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Real and apparent invariants in the transformation of the equations governing wave-motion in the general flow of a general fluid

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The ten equations are derived that govern, to the first order, the propagation of small general perturbations in the general unsteady flow of a general fluid, in three space-variables and time. The condition that any hypersurface is a wave hypersurface of these equations is obtained, and the envelope of all such wave hypersurfaces that pass through a given point at a given time, i.e. the wave hyperconoid, is determined. These results, which are all invariant under Galilean transformation, are progressively specialized, through homentropic flow and irrotational homentropic flow, to steady uniform flow, for which both the convected wave-equation and the standard wave-equation, with their wave hypersurfaces, are finally recovered.

A special class of reference-frames is considered, namely those whose origins move with the fluid. It is then shown that, for observers at the origins of all such reference frames, the wave hypersurfaces satisfy specially simple equations locally. These equations are identical with those for waves in a uniform fluid at rest relative to the reference frame, except that the wave speed is not constant but varies with position and time in accordance with the variable mean flow. These specially simple equations appear to be invariant for Galilean transformations between all such observers.

These results are briefly applied, in reverse order, to Maxwell's equations, and to equations more general than Maxwell's, for the electric and magnetic field-strengths.

1. Introduction

Much has been written about the generalization of small-amplitude wave-motion, or the propagation of small unsteady perturbations, in fluids, including viscous effects, vortical or entropic perturbations, etc. These generalizations have, however, been almost entirely restricted to fluids that satisfy the ideal gas equations, $p=\rho RT$, $\overline{\gamma}=\overline{c}_p/\overline{c}_v=$ const., and assume, for instance, sometimes that the mean flow is steady and uniform, sometimes that the reference frame is that relative to which the uniform fluid is at rest, sometimes that the mean velocity $\mathbf{u}$ is such that $M^2=u^2/c^2 << 1$, so that the mean flow may be regarded as 'incompressible', sometimes that the disturbance to the flow from the steady state is irrotational, etc.

In contrast, very little has been written about general wave motion in the general unsteady flow of a general fluid. Some attention is given to this problem here, in the case, at least, when the effects of viscosity and heat-conduction can be neglected, and when the effects on the flow, of the gravitational field and any acceleration of the reference-frame, are also negligible.

The ten simultaneous linear partial differential equations that govern, to the first order, small-amplitude wave-motions in the general unsteady flow of a general fluid in three space-variables and time are derived. The mathematical condition for any hypersurface to be a characteristic hypersurface of these ten equations is then obtained. This leads to the condition for any hypersurface to be a wave hypersurface, and to the determination of the envelope of all such wave hypersurfaces that pass through a given point at a given time, namely the characteristic wave-hyperconoid through a point.

These results are progressively specialized, first to homentropic flow, then to irrotational homentropic flow and finally to steady uniform flow (recovering the convected or progressive wave-equation and its wave hypersurfaces) and to the reference-frame relative to which this steady uniform flow is at rest (recovering the standard wave-equation and its wave hypersurfaces). Because the general solution of the convected or progressive wave-equation does not appear to have been given previously in the literature, it is given here in an Appendix.

A special class of reference frames is then considered, namely those whose origins move with the fluid. The equations governing the motion, the condition for any hypersurface to be a wave hypersurface and the equation of the wave hyperconoid all take special simple forms at the origins of all such reference frames. Moreover, if the effects of the local acceleration of the fluid, and therefore of the reference frame, can be neglected, all these special simple forms of the equations and conditions appear to be invariant for Galilean transformations between observers at the origins of all such reference frames. These special simple forms of the wave hypersurfaces are shown to be identical to those for waves in a uniform fluid at rest, except that, in them, the wave speed is not constant but varies with position and time in accordance with the general unsteady mean flow.

The application of these results to electromagnetic waves is discussed briefly. For Maxwell's equations, the wave hypersurfaces and the wave hyperconoid are identical to those for waves in uniform flow at rest relative to the reference frame. When Maxwell's equations are transformed, by Galilean transformation, to any other reference frame, they become the convected or progressive form of Maxwell's equations (cf. Thornhill 1985b), and these have wave hypersurfaces and a wave hyperconoid that are identical to those for waves in a fluid in steady uniform motion relative to the reference frame. More general equations for the electric and magnetic field-strengths are constructed, in which the magnetic permeability, the permittivity, and therefore the wave speed, are not constant but vary with position and time, by allowing that there is a fluid ethereal medium in which electromagnetic waves propagate.These more general equations may be shown to have wave hypersurfaces and a wave hyperconoid that are identical to those for general waves in a general fluid in a general unsteady motion. It follows that, for all observers at the origins of rest-frames that move with this ethereal medium, the wave hypersurfaces will all have the special simple form, apparently invariant under Galilean transformation, which makes them, except for a variable rather than a constant wave speed, identical with those for Maxwell's equations. In particular, for all such observers, their local wave-hyperconoid will invariably be

$$(dx)^2 + (dy)^2 + (dz)^2 = c^2 \, (dt)^2 \, .$$

2. Small-amplitude waves in general unsteady flow

When the effects of viscosity and heat-conduction can be neglected and when the influence on the flow both of the local gravitational field and of any acceleration of the reference frame is negligible also, then the Eulerian equations governing the general unsteady motion of any general fluid, in three space variables $x_i (i = 1, 2, 3)$ and the time t are, with the summation convention,

$$\text{(mass)} \quad \frac{Dv}{Dt} - v \frac{\partial u_j}{\partial x_j} = 0 \tag{2.1}$$

$$\text{(momentum)} \quad \frac{Du_i}{Dt} + v \frac{\partial p}{\partial x_i} = 0 \tag{2.2a}$$

$$\text{(energy)} \quad \frac{DS}{Dt} = 0 \tag{2.3}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}$$

is the Eulerian total time-derivative, moving with the fluid. Here p denotes pressure, v specific volume, S specific entropy and (u_1, u_2, u_3) or $\{u_i\}$ is the velocity of the fluid.

The general thermodynamics of any fluid can be specified by a function $E(v, S)$ which expresses the specific energy per unit mass, E, as a function of v and S. If suffixes v, S are used to denote partial derivatives, p may then be written as $(-E_v)$ so that

$$\frac{\partial p}{\partial x_i} = -E_{vv} \frac{\partial v}{\partial x_i} - E_{vS} \frac{\partial S}{\partial x_i} \tag{2.4}$$

(see, for example, Swan and Thornhill 1974). The equations (2.2a) may then be re-written as

$$\frac{Du_i}{Dt} - vE_{vv} \frac{\partial v}{\partial x_i} - vE_{vS} \frac{\partial S}{\partial x_i} = 0. \tag{2.2b}$$

The five equations (2.1), (2.2b) and (2.3) are a system of linear first-order partial differential equations which determine the five dependent variables u_i, v, S as functions of the four independent variables x_i, t.

Let u_i, v, S, E_{vv} and E_{vS} refer to local values, at any time, in any solution of (2.1), (2.2b), and (2.3), and denote small perturbations in these quantities by

$$u_i^* = u_i + \epsilon_i, \quad v^* = v + \eta, \quad S^* = S + \sigma,$$

where ϵ_i, η and σ are small quantities of the first order. Then, to the first order,

$$E^*_{vv} = E_{vv} + \eta E_{vvv} + \sigma E_{vvS}$$

and

$$E^*_{vS} = E_{vS} + \eta E_{vvS} + \sigma E_{vSS}.$$

The conditions that u^*_i, v^*_i, S^*_i also satisfy the equations (2.1), (2.2b), and (2.3) reduce, to the first order, to

$$\frac{D\eta}{Dt} + \epsilon_j \frac{\partial v}{\partial x_j} - v\frac{\partial \epsilon_j}{\partial x_j} - \eta \frac{\partial u_j}{\partial x_j} = 0. \tag{2.5}$$

$$\frac{D\epsilon_i}{Dt} + \epsilon_j \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i}(v\eta E_{vv}) - \frac{\partial}{\partial x_i}\big(v\sigma E_{vS}\big) + \sigma E_{vS}\frac{\partial v}{\partial x_i} - \eta E_{vS}\frac{\partial S}{\partial x_i} = 0, \tag{2.6}$$

$$\frac{D\sigma}{Dt} + \epsilon_j \frac{\partial S}{\partial x_j} = 0. \tag{2.7}$$

Thus the five equations (2.5)$-$(2.7), together with the five equations (2.1), (2.2b), and (2.3) which govern the mean fluid-motion, form a system of ten simultaneous equations in the ten dependent variables $u_i, v, S, \epsilon_i, \eta, \sigma$, which govern, to the first order, general small-amplitude wave-motions in the general unsteady flow of any general fluid.

It is known that, in general, the perturbation equations (2.5) $-$ (2.7) do not lead to characteristics different from those of the equations for the mean motion, (2.1), (2.2b) and (2.3). However, because (2.5) $-$ (2.7) involve all ten independent variables, it is not immediately obvious how this comes about, nor does it appear to have been demonstrated previously in the literature, so the derivation of the characteristics for the full set of ten equations is given here.

The hypersurface, $\zeta(x_i;t) = \text{const.}$, is a characteristic hypersurface of these ten equations if (see, for example, Thornhill 1985b), a tenth-order determinant vanishes (see facing page). In (2.8) the rows correspond to the ten equations and the columns to the ten dependent variables, and

$$A = \eta E_{vv} + v\eta E_{vvv} + v\sigma E_{vvS}, \quad B = \eta E_{vS} + \eta v E_{vvS} + \eta\sigma E_{vSS}.$$

The condition (2.8) reduces to

$$\left(\frac{D\zeta}{Dt}\right)^6 \left\{ \left(\frac{D\zeta}{Dt}\right)^2 - v^2 E_{vv}\left[\left(\frac{\partial \zeta}{\partial x_1}\right)^2 + \left(\frac{\partial \zeta}{\partial x_2}\right)^2 + \left(\frac{\partial \zeta}{\partial x_3}\right)^2\right]\right\}^2 = 0. \tag{2.9}$$

$$
\begin{array}{r}
\\
(2.1)\\
(2.2b)\\
(2.2b)\\
(2.2b)\\
(2.3)\\
(2.5)\\
(2.6)\\
(2.6)\\
(2.6)\\
(2.7)
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
(u_1) & (u_2) & (u_3) & (v) & (S) & (\epsilon_1) & (\epsilon_2) & (\epsilon_3) & (\eta) & (\sigma)
\end{array}\\
\left|
\begin{array}{cccccccccc}
-v\frac{\partial\zeta}{\partial x_1} & -v\frac{\partial\zeta}{\partial x_2} & -v\frac{\partial\zeta}{\partial x_3} & \frac{D\zeta}{Dt} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{D\zeta}{Dt} & 0 & 0 & -vE_{vv}\frac{\partial\zeta}{\partial x_1} & -vE_{vS}\frac{\partial\zeta}{\partial x_1} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{D\zeta}{Dt} & 0 & -vE_{vv}\frac{\partial\zeta}{\partial x_2} & -vE_{vS}\frac{\partial\zeta}{\partial x_2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{D\zeta}{Dt} & -vE_{vv}\frac{\partial\zeta}{\partial x_3} & -vE_{vS}\frac{\partial\zeta}{\partial x_3} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{D\zeta}{Dt} & 0 & 0 & 0 & 0 & 0\\
-\eta\frac{\partial\zeta}{\partial x_1} & -\eta\frac{\partial\zeta}{\partial x_2} & -\eta\frac{\partial\zeta}{\partial x_3} & \epsilon_j\frac{\partial\zeta}{\partial x_j} & 0 & -v\frac{\partial\zeta}{\partial x_1} & -v\frac{\partial\zeta}{\partial x_2} & -v\frac{\partial\zeta}{\partial x_3} & \frac{D\zeta}{Dt} & 0\\
\epsilon_j\frac{\partial\zeta}{\partial x_j} & 0 & 0 & -A\frac{\partial\zeta}{\partial x_1} & -B\frac{\partial\zeta}{\partial x_1} & \frac{D\zeta}{Dt} & 0 & 0 & -vE_{vv}\frac{\partial\zeta}{\partial x_1} & -vE_{vS}\frac{\partial\zeta}{\partial x_1}\\
0 & \epsilon_j\frac{\partial\zeta}{\partial x_j} & 0 & -A\frac{\partial\zeta}{\partial x_2} & -B\frac{\partial\zeta}{\partial x_2} & 0 & \frac{D\zeta}{Dt} & 0 & -vE_{vv}\frac{\partial\zeta}{\partial x_2} & -vE_{vS}\frac{\partial\zeta}{\partial x_2}\\
0 & 0 & \epsilon_j\frac{\partial\zeta}{\partial x_j} & -A\frac{\partial\zeta}{\partial x_3} & -B\frac{\partial\zeta}{\partial x_3} & 0 & 0 & \frac{D\zeta}{Dt} & -vE_{vv}\frac{\partial\zeta}{\partial x_3} & -vE_{vS}\frac{\partial\zeta}{\partial x_3}\\
0 & 0 & 0 & 0 & \epsilon_j\frac{\partial\zeta}{\partial x_j} & 0 & 0 & 0 & 0 & \frac{D\zeta}{Dt}
\end{array}
\right|
\end{array}
= 0, \tag{2.8}
$$

It is found (loc. cit.) that the condition $D\zeta/Dt = 0$ corresponds to the world-lines of the flow, whereas the vanishing of the second factor in equation (2.9) gives the condition for $\zeta(x_i;t) = \text{const.}$ to be a wave hypersurface. The wave hypersurfaces through any point, at any time, have an envelope, the local wave hyperconoid, and this is found to be (loc. cit.)

$$\left(dx_1 - u_1 dt\right)^2 + \left(dx_2 - u_2 dt\right)^2 + \left(dx_3 - u_3 dt\right)^2 = c^2 \left(dt\right)^2, \tag{2.10}$$

where c is the local thermodynamic wave speed, defined by (cf. Swan and Thornhill 1974)

$$c^2 = v^2 E_{vv}. \tag{2.11}$$

The system of ten governing equations (2.1), (2.2*b*), (2.3) and (2.5) – (2.7), the condition (2.9) for $\zeta =$ const. to be a wave hypersurface, and the wave hyperconoid (2.10) are easily shown (Thornhill 1985b) to be invariant under Galilean transformation.

3. Small-amplitude waves in general unsteady homentropic flow

If both the mean flow and the perturbed flow are homentropic, the equations (2.3) and (2.7) are satisfied by $S = \text{const.}$, and $\sigma = 0$ everywhere at all times. The equations (2.2*b*) and (2.6) then reduce, respectively, to

$$\frac{Du_i}{Dt} - \left(\frac{c^2}{v}\right)\frac{\partial v}{\partial x_i} = 0. \tag{3.1}$$

$$\frac{D\epsilon_i}{Dt} + \epsilon_j \frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{c^2\eta}{v}\right) = 0. \tag{3.2}$$

The system of equations governing small-amplitude homentropic wave-motion in general unsteady homentropic flow now reduces to the eight equations (2.1), (2.5), (3.1) and (3.2), which, again, are invariant under Galilean transformation. The condition (2.9) for $\zeta = \text{const.}$ to be a wave hypersurface is unchanged, and the wave hyperconoid is still given by (2.10). Both of these remain, therefore, invariant under Galilean transformation.

If both the mean flow and the perturbed flow are not only homentropic, but also irrotational or non-vortical, then

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \quad \frac{\partial \epsilon_i}{\partial x_j} = \frac{\partial \epsilon_j}{\partial x_i}. \tag{3.3}$$

The second of these conditions makes it possible to introduce a potential function $\phi\left(x_i;t\right)$ such that

$$\epsilon_i = \frac{\partial \phi}{\partial x_i}. \tag{3.4}$$

Then, if $\partial u_i/\partial x_j$ is replaced by $\partial u_j/\partial x_i$ in the three equations (3.2), they may be written as

$$\frac{\partial}{\partial x_i}\left[\frac{D\phi}{Dt} - \frac{c^2\eta}{v}\right] = 0$$

and thus integrated to give

$$\frac{D\phi}{Dt} - \frac{c^2\eta}{v} = 0 \tag{3.5}$$

since the potential function ϕ can be chosen so that the arbitrary function of time, introduced by these integrations, is zero.

It then follows, very simply, from equations (2.5), (3.5) and (2.1) that

$$\begin{aligned}\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D\phi}{Dt}\right) &= \frac{1}{v}\frac{D\eta}{Dt} - \frac{\eta}{v^2}\frac{Dv}{Dt} \\ &= \frac{\partial^2\phi}{\partial x_j \partial x_j} - \frac{1}{v}\frac{\partial\phi}{\partial x_j}\frac{\partial v}{\partial x_j}\end{aligned}$$

and so

$$\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D\phi}{Dt}\right) = v\frac{\partial}{\partial x_j}\left(\frac{1}{v}\frac{\partial\phi}{\partial x_j}\right). \tag{3.6}$$

The system of equations that govern small-amplitude homentropic irrotational wave motion in general unsteady homentropic irrotational flow in a general fluid is now reduced to the five equations (2.1), (3.1) and (3.6) together with the conditions (3.3), whereas the condition (2.9) for $\zeta =$ const. to be a wave hypersurface, and the wave hyperconoid (2.10), are unchanged. All these equations and conditions remain invariant under Galilean transformation.

4. Small-amplitude waves in steady uniform flow

If the mean flow is steady and uniform, and therefore both homentropic and irrotational, the constant fluid velocity may be denoted by $\{\overline{u}_i\}$ and the constant specific volume and constant wave speed by $\overline{v}$ and $\overline{c}$ respectively. The system of equations governing small-amplitude homentropic irrotational wave motion in such a flow now reduces, by virtue of (3.6), to the single equation (cf. Temple 1953),

$$\nabla^2\phi = \frac{1}{\overline{c}^2}\frac{D^2\phi}{Dt^2}, \tag{4.1}$$

which is sometimes called the convected or progressive wave-equation. The condition for $\zeta =$ const. to be a wave hypersurface now becomes

$$\left(\frac{D\zeta}{Dt}\right)^2 = \overline{c}^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2 + \left(\frac{\partial\zeta}{\partial x_2}\right)^2 + \left(\frac{\partial\zeta}{\partial x_3}\right)^2\right] \tag{4.2}$$

and the local wave hyperconoid becomes

$$\left(dx_1 - \overline{u}_1 dt\right)^2 + \left(dx_2 - \overline{u}_2 dt\right)^2 + \left(dx_3 - \overline{u}_3 dt\right)^2 = \overline{c}^2\left(dt\right)^2. \tag{4.3}$$

The equations (4.1) – (4.3) are all invariant under Galilean transformation. The general solution of the progressive wave-equation (4.1) has not, so far as is known, appeared previously in the literature, and so it is given here in an Appendix.

There is one unique frame of reference relative to which a fluid in steady uniform motion is at rest. In this frame of reference, and in this frame only, the equations (4.1) – (4.3) reduce respectively to

$$\nabla^2\phi = \frac{1}{\bar{c}^2}\frac{\partial^2\phi}{\partial t^2}, \tag{4.4}$$

$$\left(\frac{\partial\zeta}{\partial t}\right)^2 = \bar{c}^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2 + \left(\frac{\partial\zeta}{\partial x_2}\right)^2 + \left(\frac{\partial\zeta}{\partial x_3}\right)^2\right] \tag{4.5}$$

and

$$\left(dx_1\right)^2 + \left(dx_2\right)^2 + \left(dx_3\right)^2 = \bar{c}^2\left(dt\right)^2. \tag{4.6}$$

It is clear that none of these equations can be invariant under Galilean transformation, since the fluid must be moving relative to any other frame of reference. It is easily shown that, under Galilean transformation to any other frame of reference, the equations (4.4) – (4.6) revert to the forms (4.1) – (4.3) respectively, exactly as they should and must.

5. Apparent invariants for a particular class of reference-frames

Consider a frame of reference O'(x_i') whose axes O'x_i' are parallel to Ox_i but whose origin O' is moving with the fluid. Let O' be at the point $X_i(t)$ at time t, so that the velocity of the fluid at O' is $u_i = \dot{X}_i(t)$.

The Galilean transformation between the two reference-frames O(x_i) and O'(x_i') is thus governed by the relations

$$x_i = X_i(t) + x_i', \quad t = t', \tag{5.1}$$

whence

$$\left.\begin{array}{ll} dx_i = \dot{X}_i(t')dt' + x_i', & dx_i' = dx_i - \dot{X}_i(t)dt, \\ dt = dt', & dt' = dt, \\ u_i = \dot{X}_i(t') + u_i', & u_i' = u_i - \dot{X}_i(t) \end{array}\right\} \tag{5.2}$$

and so

$$\left.\begin{array}{l} \dfrac{\partial}{\partial x_i} \equiv \dfrac{\partial}{\partial x_i'}, \quad \dfrac{\partial}{\partial x_i'} \equiv \dfrac{\partial}{\partial x_i}, \\ \dfrac{\partial}{\partial t} \equiv \dfrac{\partial}{\partial t'} - \dot{X}_i(t')\dfrac{\partial}{\partial x_i'}, \quad \dfrac{\partial}{\partial t'} \equiv \dfrac{\partial}{\partial t} + \dot{X}_i(t)\dfrac{\partial}{\partial x_i}, \\ dx_i - u_i dt = dx_i' - u_i' dt', \\ \dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + u_i\dfrac{\partial}{\partial x_i} \equiv \dfrac{\partial}{\partial t'} - \dot{X}_i(t')\dfrac{\partial}{\partial x_i'} \\ \qquad + \left[u_i' + \dot{X}_i(t')\right]\dfrac{\partial}{\partial x_i'} \equiv \dfrac{\partial}{\partial t'} + u_i'\dfrac{\partial}{\partial x_i'} \equiv \dfrac{D}{Dt'}. \end{array}\right\} \tag{5.3}$$

At O',

$$u'_i = 0, \quad u_i = \dot{X}_i(t), \quad \text{and} \quad \frac{\partial u_i}{\partial x_j} = \frac{\partial u'_i}{\partial x'_j}. \tag{5.4}$$

In the new frame of reference, O'(x'_i), the equations (2.1), (2.2a) and (2.3) become, respectively

$$\frac{Dv}{Dt'} - v\frac{\partial u'_j}{\partial x'_j} = 0, \tag{5.5}$$

$$\frac{Du'_i}{Dt} + \ddot{X}_i(t') + v\frac{\partial p}{\partial x'_i} = 0, \tag{5.6}$$

$$\frac{DS}{Dt'} = 0. \tag{5.7}$$

Thus, if the influence on the flow of the acceleration of O', namely $\ddot{X}_i(t')$, can again be neglected, the equations (2.1), (2.2a) and (2.3) are invariant under the transformation (5.3). The condition (2.9) for the hypersurface $\zeta =$ const. to be a wave hypersurface, and the equation (2.10) of the wave hyperconoid are also clearly invariant under the transformation (5.3).

At the origin O' of the new reference-frame, O'(x'_i), however, where $u'_i = 0$, the equations (5.5) – (5.7), neglecting $\ddot{X}_i(t')$, reduce respectively to

$$\frac{\partial v}{\partial t'} - v\frac{\partial u'_j}{\partial x'_j} = 0, \tag{5.8}$$

$$\frac{\partial u'_i}{\partial t} + v\frac{\partial p}{\partial x'_i} = 0, \tag{5.9}$$

$$\frac{\partial S}{\partial t'} = 0. \tag{5.10}$$

The condition for the hypersurface $\zeta =$ const. to be a wave hypersurface reduces, at O', to

$$\left(\frac{\partial \zeta}{\partial t'}\right)^2 = c^2\left[\left(\frac{\partial \zeta}{\partial x'_1}\right)^2 + \left(\frac{\partial \zeta}{\partial x'_2}\right)^2 + \left(\frac{\partial \zeta}{\partial x'_3}\right)^2\right] \tag{5.11}$$

and the equation of the wave hyperconoid through the point O', at any time t' reduces to

$$(dx'_1)^2 + (dx'_2)^2 + (dx'_3)^2 = c^2\,(dt')^2\,. \tag{5.12}$$

Equations (5.11) and (5.12) can now be contrasted with (4.5) and (4.6). Equations (4.5) and (4.6), on the one hand, are restricted to a uniform fluid with $\bar{c}$ constant, and cannot be invariant under transformation, since they are valid only in the one unique reference-frame relative to which the fluid is stationary. Equations (5.11) and (5.12), on the other hand, are valid, and appear to be invariant, to all observers at the origins O' of rest-frames which travel with the fluid, however general and unsteady the flow, and whatever the fluid, and with c, therefore, not constant, but having its local value at any time.

6. Electromagnetic waves

The general results obtained here, and then successively specialized, may be applied, in reverse order, to electromagnetic waves.

Thus, Maxwell's equations not only have characteristic hypersurfaces and a characteristic wave hyperconoid given by (4.5) and (4.6), but they reduce precisely, when there is no current or charge distribution, to (4.4), the standard wave equation. All these equations are special reduced equations which are valid in only one unique frame of reference, and are not, therefore, general equations that remain invariant under transformation to a new frame of reference.

When Maxwell's equations are transformed by Galilean transformation to a frame of reference moving with a constant relative velocity, $\{-\overline{u}_j\}$, they become the progressive form of Maxwell's equations (Thornhill 1985b) in which the operator $\partial/\partial t$ is replaced by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}. \tag{6.1}$$

This progressive form of Maxwell's equations has characteristics given by (4.2) and (4.3), and these progressive equations reduce precisely to (4.1), the progressive wave equation, when there is no current or charge distribution. All these equations are general equations which remain invariant under Galilean transformation.

If there were an ethereal fluid medium in which electromagnetic waves propagate (cf. Thornhill 1985a), then Maxwell's equations could apply only to a uniform ether, and only in the one unique frame of reference relative to which this uniform ether is at rest. If, for instance, this uniform ether were moving with a constant velocity $\{\overline{u}_i\}$ relative to the frame of reference, then, in the derivation of Maxwell's equations, the operator $\partial/\partial t$ would have to be replaced by Euler's total time-derivative moving with the fluid, namely D/Dt as given by the relation (6.1) above. Thus, again, Maxwell's equations would be replaced by their progressive form.

In such case it follows that the equations for the electromagnetic field could be extended to give a general set of equations which would apply to any general unsteady motion of such a fluid ether. In these equations the magnetic permeability, μ, of the ether, the permittivity, ϵ, of the ether, and the electromagnetic wave-speed, c, would all be functions of the local values of v, S in the ether at any time, such that

$$c(v,S) = [\epsilon(v,S)\mu(v,S)]^{-\frac{1}{2}}.$$

These general equations for the electromagnetic field would then take the form (see, for example, Bleaney and Bleaney 1976)

$$\frac{\partial(\epsilon E_i)}{\partial x_i} = 0, \tag{6.2}$$

$$\frac{\partial(\mu H_i)}{\partial x_i} = 0, \tag{6.3}$$

$$\frac{D\left(\epsilon E_i\right)}{Dt}=\varepsilon_{ijk}\frac{\partial H_k}{\partial x_j}, \tag{6.4}$$

$$\frac{D\left(\mu H_i\right)}{Dt}=-\varepsilon_{ijk}\frac{\partial E_k}{\partial x_j}. \tag{6.5}$$

Here $\{E_i\}$ is the electric field-strength, $\{H_i\}$ the magnetic field-strength, $\{u_i\}$ the velocity of the ethereal fluid, ε_{ijk} the alternating tensor, and now

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+u_j\frac{\partial}{x_j}. \tag{6.6}$$

These general equations (6.2) – (6.5) contain derivatives of v and S and so, to determine their characteristics, they must be combined with the general equations of the ethereal motion, namely (2.1), (2.2b) and (2.3), to give thirteen consistent equations in the eleven independent variables $\{u_i\}, v, S, \{E_i\}, \{H_i\}$. The characteristics of this system of thirteen equations can be shown to be given by (cf. Thornhill 1985b),

$$\begin{aligned}&\left(\frac{D\zeta}{Dt}\right)^5\left\{\left(\frac{D\zeta}{Dt}\right)^2-c_1^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2+\left(\frac{\partial\zeta}{\partial x_2}\right)^2+\left(\frac{\partial\zeta}{\partial x_3}\right)^2\right]\right\}\\ &\times\left\{\left(\frac{D\zeta}{Dt}\right)^2-c_2^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2+\left(\frac{\partial\zeta}{\partial x_2}\right)^2+\left(\frac{\partial\zeta}{\partial x_3}\right)^2\right]\right\}^2=0,\end{aligned} \tag{6.7}$$

where c_1, c_2 are, respectively, the local contemporary values of the thermodynamic and electromagnetic wave speeds. The relations (6.7) lead, in general, to two characteristic wave hyperconoids as given by (2.10), but with c_1 and c_2 instead of c. Of course, if the two wave speeds were equal, the waves would become electromagnetic condensational waves (cf. Thornhill 1985b) with a single wave speed c, a single system of characteristic hypersurfaces and a unique characteristic wave hyperconoid at every point and time, given by (2.10).

If the results of Section 5 are now applied to such waves, it becomes clear that, at the origins of all frames of reference which move with this fluid ether, the local characteristic hypersurfaces would be given by the relation (5.11), and the unique characteristic wave hyperconoid would be given by (5.12), namely

$$\left(dx_1\right)^2+\left(dx_2\right)^2+\left(dx_3\right)^2=c^2\left(dt\right)^2. \tag{6.8}$$

This simple local wave hyperconoid, in which the wave-speed c is a function of position and time, would thus appear to be invariant for Galilean transformations between all observers at the origins of reference which move with the local ether.

Appendix. The general solution of the equation of small-amplitude wave motion in a uniform medium in steady motion

The convected or progressive equation of wave motion, in any uniform medium in steady motion, is derived, equation (4.1), as

$$\nabla^2 \phi = \left(\frac{1}{\overline{c}^2}\right) \frac{D^2 \phi}{Dt^2}, \tag{A1}$$

in which

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j} \tag{A2}$$

and the fluid velocity $\{\overline{u}_i\}$ and the wave speed $\overline{c}$ are constant.

Any solution of the equation (A1) may be expressed in the form

$$\begin{aligned} \phi = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F \Big[&(x_1 - \overline{u}_1 t) \cos\theta \sin\psi + (x_2 - \overline{u}_2 t) \sin\theta \sin\psi \\ &+ (x_3 - \overline{u}_3 t) \cos\psi + \overline{c}t, \theta, \psi \Big] d\theta d\psi, \end{aligned} \tag{A3}$$

where F is any function (of three variables) which permits differentiation under the integral signs.

In the one unique frame of reference in which $\overline{u}_1 = \overline{u}_2 = \overline{u}_3 = 0$, the equation (A1) reduces to the standard form

$$\nabla^2 \phi = \left(\frac{1}{\overline{c}^2}\right) \frac{\partial^2 \phi}{\partial t^2} \tag{A4}$$

and the general solution (A3) reduces to

$$\phi = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F \Big[x_1 \cos\theta \sin\psi + x_2 \sin\theta \sin\psi + x_3 \cos\psi + \overline{c}t, \theta, \psi \Big] d\theta d\psi, \tag{A5}$$

(cf. Whittaker and Watson 1927).

In the case of a single space-variable x_3, ψ may take only the values $0, \pi$. The solution (A3) then reduces to

$$\phi = f\left(x_3 - \overline{u}_3 t - \overline{c}t\right) + g\left(x_3 - \overline{u}_3 t + \overline{c}t\right), \tag{A6}$$

where f and g are arbitrary functions.

In the one unique frame of reference in which $\overline{u}_3 = 0$, the solution (A6) further reduces to the familiar form

$$\phi = f\left(x_3 - \overline{c}t\right) + g\left(x_3 + \overline{c}t\right). \tag{A7}$$

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Appendix 2
(of original manuscript)
Electro-magnetic waves

(This appendix had to be replaced by Section 6 of *Proc. Roy. Soc. Lond.* A (1993) **442**, 492-504.)

In the derivation of Maxwell's equations, for the electro-magnetic field with a constant wave-speed $\bar{c}$, no account whatsoever is taken of the motion, relative to the frame of reference, of any possible ethereal medium in which electro-magnetic waves may propagate. It follows, therefore, that Maxwell's equations can only be valid either (a) if there is no ethereal medium, or (b) if the validity of the equations is confined to a unform ether, with constant wave-speed $\bar{c}$, and to the one unique frame of reference relative to which this uniform ether is at rest.

In the latter case, (b), there are two alternative methods of deriving the new form of Maxwell's equations appropriate to any other reference-frame, relative to which the ether is in steady uniform flow. One method is to realise that the time-derivative $\partial/\partial t$ in Maxwell's equations must be replaced, when the uniform ether is moving with constant velocity $\{\bar{u}_i\}$ relative to the reference-frame, by Euler's total time-derivative, moving with the fluid, namely

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \tag{A.2.1}$$

(cf. Thornhill 1985b). The other method is simply to transform Maxwell's equations, by Galilean transformation, to any other reference-frame relative to which the ether must be in steady uniform motion. Both of these methods yield exactly the same result.

It is easy to verify that the condition for the hypersurface $\zeta(x_i;t) =$ const. to be a wave-hypersurface of Maxwell's equations is that given by the condition (4.5), and that the wave-hyperconoid of Maxwell's equations is that given by equation (4.6), exactly as one would expect if the alternative (b) above applied. Thus, if the constant wave-speeds are equal, these same equations apply to the propagation of perturbations in the electric and magnetic field-strengths, as given by Maxwell's equations, as well as to the propagation of longitudinal condensational perturbations in any uniform ethereal medium at rest relative to the reference-frame. It follows, then, that, in any possible fluid ether, all three types of perturbation could propagate together, along the same wave-hypersurfaces, as electro-magnetic-condensational waves with a common wave-speed $\bar{c}$ (cf. Thornhill 1985b). Further, when there is no current or charge distribution, Maxwell's equations reduce precisely to the single equation (4.4), the standard wave-equation for irrotational homentropic waves in a uniform fluid at rest.

The accepted method, at present, of transforming Maxwell's equations, is to keep them invariant by means of the Lorentz transform, on the assumption that there is no ethereal medium and that the electro-magnetic wave-speed is a universal constant. The standard wave equation (4.4), however, with a constant value of the wave-speed $\bar{c}$, applies not only to electro-magnetic waves when there is no charge or current distribution, but can also be considered purely as a mathematical equation with a known general solution, as well as being the equation for small-amplitude homentropic irrotational wave-motion in any uniform fluid at rest. Now, considered purely as a mathematical equation, equation (4.4) and its general solution, equation (A5), transform quite normally, by Galilean transformation, into a progressive mathematical equation (A1) and its general solution, equation (A3), which are themselves invariant in all other frames of reference; and the Galilean transformation of equation (4.4) into the progressive wave-equation (4.1), considered as equations for sound-wave propagation in any fluid at rest or in uniform steady motion respectively, has been verified by countless observations over a very long period of time. A problem arises, therefore, as to how to justify a complete change in the way this unique mathematical equation should be transformed, namely a change from Galilean to Lorentz transformation, simply because the constant $\bar{c}$ has a particular numerical value appropriate to the local contemporary speed of light-waves, or merely because, at a particular moment, it happens to be regarded as an equation applying to the propagation of electro-magnetic waves, rather than to waves in any material fluid.

These difficulties, associated with the alternative (a) above, may be avoided by pursuing further the alternative (b), and seeking to derive more general equations for the electric and magnetic field-strengths in a fluid ether in general unsteady flow. Relaxing the restriction to a uniform ether, these general equations are, if attention is restricted to the ether alone, i.e. when there is no charge or current distribution, (see e.g. Bleaney and Bleaney, 1976)

$$\frac{\partial\left(\epsilon E_i\right)}{\partial x_i}=0, \tag{A.2.2}$$

$$\frac{\partial\left(\mu H_i\right)}{\partial x_i}=0, \tag{A.2.3}$$

$$\frac{D\left(\epsilon E_i\right)}{Dt}=\varepsilon_{ijk}\frac{\partial H_k}{\partial x_j}, \tag{A.2.4}$$

$$\frac{D\left(\mu H_i\right)}{Dt}=-\varepsilon_{ijk}\frac{\partial E_k}{\partial x_j}. \tag{A.2.5}$$

where

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+u_j\frac{\partial}{x_j}.$$

Here $\{E_i\}$ is the electric field-strength; $\{H_i\}$ is the magnetic field-strength; μ, the magnetic permeability of the ether; ϵ, the permittivity of the ether; and ε_{ijk} is the alternating tensor. ϵ and μ are no longer constants, when the ether is in general unsteady flow, but may vary with the local state (v, S) of the ether at any time. Thus, ϵ, μ may be regarded as functions of v, S such that $[\epsilon(v,S)\cdot\mu(v,S)]^{-1}$ is the square of the electro-magnetic wave-speed at any point at any time.

When the equations (A.2 : $2,3,4,5$) are written out in full, it is seen then that they involve terms in the derivatives of v and S, as well as derivatives of E_i, H_i. Thus, for instance, the first of equations (A.2.4) is, in full

$$\epsilon\frac{DE_1}{Dt}+E_1\epsilon_v\frac{Dv}{Dt}+E_1\epsilon_S\frac{DS}{Dt}=\frac{\partial H_3}{\partial x_2}-\frac{\partial H_2}{\partial x_3} \tag{A.2.6}$$

It is not possible, therefore, as with Maxwell's equations, or with the progressive form of Maxwell's equations, to regard the eight equations (A.2 : $2,3,4,5$) as eight self-consistent equations in the six new independent variables E_i, H_i. Rather, in determining the condition for the hypersurface $\zeta(x_i;t)=$ const. to be a wave-hypersurface of the equations, and in determining the envelope for the wave-hyperconoid, it is now necessary to consider simultaneously the complete set of thirteen equations, namely (A.2; $1, 2b, 3$), (A.2; $2,3,4,5$) in the eleven dependent variables u_i, v, S, E_i, H_i. Thus, for the hypersurface $\zeta=$ const. to be a characteristic hypersurface of these thirteen equations, all the eleventh-order determinants of an 11×13 matrix must vanish (cf. Thornhill 1985b). This matrix is set out below in such a way that the 11 columns correspond with the eleven independent variables, and the thirteen rows correspond with the thirteen equations.

$$
\begin{array}{c||cccccc}
 & (u_1) & (u_2) & (u_3) & (v) & (S) & (E_1) \\
(2.1) & -v\frac{\partial\zeta}{\partial x_1} & -v\frac{\partial\zeta}{\partial x_2} & -v\frac{\partial\zeta}{\partial x_3} & \frac{D\zeta}{Dt} & 0 & 0 \\
(2.2b) & \frac{D\zeta}{Dt} & 0 & 0 & -vE_{vv}\frac{\partial\zeta}{\partial x_1} & -vE_{vS}\frac{\partial\zeta}{\partial x_1} & 0 \\
(2.2b) & 0 & \frac{D\zeta}{Dt} & 0 & -vE_{vv}\frac{\partial\zeta}{\partial x_2} & -vE_{vS}\frac{\partial\zeta}{\partial x_2} & 0 \\
(2.2b) & 0 & 0 & \frac{D\zeta}{Dt} & -vE_{vv}\frac{\partial\zeta}{\partial x_3} & -vE_{vS}\frac{\partial\zeta}{\partial x_3} & 0 \\
(2.3) & 0 & 0 & 0 & 0 & \frac{D\zeta}{Dt} & 0 \\
(A.2.4) & 0 & 0 & 0 & E_1\epsilon_v\frac{D\zeta}{Dt} & E_1\epsilon_S\frac{D\zeta}{Dt} & \epsilon\frac{D\zeta}{Dt} \\
(A.2.4) & 0 & 0 & 0 & E_2\epsilon_v\frac{Dv}{Dt} & E_2\epsilon_S\frac{D\zeta}{Dt} & 0 \\
(A.2.4) & 0 & 0 & 0 & E_3\epsilon_v\frac{Dv}{Dt} & E_3\epsilon_S\frac{D\zeta}{Dt} & 0 \\
(A.2.5) & 0 & 0 & 0 & H_1\mu_v\frac{Dv}{Dt} & H_1\mu_S\frac{Dv}{Dt} & 0 \\
(A.2.5) & 0 & 0 & 0 & H_2\mu_v\frac{Dv}{Dt} & H_2\mu_S\frac{Dv}{Dt} & \frac{\partial\zeta}{\partial x_3} \\
(A.2.5) & 0 & 0 & 0 & H_3\mu_v\frac{D\zeta}{Dt} & H_3\mu_S\frac{D\zeta}{Dt} & -\frac{\partial\zeta}{\partial x_2} \\
(A.2.2) & 0 & 0 & 0 & E_i\epsilon_v\frac{\partial\zeta}{\partial x_i} & E_i\epsilon_S\frac{\partial\zeta}{\partial x_i} & \epsilon\frac{\partial\zeta}{\partial x_1} \\
(A.2.3) & 0 & 0 & 0 & H_i\mu_v\frac{\partial\zeta}{\partial x_i} & H_i\mu_S\frac{\partial\zeta}{\partial x_i} & 0
\end{array}
$$

$$
\begin{array}{ccccc||}
(E_2) & (E_3) & (H_1) & (H_2) & (H_3) \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial\zeta}{\partial x_3} & -\frac{\partial\zeta}{\partial x_2} \\
\epsilon\frac{D\zeta}{Dt} & 0 & -\frac{\partial\zeta}{\partial x_3} & 0 & \frac{\partial\zeta}{\partial x_1} \\
0 & \epsilon\frac{D\zeta}{Dt} & \frac{\partial\zeta}{\partial x_2} & -\frac{\partial\zeta}{\partial x_1} & 0 \\
-\frac{\partial\zeta}{\partial x_3} & \frac{\partial\zeta}{\partial x_2} & \mu\frac{D\zeta}{Dt} & 0 & 0 \\
0 & -\frac{\partial\zeta}{\partial x_1} & 0 & \mu\frac{D\zeta}{Dt} & 0 \\
\frac{\partial\zeta}{\partial x_1} & 0 & 0 & 0 & \mu\frac{D\zeta}{Dt} \\
\epsilon\frac{\partial\zeta}{\partial x_2} & \epsilon\frac{\partial\zeta}{\partial x_3} & 0 & 0 & 0 \\
0 & 0 & \mu\frac{\partial\zeta}{\partial x_1} & \mu\frac{\partial\zeta}{\partial x_2} & \mu\frac{\partial\zeta}{\partial x_3}
\end{array}
$$

All the eleventh-order determinants of this matrix, except for one, vanish identically, and the vanishing of the remaining determinant again reduces to the condition (loc. cit)

$$\left(\frac{D\zeta}{Dt}\right)^5\left\{\left(\frac{D\zeta}{Dt}\right)^2 - c^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2 + \left(\frac{\partial\zeta}{\partial x_2}\right)^2 + \left(\frac{\partial\zeta}{\partial x_3}\right)^2\right]\right\}^3 = 0 \qquad \text{(A.2.7)}$$

if $(\epsilon\mu)^{-1}$, which is the square of the local electro-magnetic wave-speed, is replaced by c^2, the square of the local thermodynamic wave-speed in the ether, so that electric, magnetic and condensational oscillations in the ether can all propagate contemporaneously along the same wave-hypersurfaces.

Thus, the condition for $\zeta =$ const. to be a wave-hypersurface, both for the general equations of motion and for the general electro-magnetic field equations $(\mathrm{A.2} : 2, 3, 4, 5)$ of any fluid ether, is

$$\left(\frac{D\zeta}{Dt}\right)^2 = c^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2 + \left(\frac{\partial\zeta}{\partial x_2}\right)^2 + \left(\frac{\partial\zeta}{\partial x_3}\right)^2\right] \qquad \text{(A.2.8)}$$

and the wave-hyperconoid through any point at any time is (loc. cit),

$$\left(dx_1 - u_1 dt\right)^2 + \left(dx_2 - u_2 dt\right)^2 + \left(dx_3 - u_3 dt\right)^2 = c^2\,(dt)^2 \qquad \text{(A.2.9)}$$

where $\left\{u_i\right\}$ and c are, respectively, the local values, at any time, of the velocity and the common thermodynamic and electro-magnetic wave-speed of the fluid ether.

The results of Section 5 of the main paper can now be applied to electro-magnetic-condensational waves in any fluid ether in general unsteady flow. For any observer at the origin of the rest-frame moving with the fluid ether, the equations $(\mathrm{A.2} : 8, 9)$ reduce respectively, to the equations $(\mathrm{A.2} : 5, 11, 12)$ namely

$$\left(\frac{\partial\zeta}{\partial t}\right)^2 = c^2\left[\left(\frac{\partial\zeta}{\partial x_1}\right)^2 + \left(\frac{\partial\zeta}{\partial x_2}\right)^2 + \left(\frac{\partial\zeta}{\partial x_3}\right)^2\right] \qquad \text{(A.2.10)}$$

and

$$\left(dx_1\right)^2 + \left(dx_2\right)^2 + \left(dx_3\right)^2 = c^2\,(dt)^2 \qquad \text{(A.2.11)}$$

whilst the general electro-magnetic field equations reduce to another variation on Maxwell's equations, namely

$$\frac{\partial\left(\epsilon E_i\right)}{\partial x_i} = 0; \quad \frac{\partial\left(\mu H_i\right)}{\partial x_i} = 0 \qquad \text{(A.2.12)}$$

$$\frac{\partial\left(\epsilon E_i\right)}{\partial t} = \varepsilon_{ijk}\frac{\partial H_k}{\partial x_j} \qquad \text{(A.2.13)}$$

$$\frac{\partial\left(\mu H_i\right)}{\partial t} = -\,\varepsilon_{ijk}\frac{\partial E_k}{\partial x_j}. \qquad \text{(A.2.14)}$$

All these equations are invariant for Galilean transformations between all such observers moving with the fluid ether. This apparent invariance, achieved by the Galilean transformation, contrasts remarkably with the invariance, under the Lorentz transform, of Maxwell's equations and their wave-hyperconoid

$$\left(dx_1\right)^2 + \left(dx_2\right)^2 + \left(dx_3\right)^2 = \bar{c}^2 \left(dt\right)^2$$

which depends entirely on the assumption that there is no ethereal medium and that the speed of electro-magnetic waves is universally constant.

There are two possible interpretations of the Michelson-Morley experiment, which correspond to the two alternatives (a) and (b) above. Again, the interpretation accepted, at present, is based on the alternative (a), that there is no ethereal medium, and this leads to the Lorentz transform. The alternative interpretation, namely that a null result from the Michelson-Morley experiment implies that the apparatus is moving with the local ether, has always been rejected, on the grounds that it is impossible for experimental apparati, moving with different velocities, all to be moving with the uniform ether demanded by Maxwell's equations. With the general equations $(A.2 : 2, 3, 4, 5)$ now derived, however, for the electro-magnetic field in the general unsteady flow of a fluid ether, this objection no longer holds. Rather, the amended form of Maxwell's equations $(A.2 : 12, 13, 14)$, with their wave-hyperconoid

$$\left(dx_1\right)^2 + \left(dx_2\right)^2 + \left(dx_3\right)^2 = c^2 \left(dt\right)^2 ,$$

in which the wave-speed c is variable, are invariant for Galilean transformations between all observers moving with any general flow of a fluid ether. This is entirely consistent with the alternative interpretation of the Michelson-Morley experiment.

In the rapid development of electro-magnetism in the nineteenth century, much effort was devoted to the quest for a viable ethereal medium but, in the event, none such was found. At the turn of the century, Planck obtained his well-known form for the energy distribution in black-body radiation, a form which agreed with all observations over the entire experimental range of frequencies. This form of the energy distribution placed such a highly restrictive demand on the properties of the ethereal medium that it offered an unprecedented opportunity for the quest for a viable ether to be substantially re-inforced until it was successful. The opportunity was missed, however, and all possibility of a fluid ether was totally discounted, first, since no gas with any number of different kinds of atom or molecule could both have Maxwellian statistics and satisfy Planck's energy distribution, and second, because invariance of Maxwell's equations under the Lorentz transform permitted no ethereal medium at all. In consequence, Planck's energy distribution had then to be interpreted in terms of his 'quantum' hypothesis, that the smallest amount of energy that can be absorbed, at any frequency ν is $E = h\nu$. This was a device which allowed great progress to be made which led to Einstein's 'light-quantum' hypothesis, requiring 'light-particles' or 'photons', of energy $E = h\nu$, all of which are identical.

Thus, the foundations of modern theoretical physics now rest on two overt contradictions, (i) that the basic assumption of no ethereal medium is contradicted by the existence of a 'monatomic' gas of identical 'photons', and (ii) that no such 'monatomic' gas can conform to the equipartition of energy and have Maxwellian statistics, and Planck's energy distribution. The mistake can now be seen to be the failure to examine the kinetic-theoretical possibilities of a gas mixture in the mathematical limit when the number of different kinds of atoms or molecules tends to infinity. Several writers (see e.g. Whittaker 1953; Thornhill 1985a), in the early part of the century by, for instance, expanding Planck's energy distribution in an infinite series, came to realise that it could be interpreted in terms of an infinite variety of 'photo-molecules' of energies $h\nu, 2h\nu, 3h\nu$ etc. None of these writers, however, applied this idea to the contemporary kinetic theory of a gas mixture, in order to determine whether it was possible to specify a particular infinite variety of particles, and derive an appropriate mixture of them, which would have Planck's energy distribution. Rather, in contrast to this approach, new gas statistics were sought (e.g. those suggested by Bose, and taken up by Einstein) which would enable a 'monatomic' gas of 'photons' to satisfy Planck's energy distribution.

It was not until 1975 that it was first discovered (Thornhill 1985a) that an ideal gas, with an infinite variety of particles of masses nm ($n=1$ to ∞), could both conform to Maxwellian statistics and have Planck's energy distribution. The required abundance function of the particles is $\underline{\mathrm{N}}_n \propto n^{-4}$ and the correlation derived between energy and frequency is not that given by Planck's or Einstein's hypothesis, but requires that the specific energy per unit mass, ϵ, of all the particles, whatever their masses, shall correlate with frequency according to the relation

$$\epsilon = \frac{h\nu}{m}. \tag{A.2.15}$$

Thus, radiation of frequency ν is not associated, as Planck or Einstein hypothesised, with 'quanta' or 'photons' having a particular energy $E = h\nu$, but with all ether particles of masses nm ($n=1$ to ∞) which have energies $nm\epsilon = nh\nu$ for a particular energy per unit mass, ϵ.