be formulated without lagrangians nor hamiltonians, and furthermore could not be reduced to the canonical form of conservative systems [65]. Further in common to HM and torsion geometries, is that the latter are associated to angular momentum densities [22], while in HM the isotopic unit incorporates spin-up spin-down couplings such as in the Rutherford-Animalu-Santilli model of the neutron [15] [51]. Possible relations between torsion as spin or angular momentum densities can be ventured in relation with anomalous spin interactions of the proton, and magnetic resonances [49]. Furthermore, it has been shown that completely skew-symmetric torsion can produce a spin flip of high energy fermionic matter at very high densities, and that in this situation helicity can be identified with spin [43]. An intrinsic macroscopic angular momentum would be the evidence of this phenomena.This may be of relevance when taking in consideration the time

[4]The introduction of this generalized unit, in contrast with the basic unit of mathematics and physics, establishes a relation between these new units and physical processes which is unknown to mathematics, and is presently developed in terms of an arithmetic of forms which follows from the principle of distinction previously alluded, the multivalued logics associated to it and self-reference [44].

periodicity of the fine structure of histograms and its relation to macroscopic angular momentum which we have discussed in [3] and others we shall discuss in this article.

To understand the need of carrying the extensions produced by the isotopic lifts, it is based in the fact that the isotopic lift of Relativity due to Santilli (see [18]) is applicable for the electromagnetic and weak interactions but not applicable for the case of hadrons. These have a charge radius of 1 fm ($10^{-13}cm$) which is the radius of the strong interactions. Unlike the electromagnetic and weak interactions a necessary condition to activate the strong interaction is that hadrons enter into a condition of mutual interpenetration. In view of the developments below, we would like to stress that the modification of the symmetries of particles under conditions of possible fusion, is the first step for the usual developments of fusion theories which have been represented in terms of diffusion processes that overcome the Coulomb repulsive potential which impedes the fusion [32]; Brownian motions and other stochastic processes also appear in a phenomenological approach to the many body problem in particle and nuclear physics, but with no hint as to the possibility of an underlying space-time structure [61]. The basic idea goes back to the foundational works of Smoluchowski (independently of A. Einstein's work in the subject) in Brownian motion [33]. In the case of fusion theories, we have a gas of neutrons (which have an internal structure) and electrons, or an hadron gas; in these cases the fused particles are considered to be alike a compressible fluid with an unstable neck in its fused drops which have to be stabilized to achieve effective fusion; we can see here the figure of deformed symmetries. Thus, the situation for the application of Brownian motion to fusion is a natural extension to the subatomic scale of the original theory. We finally notice that the models for fusion in terms of diffusion do not require QM nor QCD [32]. In contrast, HM stems from symmetry group transformations that describe the contact fusion processes that deform the neutron structure, and lead to the isotopic Schroedinger equation which in this article, together with the isotopic Heisenberg representation, will be applied to establish a link between the RCW geometries, fusion processes and diffusions. The reason for the use of the iso-Heisenberg representation, is that in Santilli's theory, the isotopic lift of the symmetries in carried out in terms of the iso-Heisenberg representation, where its connection with classical mechanics under the quantization rules including the isotopic lift is transparent. Similarly to QM it will turn out to be that this quantization that leads to HM can be framed in another terms, i.e. Brownian motions appear to be quantum representations with no need of a quantization of classical mechanics, which can nevertheless be achieved by taking in account the fluctuations represented by the noise tensor of these random motions.

2 RIEMANN-CARTAN-WEYL GEOMETRIES

In this section we follow [1,2]. In this article M denotes a smooth connected compact orientable n-dimensional manifold (without boundary). While in our initial works, we took for M to be space-time, there is no intrinsic reason for this limitation, in fact if can be an arbitrary configuration manifold and still a phase-space associated to a dynamical system. The paradigmatical example of the latter, is the projective space associated to a finite-dimensional Hilbert-space of a quantum mechanical system [3,5]. We shall further provide M with an affine connection, or still by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Here, the metric can be the Minkowski degenerate metric, or an arbitrary positive-definite (i.e. Riemannian) metric. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as:

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{1}$$

The first term in (1) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, i.e. $\left\{ \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}, \tag{2}$$

is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ is the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2 \nabla^2 = 1/2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{3}$$

A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g, since it is

$$H(\nabla) = 1/2 \triangle_g + \hat{Q} \equiv H(g, Q), \tag{4}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and $\hat{Q}$ is the vector field associated to Q via g (the so-called g conjugate vector field to the one-form Q, i.e.

$$\hat{Q}(f) = g(Q, df), \tag{5}$$

for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g:

$$\triangle_g = g^{\alpha\beta} \nabla^g_{\frac{\partial}{\partial x^\alpha}} \nabla^g_{\frac{\partial}{\partial x^\beta}} = g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta} \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} \frac{\partial}{\partial x^\gamma}. \tag{6}$$

In this expression the partial derivatives are taken with respect to the Levi-Civita connection. Therefore, assuming that g is non-degenerate, we have defined a one-to-one mapping

$$\nabla \rightsquigarrow H(g,Q) = 1/2\triangle_g + \hat{Q}$$

between the space of g-compatible linear connections ∇ with Christoffel coefficients of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \; + \; \frac{2}{(n-1)} \left\{ \delta^{\alpha}_{\beta} \; Q_{\gamma} \; - \; g_{\beta\gamma} \; Q^{\alpha} \right\}, n \neq 1 \tag{7}$$

and the space of elliptic second order differential operators on functions. The extensions of this laplacian to differential forms and in particular, to fluid-dynamics, has been presented in [1] and [4].

3 RIEMANN-CARTAN-WEYL DIFFUSIONS

In this section we shall recall the correspondence between RCW connections defined by (7) and diffusion processes of scalar fields having $H(g,Q)$ as its diffrential generator. Thus, naturally we have called these processes as *RCW diffusion processes.*. For the extensions to describe the diffusion processes of differential forms, see [1, 4]. For the sake of generality, in the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependent 1-form. In this setting τ is the universal time variable due to Stuckelberg [8]; for a very sharp account of the relation of this time to Einsten's time, t, we refer to Horwitz et al [28]. The stochastic flow associated to the diffusion generated by $H(g,Q)$ has for sample paths the continuous curves $\tau \mapsto x(\tau) \in M$ satisfying the Itô invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = \sigma(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{8}$$

In this expression, $\sigma : M \times R^m \rightarrow TM$ is such that $\sigma(x) : R^m \rightarrow TM$ is linear for any $x \in M$, the noise tensor, so that we write $\sigma(x) = (\sigma^{\alpha}_i(x))$ $(1 \leq \alpha \leq n$, $1 \leq i \leq m)$ which satisfies

$$\sigma^{\alpha}_i \sigma^{\beta}_i = g^{\alpha\beta}, \tag{9}$$

where $g = (g^{\alpha\beta})$ is the expression for the metric in covariant form, and $\{W(\tau), \tau \geq 0\}$ is a standard Wiener process on R^m, with zero mean with respect to the standard centered Gaussian function, and covariance given by $\mathrm{diag}(\tau, \ldots, \tau)$; finally, $dW(\tau) = W(\tau + d\tau) - W(\tau)$ is an increment. Now , it is important to remark that m can be arbitrary, i.e. we can take noise tensors defined on different spaces, and obtain the essentially the same diffusion process [26]. In regards to the equivalence between the stochastic and the geometric picture, this enhances the fact that there is a freedom in the stochastic picture, which if chosen as the originator of the equivalence, points out to a more fundamental basis of the stochastic description. This is satisfactory, since it is impossible to identify all the sources for noise, and in particular those coming from the vacuum, which we take as the source for the randomness. Note that in taking the drift and the diffusion tensor as the original objects to build the geometry, the latter is derived from objects which are associated to *collective* phenomena. Note that if we start with eq. (8), we can reconstruct the associated RCW connection by using eq.(9) and the fact that the trace-torsion is the g-conjugate of the drift, i.e., in simple words, by lowering indexes of $\hat{Q}$ to obtain Q. We shall not go into the details of these constructions, which relies heavily on stochastic analysis on smooth manifolds [26].

Observations 1. Note that in the above construction of the s.d.e. all terms corresponding to the Levi-Civita connection $\left\{ {\alpha \atop \beta\gamma} \right\}$ have disappeared completely. In fact one can start with a Laplacian written without these terms, say

$$H := 1/2 g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + \hat{Q}^\alpha \partial_\alpha, \tag{10}$$

and rewrite it as

$$\frac{1}{2}\triangle_g + \tilde{b}^\alpha \partial_\alpha \tag{11}$$

with

$$\tilde{b}^\alpha = \hat{Q}^\alpha + \frac{1}{2} g^{\beta\gamma} \left\{ {\alpha \atop \beta\gamma} \right\}; \tag{12}$$

we then redefine the connection $\nabla = (\Gamma^\gamma_{\alpha\beta})$ to be compatible with g and such that $\tilde{b}^\alpha = \frac{1}{2}[g^{\beta\gamma}\left\{ {\alpha \atop \beta\gamma} \right\} - \Gamma^\alpha_{\beta\gamma}]$ so that finally our original RCW laplacian $H(\nabla)$ takes the form $H(g, \tilde{b})$ of eq. (4) and the s.d.e. is given by (8); c.f. pages $285 - 289$ in Ideda Watanabe [26]. From this follows that we can write the laplacians either with the Levi-Civita covariant derivative or the usual derivative for characterizing the diffusion processes corresponding to the Schroedinger equation; this is also valid for the iso-Schroedinger equations, starting by producing the isotopic lift of the differential operator, or further, the isotopic lift of the covariant derivative operator, the isocovariant differential introduced in Section 3.2.9.C above.

4 RCW GEOMETRIES, BROWNIAN MOTIONS AND THE SCHROEDINGER EQUATION

We have shown that we can represent the space-time quantum geometries for the relativistic diffusion associated with the invariant distribution, so that

$Q = \frac{1}{2}d\ln\rho$, with $\rho = \psi^2$ and $H(g,Q)$ has a self-adjoint extension for which we can construct the quantum geometry on state-space and still the stochastic extension of the Schroedinger equation defined by this operator on taking the analytical continuation on the time variable for the evolution parameter [3]. In this section which retakes the solution of the Schroedinger problem of interpolation by Nagasawa [14], we shall present the equivalence between RCW geometries, their Brownian motions and the Schroedinger equation which is a different approach to taking the analytical continuation in time, which by the way, has a very important significance in terms of considering time to be an active variable ; see [3]. We shall now present the construction of non-relativistic QM with the restriction that the Hodge decomposition of the trace-torsion restricts to its exact component, excluding thus the electromagnetic potential terms of the full trace-torsion which we considered in [2,3]. So that we take $Q = Q(t,x) = d\ln f_t(x)$ where $f(t,x) = f_t(x)$ is a function defined on the configuration manifold given by $[a,b] \times M$, where M is a 3-dimensional manifold provided with a metric, g. The construction applies as well to the general case as well, as we shall show further below. The scheme to determine f will be to manifest the time-reversal invariance of the Schroedinger representation in terms of a forward in time diffusion process and its time-reversed representation for the original equations for creation and annihilation diffusion processes produced when there is no background torsion field, whose explicit form and relation to f we shall determine in the sequel. From now onwards, the exterior differential, the divergence operator and the laplacian will act on the M manifold variables only, so that we shall write their action on fields, say $df_t(x)$, to signal that the exterior differential acts only on the x variables of M. We should remark that in this context, the time-variable t of non-relativistic theory and the evolution parameter τ, are identical [28]. Let

$$L = \frac{\partial}{\partial t} + \frac{1}{2}\triangle_g = \frac{\partial}{\partial t} + H(g,0). \tag{13}$$

Let $p(s,x;t,y)$ be the weak fundamental solution of

$$L\phi + c\phi = 0. \tag{14}$$

The interpretation of this equation as one of creation (whenever $c > 0$) and annihilation ($c < 0$) of particles is warranted by the Feynman-Kac representation for the solution of this equation [14]. Then $\phi = \phi(t,x)$ satisfies the equation

$$\phi(s,x) = \int_M p(s,x;t,y)\phi(t,y)dy, \tag{15}$$

where for the sake of simplicity, we shall write in the sequel $dy = \mathrm{vol_g(y)} = \sqrt{\mathrm{det(g)}}dy^1 \wedge \ldots \wedge dy^3$. Note that we can start for data with a given function $\phi(a,x)$, and with the knowledge of $p(s,x;a,y)$ we define $\phi(t,x) = \int_M p(t,x;a,y)dy$.

Next we define

$$q(s,x;t,y) = \frac{1}{\phi(s,x)} p(s,x;t,y)\phi(t,y), \tag{16}$$

which is a transition probability density, i.e.

$$\int_M q(s,x;t,y)dy = 1, \tag{17}$$

while

$$\int_M p(s,x;t,y)dy \neq 1. \tag{18}$$

Having chosen the function $\phi(t,x)$ in terms of which we have defined the probability density $q(s,x;t,y)$ we shall further assume that we can choose a second bounded non-negative measurable function $\breve{\phi}(a,x)$ on M such that

$$\int_M \phi(a,x)\breve{\phi}(a,x)dx = 1, \tag{19}$$

We further extend it to $[a,b] \times M$ by defining

$$\breve{\phi}(t,y) = \int \breve{\phi}(a,x)p(a,x;t,y)dx, \forall (t,y) \in [a,b] \times M, \tag{20}$$

where $p(s,x;t,y)$ is the fundamental solution of eq. (14).

Let $\{X_t \in M, \mathcal{Q}\}$ be the time-inhomogeneous diffusion process in M with the transition probability density $q(s,x;t,y)$ and a prescribed initial distribution density

$$\mu(a,x) = \breve{\phi}(t=a,x)\phi(t=a,x) \equiv \breve{\phi}_a(x)\phi_a(x). \tag{21}$$

The finite-dimensional distribution of the process $\{X_t \in M, t \in [a,b]\}$ with probability measure on the space of paths which we denote as Q; for $a = t_0 < t_1 < \ldots < t_n = b$, it is given by

$$\begin{aligned} E_Q[f(X_a, X_{t_1}, \ldots, X_{t_n-1}, X_b)] &= \int_M dx_0 \mu(a,x_0) q(a,x_0;t_1,x_1)dx_1 \ldots \\ &\quad q(t_1,x_1;t_2,x_2)dx_2 \ldots q(t_{n-1},x_{n-1},b,x_n)dx_n \\ &\quad f(x_0,x_1,\ldots,x_{n-1},x_n) := [\mu_a q >> \end{aligned} \tag{22}$$

which is the Kolmogorov forward in time (and thus time-irreversible) representation for the diffusion process with initial distribution $\mu_a(x_0) = \mu(a,x_0)$, which using eq. (16) can still be rewritten as

$$\begin{aligned} &\int_M dx_0 \mu_a(x_0) \frac{1}{\phi_a(x_0)} p(a,x_0;t_1,x_1)\phi_{t_1}(x_1)dx_1 \frac{1}{\phi_{t_1}(x_1)} dx_1 p(t_1,x_1;t_2,x_2) \\ &\phi_{t_2}(x_2)dx_2 \ldots \frac{1}{\phi(t_{n-1},x_{n-1})} p(t_{n-1},x_{n-1};b,x_n)\phi_b(x_n)dx_n f(x_0,\ldots,x_n) \end{aligned} \tag{23}$$

which in account of $\mu_a(x_0) = \breve{\phi}_a(x_0)\phi_a(x_0)$ and eq. (16) can be written in the time-reversible form

$$\int_M \breve{\phi}_a(x_0)dx_0 p(a, x_0; t_1, x_1)dx_1 p(t_1, x_1; t_2, x_2)dx_2 \dots p(t_{n-1}, x_{n-1}; b, x_n)$$
$$\phi_b(x_n)dx_n f(x_0, \dots, x_n) \quad (24)$$

which we write as

$$= [\breve{\phi}_a p >><< p\phi_b]. \quad (25)$$

This is the *formally* time-symmetric Schroedinger representation with the transition (but not probability) density p. Here, the formal time symmetry is seen in the fact that this equation can be read in any direction, preserving the physical sense of transition. This representation, in distinction with the Kolmogorov representation, does *not* have the Markov property.

We define the adjoint transition probability density $\breve{q}(s, x; t, y)$ with the $\breve{\phi}$-transformation

$$\breve{q}(s, x; t, y) = \breve{\phi}(s, x)p(s, x; t, y)\frac{1}{\breve{\phi}(t, y)} \quad (26)$$

which satisfies the Chapmann-Kolmogorov equation and the time-reversed normalization

$$\int_M dx \breve{q}(s, x; t, y) = 1. \quad (27)$$

We get

$$E_{\breve{Q}}[f(X_a, X_{t_1}, \dots, X_b)] = \int_M f(x_0, \dots, x_n)\breve{q}(a, x_0; t_1, x_1)dx_1 \breve{q}(t_1, x_1; t_2, x_2)dx_2$$
$$\dots \breve{q}(t_{n-1}, x_{n-1}; b, x_n)\breve{\phi}(b, x_n)\phi(b, x_n)dx_n, \quad (28)$$

which has a form non-invariant in time, i.e. reading from right to left, as

$$<< \breve{q}\hat{\breve{\phi}}_b\phi_b] =<< \breve{q}\hat{\mu}_b], \quad (29)$$

which is the time-reversed representation for the final distribution $\mu_b(x) = \breve{\phi}_b(x)\phi_b(x)$. Now, starting from this last expression and rewriting it in a similar form that is in the forward process but now with $\breve{\phi}$ instead of ϕ, we get

$$\int_M dx_0 \breve{\phi}_a(x_0)p(a, x_0; t_1, x_1)\frac{1}{\breve{\phi}_{t_1}(x_1))}dx_1 \breve{\phi}(t_1, x_1)p(t_1, x_1; t_2, x_2)\frac{1}{\breve{\phi}_{t_2}(x_2)}dx_2$$
$$\dots dx_{n-1}\breve{\phi}(t_{n-1}, x_{n-1})p(t_{n-1}, x_{n-1}; b, x_n)\frac{1}{\breve{\phi}(b, x_n)}\breve{\phi}_b(x_n)\phi(b, x_n)dx_n f(x_0, \dots, x_n) \quad (30)$$

which coincides with the time-reversible Schroedinger representation $[\breve{\phi}_a p >><< p\phi_b]$.

We therefore have three equivalent representations for the diffusion process: The forward in time Kolmogorov representation, the backward Kolmogorov representation, which are both naturally irreversible in time, and the time-reversible Schroedinger representation, so that we can write succintly,

$$[\mu_a q >>= [\breve{\phi}_a p >><< p\phi_b] =<< \breve{q}\mu_b], \text{with } \mu_a = \phi_a\breve{\phi}_a,\ \mu_b = \phi_b\breve{\phi}_b. \tag{31}$$

In addition of this formal identity,we have to establish the relations between the equations that have led to them. We first note, that in the Schroedinger representation, which is formally time-reversible, we have an interpolation of states between the initial data $\breve{\phi}_a(x)$ and the final data, $\phi_b(x)$. The information for this interpolation is given by a filtration of interpolation $\mathcal{F}_a^r \cup \mathcal{F}_b^s$, which is given in terms of the filtration for the forward Kolmogorov representation $\mathcal{F} = \mathcal{F}_a^t, t \in [a,b]$ which is used for prediction starting with the initial density $\phi_a\breve{\phi}_a = \mu_a$ and the filtration $\mathcal{F}_t^b$ for retrodiction for the time-reversed process with initial distribution μ_b.

We observe that q and $\breve{q}$ are in time-dependent duality with respect to the measure

$$\mu_t(x)dx = \breve{\phi}_t(x)\phi_t(x)dx, \tag{32}$$

We shall now extend the state-space of the diffusion process to $[a,b]\times M$, to be able to transform the time-inhomogeneous processes into time-homogeneous processes, while the stochastic dynamics still takes place exclusively in M. This will allow us to define the duality of the processes to be with respect to $\mu_t(x)dtdx$ and to determine the form of the exact term of the trace-torsion, and ultimately, to establish the relation between the diffusion processes and Schroedinger equations, both for potential linear and non-linear in the wave-functions. If we define time-homogeneous semigroups of the processes on $\{(t,X_t) \in [a,b]\times M\}$ by

$$P_r f(s,x) = \begin{cases} Q_{s,s+r}f(s,x) & , \quad s \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{33}$$

and

$$\breve{P}_r g(t,y) = \begin{cases} gQ_{t-r,t}(t,y) & , \quad r \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{34}$$

then

$$< g, P_r f >_{\mu_t dtdx} = < \breve{P}_r g, f >_{\mu_t dtdx}, \tag{35}$$

which is the duality of $\{(t,X_t)\}$ with respect to the $\mu_t dtdx$ density. We remark here that we have an augmented density by integrating with respect to time t. Consequently, if in our spacetime case we define for $a_t(x), \hat{a}_t(x)$ time-dependent one-forms on M (to be determined later)

$$B\alpha : \quad = \quad \frac{\partial\alpha}{\partial t} + H(g,a_t)\alpha_t, \tag{36}$$

$$B^0\mu : \quad = \quad -\frac{\partial\mu}{\partial t} + H(g,a_t)^{\dagger}\mu_t, \tag{37}$$

and its adjoint operators

$$\breve{B}\beta = -\frac{\partial \beta}{\partial t} - H(g, \breve{a}_t)^\dagger \beta_t, \tag{38}$$

$$(\breve{B})^0 \mu_t = \frac{\partial \mu_t}{\partial t} - H(g, \breve{a}_t)^\dagger \mu_t, \tag{39}$$

where by $H(g, \breve{a}_t)^\dagger$ we mean the vol_g-adjoint of this operator, i.e. $H(g, \breve{a}_t)^\dagger \mu_t = \frac{1}{2}\triangle_g \mu_t - \mathrm{div_g}(\mu \breve{a}_t)$. From [3,14] follows that the duality of space-time processes

$$< B\alpha, \beta >_{\mu_t(x)dtdx} = < \alpha, \breve{B}\beta >_{\mu_t(x)dtdx}, \tag{40}$$

is equivalent to

$$\begin{aligned} a_t(x) + \breve{a}_t(x) &= d \ln \mu_t(x) \equiv d \ln (\phi_t(x)\breve{\phi}_t(x)), && (41) \\ B^0 \mu_t(x) = 0. & && (42) \end{aligned}$$

The latter equation being the Fokker-Planck equation for the diffusion with trace-torsion given by $a + A$, then the Fokker-Planck equation for the adjoint (time-reversed) process is valid, i.e.

$$(\breve{B})^0 \mu_t(x) = 0. \tag{43}$$

Substracting eqs. (39) and (40) we get the final form of the duality condition

$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(\frac{a_t - \breve{a}_t}{2})\mu_t)] \quad = \quad 0, \text{ for } \mu_t(x) = \breve{\phi}_t(x)\phi_t(x). \tag{44}$$

Therefore, we can establish that the duality conditions of the diffusion equation in the Kolmogorov representation and its time reversed diffusion lead to the following conditions on the additional elements of the drift vector fields:

$$\begin{aligned} a_t(x) + \breve{a}_t(x) &= d \ln \mu_t(x) \equiv d \ln (\phi_t(x)\breve{\phi}_t(x)), && (45) \\ \frac{\partial \mu}{\partial t} + \mathrm{div_g}[(\frac{a_t(x) - \breve{a}_t(x)}{2})\mu_t(x)] &= 0. && (46) \end{aligned}$$

If we assume that $a_t - \hat{a}_t$ is an exact one-form, i.e., there exists a time-dependent differentiable function $S(t, x) = S_t(x)$ defined on $[a, b]\times M$ such that for $t \in [a, b]$,

$$a_t(x) - \breve{a}_t(x) = d \ln \frac{\phi_t(x)}{\breve{\phi}_t(x)} = 2dS_t(x) \tag{47}$$

which together with

$$a_t(x) + \breve{a}_t(x) = d \ln \mu_t(x), \tag{48}$$

implies that on $D(t, x)$ we have

$$\begin{aligned} a_t(x) &= d \ln \phi_t(x), && (49) \\ \breve{a}_t(x) &= d \ln \breve{\phi}_t(x) && (50) \end{aligned}$$

Introduce now $R_t(x) = R(t,x) = \frac{1}{2}\ln\phi_t(x)\breve{\phi}_t(x)$ and $S_t(x) = S(t,x) = \frac{1}{2}\ln\frac{\phi_t(x)}{\breve{\phi}_t(x)}$, so that

$$\begin{aligned} a_t(x) &= d(R_t(x) + S_t(x)), && (51) \\ \breve{a}_t(x) &= d(R_t(x) - S_t(x)), && (52) \end{aligned}$$

and eq. (46) takes the form

$$\frac{\partial R}{\partial t} + \frac{1}{2}\triangle_g S_t + g(dS_t, dR_t) = 0. \tag{53}$$

Remarks. We have mentioned the fact that there is a hidden active role of time in QM [55], which in the above construction is built-in the very definition of the probability density in terms of a *final* and initial distributions. This back action of time appears to be not exclusive of QM. In the theory of growth of sea shells due to Santilli and Illert, it was shown that it cannot be explained by Minkowskian nor Euclidean geometry, but their isotopic lifts and their duals, and this requires the introduction of time duality and four-fold time [58]; this model has been further applied to diverse problems of morphology in biology by Reverberi [59]. We further note that the time-dependent function S on the 3-space manifold, is defined by eq. (47) up to addition of an arbitrary function of t, and when further below we shall take this function as defining the complex phase of the quantum Schroedinger wave, this will introduce the quantum-phase indetermination of the quantum evolution, as we discussed already in the setting of geometry of the quantum state-space [3,5].

Therefore, together with the three different time-homogeneous representations $\{(t, X_t), t \in [a,b], X_t \in M\}$ of a time-inhomogeneous diffusion process $\{X_t, Q)$ on M we have three equivalent dynamical descriptions. One description, with creation and killing described by the scalar field $c(t,x)$ and the diffusion equation describing it is given by a creation-destruction potential in the trace-torsion background given by an electromagnetic potential

$$\frac{\partial p}{\partial t} + H(g,0)(x)p + c(t,x)p = 0; \tag{54}$$

the second description has an additional trace-torsion $a(t,x)$, a 1-form on $R \times M$

$$\frac{\partial q}{\partial t} + H(g, a_t)q = 0. \tag{55}$$

while the third description is the adjoint time-reversed of the first representation given by $\breve{\phi}$ satisfying the diffusion equation on the background with no torsion, i.e.

$$-\frac{\partial \breve{\phi}}{\partial t} + H(g,0)\breve{\phi} + c\breve{\phi} = 0. \tag{56}$$

The second representation for the full trace-torsion diffusion forward in time Kolmogorov representation, we need to adopt the description in terms of the fundamental solution q of

$$\frac{\partial q}{\partial t} + H(g, a_t)q = 0, \tag{57}$$

for which one must start with the initial distribution $\mu_a(x) = \breve{\phi}_a(x)\phi_a(x)$. This is a time t-irreversible representation in the real world, where q describes the real transition and μ_a gives the initial distribution. If in addition one traces the diffusion backwards with reversed time t, with $t \in [a, b]$ running backwards, one needs for this the final distribution $\mu_b(x) = \breve{\phi}_b(x)\phi_b(x)$ and the time t reversed probability density $\hat{q}(s, x; t, y)$ which is the fundamental solution of the equation

$$-\frac{\partial \breve{q}}{\partial t} + H(g, \breve{a}_t)\hat{q} = 0, \tag{58}$$

with additional trace-torsion one-form on $R \times M$ given by $\hat{a}$, where

$$\breve{a}_t + a_t = d\ln\mu_t(x), \text{ with } \mu_t = \phi_t\breve{\phi}_t, \tag{59}$$

where the diffusion process in the time-irreversible forward Kolmogorov representation is given by the Ito s.d.e

$$dX_t^i = \sigma_j^i(X_t)dW_t^j + a^i(t, X_t)dt, \tag{60}$$

and the backward representation for the diffusion process is given by

$$dX_t^i = \sigma_j^i(X_t)dW_t^j + \breve{a}^i(t, X_t)dt, \tag{61}$$

where $a, \breve{a}$ are given by the eqs. $(51, 52)$, and $(\sigma\sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$.

We follow Schroedinger in pointing that ϕ and $\breve{\phi}$ separately satisfy the creation and killing equations, while in quantum mechanics ψ and $\bar{\psi}$ are the complex-valued counterparts of ϕ and $\hat{\phi}$, respectively, they are not arbitrary but

$$\phi\breve{\phi} = \psi\bar{\psi}. \tag{62}$$

Thus, in the following , this Born formula, once the equations for ψ are determined, will be a consequence of the constructions, and not an hypothesis on the random basis of non-relativistic mechanics.

Therefore, the equations of motion given by the Ito s.d.e.

$$dX_t^i = \mathrm{grad_g}\phi^i(t, X_t)dt + \sigma_j^i(X_t)dW_t^j, \tag{63}$$

which are equivalent to

$$\frac{\partial u}{\partial t} + H(g, a_t)u = 0 \tag{64}$$

with $a_t(x) = d\ln\phi_t(x) = d(R_t(x) + S_t(x))$, determines the motion of the ensemble of non-relativistic particles. Note that this equivalence requires only the Laplacian for the RCW connection with the forward trace-torsion full one-form

$$Q(t,x) = d\ln\phi_t(x) = d(R_t(x) + S_t(x)). \tag{65}$$

In distinction with Stochastic Mechanics due to Nelson [9], and contemporary ellaborations of this applied to astrophysics as the theory of Scale Relativity due to Nottale [11][13], we only need the form of the trace-torsion for the forward Kolmogorov representation, and this turns to be equivalent to the Schroedinger representation which interpolates in time-symmetric form between this forward process and its time dual with trace-torsion one-form given by $\breve{a}_t(x) = d\ln\breve{\phi}_t(x) = d(R_t(x) - S_t(x))$.

Finally, let us how this is related to the Schroedinger equation. Consider now the Schroedinger equations for the complex-valued wave function ψ and its complex conjugate $\bar{\psi}$, i.e. introducing $i = \sqrt{-1}$, we write them in the form

$$\begin{aligned} i\frac{\partial\psi}{\partial t} + H(g,0)\psi - V\psi &= 0 & (66)\\ -i\frac{\partial\bar{\psi}}{\partial t} + H(g,0)\bar{\psi} - V\bar{\psi} &= 0, & (67)\end{aligned}$$

which are identical to the usual forms. So, we have the imaginary factor appearing in the time t, which we confront with the diffusion equations generated by the RCW connection with null trace-torsion, i.e. the system

$$\begin{aligned} \frac{\partial\phi}{\partial t} + H(g,0)\phi + c\phi &= 0, & (68)\\ \frac{-\partial\breve{\phi}}{\partial t} + H(g,0)\breve{\phi} + c\breve{\phi} &= 0, & (69)\end{aligned}$$

and the diffusion equations determined by both the RCW connections with trace-torsion a and $\breve{a}$, i.e.

$$\begin{aligned} \frac{\partial q}{\partial t} + H(g,a_t)q &= 0, & (70)\\ \frac{-\partial\breve{q}}{\partial t} + H(g,\breve{a}_t)\hat{q} &= 0, & (71)\end{aligned}$$

which are equivalent to the single equation

$$\frac{\partial q}{\partial t} + H(g, d\ln\phi_t)q = 0. \tag{72}$$

If we introduce a complex structure on the two-dimensional real-space with coordinates (R,S), i.e. we consider

$$\psi = e^{R+iS}, \psi = e^{R-iS}, \tag{73}$$

viz a viz $\phi = e^{R+S}, \breve{\phi} = e^{R-S}$, with $\psi\bar{\psi} = \phi\breve{\phi}$, then for a wave-function differentiable in t and twice-differentiable in the space variables, then, ψ satisfies the Schroedinger equation if and only if (R, S) satisfy the difference between the Fokker-Planck equations , i.e.

$$\frac{\partial R}{\partial t} + g(dS_t, dR_t) + \frac{1}{2}\triangle_g S_t = 0, \tag{74}$$

and

$$V = -\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t). \tag{75}$$

which follows from substituting ψ in the Schroedinger equation and further dividing by ψ and taking the real part and imaginary parts, to obtain the former and latter equations, respectively.

Conversely, if we take the coordinate space given by $(\phi, \breve{\phi})$, both non-negative functions, and consider the domain $D = D(s,x) = \{(s,x) : 0 < \breve{\phi}(s,x)\phi(s,x)\} \subset [a,b] \times M$ and define $R = \frac{1}{2}\ln\phi\breve{\phi}, S = \frac{1}{2}\ln\frac{\phi}{\breve{\phi}}$, with R, S having the same differentiabilty properties that previously ψ, then $\phi = e^{R+S}$ satisfies in D the equation

$$\frac{\partial \phi}{\partial t} + H(g, 0)\phi + c\phi = 0, \tag{76}$$

if and only if

$$\begin{aligned} -c &= [-\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t)] \\ &+ [\frac{\partial R}{\partial t} + H(g, dR_t)S_t] + [2\frac{\partial S}{\partial t} + g(dS_t, dS_t)]. \end{aligned} \tag{77}$$

while $\breve{\phi} = e^{R-S}$ satisfies in D the equation

$$-\frac{\partial \phi}{\partial t} + H(g, 0)\breve{\phi} + c\breve{\phi} = 0, \tag{78}$$

if and only if

$$\begin{aligned} -c &= [-\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t)] \\ &- [\frac{\partial R}{\partial t} + H(g, dR_t)S_t] + [2\frac{\partial S}{\partial t} + g(dS_t, dS_t)]. \end{aligned} \tag{79}$$

Notice that $\phi, \breve{\phi}$ can be both negative or positive. So if we define $\psi = e^{R+iS}$, it then defines in weak form the Schroedinger equation in D with

$$V = -c - 2\frac{\partial S}{\partial t} - g(dS_t, dS_t). \tag{80}$$

Remarks. We note that from eq. (80) follows that we can choose S in a way such that either c is independent of S and thus V is a potential which is non-linear in the sense that it depends on the phase of the wave function ψ and thus the Schroedinger equation with this choice becomes non-linear dependent of ψ, or conversely, we can make the alternative choice of c depending non-linearly on S, and thus the creation-annihilation of particles in the diffusion equation is non-linear, and consequently the Schroedinger equation has a potential V which does not depend on ψ. It is important for further developments in this article that the non-linear Schroedinger equation can be turned into the iso-linear iso-Schroedinger equation by taking the non-linear terms of the potential into the isotopic generalized unit. Indeed, the recovery of linearity in isohilbert space is achieved by the embedding of the nonlinear terms in the isounit as shown in [17]; see eqs. (3.4.42) and (3.4.43).

4.1 Santilli-Lie Isotopies of the Differential Calculus and Metric Structures, and the Iso-Schroedinger Equation

To present the iso-Schroedinger equation, we need the Santilli-Lie-isotopic differential calculus [16,17] and the isotopic lift of manifolds, the so-called iso-manifolds, due to Tsagas and Sourlas [20]; we shall follow here the notations of Section 3.2 above. We start by considering the manifold M to be a vector space with local coordinates, which for simplicity we shall from now fix them to be a *contravariant* system, $x = (x^i), i = 1, \ldots, n$, unit given by $I = \mathrm{diag}(1, \ldots, 1)$ and metric g which we assumed diagonalized. We shall lift this structure to a vector space $\hat{M}$ provided with isocoordinates $\hat{x}$, isometric $\hat{G}$ and defined on the isonumber field $\hat{F}$, where F can be the real or complex numbers; we denote this isospace by $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. The isocoordinates are introduced by the transformation $x \mapsto U \times x \times U^\dagger = x \times \hat{I} := \hat{x}$. To introduce the *contravariant* isometric $\hat{G}$ we start by considering the transformation [5]

$$g \mapsto U \times g \times U^\dagger = \hat{I} \times g := \hat{g}. \tag{81}$$

Yet from the Definition 3.2.3 follows that the isometric is more properly defined by $\hat{G} = \hat{g} \times \hat{I}$. Thus we have a transformed $M(x, g, F)$ into the isospace $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. Thus the projection on $M(x, g, F)$ of the isometric in $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ is defined by a contravariant tensor, $\hat{g} = (\hat{g}^{ij})$ with components

$$\hat{g}^{ij} = (\hat{I} \times g)^{ij}. \tag{82}$$

If we take $\hat{I} = \psi^2(x) \times I$ we then retrieve the Weyl scale transformations, with ψ a scale field depending only on the coordinates of M. If we start with g being the Euclidean or Minkowski metrics, we obtain the iso-Euclidean and iso-Minkowski metrics; in the case we start with a general metric as in GR, we

[5]We shall assume, as usual, a diagonal metric.

obtain Isorelativity. We shall now proceed to identify the isotopic lift of the noise tensor σ which verifies eq. (9), i.e. $\sigma \times \sigma^\dagger = g$. The non-unitary transform of (a diagonalized) σ is given by

$$\sigma \mapsto U \times \sigma \times U^\dagger = \sigma \times \hat{I} := \hat{\sigma}. \tag{83}$$

Then,

$$\hat{\sigma} \hat{\times} \hat{\sigma} = \sigma \times \hat{I} \times \hat{T} \times (\sigma \times \hat{I})^\dagger = (\sigma \times \sigma^\dagger) \times \hat{I} = g \times \hat{I} = \hat{g}. \tag{84}$$

Thus the isotopic lift of the noise tensor defined on $\hat{M}(\hat{x}, \hat{G}, \hat{R})$ is given by $\hat{\sigma} = \sigma \times \hat{I}$ which on projection to $M(\hat{x}, \hat{G}, R)$ we retrieve σ. We know follow the notations and definitions of Section 3.2.5 for the isotopic differential, and for isofunctions. We introduce the isotopic gradient operator of the isometric $\hat{G}$ (the $\hat{G}$-gradient, for short), $\widehat{\text{grad}}_{\hat{\mathrm{G}}}$ applied to the isotopic lift $\hat{f}(\hat{x})$ of a function $f(x)$ is defined by

$$\widehat{\text{grad}}_{\hat{\mathrm{G}}} \hat{f}(\hat{x})(\hat{v}) = \hat{G}(\hat{d}\hat{f}(\hat{x}) \hat{;} \hat{v}), \tag{85}$$

for any vector field $\hat{v} \in T_{\hat{x}}(\hat{M})$, $\hat{x} \in \hat{M}$; we have denoted the inner product as $\hat{;}$ to stress that the inner product is taken with respect to the product in $\hat{F}$. Hence, the operator $\widehat{\text{grad}}_{\hat{\mathrm{G}}} \hat{f}(\hat{x})$ can be thought as the isovector field on the tangent manifold to $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ defined by

$$\hat{G}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^\alpha} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^\beta} = \hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^\alpha} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^i} \times \hat{I}. \tag{86}$$

Therefore, the projection on $\hat{M}(\hat{x}, \hat{g}, F)$ of the $\hat{G}$-gradient vector field of $\hat{f}(\hat{x})$ is the vector field with components

$$\hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^\alpha} = \hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^\alpha}. \tag{87}$$

This will be of importance for the determination of the drift vector field of the diffusion linked with the Santilli- iso-Schroedinger equation. We finally define the isolaplacian as

$$\hat{\triangle}_{\hat{g}} = \hat{g}^{\alpha\beta} \hat{\times} \hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^\alpha}} \hat{\times} \hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^\beta}} \tag{88}$$

Here $\hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^\alpha}}$ is defined accordingly with Definition 3.2.13 above, by (c.f. eq. (6) above)

$$\hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^\alpha}} \hat{X}^\beta = \frac{\hat{\partial} \hat{X}^\beta}{\hat{\partial} \hat{x}^\beta} + \widehat{\begin{Bmatrix} \beta \\ \gamma\alpha \end{Bmatrix}} \hat{\times} \hat{X}^\gamma, \tag{89}$$

and hence it is the isocovariant differential with respect to the Levi-Civita isoconnection with isoChristoffel coefficients

$$\widehat{\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}} = \frac{\hat{1}}{\hat{2}} \left(\frac{\hat{\partial}}{\hat{\partial} \hat{x}^\beta} \hat{g}_{\nu\gamma} + \frac{\hat{\partial}}{\hat{\partial} \hat{x}^\gamma} \hat{g}_{\beta\nu} - \frac{\hat{\partial}}{\hat{\partial} \hat{x}^\nu} \hat{g}_{\beta\gamma} \right) \hat{\times} \hat{g}^{\alpha\nu}. \tag{90}$$

We remark that from Observations 1 follows that alternatively we can define the more simpler laplacian by taking instead

$$\hat{\triangle}_{\hat{g}} = \hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\beta}}. \tag{91}$$

In both cases we take $\hat{\sigma}$ for the corresponding isonoise term in the isodiffusion representation. The latter definition of the isolaplacian differs from the original one introduced in [20].

4.2 Diffusions and the Heisenberg Representation

Up to now we have set our theory in terms of the Schroedinger representation, since the original setting for this theory has to do with scale transformations as introduced by Einstein in his last work [7] while it was recognized previously by London that the wave function was related to the Weyl scale transformation [48], and these scale fields turned to be in the non-relativistic case, nothing else than the wave function of Schroedinger equation, both in the linear and the non-linear cases. Historically the operator theory of QM was introduced before the Schroedinger equation, who later proved the equivalence of the two. The ensuing dispute and rejection by Heisenberg of Schroedinger's equation is a dramatic chapter of the history of QM [35]. It turns out to be the case that we can connect the Brownian motion approach to QM and the operator formalism due to Heisenberg and Jordan, and its isotopic lift presented in Section 3.4.

Let us define the position operator as usual and the momentum operator by

$$q^k = x^k, p_{\mathcal{D}k} = \sigma \times \frac{\partial}{\partial x^k}, \tag{92}$$

which we call the diffusion quantization rule (the subscript $\mathcal{D}$ denotes diffusion) since we have a representation different to the usual quantization rule

$$p_k = -i \times \frac{\partial}{\partial x^k}, \tag{93}$$

with $\sigma = (\sigma_a^\alpha)$ the diffusion tensor verifying $(\sigma \times \sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$ and substitute into the Hamiltonian function

$$H(p,q) = \frac{1}{2}\sum_{k=1}^{d}(p_k)^2 + \mathbf{v}(q), \tag{94}$$

this yields the formal generator of a diffusion semigroup in $C^2(R^d)$ or $L^2(R^d)$ which in our previous notation is written as $H(g,0) + v$. Thus, an operator algebra on $C^2(R^n)$ or $L^2(R^n)$ together with the postulate of the commutation relation (instead of the usual commutator relation of quantum mechanics $[p, q] = -i \times I$)

$$[p_{\mathcal{D}}, q] = p_{\mathcal{D}} \times q - q \times p_{\mathcal{D}} = \sigma \times I \tag{95}$$

this yields the diffusion equation

$$\frac{\partial \phi}{\partial t} \times \phi + \frac{1}{2} \sum_{k=1}^{d} (\sigma \frac{\partial}{\partial x^\alpha})^2 \times \phi + \mathbf{v} \times \phi = 0, \tag{96}$$

which coincides with the diffusion eq. (54) provided that $c = \mathbf{v}$. Thus, in this approach, the operator formalism and the quantization postulates, allow to deduce the diffusion equation. If we start from either the diffusion process or the RCW geometry, without any quantization conditions we already have the equations of motion of the quantum system which are non other than the original diffusion equations, or equivalently, the Schroedinger equations. We stress the fact that these arguments are valid for both cases relative to the choice of the potential function V, i.e. if it depends nonlinearly on the wave function ψ, or acts linearly by multiplication on it. Further below, we shall use this modification of the Heisenberg representation of QM by the previous Heisenberg type representation for diffusion processes, to give an account of the diffusion processes that are associated with HM. This treatment differs from our original (inconsistent with respect to HM, as it turned to be proved in the later findings by Prof. Santilli) treatment of the relation between RCW geometries and diffusions presented in [29] in incorporating the isotopic lift of all structures.

Let us frame now isoquantization in terms of diffusion processes. Define isomomentum, $\hat{p}_{\mathcal{D}}$, by

$$\hat{p}_{\mathcal{D}k} = \hat{\sigma} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^k}, \text{ with } \hat{\sigma} = \sigma \times \hat{I}, \tag{97}$$

so that the kinetic term of the iso-Hamiltonian is

$$\begin{aligned} \hat{p}_{\mathcal{D}} \hat{\times} \hat{p}_{\mathcal{D}}^\dagger &= \hat{\sigma} \hat{\times} \hat{\sigma}^\dagger \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} \\ &= \hat{g} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} = \hat{\triangle}_{\hat{g}} \end{aligned} \tag{98}$$

We finally check the consistency of the construction by proving that it can be achieved via the non-unitary transformation

$$\begin{aligned} p_{\mathcal{D}_j} \mapsto U \times p_{\mathcal{D}_j} \times U^\dagger &= U \times \sigma \times \frac{\partial}{\partial x^j} \times U^\dagger \\ &= \sigma \times \hat{I} \times \hat{T} \times \hat{I} \times \frac{\partial}{\partial x^j} = \hat{\sigma} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^j} = \hat{p}_{\mathcal{D}j}. \end{aligned} \tag{99}$$

Note that we have achieved this isoquantization in terms of the following transformations: Firstly, we carried out the transformation

$$p = -i \times \frac{\partial}{\partial x} \to p_{\mathcal{D}} := \sigma \times \frac{\partial}{\partial x}, \tag{100}$$

to further produce its isotopic lift

$$\hat{p}_{\mathcal{D}} = \hat{\sigma} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}}. \tag{101}$$

Whenever the original diffusion tensor σ is the identity I, from eq. (9) follows that the original metric g is Euclidean, we reach compatibility of the diffusion quantization with the Santilli-iso-Heisenberg representation given by taking the non-unitary transformation on the canonical commutation relations, which are given by

$$[\hat{q}^i \hat{,} \hat{p}_j] \quad = \quad \hat{i} \hat{\times} \hat{\delta}^i_j = i \times \delta^i_j \times \hat{I}, \tag{102}$$

together with

$$[\hat{r}^i \hat{,} \hat{r}^j] \quad = \quad [\hat{p}_i \hat{,} \hat{p}_j] = 0, \tag{103}$$

with the Santilli-iso-quantization rule [16,17]

$$\hat{p}_j = -\hat{\imath} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^j}. \tag{104}$$

Thus, from the quantization by the diffusion representation we retrieve the Santilli-iso-Heisenberg representation, with the difference that the diffusion noise tensor in the above construction need not be restricted to the identity.

Finally, we consider the isoHamiltonian operator

$$\hat{H} = \frac{\hat{1}}{\hat{2} \hat{\times} \hat{m}} \hat{\times} \hat{p}^{\hat{2}} + \hat{V}_0(\hat{t}, \hat{x}) + \hat{V}_k(\hat{t}, \hat{v}) \hat{\times} \hat{v}^k, \tag{105}$$

where $\hat{p}$ may be taken to be given either by the Santilli isoquantization rule

$$\hat{p}_k \hat{\times} |\hat{\psi} >= -\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^k} \hat{\times} |\hat{\psi} >, \tag{106}$$

or by the diffusion representation $\hat{p}_{\mathcal{D}}$. $\hat{V}_0(\hat{t}, \hat{x})$ and $\hat{V}_k(\hat{t}, \hat{v})$ are potential isofunctions, the latter dependent on the isovelocities. Then the iso-Schroedinger equation (or Schroedinger-Santilli isoequation) [16,17] is

$$\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{t}} |\hat{\psi} >= \hat{H} \hat{\times} |\psi > \ = \hat{H}(\hat{t}, \hat{x}, \hat{p}) \times \hat{T}(\hat{t}, \hat{x}, \hat{\psi}, \hat{\partial} \hat{\psi}, \ldots) \times |\hat{\psi} >, \tag{107}$$

where the wave isofunction $\hat{\psi}$ is an element in $(\hat{\mathcal{H}}, < |\hat{\times}| >, \hat{C}(\hat{c}, \hat{+}, \hat{\times}))$ satisfies

$$\hat{I} \hat{\times} |\hat{\psi} >= |\hat{\psi} > .$$

4.3 Hadronic Mechanics and Diffusion Processes

Finally, the components of drift isovector field, projected on $\hat{M}(\hat{x},\hat{g},R)$ in the isotopic lift of eq. (63) is given by eq. (87) with $\hat{f} = \hat{\ln}\hat{\phi}$, where $\hat{\phi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x})+\hat{\mathcal{S}}(\hat{x})}$ is the diffusion wave associated to the solution $\hat{\psi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x})+i\hat{\mathcal{S}}(\hat{x})}$ of the iso-Schroedinger equation, and its adjoint wave is $\check{\hat{\phi}}(x) = \hat{e}^{\hat{\mathcal{R}}(x)-\hat{\mathcal{S}}(x)}$. Hence, the drift isovector field has components

$$\hat{g}^{\alpha\beta}(\hat{x})\hat{\times}\frac{\hat{\partial}\hat{\ln}\hat{\phi}(\hat{x})}{\hat{\partial}\hat{x}^{\alpha}} = \hat{g}^{\alpha\beta}(\hat{x})\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}(\hat{\mathcal{R}}_{\hat{t}}\hat{+}\mathcal{S}_{\hat{t}})(\hat{x}), \tag{109}$$

Finally, we shall write the isotopic lift of the stochastic differential equation for the iso-Schroedinger eq. (107). Applying the non-unitary transformation to eq. (63), we obtain the iso-equation on $\hat{M}(\hat{x},\hat{G},\hat{R})$ for $\hat{X}_{\hat{t}}$ given by

$$d\hat{X}^{i}_{\hat{t}} = ((\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}(\hat{\mathcal{R}}_{\hat{t}}\hat{+}\mathcal{S}_{\hat{t}}))(\hat{X}_{\hat{t}})\hat{\times}\hat{d\hat{t}} + \hat{\sigma}^{i}_{j}(\hat{X}_{\hat{t}})\hat{\times}d\hat{W}^{j}_{\hat{t}}, \tag{110}$$

with $d\hat{W}_{\hat{t}} = \hat{W}(\hat{t}\hat{+}\hat{d\hat{t}})\hat{-}\hat{W}(\hat{t})$ the increment of a iso- Wiener process $\hat{W}_{\hat{t}} = (\hat{W}^{1}_{\hat{t}},\ldots,\hat{W}^{m}_{\hat{t}})$ with isoaverage equal to $\hat{0}$ and isocovariance given by $\hat{\delta}^{i}_{j}\hat{\times}\hat{t}$; i.e.,

$$\hat{1}\hat{/}(\hat{4}\hat{\times}\hat{\pi}\hat{\times}\hat{t})^{\hat{m}\hat{/}\hat{2}}\hat{\int}\hat{w}_{i}\hat{\times}\hat{e}^{-\hat{w}^{\hat{2}}\hat{/}\hat{4}\hat{\times}\hat{t}^{\hat{2}}}\hat{\times}\hat{d}\hat{w} = \hat{0}, \quad \forall i = 1,\ldots,m \tag{111}$$

and

$$\hat{1}\hat{/}(\hat{4}\hat{\times}\hat{\pi}\hat{\times}\hat{t})^{\hat{m}\hat{/}\hat{2}}\hat{\int}\hat{w}_{i}\hat{\times}\hat{w}_{j}\hat{\times}\hat{e}^{-\hat{w}^{\hat{2}}\hat{/}\hat{4}\hat{\times}\hat{t}^{\hat{2}}}\hat{\times}\hat{d}\hat{w} = \hat{\delta}^{i}_{j}\hat{\times}\hat{t}, \quad \forall i,j = 1,\ldots,m \tag{112}$$

and $\hat{\int}$ denotes the isotopic integral defined by $\hat{\int}\hat{d}\hat{x} = (\int\hat{T}\times\hat{I}\times dx)\times\hat{I} = (\int dx)\times\hat{I} = \hat{x}$. Thus, formally at least, we have

$$\hat{X}_{\hat{t}} = \hat{X}_{\hat{0}}\hat{+}\hat{\int}_{\hat{0}}^{\hat{t}}(\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}(\hat{\mathcal{R}}_{\hat{s}}\hat{+}\mathcal{S}_{\hat{s}}))(\hat{X}_{\hat{s}})\hat{\times}\hat{d}\hat{s} + \hat{\int}_{\hat{0}}^{\hat{t}}\hat{\sigma}^{i}_{j}(\hat{X}_{\hat{s}})\hat{\times}d\hat{W}^{j}_{\hat{s}}. \tag{113}$$

The integral in the first term of eq. (113) is an isotopic lift of the usual Riemann-Lebesgue integral [16d,20a], while the second one is the isotopic lift of a stochastic Itô integral; we shall not present here in detail the definition of this last term, which follows from the notions of convergence in the isofunctional analysis elaborated by Kadeisvili [19] (see Section 3.2.6), and the usual definition of Itô stochastic integrals [9,14,26], nor the presentation of analytical conditions for their convergence which follows in principle from the isotopic lift of the usual conditions.

4.4 The Extension to The Many-body Case

Up to know we have presented the case of the Schroedinger equation for an ensemble of one-particle systems on space-time. Of course, our previous constructions are also valid for the case of an ensemble of interacting multiparticle systems, so that the dimension of the configuration space is $3d + 1$, for indistinguishable d particles; the general case follows with minor alterations. If we start by constructing the theory as we did for an ensemble of one-particle systems (Schroedinger's cloud of electrons), we can still extend trivially to the general case, by considering a diffusion in the product configuration manifold with coordinates $X_t = (X_t^1, \ldots, X^d) \in M^d$, where M^d is the d Cartesian product of three dimensional space with coordinates $X_t^i = (x_t^{1,i}, x_t^{2,i}, x_t^{3,i}) \in M$, for all $i = 1, \ldots, d$. The distribution of this is $\mu_t = E_Q \circ X_t^{-1}$, which is a probability density in M^d. To obtain the distribution of the system on the three-dimensional space M, we need the distribution of the system X_t:

$$U_t^x := \frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}. \tag{114}$$

which is the same as

$$U_t^x(B) = \frac{1}{d} \sum_{i=1}^{d} 1_B(X_t^i), \tag{115}$$

where $1_B(X_t^i)$ is the characteristic system for a measurable set B , equal to 1 if $X_t^i \in B$, for any $i = 1 \ldots, d$ and 0 otherwise. Then, the probability density for the interacting ensembles is given by

$$\mu_t^x(B) = E_Q[U_t^x(B)], \tag{116}$$

where E_Q is the mean taken with respect to the forward Kolmogorov representation presented above, is the probability distribution in the three-dimensional space; see [14]. Therefore, the geometrical-stochastic representation in actual space is constructable for a system of interacting ensembles of particles. Thus the criticism to the Schroedinger equation by the Copenhagen school, as to the unphysical character of the wave function since it was originally defined on a multiple-dimensional configuration space of interacting system of ensembles, is invalid [35].

5 Possible Empirical Evidence and Conclusions

We have shown that the Schroedinger and isoSchroedinger equation have an equivalent representation in terms of diffusion processes. This can be further extended to hadronic chemistry, as shown in the previous section. This is an universal phenomenae since the applicability of the Schroedinger equation does not restrict to the microcospic realm, as already shown in the astrophysical theory due to Nottale [11]; this universality is associated with the fact that the Planck constant (or equivalently, the diffusion constant) is multivalued, or still, it is context dependent, inasmuch as the velocity of light has the same feature [17]. In the case of HM this can be seen transparently in the fact that the isotopic unit plays the role, upon quantization, of the Planck constant as can seen in eqs. $(107, 108)$ [6], or furthermore, by its product with the noise tensor of the underlying Brownian motions.In the galactic scales, this may explain the red-shift without introducing a big-bang hypothesis [17,18]. An identical conclusion was reached by Arp in considering as a theoretical framework the Le Sage's model of a Universe filled with a gas of particles [40], in our theory, the zero-point fluctuations described by the Brownian motions defined by the wave functions, as well as by viscous fluids, spinor fields, or electromagnetic fields [2] (and which one can speculate as related to the so-called dark energy problem). A similar view has been proposed by Santilli in which the elementary constituents are the so-called aetherinos [59], while in Sidharth's work, they appear to be elementary quantized vortices related to quantum-mechanical Kerr-Newman black holes [29]. Thus, whether we examine the domains of linear or non-linear quantum mechanics, or still of hadronic mechanics, vortices and superconductivity (which is the case of the Rutherford-Santilli model of the neutron which is derived from the previous constructions) appear as universal coherent structures; superconductivity is usually related to a non-linear Schroedinger equation with a Landau-Ginzburg potential, which is just an example of the Brownian motions related to torsion fields with further noise related to the metric. Furthermore, atoms and molecules have spin-spin interactions which will produce a contribution to the torsion field; we have seen already that the torsion geometry exists in the realm of hadronic chemistry, since we can extend the construction to the many-body case. In distinction with the usual repulsive Coulomb potential in nuclear physics, the isotopic deformations of the nuclear symmetries yield attractive potentials such as the Hulten potential, which in the range of 10^{-13} cm. yields the usual potential [15-20,51] without the need of introducing any sort of parameters or extra potentials. In contrast with the ad-hoc postulates of randomness in the fusion models which are considered in the usual approaches [32,33], in the present work randomness is intrinsic to space-time or alternatively a by product of it, and in the case of HM, these geometries incorporate at a foundational level, a generalized unit which incorporates all the features of the fusion process itself: the non-canonical, non-local and non-linear overlapping of the wave functions of the ensembles which correspond to the separate ensembles under deformable collisions in which the particles lose their point-like structure, or in a hypercondensed plasma state, where the dynamics of the process may have a random behavior; outside of the domain of 10^{-13} cm., the hadronic fluctuations associated to the isolinear isoSchroedinger equation decay

[6] See Postulate 3.4.1.

to the quantum fluctuations of the linear Schroedinger equation.

There are already empirical findings that may lead to validate the present view. In the last fifty years, a team of scientists at the Biophysics Institute of the Academy of Sciences of Russia, directed by S. Shnoll (and presently developed in a world net which includes Roger Nelson, Engineering Anomalies Research, Princeton University, B. Belousov, International Institute of Biophysics, Neuss (Germany), Dr. Wilker, Max-Planck Institute for Aeronomy, Lindau, and others), have carried out tens of thousands of experiments of very different nature and energy scales (α decay, biochemical reactions, gravitational waves antenna, etc.) in different points of the globe, and carried out a software analysis of the observed histograms and their fluctuations, to find out an amazing fit which is repeated with regularity of 24 hours, 27 days and the duration of a sidereal year. In these experiments the fine spectrum of their measurements reveal a non-random pattern. At points of Earth with the same local hour, these patterns are reproduced with the said periodicity. The only thing in common to these experiments is that they are occur in space-time, which has lead to conclude that they stem from space-time fluctuations, which may further be associated with cosmological fields. Furthermore, the histograms reveal a fractal structure; this structure is interpreted as appearing from an interference phenomena related to the cosmological field; we recall that diffusion processes present interference phenomena alike to , say, the two-slit experiment.[7]. Measurements taken with collimators show fluctuations emerging from the rotation of the Earth around its axis or its circumsolar orbit, showing a sharp anisotropy of space. Furthermore,it is claimed that the spatial heterogeinity occurs in a scale of 10^{-13} cm., coincidently with the scale of the strong interactions [62]. Contrary to common belief, the Michelson-Morley did not provide a final dismissal of the aether, while Einstein in the course of his life supported the idea of its existence [64]. Thousands of interferometry experiments were carried out by D. Miller, Allais and others, and contemporarily very diverse setups have proved that there is a space anisotropy [63]. As a closing remark we would like to recall that Planck himself proposed the existence of ensembles of random phase oscillators having the zero-point structure as the basis for quantum physics [56]. Thus, the apeiron would be related to the Brownian motions which we have presented in this work, and define the space and time geometries, or alternatively, are defined by them. So we are back to the idea due to Clifford, that there is no-thing but space and time configurations, instead of a separation between substratum and fields and particles appearing on it. Furthermore, what we perceive to be void, is the hyperdense source of actuality. The same conception has been proposed

[7]This fractal structure has been found to follow the pattern of the logarithmic Muller fractal, which is associated with the existence of a global scale for all structures in the Universe; see H. Muller, Free Energy - Global Scaling, *Raum& Zeit Special 1*, Ehlers-Verlag GmbH, ISBN 3-934-196-17-9; 2004. This leads to reinforce the thesis of time as an active field. Furthermore, the space and time Brownian motions can exist, in principle, in the different space and time scales warranted by these global scales.

by Prof. Santilli in the main body of this volume.

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