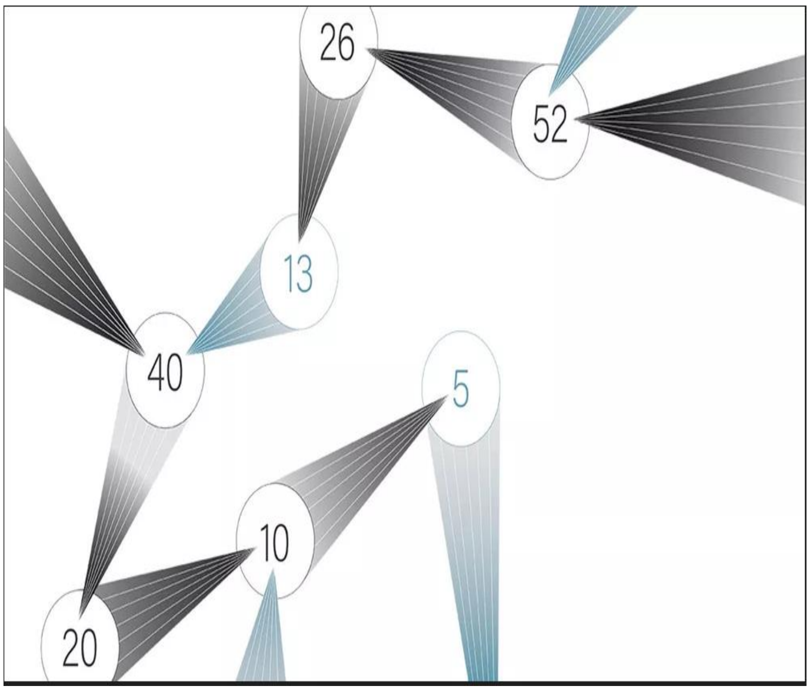


SYRACUSE-COLLATZ CONJECTURE

Exploration, Analysis and Demonstration of a Mathematical Enigma

SUMMARY

In the infinite universe of numbers, the Syracuse conjecture emerges as a captivating enigma, defying mathematical conventions and arousing the curiosity of the most daring minds.

Author

MOSTAFA SENHAJI

I- Introduction

Immerse yourself in the fascinating world of the Syracuse conjecture, more commonly known as the Collatz conjecture. Since its formulation in 1937 by mathematician Lothar Collatz, this captivating riddle continues to challenge the brightest minds in the mathematical community. Despite its apparent simplicity, this conjecture exposes enigmatic properties and deep mysteries that have piqued the interest and curiosity of researchers for decades.

In this article, we will delve into the origins of this conjecture, exploring the first approaches that were attempted to solve this mathematical puzzle. We will also examine the various strategies and results obtained so far, while probing possible extensions of this conjecture to other areas of mathematics. Prepare to be taken on a fascinating journey through the depths of the Syracuse– Collatz Conjecture, where simplicity meets complexity and mysteries abound at every turn.

II- Origins

Collatz conjecture is as captivating as the problem itself. It all started in the 1930s, when Lothar Collatz, then a young student, delved into the mysterious world of iterations in integers. Armed with graphs and hypergraphs, he explored the depths of digital sequences, without suspecting that he would soon discover a sequence that would defy intuition and captivate the imagination of mathematicians around the world.

During a visit to Hamburg in 1952, Collatz shared his enigma with Helmut Hasse, who spread it to Syracuse University in the United States. Thus, the sequence became known as the "Syracuse sequence", leaving an indelible mark on mathematical history. Meanwhile, Polish mathematician Stanislas Ulam was spreading it at Los Alamos National Laboratory, while Shizuo Kakutani broadcast it in classrooms at Yale and Chicago universities.

In the 1960s, in the midst of the Cold War, the Collatz conjecture became a subject of intense fascination for mathematicians around the world. Some even joked that she was part of a Soviet plot to slow down American research. But beyond the conjectures and rumors, the real intrigue lay in the quest for a solution to this seemingly simple, yet incredibly complex, problem.

Collatz 's conjecture continues to inspire curious minds and intrepid researchers. It is associated with an Erdös prize of 500 dollars, offered to anyone who manages to solve this mathematical mystery. This fascinating story and the challenges it pose inspire mathematicians around the world to embark on a never-ending intellectual adventure, searching for the truth hidden behind integers and infinite sequences.

III- Collatz Conjecture

Collatz conjecture, although simple to state, can be broken down into different variants or formulated in various ways. Below we'll explore a few of these variations, providing insight into the richness and complexity of this seemingly simple problem.

- **<u>Non-Divergence Conjecture (Non-Divergent Trajectories Conjecture)</u>**

All sequences generated by Collatz 's rules remain bounded, that is to say they do not move away from zero indefinitely.

- **<u>Finite Stopping Time Conjecture</u>**

Collatz 's rule eventually returns to values lower than its starting point, thus indicating a finite loop.

- **<u>Conjecture Using Semi-Groups of Affine Functions</u>**

This formulation of the conjecture involves the use of semigroups of affine functions to define sets of orbits. Using the composition of these functions, we seek to explore the trajectories generated by the iterations of the Collatz rule.

➢ **<u>Predecessor Conjecture</u>**

Focusing on a certain semigroup generated by specific affine functions, this conjecture asserts that the set of positive integers greater than 2 is contained in a particular orbit.

Collatz conjecture illustrate the complexity and diversity of possible approaches to tackle this fascinating and notoriously difficult problem.

IV- First approach to the conjecture

➢ Syracuse Suite

The Syracuse sequence of an integer N > 0 is defined by induction, as follows:

-$U_0 = N$

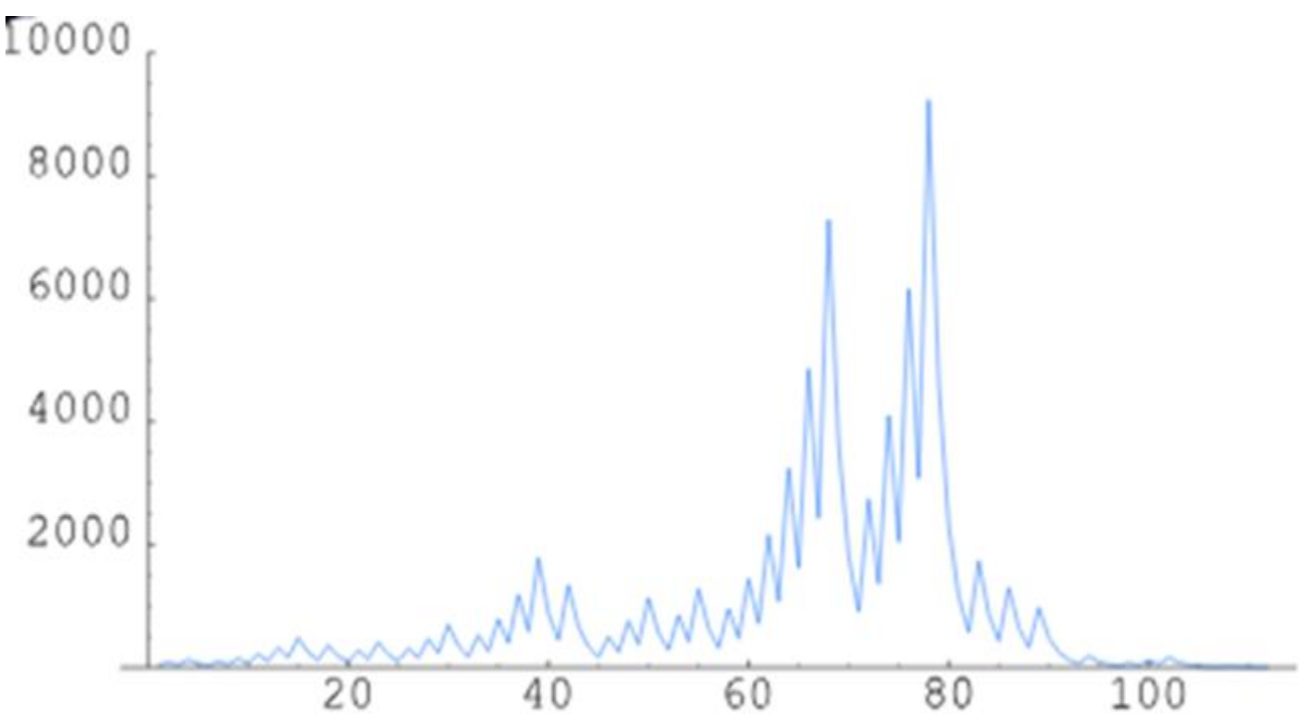
-and for any integer n, $U_{n+1} = U_n / 2$ if U_n is even or $U_n = 3U_n + 1$ if U_n is odd

➢ **Statement of the conjecture**

The conjecture states that for every integer N > 0, there exists an index n such that $U_n = 1$

➢ <u>Graphical representation and Vocabulary</u>

- **<u>Graphical representation of the sequence for N = 27</u>**



By graphically examining the evolution of the sequence for *N* = 15 and N = 27, we can see that the sequence reaches high values before returning to lower levels. The graphic representations evoke the image of an irregular fall similar to that of descending hail or the sinuous trajectory of a leaf carried by the wind. This observation inspired the use of pictorial language, thus introducing the poetic concept of the 'flight' of the s

We then introduce three parameters defined as follows:

- Flight time: represented by the smallest index $U_n = 1$. It is 17 for the Syracuse suite with *N* =15 and 111 for the Syracuse suite with *N* =27.
- Flight time at altitude: defined as the smallest index *n* such that $U_n < U_0$. It is 11 for the Syracuse suite with *N* =15 and 96 for the Syracuse suite with *N* =27.
- Maximum altitude: corresponds to the maximum value reached subsequently. It is 160 for the Syracuse suite with *N* =15 and 9,232 for the Syracuse suite with *N* =27.

V- <u>Approaches to the conjecture and results</u>

Over the years, various approaches have been explored to attempt to resolve the Syracuse–Collatz conjecture . Among the most notable, we can cite:

- **Binary approach**

The binary approach to exploring the Syracuse Conjecture immerses us in a fascinating world where odd numbers are subject to an enigmatic and elegant transformation. Imagine a machine, a kind of mathematical genius, which operates according to simple but mysterious rules. This machine, called the Syracuse function, is a true architect of odd numbers, tirelessly guiding them towards the mystical “1”.

The operation of this machine is both simple and complex. First, it takes an odd number and magically transforms it into a new number, shifting it one notch to the left and adding a "1" at the end. Then, it adds the original number to this new creation, resulting in a result three times larger, from which it subtracts any unnecessary "0" until the result is odd.

It's a delicate dance between binary and arithmetic, where each iteration of the machine reveals new mysteries and new challenges to overcome. The sequence of operations unfolds like a cascade of mathematical steps, taking us on a captivating journey through the depths of number, revealing the hidden secrets of its intrinsic nature.

Let's take a concrete example to illustrate this approach. If we choose 7 as a starting number, its binary writing is 111 (because $7 = 2^2 + 2^1 + 2^0$). The resulting sequence goes as follows:

111

1111

10110

10111

100010

100011

110100

11011

101000

1011

10000

11

100

Each step in this sequence offers a fascinating insight into how the Syracuse machine manipulates odd numbers through the prism of binary. It's an exciting journey into the world of numbers, where the simplicity of the rules hides the infinite wealth of possibilities.

- **Probabilistic approach**

The probabilistic approach has been adopted by some researchers to explore the behavior of the Syracuse sequence, a problem that has fascinated mathematicians for decades. This conjecture proposes a simple iterative sequence, but its properties remain largely mysterious.

Here is a more in-depth analysis of this approach:

1. **Statistical methods** : Researchers have used statistical methods to study the Syracuse sequence on large sets of numbers. They examined the probabilistic trends and behaviors of the terms in the sequence. This made it possible to obtain information on the growth of values and to identify possible patterns.
2. **Heuristic arguments** : Heuristic arguments are based on intuitive reasoning and empirical observations. They suggest that the terms of the sequence cannot grow indefinitely. For example, if a term reaches a high value, it is likely to go back down at some point. However, these arguments do not constitute rigorous proof.
3. **Recent work** : Terence Tao, a renowned mathematician, shed new light on the Syracuse Conjecture in 2019. His research helped to better understand the properties of the sequence and explore avenues to solve the enigma. Despite these advances, the conjecture continues to challenge researchers, and no formal proof has yet been established.

In short, the probabilistic approach offers interesting perspectives for understanding the outcome of Syracuse, but it still leaves many questions unanswered. Mathematicians continue to explore this enigma with passion and determination.

- **Calculative approach**

Passionate researchers also delved into intensive computational calculations to explore the behavior of the famous **Syracuse sequence** over extended ranges of numbers. Their goal was to detect patterns or trends that could shed light on the validity of the conjecture.

Here is a more in-depth analysis of this IT approach:

1. **Exhaustive calculations:** Researchers used powerful computers to iterate through huge sets of numbers. They applied the Syracuse sequence algorithm to each number, carefully observing the successive values generated. These calculations made it possible to explore gigantic digital territories and collect data on the behavior of the suite.
2. **Identifying Patterns:** When examining the results, the researchers looked for recurring patterns. For example, they observed whether certain numbers seemed to converge into a cycle, whether values repeated themselves, or whether peaks and troughs appeared regularly. These observations could provide clues to the validity of the conjecture.
3. **Algorithmic complexity:** The Syracuse suite is simple to define, but its behavior remains complex. The researchers studied term growth, sequence lengths, and variations in behavior. They looked for correlations between the properties of numbers and the characteristics of the sequence.

4. **Limitations of calculations:** Despite advances in computing, researchers have not yet succeeded in rigorously proving the Syracuse conjecture. Calculations can show trends, but they do not guarantee that the guess is valid for all numbers. However, these efforts continue to enrich our understanding of this mathematical conundrum.

In short, the computer exploration of the Syracuse suite is a fascinating field where the rigor of calculations blends with the beauty of numbers. Researchers persevere, hoping to one day unravel the mystery of this intriguing sequence.

- **An undecidable statement?**

Despite the considerable efforts made by mathematicians to solve the Syracuse conjecture, the results remain limited. This situation has raised questions about the very nature of the problem: is it possible that the Syracuse conjecture is undecidable within the framework of commonly accepted mathematical axioms, such as ZFC? In 1972, John Conway shed interesting light on this question by establishing algorithmic undecidability for a class of problems which naturally includes the Syracuse problem. Although this discovery does not directly solve the Syracuse problem, it suggests that some aspects of this problem may be inherently intractable. This underlines the enormous complexity of this conjecture and the fascination it continues to exert on the mathematical community.

Extension to negative numbers, real numbers and complex numbers

Besides its application to positive integers, the Syracuse– Collatz conjecture has also been extended to other sets of numbers such as negative numbers, real numbers and even complex numbers. However, the results of these extensions are still largely unknown and are the subject of ongoing research.

VI- Demonstration

1- Observation

The Syracuse sequence is defined by a simple iterative rule where each term is obtained by applying a specific operation from the previous term. This operation depends on the parity of the previous term, leading to an apparently chaotic behavior despite the simplicity of the rule.

2- **Let us demonstrate that the Syracuse sequence reaches a finiteness of terms**

Since the Syracuse sequence begins with a positive integer N , by The framing theorem for integers we prove that there exists an m such that all the terms of the sequence will be in the interval $[1, \mathbf{2}^{\boldsymbol{2m}}]$, that is to say **for there to be a $_{n}$ term of the Syracuse sequence, there exists a natural number m ≥ 2 / 1 ≤ U_n ≤ $\mathbf{2}^{\boldsymbol{2m}}$**.

In addition, we know by theorem that it there are a finite number of natural numbers in any given interval **So the Syracuse sequence is bounded and it has a finite number of terms**

3- Demonstrating that the Syracuse sequence inevitably converges towards a cycle

a) Let us show that:

∀ m ≥ 1, m ∈ N, ∃ I ∈ $[1, 2^{2m}]$, I odd integer / 3I+1=2^{2m}

- m=1 we have interval $[1, 2^2]$ and 1 ∈ $[1, 2^2]$ and 3x1 + 1 = 4 = 2^2therefore 1 versify our hypothesis

- m=2 we have interval $[1, 2^4]$ and 5 ∈ $[1, 2^4]$ 'a 3x5 + 1 = 16 = 2^4therefore 5 versify our hypothesis

We assume that (∀ m ≥ 1, m ∈ N), ∃ I ∈ $[1, \mathbf{2}^{\boldsymbol{2m}}]$, I odd integer / 3I+1= $\mathbf{2}^{\boldsymbol{2m}}$and let's show that it is true for (m+1)

We have ; 3I + 1 = =$\mathbf{2}^{\boldsymbol{2m}}$ ⇒ (3I + 1) x $\mathbf{2}^{\mathbf{2}}$= = $\mathbf{2}^{\boldsymbol{2m}}x\mathbf{2}^{\mathbf{2}}$ ⇒ 3x(4I) + 4 = =$\mathbf{2}^{\boldsymbol{2m+2}}$

⇒ 3x(4I) + 3+1 = =$\mathbf{2}^{\boldsymbol{2(m+1)}}$ ⇒ 3x(4I + 1)+1 = $\mathbf{2}^{\boldsymbol{2(m+1)}}$and I odd therefore 4I is even therefore (4I+1) is odd and **1 ≤ 4I +1 ≤**$\mathbf{2}^{\boldsymbol{2(m+1)}}$

SO ∃ (I'= 4I+1) ∈ $[1, \mathbf{2}^{\boldsymbol{2(m+1)}}]$ / 3I' + 1 =$\mathbf{2}^{\boldsymbol{2(m+1)}}$

Therefore, we have shown that for any integer m ≥ 1, m ∈ ℕ ,

∀ (m ≥ 1, m ∈ N) , ∃ I ∈ $[1, 2^{2m}]$,I odd integer / 3I+1=2^{2m}

b- **Example:** $U_1(1) = 16$, $U_2(3) = 16$, $U_1(5) = 16$, $U_{12}(7) = 16$, $U_{15}(9) = 16$, $U_3(10) = 16$, $U_{10}(11) = 16$, $U_5(12) = 16$, $U_5(13) = 16$, $U_{13}(14) = 16$, $U_{13}(15) = 16$

By studying all the Syracuse sequences which take their initial values in the interval [3,16] - {8,16} admits the term 16 which is the upper bound, we can therefore conclude that any Syracuse sequence which takes its initial value in the interval [1, $\mathbf{2^{2m}}$] admits the upper bound as one of these terms,

c- **Let us demonstrate that:**

∀ m ≥ 1, m ∈ N : ∀ $U_n(I)$ Syracuse sequence with I ∈ [1, 2^{2m}], there exists an index i ∈ N such that:

$U_I(I)$ odd, $U_{i+1}(I) = 3\,U_I(I) + 1 = 2^{2m}$

Proof by absurdity:

Suppose that for each Syracuse sequence *Un* (*I*) with I ∈ [1, 2^{2m}], there is no index i such that 3Ui(I)+1 reaches2^{2m}

However, we have already shown previously (see above) that: ∀ (m ≥ 1, m ∈ N) , ∃ I ∈ [1, 2^{2m}],I odd integer / 3I+1=2^{2m}

We construct the Syracuse sequence *Un* (*I*) which has as its initial value this specific odd integer that we have shown to exist,

So: $U_0(I) = I$ which is odd therefore $U_1(I) = 3\,I + 1 = 2^{2m}$

Therefore, our initial assumption that there is no index *i* such that (3 *Ui* (*I*)+1= 2^{2m}) for any Syracuse sequence *Un* (*I*) with I ∈ [1, 2^{2m}] is false therefore for all *m* ≥1, there exists an index *i* such that (3 *Ui* (*I*)+1= 2^{2m})for any Syracuse sequence *Un* (*I*) with I ∈ [1, 2^{2m}] .

d- Recapitulation

- For there to be U_n term of the Syracuse sequence, there exists a natural number m ≥ 2 / $1 \le U_n \le 2^{2m}$.
- ∀ m ≥ 1, m ∈ N : ∀ U_n (I) Syracuse sequence with I ∈ $[1, 2^{2m}]$, there exists an index i ∈ N such that: U_I (I) odd, U_{i+1} (I) = 3 U_I (I) + 1= 2^{2m}

SO; U_{i+1} (I) = 3 U_I (I) + 1= 2^{2m}

U_{i+2} (I) = = 2^{2m}/ 2 =2^{2m-1}

…………………………………

…………………………………

U_{i+2m} (I) = 2

U_{i+2m+1} = 2/2 =1

So we have demonstrated that any Syracuse sequence inevitably converges towards one (1) which will then call out the cycle (4, 2, 1).

In summary, the nature of the operations in the Syracuse sequence leads the sequence to inevitably converge towards one (1), i.e. towards the cycle (4,2,1)

VII- Conclusion

The conclusion of this article on the Syracuse - Collatz conjecture highlights the fundamental importance of the convergence of the Syracuse sequence to the cycle (4, 2, 1). This convergence not only demonstrates a recurring pattern in the behaviour of integers, but it also offers valuable insights into the nature of integers themselves. By understanding how this sequence inevitably converges to a cycle for any initial positive integer, we enrich our understanding of the underlying structures of integers and their arithmetic properties. This observation is not only a mathematical curiosity, but it reveals a profound aspect of the regularity hidden in integers. Thus, the convergence of the Syracuse sequence to the cycle (4, 2, 1) plays an essential role in the exploration and understanding of integers, opening the way to new questions and new discoveries in this fascinating field mathematics

Applications of the Syracuse-Collatz Conjecture

1. Cryptography

The Syracuse– Collatz conjecture has links to cryptography, particularly to integer factorization problems. Studying the properties of Collatz sequences can help to better understand the behaviors of prime numbers and to develop more robust encryption algorithms.

2. Algorithmic Complexity

The Syracuse- Collatz conjecture is an excellent example of a problem that is simple to state but difficult to solve. Its chaotic and unpredictable behavior makes it an interesting subject of study for the analysis of algorithmic complexity. Research into the limits of conjecture resolution can lead to advances in complexity theory.

3. Graph Theory

The Syracuse sequence can be represented as a graph, where each number is a node and the transitions between numbers are edges. Studying the properties of this graph can provide insights into connectivity, cycles, and underlying structures. Concepts from graph theory, such as trees and cycles, can be applied to the Syracuse conjecture.

4. System Dynamics

System dynamics studies the behavior of complex systems over time. The Syracuse suite is an example of a simple but nonlinear dynamical system. Its chaotic behavior and its properties of convergence towards the (4, 2, 1) cycle are subjects of interest to researchers in system dynamics.

5. Mathematics Education

The Syracuse- Collatz conjecture can be used as an educational tool to teach mathematical concepts such as recurrence, sequences, functions, proofs and logic. It can capture students' interest and encourage them to explore mathematics further.

Bibliographic references

In this section we present a list of bibliographical references and notes that were used to write this article on the Syracuse– Collatz conjecture

1. . Collatz, L. (1937). “Uber die Zuordnung eines Elements zu einer Reihe von Elementen in einer eindeutigen Abbildung " (in German). Journal für die Reine und Angewandte Mathematik . 177:193–212. doi:10.1515/crll.1937.177.141.

- Collatz, L. (1937). “Uber die vermutliche Nichtexistencenz gewisser Periodischer Prozesse zweiter Art." Proceedings of the 1st Congress of Yugoslav Mathematicians, Zagreb, pp. 47–50.

2. Erdös , P. (1965). “We have a Problem of Collatz.” Mathematika , 12(1), 269-271.

3. Lagarias , JC (2010). “The 3x+1 Problem: An Annotated Bibliography.” The Ultimate Challenge: The 3x+1 Problem, 33-48.

Lagarias , J. C. (1985). "The 3x+1 Problem and its Generalizations." The American Mathematical Monthly, 92(1), 3-23.

. Lagarias , JC (2010). "The 3x + 1 Problem: An Annotated Bibliography, II (2000-2009)" (PDF). arXiv:1001.5381 [math.HO].

4. Terras, A. (2006). "The Finite Collatz Problem." Bulletin of the American Mathematical Society, 43(1), 3-24.

Terras, R. (2001). Zeta Functions of Graphs: A Stroll through the Garden. Cambridge University Press. ISBN 978-0-521-80017-6.

5. Ulam, S. M. (1962). “We some mathematical problems connected with patterns of growth of figures." Proceedings of Symposia in Applied Mathematics, 14, 215-224.

6. Wirsching, G.R. (1992). "The dynamical system generated by the 3n+1 function." Lecture Notes in Mathematics, Springer, 1681.

7. Wirsching, G.R. (2010). "The dynamical system generated by the 3n+1 function". In: Moreira, JJ Almeida; Yokoyama, T. (eds.). Ergodic Theory, Dynamical Systems, and the Continuing Influence of John C. Oxtoby. American Mathematical Society. pp. 259–273.

8. Allouche, JP; Shallit, J. (2003). Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press. ISBN 978-0-521-82332-6.

9. . Conway, J. H. (1987). “Unpredictable Iterations”. In: Devaney, RL; Keen, L.; Ruelle, D. (eds.). Chaotic Dynamical Systems. American Mathematical Society. pp. 47–50.

10. . Tao, T. (2019). "Almost all orbits of the Collatz map attain almost bounded values." arXiv preprint arXiv:1910.09363.

11. Conway, J. H. (1972). " Fractran : a simple universal programming language for arithmetic." Proceedings of the Cambridge Philosophical Society. Cambridge University Press .

12. The "Bounding Theorem for Integers": - Rosen , Kenneth H. *Discrete Mathematics and Its Applications.* McGraw-Hill Education, 2018.

- Epp, Susanna S. *Discrete Mathematics with Applications.* Cengage Learning, 2018.

- Grimaldi, Ralph P. *Discrete and Combinatorial Mathematics: An Applied Introduction.* Pears2021

13. “Theorem that there are a finite number of natural numbers in any given interval”

1. Halmos , Paul R. *Naive Set Theory.* Springer, 1974.
2. Rudin, Walter. *Principles of Mathematical Analysis.* McGraw-Hill Education, 1976.
3. Hardy, GH, Wright, EM, Heath-Brown, DR, & Silverman, JH *An Introduction to the Theory of Numbers.* Oxford University Press, 2008.

****Notes: * ***

- Note 1: The original article by Collatz, published in German, presents for the first time the conjecture which today bears his name.

- Note 2: The annotated bibliography of Lagarias provides an excellent compilation of published work on the Syracuse - Collatz conjecture over the last decades.

- Note 3: Terras' book offers an interesting perspective on zeta functions associated with graphs, an area related to the study of the Syracuse conjecture.

- Note 4: Wirsching examines in detail the dynamical system generated by the function \ (3n + 1\), providing an in-depth analysis of the properties of the conjecture.

Shallit 's book explores automatic sequences, a related topic that has links to the Syracuse– Collatz conjecture.

- Note 6: Conway discusses unpredictable and chaotic iterations, illustrating how simple rules can give rise to complex behaviors.

- Note 7: The **framing theorem,** also known as the **police theorem,** is a fundamental result in mathematics concerning sequences. The framing theorem tells us that if a sequence ((u_n)) is “trapped” between two other sequences ((s_n)) and ((r_n)) whose limits are identical (i.e. they both converge to the same real ℓ), then the sequence ((u_n)) also converges to ℓ.

-Note 8: The theorem states that there exists a finite number of natural numbers in any given interval. Here are the key points:

- **Dirichlet's theorem:** Also called the **drawer principle,** it states that if you distribute **n+1** objects into **n** drawers, then at least one drawer will contain **at least two objects.**
- **Application to intervals:** If you consider an interval of natural numbers, there are a finite number of integers in that interval, because you can distribute them into a finite number of drawers (or smaller intervals).
- **Consequence:** This means that there is no infinite interval of natural numbers, and therefore, there is always a finite number of integers in any given interval.

These references and notes have been used to further our understanding of the Syracuse– Collatz conjecture and to ensure the accuracy and reliability of the information presented in this article.