ON GEOMETRIC PROBABILITY, HOLOGRAPHY, SHILOV BOUNDARIES
AND THE FOUR PHYSICAL COUPLING CONSTANTS OF NATURE

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Abstract
By recurring to Geometric Probability methods, it is shown that the coupling constants, $\alpha_{EM}; \alpha_W; \alpha_C$ associated with Electromagnetism, Weak and the Strong (color ) force are given by the ratios of the ratios of the measures of the Shilov boundaries $Q_2 = S^1 \times \mathbb{RP}^1$; $Q_3 = S^2 \times \mathbb{RP}^1$; $S^5$, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the 5D conformally compactified real Minkowski spacetime $\tilde{M}_5$ that has the same topology as the Shilov boundary $Q_5$ of the 5 complex-dimensional poly-disc $D^5$. The homogeneous symmetric complex domain $D^5 = SO(5,2)/SO(5) \times SO(2)$ corresponds to the conformal relativistic curved 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the 5D Anti de Sitter space $AdS_5$. The geometric coupling constant associated to the gravitational force can also be obtained from the ratios of the measures involving Shilov boundaries. We also review our derivation of the observed vacuum energy density based on the geometry of de Sitter ( Anti de Sitter ) spaces.

1. The Fine Structure Constant and Geometric Probability

Geometric Probability [21] is the study of the probabilities involved in geometric problems, e.g., the distributions of length, area, volume, etc. for geometric objects under stated conditions. One of the most famous problem is the Buffon’s Needle Problem of finding the probability that a needle of length $l$ will land on a line, given a floor with equally spaced parallel lines a distance $d$ apart. The problem was first posed by the French naturalist Buffon in 1733. For $l < d$ the probability is

$$P = \frac{1}{2\pi} \int_0^{2\pi} \frac{l|\cos(\theta)|}{d} d\theta = \frac{4l}{2\pi d} \int_0^{\pi/2} \cos(\theta) d\theta = \frac{2l}{\pi d} = \frac{2ld}{\pi d^2}. \quad (1)$$

Hence, the Geometric Probability is essentially the ratio of the areas of a rectangle of length $2d$, and width $l$ and the area of a circle of radius $d$. For $l > d$, the solution is slightly more complicated. [21]. The Buffon needle problem provides with a numerical experiment that determines the value of $\pi$ empirically. Geometric Probability is a vast field with profound connections to Stochastic Geometry.

Feynman long ago speculated that the fine structure constant may be related to $\pi$. This is the case as Wyler found long ago [1] . We will based our derivation of the fine structure constant based on Feynman’s physical interpretation of the electron’s charge as the probability amplitude that an electron emits (or absorbs) a photon. The clue to evaluate this probability within the context of Geometric Probability theory is provided by the electron self-energy diagram. Using Feynman’s rules, the self-energy $\Sigma(p)$ as a function of the electron’s incoming (outgoing) energy-momentum $p_\mu$ is given by the integral involving the photon and electron propagator along the internal lines

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i}{\gamma^\nu(p_\rho - k_\rho) - m - ig_{\mu\nu}}. \quad (2)$$

The integral is taken with respect to the values of the photon’s energy-momentum $k^\mu$. By inspection one can see that the electron self-energy is proportional to the fine structure constant $\alpha_{EM} = e^2$, the square of the probability amplitude (in natural units of $\hbar = c = 1$) and physically represents the electron’s emission of a virtual photon (off-shell, $k^2 \neq 0$) of energy-momentum $k_\mu$ at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the Geometric Probability that an electron emits a photon at $t = -\infty$ (infinite past) and re-absorbs it at a much later time
The electron (a particle) moving in a 4-dimensional homogeneous coset space associated with the 4-dimensional scalar curvature that can be identified with the conformal relativistic phase space associated with the origin is not necessarily transitive on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries which implement the Maldacena-T Hooft-Susskind Holographic principle [13] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the poly-disc $D_4$ of 4 complex dimensions is an 8-real-dim Hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic curved phase space associated with the electron (a particle) moving in a 4D Anti de Sitter space $\text{AdS}_4$. The poly-disc is a Hermitian symmetric homogeneous coset space associated with the 4D conformal group $\text{SO}(4,2)$ since $D_4 = \text{SO}(4,2)/\text{SO}(4) \times \text{SO}(2)$. Its Shilov boundary $\text{Shilov} (D_4) = Q_4$ has precisely the same topology as the 4D conformally compactified real Minkowski spacetime $Q_4 = M_4 = S^3 \times S^1/Z_2 = S^3 \times \mathbb{RP}^1$. For more details about Shilov boundaries, the conformal group, future tubes and holography we refer to the article by Gibbons [12] and [7,16].

In order to define the Geometric Probability associated with this process of the electron’s emission of a photon at $i^−$ ($t = -\infty$), followed by an absorption at $i^+$ ($t = +\infty$), we must take into account the important fact that the photon is on-shell $k^2 = 0$ asymptotically (at $t = \pm \infty$), but it can move off-shell $k^2 \neq 0$ in the intermediate region which is represented by the interior of the conformally compactified real Minkowski spacetime $Q_4 = M_4 = S^3 \times S^1/Z_2 = S^3 \times \mathbb{RP}^1$.

Denoting by $\hat{\mu}[Q_4]$ the measure-density (the measure-current) whose flux through the future and past celestial spheres $S^2$ (associated with the future/past light-cones) at timelike infinity $i^+, i^−$, respectively, is $V(S^2)\hat{\mu}[Q_4]$. The net flux through the two celestial spheres $S^2$ at timelike infinity $i^\pm$ requires an overall factor of 2 giving then the value of $2V(S^2)\hat{\mu}[Q_4]$. The Geometric Probability is defined by the ratio of the measures associated with the celestial spheres $S^2$ at $i^+, i^−$ timelike infinity, where the photon moves on-shell, relative to the measure of the full interior region of $Q_4 = M_4 = S^3 \times S^1/Z_2 = S^3 \times \mathbb{RP}^1$, where the photon can move off-shell, as it propagates from $i^−$ to $i^+$:

$$\alpha = \frac{2V(S^2)\hat{\mu}[Q_4]}{\mu[Q_4]}.$$  \hspace{1cm} (3)

The ratio $(\hat{\mu}[Q_4]/\mu[Q_4])$ can be re-written in terms of the ratios of the normalized measures of

$$\hat{M}_5 = Q_5 = \text{Shilov} [D_5] = S^4 \times S^1/Z_2 = S^4 \times \mathbb{RP}^1.$$ \hspace{1cm} (4)

namely, in terms of the normalized measures of the conformally compactified 5D Minkowski spacetime. This is achieved as follows [4]
resulting from the embeddings (immersions) of $D_4 \rightarrow D_5$.

The origin of the factor $V(S^4)$ in the r.h.s of (5), as one goes from the ratio of measures in $Q_4$ to the ratio of the measures in $Q_5$, is due to the reduction from the action of the isotropy group of the origin $SO(5) \times SO(2)$ on $Q_5$, to the action of the isotropy group of the origin $SO(4) \times SO(2)$ on $Q_4$, furnishing an overall reduction factor of $V[SO(5)/SO(4)] = V(S^4)$. The 5 complex-dimensional poly-disc $D_5 = SO(5,2)/SO(5) \times SO(2)$ is the 10 real-dim Hyperboloid $H^{10}$ corresponding to the conformal relativistic curved phase space of a particle moving in 5D Anti de Sitter Space $AdS_5$. This picture is also consistent with the Kaluza-Klein compactification procedure of obtaining 4D EM from pure Gravity in 5D. The $H^{10}$ can be embedded in the 11-dim pseudo-Euclidean $R^{9,2}$ space, with two-time like directions. This is where 11-dim lurks into our construction.

Next we turn to the Hermitian metric on $D_5$ constructed by Hua [8] which is $SO(5,2)$-invariant and is based on the Bergmann kernel [15] involving a crucial normalization factor of $1/V$. The standard normalized measure $\mu_N[Q_5]$ based on the Poisson kernel and involving a normalization factor of $1/V(Q_5)$ is not invariant under the full group $SO(5,2)$. It is only invariant under the isotropy group of the origin $SO(5) \times SO(2)$. In order to construct an invariant measure on $Q_5$ under the full group $SO(5,2)$ one requires to introduce a crucial factor related to the Jacobian measure involving the action of the conformal group $SO(5,2)$ on the full bulk domain $D_5$. As explained by [4] one has:

$$\frac{\mu_N[Q_5]}{\mu[Q_5]} = \frac{1}{V(Q_5)} \sqrt{\|J^1 \|} = \frac{1}{V(Q_5)} \sqrt{||J^1 G||} = \frac{1}{V(Q_5)} \sqrt{det g^{-1}} = \frac{1}{V(Q_5)} [ |det (g)| ]^{-1/4} = \frac{1}{V(Q_5)} [ V(D_5) ]^4. \quad (6)$$

the $z$ dependence of the complex Jacobian is no longer explicit because the determinant of the $SO(5,2)$ matrices is unity.

This explains very clearly the origins of the factor $[ V(D_5) ]^4$ in Wyler’s formula for the fine structure constant [1]. This reduction factor of $V(Q_5)$ is in this case given by $V(D_5)^{1/4}$. As we shall see below, the power of $1/4$ is related to the inverse of the dim($S^4$) = 4. This summarizes, briefly, the role of Bergmann kernel [15] in the construction by Hua [8], and adopted by Wyler [1], of the Hermitian metric of a bounded homogenous (symmetric) complex domain. To sum up, we must perform the reduction from $V(Q_5) \rightarrow V(Q_5)/V(D_5)^{1/4}$ in the construction of the normalized measure $\mu_N[Q_5]$. This approach is very different than the interpretation given by Smith [3] and later adopted by Smilga [5].

Hence, the Geometric Probability ratio becomes

$$\frac{\tilde{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)} \frac{\mu_N[Q_5]}{\mu[Q_5]} = \frac{1}{V(S^4)} \frac{1}{V(Q_5)} [ V(D_5) ]^4 \equiv \frac{1}{\alpha_G}. \quad (7a)$$

This last ratio, for reasons to be explained below, is nothing but the inverse of the geometric coupling strength of gravity, $1/\alpha_G$. The relationship to the gravitational constant is based on the definition of the coupling appearing in the Einstein-Hilbert Lagrangian ($R/16\pi G$), as follows

$$(16\pi G)(m^2_{Planck}) \equiv \alpha_{EM} \alpha_G = 8\pi \Rightarrow G = \frac{1}{16\pi m^2_{Planck}} = \frac{8\pi}{2m^2_{Planck}} \Rightarrow$$

$$Gm^2_{proton} = \frac{1}{2} \left( \frac{m_{proton}}{m_{Planck}} \right)^2 \approx 5.9 \times 10^{-30}. \quad (7b)$$

and in natural units $h = c = 1$ yields the physical force strength of Gravity at the Planck Energy scale $1.22 \times 10^{19}$ GeV. The Planck mass is obtained by equating the Schwarzschild radius $2Gm_{Planck}$ to the Compton wavelength $1/m_{Planck}$ associated with the mass; where $m_{Planck} \sqrt{2} = 1.22 \times 10^{19}$ GeV and the proton mass is $0.938$ GeV. Some authors define the Planck mass by absorbing the factor of $\sqrt{2}$ inside the definition of $m_{Planck} = 1.22 \times 10^{19}$ GeV.
The role of the conformal group in Gravity in these expressions (besides the holographic bulk/boundary AdS/CFT duality correspondence [13]) stems from the MacDowell-Mansouri-Chamseddine-West formulation of Gravity based on the conformal group SO(3,2) which has the same number of 10 generators as the 4D Poincare group. The 4D vielbein $e^a_{\mu}$ which gauges the spacetime translations is identified with the SO(3,2) generator $A^{[a]}_{\mu}$, up to a crucial scale factor $R$, given by the size of the Anti de Sitter (de Sitter space) throat. It is known that the Poincare group is the Wigner-Inonu group contraction of the de Sitter Group SO(4,1) after taking the throat size $R = \infty$. The spin-connection $\omega^{ab}_{\mu}$ that gauges the Lorentz transformations is identified with the SO(3,2) generator $A^{[ab]}_{\mu}$. In this fashion, the $e^a_{\mu}, \omega^{ab}_{\mu}$ are encoded into the $A^{[mn]}_{\mu}$ SO(3,2) gauge fields, where $m, n$ run over the group indices 1,2,3,4,5. A word of caution, Gravity is a gauge theory of the full diffeomorphisms group which is infinite-dimensional and which includes the translations. Therefore, strictly speaking gravity is not a gauge theory of the Poincare group. The Ogiovetsky theorem shows that the diffeomorphisms algebra in 4D can be generated by an infinity of nested commutators involving the $GL(4,R)$ and the 4D Conformal Group SO(4,2) generators.

In [17] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of Gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal group in Gravity in these expressions (besides the holographic bulk/boundary AdS/CFT duality correspondence [13]). It was also shown in [17] why the Euclideanized AdS$_{2n}$ spaces are SO(2n−1,2) instantons solutions of a non-linear sigma model obeying a double self-duality condition.

Therefore, the Geometric Probability $\alpha_{EM}$ for an electron to emit a photon at $t = +\infty$ and to absorb it at $t = -\infty$ agrees with the Wyler’s celebrated expression for the fine structure constant

$$\alpha_{EM} = \frac{2V(S^2)}{\mu(Q_4)} = \frac{1}{V(S^4)} \frac{1}{V(Q_5)} [V(D_5)]^2 = \frac{9}{8\pi^4} \frac{(\frac{n}{2})!}{(2^{n} \times 5!)^{1/4}} = \frac{1}{137.03608}. \quad (8)$$

after one inserts the values of the volumes:

$$V(D_5) = \frac{\pi^5}{2^4 \times 5!}, \quad V(Q_5) = \frac{8\pi^3}{3}, \quad V(S^4) = \frac{8\pi^2}{3}. \quad (9)$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1}n!}, \quad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \quad (10a)$$

$$V(Q_n) = V(S^{n-1} \times RP^1) = V(S^{n-1}) \times V(RP^1) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)}. \quad (10b)$$

Objections were raised to Wyler’s original expression by Robertson [2]. One of them was that the hyperboloids (discs) are not compact and whose volumes diverge since the Lobachevsky metric diverges on the boundaries of the poly-discs. Gilmore explained [2] why one requires to use the Euclideanized regularized volumes as Wyler did. Furthermore, in order to resolve the scaling problems of Wyler’s expression, Gilmore showed why it is essential to use dimensionless volumes by setting the throat sizes of the Anti de Sitter hyperboloids to $r = 1$, because this is the only choice for $r$ where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the so-called scaling objections against Wyler raised by Robertson were satisfactorily solved by Gilmore [2].

The question as to why the value of $\alpha_{EM}$ obtained in Wyler’s formula is precisely the value of $\alpha_{EM}$ observed at the scale of the Bohr radius $a_B$, has not been solved, to my knowledge. The Bohr radius is associated with the ground (most stable) state of the Hydrogen atom [3].
of the Hydrogen atom is well known to be the conformal group $SO(4,2)$ due to the fact that there are two
conserved vectors, the angular momentum and the Runge-Lentz vector. After quantization, one has two
commuting $SU(2)$ copies $SO(4) = SU(2) \times SU(2)$. Thus, it makes physical sense why the Bohr-scale should
appear in this construction. Bars [14] has studied the many physical applications and relationships of many
seemingly distinct models of particles, strings, branes and twistors, based on the (super) conformal groups in
diverse dimensions. In particular, the relevance of two-time physics in the formulation of $M,F,S$ theory has
been advanced by Bars for some time. The Bohr radius corresponds to an energy of $137.036 \times 2 \times 13.6 \ eV \sim
3.72 \times 10^3 \ eV$. It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of electron,
and the classical electron radius are all related to each other by a successive scaling in products of $\alpha_{EM}$.

2. The Fiber Bundle Interpretation of The Wyler Formula

Having found Wyler’s expression from Geometrical Probability, we shall present a Fiber Bundle
interpretation of the Wyler expression by starting with a Fiber bundle $E$ over the base curved-space $D_5 =
SO(5,2)/SO(5) \times SO(2)$. The subgroup $H = SO(5)$ of the isotropy group $K = SO(5) \times SO(2)$ acts on the
Fibers $F = S^4$ (the internal symmetry space). Locally, and only locally, the Fiber bundle $E$ is the product
$D_5 \times S^4$. However, this is not true globally. On the Shilov boundary $Q_5$, the restriction of the Fiber bundle $E$
to the Shilov boundary $Q_5$ is written by $E|_{Q_5}$, and locally it is the product of $Q_5 \times S^4$, but this is not
true globally. For this reason one has that the volume $V(E|_{Q_5}) \neq V(Q_5 \times S^4) = V(Q_5) \times V(S^4)$. But instead,
$V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$.

This is the reasoning behind the construction of the quantity $\hat{\mu}[Q_4]/\mu[Q_4]$ that has the units of a density.
Its inverse $\mu[Q_4]/\hat{\mu}[Q_4]$ is the volume associated with the restriction of the Fiber Bundle $E$ to the Shilov
boundary $Q_5 : V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$.

The reason why one embeds $D_4 \to D_5$ and $Q_4 \to Q_5$ is because the space $Q_4 = S^3 \times RP^1$ is not
large enough to implement the action of the $SO(5)$ group, the compact version of the Anti de Sitter Group
$SO(3,2)$ that is required in the MacDowell-Mansouri-Chamseddine-West formulation of Gravity. However,
the space $Q_5 = S^4 \times RP^1$ is large enough to implement the action of $SO(5)$ via the internal symmetry space
$S^4 = SO(5)/SO(4)$. This justifies the embedding procedure of $D_4 \to D_5$. This Fiber Bundle interpretation
is not very different from Smith’s interpretation [3]. Following the Fiber Bundle interpretation of
the volume $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$, we will now prove why

$$2 V(S^2) = \frac{\mu(S^1)}{\hat{\mu}(S^1)} = 8\pi. \quad (11)$$

The space $S^1$ is associated with the $U(1)$ group action and naturally encodes the $U(1)$ gauge invariance
linked to Electromagnetism (EM). The result of eq-(11) is what will allow us to define $\alpha_{EM}$ as the
ratio of the ratios of suitable measures in $S^1$ and $Q_4$, respectively,

$$\alpha_{EM} = \frac{2V(S^2) \hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat{\mu}(S^1))}{(\mu[Q_4]/\hat{\mu}[Q_4])} \quad (12)$$

We may notice that $S^1 \equiv Q_1$ (very special case) since the circle is both the Shilov and ordinary
topological boundary of the disc $D_1$. However, $Q_2 \equiv S^1 \times S^1/Z_2 = S^1 \times RP^1$. Once again, we will write the
ratio of the measures in $Q^1 = S^1$ in terms of the ratio of the normalized measures in $Q^2$ via the reduction
from $S^1 \times S^1/Z_2$ to $S^1$. This requires the embedding (immersion) of $D_1 \to D_2$ in order to construct the
measures on $D_1,Q_1$ as induced from the measures in $D_2,Q_2$ resulting from the embedding (immersion):

$$\frac{\hat{\mu}(S^1)}{\mu(S^1)} = \frac{\hat{\mu}(Q_1)}{\mu(Q_1)} = \frac{1}{V(S^1/Z_2)} \frac{\mu[Q_2]}{\mu[Q_2]} = \frac{1}{V(S^1/Z_2)} \frac{V(Q_2)/V(D_2)}. \quad (13)$$

Notice that $\hat{\mu}(S^1)$ as explained before is a measure-density on $S^1$. Likewise, $\hat{\mu}(Q_4)$ was a measure-density on
$Q_4$. We should not confuse these measure-densities with the normalized measures in one-higher dimension.

By inserting the values of the measures and using
\[ V(S^3/Z_2) = V(RP^1) = \pi. \quad V(D_2) = \frac{\pi^2}{2 \times 2!}. \quad V(Q^2) = \frac{2\pi^2}{1(1)} = 2\pi^2. \] (14)

it yields then

\[ \frac{\mu(S^1)}{\tilde{\mu}(S^2)} = (2\pi^2) \left( \frac{1}{\pi^2/2 \times 2!} \right) = 8\pi = 2 \, V(S^2) \] (15)

as claimed. Therefore, \(2V(S^2) = \mu(S^1)/\tilde{\mu}(S^1) = 8\pi\) is the crucial factor appearing in Wyler’s formula which admits a natural Geometric probability explanation which is very different from the different interpretations provided in [3,4,5].

The Fiber Bundle interpretation associated with the \(U(1) \sim SO(2)\) group is the following. The Fiber bundle \(E\) is defined over the curved space \(D_2 = SO(2,2)/SO(2) \times SO(2)\). The subgroup \(H = SO(2) \sim U(1)\) of the isotropy group \(K = SO(2) \times SO(2)\) acts on the fibers identified with the symmetry space \(S^1\) (where the \(U(1)\) group acts). The Fiber bundle \(E\) locally can be written as \(D_2 \times S^1\) but not globally. The restriction of the Fiber bundle \(E\) to the Shilov boundary \(Q_2 = S^1 \times S^1/Z_2 = S^1 \times RP^1\) is \(E|_{Q_2}\) and locally can be written as \(Q_2 \times S^1\), but not globally. This is why the volume \(V(E|_{Q_2}) \neq V(Q_2) \times V(S_1)\) but instead it equals \((V(Q_2)/V(D_2)) \times V(S^1/Z_2) = 2V(S^2) = 8\pi\).

Concluding, the Geometric Probability that an electron emits a photon at \(t = -\infty\) and absorbs it at \(t = +\infty\) is given by the ratio of the ratios of measures, and it agrees with Wheeler’s ideas that one must normalize the couplings with respect to the geometric coupling strength of Gravity:

\[ \alpha_{EM} = \frac{2V(S^2) \tilde{\mu}(Q_4)/\tilde{\mu}(Q_4)}{\mu(Q_2)/\tilde{\mu}(Q_2)} = (8\pi) \frac{1}{V(S^4) V(Q_5)} \left[ V(D_3) \right]^4 = \frac{1}{137.03608}. \] (16)

The second important conclusion that can be derived from Geometric Probability theory is the general numerical values of the exponents \(s_n\) appearing in the factors \(V(D_n)^{s_n}\). The normalization factor \(V(Q_5)/V(D_5)^{1/4}\) in the construction of the ratio of measures \(\mu_{s_n}[Q_5]/\tilde{\mu}[Q_5]\) involves in this case powers of the type \(V(D_3)^{1/4}\). The power of \(1/4\) is related to the inverse of the \(\dim(S^1) = 4\) (the internal symmetry space \(SO(5)/SO(4)\)). From eq. (13) we learnt that the reduction factor of \(V(Q_2)/V(D_2)\) was \(V(D_2)\); i.e. the exponent is unity. The power of unity is related to the inverse of the \(\dim(S^1/Z_2) = 1\). Thus, the arguments based on Geometric Probability lead to normalized measures by factors of \(V(Q_n)/V(D_n)^{s_n}\) and whose exponents \(s_n\) are given by the inverse of the dimensions of the internal symmetry spaces \(s_n = (\dim(S^{n-1}))^{-1}\). There is a different interpretation of these factors \(V(D_n)^{s_n}\) given by Smith [3]. In general, for other homogeneous complex domains, this power is given by the inverse of the dimension of the internal symmetry space.

3. The Weak and Strong Coupling Constants from Geometric Probability

We turn now to the derivation of the other coupling constants. The Fiber Bundle picture of the previous section is essential in our construction. The Weak and the Strong geometric coupling constant strength, defined as the probability for a particle to emit and later absorb a \(SU(2), SU(3)\) gauge boson, respectively, can both be obtained by using the main formula derived from Geometric Probability after one identifies the suitable homogeneous domains and their Shilov boundaries to work with. We will show why the weak and strong couplings are given by

\[ \alpha_{Weak} = \frac{\mu(Q_2)/\tilde{\mu}(Q_2)}{\mu(Q_2)/\tilde{\mu}(Q_2)} = \frac{\mu(Q_2)/\tilde{\mu}(Q_2)}{\alpha_G} = \frac{\mu(Q_2)/\tilde{\mu}(Q_2)}{(8\pi/\alpha_{EM})}. \] (17)

and

\[ \alpha_{Color} = \frac{\mu(S^4)/\tilde{\mu}(S^4)}{\mu(Q_4)/\tilde{\mu}(Q_4)} = \frac{\mu(S^4)/\tilde{\mu}(S^4)}{\alpha_G} = \frac{\mu(S^4)/\tilde{\mu}(S^4)}{(8\pi/\alpha_{EM})}. \] (18)

The Shilov boundary of \(D_2\) is \(Q_2 = S^1 \times RP^1\) but is not large enough to accommodate the action of the isospin group \(SU(2)\). One needs a Fiber Bundle over \(D_3 = SO(3,2)/SO(3) \times SO(2)\) whose subgroup
$H = SO(3)$ of the isotropy group $K = SO(3) \times SO(2)$ acts on the internal symmetry space $S^2$ (the fibers). Since the coset space $SU(2)/U(1)$ is a double-cover of the $S^2$ as one goes from the $SO(3)$ action to the $SU(2)$ action one must take into account an extra factor of $2$. This is the reason why one jumps to one-dimension higher from $Q_2$ to $Q_3 = S^2 \times RP^1$, because the coset $SU(2)/U(1)$ is a double-cover of the sphere $S^2 = SO(3)/SO(2)$ and can accommodate the action of the $SU(2)$ group.

By following the same procedure as above, i.e. by re-writing the ratio of the measures $(\hat{\mu}[Q_2]/\mu[Q_3])$ in terms of the ratio of the measures $(\mu_N[Q_3]/\bar{\mu}[Q_3])$ via the embeddings of $D_2 \rightarrow D_3$, one has

\[
(\hat{\mu}[Q_2]/\mu[Q_3]) = \frac{1}{V(SU(2)/U(1))} \frac{\mu_N[Q_3]}{\bar{\mu}[Q_3]},
\]

(19)

Notice that because $SU(2)$ is a $2 - 1$ covering of the $SO(3)$, this implies that the measure

\[
V(SU(2)/U(1)) = 2 \cdot V(SO(3)/U(1)) = 2 \cdot V(S^3) = 8\pi.
\]

As indicated above, because the dimension of the internal symmetry space is $dim(S^2) = 2$, the construction of the normalized measure $\mu_N[Q_3]$ will require a reduction of $V(Q_3)$ by a factor of $V(D_3)$ raised to the power of $(dim(S^2))^{-1} = 1/2$ :

\[
\frac{\mu_N[Q_3]}{\bar{\mu}[Q_3]} = \frac{1}{V(Q_3)/V(D_3)^{1/2}} = \frac{1}{V(Q_3)} \frac{1}{V(D_3)^{1/2}}.
\]

(21)

Therefore, the ratio of the measures is

\[
\frac{\hat{\mu}[Q_2]}{\mu[Q_2]} = \frac{1}{2V(S^2)} \frac{1}{V(Q_3)} \frac{1}{V(D_3)^{1/2}}.
\]

(22)

whose Fiber Bundle interpretation is that the volume of the Fiber Bundle over $D_3$, but restricted to the Shilov boundary $Q_3$, and whose structure group is $SU(2)$ (the double cover of $SO(3)$), is $V(E|Q_3) = 2V(S^2) \times (V(Q_3)/V(D_3)^{1/2})$. Thus, that the Geometric probability expression is

\[
\alpha_{Weak} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(\mu[Q_2]/\bar{\mu}[Q_2])} = \frac{(\mu[Q_2]/\bar{\mu}[Q_2])}{(8\pi/\alpha_{EM})} = 2V(S^2) V(Q_3) \frac{1}{V(D_3)^{1/2}} \frac{\alpha_{EM}}{8\pi} = 0.2536.
\]

(23)

that corresponds to the weak geometric coupling constant $\alpha_W$ at an energy order of

\[
E = M = 146 \text{GeV} \sim \sqrt{M_{W_+}^2 + M_{W_-}^2 + M_Z^2}.
\]

(24)

after we have inserted the expressions

\[
V(S^2) = 4\pi, \quad V(Q_3) = 4\pi^2, \quad V(D_3) = \frac{\pi^3}{24},
\]

(25a)

into the formula (23). The relationship to the Fermi coupling $G_{Fermi}$ goes as follows (after indentifying the energy scale $E = M = 146 \text{GeV}$) :

\[
G_F \equiv \frac{\alpha_W}{M^2} \Rightarrow G_F \frac{m_{proton}^2}{M^2} = \frac{\frac{\alpha_W}{M^2}}{m_{proton}^2} = 0.235 \times \frac{(146 \text{GeV})^2}{\text{GeV}^2} \approx 1.04 \times 10^{-5}.
\]

(25b)

in very good agreement with experimental observations.

Once more, it is unknown why the value of $\alpha_{Weak}$ obtained from Geometric Probability corresponds to the energy scale related to the $W_+, W_-, Z_0$ boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of $\alpha_{Color}$ following the Fiber Bundle approach. Since $S^4$ is not large enough to accommodate the action of the color group $SU(3)$ one needs to work with one-dimension higher, $S^5$, that can be interpreted as the boundary of the 6D Ball $B_6 = SU(4)/U(3) = SU(4)/SU(3) \times U(1)$. A Fiber bundle over $B_6$ has the subgroup $H = SU(3)$ of the isotropy group $K = SU(3) \times U(1)$ acting on the internal symmetry space (the fibers) associated with the $SU(3)$ color group $CP^2 = SU(3)/U(2) = SU(3)/SU(2) \times U(1)$.
In this special case the Shilov and ordinary topological boundaries of $B_6$ coincide with $S^5$ [3]. Notice also that the volume $V(CP^2) = V(S^4) = 8\pi^2/3$ and that $SU(4)$ is the compact version of the noncompact and simple 4D conformal group $SU(2, 2)$ . The 4D conformal Lie algebra $so(4, 2) \sim su(2, 2)$. This justifies the use of $S^4$ in the expressions below.

Similarly, following the same procedures above, by restricting the Fiber bundle over $B_6$ to the Shilov boundary $S^5$ of the space $B_6$, written as $E_{S^5}$, and which locally is $S^5 \times CP^2$ , but not globally, the volume is no longer given by the naive product of volumes but instead it is $V(E_{S^5}) = V(CP^2) \times (V(S^5)/V(B_6)^{1/4}) = V(S^4) \times (V(S^5)/V(B_6)^{1/4})$. Hence, the ratio of the measures in $S^4$ (boundary of $B_6$) can be re-written in terms of the ratio of the measures in $S^5$ (boundary of $B_6$) via the embeddings of $B_5 \rightarrow B_6$ as follows:

$$\frac{\tilde{\mu}[S^4]}{\mu[S^2]} = \frac{1}{V(S^4)} \frac{\mu_N[S^5]}{\tilde{\mu}[S^5]} = \frac{1}{V(S^4)} \frac{V(S^5)/V(B_6)^{1/4}}{V(S^5)/V(B_6)^{1/4}} = \frac{1}{V(S^4)} \frac{1}{V(B_6)^{1/4}}.$$ (26)

since the exponent of the reduction factor $V(B_6)^{1/4}$ is given by $(dim(S^4))^{-1} = 1/4$. Notice, again, that $\tilde{\mu}[S^4]$ is the measure-density in $S^4$ and must not be confused with the normalized measures.

Therefore, one arrives at

$$\alpha_{\text{Color}} = V(S^4) \frac{V(S^5)}{V(B_6)^{1/4}} \frac{1}{V(B_6)^{1/4}} \frac{\alpha_{\text{EM}}}{8\pi} = 0.6286.$$ (27)

that corresponds to the strong coupling constant at an energy related to the pion masses [3] :

$$E = 241 \text{ MeV} \sim \sqrt{m_{\pi^+}^2 + m_{\pi^-}^2 + m_{\pi^0}^2}.$$ (28)

and where we have used the expressions :

$$V(S^4) = V(CP^2) = \frac{8\pi^2}{3}. \quad V(S^5) = 4\pi^3. \quad V(B_6) = \frac{\pi^3}{6}.$$ (29)

The pions are the known lightest quark/antiquark pairs that feel the strong interaction [3]. Once again, it is unknown why the value of $\alpha_{\text{Color}}$ obtained from Geometric Probability (28) corresponds to the energy scale related to the masses of the three pions [3] . Masses of the fundamental particles were derived in [3] based on the definitions that mass is the probability amplitude for a particle to change direction.

To conclude, by defining the geometric coupling constants as the Geometric Probability to emit (and later absorb) a gauge boson, all the three geometric coupling constants, $\alpha_{\text{EM}}; \alpha_{\text{Weak}}; \alpha_{\text{Color}}$ are given by the ratios of the ratios of the measures of the Shilov boundaries $Q_2 = S^4 \times RP^1$; $Q_3 = S^5 \times RP^1$; $S^5$, respectively, with respect to the ratios of the measures $\tilde{\mu}[Q_5]/\mu_N[Q_3]$ associated with the 5D conformally compactified real Minkowski spacetime $M_5$ that has the same topology as the Shilov boundary $Q_5$ of the 5 complex-dimensional poly-disc $D_5$. The latter corresponds to the conformal relativistic curved 10 real-dimensional phase space $\mathcal{H}_10$ associated with a particle moving in the 5D Anti de Sitter space $AdS_5$.

It is not known whether this procedure would work for Grand Unified Theories based on the groups

$$SU(5), SO(10), E_6, E_7, E_8.$$ (30)

Beck [6] has obtained all the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dim lattices based on Stochastic Quantization methods. The results above and by Smith [3] are analytical rather than being numerical [6] and it is not clear if there is any relationship between these two approaches. Noyes has proposed an iterated numerical hierarchy based on Mersenne primes $M_p = 2^p - 1$ for certain values of $p$ = primes [18] and obtained many numerical values for the physical parameters. Pitkanen has developed methods to calculate the physical masses recurring to a p-adic hierarchy of scales based on Mersenne primes [19].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in [22]. These are numbers which can be written in terms of sums of its divisors, including unity, like $6 = 1 + 2 + 3$, and are of the form $P(p) = \frac{1}{2} 2^p(2^p - 1)$ if, and only if, $2^p - 1$ is a Mersenne prime. Not all values of $p$ = prime yields primes. The number $2^{11} - 1$ is not a Mersenne prime, for example. The
number of generators of the anomaly free groups $SO(32)$, $E_8 \times E_8$ of the 10-dim superstring is 496 which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the $SO(2^{13}) = SO(8192)$ group related to the bosonic string compactified on the $E_8 \times SO(16)$ lattice. The number of generators of $SO(8192)$ is an even perfect number since $2^{13} - 1$ is a Mersenne prime. For an introduction to p-adic numbers in Physics and String theory see [20]. A lot more work needs to be done to be able to answer the question: Is all this just a mere numerical coincidence or is it design?

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