On Novel Static Spherically Symmetric Solutions of Einstein’s equations and the Cosmological Constant Problem

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Abstract

A novel class of static spherically symmetric (SSS) solutions of Einstein’s equations are explicitly constructed in terms of a family of admissible radial functions \( R = R(r) \). The proper generalizations of the (Anti) de Sitter solutions are presented that could provide a very plausible resolution of the cosmological constant problem along with a natural explanation of the ultraviolet/infrared (UV/IR) entanglement required to solve this problem. A nonvanishing value of the vacuum energy density of the order of \( 10^{-121} M_{\text{Planck}}^4 \) is derived in perfect agreement with the experimental observations. Exact solutions of the cubic equations associated with the location of the horizons of a novel class of (Anti) de Sitter-Schwarzschild metrics are found. These solutions are very appealing because one could interpret \( M \) as the mass of an unbounded universe (since the range of values for \( r \) are \( 0 \leq r \leq \infty \)) and whose magnitude of the cosmological constant is \( \lambda = R_H^2 \) when \( R_H = R_{\text{Hubble(today)}} \). In addition we obtain a lower bound to the mass of the universe of the order \( 2M \sim 10^{61} M_{\text{Planck}} \sim 10^{80} m_{\text{proton}} \) that agrees with the Dirac-Eddington large number result.

1 On the class of Schwarzchild Solutions

We begin by writing down the class of static spherically symmetric (SSS) vacuum solutions of Einstein’s equations studied by Abrams [5] (where there are NO mass sources anywhere) given by a infinite family of solutions parametrized by a family of admissible radial functions \( R(r) \)

\[
(ds)^2 = g_{00} (dt)^2 + g_{rr} (dr)^2 - (R(r))^2 (d\Omega)^2
\]  

(1.1)
the solid angle infinitesimal element is
\[(d\Omega)^2 = \sin^2(\phi)(d\theta)^2 + (d\phi)^2.\] (1.2)

and
\[g_{00} = (1 - \frac{\alpha}{R(r)})\]
\[g_{rr} = -(1 - \frac{\alpha}{R(r)})^{-1} \left( \frac{dR(r)}{dr} \right)^2.\] (1.3)

where \(\alpha\) is an arbitrary constant that happens to have dimensions of mass when \(c = G = 1\) (but there are no masses at all in this vacuum case) and \(R(r)\) are an infinite family of functions like
\[R(r) = r + \alpha; \quad R(r) = [r^3 + \alpha^3]^{1/3}; \quad R(r) = [r^n + \alpha^n]^{1/n}; \quad R(r) = \frac{\alpha}{1 - e^{-\alpha/r}} \ldots (1.4)\]

found by Brouill\[n\] [3], Schwarzschild [2], Crothers [7] and Fiziev-Manev [12], respectively obeying the conditions that
\[R(r = 0) = \alpha; \quad \text{and when } r \gg \alpha \Rightarrow R(r) \rightarrow r \] (1.5)

Numerous authors have corroborated over the years through lengthy but straightforward calculations [5], [6], [7], [8], [9] that there exist an infinite class of solutions to the vacuum SSS Einstein’s equations \(R_{\mu\nu} = R = 0\) for an arbitrary family of radial functions \(R(r)\) of the type displayed above (but the curvature Riemann tensor \(R^\mu_{\nu\rho\sigma} \neq 0\)). This arbitrary family of radial functions \(R(r)\) resemble the travelling wave Maxwell solutions in the vacuum that are given by arbitrary functions of \(x - ct\) and \(x + ct\) like \(\Phi = f(x - ct) + f(x + ct)\).

One may notice that the Hilbert-Droste-Weyl form of the metric [4] when \(\alpha = 2M\) is obtained automatically from the solutions given by eq-(1.1) at large distances \(r\) compared to \(2M\) such that
\[r \gg 2M \Rightarrow R(r) \rightarrow r\] (1.6)
that yields
\[(ds)^2 \sim g_{00} (dt)^2 + g_{rr} (dr)^2 - (r)^2 (d\Omega)^2\] (1.7)
with
\[g_{00} \rightarrow (1 - \frac{2M}{r})\] (1.8)
\[g_{rr} \rightarrow -(1 - \frac{2M}{r})^{-1} \]

(1.9)

Because the behaviour of eqs-(1.7-1.9) is only valid for \(r \gg 2M\) it is meaningless and contradictory to claim that when one (erroneously) equates \(r = 2M\) (since this would invalidate the previous condition \(r \gg 2M\)) one has \(g_{00} = 0\) at the so-called “horizon” \(r = 2M\). Notice that when one sets \(r = 2M\) in the genuine and true Schwarzschild solution the value of \(g_{00} \neq 0\). Such behaviour \(g_{00} = 0\) in eqs-(1.1-1.3) only occurs at:
\[ r = 0 \Rightarrow R(r = 0) = 2M \Rightarrow g_{00}(r = 0) = 0 \quad (1.10) \]

It was explicitly shown in [16] why one must introduce the absolute values \(|r|\) in these solutions eqs-(1.1-1.3) in order to account for the field of a point mass source at \( r = 0 \). This is the key reason why the vacuum SSS solutions (no mass sources anywhere) differ crucially from the solutions due to a point mass source. There is a true curvature singularity at \( r = 0 \) (at the location of the point mass source) due to a delta function. This is not the case of the true vacuum solutions (no mass sources anywhere so \( \mathcal{R}_{\mu\nu} = \mathcal{R} = 0 \)).

Thus, when one goes ahead and inserts a point mass \( M \) source at \( r = 0 \) matters will change drastically. In order to find the correct solutions to the modified Einstein’s equations in units of \( G = c = 1 \) one must write those radial functions \( R(\mathcal{R}) \) by replacing the \( r \) variable by its modulus \(|r|\) in eqs-(1.1-1.3) This is what will account for the delta function terms \( \delta(r) \) in Einstein’s equations resulting from the singularity at \( r = 0 \). A different and detailed treatment of point masses, point charges, delta function sources and the physical implications of the many different choices of the radial functions \( R(r) \) in General Relativity has been given by Fiziev [12].

There is no region with \( R < 2M \) in this point mass case. This makes sense since a point mass located at \( r = 0 \) has no interior by definition. As trite as this argument may be it lies at the root of the so-called Hilbert’s ”error” [5] and whose legacy led to black-holes research. The gist of the argument by Abrams [5] lies in the fact that Hilbert projected the timelike one-dimensional worldline of the physical \( r = 0 \) singularity onto the whole null horizon surface of cylindrical topology \( R \times S^2 \) by mapping the point \( r = 0 \) onto the whole sphere \( R = 2M \).

To illustrate how relevant it is to take the proper absolute values, we recall (in flat space) that the Laplacian in spherical coordinates of \( 1/|r| \) is

\[
\frac{1}{r^2}(d/dr)[r^2(d/dr)(1/|r|)] = \frac{1}{r^2}(d/dr)[r^2(-1/|r|^2) \text{ sign}(r)] =
\]

\[-\frac{1}{r^2}(d/dr)\text{sign}(r) = -(1/r^2) \delta(r) \quad (1.11)\]

since \( r^2 = |r|^2 \), which is consistent with Poisson’s law which states that the Laplacian of the Newtonian potential \(-GM/|r|\) is \( 4\pi G \rho \). This is true here if, and only if, \( \rho = (M/4\pi r^2)\delta(r) \) that is indeed the case in Newtonian gravity. However, the Laplacian in spherical coordinates of \((1/r)\) is zero. For this reason, there is a fundamental difference in dealing with expressions involving absolute values \(|r|\) like \( 1/|r| \) from those which depend on \( r \) like \( 1/r \) [10].

In [16] it was shown that when one inserts a point mass \( M \) at \( r = 0 \), the class of SSS solutions (in \( G = c = 1 \) units) given by the genuine Schwarzschild form, and in terms of radial functions \( R(|r|) \), lead to the Einstein’s equations (for signature +, −, −, −) of the form

\[
G_{00} = \frac{2}{R(dR/d|r|)} \left(1 - \frac{2M}{R}\right)^2 \delta(r) = -8\pi T_{00} \neq 0. \quad (12a)
\]
\[ G_{rr} = -8\pi T_{rr} = 0. \]  

\[ G_{\theta\theta} = -\frac{R \sin^2 \phi}{(dR/d|r|)} \left( 1 - \frac{2M}{R} \right) \delta(r) - \frac{1}{2} \frac{2M \sin^2 \phi}{(dR/d|r|)} \delta(r) = -8\pi T_{\theta\theta} \neq 0. \]  

\[ G_{\phi\phi} = -\frac{R}{(dR/d|r|)} \left( 1 - \frac{2M}{R} \right) \delta(r) - \frac{1}{2} \frac{2M}{(dR/d|r|)} \delta(r) = -8\pi T_{\phi\phi} \neq 0. \]

meaning that the singular mass distribution at \( r = 0 \) has pressure terms in addition to a pure mass density term.

Since \( R(r = 0) = 2M \) at the location of the singularity \( r = 0 \) the scalar curvature obeys the trace condition [16]

\[ \mathcal{R} = - \left[ \frac{2M}{R^2(dR/d|r|)} + \frac{4}{R(dR/d|r|)} \left( 1 - \frac{2M}{R} \right) \right] \delta(r) = - \left[ \frac{2M}{R^2(dR/d|r|)} \right] \delta(r) = 8\pi \left[ g^{00}T_{00} + g^{rr}T_{rr} + g^{\theta\theta}T_{\theta\theta} + g^{\phi\phi}T_{\phi\phi} \right]. \]  

(1.12e)

In order to evaluate the curvature tensor, the Ricci tensor and the scalar curvature based on the solutions (1.1-1.3) obtained by replacing \( r \to |r| \) and \( \alpha = 2M \) one must use properly the rules of derivatives of quantities involving absolute values as indicated below:

\[(d|r|/dr) = \text{sign}(r). \quad \text{sign}(r) = +1, \ r > 0. \quad \text{sign}(r) = -1, \ r < 0 \]  

(1.13)

and \( \text{sign}(0) \) is ill defined since there is a discontinuity of the derivative of \( |r| \) at \( r = 0 \); i.e a discrete jump from \(-1\) to \(+1\). For this reason the second derivatives

\[ \frac{d^2|r|}{dr^2} = \frac{d\text{sign}(r)}{dr} = \delta(r) \]  

(1.14)

furnish the required delta function term. The derivatives of the metric elements w.r.t the variable \( r \) are attained by a simple use of the chain rule

\[ (dg_{00}(R(|r|)/dr) = (dg_{00}(R)/dR) \ (dR/d|r|) \ (d|r|/dr) = (dg_{00}(R)/dR) \ (dR/d|r|) \ \text{sign}(r) \]  

(1.15)

Thus the second derivative terms \( (d/dr)^2 g_{00}(|R|) \) yields a term containing a delta function stemming from the \( \text{sign}(r) \) term giving:

\[(dg_{00}(R)/dR) \ (dR/d|r|) \ \delta(r) \]  

(1.16)

etc............

Such \( \delta(r) \) terms would never appear had one used \( R = R(r) \). Unfortunately, these \( \delta(r) \) factors have not been properly accounted for by many authors [7] and erroneously misled them to conclude that there are no curvature singularities in General Relativity ( \( \mathcal{R} \) was erroneously claimed to be identically zero everywhere ) when we know this cannot be true. There is a delta function singularity at \( r = 0 \) which originates from properly using the modulus \( |r| \) in the radial functions \( R(|r|) \) instead of erroneously writing \( R = R(r) \) [10].
After a lengthy computation the scalar curvature was shown [16] to be given by:

\[ \mathcal{R} = - \left[ \frac{2M}{R^2(dR/d|r|)} \right] \delta(r). \]  

(1.17)

and since \( R(r = 0) = 2M \) it is consistent with the trace condition in eq-(1.12c). One may notice that if, and only if, the measure (Jacobian) \( \sqrt{|g|} = R^2(dR/d|r|) \) \( \sin \phi = r^2 \sin \phi \) coincides with the ordinary measure in spherical coordinates, then the scalar curvature when one sets \( R(r = 0) = 2M \) in eq-(1.18) becomes

\[ \mathcal{R} = \frac{-2M}{R^2(dR/d|r|)} \delta(r) = -\frac{2M}{r^2} \delta(r) \Rightarrow \int R^2 dR = \int r^2 d|r| = \int |r|^2 d|r| \Rightarrow \]

\[ R^3 = |r|^3 + \text{constant} \]  

(1.19)

the boundary condition \( R(|r| = 0) = 2M \) leads precisely to the genuine cubic solution \( R^3 = |r|^3 + (2M)^3 \) found by Schwarzschild [2].

In order to check the consistency of eqs-(1.12) one must solve Einstein’s equations for a point mass, and begin firstly by writing the components of \( T_{\mu \nu} \) associated with a point mass particle which is moving in its own gravitational background in terms of the appropriately defined covariantized delta function. The worldline of the point mass source is parametrized by the four functions

\[ X^0 = t(\tau), \quad X^1 = r(\tau); \quad X^2 = \theta(\tau); \quad X^3 = \phi(\tau) \]  

(1.20)

The matter action is

\[ S_{\text{matter}} = -M \int d\tau = -M \int \sqrt{|g_{\mu \nu}(dX^\mu/d\tau)(dX^\nu/d\tau)} \ d\tau = \]

\[ -M \int \sqrt{g} \ d^n x \int \delta^n(x^\mu - X^\mu(\tau)) \sqrt{g_{\mu \nu}(dx^\mu/d\tau)(dx^\nu/d\tau)} \ d\tau. \]  

(1.21)

From which we can deduce the expression for the stress energy tensor density

\[ T^\mu_\nu = 2 \frac{\delta S_{\text{matter}}}{\delta g_{\mu \nu}} = \]

\[ -M \int \frac{(dx^\mu/d\tau)(dx^\nu/d\tau)}{\sqrt{(dx^\sigma/d\tau)(dx^\tau/d\tau)}} \frac{1}{\sqrt{|g|}} \delta(r - r(\tau)) \delta(\theta - \theta(\tau)) \delta(\phi - \phi(\tau)) \delta(t - x^0(\tau)) \ d\tau. \]  

(1.22a)

The reason why there is a factor of 2 in the definition of \( T^\mu_\nu \) is due to the symmetrization of indices in the variation of \( S_m \) w.r.t \((1/2)\delta g_{\mu \nu} \).

The worldline of an inert point mass at fixed values of

\[ r = r_o = \text{constant} \neq 0; \quad \theta = \theta_o = \text{constant}, \quad \phi = \phi_o = \text{constant} \]  

(1.22b)
is determined by the temporal function $x^0 = t = x^0(\tau)$ such that

$$(d\tau)^2 = g_{00}(dt)^2 \Rightarrow \tau = \int \sqrt{g_{00}} \, dt \Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{g_{00}}} \quad (1.22c)$$

$$(dx^0/d\tau)(dx^0/d\tau) = \frac{1}{g_{00}} = g^{00}. \quad (1.22d)$$

For this particular timelike worldline history (on-shell so $(dx^\sigma/d\tau)(dx_\sigma/d\tau) = 1$) the only non-vanishing component of the stress energy tensor is

$$T_{00} = -M \int \frac{(dx_0/d\tau)^2}{\sqrt{|g|}} \delta(r-r(\tau)) \delta(\theta-\theta(\tau)) \delta(\phi-\phi(\tau)) \frac{\delta(t-x^0(\tau))}{\sqrt{(dx^\sigma/d\tau)(dx_\sigma/d\tau)}} \, d\tau =$$

$$T_{00} = -M \int \frac{g_{00}(|\vec{r} - \vec{r}_o|)}{\sqrt{|g|}} \delta(r-r_o) \delta(\theta-\theta_o) \delta(\phi-\phi_o) \delta(t-x^0(\tau)) \, d\tau =$$

$$T_{00} = -M \frac{g_{00}(|\vec{r} - \vec{r}_o|)}{\sqrt{|g|}} \delta(r-r_o) \delta(\theta-\theta_o) \delta(\phi-\phi_o) \quad (1.22e)$$

As expected, we have found that the $T_{00}$ component is just related to the mass density $\rho$ in spherical coordinates for a point mass source located at $\vec{r}_o = (x_o, y_o, z_o) \neq 0$. If the point mass source is located at the origin of the spherical coordinates system $\vec{r}_o = 0$, the Jacobian in this case becomes $\sqrt{|g|} = R^2(dR/dr) \sin\phi$, and $g_{00}(|\vec{r} - \vec{r}_o|) = g_{00}(r)$. However, since the angles are degenerate at $r = 0$ (the angles are not well defined at the origin) to cure this ambiguity one can perform the average over all solid angle directions (from 0 to 4$\pi$) and which furnishes a crucial (1/4$\pi$) factor that is deeply connected to the ubiquitous $2M$ term, as follows

$$\frac{1}{4\pi} \int T_{00} \sin(\phi) \, d\phi \, d\theta = -\frac{M}{4\pi} \int \frac{g_{00}(r)}{R^2(dR/dr) \sin\phi} \delta(r) \delta(\theta-\theta_o) \delta(\phi-\phi_o) \sin(\phi) \, d\phi \, d\theta =$$

$$< T_{00} >_{solid \ angle} = -\frac{M}{4\pi R^2(dR/dr)} g_{00}(r) \delta(r) \Rightarrow 8\pi g_{00} < T_{00} > = -\frac{2M}{R^2(dR/dr)} \delta(r)$$

(1.23)

Notice that this averaging procedure in eq-(1.23) does not yield the same answer for $T_{00}$ as indicated by eq-(1.12a) and which results from the direct evaluation of the Einstein tensor associated with the class of generalized Schwarzschild solutions. The rigorous procedure would require locating the point mass source away from the origin $r = 0$ in order to have a well-defined delta function in spherical coordinates. However, despite this fact one can verify from eq-(1.23) that $8\pi g_{00} < T_{00} > = -\frac{2M}{R^2(dR/dr)} \delta(r)$ does indeed agree with the expression $8\pi g^{\mu\nu}T_{\mu\nu}$ of eq-(1.12e). Therefore, to sum up, we have checked that the relation

$$\mathcal{R} = 8\pi g_{00} < T_{00} >_{solid \ angle} = 8\pi g^{\mu\nu}T_{\mu\nu} = -\frac{2M}{R^2(dR/dr)} \delta(r). \quad (1.24)$$
holds relating eq-(1.12e) with eq-(1.23) in the Schwarzschild class of SSS solutions of Einstein’s equations. This is a sign of consistency.

To sum up the salient features of the novel class of Schwarzschild solutions: the world line of the singularity \( r = 0 \) is timelike despite that \( g_{00}(r = 0) = 0 \) because there is a tilt of the light-cone at \( r = 0 \) such that the ingoing null lines at \( r = 0 \) coincide with the timelike worldline of the singularity and the outgoing null lines will reach an observer at asymptotic infinity. By definition, the point mass source is a naked singularity and we have shown that the spacetime background generated by this point mass is derived from eqs-(1.1-1.3) by replacing \( \alpha = 2M \) and by inserting the proper modulus function \( |r| \) in the infinite family of admissible radial functions \( R = R(|r|) \) with the provision that \( R(r = 0) = 2M \) and \( R(|r| \to \infty) \to |r| \). This completes the review of [16].

### 2 On the Cosmological Constant Problem

In this final and main section we shall study some of the most pertinent cosmological implications of introducing radial functions \( R(r) \neq r \) in the (Anti) de Sitter-Schwarzschild solutions as follows

\[
 g_{00} = \left( 1 - \frac{2M}{R(|r|)} - \lambda \right) R(|r|)^2, \quad g_{rr} = -\left( 1 - \frac{2M}{R(|r|)} - \lambda \right) R(|r|)^2 \left( dR(|r|)/dr \right)^2 \tag{2.1}
\]

The angular part is given as usual in terms of the solid angle by \( -(R(|r|))^2(d\Omega)^2 \). We choose the parameter \( \lambda = \Lambda/3 \) where \( \Lambda \) is the cosmological constant. The \( \lambda < 0 \) case corresponds to Anti de Sitter-Schwarzschild solution and \( \lambda > 0 \) corresponds to the de Sitter-Schwarzschild solution. The physical interpretation of these solutions is that they correspond to “black holes” in curved backgrounds that are not asymptotically flat. For very small values of \( R \) one recovers the ordinary Schwarzschild solution. For very large values of \( R \) one recovers asymptotically the (Anti) de Sitter backgrounds of constant scalar curvature.

These are the SSS solutions to Einstein’s equations with a cosmological constant. These solutions were studied earlier by [7] but unfortunately this author performed an erroneous analysis of these cosmological models. Thus, contrary to the claims [7], we will show below that there are nontrivial solutions with a nonvanishing cosmological constant \( \lambda \) when the correct expression for the radial functions \( R(r) \) are introduced.

One particular expression for the radial function in the de Sitter-Schwarzschild (\( \lambda > 0 \)) case is

\[
 \frac{1}{R^2 - (2M)^2} = \frac{1}{r^2} + \lambda. \tag{2.2}
\]

since \( r^2 = |r|^2 \) there is no need to explicitly write the modulus sign in (2.2) and in the discussion below. When \( \lambda = 0 \) one recovers \( R^2 = r^2 + (2M)^2 \) as before in the pure Schwarzschild case given by a family of admissible radial functions obeying \( R(r = 0) = 2M \) and asymptotically tending to \( R \sim r \) for large values of \( r \) compared to \( 2M \). When \( M = 0 \) the radial function becomes
In this case, one encounters the reciprocal situation (the "dual" picture) of the Schwarzschild solutions: (i) when \( r \) tends to zero (instead of \( r = \infty \)) the radial function behaves \( R(r \to 0) \to r \); in particular \( R(r = 0) = 0 \) and (ii) when \( r = \infty \) (instead of \( r = 0 \)) the value of \( R(r = \infty) = R_{\text{Horizon}} = \sqrt{\frac{r}{\lambda}} \) and one reaches the location of the horizon given by the condition \( g_{00}[R(r = \infty)] = 0 \).

The proper radius \( R_p(r) \) is given by the integral

\[
R_p(r) = \int \frac{dR}{\sqrt{1 - \lambda R^2}} = \frac{1}{\sqrt{\lambda}} \arcsin \left[ R(r)\sqrt{\lambda} \right] \Rightarrow
\]

\[
R_p(r = 0) = 0 \quad \text{since} \quad R(r = 0) = 0; \quad \text{and} \quad R_p(r = \infty) = \frac{\pi}{2} \frac{1}{\sqrt{\lambda}} = \frac{\pi}{2} R_{\text{Horizon}}. \tag{2.4}
\]

When \( M \neq 0 \) one has for the de Sitter case

\[
g_{00}(r_*) = 0 \Rightarrow 1 - \frac{2M}{R(r_*)} - \lambda R(r_*)^2 = 0 \tag{2.5}
\]

a cubic equation whose solutions \( R_* \) will restrict the values of the radial function \( R_* = R(r_*) \) at \( r = r_* \neq \infty \), in terms of the mass parameters \( M \) and the cosmological constant \( \lambda = 16\pi G \rho_{\text{vacuum}} \). The cubic equation will be solved exactly as shown below contrary to the assertions of [7] that it cannot be solved exactly.

Let us begin with the de Sitter case (by setting \( M = 0 \)), the condition

\[
g_{00}(r = \infty) = 0 \Rightarrow 1 - \lambda R(r = \infty)^2 = 0 \tag{2.6}
\]

has a real valued solution

\[
R(r = \infty) = \sqrt{\frac{1}{\lambda}} = R_{\text{Horizon}}. \tag{2.7}
\]

the correct order of magnitude of the observed cosmological constant can be derived from eq-(2.7) by equating \( R(r = \infty) = R_{\text{Horizon}} = \text{Hubble Horizon Radius as seen today of the order of } 10^{60} L_{\text{Planck}} \) and setting \( G = L_{\text{Planck}}^2 \) (\( \hbar = c = 1 \) units) in

\[
16\pi G \rho_{\text{vacuum}} = \Lambda = 3\lambda = \frac{3}{R(r = \infty)^2} = \frac{3}{R_H^2} \Rightarrow
\]

\[
\rho_{\text{vacuum}} = \frac{3}{16\pi} \frac{1}{L_P^2} \frac{1}{R_H^2} = \frac{3}{16\pi} \frac{1}{L_P^2} \left( \frac{L_P}{R_H} \right)^2 \sim 10^{-121} \left( M_{\text{Planck}} \right)^4. \quad \text{when} \quad R_H \sim 10^{60} L_P. \tag{2.8}
\]

which agrees with the experimental observations.

We continue with a relevant analysis of the UV/IR (ultraviolet-infrared) entanglement involving the interaction of small-large scales within the context of the cosmological constant problem. The transformation


\[ r \to \frac{1}{Ar}; \quad \lambda \neq 0. \quad (2.9) \]

exchanges small distances with large distances and vice versa, reminiscent of the T-duality in string theory compactifications, and leads to a dual radial function of the form

\[ \frac{1}{R^2} = (\lambda r)^2 + \lambda. \quad (2.10a) \]

where now one has the reciprocal ("dual") behaviour as that of eq-(2.7)

\[ \tilde{R}(r = \infty) = 0; \quad \tilde{R}(r = 0) = \frac{1}{\sqrt{\lambda}}. \quad (2.10b) \]

and the horizon condition \( g_{00}(R_{\text{Horizon}}) = 0 \) is now attained at \( r = 0 \) (due to the small-large scales exchange)

\[ g_{00}(r = 0) = 0 \Rightarrow 1 - \lambda \tilde{R}(r = 0)^2 = 0 \Rightarrow \tilde{R}(r = 0) = \sqrt{1/\lambda} = R_{\text{Horizon}}. \quad (2.11) \]

and once again we get the same result as before in (2.8).

It is clear now why if one had written \( \tilde{R}(r) = r \) in eq-(2.11) and introduced the Planck scale as an ultraviolet cutoff, instead of setting \( r = 0 \), one would have obtained an answer in eq-(2.11) that is off by 120 orders of magnitude! (which is the cosmological constant problem). What the dual radial function \( \tilde{R}(r) \) achieves in eqs-(2.10a, 2.11) is to map the extreme ultraviolet (UV) region \( r = 0 \) onto the infrared (IR) region \( \tilde{R}(r = 0) = R_{\text{Hubble}} \).

Hence, the presence of the dual radial function \( \tilde{R}(r) \) implements the necessary UV/IR entanglement associated with the resolution of the cosmological constant problem.

In [15] we have shown why AdS4 gravity with a topological term; i.e. an Einstein-Hilbert action with a cosmological constant plus Gauss-Bonnet terms can be obtained from the vacuum state of a BF-Chern-Simons-Higgs theory without introducing by hand the zero torsion condition imposed in the MacDowell-Mansouri-Chamsedine-West construction. One of the most salient features of [15] was that a geometric mean relationship was derived (from scratch, instead of postulating it) among the vacuum energy density \( \rho \), the Planck area \( L_P^2 \) and the AdS4 throat size squared \( R^2 \) given by \( \rho = (L_P)^{-2} R^{-2} \).

Upon setting the throat size to coincide with the Hubble scale \( R_H \) (since the throat size of de Sitter and Anti de Sitter is the same) one obtains the observed value of the vacuum energy density \( \rho = L_{\text{Planck}}^2 R_H^{-2} = L_P^{-4} (L_P/R_H)^2 \sim 10^{-120} (M_{\text{Planck}})^4 \).

To finalize we will analyze in detail the exact solutions to the cubic equation in the (Anti) de Sitter-Schwarzschild solutions. Let us begin with de Sitter-Schwarzschild case. The cubic equation that sets the location \( R_* \) of the horizon \( g_{00}(R = R_*) = 0 \) is given by

\[ R_*^3 - \frac{R_*}{\lambda} + \frac{2M}{\lambda} = 0. \quad \lambda > 0. \quad (2.12) \]

whose 3 solutions are

\[ R_1 = (S + T). \quad (2.13a) \]
\[ R_2 = -\frac{1}{2}(S + T) + \frac{i\sqrt{3}}{2} (S - T). \]  
\[ R_3 = -\frac{1}{2}(S + T) - \frac{i\sqrt{3}}{2} (S - T). \]  
\[ (2.13b) \]

where

\[ S = \left[ \frac{-M}{\lambda} + \sqrt{\frac{M^2}{\lambda^2} - \frac{1}{27\lambda^3}} \right]^{1/3}. \]  
\[ (2.14a) \]

\[ T = \left[ \frac{-M}{\lambda} - \sqrt{\frac{M^2}{\lambda^2} - \frac{1}{27\lambda^3}} \right]^{1/3}. \]  
\[ (2.14b) \]

If we don’t wish to have complex roots one has two cases to study. One case is when \( S = T \) and the other case is when \( S \neq T \) by disregarding the complex roots and keeping only the real root \( R_1 \). Let us focus now on the \( S = T \) case:

\[ S = T \Rightarrow \frac{M^2}{\lambda^2} - \frac{1}{27\lambda^3} = 0 \Rightarrow \frac{M}{\lambda} = \frac{1}{\sqrt{27}\lambda^3}. \]  
\[ (2.15) \]

the roots become

\[ R_1 = -2 \left[ \frac{M}{\lambda} \right]^{1/3} < 0. \]  
\[ (2.16a) \]

\[ R_2 = R_3 = -\frac{1}{2}(-2) \left[ \frac{M}{\lambda} \right]^{1/3} = \left[ \frac{M}{\lambda} \right]^{1/3} = \frac{1}{\sqrt{27}\lambda^3} = \frac{1}{\sqrt{3}\lambda} = 0.5773 \, R_H. \]  
\[ (2.16b) \]

The fact that we have found one negative root for the radial function \( R(r_1) \) does not necessarily mean that the value of \( r_1 \) is negative. We will discuss this \( R_1 < 0 \) case in detail below. There are two equal positive roots \( R_2 = R_3 \) whose value is less than the Hubble scale \( R_H \)

\[ R_* = R_2 = R_3 = \frac{1}{\sqrt{3}\lambda} < \frac{1}{\sqrt{\lambda}} = R_H. \]  
\[ (2.17) \]

as it should, otherwise there would not have been a real valued solution for \( r_* \) such that \( R(r_*) = R_2 = R_3 \). Plugging the value of \( R_* = R_2 = R_3 = (3\lambda)^{-1/2} \) into the defining relation for the radial function in eq-(2.2) yields the finite value of \( r_* \) ( compared to the \( r = \infty \) value when \( M = 0 ) \) after one uses the relation \( M^2 = (1/27\lambda) \) of eq-(2.15) in

\[ \frac{1}{R_*^2 - (2M)^2} = \frac{1}{r_*^2} + \lambda \Rightarrow r_* = \sqrt{\frac{R_*^2 - (2M)^2}{1 - \lambda((R_*^2 - (2M)^2)}} = \]

\[ r_* = \sqrt{\frac{15}{66}} \frac{1}{\sqrt{\lambda}} = 0.4767 \, R_H. \]  
\[ (2.18) \]

To sum up, the solutions to the cubic equation yield in the \( S = T \) case the following numerical relations
\[ R(r = 0) = 2M; \quad R(r = \infty) = \sqrt{(2M)^2 + \frac{1}{\lambda}} > 2M. \]  

(2.19)

and

\[ 2M = \sqrt{\frac{4}{27}} \frac{1}{\sqrt{\lambda}} < R(r_\ast) = R_\ast = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{\lambda}} < \sqrt{(2M)^2 + \frac{1}{\lambda}} = \sqrt{\frac{31}{27}} \frac{1}{\sqrt{\lambda}}. \]  

(2.20)

The case \( S \neq T \) is obtained by disregarding the two complex roots while maintaining the real root \( R_1 \). However, one ends up with another negative root \( R_1 \)

\[
R_1 = \left[ -\frac{M}{\lambda} + \sqrt{\frac{M^2}{\lambda^2} - \frac{1}{27\lambda^3}} \right]^{1/3} + \left[ -\frac{M}{\lambda} - \sqrt{\frac{M^2}{\lambda^2} - \frac{1}{27\lambda^3}} \right]^{1/3} < 0. 
\]  

(2.21)

because one is required to choose in this \( S \neq T \) case the condition

\[
\frac{M^2}{\lambda^2} - \frac{1}{27\lambda^3} > 0. 
\]  

(2.22)

that, in turn, will force \( R_1 < 0 \). Despite the fact that \( R_1(r = r_1) < 0 \) this does not necessarily mean that the value of \( r_1 \) is negative. If \( R_1^2 > (2M)^2 \) there are real-valued solutions for \( r = r_1 \) that could still be positive by direct inspection of eq-(2.18). The inequality \( R_1(M, \lambda)^2 > (2M)^2 \) obtained from eq-(2.21) in conjunction with the other inequality given by eq-(2.22) will yield the constraint relation of the values \( M, \lambda \) in the \( M - \lambda \) parameter space that would determine whether or not there exists a real-valued and positive \( r_1 > 0 \) despite having \( R_1 < 0 \). Whether or not such conditions can be met simultaneously for the values \( M > 0; \lambda > 0 \) needs to be studied further. Unfortunately the expressions are rather unwieldy. If one had chosen the radial function to be \( R(r) = r \) then one would immediately conclude that \( r_1 < 0 \). But since \( R(r) \neq r \) one can still have \( r_1 > 0 \) for \( R_1 < 0 \) which is a very interesting possibility that warrants further investigation.

It is important to remark at this point that

\[
g_{00} = (1 - \frac{2M}{R} - \lambda R^2) \leq 0 
\]  

(2.23)

not only when \( 2M \leq R \leq R_\ast \) but also when \( R > R_\ast \) due to the double-root nature of the solutions to the cubic equation given by eq-(2.16b). Because the component \( g_{00} \) does not change sign as one crosses \( R_\ast \), strictly speaking, one does not have a horizon as such for \( R_\ast \) because \( g_{00} \leq 0 \) in the domain of values of the radial function defined by \( 2M \leq R \leq \sqrt{(2M)^2 + \frac{1}{\lambda}} \) that is associated, respectively, with the values of \( r \) in the domain \( 0 \leq r \leq \infty \).

However, there is a horizon in the case of the simple real root \( R_1 < 0 \) ( when \( S \neq T \) ) in eq-(2.21) because \( g_{00} \geq 0 \) when \( R_1 < R < 0 \) provided \( (2M)^2 < R^2 < R_1^2 \); and \( g_{00} \leq 0 \) when \( R < R_1 < 0 \). Thus \( g_{00} \) does change sign when one crosses \( R_1 < 0 \). The same conclusions apply to the negative simple root \( R_1 < 0 \) found earlier for the \( S = T \) case.
and given by eq-(2.16a). One has a true horizon since $g_{00}$ changes sign as one crosses $R_1$. Since the solution of eq-(2.16a) obeys the requirement $R_1^2 = (4/3\lambda) > (2M)^2 = (4/27\lambda)$ one could have real-valued and positive $r_1 > 0$ solutions by inspection of eq-(2.18).

Let us study now the Anti de Sitter-Schwarzschild case. The location of the horizon involves finding solutions of the cubic equation

$$g_{00}(r_*) = 0 \Rightarrow 1 - \frac{2M}{R(r_*)} + \lambda R(r_*)^2 = 0 \quad (2.24)$$

It is very important to emphasize that one has already taken into account the fact $\lambda_{AdS} = -\lambda_{dS}$ in eq-(2.24). Therefore in eq-(2.24), and all the expressions that follow, when we write $\lambda$ it should be understood as $|\lambda|$ and hence it is a positive quantity. The unique real-valued positive solution (obtained by replacing $\lambda \rightarrow -\lambda$ in the above solutions of the de Sitter case) is:

$$R_* = \left[ \frac{M}{\lambda} + \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}} \right]^{1/3} + \left[ \frac{M}{\lambda} - \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}} \right]^{1/3} > 0. \quad (2.25)$$

We must disregard the two complex roots. There are no double roots in the AdS case because $\frac{4M^2}{\lambda^2} + \frac{1}{27\lambda^3} \neq 0$. A careful study reveals that the radial function $R(r)$ in the Anti de Sitter case must differ from the de Sitter case and is obtained from eq-(2.2) by replacing $\lambda \rightarrow -\lambda$

$$\frac{1}{R^2 - (2M)^2} = \frac{1}{r^2} - \lambda \Rightarrow R(r = 0) = 2M; \quad R(r = \infty) = \sqrt{(2M)^2 - \frac{1}{\lambda}} < 2M. \quad (2.26)$$

and it leads to the inequality $2M > R_* > R(r = \infty)$ because it is a decreasing function of $r$ and which can be recast explicitly as

$$2M > \left[ \frac{M}{\lambda} + \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}} \right]^{1/3} + \left[ \frac{M}{\lambda} - \sqrt{\frac{M^2}{\lambda^2} + \frac{1}{27\lambda^3}} \right]^{1/3} > \sqrt{(2M)^2 - \frac{1}{\lambda}} \geq 0 \quad (2.27)$$

Hence, eq-(2.27) defines the explicit constraint relation between the allowed values of $M$ and $\lambda$ in the $M - \lambda$ parameter space. In this case one has a true horizon since the metric component

$$g_{00}(R(r)) = 1 - \frac{2M}{R(r)} + \lambda R(r)^2 \geq 0; \quad \text{when} \quad 2M \geq R \geq R_* \quad (2.28)$$

will change sign

$$g_{00}(R(r)) = 1 - \frac{2M}{R(r)} + \lambda R(r)^2 \leq 0; \quad \text{when} \quad R_* \geq R \geq \sqrt{(2M)^2 - \frac{1}{\lambda}} \geq 0 \quad (2.29)$$
The UV/IR entanglement map \( r \rightarrow 1/\lambda r \) in eq-(2.26) yields the dual version of the radial function \( \tilde{R}(r) \)

\[
\frac{1}{\tilde{R}^2 - (2M)^2} = (\lambda r)^2 - \lambda \Rightarrow \tilde{R}(r = \infty) = 2M; \quad \tilde{R}(r = 0) = \sqrt{(2M)^2 - \frac{1}{\lambda}} < 2M.
\]

which is an increasing function of \( r \), instead of a decreasing function like \( R(r) \) in eq-(2.26). In this dual case the metric component

\[
g_{00}(\tilde{R}(r)) = 1 - \frac{2M}{\tilde{R}(r)} + \lambda \tilde{R}(r)^2 \leq 0; \quad \text{when } \sqrt{(2M)^2 - \frac{1}{\lambda}} \leq \tilde{R} \leq R_*
\]

will change sign and become

\[
g_{00}(\tilde{R}(r)) = 1 - \frac{2M}{\tilde{R}(r)} + \lambda \tilde{R}(r)^2 \geq 0; \quad \text{when } R_* \leq \tilde{R} \leq 2M
\]

One notices that the \( g_{00} > 0 \) behaviour occurs when \( R > R_* = R_{\text{Horizon}} \) and/or \( \tilde{R} > R_* = R_{\text{Horizon}} \) and it is similar to the behaviour of \( g_{00} \) in the exterior region of a "black hole" horizon. From eqs-(2.29, 2.31) one can infer from the condition

\[
\sqrt{(2M)^2 - \frac{1}{\lambda}} \geq 0 \Rightarrow 2M \geq \frac{1}{\sqrt{\lambda}}.
\]

If one were to interpret \( 2M = \frac{1}{\sqrt{\lambda}} = R_{\text{Hubble}} \) as the lower bound for the mass of the universe and take a value of \( R_{\text{Hubble}} \sim 10^6 L_{\text{Planck}} \) one would have in the appropriate units the following

\[
2M \sim 10^{61} M_{\text{Planck}} \sim 10^{80} m_{\text{proton}}.
\]

that agrees with the Dirac-Eddington large number coincidences

\[
N = 10^{80} \sim \left( \frac{F_e}{F_G} \right)^2 \sim \left( \frac{R_{\text{Hubble}}}{r_e} \right)^2.
\]

where \( F_e = e^2/r \) is the electrostatic force between an electron and a proton; \( F_G = Gm_e m_p/r^2 \) is the corresponding gravitational force and \( r_e = e^2/m_e \sim 10^{-13} cm \) is the classical electron radius in natural units of \( \hbar = c = 1 \). Of course, this is not to say that the AdS-Schwarzchild case is the same as the Friedman-Robertson-Walker model, but only that one could equate the net mass (inside \( R_H \)) of the latter with the \( 2M \) parameter of the former to get an estimate of the lower bound of the mass of the observable universe. To match the observational data requires further work since it is more likely that \( 2M > \frac{1}{\sqrt{\lambda}} = R_{\text{Hubble}} \) due to the presence of dark matter.

By inspection one can verify that the lower bound \( 2M = \frac{1}{\sqrt{\lambda}} \) obeys the condition given by eq-(2.27). The latter becomes
\[ 2M = \frac{1}{\sqrt{\lambda}} > (\left[ \frac{1}{2} + \sqrt{\frac{31}{108}} \right]^{1/3} + \left[ \frac{1}{2} - \sqrt{\frac{31}{108}} \right]^{1/3}) \frac{1}{\sqrt{\lambda}} = 0.6823 \frac{1}{\sqrt{\lambda}} \geq 0 \ (2.35) \]

It is clear that a lot of work and re-thinking remains to be done pertaining the proper use of the radial functions \( R(r) \) in the class of SSS solutions to Einstein’s equations with and without a cosmological constant. The fact that we were able to obtain the correct magnitude of the observed cosmological constant and the correct lower estimate of the mass of the universe related to the Dirac-Eddington’s large number \( N = 10^{80} \) is a positive sign that one should use the solutions displayed in this work based on a suitable class of radial functions \( R(r) \) rather than the naive choice \( R = r \) we have been familiar with during all these decades!

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