Anomalous Spacetimes

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The usual interpretations of solutions for Einstein's gravitational field satisfying the static vacuum conditions contain anomalies that are not mathematically permissible. It is shown herein that the usual solutions must be modified to account for the intrinsic geometry associated with the relevant line-elements.

1 Introduction

The standard solution in the case of the static vacuum field of a single gravitating body, satisfying Einstein's field equations $R_{\mu\nu} = 0$, is (using G = c = 1),

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.1}$$

upon which it is routinely claimed that $2m < r < \infty$ is an exterior region and $0 < r < 2m$ is an interior region. Notwithstanding the inequalities it is routinely allowed that $r = 2m$ and $r = 0$ by which it is also routinely claimed that $r = 2m$ marks a "removable" or "coordinate" singularity and that $r = 0$ marks a "true" or "physical" singularity [1].

The standard treatment of the foregoing line-element proceeds from simple inspection of (1.1) and thereby upon the following assumptions:

(a) that there is only one radial quantity defined on (1.1);

(b) that r can approach zero, even though the line-element (1.1) is singular at $r = 2m$;

(c) that r is the radial quantity in (1.1).

With these unstated assumptions, but assumptions nonetheless, it is usual procedure to develop and treat of black holes. However, all three assumptions are demonstrably false at an elementary level.

2 That assumption (a) is false

Consider standard Minkowski space (using c = G = 1) described by

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2, \tag{2.1}$$

$$0 \leq r < \infty,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. The spatial components of (2.1) describe a sphere centred at the origin of the coordinate system, i.e. at $r = 0$.

In relation to (2.1) calculate the radius R of the sphere:

$$R = \int_0^r dr = r. \tag{2.2}$$

Calculate the surface area of the sphere:

$$A = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi = 4\pi r^2 = 4\pi R^2. \tag{2.3}$$

Calculate the volume of the sphere:

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} r^2 \sin\theta dr d\theta d\varphi = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3. \tag{2.4}$$

Call the square root of the coefficient of $d\Omega^2$ the radius of curvature, R_c. Then on (2.1), $R_c = r$. Call the integral of the square root of the term containing the square of the differential element of the radius of curvature the proper radius, R_p. Then for (2.1), according to (2.2),

$$R_p = r \equiv R_c. \tag{2.5}$$

Thus, for Minkowski space, $R_p \equiv R_c$. This is because Minkowski space is pseudo-Efcleethean*.

Now consider (1.1). There

$$R_c = r,$$

$$R_p = \int \sqrt{\frac{r}{r - 2M}} dr \neq r = R_c.$$

Hence, $R_P \neq R_c$ in (1) in general. This is because (1.1) is non-Efcleethean (it is pseudo-Riemannian). Thus, assumption (a) is false.

*For the geometry due to Efcleethees – usually and abominably rendered Euclid.

3 That assumption (b) is false

On (1.1),

$$R_p = R_p(r) = \int \sqrt{\frac{r}{r-2m}}dr$$

$$= \sqrt{r\,(r-2m)} + 2m\ln\left|\sqrt{r}+\sqrt{r-2m}\right| + K, \quad (3.1)$$

where K is a constant of integration.

For some value r_0, $R_p(r_0) = 0$, where r_0 is to be determined from (3.1). According to (3.1), $R_p(r_0) = 0$ when $r = r_0 = 2m$ and $K = -m\ln 2m$. Hence,

$$R_p = R_p(r) = \int \sqrt{\frac{r}{r-2m}}dr =$$

$$= \sqrt{r\,(r-2m)} + 2m\ln\left(\frac{\sqrt{r}+\sqrt{r-2m}}{\sqrt{2m}}\right). \quad (3.2)$$

Therefore, $2m < r < \infty \Rightarrow 0 < R_p < \infty$, where $r \equiv R_c$. The inequality is required to maintain Lorentz signature, since the line-element is undefined at $r = 2m$, which is the only possible singularity on the line element. Thus, assumption (b) is false.

4 That assumption (c) is false

Generalise (2.1) so that the centre of a sphere can be located anywhere in Minkowski space, thus

$$ds^2 = dt^2 - \left(d\left|r-r_0\right|\right)^2 - \left|r-r_0\right|^2 d\Omega^2$$

$$= dt^2 - \frac{(r-r_0)^2}{\left|r-r_0\right|^2}dr^2 - \left|r-r_0\right|^2 d\Omega^2$$

$$= dt^2 - dr^2 - \left|r-r_0\right|^2 d\Omega^2, \quad (4.1)$$

$$0 \le \left|r-r_0\right| < \infty,$$

which is well-defined for all real r. The value of r_0 is arbitrary. The spatial components of (4.1) describe a sphere of radius $D = |r-r_0|$ centred at some point r_0 on a common radial line through r and the origin of coordinates at $r = 0$ (i.e. centred at the point of intersection of the common radial line with the spherical surface $r = r_0$). If $r_0 = 0$, (1.1) is recovered. One does not need to make $r_0 = 0$ so that the centre of the sphere coincides with the origin of the coordinate system itself, at $r = 0$.

Then on (4.1),

$$R_c = \left|r-r_0\right|,$$

$$R_p = \int_0^{|r-r_0|} d\left|r-r_0\right| = \int_{r_0}^{r} \frac{(r-r_0)}{\left|r-r_0\right|}dr = \left|r-r_0\right| \equiv R_c, \quad (4.2)$$

and so $R_p \equiv R_c$ on (4.1), since (4.1) is pseudo-Efcleethean. Setting $D = |r-r_0|$ for convenience, generalise (4.1) thus,

$$ds^2 = A(C(D))dt^2 - B(C(D))d\sqrt{C(D)}^2 - C(D)d\Omega^2, \quad (4.3)$$

where $A(C(D)), B(C(D)), C(D) > 0$. Then for $R_{\mu\nu} = 0$, metric (4.3) has the solution,

$$ds^2 = \left(1 - \frac{\alpha}{\sqrt{C(D)}}\right)dt^2 -$$

$$-\left(1 - \frac{\alpha}{\sqrt{C(D)}}\right)^{-1} d\sqrt{C(D)}^2 - C(D)d\Omega^2, \quad (4.4)$$

where α is a function of the mass generating the gravitational field. Then

$$R_c = R_c(D) = \sqrt{C(D)},$$

$$R_p = R_p(D) = \int \sqrt{\frac{\sqrt{C(D)}}{\sqrt{C(D)}-\alpha}}\,d\sqrt{C(D)}$$

$$= \int \sqrt{\frac{R_c(D)}{R_c(D)-\alpha}}\,dR_c(D)$$

$$= \sqrt{R_c(D)\left(R_c(D)-\alpha\right)} +$$

$$+\alpha\ln\left(\frac{\sqrt{R_c(D)}+\sqrt{R_c(D)-\alpha}}{\sqrt{\alpha}}\right), \quad (4.5)$$

where $R_c(D) \equiv R_c\left(|r-r_0|\right) = R_c(r)$. Clearly r is a parameter, located in Minkowski space according to (4.1), (4.2) and (4.3).

Now $r = r_0 \Rightarrow D = 0$, and so by (4.5), $R_c(D = 0) = \alpha$ and $R_p(D = 0) = 0$. One must ascertain the admissible form of $R_c(D)$ subject to the conditions $R_c(D = 0) = \alpha$ and $R_p(D = 0) = 0$ and $dR_c(D) > 0$, along with the requirements that $R_c(D)$ must produce (1.1) from (4.4) at will, must yield Schwarzschild's original solution at will (which is not the line-element (1.1) [2]), must produce Brillouin's solution at will [3], and must yield an infinite number of equivalent metrics [4]. The only admissible form satisfying these conditions is,

$$R_c = R_c(D) = \left(D^n + \alpha^n\right)^{\frac{1}{n}} \equiv \left(|r-r_0|^n + \alpha^n\right)^{\frac{1}{n}} = R_c(r), \quad (4.6)$$

$$D > 0, \quad r \in \Re, \quad n \in \Re^+, \quad r \neq r_0,$$

where r_0 and n are entirely arbitrary constants.

Choosing $r_0 = 0$, $r > 0$, $n = 3$,

$$R_c(r) = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}, \quad (4.7)$$

and putting (4.7) into (4.4) gives Schwarzschild's original solution, defined on $0 < r < \infty$.

Choosing $r_0 = 0,\ r > 0,\ n = 1,$

$$R_c(r) = r + \alpha, \tag{4.8}$$

and putting (4.8) into (4.4) gives Marcel Brillouin's solution, defined on $0 < r < \infty$.

Choosing $r_0 = \alpha,\ r > \alpha,\ n = 1,$

$$R_c(r) = (r - \alpha) + \alpha = r, \tag{4.9}$$

and putting (4.9) into (4.4) gives line-element (1.1), but defined on $\alpha < r < \infty$, as found by Johannes Droste in May 1916 [5]. Note that according to (4.9) (and in general by (4.6)), r is not a radial quantity in the gravitational field, because $R_c(r) = (r-\alpha)+\alpha = D+\alpha$ is really the radius of curvature in (1.1), defined for $0 < D < \infty$. Thus, assumption (c) is false.

5 That the manifold is inextendable

That the singularity at $R_p(r_0) \equiv 0$ is insurmountable is clear by the following ratio,

$$\lim_{r\to r_0^{\pm}} \frac{2\pi R_c(r)}{R_p(r)} = \infty.$$

Hagihara [6] has shown that all radial geodesics that do not run into the boundary at $R_c(r_0) = \alpha$ (i.e. that do not run into the boundary at $R_p(r_0) = 0$) are geodesically complete.

Doughty [7] has shown that the acceleration a of a test particle approaching the centre of mass at $R_p = 0$ is given by,

$$a = \frac{\sqrt{-g_{00}}\left(-g^{11}\right)|g_{00,1}|}{2g_{00}}.$$

By (4.4) and (4.6), this gives,

$$a = \frac{\alpha}{2R_c^{\frac{3}{2}}\sqrt{R_c(r) - \alpha}}.$$

Then clearly as $r \to r_0^{\pm}$, $a \to \infty$, independently of the value of r_0.

J. Smoller and B. Temple [8] have shown that the Oppenheimer-Volkoff equations do not permit gravitational collapse to form a black hole and that the alleged interior of the Schwarzschild spacetime (i.e. $0 \le R_c(r) < \alpha$ is therefore disconnected from Schwarzschild spacetime and so does not form part of the solution space.

N. Stavroulakis [9, 10, 11, 12] has shown that an object cannot undergo gravitational collapse into a singularity, or to form a black hole.

Suppose $0 \le \sqrt{C(D(r))} < \alpha$. Then (4.4) becomes

$$ds^2 = -\left(\frac{\alpha}{\sqrt{C}} - 1\right)dt^2 + \left(\frac{\alpha}{\sqrt{C}} - 1\right)^{-1} d\sqrt{C}^2 -$$

$$-C(d\theta^2 + \sin^2\theta d\varphi^2),$$

which shows that there is an interchange of time and length. To amplify this set $r = \bar{t}$ and $t = \bar{r}$. Then

$$ds^2 = \left(\frac{\alpha}{\sqrt{C}} - 1\right)^{-1}\frac{\dot{C}^2}{4C}d\bar{t}^2 - \left(\frac{\alpha}{\sqrt{C}} - 1\right)d\bar{r}^2$$

$$-C(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $C = C(\bar{t})$ and the dot denotes $d/d\bar{t}$. This is a time dependent metric and therefore bears no relation to the problem of a static gravitational field.

Thus, the Schwarzschild manifold described by (4.4) with (4.6) (and hence by (1.1)) is inextendable.

6 That the Riemann tensor scalar curvature invariant is everywhere finite

The Riemann tensor scalar curvature invariant (the Kretschmann scalar) is given by $f = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. In the case of (4.4) with (4.6) this is

$$f = \frac{12\alpha^2}{R_c^6(r)} = \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{\frac{6}{n}}}.$$

A routine attempt to justify the standard assumptions on (1.1) is the a posteriori claim that the Kretschmann scalar must be unbounded at a singularity [13, 1]. Nobody has ever offered a proof that General Relativity necessarily requires this. That this additional ad hoc assumption is false is clear from the following ratio,

$$f(r_0) = \frac{12\alpha^2}{\left(|r_0 - r_0|^n + \alpha^n\right)^{\frac{6}{n}}} = \frac{12}{\alpha^4}.$$

In addition,

$$\lim_{r\to\pm\infty} \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{\frac{6}{n}}} = 0,$$

and so the Kretschmann scalar is finite everywhere.

7 That the Gaussian curvature is everywhere finite

The Gaussian curvature K of (4.4) is,

$$K = K(R_c(r)) = \frac{1}{R_c^2(r)},$$

where $R_c(r)$ is given by (4.6). Then,

$$K(r_0) = \frac{1}{\alpha^2} \quad \forall \quad r_0,$$

and

$$\lim_{r\to\pm\infty} K(r) = 0,$$

and so the Gaussian curvature is everywhere finite.

8 Conclusions

Using the spherical-polar coordinates, the general solution to $R_{\mu\nu} = 0$ is (4.4) with (4.6), which is well-defined on

$$-\infty < r < \infty,$$

for any r_0, $-\infty < r_0 < \infty$, where r_0 is entirely arbitrary, and corresponds to

$$0 < R_p(r) < \infty, \quad \alpha < R_c(r) < \infty,$$

for the gravitational field. The only singularity that is possible occurs at $g_{00} = 0$. It is impossible to get $g_{11} = 0$ because there is no value of the parameter r by which this can be attained. No interior exists in relation to (4.4) with (4.6), which contain the usual metric (1.1).

The radius of curvature $R_c(r)$ does not in general determine the radial geodesic distance to the centre of curvature of Einstein's gravitational field and is only to be interpreted in relation to the Gaussian curvature by the equation $K = 1/R_c^2(r)$. The radial geodesic distance from the spherical geodesic surface with Gaussian curvature K, to the centre of curvature, is given by the proper radius, $R_p(R_c(r))$. The centre of curvature is located at the point $R_p(r_0) = 0$.

Expression (4.4) with (4.6) (and hence (1.1)) describes only a centre of mass located at $R_p(r_0) = 0$ in the gravitational field, $\forall\, r_0$. As such it does not take into account the distribution of matter and energy in a gravitating body, since $\alpha(M)$ is indeterminable in this limited situation. One cannot generally just utilise a potential function in comparison with the Newtonian potential to determine α because α is subject to the distribution of the matter of the source of the gravitational field. The value of α must be calculated from a line-element describing the interior of the gravitating body, satisfying $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \neq 0$. The interior line-element is necessarily different to the exterior line-element of an object such as a star. A full description of the gravitational field of a star therefore requires two line-elements, not one as is routinely assumed, and when this is done, there are no singularities anywhere. The assumption that one line-element is sufficient is false. Outside a star, (4.4) with (4.6) describes the gravitational field in relation to the centre of mass of the star, but α is determined by the interior metric, which, in the case of the usual treatment of (1.1), has gone entirely unrecognised, so that the value of α is instead determined by a comparison with the Newtonian potential.

Black holes are not predicted by General Relativity. The Kruskal-Szekeres coordinates do not describe a coordinate patch that covers a part of the gravitational manifold that is not otherwise covered - they describe a completely different pseudo-Riemannian manifold that has nothing to do with Einstein's gravitational field.

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