

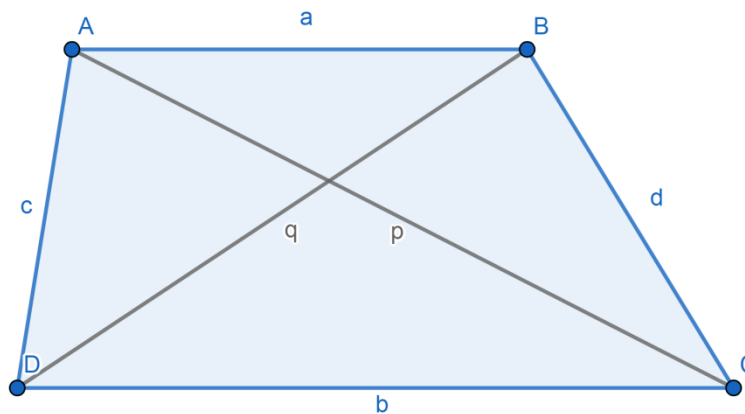
# A Theorem in the Trapezoid

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**Abstract** : In this note, we prove a remarkable theorem in the trapezoid.

## Theorem

Let ABCD be the trapezoid shown below:



We have :

$$\left\{ \begin{array}{l} b = \sqrt{\frac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2(c^2 + p^2) - 2(d^2 + q^2)}} \quad (1) \\ a = \sqrt{\frac{(d^2 - p^2)^2 - (c^2 - q^2)^2}{2(d^2 + p^2) - 2(c^2 + q^2)}} \quad (2) \end{array} \right.$$

## Proof

Heron's formula for the area of a triangle with lengths  $a, b, c$  is given by :

$$\frac{1}{4} \sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)} \quad (3)$$

One can easily prove that the expression (3) is equivalent to the following expression :

$$\frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2} \quad (4)$$

Now, using formula (4), the area  $K_1$  of the triangle  $ACD$  in the above trapezoid is:

$$K_1 = \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - p^2)^2}$$

Similarly the area  $K_2$  of the triangle  $BCD$  is:

$$K_2 = \frac{1}{4}\sqrt{4b^2d^2 - (b^2 + d^2 - q^2)^2}$$

Since the two triangles  $ACD$  and  $BCD$  have the same base and altitude, their areas are equal, therefore :

$$\begin{aligned} K_1 = K_2 &\Leftrightarrow \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - p^2)^2} = \frac{1}{4}\sqrt{4b^2d^2 - (b^2 + d^2 - q^2)^2} \\ &\Leftrightarrow 4b^2c^2 - (b^2 + c^2 - p^2)^2 = 4b^2d^2 - (b^2 + d^2 - q^2)^2 \\ &\Leftrightarrow 4b^2c^2 - 4b^2d^2 = -(b^2 + d^2 - q^2)^2 + (b^2 + c^2 - p^2)^2 \\ &\Leftrightarrow 4b^2c^2 - 4b^2d^2 = (b^2 + c^2 - p^2 + b^2 + d^2 - q^2)(b^2 + c^2 - p^2 - b^2 - d^2 + q^2) \\ &\Leftrightarrow 4b^2c^2 - 4b^2d^2 = (2b^2 + c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2) \\ &\Leftrightarrow 4b^2c^2 - 4b^2d^2 = 2b^2(c^2 - p^2 - d^2 + q^2) + (c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2) \\ &\Leftrightarrow 4b^2c^2 - 4b^2d^2 - 2b^2(c^2 - p^2 - d^2 + q^2) = (c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2) \\ &\Leftrightarrow b^2(4c^2 - 4d^2 - 2c^2 + 2p^2 + 2d^2 - 2q^2) = (c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2) \\ &\Leftrightarrow b^2(2c^2 + 2p^2 - 2d^2 - 2q^2) = (c^2 - p^2)^2 - (d^2 - q^2)^2 \\ &\Leftrightarrow b^2 = \frac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2c^2 + 2p^2 - 2d^2 - 2q^2} \\ &\Leftrightarrow b = \sqrt{\frac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2(c^2 + p^2) - 2(d^2 + q^2)}} \end{aligned}$$

Similarly we prove formula (2) by considering the triangles  $ABD$  and  $ABC$ . ■